AMR on quadtrees
CWENO-Adaptive order

Course: High order reconstructions in hyperbolic conservation and balance laws

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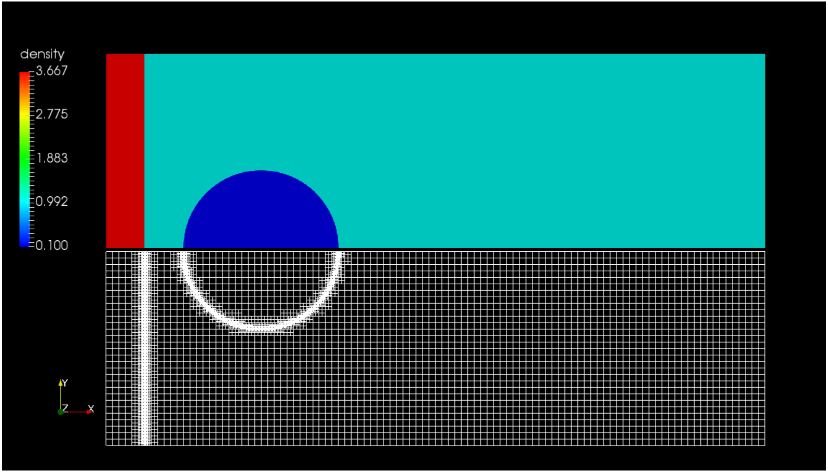
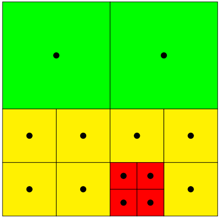
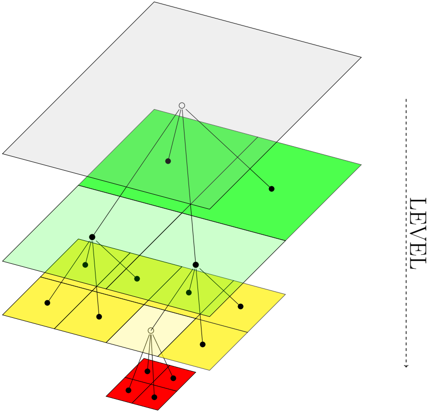
Shock-bubble interaction: initial data and mesh



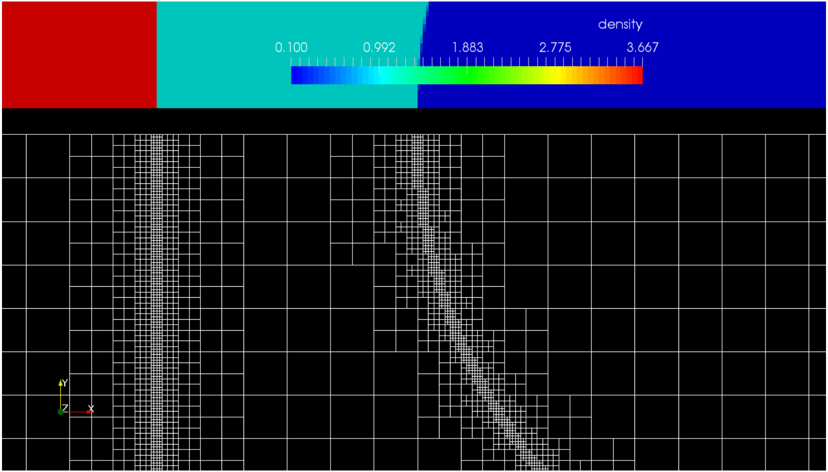
Evolution: the right-moving shock impinges on the low-density bubble and breaks it, while being deformed; weaker shocks move towards the wall, are then reflected back and interact with the flow.

Binary/quad/oct-tree grids





Shock-bubble interaction: initial data and mesh (zoom)

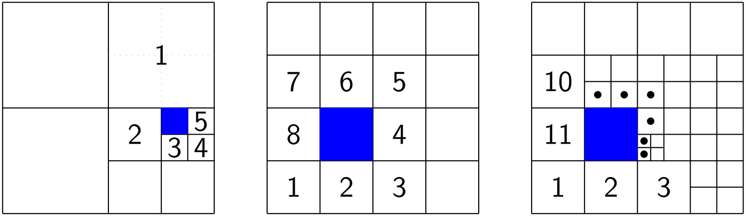


A quad-tree mesh with 4 levels of refinement.

First neighbours in quad-tree meshes (examples)

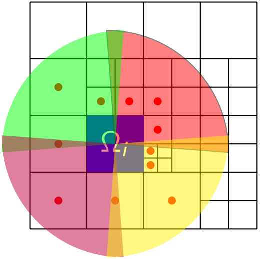
The number, relative size and relative position of neighbours

- is not fixed



- it is never less than 5
- even on uniform mesh, the number does not match the degrees of freedom of a $\mathbb{P}_2$ polynomial
- In fact typically one allows only a ratio 1/2 of nearby cells, so the number is between 6 and 12 in most quad-tree implementations

CWENO3 on a 2d quad-tree mesh



- all polynomials s.t. $\frac{1}{|\Omega_i|}\int_{\Omega_i} P = \overline{u}_i$
- stencil = first neighbours
- Compute P_{opt} of degree 2 (least squares)
- Compute $P_{NE}, P_{NW}, P_{SW}, P_{SE}$ of degree 1 (least squares)
- $d_0 = \frac{1}{2}$, $d_{NE} = d_{NW} = d_{SE} = d_{SW} = \frac{1}{8}$
- $P_0 := \frac{1}{d_0}\left(P_{\mathrm{opt}} - d_{NE}P_{NE} - d_{NW}P_{NW} - d_{SW}P_{SW} - d_{SE}P_{SE}\right)$
- Compute indicators and nonlinear weights: $d_k \rightsquigarrow \omega_k$
- $\mathcal{R} := \omega_0 P_0 + \omega_{NE}P_{NE} + \omega_{NW}P_{NW} + \omega_{SW}P_{SW} + \omega_{SE}P_{SE}$
- evaluate $\mathcal{R}$ where required by the scheme

Note: this was my first CWENO implementation ever, that motivated all the other work; if I were to do this today, I'd use CWENOZ and $d_0 = 3/4$.

M.S., A. Coco, G. Russo - J. Sci. Comput. (2016)

CWENO3 in 2d: choice of basis

$$\text{in cell } \Omega_i : \qquad P_{\text{opt}}(x,y) = \overline{u}_i + \sum_{k=1}^{5} a_{i,k}\varphi_{i,k}(x,y)$$

where

$$\begin{aligned}
\varphi_{i,1}(x,y) &= (x-x_i),\\
\varphi_{i,2}(x,y) &= (y-y_i),\\
\varphi_{i,3}(x,y) &= (x-x_i)^2 - \Delta\mathrm{x}^2/12,\\
\varphi_{i,4}(x,y) &= (y-y_i)^2 - \Delta\mathrm{y}^2/12,\\
\varphi_{i,5}(x,y) &= (x-x_i)(y-y_i),
\end{aligned}$$

so that

- $\langle \varphi_{i,k} \rangle_{\Omega_i} = 0$
- $\forall a_{i,k} : \langle P_{\text{opt}} \rangle_{\Omega_i} = \overline{u}_i$
- solve only an unconstrained $N \times 5$ least-squares problem for $a_{i,k}$, where N is the number of neighbours.

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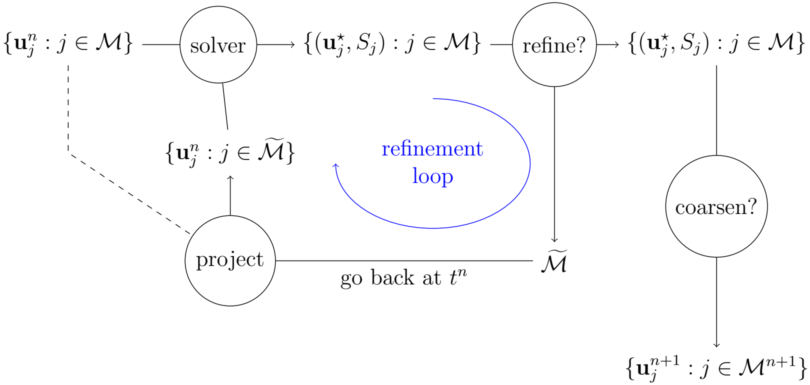
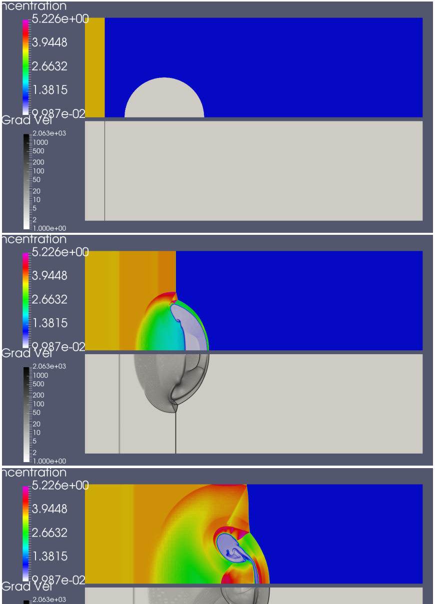
AMR loop

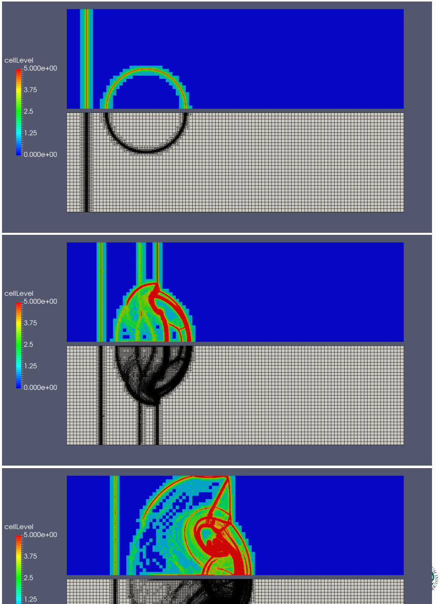


Note: after refinement, recomputation is local and conservative. Only cells in the cone of influence of newly created ones are recomputed.

Shock-bubble interaction for 2D Euler equations

coarse 30×102 mesh, $\ell = 6$ levels of refinement





So far we have seen

Notation

$$\mathrm{CWENO}(P_{\mathrm{opt}}\,;\ P_1,\dots,P_n)$$
$$\mathrm{CWENOZ}(P_{\mathrm{opt}}\,;\ P_1,\dots,P_n)$$

Examples:

CWENO 3 $\mathrm{CWENO}(P_C^{(2)}\,;\ P_L^{(1)}, P_R^{(1)})$

CWENO 5 $\mathrm{CWENO}(P_C^{(4)}\,;\ P_L^{(2)}, P_C^{(2)}, P_R^{(2)})$

CWENO 7 $\mathrm{CWENO}(P_C^{(6)}\,;\ P_{LL}^{(3)}, P_L^{(3)}, P_R^{(3)}, P_{RR}^{(3)})$

CWENO $3-2D$ $\mathrm{CWENO}((P_C^{(2)}\,;\ P_{NE}^{(1)}, P_{NW}^{(1)}, P_{SE}^{(1)}, P_{SW}^{(1)})$

CWENOZ versions of the above

CWENO $3-2D(AMR)$ $\mathrm{CWENO}(P_C^{(2)}\,;\ P_{NE}^{(1)}, P_{NW}^{(1)}, P_{SE}^{(1)}, P_{SW}^{(1)})$

Cravero, Puppo, M.S., Visconti - Math. of Comp. (2018)
Cravero, M.S., Visconti - SINUM (2019)
M.S., Coco, Russo - J. Sci. Comput. (2016)

The theory so far

Theorems: For $P_{\text{opt}}(\mathbf{x}) \in \mathbb{P}_G$ and $P_1(\mathbf{x}), \ldots, P_m(\mathbf{x}) \in \mathbb{P}_g$

…For the classical choices with $G \geq 2g$, sufficient conditions on ϵ, ℓ for CWENO and CWENOZ to achieve the optimal order $G+1$ as $\Delta\mathrm{x} \to 0$.

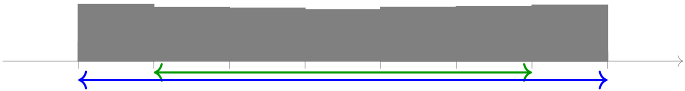
- ☺ there exists choices of $\epsilon = \Delta\mathrm{x}^{\hat{m}}$ and ℓ that satisfy the sufficient conditions

Remark. The theory really relies on the Taylor expansions of the indicators, so its is applicable also outside the context of reconstructions from cell averages; reconstructions from point values can be designed using the same theorems.

- ☹ For very large G, the stencil of the polynimials of degree $G/2$ are still quite large and it may be difficult to avoid discontinuities, expecially in multiD.

Cravero, Puppo, M.S., Visconti - Math. of Comp. (2018)
Cravero, M.S., Visconti - SINUM (2019)

Hierarchic approach



$$\text{CWENOZ}\left(\text{CWENOZ}\left(P^{(6)}; P^{(3)}_{1,2,3,4}\right); \text{CWENOZ}\left(P^{(4)}; P^{(2)}_{1,2,3}\right)\right)$$

- on smooth flows,

$$\text{CWENOZ} \approx \text{CWENOZ}\left(P^{(6)}; P^{(3)}_{1,2,3,4}\right) \approx P^{(6)}$$

- if a shock enters the stencil of $P^{(6)}$,
  until the stencil of $P^{(4)}$ is still smooth,

$$\text{CWENOZ} \approx \text{CWENOZ}\left(P^{(4)}; P^{(2)}_{1,2,3}\right) \approx P^{(4)}$$

- as the shock moves towards the central cell,

$$\text{CWENOZ} \approx \text{CWENOZ}\left(P^{(4)}; P^{(2)}_{1,2,3}\right) \approx P^{(2)}_{\star}$$

"Adaptive order" WENO

- most can be recast as hierarchic *CWENO* or hierarchic CWENOZ
- example WAO(7,5,3) by Balsara, Garain, Shu (2016)
- example Arbogast, Huang, Zhao (2018)

Common problem

Hierarchic $\Rightarrow$ multiple nonlinear weights computations

For example

$$\mathrm{CWENOZ}\left(\mathrm{CWENOZ}\left(P^{(6)}; P^{(3)}_{1,2,3,4}\right); \mathrm{CWENOZ}\left(P^{(4)}; P^{(2)}_{1,2,3}\right)\right)$$

1. compute $P^{(6)}, P^{(3)}_{1,2,3,4}, P^{(4)}, P^{(2)}_{1,2,3}$
2. compute their oscillation indicators
3. compute nonlinear weights for CWENOZ
4. compute nonlinear weights for CWENOZ
5. compute nonlinear weights for CWENOZ

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Question

Instead of

$$\mathrm{CWENOZ}\left(\mathrm{CWENOZ}\left(P^{(6)}; P^{(3)}_{1,2,3,4}\right); \mathrm{CWENOZ}\left(P^{(4)}; P^{(2)}_{1,2,3}\right)\right)$$

can't we simply compute

$$\mathrm{CWENOZ}\left(P^{(6)}; P^{(4)}, P^{(2)}_{1,2,3}\right) \quad ?$$

If it worked,
we would save a number of nonlinear weight computation!

CWENO(Z) with high degree gap

Let us consider on (or more) polynomials with very low degree:

$$\boxed{\mathrm{CWENOZ}\left(P_{\mathrm{opt}}; P_1, \ldots, P_n; Q\right) = \omega_0 P_0 + \sum_{i=1}^{n} \omega_k P_k + \boldsymbol{\omega} Q}$$

where

$$P_0 = \frac{1}{d_0}\left[P_{\mathrm{opt}} - \sum_{i=1}^{n} d_i P_i - \boldsymbol{\delta} Q\right]$$

and

$$\deg P_{\mathrm{opt}} = 2g \qquad \deg P_k \geq g \qquad \deg Q = \boldsymbol{\gamma} < g$$

Convergence problem

- sufficient conditions of the theorem not satisfied
- convergence rate degrades also in practice

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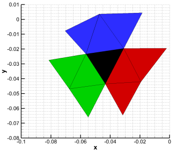
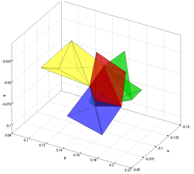
Examples CWENO(Z) with high degree gap

`Dumbser, Boscheri, M.S., Russo - J. Sci. Comput. (2017)`

CWENO on simplices with $P^{(G)}$ and $P^{(1)}$'s which needs $d_0 = 0.999$ for large n





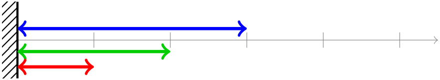
`Zhang, Qiu et al. (2016-18)`

WENO-ZQ's also employ a very large d_0 (with Z-weights)

`Naumann, Kolb, M.S. - J. Appl. Math. & Comput. (2018)`

CWENO($P_R^{(2)}$; $P_R^{(1)}$; $P^{(0)}$) with optimal weights

$0.75 - \Delta \mathrm{x}, 0.25, \Delta \mathrm{x}$



Convergence of CWENO(Z) with high degree gap

If $\deg P_{\text{opt}} = 2g$, $\deg P_k \geq g$, but $\deg Q = \gamma < g$:

$$u(\mathbf{x}) - R(\mathbf{x}) =$$

$$\underbrace{(u(\mathbf{x}) - P_{\text{opt}}(\mathbf{x}))}_{\mathcal{O}(\Delta x^{2g+1})} + (d_0 - \omega_0)\underbrace{(P_0(\mathbf{x}) - u(\mathbf{x}))}_{\mathcal{O}(\Delta x^{\gamma+1})}$$

$$+ \sum_{k=1}^{n} (d_k - \omega_k)\underbrace{(P_k(\mathbf{x}) - u(\mathbf{x}))}_{\mathcal{O}(\Delta x^{g+1})} + (d - \omega)\underbrace{(Q(\mathbf{x}) - u(\mathbf{x}))}_{\mathcal{O}(\Delta x^{\gamma+1})}$$

Accuracy on smooth data depends on

$$d_k - \omega_k = \mathcal{O}(\Delta x^{2g-g}) \qquad d - \omega = \mathcal{O}(\Delta x^{2g-\gamma})$$

Theorem

Optimal convergence can be achieved if $d = \mathcal{O}(\Delta x^{g-\gamma})$.

New proposals for CWENO(Z) with high degree gap

- theorem with sufficient conditions on the reconstruction parameters that guarantee the convergence, for general choice of degrees
- detailed study of

$$\boxed{\mathrm{CWZ753} = \mathrm{CWENOZ}\left(P_C^{(6)}; P_C^{(4)}; P_L^{(2)}, P_C^{(2)}, P_R^{(2)}\right)}$$

which is shown to have similar (slightly better) performance of WAO(7,5,3) or WENO by Arbogast
with 10% less CPU time

M.S., Visconti - J. Sci. Comp. (2020)

Convergence of CWZ753 – theory

$$\tau := \left|\mathsf{OSC}[P_6] - \mathsf{OSC}[P_4]\right| = \mathcal{O}\left(\Delta\mathrm{x}^6\right)$$

$\hat{m}$	Summary
1	$\ell \geq 1, r \geq 1$
2	$\ell \geq 1, r \geq 1$
3	$\ell \geq 1, r \geq 1$
4	$\ell \geq 1, r \geq 1$
5	$\ell \geq 1, r \geq 1$
6	$\ell \geq 2, r \geq 1$ or $\ell \geq 1, r \geq 2$
7	$\ell \geq 3, r \geq 1$ or $\ell \geq 2, r \geq 2$

power exponent

$$\alpha_\star = d_\star \left(1 + \left(\frac{\tau}{\mathsf{OSC}[P_\star]+\epsilon}\right)^{\boldsymbol{\ell}}\right)$$

for $\star \in \{6, 4, 2L, 2C, 2R\}$

size of ϵ

$$\boldsymbol{\epsilon} = \Delta\mathrm{x}^{\hat{\boldsymbol{m}}}$$

linear coefficients of $P^{(2)}_{L,C,R}$

$$d_{2L} = d_{2C} = d_{2R} = \Delta\mathrm{x}^{\boldsymbol{r}}$$

General guidelines: use the smallest possibile ℓ and ϵ
(but watch out the machine precision when choosing $\hat{m}$!)

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Convergence of CWZ753 – experiments

Parameter set that does not satisfy the conditions

$$\hat{m} = 6, \ell = 1, r = 1$$

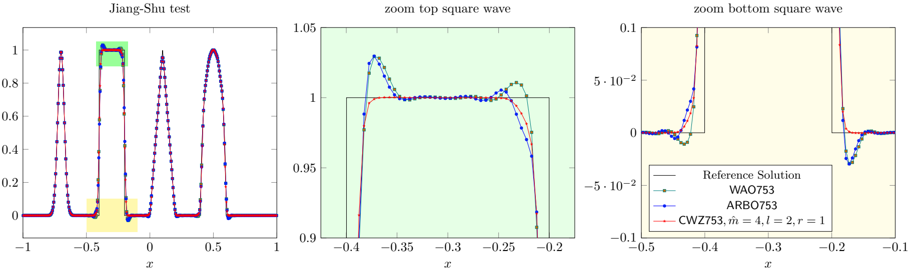
Δx	error	rate
0.1	$8.14 \cdot 10^{-3}$	–
$5.00 \cdot 10^{-2}$	$4.88 \cdot 10^{-4}$	4.06
$2.50 \cdot 10^{-2}$	$1.76 \cdot 10^{-5}$	4.79
$1.25 \cdot 10^{-2}$	$5.59 \cdot 10^{-7}$	4.98
$6.25 \cdot 10^{-3}$	$1.10 \cdot 10^{-8}$	5.67
$3.13 \cdot 10^{-3}$	$1.71 \cdot 10^{-10}$	6.00
$1.56 \cdot 10^{-3}$	$2.67 \cdot 10^{-12}$	6.00
$7.81 \cdot 10^{-4}$	$4.18 \cdot 10^{-14}$	6.00

Parameter set that satisfies the conditions

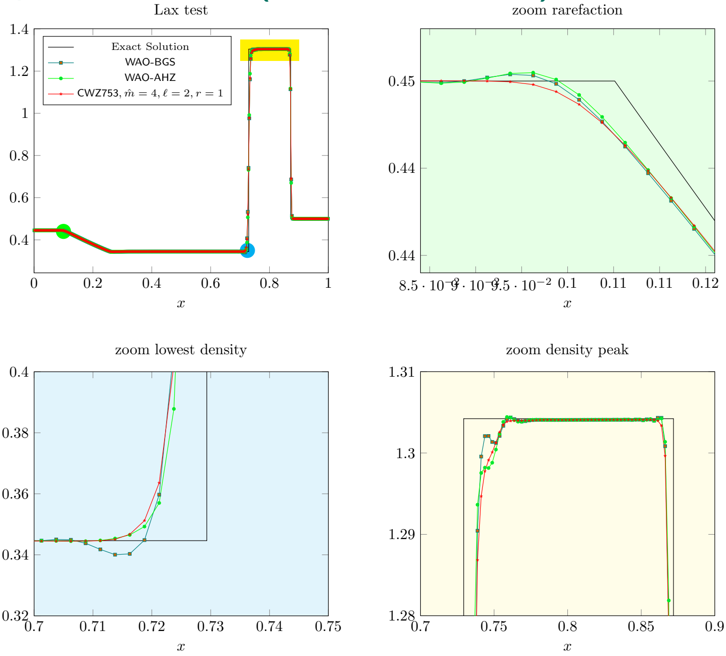
$$\hat{m} = 6, \ell = 1, r = 2$$

Δx	error	rate
0.1	$8.14 \cdot 10^{-3}$	–
$5.00 \cdot 10^{-2}$	$1.54 \cdot 10^{-4}$	5.72
$2.50 \cdot 10^{-2}$	$1.21 \cdot 10^{-6}$	6.99
$1.25 \cdot 10^{-2}$	$9.17 \cdot 10^{-9}$	7.05
$6.25 \cdot 10^{-3}$	$7.09 \cdot 10^{-11}$	7.02
$3.13 \cdot 10^{-3}$	$5.52 \cdot 10^{-13}$	7.00
$1.56 \cdot 10^{-3}$	$4.31 \cdot 10^{-15}$	7.00
$7.81 \cdot 10^{-4}$	$3.37 \cdot 10^{-17}$	7.00

Linear transport (Jiang-Shu initial data)



Euler gas dynamics (Lax shock tube)



CPU times

Jiang-Shu test.

	Core i7-6600U @ 2.60GHz			Core i3-2100T @ 2.50GHz		
Cells	CWZ753	WAO753	ARBO753	CWZ753	WAO753	ARBO753
200	9.987 s	+7.87%	+11.15%	14.39 s	+9.95%	+13.00%
400	38.45 s	+9.25%	+11.93%	57.16 s	+10.07%	+12.84%
800	153 s	+8.93%	+11.97%	229.2 s	+9.80%	+12.28%

Shu-Osher test with reconstruction along conservative variables

	Core i7-6600U @ 2.60GHz			Core i3-2100T @ 2.50GHz		
Cells	CWZ753	WAO753	ARBO753	CWZ753	WAO753	ARBO753
200	3.06 s	+10.30%	+17.29%	4.094 s	+11.13%	+18.70%
400	12.42 s	+10.52%	+17.75%	16.54 s	+11.27%	+18.76%
800	49.09 s	+9.89%	+15.64%	66.79 s	+10.22%	+17.56%

Lax test with reconstruction along characteristic variables

	Core i7-6600U @ 2.60GHz			Core i3-2100T @ 2.50GHz		
Cells	CWZ753	WAO753	ARBO753	CWZ753	WAO753	ARBO753
200	3.108 s	+10.15%	+16.04%	10.82 s	+9.61%	+11.67%
400	12.11 s	+13.81%	+15.31%	43 s	+9.00%	+10.32%
800	47.92 s	+13.16%	+19.70%	172.2 s	+9.22%	+9.93%

Thank you for your kind attention!

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