CWENO-boundary and sundry applications

Course: High order reconstructions in hyperbolic conservation and balance laws

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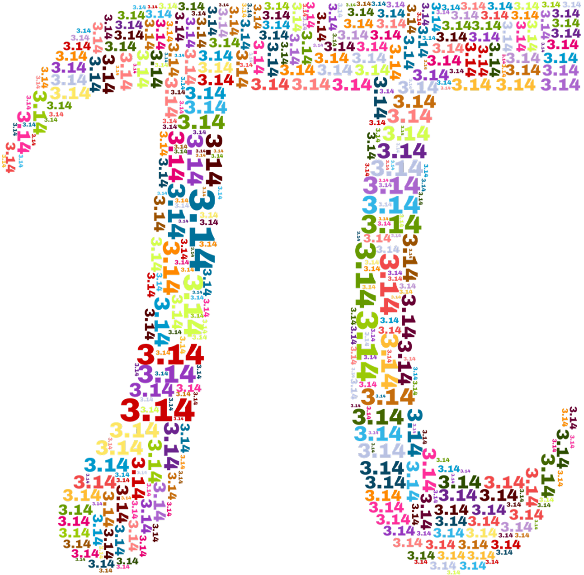
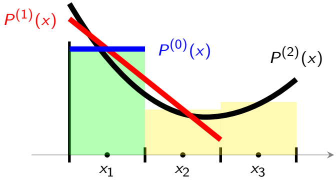
Happy π day!



CWENOZ-boundary: motivation



The no-ghost reconstruction of Naumann, Kolb, M.S. (2018)

$$\mathrm{CWENO}(P^{(2)}; P^{(1)}; P^{(0)}) \qquad \text{with } \delta = \Delta \mathrm{x}^2$$

converges at the correct accuracy (for small $\Delta \mathrm{x}$), but:

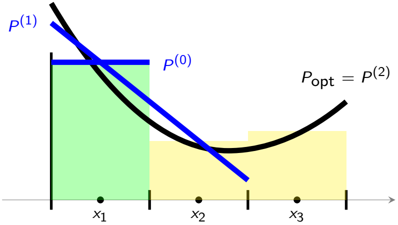
- it has rather low accuracy on coarse meshes
- caused by un-necessarily large $P^{(0)}$ non-linear coefficient on smooth flows
- this can likely be cured by using CWENOZ instead
- the only issue is how to define τ for the boundary cell

Borrowed τ for the boundary cell

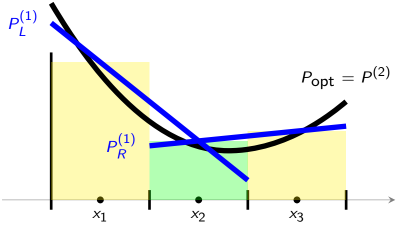
For CWENOZ3 we have that
$\tau = \|2\,\mathrm{OSC}[P_{\mathrm{opt}}] - \mathrm{OSC}[P_L] - \mathrm{OSC}[P_R]\| = \mathcal{O}(\Delta\mathrm{x}^4)$.

On the first computational cell

- Search for $\tau_1 = \lambda_{opt} OSC[P_{\mathrm{opt}}] + \lambda_1 OSC[P^{(1)}]$ with λ's chosen such that $\tau_1 = \mathrm{o}(\Delta\mathrm{x}^r)$ for large r on smooth data.
- Any linear combinaction of $OSC[P_{\mathrm{opt}}]$ and $OSC[P^{(1)}]$ is $\mathcal{O}(\Delta\mathrm{x}^3)$,
- however, τ small $\Leftrightarrow$ we should use P_{opt}.



just use $\tau_1 = \tau_2$!



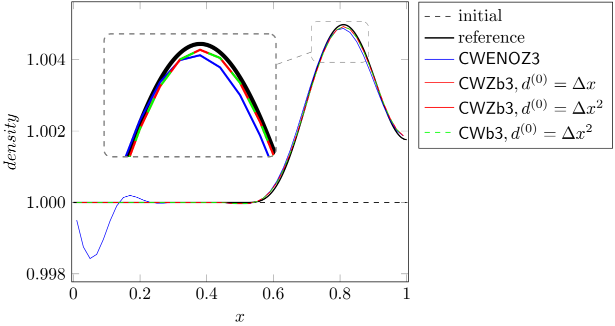
$\tau_2 = \mathcal{O}(\Delta\mathrm{x}^4)$ on smooth flows

M.S., Travaglia, Puppo - Commun. Appl. Math. Comput. (2023)

Smooth solution of Euler gasdynamics

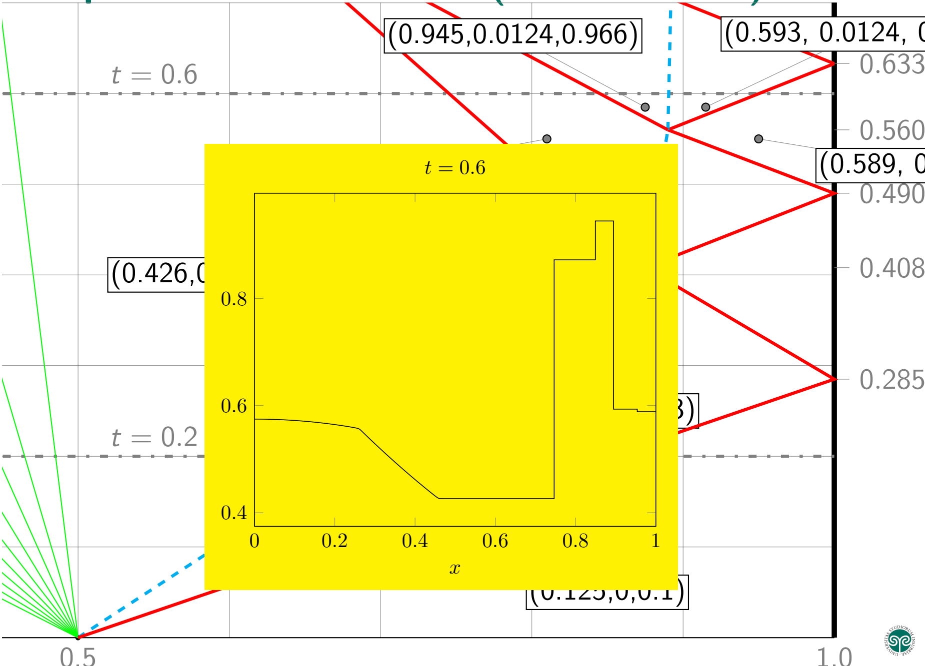
Gas initially at rest with $\rho = 1, p = 1, v = 0$.
Time-dependent Dirichlet boundary condition on the left

$$\rho(t,0) = 1.0+\delta(t) \quad p(t,0) = 1.0+\gamma\delta(t) \quad \delta(t) = \begin{cases} 0.01(\sin(2\pi t))^3 & , t \in [\\ 0 & , t > \end{cases}$$

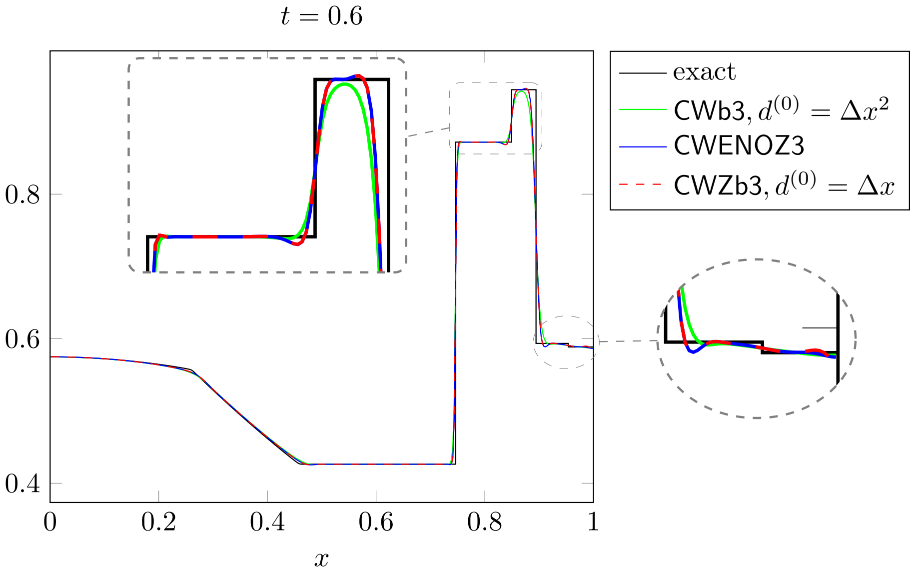


No spurious wave reflection at Dirichlet boundary is observed with no-ghost reconstructions.

Sod problem with wall b.c. (exact solution)

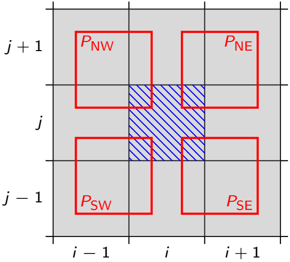


Sod problem with wall b.c. (numerical solution)



CWENOZ3 in two space dimensions

$$\mathrm{CWENOZ}(P_{\mathrm{opt}}; P_{\mathrm{NE}}, P_{\mathrm{SE}}, P_{\mathrm{SW}}, P_{\mathrm{NW}}) = \sum_{k=0}^{4} \omega_k P_k$$



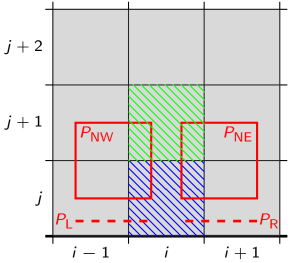
- $P_{\mathrm{opt}}(x) \in \mathbb{P}_2(x, y)$ on the central 3×3 stencil
- $P_{\mathrm{NE}}, P_{\mathrm{SE}}, P_{\mathrm{SW}}, P_{\mathrm{NW}} \in \mathbb{P}_1(x, y)$ on the 2×2 sub-stencil
- $d_0 = \frac{3}{4}$ and $d_{1,\dots,4} = \frac{1}{16}$

Global smoothness indicator: optimal choice

$$\tau = |4\mathrm{OSC}[P_{\mathrm{opt}}] - \mathrm{OSC}[P_{\mathrm{NE}}] - \mathrm{OSC}[P_{\mathrm{SE}}] - \mathrm{OSC}[P_{\mathrm{SW}}] - \mathrm{OSC}[P_{\mathrm{NW}}]|$$

Cravero, M.S., Visconti - SINUM (2019)

Boundary reconstruction



- $P_{\mathrm{opt}} \in \mathbb{P}_2(x,y)$ on the 3×3 stencil
- $P_{\mathrm{NE}}, P_{\mathrm{NW}} \in \mathbb{P}_1(x,y)$ on the 2×2 sub-stencil
- $P_{\mathrm{R}}, P_{\mathrm{L}} \in \mathbb{P}_1(x)$

Using $d_0 = \frac{3}{4}$ and $d_{\mathrm{NE}} = d_{\mathrm{NW}} = d_{\mathrm{R}} = d_{\mathrm{L}} = \frac{1}{16}$,

$$\mathrm{CWENOZ}(P_{\mathrm{opt}}; P_{\mathrm{NE}}, P_{\mathrm{SE}}, P_{\mathrm{SW}}, P_{\mathrm{NW}})$$

with τ borrowed from the cell above (green in the picture)

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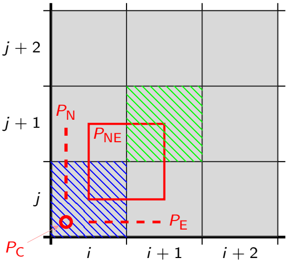
Corner recontruction



- $P_{opt} \in \mathbb{P}_2(x, y)$ on the 3×3 stencil
- $P_{NE} \in \mathbb{P}_1(x, y)$ on the 2×2 sub-stencil
- $P_E \in \mathbb{P}_1(x)$
- $P_N \in \mathbb{P}_1(y)$
- P_C constant polynomial

Using $d_{NE} = \frac{1}{16}, d_E = d_N = d_C = \Delta x^2, d_0 = 1 - \sum_{i=1}^{4} d_i$ the reconstruction is

$$\mathrm{CWENOZ\,AO}(P_{opt}; P_{NE}; P_E, P_N\ ;\ P_C)$$

with τ borrowed from the north-east neighbour (green in picture).

 M.S., Travaglia, Puppo - Commun. Appl. Math. Comput. (2023)

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Isentropic vortex test

(Silly) robustness test, implementing periodic b.c. without using the periodic data for the reconstruction.

	CWENOZ				CWZb			
N	density	rate	energy	rate	density	rate	energy	rate
50	0.33	–	1.83	–	0.3	–	1.71	–
100	$6.41e{-2}$	2.36	0.31	2.57	$6.21e{-2}$	2.29	0.3	2.52
200	$9.03e{-3}$	2.83	$4.24e{-2}$	2.86	$8.89e{-3}$	2.80	$4.17e{-2}$	2.83
400	$1.15e{-3}$	2.97	$5.39e{-3}$	2.97	$1.14e{-3}$	2.96	$5.37e{-3}$	2.96
800	$1.44e{-4}$	3.00	$6.82e{-4}$	2.98	$1.44e{-4}$	2.99	$6.84e{-4}$	2.97
1,600	$1.80e{-5}$	3.00	$9.12e{-5}$	2.90	$1.81e{-5}$	3.00	$9.15e{-5}$	2.90

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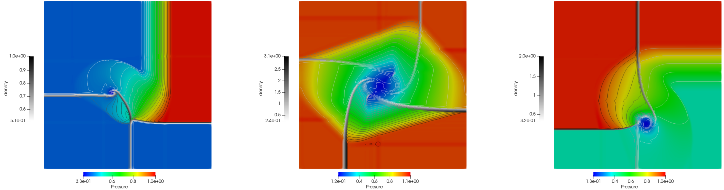
2D Riemann Problems

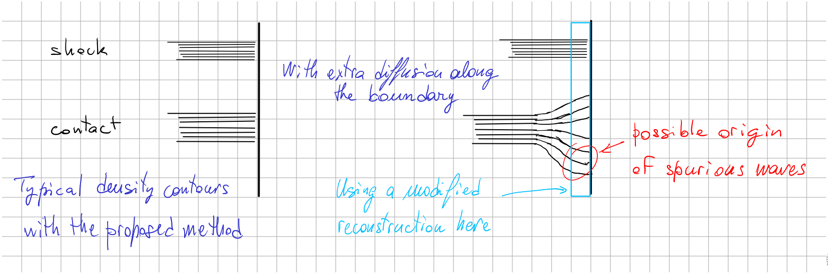
- free-flow b.c. and initial data in 4 quadrants:

U_{NW}	U_{NE}
U_{SW}	U_{SE}

- typically involve waves orthogonal to the boundary

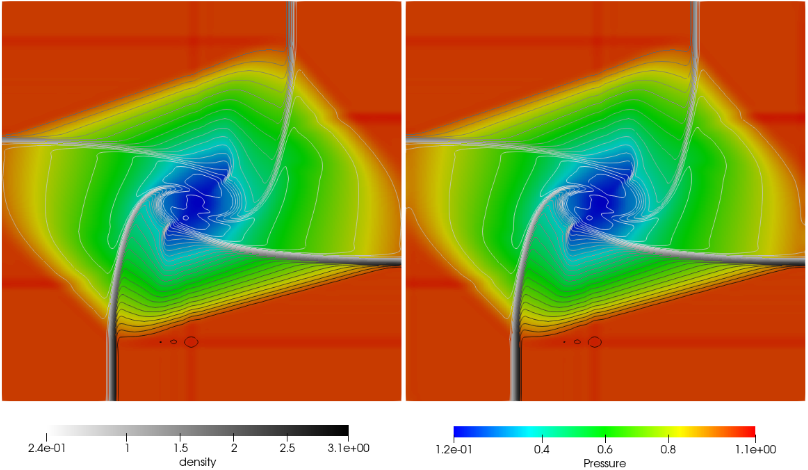


- must to control the numerical diffusion at/along the boundary

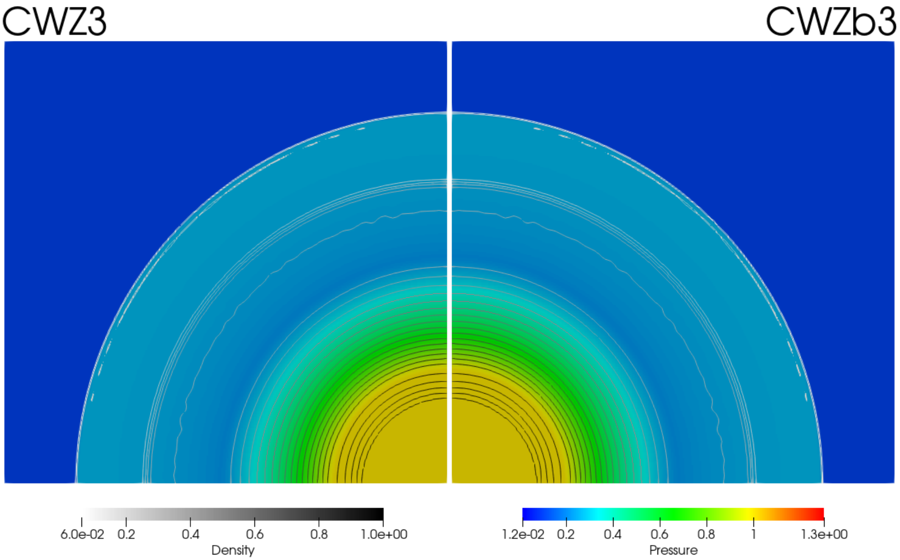


Riemann Problem with 4 contacts

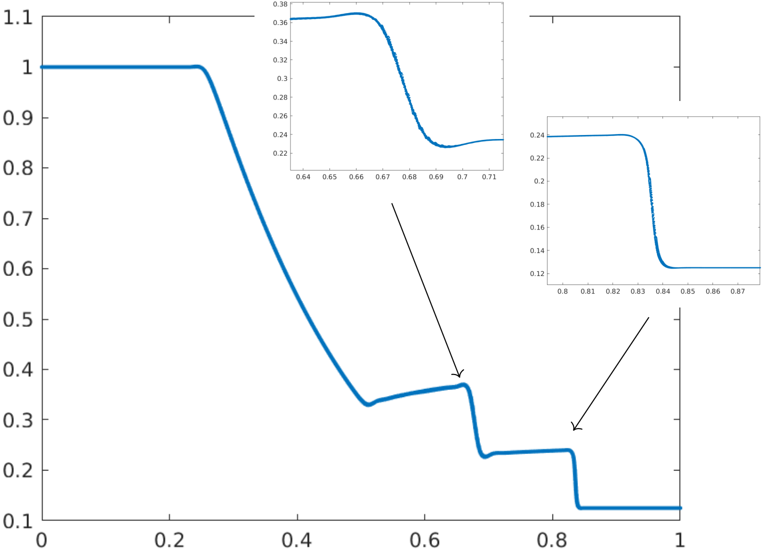
CWZ3 CWZb3



Radial Sod problem at t=0.2

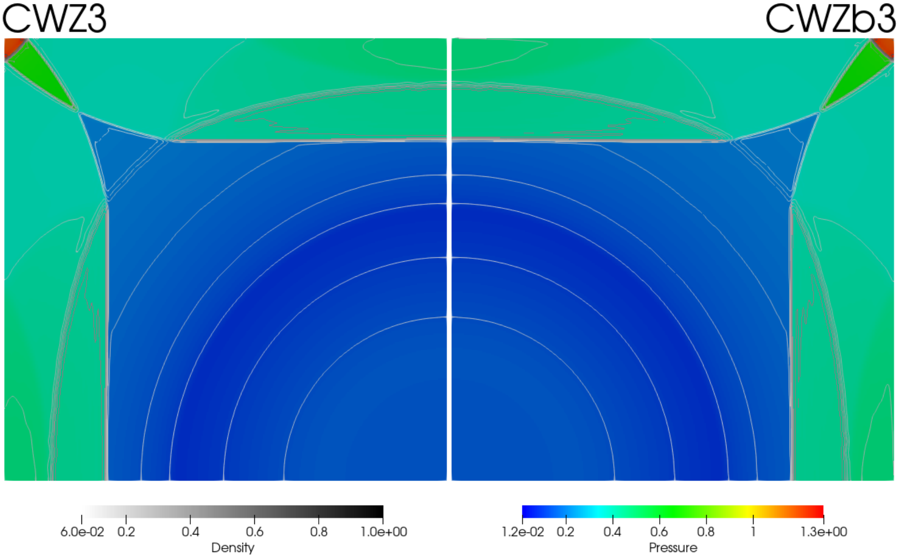


Radial Sod problem at t=0.2: $(\|\mathbf{x}_i\|_2, \overline{\rho}_i)\ \forall i \in$ grid



Radial Sod problem at t=0.6

Using wall b.c.



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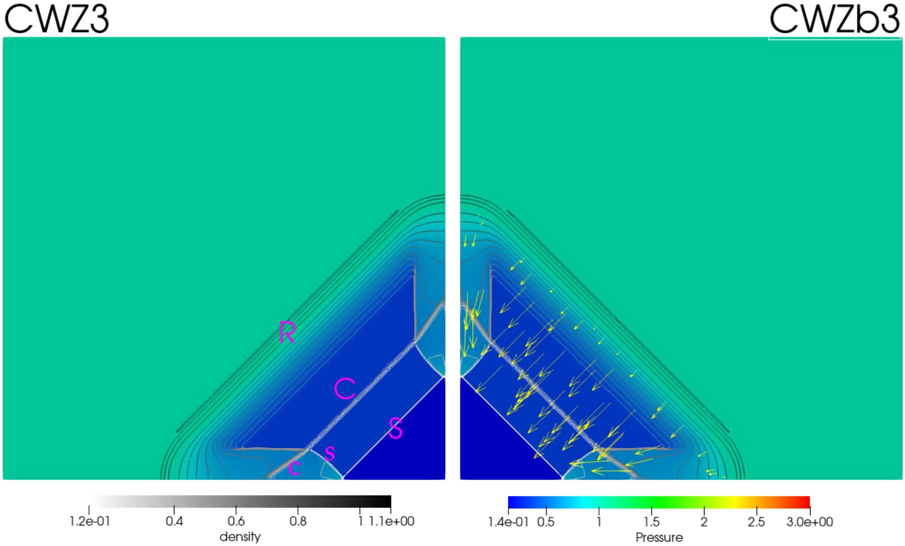
Implosion problem (t=0.03)



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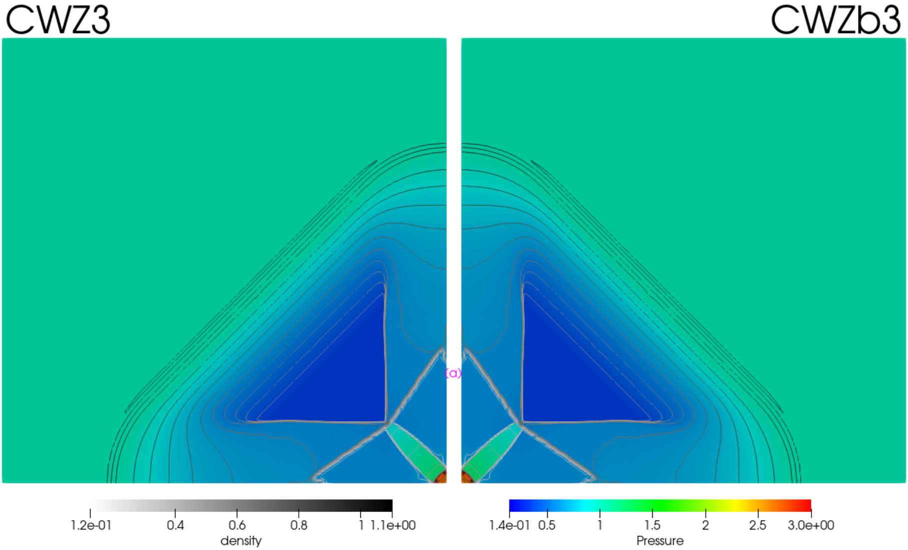
Implosion problem (t=0.06)



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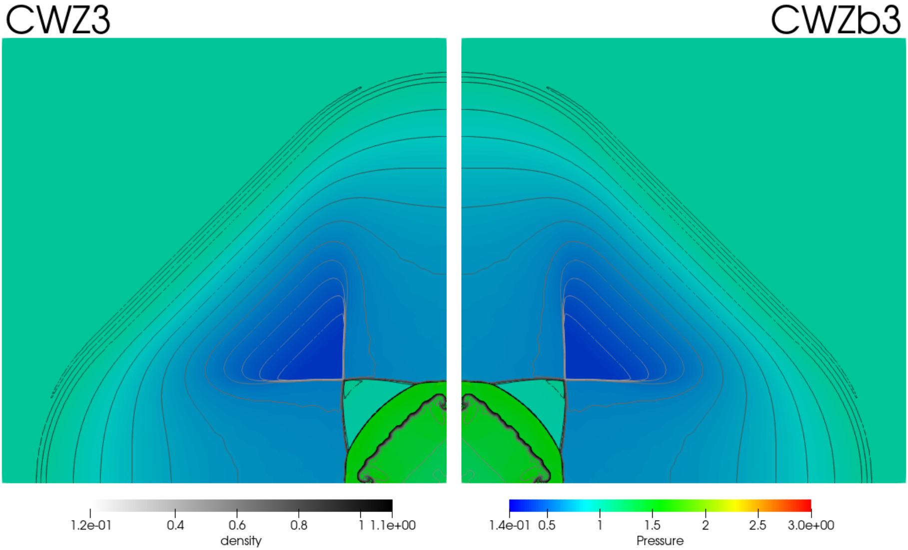
Implosion problem (t=0.1)



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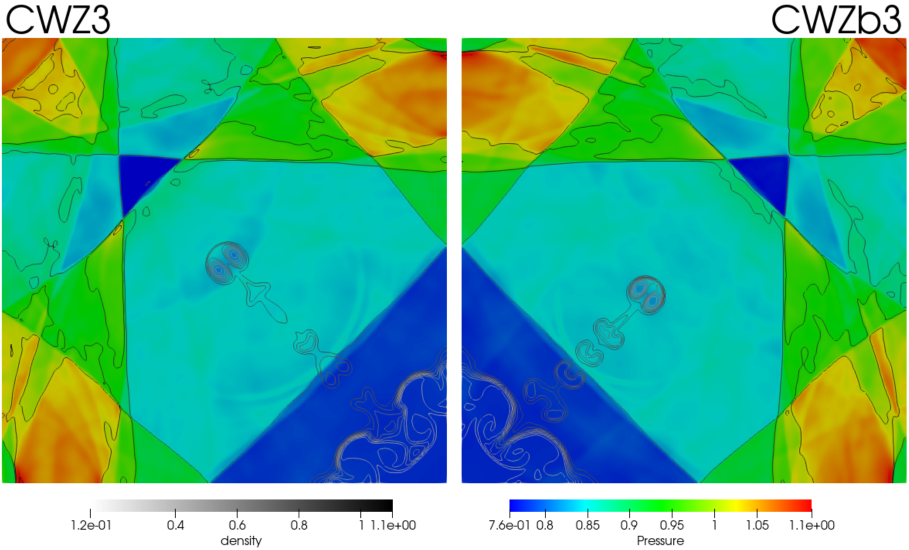
Implosion problem (t=2.5)



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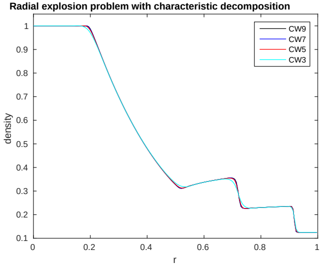
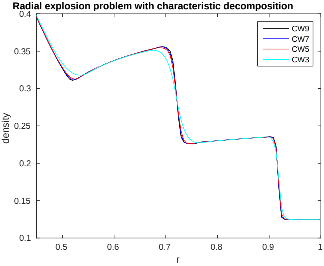
Source terms

Euler equations in $\mathbb{R}^n$ for radially symmetric data:

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} + \partial_r \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{pmatrix} = -\frac{n-1}{r} \begin{pmatrix} \rho u \\ \rho u^2 \\ up \end{pmatrix}.$$

Source term integrated with Gaussian quadrature
(1 to 4 internal nodes), up to order 9:





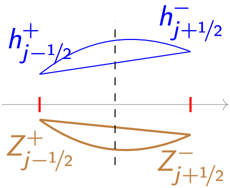
Cravero, Puppo, M.S., Visconti - Math. of Comp. (2018)

High order well-balanced on lake at rest

Well balanced second order with hydrostatic reconstruction

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{U}_j = -\frac{F^{\mathrm{hyd},-}_{j+1/2} - F^{\mathrm{hyd},+}_{j-1/2}}{\Delta x} + \underbrace{-g\frac{h^-_{1+1/2} + h^+_{1+1/2}}{2}\frac{z^-_{1+1/2} - z^+_{1+1/2}}{\Delta x}}_{\mathcal{Q}^{\mathrm{h.o.}}_j}$$

Use higher order reconstructions,
virtually split the cell into 2 pieces and



- $\mathcal{Q}^{\mathrm{h.o.},(2)}_j = \mathcal{Q}^{\mathrm{h.o.}}_{j,L} + \mathcal{Q}^{\mathrm{h.o.}}_{j,R}$
  is also well balanced
  (no jump at middle & fluxes telescope)

$\Rightarrow$ is well-balanced also the 4^{th} order accurate

$$\frac{4\mathcal{Q}^{\mathrm{h.o.},(2)}_j - \mathcal{Q}^{\mathrm{h.o.}}_j}{3}$$

- Romberg's trick gives well-balanced quadratures of any order

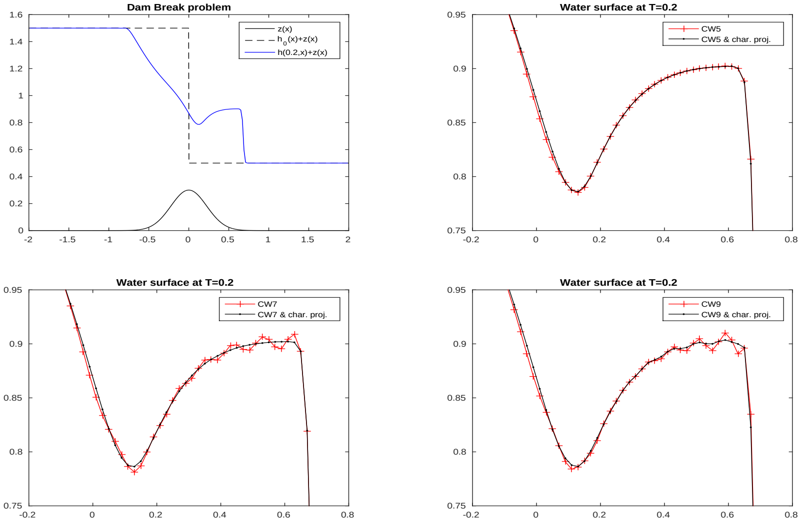
Audusse et al. - JCP (2004)
Noelle, Pankratz, Puppo, Natvig - JCP(2006)

Well-balanced hydrostatic Romberg quadratures

order	formula	nodes	
2	$\mathcal{Q}_j^{\text{h.o.},(1)}$	2	$-\frac{1}{2}, \frac{1}{2}$
4	$\dfrac{4\mathcal{Q}_j^{\text{h.o.},(2)} - \mathcal{Q}_j^{\text{h.o.},(1)}}{3}$	3	$-\frac{1}{2}, 0, \frac{1}{2}$
6	$\dfrac{64\mathcal{Q}_j^{\text{h.o.},(4)} - 20\mathcal{Q}_j^{\text{h.o.},(2)} + \mathcal{Q}_j^{\text{h.o.},(1)}}{45}$	5	$-\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$
8	$\dfrac{4096\mathcal{Q}_j^{\text{h.o.},(8)} - 1344\mathcal{Q}_j^{\text{h.o.},(4)} + \ldots - \mathcal{Q}_j^{\text{h.o.},(1)}}{2835}$	9	$-\frac{1}{2}, -\frac{3}{8}, \ldots, \frac{1}{2}$
10	$\ldots$	17	$-\frac{1}{2}, -\frac{7}{16}, \ldots, \frac{1}{2}$
$\vdots$			

Because of the many inner reconstruction points, Central WENO reconsructions are advantageous.

The schemes at work on a wet dam break



Cravero, Puppo, M.S., Visconti - Math. of Comp. (2018)

Well-balancing schemes for Euler&gravity

Framework

As in hydrostatic reconstruction, reconstruct the fluctuations around a (known) steady state. Suppose that, given $\overline{U}_j$, we know the steady state $u(x)$ that it could come from; let $\widehat{\mathcal{R}}$ be the reconstruction of the data $\{\overline{U}_{j+\alpha} - \langle u(x) \rangle_{j+\alpha}\}$ and set $\mathcal{R} = \widehat{\mathcal{R}} + u(x)$.

Problem 1

In Euler&gravity, the steady states are known via their $\rho(x)$ and $p(x)$. At best, from $\overline{U}_j$ we know the cell averages $\overline{\rho}_{j+\alpha}$ and $\overline{p}_{j+\alpha}$ of an equilibrium around the cell.

Solution !?

We need to compute $\overline{p}_j$ from $(\overline{\rho}_j, \overline{\rho u}_j, \overline{E}_j)$

Problem 2

We need to do so with high order accuracy.

Well-balanced schemes for Euler+gravity

- compute an accurate reconstruction $\widehat{\mathcal{R}}_\rho(x)$ of the fluctuations of ρ and then $\mathcal{R}_\rho(x) = \rho_{\mathsf{eq}}(x) + \widehat{\mathcal{R}}_\rho(x)$.
- reconstruct $\mathcal{R}_{\rho u}(x)$ and $\widehat{\mathcal{R}}_E(x)$ from the cell averages
- compute $\overline{p}_j = \left\langle p_j(x) \right\rangle_j = \left\langle (\gamma-1)\left(\widehat{\mathcal{R}}_E(x) - \frac{1}{2}\mathcal{R}_\rho(x)\mathcal{R}_{\rho u}(x)\right)\right\rangle_j$

  Note: one really just needs the p values at quadrature nodes, so this could be extended to more complex consitutive laws.
- reconstruct fluctuations $\mathcal{R}_p(x)$ of pressure
- set $\mathcal{R}_E(x) = \frac{1}{2}\mathcal{R}_\rho(x)\mathcal{R}_{\rho u}(x) + \frac{\mathcal{R}_p(x)}{\gamma-1}$

In the paper, this is done in 2D, for isentropic and isothermal atmospheres, Cartesian grids; also some equilibria with constact horizontal wind are preserved

Klingenberg, Puppo, M.S. - SIAM J. Sci. Comput. (2019)

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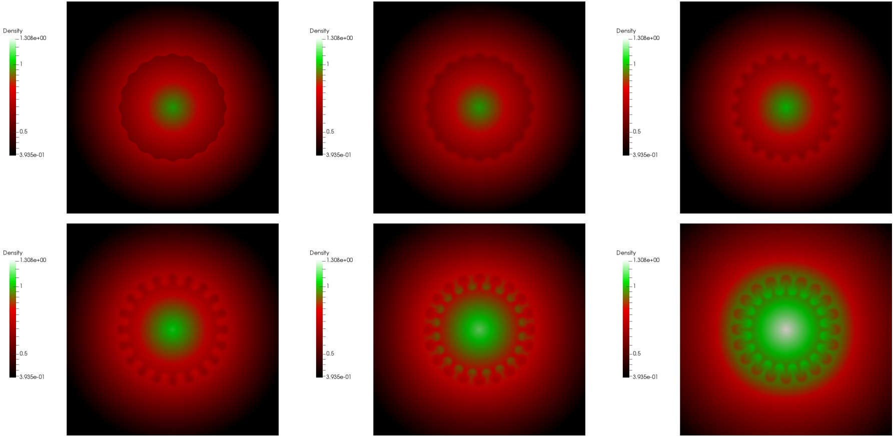
Radial Rayleigh-Taylor instability

On a 400×400 Cartesian grid



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List of my CWENO papers

- M.S., Coco, Russo - J. Sci. Comput. (2016) CWENO3 on quad-trees
- Dumbser, Boscheri, M.S., Russo - J. Sci. Comput. (2017) CWENO reconstructions for high order ADER schemes
- Cravero, Puppo, M.S., Visconti - Math. of Comp. (2018): accuracy of CWENO reconstructions; 1d examples, including Romberg quadrature for shallow water
- Castro, M.S.- Int. J. Numer. Meth. Fluids (2018) CWENO3 and CWENO4 on 2d Cartesian meshes for shallow water
- Naumann, Kolb, M.S. - J. Appl. Math. & Comput. (2018) CWENO3 for boundary cells; 1d
- M.S., Loubére - JCP (2018) comparison with MOOD approach on AMR schemes
- Cravero, M.S., Visconti - SINUM (2019): accuracy of CWENOZ reconstructions; 1 and 2d examples
- Boscheri, M.S., Dumbser - Comm. Comput. Phys. (2019) CWENO schemes as FV limiter for DG
- Klingenberg, Puppo, M.S. - SIAM J. Sci. Comput. (2019) CWENO in well-balanced Euler&gravity
- M.S., Visconti - J. Sci. Comp. (2020) CWENO with “high degree gap”
- M.S., Travaglia, Puppo - Commun. Appl. Math. Comput. (2023) CWENOZ3 for boundary cells; 1 and 2d

Thank you for your kind attention!

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