CWENO

Course: High order reconstructions in hyperbolic conservation and balance laws

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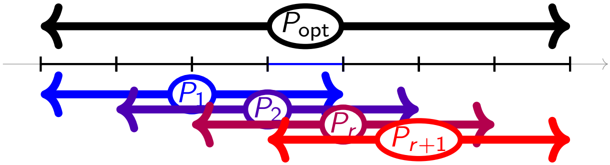
Essentially non-oscillatory reconstructions

Given the cell averages $\overline{u}_{j-r}, \dots, \overline{u}_{j+r}$ of a bounded function $u(x)$,



P_{opt} s.t. $\forall i = -r, \dots, r$:
$$\frac{1}{|\Omega_{j+i}|}\int_{\Omega_{j+i}} P_{\text{opt}}(x)\mathrm{d}x = \overline{u}_{j+i}$$

1. has accuracy $\mathcal{O}(h^{2r+1})$ in smooth regions
2. is however oscillatory if a discontinuity is present in its stencil
3. is best replaced by a (lower accuracy) non-oscillatory alternative, e.g. one of the P_k's

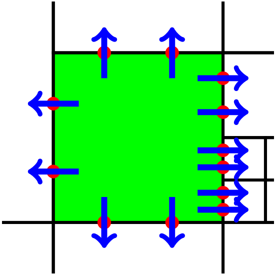
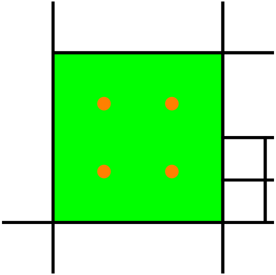
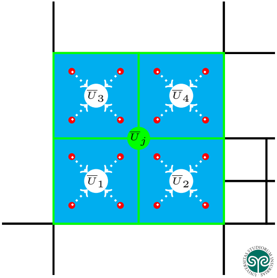
Requirements for the reconstruction procedure

For a FV scheme in 2(3)D, with AMR, source terms, ...
the reconstruction must be:

- high-order accurate and non-oscillatory

For high order finite volume methods, it should also be efficient at:

- reconstructing point values at many locations on $\partial\Omega_j$
  Mesh topology ($\Rightarrow$ quadrature nodes) is changing in time
- reconstructing point values at locations inside Ω_j
  for source terms or for numerical entropy indicator
- computing sub-cell averages in refinement or moving mesh schemes

Linear coefficients WENO3 and WENO5

WENO3 on the boundary of the cell

$$d_L(x_{j+1/2}) = \frac{1}{3} \quad d_R(x_{j+1/2}) = \frac{2}{3} \qquad d_L(x_{j-1/2}) = \frac{2}{3} \quad d_R(x_{j-1/2}) = \frac{1}{3}$$

and on nonuniform meshes:

$$d_L(x_{j+1/2}) = \frac{h_{j-1}}{h_{j-1}+h_j+h_{j+1}}, \; d_R(x_{j+1/2}) = \frac{h_j+h_{j+1}}{h_{j-1}+h_j+h_{j+1}} \dots$$

WENO5 on the right boundary of the cell

$$d_{-1} = \frac{h_1(h_1+h_2)}{(h_{-2}+\ldots+h_2)(h_{-2}+\ldots+h_1)} > 0$$

$$d_0 = \frac{(h_{-2}+h_{-1}+h_0)(h_1+h_2)(h_{-2}+2h_{-1}+2h_0+2h_1+h_2)}{(h_{-2}+\ldots+h_2)(h_{-1}+\ldots+h_2)(h_{-2}+\ldots+h_1)} > 0$$

$$d_1 = \frac{(h_{-2}+h_{-1}+h_0)(h_{-1}+h_0)}{(h_{-2}+\ldots+h_2)(h_{-1}+\ldots+h_2)} > 0$$

WENO5 at cell center

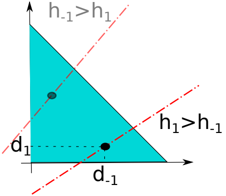
For the cell center,

$$d_{-1} = -\frac{9}{80} \; d_0 = \quad \frac{49}{40} \; d_1 = -\frac{9}{80}$$

and must treat them with the special technique for negative weights

Or reduce to 4-th order and find that:

$$d_{-1}(h_{-2} + \ldots + h_1) - d_1(h_{-1} + \ldots + h_2) = h_1 - h_{-1}$$



(d_{-1}, d_1) must be in the interior of the blue region, so that

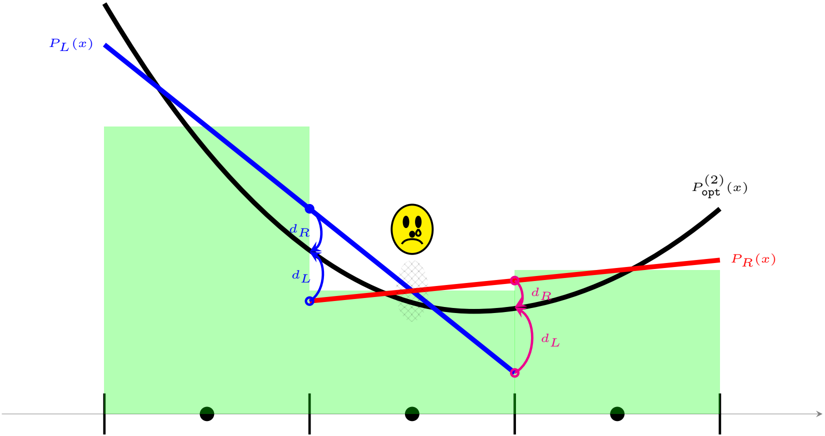
$$\sum d_k = 1, \qquad d_k > 0.$$

Shi, Hu, Shu - JCP (2002)
Puppo, M.S. - J Sci Comp (2016)

WENO3 cell center

It is impossible to find d_L, d_R s.t.

$$P^{(2)}(x_j) = d_L P_L^{(1)}(x_j) + d_R P_R^{(1)}(x_j)$$



An extra candidate polynomial: P_0

Motivated from central schemes,

$$\cancel{\text{Given } \hat{x} \in \Omega,\ P_{\text{opt}}(\hat{x}) = d_L(\hat{x}) P_L(\hat{x}) + d_R(\hat{x}) P_R(\hat{x})} \qquad \text{(WENO3)}$$

was replaced by

$$\boxed{\forall x:\ P_{\text{opt}}(x) = d_0\, P_0(x) + d_L P_L(x) + d_R P_R(x)} \qquad \text{(CWENO3)}$$

how? $$P_0(x) := \frac{P_{\text{opt}}(x) - d_L P_L(x) - d_R P_R(x)}{d_0}$$

why?
- valid at all points in cell
- d_k do not depend on the reconstruction point

$\Rightarrow$ no dependence on mesh topology,
not even in 2d/3d, AMR, ...

Levy, Puppo, Russo - M2AN (1999)

1D CWENO in the literature

- Levy, Puppo, Russo - M2AN (1999) order 3, uniform mesh
- Capdeville - JCP (2008) order 5, non uniform mesh
- Zaharan - Appl. Math. Comp. (2009) CWENO with WENO-Z nonlinear weights
- Kolb - SINUM (2015) Analysis of CWENO3 on uniform meshes (choice of ϵ)
- Cravero, M.S. - J Sci Comput (2016) Analysis of CWENO3 on non-uniform meshes (choice of ϵ)
- Balsara, Garain, Shu - JCP (2016) WAO reconstruction: CWENO-like, Z-weights, adaptive order
- Cravero, Puppo, M.S., Visconti - Math. Comp. (2019) general CWENO framework, order up to 9
- Zhou, Qiu, ... and many more reconstructions not always called CWENO

The CWENO operator

$$\mathcal{R}(x) = \boxed{\text{CWENO}\left(P_{\text{opt}}(x)\,;P_1(x),\dots,P_n(x)\right)}$$

1. choose $d_0, d_1, d_n \in (0,1)$ such that $\sum_{\xi=0}^{n} d_\xi = 1$:
2. compute $P_0(x) := \frac{1}{d_0}\left(P_{\text{opt}}(x) - \sum_{\xi=1}^{n} d_\xi P_\xi(x)\right)$
3. Compute nonlinear weights $d_k \rightsquigarrow \alpha_k \rightsquigarrow \omega_k$
(no x dependence!)
4. Reconstruction polynomial (unif. accurate in the cell!)

$$\forall x \in \text{cell} \qquad \mathcal{R}(x) = \sum_{\xi=0}^{n} \omega_\xi P_\xi(x)$$

5. evaluate $\mathcal{R}(x)$ once per reconstruction point

Cravero, Puppo, M.S., Visconti - Math. Comp. (2018)

CWENO master equation

CWENO employs

$$\forall x : P_{\text{opt}}(x) = d_0 \frac{P_{\text{opt}} - \sum_{k=1}^{n} d_k P_k(x)}{d0} + \sum_{k=1}^{n} d_k P_k(x)$$

WENO uses instead

$$\exists d_k(\hat{x}) \text{ s.t. } P_{\text{opt}}(\hat{x}) = \sum_{k=1}^{n} d_k(\hat{x}) P_k(\hat{x})$$

BOTH obtain the reconstruction replacing the linear coefficients d_* with the nonlinear ones ω_*.

CWENO accuracy on smooth data

Theorem

Consider a CWENO construction with

- $P_{\text{opt}} \in \mathbb{P}_G$
- $P_k \in \mathbb{P}_g, k = 1, \dots, \hat{m}$
- $d_0 \geq \delta > 0$
- $\epsilon = \hat{\epsilon} h^2$ (or $\epsilon = \hat{\epsilon} h$)

Then, the reconstruction is of order G, provided that $g \geq G/2$.

The proof is very similar to the WENO case.

Cravero, Puppo, M.S., Visconti - Math. Comp. (2018)

Accuracy on smooth data

Goal: show that $u(x) - \mathcal{R}(x) = \mathcal{O}(h^{2r+1})$ when $P_{\text{opt}} \in \mathbb{P}_{2r}$ and $P_k \in \mathbb{P}_r$.

$$
\begin{aligned}
\overbrace{u(x) - \mathcal{R}(x)}^{\text{reconstruction error}} &= \underbrace{u(x) - P_{\text{opt}}(x)}_{\mathcal{O}(h^{2r+1})} + P_{\text{opt}}(x) - \mathcal{R}(x) \\
&= \mathcal{O}(h^{2r+1})) + \sum_{k\geq 0} (d_k - \omega_k) P_k(\hat{x}) + \underbrace{\sum_{k\geq 0} d_k}_{=1} u(x) - \underbrace{\sum_{k\geq 0} \omega_k}_{=1} u(x) \\
&= \mathcal{O}(h^{2r+1}) + \sum_k \underbrace{(d_k - \omega_k)}_{\mathcal{O}(h^r)?} \underbrace{(P_k(\hat{x}) - u(x))}_{\mathcal{O}(h^{r+1})}
\end{aligned}
$$

So it is enough to show that $d_k - \omega_k$ is small on smooth data.

Sketch of accuracy proof

- on smooth data, Taylor expansion

$$\mathsf{OSC}[P_*] = \Delta \mathrm{x}^2 \left(u'(x_j)\right)^2 + \mathsf{o}(\Delta \mathrm{x}^2)$$

  holds for all candidate polynomials
- then

$$\alpha_j = \frac{d_j}{(\mathsf{OSC}[P_j] + \epsilon)^2} = C_j d_j$$

  for $C_0 \approx C_1 \approx \ldots \approx C_n$
- then $\omega_j = \alpha_j / \sum_k \alpha_k$ and

$$\forall j : \omega_j - d_j = \mathcal{O}(\Delta \mathrm{x}^r)$$

Cravero, Puppo, M.S., Visconti - Math. Comp. (2018)

Accuracy on smooth data (cont.)

Following Arandiga et al.: compare each $I_k = \mathrm{OSC}[P_k]$ with some fixed $I_{\hat{k}}$

$$\omega_k = d_k \left[1 - d_k \frac{I_k - I_{\hat{k}}}{\epsilon + I_{\hat{k}}} \sum_{s=0}^{t-1} \left(\frac{\epsilon + I_k}{\epsilon + I_{\hat{k}}} \right)^s + o\left(\frac{I_k - I_{\hat{k}}}{\epsilon + I_{\hat{k}}} \right) \right]$$

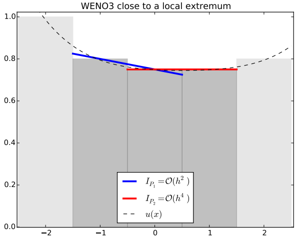
and use the fact that

$$\mathrm{OSC}[P] = u'^2 h^2 + c_1 u' u'' h^3 + (c_2 u' u''' + c_3 u''^2) h^4 + \dots$$

for any candidate P

Obstructions to $d_k - \omega_k = \mathcal{O}(h^r)$ come from

- differences between the indicators of two candidate polynomials
- ϵ too small



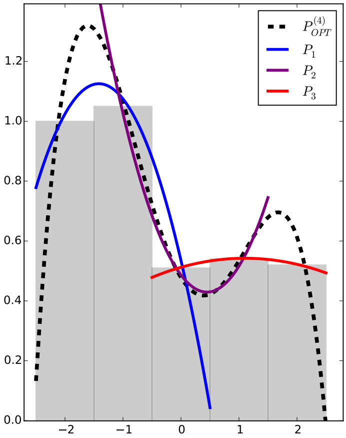
Arandiga et al. - SINUM (2012)
Cravero, Puppo, M.S., Visconti - Math. Comp. (2018)

WENO: discontinuous data

In this example:

$$I_{P_1} \asymp 1, \quad I_{P_2} \asymp 1, \quad I_{P_3} \asymp h^2$$
$$\alpha_1 \asymp 1, \quad \alpha_2 \asymp 1, \quad \alpha_3 \asymp {}^1/_{h^2}$$
$$\alpha_1 + \alpha_2 + \alpha_3 \asymp {}^1/_{h^2}$$
$$\omega_1 \asymp h^2, \quad \omega_2 \asymp h^2, \quad \omega_3 \asymp 1$$
$$\Downarrow$$
$$\mathcal{R} \simeq P_3$$
$$\Downarrow$$

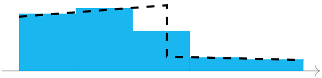
TVB reconstruction



Note!

It is crucial that at least one of $I_{P_1}, \ldots, I_{P_{r+1}}$ be $\mathcal{O}(h^2)$!

WENO: discontinuity in the central cell



- all candidate polynomials contain a discontinuity
- $\Rightarrow \quad \forall k : \mathcal{I}[P_k] \asymp 1$

- in the case of finite differences,
  all candidate polynomials are monotone in the cell
  $\Rightarrow$ monotone reconstructed values

- in the case of finite volumes,
  very small over/undershoots can be created
  $\Rightarrow$ TVB reconstructed values

Harten, Osher, Engquist, and Chachravarty. *Uniformly high order accurate essentially non-oscillatory schemes III*, NASA ICASE report 86-22 (1986).

CWENO on discontinuous data (jump not in central cell)

CWENO argument?

For $k = 1, \ldots, r$:

- $P_{\hat{k}}$ iterpolates smooth data, $\Rightarrow \mathcal{I}[P_{\hat{k}}] = \mathcal{O}(h^2)$
- P_k interpolates across the jump $\Rightarrow \mathcal{I}[P_k] \asymp 1$

$\Rightarrow$ $\omega_k \approx 0$, while $\omega_{\hat{k}} = \mathcal{O}(1)$ and the reconstruction (essentially) uses only information from the smooth part only if $\mathcal{I}[P_0] \asymp 1$!

Property R holds true for a CWENO reconstruction if

$$\text{discontinuous data anywhere in stencil} \quad \Rightarrow \quad \mathcal{I}[P_0] \asymp 1$$

- Property R is sufficient to prove "ENO" property of CWENO.
- easily proven by comparing high degree terms of P_0 and P_{opt}

Property R

- it is sufficient to prove "ENO" property of CWENO
- it looks trivial, but

$$P_0 := \tfrac{1}{d_0}\left(P_{\mathrm{opt}} - \sum_{k=1}^{r} d_k P_k\right)$$

$$\mathcal{I}[P_0] := \sum_{\ell=1}^{N} h^{2\ell-1} \int_{\Omega_j} \left[\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} P_0\right]^2 \mathrm{d}x$$

$$= \tfrac{1}{d_0}\mathcal{I}[P_{\mathrm{opt}}] + \sum_{k=1}^{r} \tfrac{d_k}{d_0}\mathcal{I}[P_k] + \text{cross terms}$$

- and the cross terms do not have a definite sign

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Proofs of Property R (case by case)

Proposition (CWENO3)

For any choice of weights, explicit computation yields

$$\frac{\mathcal{I}[P_0]}{\mathcal{I}[P_{\text{opt}}]} = \frac{3d_0^2 - 6d_0 + 16}{16d_0^2} > \frac{13}{16}$$

Proposition (CWENO5)

For any (symmetric) choice of weights, then $\frac{\mathcal{I}[P_0]}{\mathcal{I}[P_{\text{opt}}]}$ attains its minimum for $d_0 = 1$ and

$$\left.\frac{\mathcal{I}[P_0]}{\mathcal{I}[P_{\text{opt}}]}\right|_{d_0=1} > 0.6.$$

Proposition (CWENO7)

A similar result holds true...

Proof of Property R (in general)

$$P_{\text{opt}}(x) = \sum_{i=0}^{G} b_i x^i$$

$$P_k(x) = \sum_{i=0}^{g} a_{k,i} x^i, \quad \text{where } g < G$$

$$\Rightarrow P_0(x) = \frac{1}{d_0}\left(P_{\text{opt}}(x) - \sum_k d_k P_k(x)\right) = \sum_{i=0}^{g} a_{0,i} x^i + \sum_{i=g+1}^{G} \frac{b_i}{d_0} x^i$$

If there is a discontinuity in the stencil of P_{opt},

$$\Rightarrow b_G \sim Ch^{-G} \Rightarrow I[P_0] > (G!C/d_0)^2$$

and can not scale with h.

Cravero, Puppo, M.S., Visconti - Math. Comp. (2018)

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Comparision of WENO and CWENO

WENO

- × d_k depend on reconstruction point
- × for each reconstruction point $\hat{x}$, $d_k(\hat{x}) \rightsquigarrow \omega_k(\hat{x})$
- ✓ uses only lower degree polynomials
- ✓ uses only point values of polynomials
- great for conservation laws on structured cartesian meshes

CWENO

- ✓ d_k "arbitrary" (e.g. $d_0 = 3/4, d_k = 1/4n$)
- ✓ $d_k \rightsquigarrow \omega_k$ only once per cell
- × needs also the high order P_0 and $\mathcal{I}[P_0]$
- × tracks individual coefficients
- more suitable for balance laws, AMR, unstructured meshes, ADER, ...

Thank you for your kind attention!

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