CWENOZ and 2D

Course: High order reconstructions in hyperbolic conservation and balance laws

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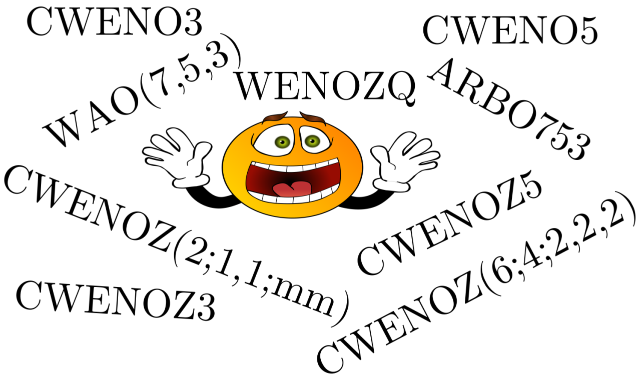
Como
Lake
Torino
Milano
Venezia
Firenze
Roma
Napoli
Catania

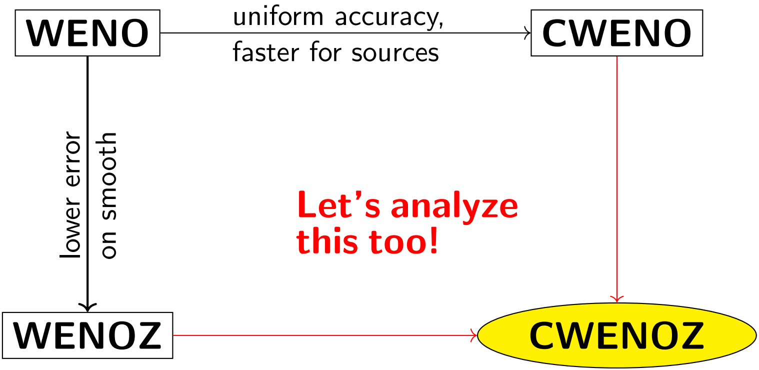
Disclaimer

You might feel like this:



but we'll try to fix this!

Why CWENOZ?



Also, some CWENOZ constructions were already employed under names like "generalized WENO" and WENO-AO.

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The CWENO(Z) master equation

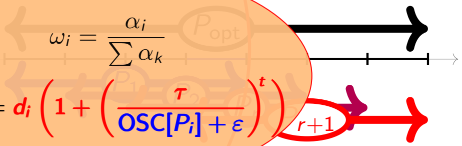
Freely choose $d_0, \ldots, d_n \in (0,1)$ such that $\sum_0^n d_k = 1$

$$\boxed{\forall x \in \text{cell}: \; P_{\text{opt}}(x) = d_0 \; P_0(x) + \sum_{k=1}^{M} d_k P_k(x)} \qquad \text{(linear)}$$

$$P_0(x) = \frac{P_{\text{opt}} - \sum_{k=1}^{n} d_k P_k}{d_0}$$

$$\omega_i = \frac{\alpha_i}{\sum \alpha_k}$$

$$\boldsymbol{\alpha_i} = \boldsymbol{d_i} \left(\mathbf{1} + \left(\frac{\boldsymbol{\tau}}{\mathbf{OSC}[\boldsymbol{P_i}] + \boldsymbol{\varepsilon}} \right)^{\boldsymbol{t}} \right)$$



$$\boxed{\forall x \in \text{cell}: \; R(x) = \omega_0 \; P_0(x) + \sum_{k=1}^{n} \omega_k P_k(x)} \qquad \text{(nonlinear)}$$

Taylor expansions of multidimensional smoothness indicators

Using the multi-index notation $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)$, when $\mathbf{x} \in \mathbb{R}^d$,

$$OSC[q] := \sum_{|\boldsymbol{\beta}| \geq 1} \Delta\mathbf{x}^{2\boldsymbol{\beta}-\mathbf{1}} \int_{\Omega_0} (\partial_{\boldsymbol{\beta}} q(\mathbf{x}))^2 \mathrm{d}x.$$

Proposition

Let $\mathcal{S}$ be a stencil around the reconstruction cell and let $q(\mathbf{x})$ be a polynomial approximating a regular function $u(\mathbf{x})$ with accuracy $\geq g$,

$$\Rightarrow \; OSC[q] = B_g + R[q]$$

- B_g depends on g, but not on $\mathcal{S}$.
- $R[q] = \mathrm{o}(B_g)$ and depends on $\mathcal{S}$.

B_g can be given an explicit expression in terms of the derivatives of u at the reconstruction cell center.

Example: CWENO(Z)3 indicators

For CWENOZ3 we have

$$I_1 = \mathrm{OSC}[P_L] \quad I_2 = \mathrm{OSC}[P_R] \quad I_0 = \mathrm{OSC}[P_{\mathrm{opt}}]$$

$$\begin{aligned}
I_1 &= B_1 - u'(0)u''(0)\Delta x^3 + \left(\frac{5}{12}u'(0)u'''(0) + \frac{1}{4}u''(0)^2\right)\Delta x^4 + \mathcal{O}(\Delta x^5),\\
I_2 &= B_1 + u'(0)u''(0)\Delta x^3 + \left(\frac{5}{12}u'(0)u'''(0) + \frac{1}{4}u''(0)^2\right)\Delta x^4 + \mathcal{O}(\Delta x^5),\\
I_0 &= B_1 \qquad\qquad\qquad\qquad\quad + \left(\frac{5}{12}u'(0)u'''(0) + \frac{13}{12}u''(0)^2\right)\Delta x^4 + \mathcal{O}(\Delta x^5).
\end{aligned}$$

for

$$B_1 = (u'(0))^2\Delta\mathrm{x}^2$$

Example: CWENO(Z)5 indicators

$$
\begin{aligned}
I_1 &= u'(0)^2\Delta x^2 + \tfrac{13}{12}u''(0)^2\Delta x^4 - \tfrac{7}{12}u'(0)u'''(0)\Delta x^4 + \mathcal{O}(\Delta x^5)\\
I_2 &= u'(0)^2\Delta x^2 + \tfrac{13}{12}u''(0)^2\Delta x^4 + \tfrac{5}{12}u'(0)u'''(0)\Delta x^4 + \mathcal{O}(\Delta x^6)\\
I_3 &= u'(0)^2\Delta x^2 + \tfrac{13}{12}u''(0)^2\Delta x^4 - \tfrac{7}{12}u'(0)u'''(0)\Delta x^4 + \mathcal{O}(\Delta x^5)\\
I[P_{\mathrm{opt}}] &= \underbrace{u'(0)^2\Delta x^2 + \tfrac{13}{12}u''(0)^2\Delta x^4}_{B_2} + \tfrac{1}{12}u'(0)u'''(0)\Delta x^4 + \mathcal{O}(\Delta x^6)
\end{aligned}
$$

Remark: $R = o(B_2)$ also at critical points (i.e. $u'(0) = 0$)

For more terms in the exapansions and for the 7th and 9th order case, see the Supplementary Material of the paper.

Cravero, M.S., Visconti - SINUM (2019)

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Main theorem

$$\alpha_i = d_i \left(1 + \left(\frac{\tau}{\mathrm{OSC}[P_i] + \Delta \mathrm{x}^{\hat{m}}} \right)^{\ell} \right)$$

Theorem

Assume that

- $P_1(\mathbf{x}), \ldots, P_m(\mathbf{x}) \in \mathbb{P}_g$ *and* $P_{\mathrm{opt}}(\mathbf{x}) \in \mathbb{P}_G$ *in the* CWENOZ *scheme have accuracy* $\geq M$
- M, $\hat{m}$, ℓ *satisfy*

$$\hat{m} \leq 2M + 1 \tag{1a}$$

$$\ell(2M + 2 - \hat{m}) \geq G - g - 1 \tag{1b}$$

$$\ell\left[\theta(\tau) - \min(\hat{m}, 2M)\right] \geq G - g - 1 \tag{1c}$$

Then, on smooth data, the CWENOZ *scheme achieves the optimal order* $G + 1$ *as* $\Delta \mathrm{x} \to 0$.

$\theta(f) = n \Leftrightarrow f = C\Delta \mathrm{x}^n + o(\Delta \mathrm{x}^n)$

In practice...

$$\alpha_i = d_i \left(1 + \left(\frac{\tau}{\mathrm{OSC}[P_i] + \Delta x^{\hat{m}}}\right)^{\ell}\right)$$

- if $\theta(\tau) > 2M$,
  there exist ℓ for optimal convergence for any $\hat{m} \leq 2M+1$
- else,
  there exist ℓ for optimal convergence for all $\hat{m}$ s.t. $\hat{m} < \theta(\tau) < 2M+1$
- in any case, the larger is $\theta(\tau)$, the smaller ϵ and smaller ℓ are needed to achieve optimal convergence.

Take home message:

Choose λ_k s.t.

$$\tau = \sum_{k=0}^{m} \lambda_k I_k$$

is as small as possible!

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The "squeeze τ" game

Since

$$I_k = B_M + R$$

with $R = \mathrm{o}(B_M)$ then

For free, any combination

$$\tau = \sum_k \lambda_k I_k$$

with $\sum_k \lambda_k = 0$ will grant that $\tau = \mathrm{o}(B_M)$.

But look closely at the CWENOZ3 case:

$$\tau = 2I_0 - I_1 - I_2 = \mathcal{O}(\Delta x^4)$$

$$I_1 = B_1 - u'(0)u''(0)\Delta x^3 + \left(\frac{5}{12}u'(0)u'''(0) + \frac{1}{4}u''(0)^2\right)\Delta x^4 + \mathcal{O}(\Delta x^5),$$

$$I_2 = B_1 + u'(0)u''(0)\Delta x^3 + \left(\frac{5}{12}u'(0)u'''(0) + \frac{1}{4}u''(0)^2\right)\Delta x^4 + \mathcal{O}(\Delta x^5),$$

$$I_0 = B_1 + \left(\frac{5}{12}u'(0)u'''(0) + \frac{13}{12}u''(0)^2\right)\Delta x^4 + \mathcal{O}(\Delta x^5).$$

CWENOZ optimal τ in 1D (uniform grids)

- CWENOZ 3

$$\boxed{\hat{\tau}_3 = \left| OSC[P^1_L] + OSC[P^1_R] - 2\, OSC[P_{\text{opt}}] \right| = \mathcal{O}(\Delta x^4)}$$

instead of $\tau_3 = \mathcal{O}(\Delta x^3)$ without using $OSC[P_{\text{opt}}]$.

- CWENOZ 5

$$\boxed{\hat{\tau}_5 = \left| OSC[P^2_L] + 4\, OSC[P^2_C] + OSC[P^2_R] - 6\, OSC[P_{\text{opt}}] \right| = \mathcal{O}(\Delta x^6)}$$

instead of $\tau_5 = \mathcal{O}(\Delta x^5)$ without using $OSC[P_{\text{opt}}]$.

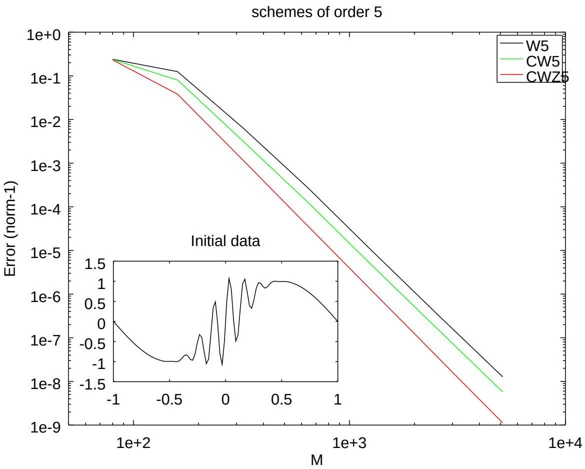
- for higher orders the optimal definition for WENOZ is also optimal for CWENOZ, but you may still define τ involving also $OSC[P_{\text{opt}}]$

Optimal CWENOZ: Cravero, M.S., Visconti - SINUM (2019)
Optimal WENOZ: Don, Borges - JCP (2013)

Is CWENOZ really better? (1)

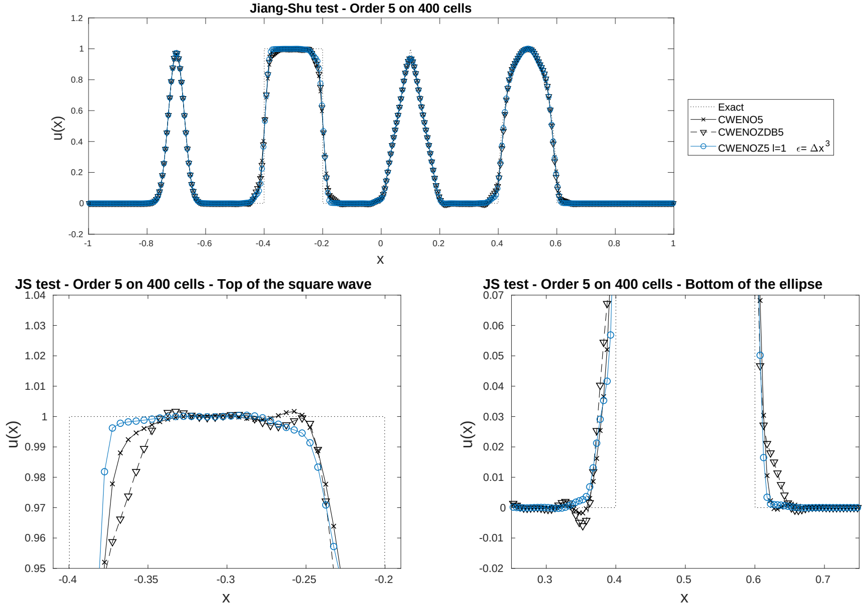
- more versatile than WENO
- little extra effort than CWENO
- but yields better errors...

$$u_t + u_x = 0$$



Jiang-Shu linear transport test with CWENOZ5

- . . . and often even less oscillations!



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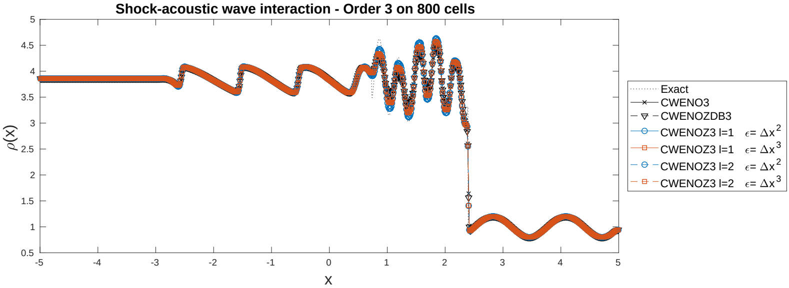

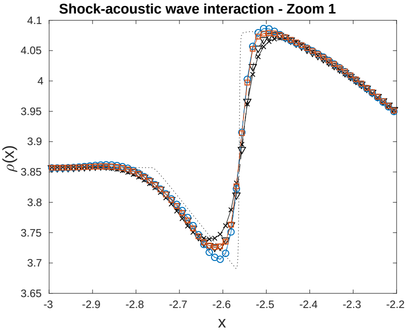

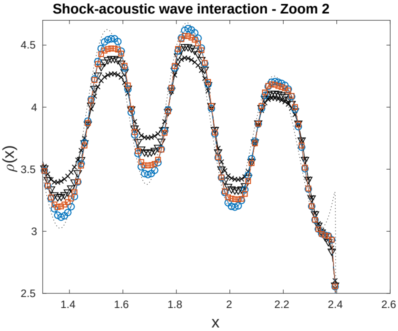

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Open-source implementation

claw1dArena

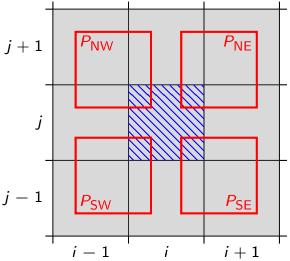
- Downloadable from `zenodo.org`
  DOI 10.5281/zenodo.2641724
- Developed with numerical experimentation in mind:
  - → C++ implementation with very few required libraries
  - → choose conservation law at compile time
  - → choose any combination of reconstruction, timestepper, numerical flux, well-balancing, discretization parameters, etc at run-time
  - → GPL licence: just give credit, if you use it

Let's you experiment with all my 1d reconstructions: CWENO, CWENOZ, CWENO-AO, CWENO-boundary

Currently: linear advection, Burgers, SWE (w.b.), Euler, easily extendible to other conservation laws

CWENOZ3 in two space dimensions

$$\mathrm{CWENOZ}(P_{\mathrm{opt}}; P_{\mathrm{NE}}, P_{\mathrm{SE}}, P_{\mathrm{SW}}, P_{\mathrm{NW}}) = \sum_{k=0}^{4} \omega_k P_k$$



- $P_{\mathrm{opt}}(x) \in \mathbb{P}_2(x,y)$ on the central 3×3 stencil
- $P_{\mathrm{NE}}, P_{\mathrm{SE}}, P_{\mathrm{SW}}, P_{\mathrm{NW}} \in \mathbb{P}_1(x,y)$ on the 2×2 sub-stencil
- $d_0 = \frac{3}{4}$ and $d_{1,\dots,4} = \frac{1}{16}$

All polynomial are defined in a least square sense:

$$P(x) = \overline{U}_j + \sum_k c_k \phi_k(x)$$

where the basis is such that $\int_{\Omega_j} \phi_k = 0$ and solve least squares for $\mathbf{c}$.

CWENOZ3 in two space dimensions

Recall that the theorem $\mathrm{OSC}[P] = B_M + R$ holds also in $\mathbb{R}^d$.

On a 2d uniform Cartesian grid with $\Delta\mathrm{x} = \Delta\mathrm{y} = h$,

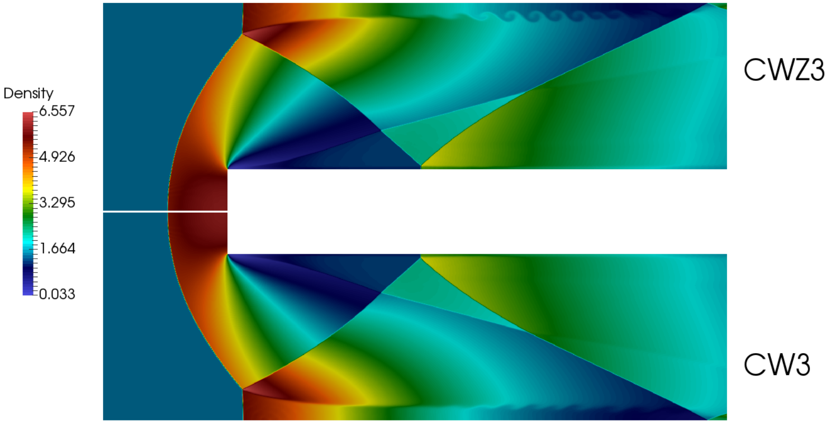
$$\begin{aligned}
\mathrm{OSC}[P^{(1)}_{NE}] &= (u_x^2 + u_y^2)h^2 + \left(+u_x u_{xx} + \tfrac{2}{3} u_x u_{xy} + \tfrac{2}{3} u_y u_{xy} + u_y u_{yy}\right) h^3 + \mathcal{O}(h^4)\\
\mathrm{OSC}[P^{(1)}_{NW}] &= (u_x^2 + u_y^2)h^2 + \left(-u_x u_{xx} + \tfrac{2}{3} u_x u_{xy} - \tfrac{2}{3} u_y u_{xy} + u_y u_{yy}\right) h^3 + \mathcal{O}(h^4)\\
\mathrm{OSC}[P^{(1)}_{SE}] &= (u_x^2 + u_y^2)h^2 + \left(+u_x u_{xx} - \tfrac{2}{3} u_x u_{xy} + \tfrac{2}{3} u_y u_{xy} - u_y u_{yy}\right) h^3 + \mathcal{O}(h^4)\\
\mathrm{OSC}[P^{(1)}_{SW}] &= (u_x^2 + u_y^2)h^2 + \left(-u_x u_{xx} - \tfrac{2}{3} u_x u_{xy} - \tfrac{2}{3} u_y u_{xy} - u_y u_{yy}\right) h^3 + \mathcal{O}(h^4)\\
\mathrm{OSC}[P_{\mathrm{opt}}] &= (u_x^2 + u_y^2)h^2 + \mathcal{O}(h^4)
\end{aligned}$$

In general, $\sum_{k=0}^{m} \lambda_k = 0, \quad \Rightarrow \boxed{\tau = \lambda_0 \mathrm{OSC}[P_{\mathrm{opt}}] + \sum_{k=1}^{4} \lambda_k \mathrm{OSC}[P^{(1)}_k] = \mathcal{O}(h^3)}$

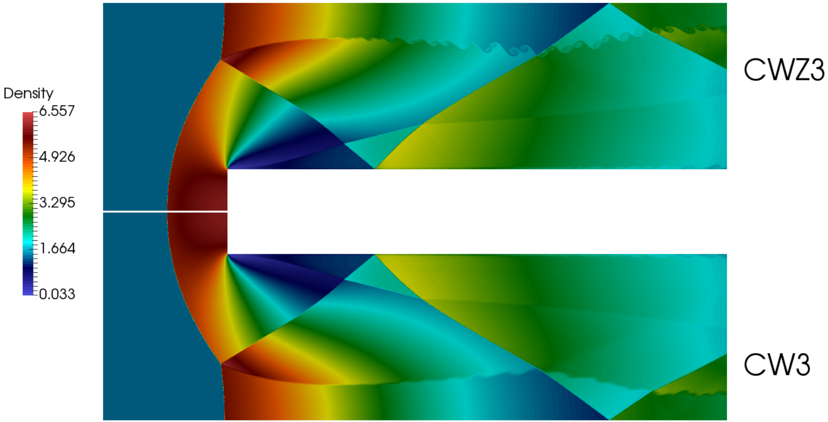
but the symmetries allow an even better definition of τ:

$$\tau = \mathrm{OSC}[P^{(1)}_{NE}] + \mathrm{OSC}[P^{(1)}_{NW}] + \mathrm{OSC}[P^{(1)}_{SE}] + \mathrm{OSC}[P^{(1)}_{SW}] - 4\mathrm{OSC}[P_{\mathrm{opt}}] = \mathcal{O}(h^4)$$

Forward-facing step at $t = 2.4$ with 1M dofs



Forward-facing step at $t = 4.0$ with 4M dofs



Thank you for your kind attention!

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