Introductory lecture

Course: High order reconstructions in hyperbolic conservation and balance laws

Matteo Semplice

Dipartimento di Scienza e Alta Tecnologia
Università dell'Insubria

Shenzhen, March 2024

matteo.semplice@uninsubria.it

Plan of the course

1. review classical reconstructions:
   TVD, ENO, WENO, ...
2. Central WENO reconstructions in 1d
3. CWENOZ, Non-uniform meshes, higher dimensions
4. Adaptive-order reconstructions (CWAO) and
   boundary reconstruction (CWB)
5. Applications to well-balanced schemes and
   Adaptive Mesh Refinement

   (time permitting) "towards implicit CWENO schemes" (QUINPI)

matteo.semplice@uninsubria.it

Conservation and balance laws

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{u}) = s(\mathbf{u}, x)}$$

- systems of conservation laws: $\mathbf{u} \in \mathbb{R}^p$
- multidimensional: $x \in \mathbb{R}^d$
- with source terms and steady states $\partial_t \mathbf{u} = 0 = -\nabla \cdot \mathbf{F}(\mathbf{u}) + s(\mathbf{u}, x)$

matteo.semplice@uninsubria.it

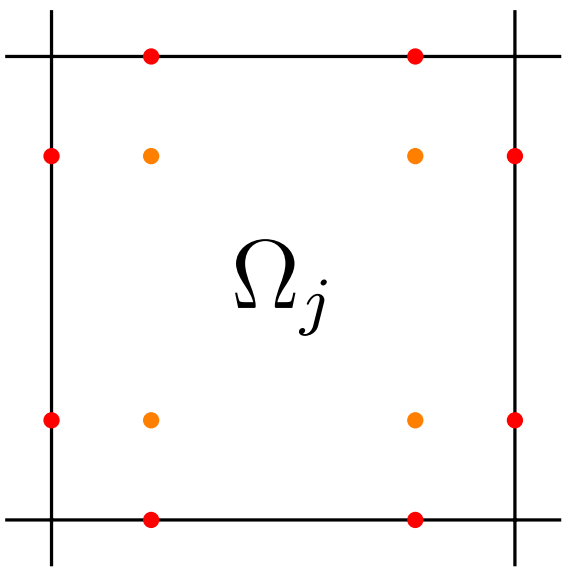
Semidiscrete finite volume schemes

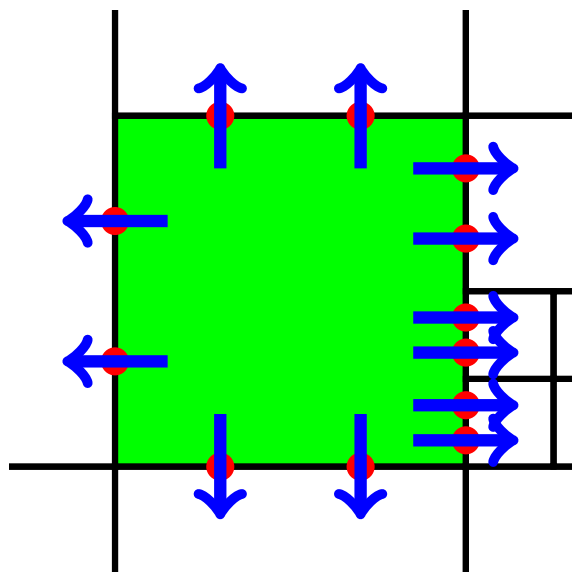
For every cell Ω_j in the mesh, semidiscrete formulation

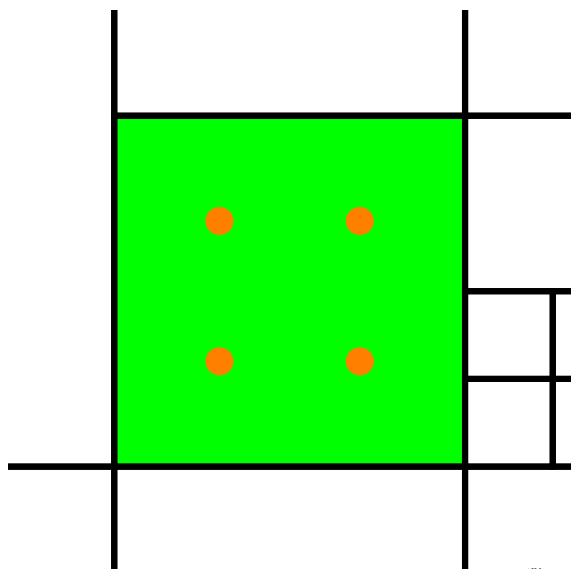
$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{u}}_j = -\frac{1}{|\Omega_j} \int_{\partial\Omega_j|} \mathcal{F}(\hat{u}_{\text{in}}, \hat{u}_{\text{ext}}) \cdot \mathbf{n} + \frac{1}{|\Omega_j|} \int_{\Omega_j} s(\mathbf{u}, x)$$

and use

- Runge-Kutta timestepper (SSP-ERK)
- quadrature rule on $\partial\Omega_j$ and quadrature rule on Ω_j
- must obtain point values at quadrature points from the cell averages







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ADER-FV schemes

For every cell in the mesh,

- compute a Galerkin predictor $\hat{u}_j(\xi,\tau)$ in each space-time volume $\Omega_j \times [t^n, t^{n+1}]$

  $\hat{u}_j \in \mathbb{P}_r(\xi,\tau)$ s.t.

  $$\forall \psi \in \mathbb{P}_r(\xi,\tau) : \int_{t^n}^{t^{n+1}} \int_{\Omega_j} \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{u}) - s(\mathbf{u},x) \right) \psi(\xi,\tau) = 0$$

  and integrate by parts only in time

$\Rightarrow$ need a starting value $\hat{u}_j(\xi,0) \in \mathbb{P}_r$ that is computed from the cell averages at time t^n

- fully-discrete update

$$\overline{\mathbf{u}}_j^{n+1} = \overline{\mathbf{u}}_j^n - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_j} \mathcal{F}(\hat{u}_j, \hat{u}_{\text{ext}}) \cdot \mathbf{n} + \int_{t^n}^{t^{n+1}} \int_{\Omega_j} s(\hat{u}_j, x)$$

Other discretizations

In finite difference schemes,

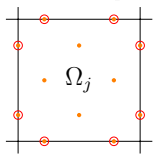
- need to reconstruct face values of fluxes from point values at cell centers

In $\mathbb{P}_n - \mathbb{P}_m$ schemes (hybrid between ADER-FV and DG)

- need to reconstruct $\mathbb{P}_m$ in each cell, using $\mathbb{P}_n$ data in the neighbours

In well-balanced schemes,

- need to reconstruct at specific quadrature points in the cell

  This is an example for Euler+gravity: 

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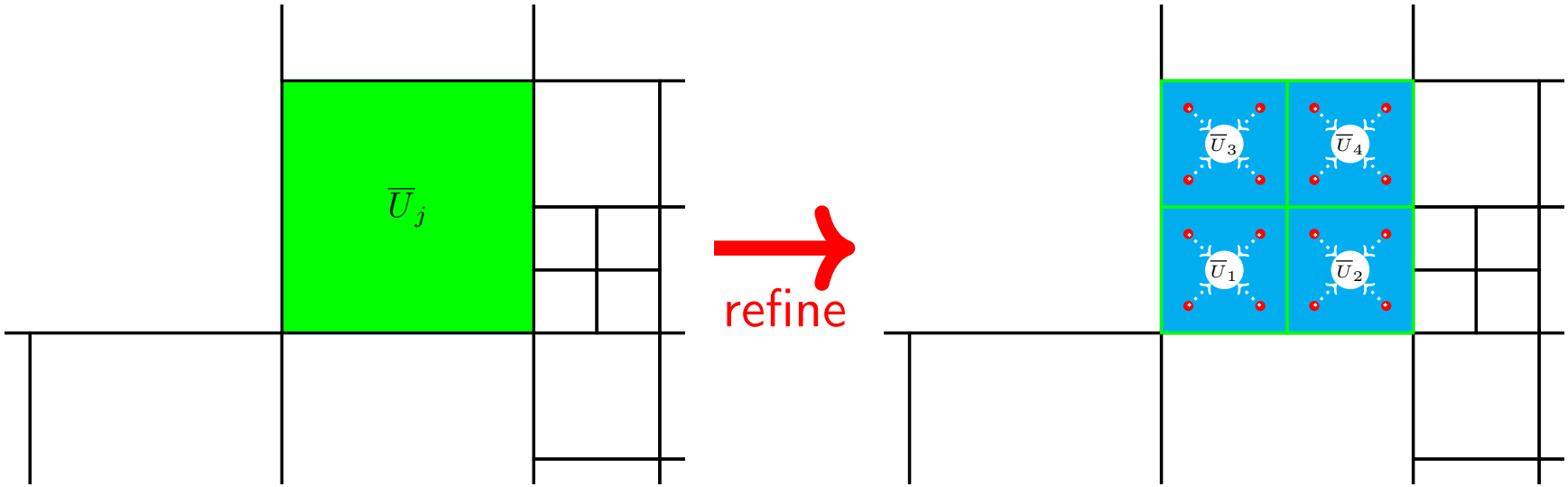
Adaptive Mesh Refinement

When a cell is refined



one needs to compute the sub-cell averages s.t.

$$\frac{\overline{U}_1 + \overline{U}_2 + \overline{U}_3 + \overline{U}_4}{4} = \overline{U}_j$$

- For accuracy,

$$\overline{U}_1 = \frac{1}{|\Omega_1|} \int_{\Omega_1} u_j(x)$$

where $u_j(x)$ is an accurate reconstruction

matteo.semplice@uninsubria.it

Reconstruction procedure

A reconstruction from cell averages in cell Ω_j is

$$\mathcal{R}_j[\overline{u}_{j-p},\dots,\overline{u}_{j+q}](x)$$

a procedure that computes a function (polynomial) $\mathcal{R}_j(x)$ such that

- $\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathcal{R}_j(x)\mathrm{d}x=\overline{u}_j$
- is of accuracy $r\geq 1$ in the sense that whenever the data $\overline{u}_{j-p},\dots,\overline{u}_{j+q}$ are sampled from a smooth enough function $u(x)$, then $\mathcal{R}_j(x)-u(x)=\mathcal{O}(\Delta \mathrm{x}^r)$

Examples:

- (constant) $\mathcal{R}_j(x)=\overline{u}_j$
- (central) $\mathcal{R}_j(x)\in\mathbb{P}_{2r}$ such that

$$\forall k=-r,\dots,r\ :\ \frac{1}{|\Omega_{j-k}|}\int_{\Omega_{j-k}}\mathcal{R}_j(x)\mathrm{d}x=\overline{u}_{j-k}$$

matteo.semplice@uninsubria.it

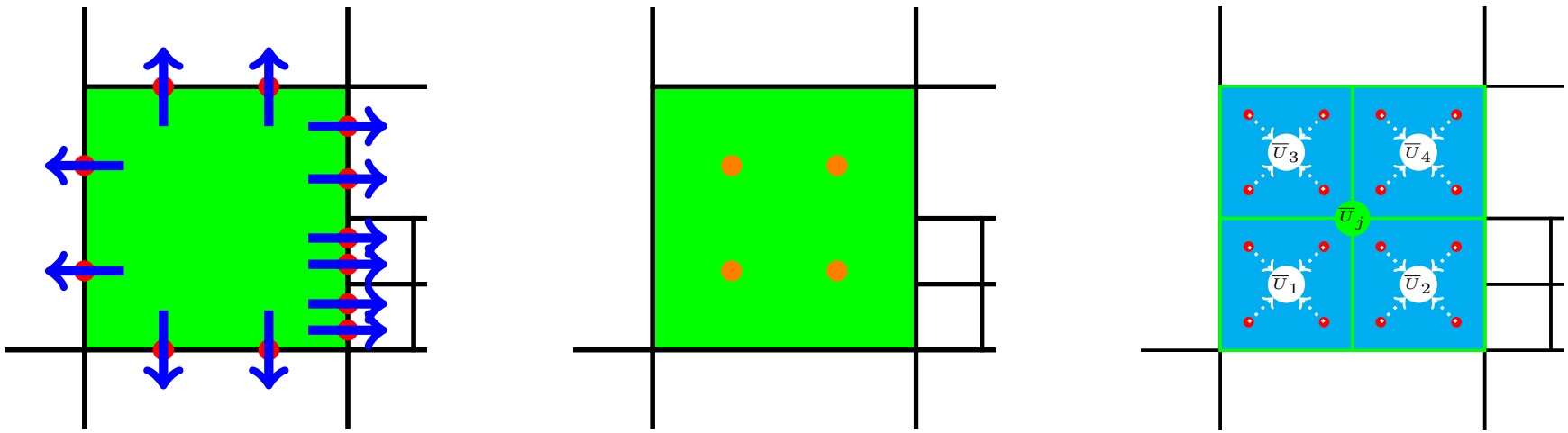
Requirements for the reconstruction procedure

For a FV scheme, the reconstruction must be:

- high-order accurate and non-oscillatory

For high order finite volume methods, it should also be efficient at:

- reconstructing point values at many locations on $\partial\Omega_j$
  Mesh topology ($\Rightarrow$ quadrature nodes) is changing in time
- reconstructing point values at locations inside Ω_j
  for source terms, refinement, moving mesh schemes



- For the ADER predictors, one really needs a polynomial defined in the whole cell Ω_j

matteo.semplice@uninsubria.it

TVD flux/slope limiters

For second order 1D schemes, choose reconstruction of the form

$$\mathcal{R}_j(x) = \overline{U}_j + \sigma_j(x - x_j)$$

- σ_j is the limited slope

$$\sigma_j = \frac{\overline{U}_{j+1} - \overline{U}_j}{\Delta x}\Phi(\theta_j)$$

- Φ is the slope limiter
- θ_j is the regularity indicator

$$\theta_j = \frac{\overline{U}_j - \overline{U}_{j-1}}{\overline{U}_{j+1} - \overline{U}_j} \sim \frac{u'(x_j) - u''(x_j)\Delta x/2}{u'(x_j) + u''(x_j)\Delta x/2} \sim 1$$

- θ_j chosen as the ratio of left/right slope

Sweby - SINUM (1984)

matteo.semplice@uninsubria.it

Sweby region

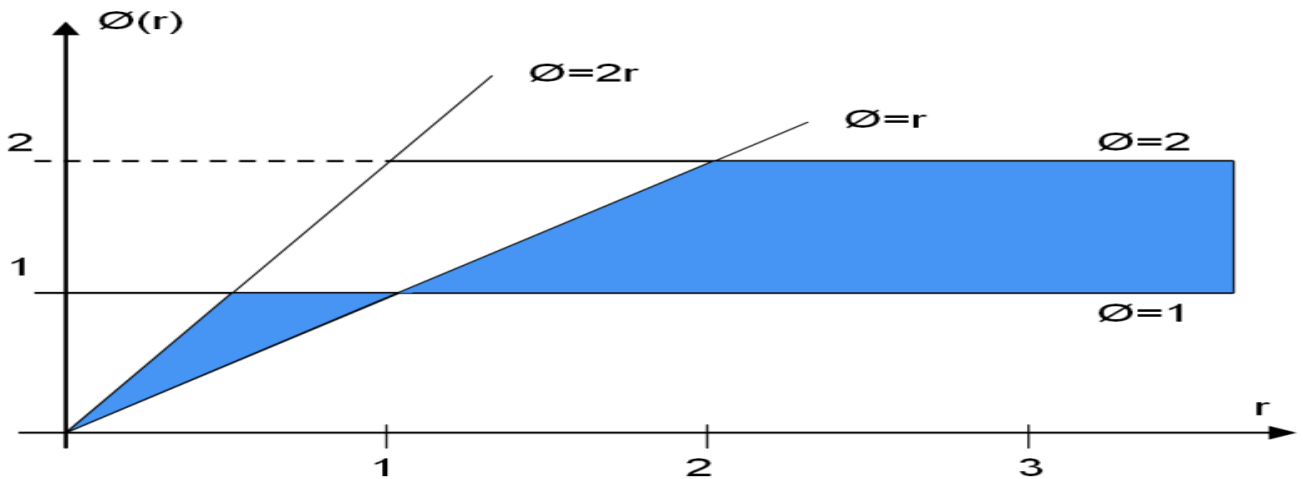
For second order accuracy

Since $\theta_j = 1 + o(1)$ on smooth data, we need a condition on $\Phi(1)$.
Precisely

$$\Phi(1) = 1$$

For TVD

We expect TVD issues to arise when the left/right slopes are very different, so for $\theta \approx 0$ and $\theta \approx \infty$

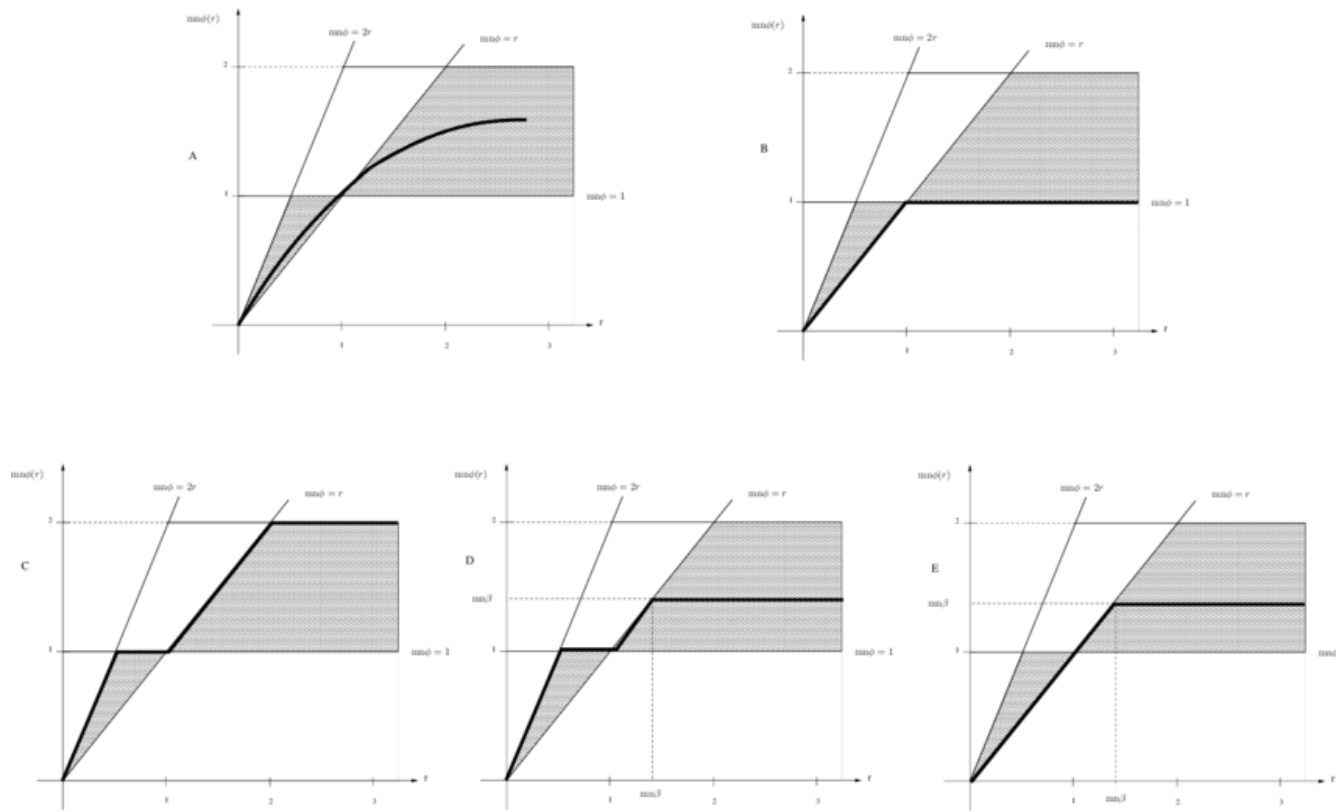


Admissible limiter region for second-order TVD schemes
(Sweby, 1984)

matteo.semplice@uninsubria.it

Some limiters

Second-oder TVD limiters:
they differ in diffusivity, accuracy, "wave deformation", etc



And more recently, also third order (TVB) schemes based on limiting were introduced, e.g. Schmidtmann et al. - J. Sci. Comput. (2016)

TVD or TVB?

For a second order scheme, let

$$\sigma_L = \frac{\overline{U}_j - \overline{U}_{j-1}}{\Delta x} \qquad \sigma_R = \frac{\overline{U}_{j+1} - \overline{U}_j}{\Delta x}$$

Then,

$$\sigma_{\text{minmod}} = \begin{cases} 0 & \text{if } \sigma_L \sigma_R <= 0 \\ \sigma_L & \text{if } \sigma_L \sigma_R > 0 \text{ and } |\sigma_L| < |\sigma_R| \\ \sigma_R & \text{if } \sigma_L \sigma_R > 0 \text{ and } |\sigma_L| > |\sigma_R| \end{cases}$$

Remark Using

$$\sigma_{\text{ENO2}} = \begin{cases} \sigma_L & \text{if } |\sigma_L| < |\sigma_R| \\ \sigma_R & \text{if } |\sigma_L| > |\sigma_R| \end{cases}$$

yields a scheme which

- it TVB (not TVD)
- has a much better resolution, does not "clip extrema"

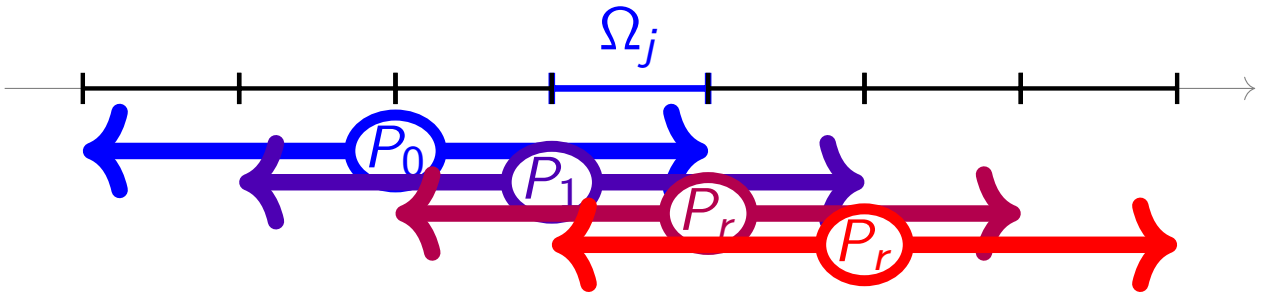
matteo.semplice@uninsubria.it

Essentially Non Oscillatory (ENO) schemes

For a reconstruction of order r, in cell Ω_j:

- form polynomials $P_0, \dots, P_r \in \mathbb{P}_r$, each interpolating data in

$$\text{stencil for } P_k : \{\Omega_{j-r+k}, \dots, \Omega_{j+k}\}$$



- compute some "oscillation indicators" for every polynomial

$$\text{OSC}[P_k] = \begin{cases} \text{small if } P_k \text{ interpolates smooth data} \\ \text{large if } P_k \text{ interpolates discontinous data} \end{cases}$$

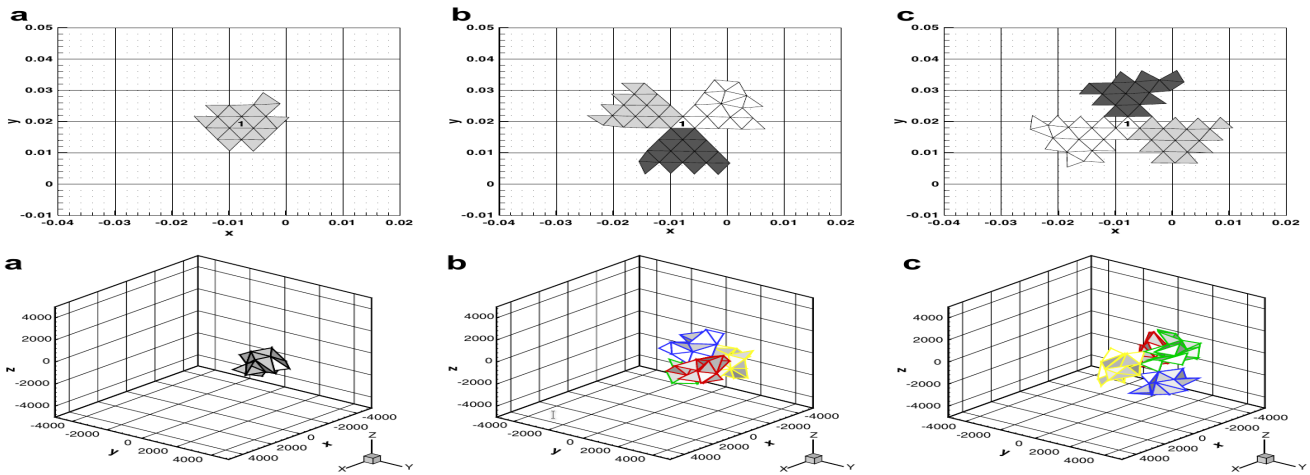
- set $\mathcal{R}_j(x) = P_i(x)$, where P_i is the polynomial with the smallest indicator

Harten, Osher - SINUM (1987)

matteo.semplice@uninsubria.it

Multi-dimensional (ENO) schemes

1. Dimensional splitting
2. Stencil search



Remark One-dimensional ENO and multi-dimensional ones (except dimensional splitting) yield a reconstruction polynomial which is defined in the whole cell. Thus

- initial effort to choose $\mathcal{R}_j$
- easy evaluation of $\mathcal{R}_j$ at any point in cell Ω_j

matteo.semplice@uninsubria.it

Jiang-Shu oscillation indicators

$$\mathrm{OSC}[P] = \sum_{\ell \geq 1} \Delta \mathrm{x}^{2\ell-1} \int_{\Omega_j} \left(\frac{\mathrm{d}^\ell P}{\mathrm{d}x^\ell} \right)^2$$

Remark The factors $\Delta \mathrm{x}^{2\ell-1}$ are there to ensure that $\mathrm{OSC}[P] = \mathcal{O}(1)$ in the worst case scenario, which is a jump discontinuity in the case of conservation laws.
In fact, if P interpolates across a jump, then $\frac{\mathrm{d}^\ell P}{\mathrm{d}x^\ell} \asymp \Delta \mathrm{x}^{-\ell}$
Remark When dealing with Hamilton-Jacobi, or in general with equations with globally continous solutions, one should change the definition into

$$\mathrm{OSC}[P] = \sum_{\ell \geq 2} \Delta \mathrm{x}^{2\ell-3} \int_{\Omega_j} \left(\frac{\mathrm{d}^\ell P}{\mathrm{d}x^\ell} \right)^2$$

Remark In multi-D, one gives a similar definition, involving the diameter of the cell in the scaling factor

Jiang, Shu - JCP (1996)

matteo.semplice@uninsubria.it

Jiang-Shu oscillation indicators: properties

$$\mathrm{OSC}[P] = \sum_{\ell \geq 1} \Delta x^{2\ell-1} \int_{\Omega_j} \left(\frac{\mathrm{d}^\ell P}{\mathrm{d}x^\ell} \right)^2$$

On smooth data, the oscillation is dominated by the first term:

$$\mathrm{OSC}[P] = \Delta x^2 (u_x)^2 + \mathrm{o}(\Delta x^2)$$

so

- $\mathrm{OSC}[P] \asymp 1$ on jump discontinuities (slide before)
- $\mathrm{OSC}[P] \asymp \Delta x^2$ on regular solutions, away from local extrema
- $\mathrm{OSC}[P] = \mathrm{o}(\Delta x^2)$ on local extrema ("critical points")

matteo.semplice@uninsubria.it

Jiang-Shu oscillation indicators: computation

$$\mathrm{OSC}[P] = \sum_{\ell \geq 1} \Delta x^{2\ell-1} \int_{\Omega_j} \left(\frac{\mathrm{d}^\ell P}{\mathrm{d}x^\ell} \right)^2$$

Writing $P(x) = \sum_{k \geq 0} a_k (x - x_j)^k$,

- $\mathrm{OSC}[P]$ is a quadratic form on the coefficients $\mathbf{a}$ of P

Since $\mathbf{a}$ depends linearly on the data $\mathbf{u} = [\overline{U}_{j-r+k}, \ldots, \overline{U}_{j+k}]^T$ in the stencil of P,

- $\mathrm{OSC}[P]$ is a quadratic form on the data $\mathbf{u}$ which are interpolated by P

Remark In any case, the computation of $\mathrm{OSC}[P_1], \ldots$ can become the bottleneck of the algorithm, espacially in multi-D. There have been proposals of special polynomial basis to speed up their evaluation

Balsara, Garain, Shu - JCP (2106)

Pros/cons of ENO

- provides a reconstruction polynomial in the whole cell, can be evauated afterwards at little cost
- polynomial selection can be affected by roundoff errors on very smooth/flat areas (where $\mathrm{OSC}[P] = \mathcal{O}(\Delta x^4)$ and maybe much smaller)
- on smooth flows, uses $2r+1$ data, but gets only order $r+1$ accuracy. With $2r+1$ data, one would expect $2r+1$ instead.

matteo.semplice@uninsubria.it

Weighted Essentially Non Oscillatory (WENO)

The ideas

- blend, instead of choosing, all ENO candidate polynomials
- do so to "mimick" the behaviour of the $P_{\text{opt}} \in \mathbb{P}_{2r}$ interpolating all data
- let the above happen only on smooth data, revert to ENO otherwise

The bad news

$$\mathbb{P}_{2r} \ni P_{\text{opt}}(x) = \sum_{k=0}^{r} d_k \underbrace{P_k(x)}_{\in \mathbb{P}_r}$$

So d_k must depend on x!

The realization To avoid oscillations,

$$\mathcal{R}_j(x) = \sum_{k=0}^{r} \omega_k \underbrace{P_k(x)}_{\in \mathbb{P}_r} \quad \text{with} \quad \omega_k \sim \frac{1}{(\text{OSC}[P])^m}$$

matteo.semplice@uninsubria.it

WENO implementation

The original Jiang-Shu definition was

$$\omega_k = \frac{\alpha_k}{\sum_j \alpha_j} \qquad \alpha_k = \frac{d_k}{(\mathrm{OSC}[P_k] + \epsilon)^2}$$

On local extrema, the interplay between the very small OSC's and the ϵ can lead to uneven convergence or even to loss of accuracy. Thus "mapped WENO" was proposed:

- compute ω_k as in WENO
- but use in the reconstruction the "mapped weights"

$$\hat{\omega}_k = \psi_k(\omega_k)$$

where $\psi_k : [0,1] \to [0,1]$ with a fixed point in (d_k, d_k) and flat derivative there

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Jiang, Shu - JCP (1996)

Henrick, Aslam, Powers - JCP (2005)

matteo.semplice@uninsubria.it

Variable ϵ and the Z-weights

Another solution is to choose

$$\epsilon \sim \Delta x^2$$

Yet another to combine it with a different definition of weights

$$\omega_k = \frac{\alpha_k}{\sum_j \alpha_j} \qquad \alpha_k = d_k \left(1 + \left(\frac{\tau}{\mathrm{OSC}[P_k] + \epsilon}\right)^2\right)$$

where τ is a global smoothness indicator.
For efficiency,

- τ is chosen as a linear combination of the OSC's
- optimal coeffcients are in the 2013 paper

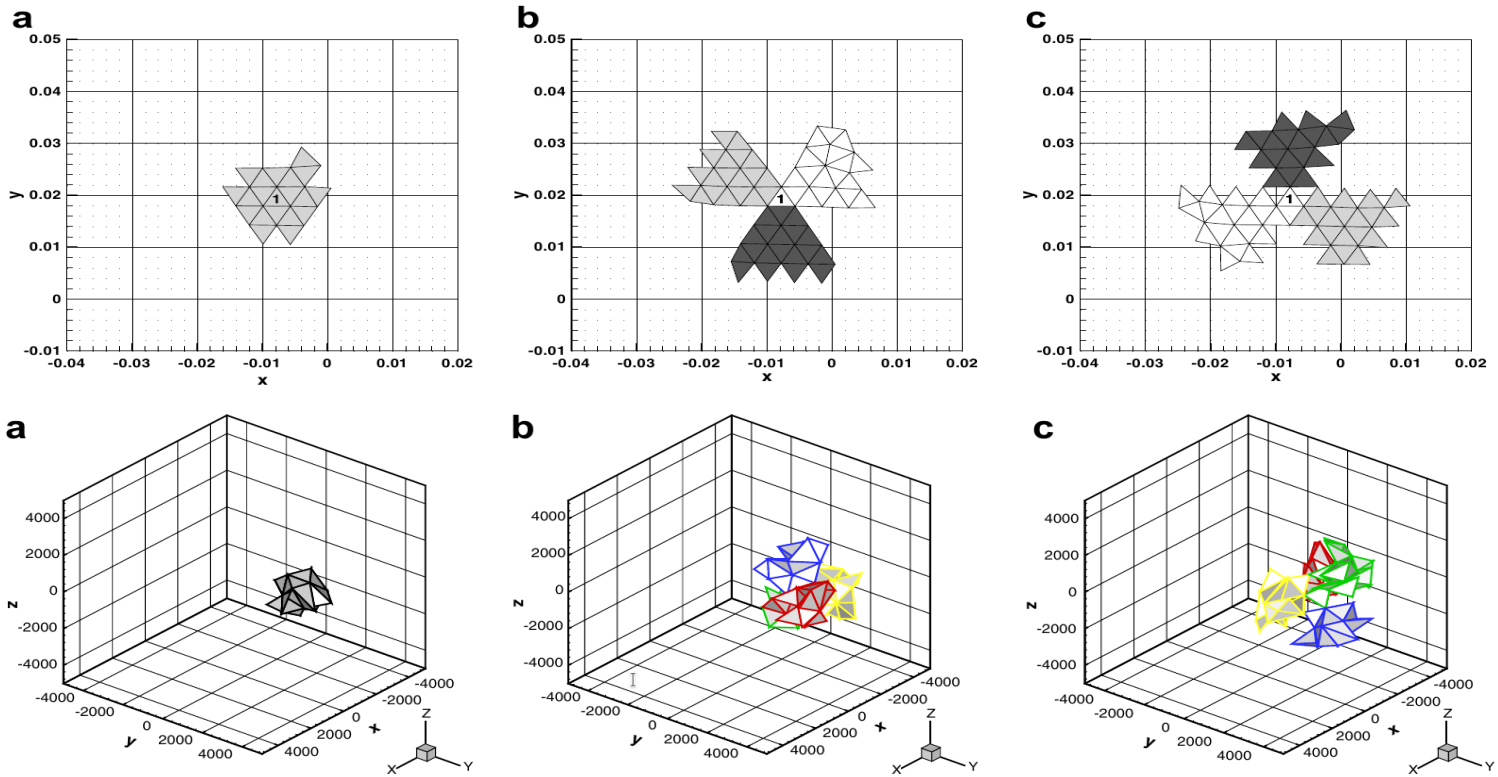
Hint: on smooth data, $\mathrm{OSC}[P] \sim \Delta x^2 (u_x)^2$ so that any combination $\tau = \sum_{k=0}^{r} \lambda_k \mathrm{OSC}[P_k]$ with $\sum_{k=0}^{r} \lambda_k = 0$ will yield $\tau = \mathcal{O}(\Delta x^3)$

Arandiga, Baeza, Belda, Mulet - SINUM (2011)
Don, Borges, ...(2008-2013)

matteo.semplice@uninsubria.it

Multi-dimensional WENO

1. Dimensional splitting: easy and fast!
2. Stencil search: a truly multi-dimnsional WENO, using $P_k(x, y, \ldots)$ in stencils $\mathcal{S}_k \ni \Omega_k$, is very complicated because one has to find the d_k coefficients for every reconstruction point



Figures from Dumbser, Kaser (2007)

Thank you for your kind attention!

Matteo Semplice
matteo.semplice@uninsubria.it

Dipartimento di Scienza e Alta Tecnologia
Università dell'Insubria
Via Valleggio, 11
Como