Lagrangian theory of nonlinear gravity waves in shallow-water and related models

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Plan

Shock formation in Hopf equation

Hopf equation (obtained at no rotation and weak nonlinearity) :

$$U_T + UU_x = 0 \tag{1}$$

Characteristic description :

$$U = \dot{X}, \Rightarrow \ddot{X} = 0, \Rightarrow \dot{X} = U_I(x), \Rightarrow X(x, T) = x + U_I(x)t. \tag{2}$$

U_I - initial distribution of U. Characteristic curves $X(x, t) \leftrightarrow$ "Lagrangian" trajectories

Breaking :

$$\forall x_1, x_2 : x_2 > x_1, \; U_I(x_2) < U_I(x_1), \tag{3}$$

intersection of characteristics $\equiv$ breaking.

$1d$ quasi-linear systems

Definition :

$$\partial_t V_i(x,t)+\sum_{j=1}^{N} A_{ij}\left(\vec{V}\right)\partial_x V_j(x,t) = B_i\left(\vec{V}\right),\ i=1,2,...,N. \tag{4}$$

Eigenvectors and eigenvalues :

Let $\vec{l}^{(a)}$ - left eigenvectors and $\xi^{(a)}$ - corresponding eigenvalues, $a = 1, 2, \dots$:

$$\vec{l}^{(a)} \cdot A = \xi^{(a)} \vec{l}^{(a)},\ \Rightarrow \tag{5}$$

$$\vec{l}^{(a)} \cdot \left(\partial_t \vec{V} + A \circ \partial_x \vec{V}\right) = \vec{l}^{(a)} \cdot \left(\partial_t \vec{V} + \xi^{(a)} \partial_x \vec{V}\right). \tag{6}$$

Characteristics :

$$\frac{dx}{dt} = \xi^{(a)} \tag{7}$$

Advection along a characteristic :

$$\dot{\vec{V}} \equiv \frac{d\vec{V}}{dt} = \left(\partial_t + \xi^{(a)}\partial_x\right)\vec{V}. \tag{8}$$

$$\vec{l}^{(a)} \cdot \dot{\vec{V}} = \vec{l}^{(a)} \cdot \vec{B} \tag{9}$$

- ordinary differential equations (easy to integrate).

Hyperbolic systems :

N real and different eigenvalues $\xi^{(a)}$.

Riemann invariants :

If $\vec{l}^{(a)} = const$ (or integrating multiplier exists) - Riemann variables (invariants if $\vec{B} = 0$) :

$$r^{(a)} = \vec{l}^{(a)} \cdot \vec{V} :, \quad \frac{dr^{(a)}}{dt} = \vec{l}^{(a)} \cdot \vec{B} \tag{10}$$

Shocks :

Intersection of characteristics $\leftrightarrow$ derivatives of Riemann invariants become infinite in finite time.

Example : 1D SW

Quasi - linear form of 1D SW equations :

$$\partial_t \begin{pmatrix} u \\ h \end{pmatrix} + \begin{pmatrix} u & 1 \\ h & u \end{pmatrix} \partial_x \begin{pmatrix} u \\ h \end{pmatrix} = 0, \tag{11}$$

Eigenvectors and eigenvalues :

$$\vec{l}^{\pm} = (\pm\sqrt{h}, 1), \quad \xi^{\pm} = u \pm \sqrt{h}. \tag{12}$$

Riemann invariants :

$$r^{\pm} = u \pm 2\sqrt{h}, \quad \frac{dr^{\pm}}{dt^{\pm}} = 0, \ \frac{d}{dt^{\pm}} \equiv \partial_t + \xi^{\pm}\partial_x. \tag{13}$$

Wave-breaking in 1D SW

Equation for derivatives of Riemann invariants :

$D^{\pm} \equiv \partial_x r^{\pm}$

$$\frac{dD^{\pm}}{dt^{\pm}} + \partial_x \xi^{\pm} D^{\pm} = 0, \ \xi^{\pm} = \frac{3}{4} r^{\pm} + \frac{1}{4} r^{\mp}, \Rightarrow \tag{14}$$

$$\frac{dD^{\pm}}{dt^{\pm}} + \frac{3}{4}\left(D^{\pm}\right)^2 + \frac{1}{4} D^{\pm} D^{\mp} = 0. \tag{15}$$

Suppose one of the invariants is identically zero $\Rightarrow$ Riccati equation along the characteristic for remaining D :

$$\frac{dD}{dt} + \frac{3}{4}(D)^2 = 0, \ \rightarrow D = (D_I^{-1} + \frac{3}{4} t)^{-1} \tag{16}$$

$\Rightarrow$ singularity in finite time, if initial D is negative.

1.5D RSW vs 1D SW

Eulerian 1D SW - 2 dependent variables (u, h). Eulerian 1.5D RSW - 3 variables (u, v, h). Pfaff theorem guarantees finding Riemann invariants only for 2 dependent variables.
Simplification in Lagrangian form - use of conservation of geostrophic momentum $M = v + fX \to v$ not independent, expressed in terms of X.

Quasi-linear Lagrangian 1.5D RSW

Mass-weighted variable a : $J = \frac{\partial X}{\partial a} = \frac{H}{h(X,t)}$. $\frac{\partial h}{\partial X} = \frac{\partial P}{\partial a}$, where $P = \frac{gH}{2J^2}$. Rewriting Lagrangian equations i terms of (u, J) :

$$\dot{u} - fv + gH\frac{\partial}{\partial a}\left(\frac{1}{2J^2}\right) = 0, \tag{17}$$

$$\dot{M} \equiv \dot{v} + fu = 0, \tag{18}$$

$$\dot{J} - \frac{\partial u}{\partial a} = 0, \tag{19}$$

Below : nondimensionalization $\to g, H$ out.

Hyperbolic structure

Quasi-linear system :

$$\begin{pmatrix} u \\ J \end{pmatrix}_t + \begin{pmatrix} 0 & -J^{-3} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ J \end{pmatrix}_a = \begin{pmatrix} v \\ 0 \end{pmatrix} . \qquad (20)$$

Eigenvalues of the matrix in the l.h.s. : $\mu_\pm = \pm J^{-\frac{3}{2}}$, corresponding left eigenvectors : $\left(1 \, , \pm J^{-\frac{3}{2}}\right)$.

Riemann invariants : $r_\pm = u \pm 2J^{-\frac{1}{2}}$

$$\partial_t r_\pm + \mu_\pm \partial_a r_\pm = v \, . \qquad (21)$$

Original variables in terms of $r_\pm$:

$$u = \frac{1}{2}(r_+ + r_-), \; J = \frac{16}{(r_+ - r_-)^2} > 0, \; \mu_\pm = \pm \left(\frac{r_+ - r_-}{4}\right)^3 . \qquad (22)$$

Equations for derivatives $D_{\pm} = \partial_a r_{\pm}$ of Riemann invariants

$$\partial_t D_{\pm} + \mu_{\pm}\partial_a D_{\pm} + \frac{\partial \mu_{\pm}}{\partial r_+} D_+ D_{\pm} + \frac{\partial \mu_{\pm}}{\partial r_-} D_- D_{\pm} = \partial_a v = Q(a) - J\,, \tag{23}$$

where potential vorticity (PV) $Q = \partial_a v + J$, a Lagrangian invariant.

Using derivatives along characteristics $\frac{d}{dt_{\pm}} = \partial_t + \mu_{\pm}\partial_a$:

$$\frac{dD_{\pm}}{dt_{\pm}} + \frac{\partial \mu_{\pm}}{\partial r_+} D_+ D_{\pm} + \frac{\partial \mu_{\pm}}{\partial r_-} D_- D_{\pm} = Q(a) - J\,. \tag{24}$$

Breaking corresponds to $D_{\pm} \to \pm\infty$ in finite time.

Conditions of shock formation

New variables $\mathcal{D}_\pm = e^\lambda D_\pm$, with $\lambda = \frac{3}{128} \log |r_+ - r_-| \rightarrow$

$$\frac{d\mathcal{D}_\pm}{dt_\pm} = -e^{-\lambda} \frac{\partial \mu_\pm}{\partial r_\pm} \mathcal{D}_\pm^2 + e^\lambda \left(Q(a) - J \right) , \qquad (25)$$

where $\frac{\partial \mu_\pm}{\partial r_\pm} = \frac{3}{64}(r_+ - r_-)^2 > 0$.
This is a generalized Ricatti equation, and from its qualitative analysis it follows that :

1. if initial relative vorticity $Q - J = \partial_a v$ is sufficiently negative, **breaking** takes place whatever initial conditions are
2. if the relative vorticity is positive as well as the derivatives of the Riemann invariants at the initial moment, there is **no breaking**.

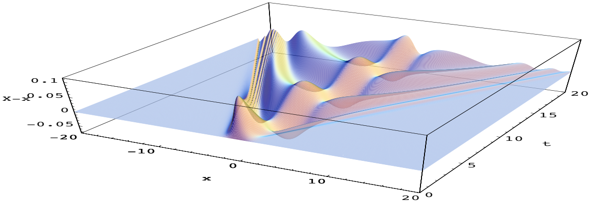
Geostrophic adjustment

Master equation for initial-value problem in terms of $\chi(x,t) = X(x,t) - x$:

$$\ddot{\chi} + f^2\chi + gh_I' \frac{1}{(1+\chi')^2} + \frac{1}{2} g h_I \left[\frac{1}{(1+\chi')^2}\right]' = f v_I \; . \quad (26)$$

$\dot{\chi}(t=0) = u_I(x)$

If $gh_I' = fv_I$, a geostrophic equilibrium, $\Rightarrow \chi \equiv 0$ solution $\leftrightarrow$ steady state. Relaxation to geostrophic equilibrium by minimizing energy by wave emission = geostrophic adjustment. Example : initial unbalanced localized bump in h.



Existence of adjusted state

Reduced of quasi-linear system to a single equation :

$$\ddot{J} + f^2 J + \frac{\partial^2 P}{\partial a^2} = fHQ\,, \qquad (27)$$

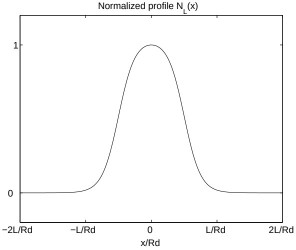
where $Q(a) = \frac{1}{H}\left(\frac{\partial v}{\partial a} + fJ\right) = \frac{1}{H}\left(\frac{\partial v_I}{\partial a} + fJ_I\right)$.
Adjusted state $\equiv$ stationary solution of (27). Re-introducing the h and X variables :

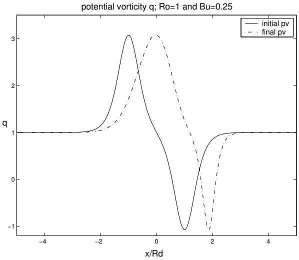
$$-\frac{g}{f}\,\frac{d^2 h(X)}{dX^2} + h(X)\,Q(X) = -f. \qquad (28)$$

Potential vorticity in terms of initial height and velocity :
$Q(X(x)) = \frac{f + \frac{\partial v_I}{\partial x}}{h_I}$.
Theorem. *Equation (28) has unique bounded and everywhere positive solution $h(X)$ on R for positive $Q(X)$ with compact support and constant asymptotics.*

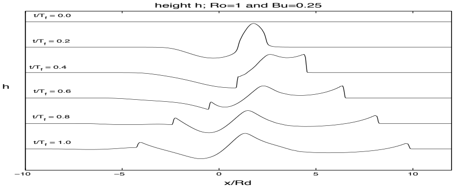
Numerical simulations of Rossby adjustment

Initial state and evolution of potential vorticity





Adjustment process



Master equation for stationary waves

Stationary-wave solutions of (17, 18, 19) :
$u = u(\xi)\,, v = v(\xi), J = J(\xi),\ \xi = a - c\,t$.
Eliminating u : $u = \frac{c}{f} v'$ ("$'$" $= \frac{d}{d\xi}$) $\Rightarrow$
$fJ + v' = \text{const} = QH$. Elimination of u and $J \rightarrow$

$$v'' + \frac{f^2}{c^2} v + \frac{gH}{2c^2} f^3 \left(\frac{1}{(f - v')^2} \right)' = 0 \qquad (29)$$

Integrating once, after multiplying it by $(c^2/f^2)v' \Rightarrow$

$$\mathcal{H} = \frac{1}{2} \left(\frac{c^2}{f^2} v'^2 + v^2 - gH \frac{v'^2}{(f - v')^2} \right) = const. \qquad (30)$$

Using $v' = f(1 - J)$ and $fv = c^2 J' + gH \left(1/2J^2\right)'$

$$\mathcal{H} = \frac{1}{2} \left[R_d^2 \left[M^2 J' + \left(\frac{1}{2J^2} \right)' \right]^2 + M^2 (1 - J)^2 - \frac{(1 - J)^2}{J^2} \right], \qquad (31)$$

$M = c/c_0,\ c_0 = \sqrt{gH},\ R_d = c_0/f$.

Equivalent particle-in-a-well problem

A 'particle' moving on the zero-energy level :

$$\frac{J'^2}{2} + \mathcal{U}(J) = 0\,, \tag{32}$$

J - 'particle's' coordinate, ξ - 'time',

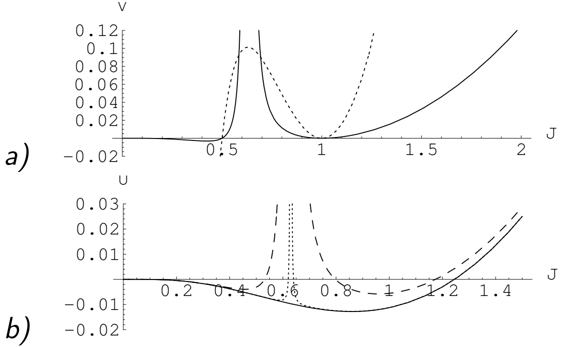
$$\mathcal{U}(J) = \frac{1}{R_d^2} \frac{V(J) - \mathcal{H}}{\left(M^2 - J^{-3}\right)^2}$$

is a singular 'potential' built from 'pre-potential'

$$V(J) = \frac{(1-J)^2}{2}\left(M^2 - J^{-2}\right)$$

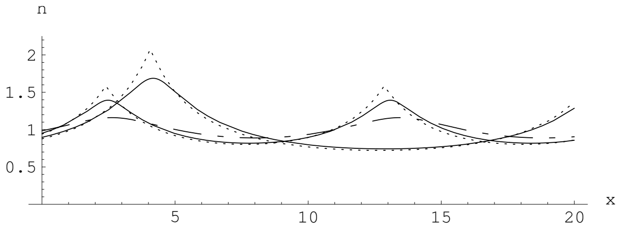
Turning points $\leftrightarrow$ zeros of the potential.
Stationary-wave solution $\Leftrightarrow$ potential well bounded by two positive zeros.

Pre-potential and potential for different $\mathcal{H}$



a) 'Prepotential' $V(J)$ (dashed), for $M = 2$, 'potential' $\mathcal{U}(J)$ (solid) for $\mathcal{H} = 0$ and $R_d = 1$.
b) 'Potential' $\mathcal{U}(J)$ for three values of the constant $\mathcal{H}$: the critical value $\mathcal{H}_c = 0.1013\ldots$ (solid), 0.101 (dotted curve), and 0.05 (dashed curve). A nonlinear wave can exist for values of $\mathcal{H}$ such that the potential has two zeros for strictly positive values of J. For $\mathcal{H} = \mathcal{H}_c$ this is no longer the case.

Profiles of stationary waves



Height profiles of stationary nonlinear waves in physical space for various values of M and $\mathcal{H}$, with $R_d = 1$. Shorter wavelength : $M = 2$, longer wavelength : $M = 3$. Limiting asymptotics ($\mathcal{H} = \mathcal{H}_c$) : dotted ; solid lines $\leftrightarrow$ $\mathcal{H} = 0.9\mathcal{H}_c$ in both cases. Wave with $M = 2$, $\mathcal{H} = 0.5\mathcal{H}_c$: dash-dotted curve. Maximum amplitude and wavelength increase with M.

Periodic nonlinear waves in OH equation

Renormalized equation for steady-moving waves $\eta = \eta(x - cT)$:

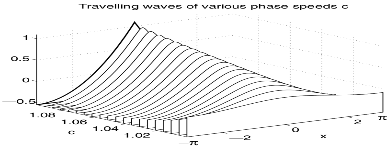
$$\left(\eta_T + \eta\eta'\right)' - \eta = 0 \;\rightarrow\; \left(-c\,\eta + \eta\eta'\right)' - \eta = 0, \tag{33}$$

Solvable in elliptic functions. Continuous 2π-periodic solutions exist only if

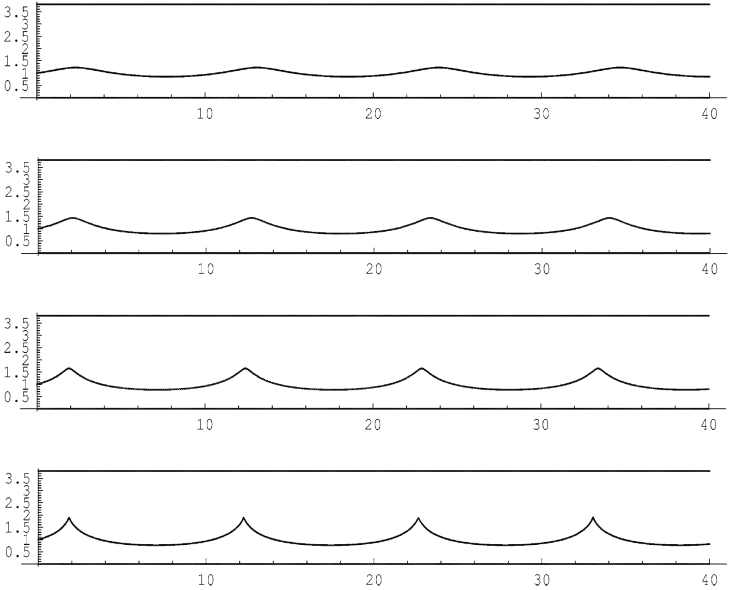
$$1 \leq c \leq \frac{\pi^2}{9}, \tag{34}$$

with the limiting-amplitude cusp wave

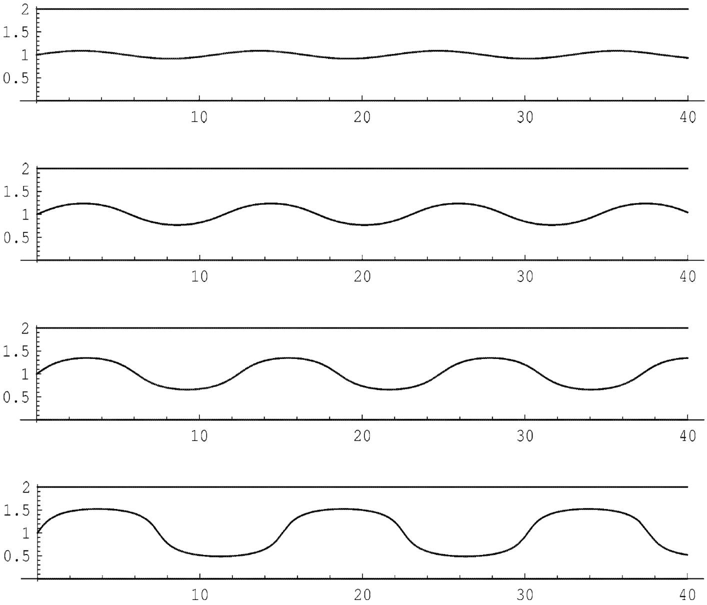
$$\eta(x) = \frac{\pi^2}{9} - \frac{\pi}{3}|x| + \frac{1}{2}x^2 \tag{35}$$



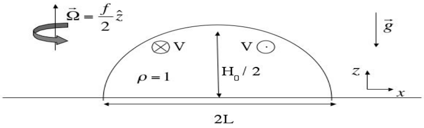
Waves of increasing amplitude : family 1



Waves of increasing amplitude : family 2



Double outcropping front



Adjustment : no outward wave emission. The "pulson" ansatz

$$X(x,t) = x\chi(t), \quad h_I(x) = \frac{h_0}{2}\left(1 - \frac{x^2}{L^2}\right), \quad v_I(x) = xV, \tag{36}$$

h_0, V, L are constants. Plugging it in the master equation :

$$\ddot{X} + f^2 X + g h_I' \frac{1}{(X')^2} + \frac{g h_I}{2}\left[\frac{1}{(X')^2}\right]' = fM,$$

and non-dimensionalizing results in the ODE for χ :

$$\ddot{\chi} + \chi - \frac{\gamma}{\chi^2} = \mu, \tag{37}$$

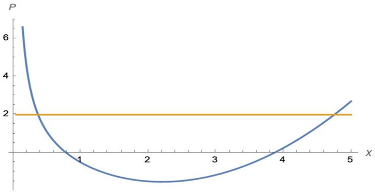
γ - Burger number $\frac{gh_0}{f^2L^2}$, $\mu = 1 + \frac{V}{f}$.

Equivalent particle-in-a-well problem

Integrating once :

$$\frac{\dot{\chi}^2}{2} + P(\chi) = E, \quad P(\chi) = \frac{\chi^2}{2} - \mu\chi + \frac{\gamma}{\chi}, \qquad (38)$$

integration constant E - from initial conditions $\chi(t=0) = 1, \dot{\chi}(t=0) = U : E = \frac{U^2}{2} + \frac{1}{2} - \mu + \gamma$. Equation (38) can be integrated in elliptic functions.
Example : "potential" $P(\chi)$ for $\mu = 2$, $\gamma = 1$:



Solution : finite-amplitude, oscillating with supra-inertial frequency.

Master equation for 1.5D TRSW

Lagrangian 1.5D TRSW

$$\dot{v} + f\dot{X} = 0, \quad \dot{b} = 0 \tag{39}$$

$$\ddot{X} - fv + bh_X + \frac{h}{2} b_X = 0, \tag{40}$$

with $h(X)\,\mathrm{d}X = h_0(x)\,\mathrm{d}x \quad \Longleftrightarrow \quad h(X)X' = h_I(x)$. Direct integration of (39) $\Rightarrow$

$$v(X,t) = v_0(x) - f(X(x,t) - x) \quad \text{and} \quad b(X,t) = b_I(x), \rightarrow \tag{41}$$

Master equation :

$$\ddot{X} + f^2 X + \frac{b_I}{X'}\left(\frac{h_I}{X'}\right)' + \frac{h_I b_I'}{2(X')^2} = f(v_I + fx), \tag{42}$$

h_I, v_I - initial values. In terms of parcel deviations : $\chi(y,t) = X(x,t) - x$:

$$\ddot{\chi} + f^2\chi + \frac{b_0}{1+\chi'}\left(\frac{h_0}{1+\chi'}\right)' + \frac{h_0 b_0'}{2(1+\chi')^2} = fv_0. \tag{43}$$

Hyperbolic structure of Lagrangian equations

Rewriting Lagrangian equations :

$$\begin{cases} \dot{u} + \dfrac{1}{h_I} P' = fv, \\ \dot{J} - u' = 0, \end{cases} \tag{44}$$

with $P = \frac{b_I h_I^2}{2(Y')^2}$. and $h_I = H$.

Quasi-linear form :

$$\begin{pmatrix} \dot{u} \\ \dot{J} \end{pmatrix} + A \cdot \begin{pmatrix} u \\ J \end{pmatrix}_y = \begin{pmatrix} fv + \frac{Hb_I'}{2J^2} \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -Hb_I J^{-3} \\ -1 & 0 \end{pmatrix};$$

Put $H = f = 1$ and proceed as in RSW.
Eigenvalues of A : $\mu_{\pm} = \pm\sqrt{b_I} J^{-3/2}$,
Left eigenvectors : $(1, \mp\sqrt{b_I} J^{-3/2}) \Rightarrow$
Riemann invariants : $r_{\pm} = u \pm 2\sqrt{b_I} J^{-1/2}$.

Derivatives of Riemann invariants

Differentiating in time :

$$\dot{r}_\pm + \mu_\pm (r_\pm)_x = v + \frac{b_l'}{2J^2}. \quad (45)$$

Differentiating with respect to $x \Rightarrow$ equations for $D_\pm := (r_\pm)_x$:

$$\dot{D}_\pm + \mu_\pm (D_\pm)_x + (\mu_\pm)_x D_\pm = v_x + \left(\frac{b_l'}{2J^2}\right)_x. \quad (46)$$

Reverse : $r_+ - r_- = 4\sqrt{b_l}J^{-1/2} \rightarrow \mu_\pm = \pm \frac{1}{b_l}\left(\frac{r_+ - r_-}{4}\right)^3, \Rightarrow$

$$\begin{aligned}\dot{D}_\pm \quad &+ \quad \mu_\pm (D_\pm)_x \mp \frac{b_l'}{b_l^2}\left(\frac{r_+ - r_-}{4}\right)^3 D_\pm + \frac{\partial \mu_\pm}{\partial r_+} D_+ D_\pm \\ &+ \quad \frac{\partial \mu_\pm}{\partial r_-} D_- D\pm = v_x + \left(\frac{b_l' h^2}{2}\right)_x, \quad (47)\end{aligned}$$

generalized Riccati equation.

Breaking criteria

Differences with RSW :

- First term (vorticity) on the r.h.s. of (47) acquires addition $\left(\frac{b_l' h^2}{2}\right)_x$ $\Rightarrow$ vorticity plus the new term depending on initial distributions of buoyancy and thickness should be sufficiently negative for breakdown to take place.
- Breakdown conditioned by signs of derivatives of Riemann invariants which depend not only on the signs of derivatives of v and h (as in RSW), but also on the sign of the derivative of b_0 $\Rightarrow$

Sign of b_l' is of crucial importance.

Thermo-geostrophic adjustment

Adjusted state - stationary solution of the equivalent master equation :

$$\ddot{J} + fJ + \left(\frac{1}{h_0} P'\right)' = 1 + f v_I', \tag{48}$$

Lagrangian pressure in TRSW :

$$P := \frac{b_I h_I^2}{2(X')^2}$$

Back to $h(X) = J^{-1}$, for $h_I = H = 1$:

$$-\frac{\mathrm{d}}{\mathrm{d}X}\left(\sqrt{b_I}\,\frac{\mathrm{d}\left(\sqrt{b_I}\,h\right)}{\mathrm{d}X}\right) - Q(X)h(X) = -f, \tag{49}$$

$$Q := \frac{f + v'(X)}{h(X)} \text{ -- potential vorticity} \tag{50}$$

Existence of the adjusted state

Change of variables : $h \to \widehat{h} = \sqrt{b_I}\, h$ and $X \to \xi = \int \sqrt{b_I}\, \mathrm{d}X$

$$-\frac{1}{f}\frac{\mathrm{d}^2\widehat{h}}{\mathrm{d}\xi^2} + Q(\xi)\widehat{h} = f\sqrt{b_I} \Rightarrow \qquad (51)$$

Theorem

For positive monotone b_I with constant asymptotics at infinities (density/temperature front), and nonnegative potential vorticity $Q \geq 0$, there exist a unique solution of (51) *decaying at* $\pm\infty$

Master equation for stationary waves

Steady propagating waves : functions of $y - V\tau,\ V = \text{const}$

$$\left[\left[J + \frac{1}{2J^2}\right]' + \bar{\delta}\left[\frac{V^2 - c_a^2}{J^2}\left(\frac{1}{J}\right)'\right]'\right]'' + \gamma^2 J = 0. \qquad (52)$$

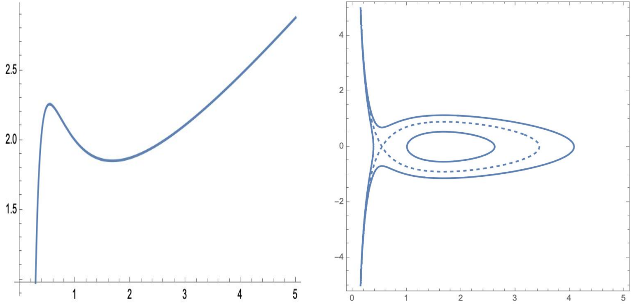
Prime : derivative with respect to $y - V\tau, \delta = 3\bar{\delta}$.
Non-rotating limit, integrating once $\rightarrow$

$$J + \frac{1}{2J^2} + \bar{\delta}\frac{1}{J^2}\left(\frac{1}{J}\right)'' = A = \text{const.} \quad h = J^{-1} \rightarrow \qquad (53)$$

$$\delta h'' + \frac{1}{2V^2} + \frac{1}{h^3} - \frac{A}{h^2} = 0 \leftrightarrow \bar{\delta}\frac{h'^2}{2} + \frac{h}{2V^2} - \frac{1}{2h^2} + \frac{A}{h} = E, \quad (54)$$

Particle-in-a-well problem, $\bar{\delta}$ - mass, h - position, E - energy, $\mathcal{V}(A, V; h) = \frac{h}{2V^2} - \frac{1}{2h^2} + \frac{A}{h}$ - potential.

Periodic and solitary stationary waves

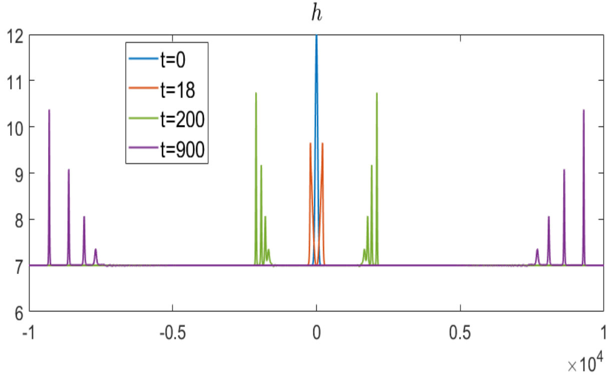


Potential $\mathcal{V}(A, V; h)$ at $A = 2, V = 1$ (left panel), and the corresponding phase portrait of the system (54) in the h, h' plane (right panel).

Solid, closed trajectory : periodic waves.

Dashed : separatrix trajectory, solitary wave.

Solitons in SGN system



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