

Global solutions of quasilinear wave equations

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1 Introduction

We show that the Cauchy problem in \mathbf{R}^{1+3} :

$$(1.1) \quad \tilde{\square}_{g(\phi)}\phi = 0, \quad \phi|_{t=0} = \phi_0, \quad \partial_t\phi|_{t=0} = \phi_1$$

has a global solution for all $t \geq 0$ if initial data are sufficiently small. Here the curved wave operator is $\tilde{\square}_g = g^{\alpha\beta}\partial_\alpha\partial_\beta$, where we used the convention that repeated upper and lower indices are summed over $\alpha, \beta = 0, 1, 2, 3$, and $\partial_0 = \partial/\partial t$, $\partial_i = \partial/\partial x^i$, $i = 1, 2, 3$. We assume that $g^{\alpha\beta}(\phi)$ are smooth functions of ϕ such that $g^{\alpha\beta}(0) = m^{\alpha\beta}$, where $m^{00} = -1$, $m^{11} = m^{22} = m^{33} = 1$ and $m^{\alpha\beta} = 0$, if $\alpha \neq \beta$. The result holds for vector valued ϕ , in particular for the principal part of Einstein's equations; $\phi^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$.

This result was conjectured in [L2] where it was also shown in the spherically symmetric case for

$$(1.2) \quad -\partial_t^2\phi + c(\phi)^2\Delta_x\phi = 0, \quad \text{where} \quad c(0) = 1.$$

In [L2] there was also a heuristic argument for why the conjecture should be true in general: Consider

$$(1.3) \quad \square\phi = a_{\alpha\beta}\partial^\alpha\phi\partial^\beta\phi + \text{cubic terms},$$

where here we used multiindex notation and the sum is over $0 \leq |\alpha| \leq |\beta| \leq 2$ and $a_{\alpha\beta}$ are constants. If we neglect derivatives tangential to the outgoing Minkowski light cones and cubic terms, that are known to decay faster, we get the asymptotic equation for $\Phi = r\phi$, introduced by Hörmander [H1, H2, H3]:

$$(1.4) \quad (\partial_t + \partial_r)(\partial_t - \partial_r)\Phi \sim r^{-1}A_{mn}(\partial_t - \partial_r)^m\Phi (\partial_t - \partial_r)^n\Phi, \quad A_{mn} = \frac{1}{4}\sum_{|\alpha|=m, |\beta|=n} a_{\alpha\beta}\hat{\omega}^\alpha\hat{\omega}^\beta, \quad \hat{\omega} = (-1, \omega).$$

Here we have introduced polar coordinates $x = r\omega$, $\omega \in \mathbf{S}^2$. The classical *null condition* introduced by Klainerman [K1] is that $A_{nm} \equiv 0$ under which Klainerman [K2] and Christodoulou [C] proved global existence. In [L2] it was observed that the asymptotic equation corresponding to (1.1) has global solution¹, contrary to other cases like $\square\phi = \phi_t\Delta\phi$ or $\square\phi = \phi_t^2$, where solutions are known to blow up for all small data, see John [J1, J2]. However, unlike for the classical null condition, the solution of (1.1) do not behave asymptotically like a solution of a free linear wave equation.

The method of proof of [L2] is integration along characteristics so it does not directly generalize to the non-symmetric case. However, as observed in [L1], the method of integration along characteristics

¹In [L-R1] we in general say that (1.3) satisfy the *weak null condition*, if (1.4) has global solution with some decay.

can still be used to obtain sharp decay estimates assuming weaker decay estimates that can be obtained from energy estimates for vector fields applied to the solution. This then has to be combined with some refined energy estimates that take into account that the characteristic surfaces curve asymptotically, since the solution do not decay as much as a solution of a free linear wave equation.

Recently Alinhac [A2] generalized the result in [L2] to general data for the special case (1.2)². [A2] combines ideas from [L1, L2] of how to obtain decay estimates with ideas from [A1] for energy estimates with weights. Because the asymptotic behavior is different from that of solutions to a free linear wave equation, [A2] constructs vector fields adapted to the characteristic surfaces at infinity, which in spirit is similar to the work of Christodoulou-Klainerman [C-K]. Since these depend on the solution itself, commuting the vector fields with the wave operator leads to a loss of regularity so it has to be combined with a smoothing procedure, which leads to long schematic commutator estimates.

There is however no need to construct vector fields adapted to the geometry at infinity. In fact we just use the vector fields for the Minkowski space time. In [L-R3], for Einstein's equations, we also got away with just using the regular vector fields, but only because we got additional control from the wave coordinate condition. The observations here will hopefully will lead to a proof that uses less of the special structure and applies to a more general class of equations, which is useful in applications.

As mentioned above the proof involves obtaining sharp decay estimates for low derivatives just assuming a weak decay estimate that later will be obtained from energy estimates for higher derivatives. The sharp decay estimates uses integration along characteristics as in [L1, L2, L-R2, L-R3]. We adopt the energy method with weights of [A2], depending on the solution of an approximate eikonal equation. This is a much easier substitute for energies on characteristic surfaces as I originally planned to use. The construction of vector fields adapted to the asymptotic behavior of the characteristic surfaces of [A2] is avoided by considering a family of energy and decay estimates for the vector fields of flat Minkowski space time, with different decays for different types of derivatives. We prove the following:

Theorem 1.1. *Suppose that ϕ_0 and ϕ are smooth functions such that $\phi_0(x) = \phi_1(x) = 0$, when $|x| \geq 1$, and let $N \geq 14$. Then there is a constant $\varepsilon_0 > 0$, such that if $\sum_{|\alpha| \leq N} (\|\partial \partial^\alpha \phi_0\|_{L^2} + \|\partial^\alpha \phi_1\|_{L^2}) = \varepsilon \leq \varepsilon_0$, the Cauchy problem (1.1) has a global solution ϕ for all $t \geq 0$. The solution satisfies the decay estimates in Proposition 6.1 with $\nu = 1 - c\varepsilon$, for some $c > 0$, and the energy estimates in Proposition 9.1.*

We remark that the result is true also for systems $\phi = (\phi_1, \dots, \phi_M)$, in particular the principal quasilinear part of Einstein's equations. We also remark that the assumptions on compact support is not needed and we can include decaying data using energy norms with weights as in [L-R2, L-R3].

Let us now give the strategy of the proof and the main ideas. The proof involves getting sharp decay estimates for low derivatives assuming weak decay estimates, and energy estimates for high derivatives assuming sharp decay estimates for low derivatives. The weak decay estimates can then be obtained from energy estimates using a bootstrap or continuity argument that we describe below. Let

$$(1.5) \quad E_N(t) = \sum_{|I| \leq N} \int |\partial Z^I \phi(t, x)|^2 dx,$$

where Z^I is a product of $|I|$ of the vector fields, $\Omega_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha$, $S = x^\alpha \partial_\alpha$ that span the tangent space of the forward light cone and have good commutators with the wave operator, and ∂_α . (Here $x_i = x^i$, $i \geq 1$, $x_0 = -x^0 = -t$.) In view of local existence results it suffices to give a bound for $E_N(t)$,

²As mentioned in [A3] the method of [A2] use the special structure of (1.2) and is unlikely to work in general.

which will be obtained through a **continuity argument**, see section 11. Fix $0 < \delta < 1$. Assuming that

$$(1.6) \quad E_N(t) \leq 16N\varepsilon^2(1+t)^\delta,$$

for $0 \leq t \leq T$, which holds for $T = 0$, we will show that this bound, implies the same bound with 16 replaced by 8 if ε is sufficient small (independently of T). Using Klainerman-Sobolev inequality and the assumption of compactly supported data, see sections 10, 11, this gives **weak decay estimates**:

$$(1.7) \quad |Z^I \phi(t, x)| \leq c_0 \varepsilon (1+t)^{-\nu}, \quad \nu > 0, \quad |I| \leq N-2.$$

These weak decay estimates imply the sharp decay estimates in Proposition 6.1, as well as the estimates for the approximate radial characteristic surfaces in Proposition 5.1, Lemma 5.2 and Lemma 5.3. These sharp decay estimates for low derivatives are sufficient for the energy estimate in Proposition 9.1 to hold and we therefore get back a stronger energy estimate if $\varepsilon > 0$ is sufficient small:

$$(1.8) \quad E_N(t) \leq 8N\varepsilon^2(1+t)^{C_{0,N}\varepsilon}.$$

We now give the main ideas for the **sharp decay estimates**. We will try to mimic the integration along characteristic that was done in the radial case in [L2], by expressing the wave operator in spherical coordinates and a null-frame, using the weak decay estimates to control the angular derivatives. The discussion below will be a bit technical, but it is useful to get a feeling for how the different kind of terms are dealt with since the structure of the argument is the same also for the energy estimates.

In section 2 we express the inverse of the metric in terms of a nullframe:

$$(1.9) \quad g^{\alpha\beta} = -\frac{1}{2}(L_1^\alpha \underline{L}^\beta + \underline{L}^\alpha L_1^\beta) + \gamma^{\alpha\beta},$$

where

$$(1.10) \quad L_1^\alpha = L^\alpha - H^{\underline{L}\underline{L}}\underline{L}^\alpha - 2H^{\underline{L}L}L^\alpha - 2H^{\underline{L}A}A^\alpha,$$

$$(1.11) \quad \gamma^{\alpha\beta} = \delta^{AB}A^\alpha B^\beta + H^{LL}L^\alpha L^\beta + H^{AL}A^\alpha \underline{L}^\beta + H^{AL}L^\alpha A^\beta + H^{AB}A^\alpha B^\beta.$$

Here $H^{\underline{L}\underline{L}}$ etc. are the components of $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ in the Minkowski null frame

$$(1.12) \quad \underline{L} = (1, -\omega), \quad L = (1, \omega), \quad A, B \in \mathbf{S}^2, \quad \delta_{\alpha\beta}A^\alpha B^\beta = \delta_{AB}.$$

In section 3 we use (1.9) to decompose the wave operator:

$$(1.13) \quad \left| \left(2L_1^\alpha \partial_\alpha + \frac{\ell}{r} \right) (r \partial_q \phi) - r \tilde{\square}_g \phi \right| \lesssim \sum_{1 \leq |k| \leq 2} r^{|k|-1} |\bar{\partial}^k \phi|, \quad \partial_q = \frac{1}{2}(\partial_r - \partial_t),$$

where $\ell = \delta_{AB}H^{AB} + 4H^{\underline{L}L} - 2H^{LL}$, and $\bar{\partial} \in \{L, A, B\}$ are derivatives tangential to the outgoing Minkowski light cones, that can be estimated in terms of the vector fields:

$$(1.14) \quad (1 + |t - r|)|\partial\phi| + (1 + t + r)|\bar{\partial}\phi| \lesssim \sum_{|I|=1} |Z^I \phi|.$$

Note that when $|t - r| > t/2$ this together with (1.7) gives the sufficient εt^{-1} decay for all derivatives but when $|t - r|$ is close to the light cone we are missing one derivative perpendicular to the light cone.

In section 4 we integrate (1.13) along the flow lines of the vector field $2L_1^\alpha \partial_\alpha$, from $|t - r| = t/2$, to also get an estimate for a derivative perpendicular to the outgoing light cones $r \partial_q \phi$ which yields

$$(1.15) \quad (1 + t + r) |\partial \phi(t, x)| \leq C \sup_{0 \leq \tau \leq t} \sum_{|I| \leq 1} \|Z^I \phi(\tau, \cdot)\|_{L^\infty} \\ + C \int_0^t \left((1 + \tau) \|\tilde{\square}_g \phi(\tau, \cdot)\|_{L^\infty(D_\tau)} + \sum_{|I| \leq 2} (1 + \tau)^{-1} \|Z^I \phi(\tau, \cdot)\|_{L^\infty(D_\tau)} \right) d\tau,$$

If $H = 0$, (1.13) is the decomposition in radial and spherical coordinates and (1.15) was used in [L1].

In section 6.1 we use the weak decay estimates (1.7) in (1.15) to get the **sharp decay estimates**

$$(1.16) \quad |\partial \phi| \leq c_1 \varepsilon (1 + t)^{-1}, \quad |\phi| \lesssim c_1 \varepsilon (1 + |t - r|) (1 + t)^{-1}.$$

The last inequality follows by integrating the first from $r = t + 1$ where ϕ vanishes. If (1.16) had been true also for $Z\phi$ it would have been easy, but there is a small loss that requires a delicate analysis.

With the sharper decay (1.16) for $H(\phi)$ the decomposition of the wave operator (1.13) simplifies to

$$(1.17) \quad \left| L_2^\alpha \partial_\alpha (p \partial_q \phi) - r \tilde{\square}_g \phi \right| \leq \frac{C}{1 + t} \sum_{|I| \leq 2} |Z^I \phi|,$$

where $p = r + t$ and

$$(1.18) \quad L_2^\alpha \partial_\alpha = (L^\alpha - H^{\underline{L}\underline{L}} L^\alpha) \partial_\alpha = (\partial_t + \partial_r) + H^{\underline{L}\underline{L}} (\partial_r - \partial_t).$$

In section 5 we study the integral curves of the vector field (1.18) since we will integrate (1.17). Let $q = r - t$, $p = r + t$ and $\omega = x/|x|$, and introduce the radial characteristics $q = q(s, \rho, \omega)$ by

$$(1.19) \quad dq/ds = 2H^{\underline{L}\underline{L}}, \quad \text{when } |t - r| \leq t/2, \quad q = \rho, \quad \text{when } |t - r| \geq t/2, \quad p = 2s.$$

Equivalently let ρ be the solution of a radial eikonal equation:

$$(1.20) \quad L_2^\alpha \partial_\alpha \rho = 0, \quad \text{when } |t - r| \leq t/2, \quad \rho = r - t, \quad \text{when } |t - r| > t/2.$$

ρ behaves roughly like q :

$$(1.21) \quad \left(\frac{1 + t}{1 + |\rho|} \right)^{-c_1 \varepsilon} \leq \frac{\partial \rho}{\partial q} \leq \left(\frac{1 + t}{1 + |\rho|} \right)^{c_1 \varepsilon}, \quad \left(\frac{1 + t}{1 + |\rho|} \right)^{-c_1 \varepsilon} \leq \frac{1 + |\rho|}{1 + |q|} \leq \left(\frac{1 + t}{1 + |\rho|} \right)^{c_1 \varepsilon}.$$

This comes from differentiating (1.20): $L_2^\alpha \partial_\alpha \partial_q \rho + 2 \partial_q H^{\underline{L}\underline{L}} \partial_q \rho = 0$, and multiplying by the integrating factor using the estimate (1.16) for $H(\phi)$, as was observed in the spherically symmetric case in [L2].

In section 6.2 we prove the following **sharp decay estimates** for second derivatives:

$$(1.22) \quad |\partial \phi| \leq \frac{c_1 \varepsilon}{1 + t} \frac{1}{(1 + |\rho|)^\nu}, \quad |\partial^2 \phi| \leq \frac{c_2 \varepsilon}{1 + t} \left| \frac{\partial \rho}{\partial q} \right| \frac{1}{(1 + |\rho|)^{1 + \nu}}, \quad \nu > 0.$$

The first estimate follows from integrating (1.17) along the integral curves of L_2 from $t = 2|\rho|$ using (1.7). For the proof of the second we note that since $[\partial_\rho, L_2^\alpha \partial_\alpha] = 0$ it follows from (1.17) (c.f. [L2])

$$(1.23) \quad \left| L_2^\alpha \partial_\alpha (p \partial_\rho \partial_q \phi) - r \partial_\rho \tilde{\square}_g \phi \right| \leq \frac{C \rho_q^{-1}}{1+t+r} \sum_{|I| \leq 2} |\partial Z^I \phi|, \quad \text{where } \partial_\rho = \rho_q^{-1} \partial_q.$$

The second estimate in (1.22) follows from integrating (1.23) using (1.14) and (1.21).

For vector fields we are not quite as lucky and there is a loss in the **strong decay estimate**:

$$(1.24) \quad |Z\phi| \leq c_2 \varepsilon (1+t)^{-1+c_2\varepsilon} (1+|q|)^{1-\nu} \leq \frac{c_2 \varepsilon}{1+t} \left((1+|q|) + (1+t)^{c_2\varepsilon/\nu} \right).$$

In fact, by (1.22) the commutator is

$$(1.25) \quad \sum_{|I| \leq 1} |\tilde{\square}_g Z^I \phi| \leq C \sum_{|I| \leq 1} |Z^I H| |\partial^2 \phi| \leq \frac{c_2 \varepsilon \rho_q}{1+t} \sum_{|I| \leq 1} \frac{|Z^I \phi|}{1+|\rho|}.$$

If we use (1.17) applied to $Z^I \phi$ in place of ϕ and also commute ρ_q^{-1} through (1.17), we get an extra term due to that $L_2^\alpha \partial_\alpha \rho_q = -2\rho_q \partial_q H^{LL}$:

$$(1.26) \quad \sum_{|I| \leq 1} \left| L_2^\alpha \partial_\alpha (p \partial_\rho Z^I \phi) \right| \leq c_2 \varepsilon \sum_{|I| \leq 1} \left(\frac{|Z^I \phi|}{1+|\rho|} + |\partial_\rho Z^I \phi| \right) + \frac{C \rho_q^{-1}}{1+t+r} \sum_{|I| \leq 3} |Z^I \phi|.$$

Since $\phi = 0$ when $\rho \leq -1$ we can estimate $|Z^I \phi|/(1+|\rho|)$ by the derivative $|\partial_\rho Z^I \phi|$ we get if we first integrate (1.26) from $t = 2|\rho|$ where we can use (1.7) and (1.14):

$$(1.27) \quad M(t) \leq \int_1^t \frac{c_2 \varepsilon}{1+\tau} M(\tau) d\tau + c_0 \varepsilon \quad \text{where } M(t) = (1+t) \sup_{|\rho| \leq t/2} (1+|\rho|)^\nu \sum_{|I| \leq 1} |\partial_\rho Z^I \phi|$$

which by a Gronwall type of argument implies that $M(t) \leq c_0 (1+t)^{c_2\varepsilon}$ from which (1.24) follows.

For more vector fields there is a problem with the most straightforward approach. We have

$$(1.28) \quad |\tilde{\square}_g Z^I \phi| \lesssim |Z^I \phi| |\partial^2 \phi| + \sum_{|J| \leq 1, |K|=|I|-1} |Z^J \phi| |\partial^2 Z^K \phi| + \sum_{|J|+|K| \leq |I|, |J| < |I|, |K| < |I|-1} |Z^J \phi| |\partial^2 Z^K \phi|.$$

The first term can be handled as above and the terms in the last are lower order. However the problem is the term with $|K| = |I| - 1$ and $|J| = 1$ which is highest order. Using (1.14) and (1.24)

$$(1.29) \quad |Z\phi| \sum_{|K|=|I|-1} |\partial^2 Z^K \phi| \leq \frac{c_2 \varepsilon}{1+t} \sum_{|J|=|I|} |\partial Z^J \phi| + \frac{c_2 \varepsilon}{(1+t)^{1-c_2\varepsilon/\nu}} \sum_{|K|=|I|-1} |\partial^2 Z^K \phi|.$$

The first sum can be handled as above, however the lack of decay in the last sum cause a problem. The estimate (1.24) can not be improved to get the needed $\varepsilon(1+t)^{-1}$ decay close to the light cone. One could use modified vector fields that take into account the bending of the light cones at infinity. The modified rotations are defined by $(\tilde{\Omega}\phi)(q, p, \omega) = \Omega(\phi(q(\rho, s, \omega), 2s, \omega))$, where $q(\rho, s, \omega)$ is as in (1.19). This however leads to the loss of regularity encountered by [A2] for the energy estimate. We will

take a different approach. We can handle a $t^{c\varepsilon}$ loss in terms of quantities we have already estimated. Therefore we will first estimate $\partial^2 Z^K \phi$ in (1.29) before we estimate $\partial Z^I \phi$. In estimating $\partial^2 Z^K \phi$ a commutator term of the same form shows up with $\partial^2 Z^K \phi$ replaced by $\partial^3 Z^L \phi$, where $|L| = |K| - 1$, so we must first estimate $\partial^3 Z^L \phi$ and so on until finally we are left with $\partial^{1+|I|} \phi$.

In section 6.4 we use induction to prove the following **sharp decay estimate** for higher derivatives:

$$(1.30) \quad |\partial^k \phi| \leq \frac{c_k \varepsilon}{1+t} \left(\frac{1+t}{1+|\rho|} \right)^{c_k \varepsilon} \frac{1}{(1+|\rho|)^{k-1+\nu}}.$$

In fact, the commutator is

$$(1.31) \quad |\tilde{\square}_g \partial^n \phi| \leq C |\partial \phi| |\partial^{1+n} \phi| + C \sum_{k_1 + \dots + k_\ell = n+2, 1 \leq k_j \leq n, \ell \geq 2} |\partial^{k_1} \phi| \dots |\partial^{k_\ell} \phi|$$

where the first term has sufficient decay since $|\partial \phi| \leq c_1 \varepsilon (1+t)^{-1}$ and the second sum is lower order.

Finally in section 6.5 we use induction in k as described above to show that

$$(1.32) \quad |\partial^{i-k} Z^K \phi| \leq c_{k,i} \varepsilon (1+t)^{-1+c_{k,i} \varepsilon} (1+|\rho|)^{1-(i-k)-\nu}; \quad i = |I|, k = |K|.$$

2 Expressing the metric and the wave operator in the null frame

We introduce a nullframe for the Minkowski metric, $\mathcal{U} = \{L, \underline{L}, S_1, S_2\}$, where

$$(2.1) \quad L^0 = 1, \quad L^i = \omega^i, \quad \underline{L}^0 = 1, \quad \underline{L}^i = -\omega^i, \quad \omega^i = \frac{x^i}{|x|}, \quad i = 1, 2, 3,$$

and S_1 and S_2 are two smooth orthonormal vector fields on the tangent space of the sphere $T(S^2)$. (We remark that these only exist locally so one has to work in a coordinate chart.) We will raise and lower the indices with respect to the Minkowski metric $V_\alpha = m_{\alpha\beta} V^\beta$, $V^\alpha = m^{\alpha\beta} V_\beta$, $m^{\alpha\beta} = m_{\alpha\beta}$. (Here $m_{00} = -1$, $m_{ii} = 1$, $i = 1, 2, 3$, $m_{\alpha\beta} = 0$, if $\alpha \neq \beta$.) We can express a vector field X or the corresponding one form in the nullframe

$$(2.2) \quad X^\alpha = X^L L^\alpha + X^{\underline{L}} \underline{L}^\alpha + X^A A^\alpha, \quad X_\alpha = X^L L_\alpha + X^{\underline{L}} \underline{L}_\alpha + X^A A_\alpha.$$

Here and in what follows A, B, C, \dots denotes any of the vectors S_1, S_2 , and we used the convention that we sum over repeated upper and lower indices;

$$(2.3) \quad X^A A^\alpha = X^{S_1} S_1^\alpha + X^{S_2} S_2^\alpha.$$

The components can be calculated from the contractions:

$$(2.4) \quad X^L = -\frac{1}{2} X_{\underline{L}}, \quad X^{\underline{L}} = -\frac{1}{2} X_L, \quad X^A = X_A, \quad \text{where} \quad X_Y = X_\alpha Y^\alpha = m_{\alpha\beta} X^\alpha Y^\beta = Y_X.$$

(This follows since $L^\alpha \underline{L}_\alpha = -2$, $L^\alpha L_\alpha = \underline{L}^\alpha \underline{L}_\alpha = 0$, $L^\alpha A_\alpha = \underline{L}^\alpha A_\alpha = 0$, and $A^\alpha B_\alpha = \delta_{AB}$.) Recall that the inverse of the Minkowski metric $m^{\alpha\beta}$ can be expressed in a nullframe

$$m^{\alpha\beta} = -\frac{1}{2} (L^\alpha \underline{L}^\beta + \underline{L}^\alpha L^\beta) + \delta_{AB} A^\alpha B^\beta,$$

where $\delta_{AB}A^\alpha B^\beta = S_1^\alpha S_1^\beta + S_2^\alpha S_2^\beta$. We make a similar decomposition for the bilinear form $g^{\alpha\beta}$:

$$(2.5) \quad g^{\alpha\beta} = g^{UV}U^\alpha V^\beta.$$

Here and in what follows U, V, W, \dots denotes any vector in $\mathcal{U} = \{L, \underline{L}, S_1, S_2\}$, and we used the convention that we sum over repeated upper and lower indices. The components can be calculated in terms of the contractions as follows:

$$(2.6) \quad g^{L\underline{L}} = \frac{1}{4}g_{\underline{L}L}, \quad g^{LL} = \frac{1}{4}g_{L\underline{L}}, \quad g^{\underline{L}\underline{L}} = \frac{1}{4}g_{LL}, \quad g^{LA} = -\frac{1}{2}g_{\underline{L}A}, \quad g^{\underline{L}A} = -\frac{1}{2}g_{LA}, \quad g^{AB} = g_{AB},$$

where

$$(2.7) \quad g_{UV} = g^{\alpha\beta}U_\alpha V_\beta = g_{\alpha\beta}U^\alpha V^\beta, \quad \text{if } g_{\alpha\beta} = m_{\alpha\alpha'}m_{\beta\beta'}g^{\alpha'\beta'}$$

denotes the lowering of indices with respect to the Minkowski metric and not the inverse of $g^{\alpha\beta}$.

Lemma 2.1. *Suppose that $g^{\alpha\beta}$ is symmetric. Then*

$$(2.8) \quad g^{\alpha\beta} = -\frac{1}{2}(L_1^\alpha \underline{L}^\beta + \underline{L}^\alpha L_1^\beta) + \gamma^{\alpha\beta},$$

where

$$(2.9) \quad L_1^\alpha = g^{\mu\alpha}L_\mu + \frac{1}{4}g^{\mu\nu}L_\mu L_\nu \underline{L}^\alpha = -\frac{1}{2}g_{L\underline{L}}L^\alpha - \frac{1}{4}g_{LL}\underline{L}^\alpha + g_L^A A^\alpha,$$

$$(2.10) \quad \gamma^{\alpha\beta} = g^{LL}L^\alpha L^\beta + g^{AL}A^\alpha L^\beta + g^{AL}L^\alpha A^\beta + g^{AB}A^\alpha B^\beta.$$

Proof. Using (2.2) and (2.5) we can write

$$(2.11) \quad g^{\alpha\beta} = g^{\underline{L}\underline{L}}\underline{L}^\alpha \underline{L}^\beta + g^{LL}L^\alpha \underline{L}^\beta + g^{\underline{L}L}\underline{L}^\alpha L^\beta + g^{\underline{L}A}\underline{L}^\alpha A^\beta + g^{AL}A^\alpha \underline{L}^\beta + \gamma^{\alpha\beta},$$

$$(2.12) \quad g^{\alpha\beta} = g^{\underline{L}\beta}\underline{L}^\alpha + g^{\alpha\underline{L}}\underline{L}^\beta - g^{\underline{L}\underline{L}}\underline{L}^\alpha \underline{L}^\beta + \gamma^{\alpha\beta},$$

and the lemma follows from using (2.4) and (2.6). □

Let us introduce some further notation

$$(2.13) \quad \partial_q = \frac{1}{2}(\partial_r - \partial_t) = -\frac{1}{2}\underline{L}^\alpha \partial_\alpha, \quad \partial_p = \frac{1}{2}(\partial_r + \partial_t) = \frac{1}{2}L^\alpha \partial_\alpha,$$

and

$$(2.14) \quad \bar{\partial} = \{\partial_L, \partial_{S_1}, \partial_{S_2}\}, \quad \text{where } \partial_Y = Y^\alpha \partial_\alpha.$$

Lemma 2.2. *Suppose that $g^{\alpha\beta}$ is a symmetric and bounded. Then*

$$(2.15) \quad |g^{\alpha\beta} \partial_\alpha \partial_\beta \phi - g_{LL} \partial_q^2 \phi| \leq C |\bar{\partial} \partial \phi|$$

and with $\overline{\text{tr}} g = \delta_{AB} g^{AB}$ we have

$$(2.16) \quad \left| 2L_1^\alpha \partial_\alpha \partial_q \phi + \frac{\overline{\text{tr}} g}{r} \partial_q \phi - g^{\alpha\beta} \partial_\alpha \partial_\beta \phi \right| \leq C \sum_{1 \leq |k| \leq 2} r^{|k|-2} |\bar{\partial}^k \phi|.$$

Furthermore

$$(2.17) \quad \left| g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho + (\partial_q \rho) L_1^\alpha \partial_\alpha \rho - \delta^{AB} \partial_{A\rho} \partial_{B\rho} \right| \lesssim (|H_{LL}| + |H_{LA}| + |H_{AB}|) |\bar{\partial} \rho|^2,$$

where $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$.

Proof. (2.15) and (2.17) follow directly from (2.8). By (2.8)

$$(2.18) \quad g^{\alpha\beta} \partial_\alpha \partial_\beta \phi = -L_1^\alpha \underline{L}^\beta \partial_\alpha \partial_\beta \phi + \gamma^{\alpha\beta} \partial_\alpha \partial_\beta \phi.$$

Now

$$(2.19) \quad -L_1^\alpha \underline{L}^\beta \partial_\alpha \partial_\beta \phi = -L_1^\alpha \partial_\alpha (\underline{L}^\beta \partial_\beta \phi) + (L_1^\alpha \partial_\alpha \underline{L}^\beta) \partial_\beta \phi.$$

Note that if $\omega_j = x_j/|x|$ then

$$(2.20) \quad \partial_i \omega_j = \frac{1}{r} (\delta_{ij} - \omega_i \omega_j), \quad \text{so} \quad L^i \partial_i \omega^j = \underline{L}^i \partial_i \omega^j = 0, \quad A^i \partial_i \omega^j = \frac{1}{r} A^j.$$

Hence $L_1^\alpha \partial_\alpha \underline{L}^\beta = -L_1^A A^\beta / r = -g_L^A A^\beta / r$ and it follows that

$$(2.21) \quad -L_1^\alpha \underline{L}^\beta \partial_\alpha \partial_\beta \phi = 2L_1^\alpha \partial_\alpha \partial_q \phi + g_L^A r^{-1} A^\alpha \partial_\alpha \phi.$$

We have

$$(2.22) \quad \begin{aligned} \gamma^{\alpha\beta} \partial_\alpha \partial_\beta \phi &= g^{LL} L^\alpha L^\beta \partial_\alpha \partial_\beta \phi + 2g^{LA} A^\alpha L^\beta \partial_\alpha \partial_\beta \phi + g^{AB} A^\alpha B^\beta \partial_\alpha \partial_\beta \phi \\ &= g^{LL} L^\alpha \partial_\alpha (L^\beta \partial_\beta \phi) + 2g^{LA} A^\alpha \partial_\alpha (L^\beta \partial_\beta \phi) + g^{AB} A^\alpha \partial_\alpha (B^\beta \partial_\beta \phi) - g^{AB} (A^\alpha \partial_\alpha B^\beta) \partial_\beta \phi \end{aligned}$$

Since $\omega_j B^j = 0$ it follows that

$$(2.23) \quad (A^i \partial_i B^j) \omega_j = -A^i B^j \partial_i \omega_j = -A_j B^j \frac{1}{r} = -\frac{1}{r} \delta_{AB}, \quad \partial_i = \omega_i \partial_r + \bar{\partial}_i,$$

where $\bar{\partial}_i$ is tangential it follows that

$$(2.24) \quad -g^{AB} (A^\alpha \partial_\alpha B^\beta) \partial_\beta \phi = g^{AB} \delta_{AB} \frac{1}{r} \partial_r \phi - g^{AB} (A^i \partial_i B^j) \bar{\partial}_j \phi.$$

Since B^j are smooth functions of ω it follows that $|A^i \partial_i B^j| \leq C/r$. The lemma therefore follows from the above identities. \square

3 The vector fields associated with the wave operator, commutators.

Let $Z \in \mathcal{Z}$ be any of the vector fields

$$\Omega_{\alpha\beta} = -x_\alpha\partial_\beta + x_\beta\partial_\alpha, \quad S = t\partial_t + r\partial_r, \quad \partial_\alpha,$$

where $x_0 = -t$ and $x_i = x^i$, for $i \geq 1$. Let $I = (\iota_1, \dots, \iota_k)$, where $|\iota_i| = 1$, be an ordered multiindex of length $|I| = k$ and let $Z^I = Z^{\iota_1} \cdots Z^{\iota_k}$ denote a product of $|I|$ such derivatives. With a slight abuse of notation we will also identify the index set with vector fields, so $I = Z$ means the index I corresponding to the vector field Z . Furthermore, by a sum over $I_1 + I_2 = I$ we mean a sum over all possible order preserving partitions of the ordered multiindex I into two ordered multiindices I_1 and I_2 , i.e. if $I = (\iota_1, \dots, \iota_k)$, then $I_1 = (\iota_{i_1}, \dots, \iota_{i_n})$ and $I_2 = (\iota_{i_{n+1}}, \dots, \iota_{i_k})$, where i_1, \dots, i_k is any reordering of the integers $1, \dots, k$ such that $i_1 < \dots < i_n$ and $i_{n+1} < \dots < i_k$ and i_1, \dots, i_k . With this convention Leibnitz rule becomes $Z^I(fg) = \sum_{I_1+I_2=I} (Z^{I_1}f)(Z^{I_2}g)$.

We recall that the family \mathcal{Z} possesses special commutation properties: for any vector field $Z \in \mathcal{Z}$ $[Z, \square] = -C_Z \square$, where the constant C_Z is only different from zero in the case of the scaling vector field $C_S = 2$. Moreover $[Z, \partial_\alpha] = C_{Z\alpha}^\beta \partial_\beta$, for some constants $C_{Z\alpha}^\beta$. It is easy to show the following identities

$$(3.1) \quad \partial_t = \frac{tS - x^i\Omega_{0i}}{t^2 - r^2}, \quad \partial_i = \frac{-x^j\Omega_{ij} + t\Omega_{0i} - x_i S}{t^2 - r^2},$$

and for some smooth functions $f_A^{ij}(\omega)$;

$$(3.2) \quad \partial_L = \frac{S + \omega^i\Omega_{0i}}{t+r}, \quad \partial_A = \frac{f_A^{ij}(\omega)\Omega_{ij}}{r}, \quad \Omega_{ij} = \frac{r}{t}(\omega_j\Omega_{0i} - \omega_i\Omega_{0j}).$$

Recall that $\bar{\partial}$ denotes the tangential derivatives, i.e., ∂_T , where $T \in \mathcal{T} = \{L, S_1, S_2\}$.

Lemma 3.1. *For any function ϕ ;*

$$(3.3) \quad (1+t+r)|\bar{\partial}\phi| + (1+|t-r|)|\partial\phi| \leq C \sum_{|I|=1} |Z^I\phi|,$$

$$(3.4) \quad (1+t+r)|\partial\phi| \leq Cr|\partial_q\phi| + C \sum_{|I|=1} |Z^I\phi|,$$

$$(3.5) \quad |\bar{\partial}^2\phi| + r^{-1}|\bar{\partial}\phi| \leq \frac{C}{r} \sum_{|I|\leq 2} \frac{|Z^I\phi|}{1+t+r}, \quad \text{where } |\bar{\partial}^2\phi|^2 = \sum_{S,T \in \mathcal{T}} |\partial_S\partial_T\phi|^2,$$

$$(3.6) \quad (1+|t-r|)^k |\partial^k\phi| \leq C \sum_{|I|\leq |k|} |Z^I\phi|.$$

Proof. First we note that if $r+t \leq 1$ then (3.3) holds since the standard derivatives ∂_α are included in the sum on the right. The inequality for $|\bar{\partial}\phi|$ in (3.3) follows directly from (3.2). The inequality for $|\partial\phi|$ in (3.3) follows from (3.1). The inequality (3.4) follows similarly from (3.1)-(3.2). The proof of (3.5) follows immediately from (3.2) and the inequality $|\partial_i\omega_j| \leq Cr^{-1}$. The inequality (3.6) follows from repeated use of (3.3) and the commutator identity $[Z, \partial_i] = c_i^\alpha \partial_\alpha$, where c_i^α are constants. \square

Let $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ and

$$(3.7) \quad L_1^\alpha = L^\alpha - \frac{1}{2}H_{LL}L^\alpha - \frac{1}{4}H_{LL}\underline{L}^\alpha + H_L^A A^\alpha,$$

$$(3.8) \quad L_2^\alpha \partial_\alpha = (L^\alpha - \frac{1}{4}H_{LL}\underline{L}^\alpha) \partial_\alpha = 2\partial_p + \frac{1}{2}H_{LL}\partial_q.$$

Lemma 3.2. *Let $\tilde{\square}_g = g^{\alpha\beta} \partial_\alpha \partial_\beta$, $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ and suppose that $|H| \leq 1/4$. Then*

$$(3.9) \quad \left| \left(2L_1^\alpha \partial_\alpha + \frac{\ell}{r} \right) (r \partial_q \phi) - r \tilde{\square}_g \phi \right| \leq \frac{C}{1+t+r} \sum_{|I| \leq 2} |Z^I \phi|, \quad \ell = \overline{\text{tr}} H + H_{LL} - \frac{1}{2}H_{LL},$$

where $\overline{\text{tr}} H = \delta^{AB} H_{AB}$. Suppose also that

$$(3.10) \quad |H_{LL}| + |H_{LA}| + |H_{AA}| + |H_{LL}| \leq \frac{1+|t-r|}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a, \quad a \geq 0.$$

Then

$$(3.11) \quad \left| 2L_2^\alpha \partial_\alpha (r \partial_q \phi) - r \tilde{\square}_g \phi \right| \leq \frac{C}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a \sum_{|I| \leq 2} |Z^I \phi|.$$

Proof. (3.9) follows from (2.16) using (3.5) and

$$(3.12) \quad \overline{\text{tr}} g = 2 + \overline{\text{tr}} H, \quad 2L_1^\alpha \partial_\alpha r = 2 - H_{LL} + \frac{1}{2}H_{LL}.$$

(3.11) follows from (3.9) and (3.3) using that

$$(3.13) \quad r |L^\alpha \partial_\alpha (\underline{L}^\beta \partial_\beta \phi)| = r |L^\alpha \underline{L}^\beta \partial_\alpha \partial_\beta \phi| \lesssim \sum_{|I|=1, \beta=0,1,2,3} |Z^I \partial_\beta \phi| \lesssim \sum_{|I| \leq 1} |\partial Z^I \phi| \lesssim \frac{1}{1+|t-r|} \sum_{|I| \leq 2} |Z^I \phi|.$$

□

Lemma 3.3. *Suppose that H satisfy the assumptions of Lemma 3.2 and*

$$(3.14) \quad |\partial H| \leq \frac{1}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a, \quad a \geq 0.$$

Then

$$(3.15) \quad \left| 2L_2^\alpha \partial_\alpha (r \partial_q^2 \phi) + r (\partial_q H_{LL}) \partial_q^2 \phi - r \partial_q \tilde{\square}_g \phi \right| \leq \frac{C}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a \sum_{|I| \leq 2} |\partial Z^I \phi|.$$

Proof. To prove (3.15) we first commute $\partial_\gamma \tilde{\square}_g \phi = \tilde{\square}_g \partial_\gamma \phi + (\partial_\gamma H^{\alpha\beta}) \partial_\alpha \partial_\beta \phi$, use (2.15) applied to $\partial_\gamma H^{\alpha\beta}$;

$$(3.16) \quad |(\partial_\gamma H^{\alpha\beta}) \partial_\alpha \partial_\beta \phi - (\partial_\gamma H_{LL}) \partial_q^2 \phi| \lesssim |\partial H| |\bar{\partial} \phi| \lesssim |\partial H| \sum_{|I|=1} |\partial Z^I \phi|.$$

Using (3.11) applied to $\partial_\gamma \phi$ in place of ϕ now gives

$$(3.17) \quad \left| 2L_2^\alpha \partial_\alpha (r \partial_q \partial_\gamma \phi) + r (\partial_\gamma H_{LL}) \partial_q^2 \phi - r \partial_\gamma \tilde{\square}_g \phi \right| \leq \frac{C}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a \sum_{|I| \leq 2} |\partial Z^I \phi|.$$

and contracting with \underline{L}^γ , using that it commutes with L_2 and ∂_q , gives (3.15) □

Let us now calculate the commutators of vector fields with $\tilde{\square}_g = g^{\alpha\beta} \partial_\alpha \partial_\beta = \square + H^{\alpha\beta} \partial_\alpha \partial_\beta$:

$$\begin{aligned}
(3.18) \quad Z\tilde{\square}_g\psi &= Z\square\psi + Z(H^{\alpha\beta} \partial_\alpha \partial_\beta \psi) \\
&= \square Z\psi - C_Z \square\psi + H^{\alpha\beta} \partial_\alpha \partial_\beta Z\psi + 2H^{\alpha\beta} C_{Z\alpha}^\gamma \partial_\gamma \partial_\beta \psi + (ZH^{\alpha\beta}) \partial_\alpha \partial_\beta \psi \\
&= \tilde{\square}_g Z\psi - C_Z \tilde{\square}_g \psi + C_Z H^{\alpha\beta} \partial_\alpha \partial_\beta \psi + 2H^{\alpha\beta} C_{Z\alpha}^\gamma \partial_\gamma \partial_\beta \psi + (ZH^{\alpha\beta}) \partial_\alpha \partial_\beta \psi
\end{aligned}$$

and hence with $\widehat{Z} = Z + C_Z$;

$$(3.19) \quad \tilde{\square}_g Z\psi = \widehat{Z} \tilde{\square}_g \psi - (ZH^{\alpha\beta} + 2C_{Z\gamma}^\alpha H^{\gamma\beta} + C_Z H^{\alpha\beta}) \partial_\alpha \partial_\beta \psi.$$

In general we have

$$(3.20) \quad \tilde{\square}_g Z^I \psi = \widehat{Z}^I \tilde{\square}_g \psi + \sum_{|J|+|K|\leq|I|, |K|<|I|} C_{JK\alpha\beta}^I \delta^\gamma (Z^J H^{\alpha\beta}) \partial_\gamma \partial_\delta Z^K \psi,$$

where $C_{JK\alpha\beta}^I \delta^\gamma$ are constants. The same formula holds for usual derivatives ∂_α in place of Z even without the lower order terms with $|J| + |K| < |I|$, but we will need to separate these from the vector fields since they will behave better. Let $\mathbf{k} = (k_1, \dots, k_n)$ be a multindex and $\partial^{\mathbf{k}} = \partial_{k_1} \dots \partial_{k_n}$. Then

$$(3.21) \quad \tilde{\square}_g \partial^{\mathbf{k}} Z^I \psi = \partial^{\mathbf{k}} \widehat{Z}^I \tilde{\square}_g \psi + \sum_{|J|+|K|\leq|I|, \mathbf{m}+\mathbf{n}=\mathbf{k}, |\mathbf{n}|+|K|<|\mathbf{k}|+|I|} C_{JK\alpha\beta}^I \delta^\gamma (\partial^{\mathbf{m}} Z^J H^{\alpha\beta}) \partial_\alpha \partial_\beta \partial^{\mathbf{n}} Z^K \psi.$$

Moreover

$$(3.22) \quad |\partial^{\mathbf{m}} Z^J H^{\alpha\beta}(\phi)| \leq C \sum_{\mathbf{m}_1+\dots+\mathbf{m}_\ell=\mathbf{m}, J_1+\dots+J_\ell=J, \ell\geq 1} |\partial^{\mathbf{m}_1} Z^{J_1} \phi| \dots |\partial^{\mathbf{m}_\ell} Z^{J_\ell} \phi|.$$

We have the following:

Lemma 3.4. *Suppose that $\tilde{\square}_g \phi = 0$ and $|\partial^{\mathbf{n}} Z^K \phi| \leq 1$, for $|\mathbf{n}| + |K| \leq N - 5$. Then for $|\mathbf{k}| + |I| \leq N$ we have*

$$(3.23) \quad |\tilde{\square}_g \partial^{\mathbf{k}} Z^I \phi| \lesssim \sum_{|\mathbf{n}|\leq|\mathbf{k}|, |J|+|K|\leq|I|, |K|<|I|} |Z^J \phi| |\partial^2 \partial^{\mathbf{n}} Z^K \phi| + \sum_{|\mathbf{m}|+|\mathbf{n}|\leq|\mathbf{k}|, |J|+|K|\leq|I|} |\partial \partial^{\mathbf{m}} Z^J \phi| |\partial \partial^{\mathbf{n}} Z^K \phi|.$$

If $|\mathbf{k}| = 0$ then only the first sum is present and if $|I| = 0$ then only the second sum is present.

Proof. If $|K| = |I|$ in the sum (3.21) then $|J| = 0$, $|\mathbf{m}| \geq 1$ and $|\mathbf{n}| < |\mathbf{k}|$ so using (3.22) we see that this term can be bounded by a term of the form in the second sum. On the other hand if $|\mathbf{n}| = |\mathbf{k}|$ in (3.21) then $|\mathbf{m}| = 0$ and $|K| < |I|$, and this term can be bounded by a term in the first sum above. Finally a term with $|\mathbf{n}| < |\mathbf{k}|$ and $|K| < |I|$ in (3.21) can be bounded by a term contained in one of the sums above since under the assumptions of the lemma

$$(3.24) \quad |\partial^{\mathbf{m}} Z^J H^{\alpha\beta}(\phi)| \leq C \sum_{|\mathbf{n}|\leq|\mathbf{m}|, |K|\leq|J|} |\partial^{\mathbf{n}} Z^K \phi|.$$

□

4 Decay estimates for the wave equation on a curved background

For $(t, x) \in D = \{(t, x) \in \mathbf{R} \times \mathbf{R}^3; t/2 < |x| < 3t/2\}$, let $X_i(s) \in \mathbf{R}^{1+3}$, $i = 1, 2$, be the backward integral curve

$$(4.1) \quad \frac{d}{ds} X_i^\alpha = L_i^\alpha(X_i), \quad s \leq 0, \quad X_i(0) = (t, x)$$

of the vector fields (3.7). Let $s_i < 0$ be the largest number such that $X(s_i) \in \partial D$, $X(s) \in D$, $s > s_i$. Let $\tau_i = \tau_i(t, x) = X^0(s_i)$. Assuming that $|H| \leq 1/4$ the integral curve will in fact intersect ∂D .

The following lemma is a generalization of a lemma in [L1, L-R1, L-R2].

Lemma 4.1. *Suppose that $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ satisfies $|H| \leq 1/16$ and either of the following*

$$(4.2) \quad (1) \quad \int_0^T \|H(t, \cdot)\|_{L^\infty(D_t)} \frac{dt}{1+t} \leq 1,$$

where $D_t = \{x \in \mathbf{R}^3; t/2 < |x| < 3t/2\}$, or

$$(4.3) \quad (2) \quad |H_{LL}| + |H_{LA}| + |H_{AA}| + |H_{L\underline{L}}| \leq \frac{1}{4} \frac{1+|t-r|}{1+t+r} \left| \frac{1+t+r}{1+|t-r|} \right|^a, \quad \text{in } D,$$

where $D = \{(t, x) \in \mathbf{R} \times \mathbf{R}^3; t/2 < |x| < 3t/2\}$ and $a \geq 0$. Then for any $a \geq 0$;

$$(4.4) \quad (1+t+r)|\partial\phi(t, x)| \leq C \sup_{\tau_i \leq \tau \leq t} \sum_{|I| \leq 1} \|Z^I \phi(\tau, \cdot)\|_{L^\infty} \\ + C \int_{\tau_i}^t \left((1+\tau) \|\tilde{\square}_g \phi(\tau, \cdot)\|_{L^\infty(D_\tau)} + \sum_{|I| \leq 2} (1+\tau)^{-1+a} \|(1+|q(\tau, \cdot)|)^{-a} Z^I \phi(\tau, \cdot)\|_{L^\infty(D_\tau)} \right) d\tau,$$

where $q(t, x) = |x| - t$. Here $\tau_i = \tau_i(t, x)$, where $i=1$ if condition (1) hold and $i=2$ if condition (2) hold, is defined as follows. If $(t, x) \notin D$ then $\tau_i = t$. If $(t, x) \in D$ then τ_i is the first time the backward integral curve from (t, x) for the vector field L_i , in (3.7)- (3.8), leaves the region D . In general $0 \leq \tau_i \leq t$.

Proof. By (3.4) we only need to show that $\psi = r \partial_q \phi$ is bounded by the right hand side. Lemma 3.2 can be summarized

$$(4.5) \quad \left| \left(2L_i^\alpha \partial_\alpha + \frac{\ell_i}{r} \right) \psi - r \tilde{\square}_g \phi \right| \leq \frac{C}{1+t+r} \left| \frac{1+t+r}{1+|r-t|} \right|^a \sum_{|I| \leq 2} |Z^I \phi|, \quad \text{where } \begin{cases} \ell_1 = \overline{\text{tr}} H + H_{L\underline{L}} - \frac{1}{2} H_{LL}, \\ \ell_2 = 0 \end{cases}$$

With the integrating factor

$$(4.6) \quad G_i(s) = -2^{-1} \int_s^0 \ell_i(X_i(\sigma))/r(X_i(\sigma)) d\sigma$$

we have along the integral curves (4.1)

$$(4.7) \quad \frac{d}{ds} \left(\psi(X_i(s)) e^{G_i(s)} \right) = \frac{1}{2} e^{G_i(s)} \left(\left(2L_i^\alpha \partial_\alpha + \frac{\ell_i}{r} \right) \psi \right) (X_i(s))$$

It follows from the assumption (4.2) that $|G_1| \leq C$ independently of s and $G_2 = 0$. Hence it follows from integrating this from s_i to 0 that

$$(4.8) \quad |\psi(t, x)| \lesssim |\psi(X_i(s_i))| + \int_{s_2}^0 \left| \left((2L_i^\alpha \partial_\alpha + \frac{\ell_i}{r}) \psi \right) (X_i(s)) \right| ds \\ \lesssim \sum_{|I| \leq 1} |Z^I \phi(X_i(s_i))| + \int_{s_i}^0 r |\tilde{\square}_g \phi(X_i(s))| + \sum_{|I| \leq 2} (1+t+r)^{-1} |Z^I \phi(X_i(s))| ds.$$

Since $t = X_i^0$ and $dX_i^0/ds = L_i^0$, where $L_2^0 = 1 - \frac{1}{4}H_{LL}$ and $L_1^0 = 1 - \frac{1}{4}H_{LL} - \frac{1}{2}H_{L\bar{L}}$ it follows that $1/2 \leq |dt/ds| \leq 2$ and the lemma follows. \square

Next we define substitutes ρ_i for $r - t$. Let $\rho_i = \rho_i(t, x)$ be constant along the integral curves of L_i and equal to $r - t$ outside a neighborhood of the forward light cone:

$$(4.9) \quad L_i^\alpha \partial_\alpha \rho_i = 0, \quad \text{when } |t - r| \leq t/2, \quad \rho_i = r - t, \quad \text{when } |t - r| > t/2$$

Note that (t_i, x_i) is the first point the backward integral curve intersects $|r - t| = t/2$ then $|\rho_i(t, x)| = |t_i - |x_i|| = t_i/2 = \tau_i(t, x)/2 \leq t/2$, since t is increasing along the forward integral curves. Here τ_i was defined to be the smallest t along the integral curve with $|r - t| \leq t/2$. Hence

$$(4.10) \quad |\rho_i(t, x)| = \tau_i(t, x)/2 \leq t/2, \quad \text{when } |t - |x|| \leq t/2$$

We have

Lemma 4.2. *Suppose that either of the conditions in Lemma 4.1 hold. Then for $\nu > \mu \geq 0$ and any $a \geq 0$ we have*

$$(4.11) \quad (1+t+|x|)(1+|\rho_i(t, x)|)^\nu |\partial \phi(t, x)| \leq C \sup_{\tau_i \leq \tau \leq t} (1+\tau)^{\nu-\mu+b} \sum_{|I| \leq 2} \|(1+|\rho_i(\tau, \cdot)|)^\mu (1+|q(\tau, \cdot)|)^{-a} Z^I \phi(\tau, \cdot)\|_{L^\infty} \\ + C \int_{\tau_i}^t (1+\tau) \|(1+|\rho_i(\tau, \cdot)|)^\nu \tilde{\square}_g \phi(\tau, \cdot)\|_{L^\infty(D_\tau)} d\tau,$$

where $q(t, x) = |x| - t$. Here $i = 1$ if condition (1) holds and $i = 2$ if condition (2) holds.

Proof. This follows from Lemma 4.1 using that $\rho_i(t, x) = \tau_i(t, x)/2 \leq \tau/2$ along the integral curves and ρ_i are constant along the integral curves. We also use that $\int_{2\rho_i}^\infty (1+\tau)^{-1-\nu+\mu} d\tau = C(1+\rho_i)^{\mu-\nu}$. \square

5 Estimates for the radial characteristics and eikonal equation

We will use a curved substitute $\rho(t, x)$, for the distance to the forward light cone $r - t$. Let $\rho = \rho(t, x)$ be equal to $r - t$ outside a neighborhood of the forward light cone constant along the integral curves of the radial vector field L_2 close to the light cone:

$$(5.1) \quad L_2^\alpha \partial_\alpha \rho = 0, \quad \text{when } |t - r| \leq t/2, \quad \rho = r - t, \quad \text{when } |t - r| \geq t/2,$$

where

$$(5.2) \quad L_2^\alpha \partial_\alpha = 2\partial_\rho + \frac{1}{2}H_{LL}\partial_q,$$

and we think of $\rho = \rho(q, p, \omega)$ as a function of $q = r - t$, $p = r + t$ and $\omega = x/|x|$. We call (5.1) the radial eikonal equation. Alternatively, let $X_2(s)$ be the integral curves of the vector field L_2 , i.e. $\dot{X}_2 = L_2$. Then we can choose the initial conditions when $|t - r| = t/2$ so that

$$(5.3) \quad q = -X_{2L}(s, \rho, \omega), \quad \text{and} \quad p = X_{2L} = 2s,$$

where

$$(5.4) \quad \frac{d}{ds}X_{2L} = -\frac{1}{2}H_{LL}, \quad \text{when } |t - r| \leq t/2, \quad q = \rho, \quad \text{when } |t - r| \geq t/2.$$

We call these the radial characteristics.

We now state the main estimate for ρ assuming some estimates for H_{LL} that will be proven later. We will assume that $H_{LL} = 0$, when $r > t + 1$ and $t > 0$ so in fact $\rho = r - t$, when $r > t + 1$.

Since, as we show below, $0 < \partial\rho/\partial q < \infty$, ρ is an invertible function of q for fixed (p, ω) , satisfying $d\rho/dq = \partial\rho/\partial q$, and q is an invertible function of ρ , satisfying $dq/d\rho = (d\rho/dq)^{-1}$. We have thus introduced a change of variables $(\rho, s, \omega) \rightarrow (q(\rho, s, \omega), 2s, \omega)$. Note that multiplication by any function of ρ commutes with $L_2^\alpha \partial_\alpha$ and a calculation using that shows that

$$(5.5) \quad [L_2^\alpha \partial_\alpha, \partial_q] = -\frac{\partial_q H_{LL}}{2} \partial_q, \quad \text{and} \quad [L_2^\alpha \partial_\alpha, \partial_\rho] = 0, \quad \text{if } \partial_\rho = \rho_q^{-1} \partial_q.$$

The following lemma was essentially proven in [L1] in the spherically symmetric case:

Proposition 5.1. *Let $\rho(t, x)$ be as in (5.1) and suppose that H_{LL} satisfies*

$$(5.6) \quad |\partial H_{LL}| \leq \frac{c_1 \varepsilon}{1+t} \frac{1}{(1+|\rho|)^\nu}, \quad \text{and} \quad |H_{LL}| \leq c_1 \varepsilon \frac{1+|q|}{1+t}$$

for some $\nu \geq 0$. Then

$$(5.7) \quad \left(\frac{1+t}{1+|\rho|} \right)^{-c_1 \varepsilon V(\rho)} \leq \frac{\partial \rho}{\partial q} \leq \left(\frac{1+t}{1+|\rho|} \right)^{c_1 \varepsilon V(\rho)}, \quad V(\rho) = (1+|\rho|)^{-\nu}$$

and

$$(5.8) \quad \left(\frac{1+t}{1+|\rho|} \right)^{-c_1 \varepsilon} \leq \frac{1+|q|}{1+|\rho|} \leq \left(\frac{1+t}{1+|\rho|} \right)^{c_1 \varepsilon}$$

Proof. We have

$$(5.9) \quad L_2^\alpha \partial_\alpha \partial_q \rho + \frac{1}{2} \partial_q H_{LL} \partial_q \rho = 0.$$

Let $X(s)$ be a backward integral curve of the vector fields L_2 :

$$(5.10) \quad \frac{d}{ds} X^\alpha = L_2^\alpha(X), \quad s \leq 0, \quad X(0) = (t, x)$$

and let $s_2 < 0$ be the largest number such that $X(s_2) = (t_2, x_2)$ satisfies $|t_2 - |x_2|| = t_2/2$. If we multiply by the integrating factor

$$(5.11) \quad G(s) = - \int_s^0 \frac{1}{2} (\partial_q H_{LL})(X(\tau)) d\tau$$

we get

$$(5.12) \quad \frac{d}{ds} \left(\frac{\partial \rho}{\partial q}(X(s)) e^{G(s)} \right) = 0.$$

Integrating this from s_2 (where $\partial \rho / \partial q = 1$) to 0 gives

$$(5.13) \quad \frac{\partial \rho}{\partial q}(t, x) = e^{G(s_2)}$$

Since ρ is constant along the integral curves $X(s)$ it follows from (5.6)

$$(5.14) \quad |G(s_2)| \leq \frac{c_1 \varepsilon}{2(1+|\rho|)^\nu} \int_{s_2}^0 \frac{ds}{1+X^0(s)} \leq \frac{c_1 \varepsilon}{(1+|\rho|)^\nu} \int_{\tau_2}^t \frac{dt'}{1+t'} \leq \frac{c_1 \varepsilon}{(1+|\rho|)^\nu} \ln \left| \frac{1+t}{1+|\rho|} \right|,$$

since $t = X^0$, $dX^0/ds = L_2^0 = 1 - \frac{1}{4}H_{LL} \geq 1/2$ and $\tau_2 = t_2 = 2|\rho|$. This proves (5.7).

(5.8) follows as above from integrating (5.4) in the form

$$(5.15) \quad \left| \frac{d}{ds} \ln |1+|q|| \right| = \frac{1}{2} \frac{|H_{LL}|}{1+|q|} \leq \frac{c_1 \varepsilon}{1+t}.$$

□

We can write (5.9) as

$$(5.16) \quad L_2^\alpha \partial_\alpha \ln |\rho_q| = -\frac{1}{2} \partial_q H_{LL}.$$

Integrating along the integral curves from the boundary where $|t-r| = t/2$ and $\rho_q = 1$ gives

$$(5.17) \quad \ln |\rho_q| = -\frac{1}{2} \int_{2|\rho|-\rho/2}^{p/2} \partial_q H_{LL} ds$$

We now give some further estimates for the approximate solution of the eikonal equation:

Lemma 5.2. *Suppose that the assumption of Proposition 5.1 hold and*

$$(5.18) \quad |\partial^2 H_{LL}| \leq \frac{c_2 \varepsilon}{1+t} \left| \frac{\partial \rho}{\partial q} \right| \frac{1}{(1+|\rho|)^{1+\nu}},$$

for some $\nu \geq 0$. Then with $\partial_\rho = \rho_q^{-1} \partial_q$ we have

$$(5.19) \quad |\partial_\rho \partial_q \rho| \leq \frac{c_2 \varepsilon}{(1+|\rho|)^{1+\nu}} \partial_q \rho \ln \left| \frac{1+t}{1+|\rho|} \right|.$$

Proof. Since $[L_2^\alpha \partial_\alpha, \partial_\rho] = 0$ we obtain by differentiating (5.16):

$$(5.20) \quad L_2^\alpha \partial_\alpha \partial_\rho \ln |\rho_q| = -\frac{1}{2} \partial_\rho \partial_q H_{LL}$$

Hence by (5.18),

$$(5.21) \quad |L_2^\alpha \partial_\alpha \partial_\rho \ln |\rho_q|| \leq \frac{c_2 \varepsilon}{(1+t)(1+|\rho|)^{1+\nu}}.$$

Moreover by differentiating (5.17) we see that the initial condition on $|t-r| = t/2$ are $\partial_q \ln |\rho_q| = (\text{sign } \rho - 1/4) \partial_q H_{LL}$. It therefore follows from integrating (5.21) from the boundary where $t = 2|\rho|$:

$$(5.22) \quad |\partial_q \ln |\rho_q|| \leq \frac{c_2 \varepsilon}{(1+|\rho|)^{1+\nu}} \ln \left| \frac{1+t}{1+|\rho|} \right| + \frac{c_1 \varepsilon}{(1+t)(1+|\rho|)^\nu}$$

Since $t \geq 2|\rho|$ in the domain where $\rho_q \neq 1$, (5.19) follows from this. \square

Lemma 5.3. *Suppose the assumption of Proposition 5.1 hold and*

$$(5.23) \quad |H_{LL}| + |H_{LA}| + |H_{AB}| + |H_{LL}| + |\Omega H_{LL}| \leq \frac{c_2 \varepsilon (1+|\rho|)^{1-\nu}}{(1+t)^{1-c_2' \varepsilon}},$$

for some $\nu'' \geq 0$. Then

$$(5.24) \quad |\bar{\partial} \rho| \leq \frac{c_2' \sigma}{1+\sigma} \frac{(1+|\rho|)^{1-\nu}}{(1+t)^{1-c_2' \varepsilon}}, \quad \sigma = c_1 \varepsilon \ln |1+t|,$$

and

$$(5.25) \quad |g^{\alpha\beta} \rho_\alpha \rho_\beta - \delta^{AB} \partial_{A\rho} \partial_{B\rho}| \leq c_2' \varepsilon \frac{(1+|\rho|)^{2-2\nu}}{(1+t)^{2-c_2' \varepsilon}}.$$

Proof. We have $|\bar{\partial} \rho| \lesssim |\partial_p \rho| + \sum_{0 < i < j} |\Omega_{ij} \rho| / (1+t)$, where

$$(5.26) \quad \partial_p \rho = -\frac{H_{LL}}{4} \partial_q \rho,$$

and

$$(5.27) \quad L_2^\alpha \partial_\alpha \Omega \rho = -\frac{1}{2} (\Omega H_{LL}) \partial_q \rho.$$

Using (5.23) and integrating the last equation from $t = 2|\rho|$, where $\Omega_{ij} \rho = 0$, gives (5.24). (Here we wrote $\int_0^t c\varepsilon(1+t)^{c\varepsilon-1} dt = (1+t)^{c\varepsilon} - 1 \leq e^{c\varepsilon \ln |1+t|} - 1$ and used the inequality $e^a - 1 \leq C e^a a / (1+a)$.)

By (2.17)

$$(5.28) \quad |g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho - \delta^{AB} \partial_{A\rho} \partial_{B\rho}| \leq 2|\partial_q \rho| |L_1^\alpha \partial_\alpha \rho| + C(|H_{LL}| + |H_{LA}| + |H_{AB}|) |\bar{\partial} \rho|^2.$$

Here

$$(5.29) \quad |L_1^\alpha \partial_\alpha \rho| \leq |L_2^\alpha \partial_\alpha \rho| + \left| \left(\frac{1}{2} H_{LL} L^\alpha + H_L^A A^\alpha \right) \partial_\alpha \rho \right| \leq (|H_{LL}| + |H_{LA}|) |\bar{\partial} \rho|$$

Hence

$$(5.30) \quad |g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho - \delta^{AB} \partial_{A\rho} \partial_{B\rho}| \lesssim (|H_{LL}| + |H_{LA}|) |\partial_q \rho| |\bar{\partial} \rho| + (|H_{LL}| + |H_{LA}| + |H_{AB}|) |\bar{\partial} \rho|^2.$$

These estimates together with (5.24) gives also (5.25). \square

6 The sharp decay estimates for the nonlinear problem

In this section we start by assuming the weaker decay estimates

$$(6.1) \quad |Z^I \phi| \leq c_0 \varepsilon (1+t)^{-\nu}, \quad |I| \leq N-3, \quad c_0 \varepsilon \leq 1,$$

for some $0 < \nu < 1$ and some sufficiently large N . We also assume that ϕ is a solution of the nonlinear problem with compactly supported data in the set $|x| \leq 1$, which means that

$$(6.2) \quad \phi(t, x) = 0, \quad \text{when } |x| \geq t+1, \quad \text{and } t \geq 0.$$

(6.1) can be obtained from energy estimates using the Klainerman-Sobolev inequalities. From the weak decay estimates we will derive stronger decay estimates. The stronger estimates will be derived in several steps. Since our metric $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$, where $H^{\alpha\beta} = H^{\alpha\beta}(\phi)$ are smooth functions of ϕ vanishing at the origin and by scaling we may assume that (so that (6.1) holds also for $H^{\alpha\beta}$)

$$(6.3) \quad \sum_{\alpha\beta} |H^{\alpha\beta}(\phi)| \leq \frac{1}{64} |\phi|$$

In what follows C will denote universal constants that depend only on the particular functions $H^{\alpha\beta}(\phi)$, but are independent of ϕ . c'_0 will denote a constant that is multiple of c_0 i.e. Cc_0 . Constants of the form c_k and $c'_k = Cc_k$ depend only on c_{k-1} and universal constants. The estimates (6.4)-(6.6) below, were used already in the spherically symmetric case in [L1].

Proposition 6.1. *Suppose that ϕ is a solution of the nonlinear equation for which (6.1) and (6.2) hold. Let $\rho = \rho_2$ be as in the previous section. Then there are constants $c_1 = Cc_0$ and $c_2 = Cc_0$, for some universal constant C , independent of ϕ if $c_0 \varepsilon \leq 1/C$, such that*

$$(6.4) \quad |\phi| \leq \begin{cases} c_1 \varepsilon (1+t)^{-1} (1+|q|) \\ c_1 \varepsilon (1+t)^{-1+c_1 \varepsilon} (1+|\rho|)^{1-\nu-c_1 \varepsilon} \end{cases}$$

$$(6.5) \quad |\partial \phi| \leq c_1 \varepsilon (1+t)^{-1} (1+|\rho|)^{-\nu},$$

$$(6.6) \quad |\partial^2 \phi| \leq c_2 \varepsilon (1+t)^{-1} (1+|\rho|)^{-1-\nu} |\partial \rho / \partial q|,$$

and

$$(6.7) \quad |Z \phi| \leq \begin{cases} c_2 \varepsilon (1+t)^{-1+c_2 \varepsilon} (1+|\rho|)^{1-\nu} \\ c_2 \varepsilon (1+t)^{-1} (|q| + (1+t)^{c_2 \varepsilon}). \end{cases}$$

Moreover, there are constants c_k depending only on c_{k-1} such that

$$(6.8) \quad |\partial^{\mathbf{k}} Z^I \phi| \leq c_k \varepsilon (1+t)^{-1+c_k \varepsilon} (1+|q|)^{1-k-\nu}, \quad \max(1, |\mathbf{k}|) + |I| \leq N-4,$$

where $k = |\mathbf{k}|$.

6.1 The decay of the first order derivatives (6.4) and (6.5)

Since by (6.1) condition (1) in Lemma 4.1 hold and it follows from (6.1) that the right hand side of (4.4) is bounded so

$$(6.9) \quad |\partial\phi| \leq c'_0\varepsilon(1+t)^{-1}$$

The first estimate in (6.4) follows from integrating this from $r = t + 1$ where $\phi = 0$. Hence

$$(6.10) \quad |\partial H| + (1 + |q|)^{-1}|H| \leq \frac{c_1\varepsilon}{1+t},$$

it follows that in fact condition (2) in Lemma 4.1 also hold. (6.5) therefore follows from Lemma 4.2 with $\mu = a = 0$. The second estimate in (6.4) follows from integrating (6.5) and using Proposition 5.1, which hold since we just showed that (6.5) hold.

6.2 The sharp decay estimates for second order derivatives (6.6)

The estimate (6.6) essentially comes from that ∂_ρ commutes with L_2 . Since (6.10) hold:

Lemma 6.2. *Suppose that H satisfy (6.10) and let ρ be as in the previous section. Then*

$$(6.11) \quad \left| 2L_2^\alpha \partial_\alpha (r \partial_\rho \partial_q \phi) - r \partial_\rho \tilde{\square}_g \phi \right| \leq \frac{C\rho_q^{-1}}{1+t+r} \sum_{|I| \leq 2} |\partial Z^I \phi|, \quad \text{where } \partial_\rho = \rho_q^{-1} \partial_q.$$

and

$$(6.12) \quad \left| 2L_2^\alpha \partial_\alpha (r \partial_\rho \psi) - r \rho_q^{-1} \tilde{\square}_g \psi \right| \leq c_1 \varepsilon |\partial_\rho \psi| + \frac{C\rho_q^{-1}}{1+t+r} \sum_{|I| \leq 2} |Z^I \psi|,$$

Proof. Since $2L_2^\alpha \partial_\alpha \rho = (\partial_\rho + H_{LL} \partial_q) \rho = 0$ it follows that $2L_2^\alpha \partial_\alpha \partial_q \rho = -\partial_q H_{LL} \partial_q \rho$ and $2L_2^\alpha \partial_\alpha \rho_q^{-1} = \rho_q^{-1} \partial_q H_{LL}$. Hence

$$(6.13) \quad 2L_2^\alpha \partial_\alpha (r \rho_q^{-1} \partial_q^2 \phi) = \rho_q^{-1} (2L_2^\alpha \partial_\alpha (r \partial_q^2 \phi) + (r \partial_q H_{LL}) \partial_q^2 \phi)$$

and (6.11) follows from (3.15). (6.12) follows in the same way. \square

It follows from (6.11) using (5.7) and (5.8) that

$$(6.14) \quad \left| L_2^\alpha \partial_\alpha \left(\rho_q^{-1} r \partial_q^2 \phi \right) \right| \leq \frac{C\rho_q^{-1}}{1+t} \sum_{|I| \leq 2} |\partial Z^I \phi| \leq \frac{C\rho_q^{-1}}{(1+t)(1+|q|)} \sum_{|I| \leq 3} |Z^I \phi| \leq \frac{c'_0\varepsilon}{(1+t)^{1+\nu-2c_1\varepsilon} (1+|\rho|)^{1+2c_1\varepsilon}}.$$

If we as in the proof of Lemma 4.1 integrate from $r = t/2$, where $t \sim \rho$ and $|\rho_q^{-1} r \partial_q^2 \phi| \leq c_0\varepsilon(1+|\rho|)^{-\nu}$ we get

$$(6.15) \quad |\rho_q^{-1} r \partial_q^2 \phi| \leq c'_1\varepsilon(1+|\rho|)^{-1-\nu},$$

since we assumed that $c_0\varepsilon \leq 1$. (6.6) follows from this using Lemma 3.1 and (6.1).

6.3 The decay estimate for one vector field (6.7)

Since $\tilde{\square}_g \phi = 0$ we have by (3.19):

$$(6.16) \quad |\tilde{\square}_g Z \phi| = |(ZH^{\alpha\beta} + 2C_{Z\gamma}^\alpha H^{\gamma\beta} + C_Z H^{\alpha\beta}) \partial_\alpha \partial_\beta \phi| \leq C(|Z\phi| + |\phi|) |\partial^2 \phi|,$$

and hence by (6.12) applied to $\psi = Z\phi$;

$$(6.17) \quad |L_2^\alpha \partial_\alpha (r \partial_\rho Z \phi)| \leq Cr(|Z\phi| + |\phi|) \frac{|\partial^2 \phi|}{\rho_q} + c_1 \varepsilon |\partial_\rho Z \phi| + \frac{C \rho_q^{-1}}{1+t} \sum_{|J| \leq 3} |Z^J \phi|.$$

Hence using (6.6), (6.5), (6.1) and (6.4) we get

$$(6.18) \quad \left| L_2^\alpha \partial_\alpha (r \partial_\rho Z \phi) \right| \leq c'_2 \varepsilon \left(\frac{|Z\phi|}{1+|\rho|} + |\partial_\rho Z \phi| \right) + \frac{c'_0 \varepsilon (1+|\rho|)^{-c_1 \varepsilon}}{(1+t)^{1+\nu-c_1 \varepsilon}} + \frac{c'_2 \varepsilon^2}{(1+t)^{1-c_1 \varepsilon} (1+|\rho|)^{2\nu+c_1 \varepsilon}}.$$

Since also

$$(6.19) \quad |Z\phi| + (1+|\rho|) |\partial_\rho Z \phi| = |Z\phi| + (1+|q|) |\partial Z \phi| \lesssim \sum_{|I| \leq 2} |Z^I \phi| \lesssim \varepsilon (1+t)^{-\nu}, \quad \text{when } |t-r| = t/2,$$

it follows from Lemma 6.3 below that

$$(6.20) \quad |Z\phi| (1+|\rho|)^{-1} + |\partial_\rho Z \phi| \leq c \varepsilon (1+t)^{-1+c\varepsilon} (1+|\rho|)^{-\nu}, \quad c = 16(c'_2 + c'_0)$$

The desired inequality (6.7) follows from this and Lemma 6.4, since $(1+|\rho|)^{c_1 \varepsilon} \lesssim (1+t)^{c_1 \varepsilon}$.

Lemma 6.3. *Suppose that for some $\nu > 0$ we have*

$$(6.21) \quad (1+|\rho|) |\partial_\rho \psi| + |\psi| \leq c'_0 (1+|\rho|)^{-\nu}, \quad \text{when } |t-r| = t/2 \text{ or } t+r \leq 2.$$

and

$$(6.22) \quad \psi = 0, \quad \text{when } r > t+1 \text{ and } t > 0.$$

Suppose also that

$$(6.23) \quad \left| L_2^\alpha \partial_\alpha (r \partial_\rho \psi) \right| \leq c'_2 \varepsilon \left(\frac{|\psi|}{1+|\rho|} + |\partial_\rho \psi| \right) + \frac{c'_0 \varepsilon (1+|\rho|)^{-c_1 \varepsilon}}{(1+t)^{1+\nu-c_1 \varepsilon}} + \frac{\gamma \varepsilon^2}{(1+t)^{1-\gamma \varepsilon} (1+|\rho|)^{\nu+\gamma \varepsilon}}.$$

Then

$$(6.24) \quad |\psi| (1+|\rho|)^{-1} + |\partial_\rho \psi| \leq c \varepsilon (1+t)^{-1+c\varepsilon} (1+|\rho|)^{-\nu}, \quad c = 16(c'_2 + c'_0 + \gamma)$$

Proof. If we now introduce the new variables $p = t+r = 2s$, $q = r-t = q(\rho, s, \omega)$ given by (5.3): $\tilde{v}(\rho, s, \omega) = v(q, p, \omega)$. Then

$$(6.25) \quad L_2^\alpha \partial_\alpha v(q, p, \omega) = \left[\partial_s \tilde{v}(\rho, s, \omega) \right] \Big|_{\rho=\rho(q,p,\omega), p=2s}.$$

It is also easy to see that if we substitute $r = (p + q)/2$ in the left of then the term with $q/2$ in place of r in the left can be bounded by terms of the form already included in the right so with $\psi(\rho, s, \omega) = (Z^I \phi)(q(\rho, s, \omega), 2s, \omega)$ we have

$$(6.26) \quad \left| \partial_s (s \partial_\rho \psi) \right| \leq c'_2 \varepsilon \left(\frac{|\psi|}{1 + |\rho|} + |\partial_\rho \psi| \right) + \frac{c'_0 \varepsilon (1 + |\rho|)^{-c_1 \varepsilon}}{(1 + s)^{1 + \nu - c_1 \varepsilon}} + \frac{\gamma \varepsilon^2}{(1 + s)^{1 - \gamma \varepsilon} (1 + |\rho|)^{\nu + \gamma \varepsilon}}$$

If we integrate this from the boundary of $D = \{(r, t); t/2 < r < 3t/2\} = \{(\rho, p); -2s/3 < \rho < 2s/5\}$ (since $q = \rho$ and $s = 3\rho/2$ or $s = 5\rho/2$ on the boundary) using the bound (6.21) on the boundary we get

$$(6.27) \quad s(1 + |\rho|)^\nu |\partial_\rho \psi| \leq c'_2 \varepsilon \int_{c_\pm |\rho|}^s \left(\frac{|\psi|}{1 + |\rho|} + |\partial_\rho \psi| \right) ds + c'_0 \varepsilon + \varepsilon(1 + p)^{\gamma \varepsilon} (1 + |\rho|)^{-\gamma \varepsilon}$$

For any $0 \leq a < 1$ we have

$$(6.28) \quad |\psi(\rho, s, \omega) - \psi(1, s, \omega)| \leq \int_\rho^1 |\partial_\rho \psi(\rho, s, \omega)| d\rho \leq (1 + |\rho|)^{1-a} \sup_{\rho \leq \varrho \leq 1} (1 + |\varrho|)^a |\partial_\rho \psi(\varrho, s, \omega)|.$$

By (6.22) $\psi(1, s, \omega) = 0$, when $s \geq 1$ and by (6.21) $|\psi(1, s, \omega)| + |\partial_\rho \psi(1, s, \omega)| \leq c_0 \varepsilon$ for all $|s| \leq 1$. With

$$(6.29) \quad M(s) = \sup_{\rho \in D_s} (1 + |s|) (1 + |\rho|)^\nu |\partial_\rho \psi(\rho, s, \omega)|, \quad \text{where } D_s = \{\rho; (s, \rho, \omega) \in D\}.$$

we hence have

$$(6.30) \quad M(s) \leq 2c'_2 \varepsilon \int_1^s \frac{M(s')}{1 + |s'|} ds' + c'_0 \varepsilon + \varepsilon(1 + |s|)^{\gamma \varepsilon}$$

If $G(s)$ denotes the integral we hence have $(1 + |s|)G'(s) = M(s) \leq 2c'_2 \varepsilon G(s) + c'_0 \varepsilon + \varepsilon(1 + |s|)^{\gamma \varepsilon} (1 + |\rho|)^{-\gamma \varepsilon}$, and if we multiply by the integrating factor $(1 + |s|)^{-c\varepsilon}$ we get

$$(6.31) \quad d(G(s)(1 + |s|)^{-c\varepsilon})/ds \leq (1 + |s|)^{-c\varepsilon - 1} (c'_0 \varepsilon + \varepsilon(1 + |s|)^{\gamma \varepsilon}).$$

If we integrate this from 1 to s we get $G(s) \leq 4(1 + |s|)^{c\varepsilon}$, and hence

$$(6.32) \quad M(s) \leq 2^{-1} c\varepsilon (1 + |s|)^{c\varepsilon}.$$

We conclude that

$$(6.33) \quad |\psi|(1 + |\rho|)^{-1} + |\partial_\rho \psi| \leq c\varepsilon(1 + t)^{-1 + c\varepsilon} (1 + |\rho|)^{-\nu}.$$

□

Lemma 6.4. *If $\nu > 0$ then*

$$(6.34) \quad (1 + t)^{c\varepsilon} (1 + |q|)^{1 - \nu} \leq (1 + t)^{c\varepsilon/\nu} + (1 + |q|)$$

Proof. The follows from the inequality

$$(6.35) \quad a^\nu b^{1 - \nu} \leq a + b, \quad 0 < \nu < 1, \quad a > 0, \quad b > 0.$$

□

6.4 The decay estimates for higher order derivatives

Let us first prove that:

Lemma 6.5. *For $1 \leq k \leq N - 4$ we have*

$$(6.36) \quad |\partial^{\mathbf{k}}\phi| \leq \frac{c_k \varepsilon}{1+t} (1+|\rho|)^{1-k-\nu} \left(\frac{1+t}{1+|\rho|} \right)^{c_k \varepsilon V(\rho)}, \quad V(\rho) = (1+|\rho|)^{-\nu}, \quad |\mathbf{k}| = k,$$

if ε is sufficiently small.

Proof. We will prove the lemma by induction. If $k = 1$ we already proved a stronger estimate and differentiating the equation $\tilde{\square}_g \phi = m^{\alpha\beta} \partial_\alpha \partial_\beta \phi + H^{\alpha\beta} \partial_\alpha \partial_\beta \phi = 0$ gives

$$(6.37) \quad \tilde{\square}_g \partial^{\mathbf{n}} \phi = - \sum_{\mathbf{m}+\mathbf{k}=\mathbf{n}, |\mathbf{m}| \geq 1} \partial^{\mathbf{m}} H^{\alpha\beta} \partial_\alpha \partial_\beta \partial^{\mathbf{k}} \phi.$$

Hence by (3.22)

$$(6.38) \quad |\tilde{\square}_g \partial^{\mathbf{n}} \phi| \leq C |\partial \phi| \sum_{|\mathbf{m}|=|\mathbf{n}|+1} |\partial^{\mathbf{m}} \phi| + C \sum_{|\mathbf{k}_1|+\dots+|\mathbf{k}_\ell|=|\mathbf{n}|+2, 1 \leq |\mathbf{k}_j| \leq |\mathbf{n}|, \ell \geq 2} |\partial^{\mathbf{k}_1} \phi| \dots |\partial^{\mathbf{k}_\ell} \phi|.$$

Using (6.5) and (6.36) for $|\mathbf{k}| \leq n = |\mathbf{n}|$ (and the fact that $|\rho| \lesssim t$), we hence obtain

$$(6.39) \quad |\tilde{\square}_g \partial^{\mathbf{n}} \phi| \leq \frac{c'_1 \varepsilon}{(1+t)(1+|\rho|)^\nu} \sum_{|\mathbf{m}|=n+1} |\partial^{\mathbf{m}} \phi| + \frac{(c'_n)^2 \varepsilon^2}{(1+t)^2 (1+|\rho|)^{n+2\nu}} \left(\frac{1+t}{1+|\rho|} \right)^{c'_n \varepsilon V(\rho)},$$

if ε is so small that $c_n \varepsilon \leq 1$. By (3.6), (5.8) and (6.1)

$$(6.40) \quad |Z^I \partial^{\mathbf{n}} \phi| \leq C (1+|q|)^{-n} \sum_{|J| \leq n+|I|} |Z^J \phi| \leq c'_0 \varepsilon (1+|q|)^{-n} (1+t)^{-\nu} \leq c'_0 \varepsilon (1+t)^{-\nu+n c_1 \varepsilon} (1+|\rho|)^{-n-n c_1 \varepsilon}.$$

The lemma now follows from the following lemma: □

Lemma 6.6. *Suppose that*

$$(6.41) \quad \sum_{|\mathbf{n}|=n, |I| \leq 2} |Z^I \psi_{\mathbf{n}}| \leq c'_0 \varepsilon (1+t)^{-\nu+n c_1 \varepsilon} (1+|\rho|)^{-n-n c_1 \varepsilon}$$

and

$$(6.42) \quad \sum_{|\mathbf{n}|=n} |\tilde{\square}_g \psi_{\mathbf{n}}| \leq \frac{c'_1 \varepsilon}{(1+t)(1+|\rho|)^\nu} \sum_{|\mathbf{n}|=n} |\partial \psi_{\mathbf{n}}| + \frac{(c'_n)^2 \varepsilon^2}{(1+t)^2 (1+|\rho|)^{n+2\nu}} \left(\frac{1+t}{1+|\rho|} \right)^{c'_n \varepsilon V(\rho)}.$$

Then

$$(6.43) \quad \sum_{|\mathbf{n}|=n} |\partial \psi_{\mathbf{n}}| \leq \frac{c_{n+1} \varepsilon}{1+t} \left(\frac{1+t}{1+|\rho|} \right)^{c_{n+1} \varepsilon V(\rho)} (1+|\rho|)^{-n-\nu}$$

Proof. Using (3.11) as in the proof of Lemma 4.1 we have

$$(6.44) \quad 2|L_2^\alpha \partial_\alpha (r \partial_q \psi_{\mathbf{n}})| \leq r |\tilde{\square}_g \psi_{\mathbf{n}}| + \frac{C}{1+t+r} \sum_{|I| \leq 2} |Z^I \psi_{\mathbf{n}}|,$$

Using that

$$(6.45) \quad |\partial \psi_{\mathbf{n}}| \leq C |\partial_q \psi_{\mathbf{n}}| + C(1+t+r)^{-1} \sum_{|I|=1} |Z^I \psi_{\mathbf{n}}|$$

we have

$$(6.46) \quad |L_2^\alpha \partial_\alpha (r \partial_q \psi_{\mathbf{n}})| \lesssim \frac{c'_1 \varepsilon \sum_{|\mathbf{m}|=|\mathbf{n}|} |r \partial_q \psi_{\mathbf{m}}|}{(1+t)(1+|\rho|)^\nu} + \frac{(c'_n)^2 \varepsilon^2}{(1+t)(1+|\rho|)^{n+2\nu}} \left(\frac{1+t}{1+|\rho|} \right)^{c'_n \varepsilon V(\rho)} + \frac{c'_0 \varepsilon (1+|\rho|)^{-n-nc_1 \varepsilon}}{(1+t)^{1+\nu-nc_1 \varepsilon}}.$$

Since $|L_2^\alpha \partial_\alpha \sum_{|\mathbf{n}|=n} |r \partial_q \psi_{\mathbf{n}}| \leq \sum_{|\mathbf{n}|=n} |L_2^\alpha \partial_\alpha (r \partial_q \psi_{\mathbf{n}})|$, and $L_2^\alpha \partial_\alpha \rho = 0$, we have

$$(6.47) \quad L_2^\alpha \partial_\alpha M_n \leq \frac{c'_1 \varepsilon M_n}{(1+t)(1+|\rho|)^\nu} + \frac{(c'_n)^2 \varepsilon^2}{(1+t)(1+|\rho|)^\nu} \left(\frac{1+t}{1+|\rho|} \right)^{c'_n \varepsilon V(\rho)} + \frac{c'_0 \varepsilon (1+|\rho|)^{\nu-nc_1 \varepsilon}}{(1+t)^{1+\nu-nc_1 \varepsilon}},$$

where

$$(6.48) \quad M_n = \sum_{|\mathbf{n}|=n} |r \partial_q \psi_{\mathbf{n}}| (1+|\rho|)^{n+\nu}.$$

Let $c_{n+1} = 2(c_1 + c'_n + 1 + 4nc_0)$ and

$$(6.49) \quad N_n = \sum_{|\mathbf{n}|=n} |r \partial_q \psi_{\mathbf{n}}| (1+|\rho|)^{n+\nu} \left(\frac{1+t}{1+|\rho|} \right)^{-c_{n+1} \varepsilon V(\rho)}.$$

Then

$$(6.50) \quad L_2^\alpha \partial_\alpha N_n \leq \frac{(c'_n)^2 \varepsilon^2}{(1+t)(1+|\rho|)^\nu} \left(\frac{1+t}{1+|\rho|} \right)^{-\varepsilon V(\rho)} + \frac{c'_0 \varepsilon (1+|\rho|)^{\nu-nc_1 \varepsilon}}{(1+t)^{1+\nu-nc_1 \varepsilon}},$$

If we integrate along the integral curves of the vector field $L_2^\alpha \partial_\alpha$ from a point (t_2, x_2) with $|t_2 - |x_2|| = t_2/2 = \rho$ to a point (t, x) as in the proof of Lemma 4.1 we get

$$(6.51) \quad N_n(t, x) \leq N_n(t_2, x_2) + \int_{2\rho}^t \left(\frac{(c'_n)^2 \varepsilon^2}{(1+t)(1+|\rho|)^\nu} \left(\frac{1+t}{1+|\rho|} \right)^{-\varepsilon V(\rho)} + \frac{c'_0 \varepsilon (1+|\rho|)^{\nu-nc_1 \varepsilon}}{(1+t)^{1+\nu-nc_1 \varepsilon}} \right) dt \leq (c'_n)^2 \varepsilon,$$

since by (6.41) $N_n(t, x) \leq c'_0 \varepsilon$ when $|t - |x|| = t/2$. The lemma now follows from the bound for N_n , (6.45) and (6.41). \square

6.5 The decay estimates for more vector fields

We will use induction to prove that

$$(6.52) \quad |\partial^{\mathbf{k}} Z^I \phi| \leq c_{k,i} \varepsilon (1+t)^{-1+c_{k,i}\varepsilon} (1+|\rho|)^{1-k-\nu}; \quad \max(1, k) + i \leq N-4, \quad k=|\mathbf{k}|, \quad i=|I|.$$

Note that by (5.8) $(1+|q|)(1+t)^{-c_1\varepsilon} \leq (1+|\rho|) \leq (1+|q|)(1+t)^{c_1\varepsilon}$ so we could just as well have stated (6.52) with ρ replaced by q .

We will use induction in $|I|$, and for fixed $|I|$ induction in $|\mathbf{k}|$. We will start by proving (6.52) for $|I|=0$ and all $|\mathbf{k}|$. Then we prove (6.52) for $|I|=m \geq 1$ and $|\mathbf{k}| \leq 1$ assuming (6.52) for $|I| \leq m-1$ and all $|\mathbf{k}|$. Finally we prove (6.52) for $|I|=m$ and $|\mathbf{k}|=n+1 \geq 2$ assuming (6.52) for $|I|=m$ and $|\mathbf{k}| \leq n$ and (6.52) for $|I| \leq m-1$ and all $|\mathbf{k}|$.

Proof of (6.52) for $|I|=0$ and all $|\mathbf{k}|$. In (6.36) we have already proven a stronger estimate than (6.52) for $|I|=0$ apart from the case of $|\mathbf{k}|=0$ which follows from integrating the same estimate for $|\mathbf{k}|=1$ in the $t-r$ direction, using that ϕ vanishes when $r-t \geq 1$ and $t > 0$.

Proof of (6.52) for $|I|=m \geq 1$ and $|\mathbf{k}| \leq 1$ assuming (6.52) for $|I| \leq m-1$ and all $|\mathbf{k}|$. By (3.20) and (3.22) and the fact that $|Z^J \phi| \leq 1$ by (6.1) we have

$$(6.53) \quad |\tilde{\square}_g Z^I \phi| \leq C |Z^I \phi| |\partial^2 \phi| + C \sum_{|J|+|K| \leq |I|, |J| \leq |I|-1, |K| \leq |I|-1} |Z^J \phi| |\partial^2 Z^K \phi|$$

and hence by (6.12) applied to $\psi = Z^I \phi$;

$$(6.54) \quad |L_2^\alpha \partial_\alpha (r \partial_\rho Z^I \phi)| \leq Cr |Z^I \phi| \frac{|\partial^2 \phi|}{\rho_q} + c'_1 \varepsilon |\partial_\rho Z^I \phi| + \frac{C \rho_q^{-1}}{1+t} \sum_{|J| \leq |I|+2} |Z^J \phi| + \frac{C}{\rho_q} \sum_{|J|, |K| \leq |I|-1} r |Z^J \phi| |\partial^2 Z^K \phi|$$

Hence using (6.6), (6.5) and (6.52) for $|I|$ replaced by $|I|-1$ we get

$$(6.55) \quad \left| L_2^\alpha \partial_\alpha (r \partial_\rho Z^I \phi) \right| \leq c'_2 \varepsilon \left(\frac{|Z^I \phi|}{1+|\rho|} + |\partial_\rho Z^I \phi| \right) + \frac{c'_0 \varepsilon (1+|\rho|)^{-c_1\varepsilon}}{(1+t)^{1+\nu-c_1\varepsilon}} + \frac{c'_{0,m-1} c'_{2,m-1} \varepsilon^2}{(1+t)^{1-c\varepsilon} (1+|\rho|)^{2\nu}}$$

It follows from Lemma 6.3 that with $c = 16(c'_2 + c'_0 + c'_{0,m-1} c'_{2,m-1})$ we have

$$(6.56) \quad |Z^I \phi| (1+|\rho|)^{-1} + |\partial_\rho Z^I \psi| \leq c \varepsilon (1+t)^{-1+c\varepsilon} (1+|\rho|)^{-\nu}$$

Proof of (6.52) for $|I|=m \geq 1$ and $|\mathbf{k}|=n+1 \geq 2$ assuming (6.52) for $|I| \leq m$ and all $|\mathbf{k}| \leq n$ and (6.52) for $|I| \leq m-1$ and all $|\mathbf{k}|$.

It follows from (3.21) and (3.22) that

$$(6.57) \quad |\tilde{\square}_g \partial^{\mathbf{n}} Z^I \phi| \lesssim \sum_{|\mathbf{m}|=|\mathbf{n}|+1} |\partial \phi| |\partial^{\mathbf{m}} Z^I \phi| + \sum_{|\mathbf{k}_1|+\dots+|\mathbf{k}_\ell|+|\mathbf{m}|=|\mathbf{n}|+2, |\mathbf{k}_j| \leq |\mathbf{n}|, \ell \geq 1, |J_1|+\dots+|J_\ell|+|K| \leq |I|, |\mathbf{m}| \leq |\mathbf{n}| \text{ or } |K| < |I|} |\partial^{\mathbf{k}_1} Z^{J_1} \phi| \dots |\partial^{\mathbf{k}_\ell} Z^{J_\ell} \phi| |\partial^{\mathbf{m}} Z^K \phi|.$$

Using (6.5) and (6.52) for $|\mathbf{k}| \leq n = |\mathbf{n}|$ and $|I| \leq m$, and $|\mathbf{k}| \leq n+2$ and $|I| \leq m-1$ we hence obtain

$$(6.58) \quad |\tilde{\square}_g \partial^{\mathbf{n}} Z^I \phi| \leq \frac{c_1 \varepsilon}{(1+t)(1+|\rho|)^\nu} \sum_{|\mathbf{m}|=|\mathbf{n}|+1} |\partial^{\mathbf{m}} Z^I \phi| + \frac{(c'_n)^2 \varepsilon^2}{(1+t)^{2-c\varepsilon} (1+|\rho|)^{|\mathbf{n}|+2\nu}}.$$

(6.52) for $|I|=m$ and $|\mathbf{k}|=n+1$ now follows as in the proof of (6.36).

7 Weighted Energy estimates for the wave equation on a curved background

We now establish the basic energy identities with weight for solutions of the equation

$$(7.1) \quad \tilde{\square}_g \phi = F$$

The weight will be of the form

$$(7.2) \quad w = e^{\sigma V(\rho)}, \quad \sigma = \kappa \varepsilon \ln |1+t|, \quad V(\rho) = |\rho - 2|^{-\nu'}, \quad \rho \leq 1, \quad \nu', \kappa \geq 0$$

We note that by (5.19):

$$(7.3) \quad g^{\alpha\beta} \rho_\alpha \rho_\beta \geq \delta^{AB} \partial_{A\rho} \partial_{B\rho} - |g^{\alpha\beta} \rho_\alpha \rho_\beta - \delta^{AB} \partial_{A\rho} \partial_{B\rho}| \geq -c'_2 \varepsilon \frac{(1+|\rho|)^{2-2\nu''}}{(1+t)^{2-c_2\varepsilon}},$$

so ρ satisfies the assumption below if $c'_2 \varepsilon \leq 1/(\kappa\nu')$. The following lemma was essentially proven in [A2]:

Lemma 7.1. *Let ϕ be a solution of the equation (7.1) decaying sufficiently fast as $|x| \rightarrow \infty$, with a metric g and a weight function w as in (7.2), with ρ , satisfying the conditions*

$$(7.4) \quad |g - m| \leq \frac{1}{2}, \quad |\partial g| \leq \frac{c_1 \varepsilon}{1+t}, \quad \rho_t < 0, \quad \frac{g^{\alpha\beta} \rho_\alpha \rho_\beta}{\rho_t (1+|\rho|)^{1+\nu'}} \geq -\frac{1/(\kappa\nu')}{(1+t) \ln |1+t|}$$

and $g^{\alpha\beta} = m^{\alpha\beta}$, the Minkowski metric, when $r > t + 1$. Then for functions ϕ vanishing for $r > t + 1$ we have, with $c = c_1 + \kappa$;

$$(7.5) \quad \int_{\Sigma_t} |\partial\phi|^2 w \, dx \leq 4 \int_{\Sigma_0} |\partial\phi|^2 w \, dx + \int_0^t \frac{4c\varepsilon}{1+\tau} \int_{\Sigma_\tau} |\partial\phi|^2 w \, dx \, d\tau + \frac{4}{c\varepsilon} \int_0^t \int_{\Sigma_\tau} (1+\tau) |\tilde{\square}_g \phi|^2 w \, dx \, d\tau.$$

Proof. Let $\phi_i = \partial_i \phi$, $i = 1, 2, 3$, and $\phi_t = \partial_t \phi$. If we differentiate below the integral sign and integrate by parts we get

$$(7.6) \quad \begin{aligned} & \frac{d}{dt} \int (-g^{00} \phi_t^2 + g^{ij} \phi_i \phi_j) w \, dx - \int 2\partial_j (g^{0j} \phi_t^2 w) \, dx = \int 2(-g^{00} \phi_t \phi_{tt} + g^{ij} \phi_i \phi_{tj} - 2g^{0j} \phi_t \phi_{tj}) w \, dx \\ & + \int (-(\partial_t g^{00}) \phi_t^2 + (\partial_t g^{ij}) \phi_i \phi_j - 2(\partial_j g^{0j}) \phi_t^2) w + (-g^{00} \phi_t^2 w_t + g^{ij} \phi_i \phi_j w_t - 2g^{0j} \phi_t^2 w_j) \, dx \\ & = \int 2(-g^{00} \phi_t \phi_{tt} - g^{ij} \phi_t \phi_{ij} - 2g^{0j} \phi_t \phi_{tj}) w \, dx \\ & + \int (-(\partial_t g^{00}) \phi_t^2 + (\partial_t g^{ij}) \phi_i \phi_j - 2(\partial_j g^{0j}) \phi_t^2 - 2(\partial_i g^{ij}) \phi_t \phi_j) w \, dx \\ & + \int (-g^{00} \phi_t^2 w_t + g^{ij} \phi_i \phi_j w_t - 2g^{0j} \phi_t^2 w_j - 2g^{ij} \phi_t \phi_j w_i) \, dx \end{aligned}$$

Hence, since we also have assume that ϕ_t and g^{0j} decay fast enough that the boundary term vanishes at infinity

$$(7.7) \quad \frac{d}{dt} \int (-g^{00}\phi_t^2 + g^{ij}\phi_i\phi_j)w \, dx = \int w(\phi_t\tilde{\square}_g\phi + (\partial_t g^{\alpha\beta})\phi_\alpha\phi_\beta - 2(\partial_\alpha g^{\alpha\beta})\phi_\beta\phi_t) \, dx \\ + \int g^{\alpha\beta}\phi_\alpha\phi_\beta w_t - 2\phi_t g^{\alpha\beta}\phi_\alpha w_\beta \, dx$$

Now

$$(7.8) \quad w_t = \frac{\kappa\nu'\varepsilon \ln|1+t|}{|\rho-2|^{1+\nu'}}\rho_t w + \frac{\kappa\varepsilon}{(1+t)|\rho-2|^{\nu'}}w, \quad w_i = \frac{\kappa\nu'\varepsilon \ln|1+t|}{|\rho-2|^{1+\nu'}}\rho_i w$$

If we set $\hat{\phi}_\alpha = \phi_\alpha/\phi_t$ and $\hat{\rho}_\alpha = \rho_\alpha/\rho_t$ we get

$$(7.9) \quad g^{\alpha\beta}\phi_\alpha\phi_\beta\rho_t - 2\phi_t g^{\alpha\beta}\phi_\alpha\rho_\beta = \phi_t^2\rho_t(g^{\alpha\beta}\hat{\phi}_\alpha\hat{\phi}_\beta - 2g^{\alpha\beta}\hat{\phi}_\alpha\hat{\rho}_\beta) = \phi_t^2\rho_t(g^{\alpha\beta}(\hat{\phi}_\alpha - \hat{\rho}_\alpha)(\hat{\phi}_\beta - \hat{\rho}_\beta) - g^{\alpha\beta}\hat{\rho}_\alpha\hat{\rho}_\beta) \\ = g^{ij}(\phi_i - \hat{\rho}_i\phi_t)(\phi_j - \hat{\rho}_j\phi_t)\rho_t - g^{\alpha\beta}\rho_\alpha\rho_\beta\phi_t^2/\rho_t$$

Moreover

$$(7.10) \quad g^{\alpha\beta}\phi_\alpha\phi_\beta - 2\phi_t g^{\alpha 0}\phi_\alpha = -g^{00}\phi_t^2 + g^{ij}\phi_i\phi_j$$

Hence

$$(7.11) \quad g^{\alpha\beta}\phi_\alpha\phi_\beta w_t - 2\phi_t g^{\alpha\beta}\phi_\alpha w_\beta = \frac{\kappa\nu'\varepsilon \ln|1+t|}{|\rho-2|^{1+\nu'}}w \left(g^{ij}(\phi_i - \hat{\rho}_i\phi_t)(\phi_j - \hat{\rho}_j\phi_t)\rho_t - g^{\alpha\beta}\rho_\alpha\rho_\beta\phi_t^2/\rho_t \right) \\ + \frac{\kappa\varepsilon}{(1+t)|\rho-2|^{\nu'}}w \left(-g^{00}\phi_t^2 + g^{ij}\phi_i\phi_j \right)$$

Since $|H| < 1/2$ it also follows that

$$\frac{1}{2}(\phi_t^2 + \delta^{ij}\phi_i\phi_j) \leq -g^{00}\phi_t^2 + g^{ij}\phi_i\phi_j \leq 2(\phi_t^2 + \delta^{ij}\phi_i\phi_j)$$

Moreover;

$$(7.12) \quad \int_0^t \int \phi_t\tilde{\square}_g\phi w \, dx d\tau \leq \int_0^t \frac{c\varepsilon}{1+\tau} \int_{\Sigma_\tau} |\partial\phi|^2 w \, dx d\tau + \frac{1}{c\varepsilon} \int_0^t \int_{\Sigma_\tau} (1+\tau) |\tilde{\square}_g\phi|^2 w \, dx d\tau.$$

□

8 Poincaré lemmas with weights

We note that $\partial_r\rho = \partial_p\rho + \partial_q\rho = (1 - H_{LL}/4)\partial_q\rho$, since $L_2^s\partial_\alpha\rho = 0$, so the estimate (5.19) for $\partial_q\rho$ also hold for $\partial_r\rho$ with c_2 replaced by $2c_2$. The following lemma was essentially proven in [A2]:

Lemma 8.1. *Suppose that w is as in (7.2) with $\kappa > 2c_2/\nu'$, and that with $\nu' > 0$ as in (7.2)*

$$(8.1) \quad |\partial_\rho \partial_r \rho| \leq \frac{2c_2 \varepsilon \ln |1+t|}{(1+|\rho|)^{1+\nu'}} \partial_r \rho, \quad 0 < \partial_r \rho < \infty$$

Then for functions supported in $r \leq t+1$ we have

$$(8.2) \quad \int \left(\frac{|\phi|}{1+|\rho|} \frac{\partial \rho}{\partial r} \right)^2 w dx + \int \left(\frac{|\phi|}{1+|r-t|} \right)^2 w dx \leq 32 \int |\partial \phi|^2 w dx$$

Proof. It suffices to prove the estimate for the first integral since the second estimate is a special case of the first with $\rho = r-t$. If we introduce polar coordinates $\rho = \rho(r, t, \omega)$ and change variables $r = r(\rho)$ for fixed (t, ω) we get

$$(8.3) \quad \begin{aligned} \int_0^\infty \left(\frac{|\phi|}{|\rho-2|} \frac{\partial \rho}{\partial r} \right)^2 w r^2 dr &= \int_{-\infty}^1 \left(\frac{|\phi|}{|\rho-2|} \right)^2 \frac{\partial \rho}{\partial r} w r^2 d\rho = \int_{-\infty}^1 |\phi|^2 \frac{\partial \rho}{\partial r} w r^2 \left(\frac{\partial}{\partial \rho} \frac{1}{|\rho-2|} \right) d\rho \\ &= -2 \int_{-\infty}^1 \frac{\phi}{|\rho-2|} \frac{\partial \phi}{\partial \rho} \frac{\partial \rho}{\partial r} w r^2 d\rho - \int_{-\infty}^1 \left(\frac{\phi}{|\rho-2|} \right)^2 |\rho-2| \frac{\partial}{\partial \rho} \left(\frac{\partial \rho}{\partial r} w r^2 \right) d\rho \end{aligned}$$

Because of the conditions above

$$(8.4) \quad \begin{aligned} \frac{\partial}{\partial \rho} \left(\frac{\partial \rho}{\partial r} w r^2 \right) &= \left(\frac{\partial}{\partial \rho} \frac{\partial \rho}{\partial r} \right) w r^2 + \frac{\partial \rho}{\partial r} ((\partial_\rho w) r^2 + 2r w \partial_\rho r) \\ &\geq -\frac{c_2 \nu' \varepsilon \ln |1+t|}{(1+|\rho|)^{1+\nu'}} \partial_r \rho w r^2 + \frac{\kappa \nu' \varepsilon \ln |1+t|}{|\rho-2|^{1+\nu'}} \partial_r \rho w r^2 + 2r w \geq 0. \end{aligned}$$

Therefore

$$(8.5) \quad \int_0^\infty \left(\frac{|\phi|}{|\rho-2|} \frac{\partial \rho}{\partial r} \right)^2 w r^2 dr \leq 2 \left(\int_0^\infty \left(\frac{|\phi|}{|\rho-2|} \frac{\partial \rho}{\partial r} \right)^2 w r^2 dr \right)^{1/2} \left(\int_0^\infty \left(\frac{\partial \phi}{\partial \rho} \frac{\partial \rho}{\partial r} \right)^2 w r^2 dr \right)^{1/2}$$

and it follows that

$$(8.6) \quad \int_0^\infty \left(\frac{|\phi|}{|\rho-2|} \frac{\partial \rho}{\partial r} \right)^2 w r^2 dr \leq 4 \int_0^\infty \left(\frac{\partial \phi}{\partial \rho} \frac{\partial \rho}{\partial r} \right)^2 w r^2 dr = 4 \int_0^\infty \left(\frac{\partial \phi}{\partial r} \right)^2 w r^2 dr$$

and the lemma follows from also integrating over the angular variables. \square

9 Energy estimates for the nonlinear problem

We will now show energy bounds assuming the strong decay estimates. Let

$$(9.1) \quad E_{k,i}(t) = \sum_{|\mathbf{k}| \leq k, |I| \leq i} \int |\partial \partial^{\mathbf{k}} Z^I \phi|^2 w dx,$$

where w is as in Proposition 7.1 with $\kappa = 2c_2/\nu'$ so the conditions in Proposition 7.1 and Lemma 8.1 hold if $\varepsilon > 0$ is sufficiently small.

Proposition 9.1. *Let $N \geq 14$ and set $N' = [N/2] + 2$. Suppose that ϕ is a solution of $\tilde{\square}_g(\phi)\phi = 0$ for $0 \leq t < T$ such that $\phi(t, x) = 0$ when $|x| \geq t + 1$. Suppose also that*

$$(9.2) \quad |\partial\phi| \leq \frac{c_1\varepsilon}{1+t},$$

$$(9.3) \quad |\partial^2\phi| \leq \frac{c_2\varepsilon}{1+t} \left| \frac{\partial\rho}{\partial q} \right| \frac{1}{(1+|\rho|)^{1+\nu'}}, \quad \nu' > 0,$$

$$(9.4) \quad |\phi| + |Z\phi| \leq \frac{c_2\varepsilon}{1+t} ((1+|q|) + (1+t)^{c_2\varepsilon}),$$

$$(9.5) \quad |\partial Z^I\phi| + (1+|q|)^{-1}|Z^I\phi| \leq \frac{c_{N'}\varepsilon}{1+t} (1+t)^{c_{N'}\varepsilon}, \quad \text{for } |I| \leq N'.$$

Then there are constants $C_{k,i}$, depending only on the constant above, such that for $0 \leq t < T$;

$$(9.6) \quad E_{k,i}(t) \leq 8 \sum_{\ell=0}^i E_{k+\ell, i-\ell}(0) (1+t)^{C_{k,i}\varepsilon}, \quad k+i \leq N.$$

(9.6) will follow from (9.7) below using induction and a Gronwall type of argument that we postpone.

Proposition 9.2. *Suppose that the assumptions in Proposition 9.1 hold. Then for $k+i \leq N$, $k, i \geq 0$;*

$$(9.7) \quad E_{k,i}(t) \leq 4E_{k,i}(0) + \int_0^t \frac{c'_2\varepsilon}{1+\tau} E_{k,i}(\tau) d\tau + 4 \int_0^t \frac{c_{N'}^2\varepsilon(1+\tau)^{c_{N'}\varepsilon}}{1+\tau} (E_{k+1, i-1}(\tau) + E_{k-1, i}(\tau)) d\tau,$$

where $E_{-1, n} = 0$, $E_{m, -1} = 0$.

By Proposition 7.1 with $\kappa = 2c_2/\nu'$

$$(9.8) \quad E_{k,i}(t) \leq 4E_{k,i}(0) + \int_0^t \frac{8(c_1+c_2)\varepsilon}{1+\tau} E_{k,i}(\tau) d\tau + 4 \sum_{|\mathbf{k}| \leq k, |I| \leq i} \int_0^t \frac{1+\tau}{c_2\varepsilon} \int |\tilde{\square}_g \partial^{\mathbf{k}} Z^I \phi|^2 w dx d\tau.$$

9.1 Proof of (9.7) in case $i = 0$.

If $|\mathbf{k}| = 0$ then $\tilde{\square}_g\phi = 0$ and (9.7) follows directly from (9.8), so we may assume that $1 \leq |\mathbf{k}| \leq N$. If we use (3.21) and (3.24), which holds since we assumed that $|\partial^{\mathbf{m}}\phi| \leq 1$ for $|\mathbf{m}| \leq N'$, we get

$$(9.9) \quad |\tilde{\square}_g \partial^{\mathbf{k}}\phi| \leq C|\partial\phi| \sum_{|\mathbf{n}|=|\mathbf{k}|+1} |\partial^{\mathbf{n}}\phi| + \sum_{|\mathbf{m}|+|\mathbf{n}| \leq |\mathbf{k}|+2, 1 \leq |\mathbf{m}| \leq |\mathbf{k}|, 1 \leq |\mathbf{n}| \leq |\mathbf{k}|} C|\partial^{\mathbf{m}}\phi| |\partial^{\mathbf{n}}\phi|,$$

and hence by (9.2) and (9.5)

$$(9.10) \quad |\tilde{\square}_g \partial^{\mathbf{k}}\phi| \leq \frac{c'_1\varepsilon}{1+t} \sum_{|\mathbf{n}|=|\mathbf{k}|+1} |\partial^{\mathbf{n}}\phi| + \frac{c_{N'}\varepsilon}{(1+t)^{1-c_{N'}\varepsilon}} \sum_{1 \leq |\mathbf{n}| \leq |\mathbf{k}|} |\partial^{\mathbf{n}}\phi|,$$

since either $|\mathbf{m}| \leq N'$ or $|\mathbf{n}| \leq N'$, in the second sum in (9.10). (9.7) in case $i = 0$ follows from (9.8) using (9.10).

9.2 Proof of (9.7) in case $k = 0$.

By (3.20) and (3.24) using that we assumed that $|Z^J \phi| \leq 1$ for $|J| \leq [N/2] + 2$, we have 3 types of terms:

$$(9.11) \quad |\tilde{\square}_g Z^I \phi| \lesssim |Z^I \phi| |\partial^2 \phi| + \sum_{|J| \leq 1, |K|=|I|-1} |Z^J \phi| |\partial^2 Z^K \phi| + \sum_{|J|+|K| \leq |I|, |J| < |I|, |K| < |I|-1} |Z^J \phi| |\partial^2 Z^K \phi|.$$

By (9.3) and the Poincare lemma, Lemma 8.1

$$(9.12) \quad \int (|Z^I \phi| |\partial^2 \phi|)^2 w dx \leq \int \left(\frac{c'_2 \varepsilon}{1+t} \left| \frac{\partial \rho}{\partial q} \right| \frac{|Z^I \phi|}{1+|\rho|} \right)^2 w dx \leq C \left(\frac{c'_2 \varepsilon}{1+t} \right)^2 \int |\partial Z^I \phi|^2 w dx.$$

By (9.4) and (3.3) we have

$$(9.13) \quad \int \left(\sum_{|J| \leq 1} |Z^J \phi| \sum_{|K| \leq |I|-1} |\partial^2 Z^K \phi| \right)^2 w dx \leq \int \left(\left(\frac{c_2 \varepsilon |q|}{1+t} \right)^2 \sum_{|K| \leq |I|-1} |\partial^2 Z^K \phi|^2 + \left(\frac{c_2 \varepsilon}{(1+t)^{1-c_2 \varepsilon}} \right)^2 \sum_{|K| \leq |I|-1} |\partial^2 Z^K \phi|^2 \right) w dx \\ \leq \left(\frac{c_2 \varepsilon}{1+t} \right)^2 \int \sum_{|K| \leq |I|} |\partial Z^K \phi|^2 w dx + \left(\frac{c_2 \varepsilon}{(1+t)^{1-c_2 \varepsilon}} \right)^2 \int \sum_{|K| \leq |I|-1} |\partial^2 Z^K \phi|^2 w dx$$

The remain terms are easier to handle. Again by (3.3), (9.5) and Lemma 8.1

$$(9.14) \quad \int \sum_{|J|+|K| \leq |I|, |J| < |I|, |K| < |I|-1} (|Z^J \phi| |\partial^2 Z^K \phi|)^2 w dx \leq \int \left(\sum_{|J| \leq |I|/2, |K| < |I|} \left(\frac{|Z^J \phi|}{1+|q|} |\partial Z^K \phi| \right)^2 + \sum_{|J| < |I|, |K| \leq |I|/2+1} \left(\frac{|Z^J \phi|}{1+|q|} |\partial Z^K \phi| \right)^2 \right) w dx \\ \leq \left(\frac{c_{N'} \varepsilon}{(1+t)^{1-c_{N'} \varepsilon}} \right)^2 \int \sum_{|K| < |I|} |\partial Z^K \phi|^2 w dx + \left(\frac{c_{N'} \varepsilon}{(1+t)^{1-c_{N'} \varepsilon}} \right)^2 \int \sum_{|J| < |I|} |\partial Z^J \phi|^2 w dx.$$

Summing up we get

$$(9.15) \quad \int |\tilde{\square}_g Z^I \phi|^2 w dx \\ \leq \left(\frac{c_2 \varepsilon}{1+t} \right)^2 \int \sum_{|K| \leq |I|} |\partial Z^K \phi|^2 w dx + \left(\frac{c_2 \varepsilon}{(1+t)^{1-c_2 \varepsilon}} \right)^2 \left(\int \sum_{|K| \leq |I|-1} |\partial^2 Z^K \phi|^2 w dx + \int \sum_{|K| \leq |I|-1} |\partial Z^K \phi|^2 w dx \right)$$

(9.7) in case $k = 0$ follows from this using (9.8).

9.3 Proof of (9.7) in case $k \geq 1$ and $i \geq 1$.

Since $|\partial^{\mathbf{m}} Z^J \phi| \leq 1$ for $|\mathbf{m}| + |J| \leq N - 5$, it follows from (3.23) that

$$(9.16) \quad |\tilde{\square}_g \partial^{\mathbf{k}} Z^I \phi| \lesssim \sum_{|\mathbf{n}| \leq |\mathbf{k}|, |J|+|K| \leq |I|, |K| < |I|} |Z^J \phi| |\partial^2 \partial^{\mathbf{n}} Z^K \phi| + \sum_{|\mathbf{m}|+|\mathbf{n}| \leq |\mathbf{k}|, |J|+|K| \leq |I|} |\partial \partial^{\mathbf{m}} Z^J \phi| |\partial \partial^{\mathbf{n}} Z^K \phi|,$$

for $|\mathbf{k}| + |I| \leq N$. The terms in the first sum can be dealt with as in the case $k = 0$ and the terms in the second sum can be dealt with as in the case $i = 0$.

9.4 Proof of (9.6) in case $k = i = 0$

If $k = i = 0$ then by (9.8)

$$(9.17) \quad E_{0,0}(t) \leq 4E_{0,0}(0) + \int_0^t \frac{c'_2 \varepsilon}{1+\tau} E_{0,0}(\tau) d\tau$$

and (9.6) in case $i = k = 0$ follows from this using Lemma 9.3 below.

9.5 Proof of (9.6) in case $i = 0$ and $k \geq n \geq 1$ assuming (9.6) in case $i = 0$ for $k \leq n-1$

By (9.7) using (9.6) for $E_{k-1,0}$ we have

$$(9.18) \quad \begin{aligned} E_{k,0}(t) &\leq 4E_{k,0}(0) + \int_0^t \frac{c'_2 \varepsilon}{1+\tau} E_{k,0}(\tau) d\tau + 4 \int_0^t \frac{c_{N'}^2 \varepsilon (1+\tau)^{c_{N'} \varepsilon}}{1+\tau} E_{k-1,0}(\tau) d\tau \\ &\leq 4E_{k,0}(0) + \int_0^t \frac{c'_2 \varepsilon}{1+\tau} E_{k,0}(\tau) d\tau + 32 \int_0^t \frac{c_{N'}^2 \varepsilon (1+\tau)^{(c_{N'} + C_{k-1,0}) \varepsilon}}{1+\tau} E_{k-1,0}(0) d\tau \end{aligned}$$

and again the estimate (9.6) follows from Lemma 9.3 below with $A = E_{k-1,0}(0) \leq E_{k,0}(0)$ and $B = c'_2 + 32c_{N'}^2 + C_{N'} + C_{k-1,0}$.

9.6 Proof of (9.6) in case $i = m \geq 1$ and $k = n \geq 1$ assuming (9.6) if $i = m$, for $k \leq n-1$ and if $i = m-1$ for all k , such that $i+k \leq N$.

We will prove (9.6) by induction in i and for fixed i induction in k . Since we have proven (9.6) for $i = 0$ and $k = 0$ it suffices to prove (9.6) in case $i = m \geq 1$ and $k = n \geq 1$ assuming (9.6) if $i = m$, for $k \leq n-1$ and if $i = m-1$ for all k , such that $i+k \leq N$. By (9.7) using (9.6) for $E_{k-1,i}$ and for $E_{k+1,i-1}$, we have

$$(9.19) \quad \begin{aligned} E_{k,i}(t) &\leq 4E_{k,i}(0) + \int_0^t \frac{c'_2 \varepsilon}{1+\tau} E_{k,i}(\tau) d\tau + 4 \int_0^t \frac{c_{N'}^2 \varepsilon (1+\tau)^{c_{N'} \varepsilon}}{1+\tau} (E_{k+1,i-1}(\tau) + E_{k-1,i}(\tau)) d\tau \\ &\leq 4E_{k,i}(0) + \int_0^t \frac{c'_2 \varepsilon}{1+\tau} E_{k,i}(\tau) d\tau + 32 \int_0^t \frac{c_{N'}^2 \varepsilon (1+\tau)^{(c_{N'} + C_{k+1,i-1} + C_{k-1,i}) \varepsilon}}{1+\tau} (\tilde{E}_{k+1,i-1}(0) + \tilde{E}_{k-1,i}(0)) d\tau, \end{aligned}$$

where $\tilde{E}_{k,i} = \sum_{\ell=0}^i E_{k+\ell,i-\ell}$. Using Lemma 9.3 with $A = \tilde{E}_{k-1,i}(0) + \tilde{E}_{k+1,i-1}$ and $B = c'_2 + 32c_{N'}^2 + C_{N'} + C_{k-1,i} + C_{k+1,i-1}$ we get with $C_{k,i} = 2B$;

$$(9.20) \quad E_{k,i}(t) \leq (4E_{k,i}(0) + \tilde{E}_{k-1,i}(0) + \tilde{E}_{k+1,i-1}(0))(1+t)^{C_{k,i} \varepsilon} \leq 8\tilde{E}_{k,i}(0)(1+t)^{C_{k,i} \varepsilon}$$

We conclude by giving the Gronwall type of lemma used above:

Lemma 9.3. *Suppose that for some constants $A, B \geq 0$*

$$(9.21) \quad E(t) \leq 4E(0) + \int_0^t \frac{B\varepsilon}{1+\tau} (E(\tau) + A(1+\tau)^{B\varepsilon}) d\tau.$$

Then

$$(9.22) \quad E(t) \leq (4E(0) + A)(1+t)^{2B\varepsilon}.$$

Proof. If $G(t)$ denotes the integral in the right of (9.21) then we have

$$G'(t) \leq \frac{B\varepsilon}{1+t}G(t) + \frac{B\varepsilon}{(1+t)^{1-B\varepsilon}}A.$$

If we Multiply with the integrating factor

$$\frac{d}{dt}\left(G(t)(1+t)^{-B\varepsilon}\right) \leq \frac{B\varepsilon}{1+t}A,$$

and integrate we get

$$(9.23) \quad G(t)(1+t)^{-B\varepsilon} \leq G(0) + B\varepsilon A \ln|1+t| \leq G(0) + A(1+t)^{B\varepsilon},$$

and hence

$$(9.24) \quad E(t) \leq G(t) \leq G(0)(1+t)^{B\varepsilon} + A(1+t)^{2B\varepsilon} \leq (4E(0) + A)(1+t)^{2B\varepsilon}.$$

□

10 Klainerman-Sobolev inequalities and $L^1 - L^\infty$ estimates

First we state the Klainerman-Sobolev inequality:

Proposition 10.1. *We have*

$$(10.1) \quad (1+t+|x|)(1+||t|-|x||)^{1/2}|\phi(t,x)| \leq C \sum_{|I|\leq 2} \|Z^I \phi(t, \cdot)\|_{L^2}.$$

Next we state an inequality due to Hörmander:

Proposition 10.2. *Suppose that $w(0,x) = \partial_t w(0,x) = 0$. Then*

$$(10.2) \quad |w(t,x)|(1+t+|x|) \leq C \sum_{|I|\leq 2} \int_0^t \int \frac{|(Z^I \square w)(\tau,y)|}{1+\tau+|y|} dy d\tau.$$

Corollary 10.3. *Suppose that $\phi(0,x) = \partial_t \phi(0,x) = 0$, when $|x| \geq 1$. Then*

$$(10.3) \quad |\phi(t,x)|(1+t+|x|) \leq C \sum_{|I|\leq 2} \int_0^t \int \frac{|(Z^I \square \phi)(\tau,y)|}{1+\tau+|y|} dy d\tau + C \sum_{|I|\leq 2} \|\partial Z^I \phi(0, \cdot)\|_{L^2}.$$

Proof. The inequality follows from writing $\phi = v + w$, where $\square w = \square \phi$, $w(0,x) = \partial_t w(0,x) = 0$, and $\square v = 0$, $v(0,x) = \phi(0,x)$, $\partial_t v(0,x) = \partial_t \phi(0,x)$. The inequality for w follows from Proposition 10.2 and we will argue that the inequality for v also follows from Proposition 10.2. The inequality for $0 \leq t \leq 1$ follows from the usual Sobolev's lemma so it remains to prove it for $t \geq 1$. Let $\chi(t)$ be a smooth cutoff function so that $\chi(t) = 0$ when $t \leq 0$ and $\chi(t) = 1$ when $t \geq 1$. Then $\square(\chi v) = \chi''v + 2\chi'v_t$ is supported in the set where $0 \leq t \leq 1$ and $|x| \leq 2$ and it has vanishing initial data. It therefore follows Proposition 10.2 applied to χv that for $t \geq 1$; $|v(t,x)|(1+t+|x|) = |\chi v(t,x)|(1+t+|x|) \leq C \|\square(\chi v)\|_{L^1} \leq C \sup_{0 \leq t \leq 1} \|\partial v(t, \cdot)\|_{L^2} = C \|\partial v(0, \cdot)\|_{L^2}$. □

11 The continuity argument

Let $N \geq 14$ and set

$$(11.1) \quad E_N(t) = \sum_{|I| \leq N} \int |\partial Z^I \phi(t, x)|^2 dx.$$

In view of local existence results it suffices to give a bound for $E_N(t)$. We assume that initial data are so small that

$$(11.2) \quad E_N(0) \leq \varepsilon^2.$$

Fix $0 < \delta < 1$. We will argue by continuity. We assume the bound

$$(11.3) \quad E_N(t) \leq 16N\varepsilon^2(1+t)^\delta,$$

for $0 \leq t \leq T$, which holds for $T = 0$, and we will show that this bound implies the same bound with 16 replaced by 8 if ε is sufficient small (independently of T).

Using Proposition 10.1 and (11.3) gives

$$(11.4) \quad |\partial Z^I \phi(t, x)| \leq \frac{C\varepsilon}{(1+t)^{1-\delta/2}(1+|t-r|)^{1/2}}, \quad |I| \leq N.$$

Integrating this in the $t-r$ direction from $r-t=1$, where $\phi=0$ gives

$$(11.5) \quad |Z^I \phi(t, x)| \leq \frac{C\varepsilon(1+|t-r|)^{1/2}}{(1+t)^{1-\delta/2}} \leq \frac{C\varepsilon}{(1+t)^{(1-\delta)/2}}, \quad |I| \leq N-2.$$

Since $\delta < 1$ the weak decay estimate (6.1) hold and hence the decay estimates in Proposition 6.1 as well as the estimates for the solution of the approximate eikonal equation in Proposition 5.1 and the lemmas in the same section. It therefore follows that the assumptions of Proposition 9.1 hold and that we therefore have

$$(11.6) \quad E_N(t) = E_{0,N}(t) \leq 8 \sum_{\ell=0}^N E_{\ell, i-\ell}(0)(1+t)^{C_{0,N}\varepsilon} \leq 8NE_N(0)(1+t)^{C_{0,N}\varepsilon} = 8N\varepsilon^2(1+t)^{C_{0,N}\varepsilon},$$

since the family of vector fields Z also contain the usual derivatives. If ε is so small that $C_{0,N}\varepsilon \leq \delta$, then we get back the estimate (11.3) with 16 replaced by 8. This concludes the proof of (11.3) and hence of Theorem 1.1. However, the proof above at most gives the weak decay estimate (6.1), and hence the strong decay estimates in Proposition 6.1, with $\nu = 1/2 - c\varepsilon$, $c > 0$. An additional argument using Corollary 10.3 easily gives the weak decay estimate (6.1) with $\mu = 1 - c\varepsilon$. In fact, since $\square_g \phi = 0$ we have using (11.5);

$$(11.7) \quad |\square Z^I \phi| \leq C \sum_{|J|+|K| \leq |I|} |Z^J \phi| |\partial^2 Z^K \phi| \leq C \sum_{|J|+|K| \leq |I|+1, |J| \leq |I|} \frac{|Z^J \phi|}{1+|q|} |\partial Z^K \phi|,$$

and hence using Lemma 8.1 and Hölder's inequality

$$(11.8) \quad \sum_{|L| \leq 2} \int |Z^L \square Z^I \phi(t, x)| dx \leq CE_N(t), \quad \text{if } |I| \leq N-3.$$

Therefore by Corollary 10.3 we have for $|I| \leq N - 3$

$$(11.9) \quad |Z^I \phi(t, x)|(1 + t + |x|) \leq C \int_0^t \frac{E_N(\tau)}{1 + \tau} d\tau + CE_N(0) \leq C\varepsilon(1 + t)^{C_{0,N}\varepsilon}.$$

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