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GLOBAL SMALL AMPLITUDE SOLUTIONS OF NONLINEAR HYPERBOLIC SYSTEMS WITH A CRITICAL EXPONENT UNDER THE NULL CONDITION*

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Dedicated to Professor Rentaro Agemi on the occasion of his 60th birthday.

Abstract. This paper deals with the Cauchy problems of nonlinear hyperbolic systems in two space dimensions with small data. We assume that the propagation speeds differ from each other and that nonlinearities are cubic. Then it will be shown that if the nonlinearities satisfy the *null condition*, there exists a global smooth solution. To prove this kind of claim, one usually makes use of the generalized differential operators Ω_{ij}, S, and L_i, which will be introduced in section 1. But it is difficult to adopt the operators $L_i = x_i\partial_t + t\partial_{x_i}$ to our problem, because they do not commute with the d'Alembertian whose propagation speed is not equal to one. We succeed in taking L_i away from the proof of our theorem. One can apply our method to a scalar equation; hence L_i are needless in this kind of argument.

Key words. null condition, different speeds, a unique global smooth solution

AMS subject classifications. 35A05, 35B45, 35L15, 35L55

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1. Introduction and statement of main result.

We consider the initial value problem for

$$\Box_i u^i \equiv \partial_t^2 u^i - c_i^2 \Delta u^i = F^i(\partial u, \partial^2 u) \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{1.1}$$

$$u^i(x,0) = \varepsilon f^i(x), \quad \partial_t u^i(x,0) = \varepsilon g^i(x) \quad \text{in } \mathbf{R}^n, \tag{1.2}$$

where $i = 1, \dots, m$, $n = 2, 3$, c_i are positive constants and $\varepsilon > 0$ is a small parameter. Besides, $F^i \in C^\infty(\mathbf{R}^{(n+1)m} \times \mathbf{R}^{(n+1)^2 m})$ and f^i, $g^i \in C_0^\infty(\mathbf{R}^n)$. We also denoted $u = (u^1, \dots, u^m)$, $\partial = (\partial_t, \partial_1, \dots, \partial_n)$ with $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ and $\partial^2 u$ stands for the second derivatives of u. As for F^i, we assume

$$F^i(\partial u, \partial^2 u) = \sum_{l=1}^{m} \sum_{\gamma,\delta=0}^{n} H_{il}^{\gamma\delta}(\partial u)\partial_\gamma \partial_\delta u^l + K_i(\partial u), \tag{1.3}$$

where $H_{il}^{\gamma\delta}$ and $K_i \in C^\infty(\mathbf{R}^{(n+1)m})$ satisfy

$$H_{il}^{\gamma\delta}(\partial u) = O(|\partial u|^{p-1}), \quad K_i(\partial u) = O(|\partial u|^p) \quad \text{near } \partial u = 0. \tag{1.4}$$

Here p is an integer with $p > 1$. In order to derive an energy estimate we further assume

$$H_{il}^{\gamma\delta}(\partial u) = H_{li}^{\gamma\delta}(\partial u) = H_{il}^{\delta\gamma}(\partial u). \tag{1.5}$$

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Although our interest lies in the case where the system (1.1) has different propagation speeds, we start with a review of known results for the case where $m = 1$ or the system (1.1) has same propagation speeds. Indeed, such cases have been studied extensively. Set $p_c = (n+1)/(n-1)$. If $p > p_c$, then the problem (1.1) and (1.2) has a smooth global solution for sufficiently small ε. Moreover, if $p = p_c$, then the problem (1.1) and (1.2) admits an "almost" global solution for small initial data. (See F. John and S. Klainerman [12], S. Klainerman [16], and M. Kovalyov [19], for instance). On the other hand, if $1 < p \le p_c$, then the problem (1.1) and (1.2) does not admit global solutions in general. (See R. Agemi [1], S. Alinhac [3], L. Hörmander [7], A. Hoshiga [9], and F. John [10].) Therefore, we shall call the number p_c the critical exponent in the following.

In the critical case $p = p_c$, the following interesting result is known. If the nonlinearity has a special form, a global solution of (1.1) and (1.2) exists, instead of an almost global solution. (See D. Christodoulou [4], P. Godin [6], A. Hoshiga [8], F. John [11], S. Katayama [13], and S. Klainerman [17], for instance.) We shall call the restriction on the nonlinearlities *null condition*, according to S. Klainerman [15]. We will restrict ourselves to the case where $n = 2$ and $p = p_c = 3$. Then, when $c_1 = \cdots = c_m = 1$, the null condition is stated as follows: For any $i, j, k, l = 1, \ldots, m$,

$$\sum_{\alpha,\beta,\gamma=0}^{2} A_{ijkl}^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma = 0 \quad \text{and} \quad \sum_{\alpha,\beta,\gamma,\delta=0}^{2} D_{ijkl}^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma X_\delta = 0 \tag{1.6}$$

hold on the hypersurface $(X_0)^2 - c_i^2\{(X_1)^2 + (X_2)^2\} = 0$, where we have set

$$A_{ijkl}^{\alpha\beta\gamma} \equiv \left.\frac{\partial^3 K_i(\partial u)}{\partial(\partial_\alpha u^j)\partial(\partial_\beta u^k)\partial(\partial_\gamma u^l)}\right|_{\partial u=0} \quad \text{and} \quad D_{ijkl}^{\alpha\beta\gamma\delta} \equiv \left.\frac{\partial^2 H_{il}^{\gamma\delta}(\partial u)}{\partial(\partial_\alpha u^j)\partial(\partial_\beta u^k)}\right|_{\partial u=0}. \tag{1.7}$$

A role of the null condition is closely connected to the following vector fields which generate a Lie algebra with respect to the usual commutator of linear operators:

$$\partial_t, \partial_1, \partial_2, \quad S = t\partial_t + r\partial_r, \quad \Omega = x_1\partial_2 - x_2\partial_1, \tag{1.8}$$

and

$$L_i = x_i\partial_t + t\partial_i \quad (i = 1, 2),$$

where $r = |x|$. In fact, we may write

$$\partial_i = -\omega_i\partial_t + \frac{1}{t}L_i + \frac{\omega_i}{t+r}S - \sum_{j=1}^{2}\frac{r\omega_i\omega_j}{t(t+r)}L_j \quad (i = 1, 2), \tag{1.9}$$

where $\omega_i = x_i/|x|$. (See [11].) In the leading terms of F^i, replacing ∂_i with (1.9) and using the null condition (1.6), we get

$$|\Gamma^a F^i(\partial u, \partial^2 u)| \le \frac{C}{t}\sum_{|b+c+d|\le|a|+1}|\Gamma^b u||\Gamma^c\partial u||\Gamma^d\partial u| + \text{ (higher order terms)}, \tag{1.10}$$

which gives us an additional decaying factor t^{-1}. This is a crucial point to treat the critical nonlinearity.

We now turn our attention to the case where $m \ge 2$ and the propagation speeds are different from each other when $n = 2$ and $p = 3$. M. Kovalyov proved the existence of the global solution of (1.1) and (1.2) in [20] under the assumption that for each $i(=1,\dots,m)$, $A_{ijjj}^{\alpha\beta\gamma}=0$ for any $\alpha,\beta,\gamma=0,1,2$, $j=1,\dots,m$ and $H_{il}^{\gamma\delta}(\partial u)\equiv 0$ for any $\gamma,\delta=0,1,2$, $l=1,\dots,m$. In [2], R. Agemi and K. Yokoyama had the same result under the weaker assumption that for each $i(=1,\dots,m)$, $A_{iiii}^{\alpha\beta\gamma}=0$ for any $\alpha,\beta,\gamma=0,1,2$ and $D_{iiii}^{\alpha\beta\gamma\delta}=0$ for any $\alpha,\beta,\gamma,\delta=0,1,2$. Here we have used the notation in (1.7). These results imply that when the propagation speeds are distinct, the global solution of (1.1) and (1.2) exists even if the nonlinearities do not satisfy (1.6). In this paper, we would like to show more generally that when the propagation speeds are distinct, (1.1) and (1.2) has a global solution under the following condition: For each $i=1,\dots,m$,

$$\sum_{\alpha,\beta,\gamma=0}^{2} A_{iiii}^{\alpha\beta\gamma}X_\alpha X_\beta X_\gamma=0 \quad\text{and}\quad \sum_{\alpha,\beta,\gamma,\delta=0}^{2} D_{iiii}^{\alpha\beta\gamma\delta}X_\alpha X_\beta X_\gamma X_\delta=0 \tag{1.11}$$

hold on the hypersurface $(X_0)^2-c_i^2\{(X_1)^2+(X_2)^2\}=0$. Having the condition (1.11) in mind, we shall rewrite F^i in the following form:

$$F^i(\partial u,\partial^2 u)=N^i(\partial u^i,\partial^2 u^i)+R^i(\partial u,\partial^2 u)+G^i(\partial u,\partial^2 u), \tag{1.12}$$

where

$$N^i(\partial u^i,\partial^2 u^i)=\sum_{\alpha,\beta,\gamma,\delta=0}^{2} D_{iiii}^{\alpha\beta\gamma\delta}\partial_\alpha u^i\partial_\beta u^i\partial_\gamma\partial_\delta u^i+\sum_{\alpha,\beta,\gamma=0}^{2} A_{iiii}^{\alpha\beta\gamma}\partial_\alpha u^i\partial_\beta u^i\partial_\gamma u^i,$$

$$R^i(\partial u,\partial^2 u)=\sum_{j,k,l=1}^{m}\sum_{\alpha,\beta,\gamma,\delta=0}^{2} E_{ijkl}^{\alpha\beta\gamma\delta}\partial_\alpha u^j\partial_\beta u^k\partial_\gamma\partial_\delta u^l + \sum_{j,k,l=1}^{m}\sum_{\alpha,\beta,\gamma=0}^{2} B_{ijkl}^{\alpha\beta\gamma}\partial_\alpha u^j\partial_\beta u^k\partial_\gamma u^l,$$

and

$$G^i(\partial u,\partial^2 u)=\sum_{l=1}^{m}\sum_{\gamma,\delta=0}^{2} H_{il}(\partial u)\partial_\gamma\partial_\delta u^l+M_i(\partial u).$$

Here $E_{ijkl}^{\alpha\beta\gamma\delta}$ and $B_{ijkl}^{\alpha\beta\gamma}$ are defined by

$$E_{ijkl}^{\alpha\beta\gamma\delta}=\begin{cases} D_{ijkl}^{\alpha\beta\gamma\delta} & (j,k,l)\neq(i,i,i),\\ 0 & (j,k,l)=(i,i,i),\end{cases} \tag{1.13}$$

$$B_{ijkl}^{\alpha\beta\gamma}=\begin{cases} A_{ijkl}^{\alpha\beta\gamma} & (j,k,l)\neq(i,i,i),\\ 0 & (j,k,l)=(i,i,i).\end{cases}$$

Also, we assume H_{il} and $M_i\in C^\infty(\mathbf{R}^{3m})$ satisfy

$$H_{il}(\partial u)=O(|\partial u|^3),\quad M_i(\partial u)=O(|\partial u|^4)\quad\text{near } \partial u=0.$$

By (1.11), N^i has the usual null-form for a scalar wave equation. Its concrete form will be proposed in section 3. We shall call N^i the null-form, while R^i is the resonance-form.

Now we state our main theorem.

THEOREM 1.1. *Let $n = 2$ and $c_i \neq c_j$ if $i \neq j$. Suppose that* (1.12), (1.5), *and* (1.11) *hold. Then there exists a positive constant ε_0 such that the initial value problem* (1.1) *and* (1.2) *has a unique C^∞-solution in $\mathbf{R}^2 \times [0, \infty)$ for $0 < \varepsilon \leq \varepsilon_0$.*

Remark 1. We would like to mention here the key idea of the proof of Theorem 1.1. Compared with the case where the system (1.1) has common propagation speeds, a treatment of the null-form is much more complicated when the speeds are different. The difficulty comes from the simple fact that L_j does not commute with $\Box_i$ if $c_i \neq 1$. Therefore, it seems difficult to adopt the operator L_j (or some modification of them) for the system (1.1) with different propagation speeds. Our main idea in this paper is to use the operator S effectively. More precisely, in order to obtain a variant of (1.10) without using L_j, we shall use the following relation instead of (1.9):

$$\partial_t = -c_i\partial_r + \frac{c_i t - r}{t}\partial_r + \frac{1}{t}S \tag{1.14}$$

and

$$\nabla = \frac{x}{r}\partial_r - \frac{x^\perp}{r^2}\Omega, \tag{1.15}$$

where $\nabla = (\partial_1, \partial_2)$ and $x^\perp = (x_2, -x_1)$. Since we need an additional decaying factor only in the region near the characteristic lay, we rewrite (1.14) as

$$\partial_t = -c_i\partial_r - \frac{\delta(r,t)}{\sqrt{t}}\partial_r + \frac{1}{t}S \quad \text{for } |c_i t - r| \leq \sqrt{t}, \tag{1.16}$$

where $-1 \leq \delta(r, t) \leq 1$. This is a key point in our argument. (For the details, see section 3 below). Moreover, this approach also works when either $m = 1$ or $c_1 = \cdots = c_m$ holds.

Remark 2. The other attempts to argue within the framework of $(\partial_\alpha, \Omega, S)$ were also done by S. Klainerman and T. Sideris [18] and by T. Sideris [23]. They studied the nonlinear elastic waves with the critical exponent. They used the operator S in order to extract a decaying factor from the elastic wave operator. However, their method requires that the nonlinearity has a divergence structure. Unfortunately, we can not apply their method to our case due to the lack of such a structure. Hence, following [19], [20], and [2], we make use of L^∞-weighted estimates derived by estimating the fundamental solution of the wave operator $\partial_t^2 - \Delta$, pointwisely. (See also section 4 below.)

2. Notations. In this section we collect some notations which will be used in the following discussion. Without loss of generality, we may assume

$$c_1 > c_2 > \cdots > c_m. \tag{2.1}$$

We denote the vector fields introduced in (1.8) by Γ_i as follows:

$$\Gamma = (\Gamma_0, \Gamma_1, \dots, \Gamma_4) = (\partial, \Omega, S),$$

where

$$\partial = (\partial_0, \partial_1, \partial_2), \quad \partial_0 = \partial_t, \quad \Omega = x_1\partial_2 - x_2\partial_1, \quad \text{and} \quad S = t\partial_t + r\partial_r.$$

We can easily verify the following commutator relations:

$$[\Gamma_\sigma, \Box_i] = -2\delta_{4\sigma}\Box_i \quad \text{for} \quad \sigma = 0, \dots, 4, i = 1, \dots, m \tag{2.2}$$

and

$$\begin{aligned} &[\partial_\alpha, \partial_\beta] = 0 \quad (\alpha, \beta = 0, 1, 2), \quad [\Omega, \partial_0] = 0, \quad [\Omega, \partial_1] = -\partial_2, \quad [\Omega, \partial_2] = \partial_1, \\ &[S, \partial_\alpha] = -\partial_\alpha \quad (\alpha = 0, 1, 2), \quad [S, \Omega] = -\Omega. \end{aligned} \tag{2.3}$$

Here $[,]$ denotes the usual commutator of linear operators and $\delta_{\alpha\beta}$ is Kronecker's delta.

Next we define several norms for a vector valued function $u(x, t)$:

$$\begin{aligned} |u(t)|_k &= \sum_{|a|\le k} \sum_{i=1}^m \|\Gamma^a u^i(\cdot, t)\|_{L^\infty}, \\ [u(t)]_k &= \sum_{|a|\le k} \sum_{i=1}^m \|w_i(|\cdot|, t)\Gamma^a u^i(\cdot, t)\|_{L^\infty}, \\ \|u(t)\|_k &= \sum_{|a|\le k} \sum_{i=1}^m \|\Gamma^a u^i(\cdot, t)\|_{L^2}, \end{aligned}$$

where k is a nonnegative integer, $a = (a_0, \dots, a_4)$ is a multi-index, $\Gamma^a = \Gamma_0^{a_0} \cdots \Gamma_4^{a_4}$, and $|a| = a_0 + \cdots + a_4$. In addition, w_i is the following weight function associated with the ith component of u:

$$w_i(r, t) = (1 + r)^{\frac{1}{2}-\gamma}(1 + t + r)^\gamma(1 + |c_i t - r|)^{\frac{1}{2}} \quad \text{for } r \ge 0, t \ge 0,$$

where $1/4 < \gamma < 1/2$. Moreover, we also use

$$|u|_{k,T} = \sup_{0<t<T} |u(t)|_k, \quad [u]_{k,T} = \sup_{0<t<T} [u(t)]_k, \quad \|u\|_{k,T} = \sup_{0<t<T} \|u(t)\|_k.$$

Next we split the region $(0, \infty) \times (0, \infty)$ for each $i (i = 1, \dots, m)$ as follows:

$$\tilde\Lambda_i = \left\{(r, t) \in (0, \infty) \times (0, \infty) : \frac{1}{3}\left(2 + \frac{c_i}{c_{i-1}}\right) r \le c_i t \le \frac{1}{3}\left(2 + \frac{c_i}{c_{i+1}}\right) r \ \text{ and } \ r \ge 1\right\}$$

and $\tilde\Lambda_i^c = ((0, \infty) \times (0, \infty)) \setminus \tilde\Lambda_i$, where we have set $c_0 = 4c_1$ and $c_{m+1} = c_m/4$. Because of (2.1), this definition is meaningful. In particular, we have

$$\tilde\Lambda_i \cap \tilde\Lambda_l = \emptyset \quad \text{if } i \ne l. \tag{2.4}$$

Using the fact that $1+r$ is equivalent to $1+t+r$ for $(r, t) \in \tilde\Lambda_i$, while, so is $1+|c_i t-r|$ for $(r, t) \in \tilde\Lambda_i^c$, we easily see that

$$w_i(r, t) \ge C(1 + t + r)^{\frac{1}{2}} \quad \text{for } (r, t) \in (0, \infty) \times (0, \infty) \tag{2.5}$$

and that if $\gamma > 1/4$,

$$w_i(r, t) \ge C(1 + t + r)^{\frac{3}{4}} \quad \text{for } (r, t) \in \tilde\Lambda_i^c. \tag{2.6}$$

We conclude this section by showing an important property of the weight function based on the following other decomposition of $(0,\infty)\times(0,\infty)$ for each $i(i=1,\dots,m)$:

$$\Lambda_i = \{(r,t)\in(0,\infty)\times(0,\infty) : |c_it-r|\le\sqrt{t}\}$$

and $\Lambda_i^c = ((0,\infty)\times(0,\infty))\setminus\Lambda_i$.

PROPOSITION 2.1. *Let $1/4<\gamma<1/2$ and $i=1,2,\dots,m$. Then we have*

$$w_i(r,t)\ge C(1+t)^{\frac34}\quad \textit{for } (r,t)\in\Lambda_i^c, \tag{2.7}$$

$$w_i(r,t)\le C(1+t)^{\frac34}\quad \textit{for } (r,t)\in\Lambda_i. \tag{2.8}$$

Proof. First we shall show (2.7). If $(r,t)\in\tilde\Lambda_i^c\cap\Lambda_i^c$, we have

$$w_i(r,t)\ge C(1+t+r)^{\gamma+\frac12}\ge C(1+t+r)^{\frac34}$$

for $\gamma>1/4$. If $(r,t)\in\tilde\Lambda_i\cap\Lambda_i^c$, we have

$$w_i(r,t)\ge C(1+t+r)^{\frac12}(1+\sqrt{t})^{\frac12}\ge C(1+t)^{\frac34}.$$

We thus obtain (2.7).

Next we shall show (2.8). Note that

$$\frac{c_it}{2}\le r\le 2c_it\quad\text{for } (r,t)\in\Lambda_i\quad\text{with } t\ge\frac{4}{c_i^2}. \tag{2.9}$$

Therefore, we get

$$w_i(r,t)\le C(1+t)^{\frac12}(1+\sqrt{t})^{\frac12}\le C(1+t)^{\frac34}$$

for such (r,t). On the other hand, if $(r,t)\in\Lambda_i$ and $0\le t\le 4/c_i^2$, r is also bounded by some uniform constant, hence (2.8) follows. This completes the proof. ∎

3. An estimate for the null-form. By (1.11), one can write N^i defined in (1.12) as linear combinations of the following:

$$\begin{aligned}
N_1^i &= ((\partial_0u^i)^2-c_i^2|\nabla u^i|^2)\partial_\alpha\partial_\beta u^i,\\
N_2^i &= \partial_\alpha u^i\partial_\beta((\partial_0u^i)^2-c_i^2|\nabla u^i|^2),\\
N_3^i &= \partial_\alpha u^i\partial_\beta u^i\Box_i u^i,\\
N_4^i &= \partial_\alpha u^i(\partial_\beta u^i\partial_\gamma\partial_\delta u^i-\partial_\gamma u^i\partial_\beta\partial_\delta u^i),\\
N_5^i &= \partial_\alpha u^i((\partial_0u^i)^2-c_i^2|\nabla u^i|^2)
\end{aligned}$$

for $\alpha,\beta,\gamma,\delta=0,1,2$. As we have already discussed in introduction, we shall extract an additional decaying factor from the null-form, by making use of their special form together with the identity (1.16). This is a crucial point in our argument.

PROPOSITION 3.1. *It holds that for $i=1,\dots,m$*

$$|\Gamma^aN^i(\partial u^i,\partial^2u^i)|\le\frac{C}{\sqrt{1+t}}\Phi_a^i+\frac{C}{\sqrt{1+t}}\Theta_a^i\quad \textit{in } \Lambda_i, \tag{3.1}$$

where we have set

$$\Phi_a^i=\sum_{|b+c+d|\le|a|+1}|\partial\Gamma^bu^i||\partial\Gamma^cu^i||\partial\Gamma^du^i|,$$

$$\Theta_a^i=\sum_{\substack{|b+c+d|\le|a|+2\\ |b|,|c|,|d|\le|a|+1}}|\Gamma^bu^i||\partial\Gamma^cu^i||\partial\Gamma^du^i|.$$

Proof. It is evident that (3.1) holds for $0 \le t \le \max\{1, 4/c_i^2\}$. Therefore, we shall assume $t \ge \max\{1, 4/c_i^2\}$ in the following. For simplicity, we omit the upper index i of u^i during the proof.

First, we consider the case $N^i = N_1^i$. If we set

$$Q_1(u, v) = \partial_0 u \partial_0 v - c_i^2 \nabla u \cdot \nabla v,$$

then we may write

$$\Gamma^a N_1^i = \sum_{a'+d'=a} \binom{a}{a'} \Gamma^{a'}(Q_1(u, u))\Gamma^{d'}(\partial_\alpha \partial_\beta u). \tag{3.2}$$

By the commutator relations (2.3), we obtain

$$\Gamma Q_1(u, v) = Q_1(\Gamma u, v) + Q_1(u, \Gamma v) - 2\delta_{4\sigma} Q_1(u, v) \quad \text{for } \sigma = 0, \dots, 4.$$

Therefore, we have

$$\Gamma^{a'} Q_1(u, u) = \sum_{b+c \le a'} C_{b,c}^{a'} Q_1(\Gamma^b u, \Gamma^c u). \tag{3.3}$$

By (3.2), (3.3), and $t \ge \max\{1, 4/c_i^2\}$, it suffices to show

$$|Q_1(u, v)| \le \frac{C}{\sqrt{t}} |\partial u||\partial v| + \frac{C}{t}(|\Gamma u||\partial v| + |\partial u||\Gamma v|). \tag{3.4}$$

Setting

$$\tilde{Q}_1(u, v) = \partial_0 u \partial_0 v - c_i^2 \partial_r u \partial_r v$$

and using the formula

$$\nabla = \frac{x}{r}\partial_r - \frac{x^\perp}{r^2}\Omega, \qquad x^\perp = (x_2, -x_1), \tag{3.5}$$

we get

$$Q_1(u, v) = \tilde{Q}_1(u, v) + \frac{c_i^2}{r^2} \Omega u \, \Omega v;$$

hence

$$|Q_1(u, v)| \le |\tilde{Q}_1(u, v)| + \frac{C}{r} |\partial u||\Omega v|, \tag{3.6}$$

where we used the fact that $|\Omega u|/r \le C|\partial u|$.

If we introduce operators $S_i^\pm = \partial_t \pm c_i \partial_r$, then a simple computation yields

$$2\tilde{Q}_1(u, v) = S_i^+ u S_i^- v + S_i^- u S_i^+ v.$$

Moreover, by the formula

$$S_i^+ = \partial_t + c_i \partial_r = -\frac{\delta(r, t)}{\sqrt{t}} \partial_r + \frac{1}{t} S \quad \text{with } -1 \le \delta(r, t) \le 1 \quad \text{in } \Lambda_i, \tag{3.7}$$

we obtain

$$|\tilde{Q}_1(u,v)| \le \frac{C}{\sqrt{t}}|\partial u||\partial v| + \frac{C}{t}(|Su||\partial v| + |\partial u||Sv|) \quad \text{in} \quad \Lambda_i. \tag{3.8}$$

Thus (3.6), (3.8), and (2.9) imply (3.4).

Second, from the above argument, we immediately obtain (3.1) for the case $N^i = N_2^i$ and $N^i = N_5^i$, because of the fact that

$$N_2^i = 2Q_1(\partial_\beta u, u)\partial_\alpha u \quad \text{and} \quad N_5^i = Q_1(u,u)\partial_\alpha u.$$

Third, we consider the case $N^i = N_3^i$. It follows from (2.2) that

$$\Gamma^a \Box_i u = \sum_{b\le a} C_b \Box_i \Gamma^b u \quad \text{and} \quad \Box_i \Gamma^a u = \sum_{b \le a} C_b' \Gamma^b \Box_i u, \tag{3.9}$$

where C_b and C_b' are some constants. By (3.7), (2.9), $t \ge \max\{1, 4/c_i^2\}$, and the identity

$$\Box_i u = S_i^+ S_i^- u - \frac{c_i^2}{r^2}\Omega^2 u,$$

we obtain

$$|\Box_i u| \le \frac{C}{\sqrt{t}}|\partial^2 u| + \frac{C}{t}|\Gamma \partial u|. \tag{3.10}$$

Hence, by the first identity in (3.9) and (3.10) we have (3.1) for the case $N^i = N_3^i$.

Finally, we consider the case $N^i = N_4^i$. Using a notation

$$Q_{\alpha\beta}(u,v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v, \quad \alpha, \beta = 0,1,2,$$

we can write

$$N_4^i = Q_{\beta\gamma}(u, \partial_\delta u)\partial_\alpha u.$$

Note that $Q_{\beta\alpha} = -Q_{\alpha\beta}$. Moreover, it follows from (2.3) that

$$\begin{aligned}
\partial_\eta Q_{\alpha\beta}(u,v) &= Q_{\alpha\beta}(\partial_\eta u, v) + Q_{\alpha\beta}(u, \partial_\eta v), \quad \eta = 0,1,2,\\
SQ_{\alpha\beta}(u,v) &= Q_{\alpha\beta}(Su,v) + Q_{\alpha\beta}(u,Sv) - 2Q_{\alpha\beta}(u,v),\\
\Omega Q_{01}(u,v) &= Q_{01}(\Omega u, v) + Q_{01}(u, \Omega v) - Q_{02}(u,v),\\
\Omega Q_{02}(u,v) &= Q_{02}(\Omega u, v) + Q_{02}(u, \Omega v) + Q_{01}(u,v),\\
\Omega Q_{12}(u,v) &= Q_{12}(\Omega u, v) + Q_{12}(u, \Omega v).
\end{aligned}$$

Therefore we have

$$\Gamma^a Q_{\alpha\beta}(u,v) = \sum_{\gamma,\delta=0}^{2} \sum_{b+c\le a} C_{bc}^{\gamma\delta} Q_{\gamma\delta}(\Gamma^b u, \Gamma^c v). \tag{3.11}$$

On the other hand, by (2.9), (3.5), and the formula

$$\partial_t = -\frac{r}{t}\partial_r + \frac{1}{t}S,$$

we have

$$|Q_{\alpha\beta}(u,v)| \le \frac{C}{t}(|\partial u||\Gamma v| + |\Gamma u||\partial v|). \tag{3.12}$$

Combining (3.11) and (3.12), we have (3.1) for the case $N^i = N_4^i$. This completes the proof of Proposition 3.1. □

4. Weighted L^∞-estimates. The aim of this section is to establish weighted L^∞-estimates of a solution $u = (u^1, \dots, u^m)$ of (1.1) and (1.2) such that $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$ and satisfies

$$[\partial u]_{k,t} \le \delta_1 \quad \text{for } 0 \le t < T, \tag{4.1}$$

where k is a nonnegative integer and $\delta_1 (0 < \delta_1 < 1)$ is a real number independent of $T > 0$. A main result of this section is the following proposition.

PROPOSITION 4.1. *Suppose that* $u = (u^1, \dots, u^m)$ *is the solution of* (1.1) *and* (1.2) *and that* (1.12) *holds. Then we have for* $(|x|, t) \in \Lambda_i^c$ *with* $t < T$ *and* $|a| \le N$

$$|w_i(|x|,t)\Gamma^a \partial u^i(x,t)| \le C_N \left(\varepsilon + [\partial u]^2_{[\frac{N+2}{2}],t} \|\partial u\|_{N+4,t} \right), \tag{4.2}$$

provided (4.1) *with* $k = [(N+2)/2]$ *holds. Moreover, if* (1.11), (1.12), *and* (4.1) *with* $k = [(N+4)/2]$ *hold, we have for* $(x,t) \in \mathbf{R}^2 \times [0,T)$ *and* $|a| \le N$

$$|w_i(|x|,t)\Gamma^a \partial u^i(x,t)| \le C_N \left(\varepsilon + (\varepsilon + [\partial u]^2_{[\frac{N+4}{2}],t}) \|\partial u\|_{N+6,t} \right). \tag{4.3}$$

Here we take δ_1 *to be sufficiently small positive number and* C_N *denotes a positive constant independent of* T *and* δ_1*.*

By (3.9) and (1.1), $\Gamma^a \partial^b u^i(x,t)$ satisfies

$$\Box_i \Gamma^a \partial^b u^i(x,t) = \tilde{F}^i(\partial u, \partial^2 u) \quad \text{in } \mathbf{R}^2 \times (0,T), \tag{4.4}$$

where we have set $\tilde{F}^i(\partial u, \partial^2 u) = \sum_{d \le a} C_{a,b} \partial^b \Gamma^d F^i(\partial u, \partial^2 u)$ and a, b, and d are multi-indices. Moreover, the initial values of $\Gamma^a \partial^b u^i(x,t)$ are determined by ε, f^j, and g^j $(j = 1, \dots, m)$ by using (1.1). For instance, when $a = 0$ and $\partial^b = \partial_t$, we have

$$(\partial_t u^i)(x,0) = \varepsilon g^i(x), \quad (\partial_t^2 u^i)(x,0) = \varepsilon c_i^2 \Delta f^i(x) + F^i(\partial u, \partial^2 u)(x,0).$$

We can solve the second equation with respect to $(\partial_t^2 u^i)(x,0)$ if δ_1 is sufficiently small.

Based on this, we decompose $\Gamma^a \partial^b u(x,t)$ as follows:

$$\Gamma^a \partial^b u(x,t) = u_0(x,t) + u_1(x,t) \quad \text{with} \quad u_0 = (u_0^1, \dots, u_0^m), u_1 = (u_1^1, \dots, u_1^m), \tag{4.5}$$

where u_1^i is a solution to $\Box_i u_1^i = \tilde{F}^i(\partial u, \partial^2 u)$ with the zero initial data, while u_0^i is a solution to $\Box_i u_0^i = 0$ and $u_0^i(x,0) = (\Gamma^a \partial^b u)(x,0)$, $\partial_t u_0^i(x,0) = (\partial_t \Gamma^a \partial^b u)(x,0)$. Since $f^j(x)$, $g^j(x) \in C_0^\infty(\mathbf{R}^2)$, the initial values of u_0^i also belong to $C_0^\infty(\mathbf{R}^2)$. Therefore, when $|a| + |b| \le N$, we have

$$|u_0^i(x,t)| \le M_N \varepsilon (1+t+r)^{-\frac{1}{2}} (1 + |c_i t - r|)^{-\frac{1}{2}} \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,\infty), \tag{4.6}$$

where M_N depends on L^1-norms of f^j, g^j and their finite times derivatives. (See Lemma 1 in R. T. Glassey [5] and also Lemma 4 in [19] and [21].)

Therefore, we need to estimate only u_1^i. We may assume $c_i = 1$ without loss of generality. In the following, we shall consider the solution to an inhomogeneous wave equation $(\partial_t^2 - \Delta)u = \partial^b F$ with the zero initial data. When $F \in C^\infty(\mathbf{R}^2 \times [0,T))$, we have

$$u(x,t) = \frac{1}{2\pi} \int_{|x-y| \le t} \frac{\partial^b F(y,s)}{\sqrt{t^2 - |x-y|^2}} dy. \tag{4.7}$$

Switching to polar coordinates as $x = (r\cos\theta, r\sin\theta)$ and $y = (\lambda\cos(\theta+\psi), \lambda\sin(\theta+\psi))$ as in section 2 in [19], we have

$$u(x,t) = \frac{1}{2\pi}\iint_{D'} \lambda d\lambda ds \int_{-\varphi}^{\varphi} \partial^b F(\lambda\xi, s) K_1 d\psi + \frac{1}{2\pi} H(t-r) \iint_{D''} \lambda d\lambda ds \int_{-\pi}^{\pi} \partial^b F(\lambda\xi, s) K_1 d\psi, \tag{4.8}$$

where H is the Heaviside function and we have set

$$\begin{aligned} \xi &= (\cos(\theta+\psi), \sin(\theta+\psi)), \\ K_1 &= K_1(\lambda, s, \psi; r, t) = \{(t-s)^2 - r^2 - \lambda^2 + 2r\lambda\cos\psi\}^{-\frac{1}{2}}, \\ \varphi &= \varphi(\lambda, s; r, t) = \arccos\left[\frac{r^2+\lambda^2-(t-s)^2}{2r\lambda}\right] \quad \text{for } (\lambda, s) \in D'. \end{aligned}$$

Moreover, the domains D' and D'' are defined as follows:

$$\begin{aligned} D' &= \{(\lambda, s) \in (0,\infty)\times(0,\infty) : 0 < s < t, \lambda_- < \lambda < \lambda_+\}, \\ D'' &= \{(\lambda, s) \in (0,\infty)\times(0,\infty) : 0 < s < t-r, 0 < \lambda < \lambda_-\}, \end{aligned}$$

where

$$\lambda_- = |t-s-r|, \quad \lambda_+ = t-s+r. \tag{4.9}$$

The key point to get such estimates as in Proposition 4.1 is to integrate by parts with respect to λ and s. Following [19] and [2], we shall sketch this process briefly. To begin with, we split the regions of integration D' and D'' into subregions as follows:

$$\begin{aligned} D' &= \mathit{blue} \cup \mathit{white}, \quad D'' = \mathit{black} \cup \mathit{red}, \\ \mathit{blue} &= \{(s,\lambda) \in D' : \lambda_- < \lambda \le \lambda_- + \delta \text{ or } \lambda_+ - \delta \le \lambda < \lambda_+\}, \\ \mathit{black} &= \{(s,\lambda) \in D' : \lambda_- - \tilde{\delta} \le \lambda < \lambda_- \text{ or } 0 < \lambda \le \tilde{\delta}\}, \end{aligned} \tag{4.10}$$

where we have set $\delta = \min\{r, 1/2\}$ and $\tilde{\delta} = \min\{(t-r)/2, 1/2\}$. Notice that *white* is empty if $0 < r \le 1/2$ and that *red* is empty if $0 < t-r \le 1$.

Let $\partial^b = \partial_\alpha$ $(\alpha = 0, 1, 2)$ in (4.8). Then, according to the above decompositions, we have

$$\begin{aligned} 2\pi u(x,t) = &\iint_{blue} \lambda d\lambda ds \int_{-\varphi}^{\varphi} (\partial_\alpha F)(\lambda\xi, s) K_1 d\psi \\ &+ H\left(r - \frac{1}{2}\right) \sum_{j=0}^{1} \iint_{white} \lambda d\lambda ds \int_0^1 (\partial_\alpha F)(\lambda\Xi_j, s) K_2 d\tau \\ &+ H(t-r) \iint_{black} \lambda d\lambda ds \int_{-\pi}^{\pi} (\partial_\alpha F)(\lambda\xi, s) K_1 d\psi \\ &+ H(t-r-1) \iint_{red} \lambda d\lambda ds \int_{-\pi}^{\pi} (\partial_\alpha F)(\lambda\xi, s) K_1 d\psi, \end{aligned} \tag{4.11}$$

where we have changed the variable as $\psi = \Psi$ in the second term and set

$$\begin{aligned} \Psi &= \Psi(\lambda, s, \tau; r, t) = \arccos[1 - (1-\cos\varphi)\tau], \\ \Xi_j &= \Xi_j(\lambda, s, \tau; r, t) = (\cos(\theta + (-1)^j\Psi), \sin(\theta + (-1)^j\Psi)), \\ K_2 &= K_2(\lambda, s, \tau; r, t) = \{2r\lambda\tau(1-\tau)(2-(1-\cos\varphi)\tau)\}^{-\frac{1}{2}}. \end{aligned}$$

Carrying out the integration by parts in the second and fourth terms, we get the following proposition.

PROPOSITION 4.2. *Let $u(x,t)$ be the solution to $(\partial_t^2-\Delta)u=\partial_\alpha F$ with the zero initial data. If $F\in C^\infty(\mathbf{R}^2\times[0,T))$, then $|u(t,x)|$ is dominated by the following:*

$$I_1(F)(x,t)=\iint_{blue}\lambda d\lambda ds\int_{-\varphi}^{\varphi}|(\partial_\alpha F)(\lambda\xi,s)|K_1d\psi,$$

$$I_2(F)(x,t)=\int_{\partial(white)}\lambda d\sigma\int_0^1|F(\lambda\Xi_j,s)|K_2d\tau,$$

$$I_3(F)(x,t)=\iint_{white}d\lambda ds\int_0^1\{|F(\lambda\Xi_j,s)|+|(\Omega F)(\lambda\Xi_j,s)|\}K_2d\tau,$$

$$I_4(F)(x,t)=\iint_{white}\lambda d\lambda ds\int_0^1|F(\lambda\Xi_j,s)|\{|\partial_sK_2|+|\partial_\lambda K_2|\}d\tau,$$

$$I_5(F)(x,t)=\iint_{white}\lambda d\lambda ds\int_0^1|(\Omega F)(\lambda\Xi_j,s)|K_2\{|\partial_s\Psi|+|\partial_\lambda\Psi|\}d\tau,$$

$$J_1(F)(x,t)=\iint_{black}\lambda d\lambda ds\int_{-\pi}^{\pi}|(\partial_\alpha F)(\lambda\xi,s)|K_1d\psi,$$

$$J_2(F)(x,t)=\int_{\partial(red)}\lambda d\sigma\int_{-\pi}^{\pi}|F(\lambda\xi,s)|K_1d\psi,$$

$$J_3(F)(x,t)=\iint_{red}d\lambda ds\int_{-\pi}^{\pi}\{|F(\lambda\xi,s)|+|(\Omega F)(\lambda\xi,s)|\}K_1d\psi,$$

$$J_4(F)(x,t)=\iint_{red}\lambda d\lambda ds\int_{-\pi}^{\pi}|F(\lambda\xi,s)|\{|\partial_sK_1|+|\partial_\lambda K_1|\}d\psi.$$

Proof. It is easy to see that the first and second terms in (4.11) are dominated by $I_1(F)$ and $J_1(F)$, respectively. Since

$$(\nabla F)(\lambda\xi,s)=\xi\partial_\lambda(F(\lambda\xi,s))-\frac{\xi^\perp}{\lambda}(\Omega F)(\lambda\xi,s),\quad \xi^\perp=\sin(\theta+\psi),-\cos(\theta+\psi)),$$

we find that the fourth term in (4.11) is dominated by $J_j(F)$ $(j=2,3,4)$ by integration by parts.

To deal with the second term in (4.11), we use the following identities:

$$(\partial_sF)(\lambda\Xi_j,s)=\partial_s(F(\lambda\Xi_j,s))-(-1)^j\partial_s\Psi(\Omega F)(\lambda\Xi_j,s),$$
$$(\nabla F)(\lambda\Xi_j,s)=\Xi_j(\partial_\lambda(F(\lambda\Xi_j,s))-(-1)^j\partial_\lambda\Psi(\Omega F)(\lambda\Xi_j,s))-\frac{\Xi_j^\perp}{\lambda}(\Omega F)(\lambda\Xi_j,s),$$

where $\Xi_j^\perp=(\sin(\theta+(-1)^j\Psi),-\cos(\theta+(-1)^j\Psi))$. Again by integration by parts, we find that the second term is dominated by $I_j(F)$ $(j=2,\dots,5)$. The proof is complete. □

We shall use the following estimates of K_1 and K_2. For the proof, see Proposition 2.1 in [19] and also Proposition 5.3 in [2].

LEMMA 4.1. *It holds that for $(\lambda,s)\in D'$*

$$\int_{-\varphi}^{\varphi}K_1d\psi=2\int_0^1K_2d\tau\le\frac{C}{(r\lambda)^{\frac12}}\log\left[2+\frac{r\lambda}{(\lambda-\lambda_-)(\lambda_++\lambda)}H(t-s-r)\right], \tag{4.12}$$

$$(4.13)\quad \int_0^1 \{|\partial_s K_2| + |\partial_\lambda K_2|\} d\tau \le \frac{C}{(r\lambda)^{\frac{1}{2}}(\lambda + s + r - t)},$$

$$(4.14)\quad \int_0^1 K_2\{|\partial_s \Psi| + |\partial_\lambda \Psi|\} d\tau \le \frac{C(r+\lambda)}{\{r\lambda(\lambda^2 - \lambda_-^2)(\lambda_+^2 - \lambda^2)\}^{\frac{1}{2}}}$$

and that for $(\lambda, s) \in D''$

$$(4.15)\quad \int_{-\pi}^{\pi} K_1 d\psi \le C\{(\lambda + \lambda_-)(\lambda_+ - \lambda)\}^{-\frac{1}{2}} \log\left[2 + \frac{r\lambda}{(\lambda_- - \lambda)(\lambda_+ + \lambda)}\right],$$

$$(4.16)\quad \int_{-\pi}^{\pi} \{|\partial_s K_1| + |\partial_\lambda K_1|\} d\psi \le \frac{C}{(\lambda_- - \lambda)\{(\lambda + \lambda_-)(\lambda_+ - \lambda)\}^{\frac{1}{2}}}.$$

Now we are in a position to derive a new weighted $L^\infty - L^\infty$ estimate for the solution ∂u of (1.1) and (1.2). We introduce the following weight functions:

$$(4.17)\quad \frac{1}{\overline{w}_i(r,t)} = \frac{1}{(1+r)^{1-2\gamma}(1+t+r)^{1+2\gamma}} + \sum_{j \ne i} \frac{1}{(1+t+r)(1+|c_j t - r|)} + \frac{1}{(1+t+r)^{1+\mu}(1+|c_i t - r|)^{1-\mu}}$$

and

$$(4.18)\quad \frac{1}{\tilde{w}(r,t)} = \frac{1}{(1+r)^{1-2\gamma}(1+t+r)^{1+2\gamma}} + \sum_{j=1}^{m} \frac{1}{(1+t+r)(1+|c_j t - r|)},$$

where $1/4 < \gamma < 1/2$ and $0 < \mu < 1$.

PROPOSITION 4.3. *Let* u_1^i *be the solution to* $(\partial_t^2 - \Delta)u_1^i = \partial_\alpha F^i(\partial u, \partial^2 u)$ *with the zero initial data. Here* u *is a solution of* (1.1) *and* (1.2).

(i) *Let* $(r,t) \in \tilde{\Lambda}_i^c$ *with* $r = |x|$ *and* $t < T$. *Assume that* $w(r,t)$ *satisfies*

$$(4.19)\quad 0 < \frac{1}{w(r,t)} \le \frac{C}{\tilde{w}(r,t)}.$$

Then we have

$$(4.20)\quad w_i(r,t)|u_1^i(x,t)| \le CM_{0,1},$$

where we have set for a nonnegative integer k

$$M_{0,k} = \sum_{|a| \le k} \sup_{0<s<t} \sup_{y \in \mathbf{R}^2} \||y|^{\frac{1}{2}} w(|y|, s) \Gamma^a F^i(y,s)|.$$

(ii) *Let* $(x,t) \in \mathbf{R}^2 \times [0,T)$. *Assume* $\eta_j(r,t)$ $(j = 1,2)$ *satisfy*

$$(4.21)\quad 0 < \frac{1}{\eta_1(r,t)} \le \frac{C}{\overline{w}_i(r,t)}, \quad 0 < \frac{1}{\eta_2(r,t)} \le \frac{C}{\tilde{w}(r,t)}.$$

Then we have

$$(4.22)\quad w_i(r,t)|u_1^i(x,t)| \le C(M_{1,1} + M_{2,1} + M_{3,1}),$$

where we have set for a positive integer k

$$M_{1,k} = \sum_{|a|\le k} \sup_{0<s<t} \sup_{y\in\mathbf{R}^2} \||y|^{\frac{1}{2}}\eta_1(|y|,s)\Gamma^a(R^i(y,s)+G^i(y,s))|,$$
$$M_{2,k} = \sum_{|a|\le k} \sup_{0<s<t} \sup_{y\in\mathbf{R}^2} \||y|^{\frac{1}{2}}\eta_2(|y|,s)\Gamma^a N^i(y,s)|,$$
$$M_{3,k} = \sum_{|a|\le k} \sup_{(|y|,s)\in\Lambda_i, s<t} \||y|^{\frac{1}{2}}\eta_2(|y|,s)(1+s)^{\frac{1}{2}}\Gamma^a N^i(y,s)|.$$

Here, we have divided the function F^i into three parts: G^i, R^i, and N^i as in (1.12).

Proof. Employing Proposition 4.2, we find that $|u_1^i(t,x)|$ is dominated by $I_j(F^i)$ $(j=1,\dots,5)$ and $J_j(F^i)$ $(j=1,\dots,4)$. ∎

In the proof of Proposition 5.4 in [2], the following estimates are shown. Strictly speaking, they proved only the former part of Lemma 4.2 below. However, following their proof, we find that the assumption (4.19) is sufficient to derive (4.24) with $j=3,4$ and (4.25).

Lemma 4.2. *Set*

$$I_1' = \iint_{blue} \frac{\lambda^{\frac{1}{2}}}{w(\lambda,s)} d\lambda ds \int_{-\varphi}^{\varphi} K_1 d\psi,$$
$$I_2' = \int_{\partial(white)} \frac{\lambda^{\frac{1}{2}}}{w(\lambda,s)} d\sigma \int_0^1 K_2 d\tau,$$
$$I_3' = \iint_{white} \frac{1}{\lambda^{\frac{1}{2}} w(\lambda,s)} d\lambda ds \int_0^1 K_2 d\tau,$$
$$I_4' = \iint_{white} \frac{\lambda^{\frac{1}{2}}}{w(\lambda,s)} d\lambda ds \int_0^1 \{|\partial_s K_2| + |\partial_\lambda K_2|\} d\tau,$$
$$I_5' = \iint_{white} \frac{\lambda^{\frac{1}{2}}}{w(\lambda,s)} d\lambda ds \int_0^1 K_2\{|\partial_s \Psi| + |\partial_\lambda \Psi|\} d\tau,$$
$$I_1'' = \iint_{black} \frac{\lambda^{\frac{1}{2}}}{w(\lambda,s)} d\lambda ds \int_{-\pi}^{\pi} K_1 d\psi,$$
$$I_2'' = \int_{\partial(red)} \frac{\lambda^{\frac{1}{2}}}{w(\lambda,s)} d\sigma \int_{-\pi}^{\pi} K_1 d\psi,$$
$$I_3'' = \iint_{red} \frac{1}{\lambda^{\frac{1}{2}} w(\lambda,s)} d\lambda ds \int_{-\pi}^{\pi} K_1 d\psi,$$
$$I_4'' = \iint_{red} \frac{\lambda^{\frac{1}{2}}}{w(\lambda,s)} d\lambda ds \int_{-\pi}^{\pi} \{|\partial_s K_1| + |\partial_\lambda K_1|\} d\psi.$$

Assume $w(r,t)$ satisfies

$$0 < \frac{1}{w(r,t)} \le \frac{C}{\overline{w}_i(r,t)}. \tag{4.23}$$

Then we have for $(x,t)\in\mathbf{R}^2\times(0,\infty)$

$$w_i(r,t) I_j' \le C, \tag{4.24}$$
$$w_i(r,t) I_j'' \le C. \tag{4.25}$$

Moreover, (4.24) *with* $j = 3, 4$ *and* (4.25) *are still true, if* $w(r,t)$ *satisfies* (4.19).

First, we shall show the statement (i) in Proposition 4.3. By the definition of $M_{0,1}$, (4.19), and Lemma 4.2, we get for $(x,t) \in \mathbf{R}^2 \times (0,\infty)$

$$w_i(r,t)I_j(F^i)(x,t) \le CM_{0,1} \quad \text{for } j = 3, 4,$$
$$w_i(r,t)J_j(F^i)(x,t) \le CM_{0,1} \quad \text{for } j = 1, \dots, 4.$$

Therefore, our task becomes to prove

$$w_i(r,t)I_j(F^i)(x,t) \le CM_{0,1} \quad \text{for } j = 1, 2, 5, \tag{4.26}$$

provided (4.19) and $(r,t) \in \tilde{\Lambda}_i^c$. Since the treatment of $I_2(F^i)$ is similar to that of $I_1(F^i)$, we shall deal with only $I_1(F^i)$ and $I_5(F^i)$. If we set

$$\frac{1}{\xi(\lambda,s)} = \frac{1}{(1+s+\lambda)(1+|s-\lambda|)}, \tag{4.27}$$

then we have from (4.17) and (4.18)

$$\frac{1}{\tilde{w}(\lambda,s)} \le \frac{1}{\overline{w}_i(\lambda,s)} + \frac{1}{\xi(\lambda,s)}. \tag{4.28}$$

Hence, using (4.24) with $j = 1, 5$, (4.12), and (4.14), we have

$$I_j(F^i)(x,t) \le CM_{0,1}(\{w_i(r,t)\}^{-1} + \tilde{I}_j(\xi)) \quad \text{for } j = 1, 5,$$

where we have set

$$\tilde{I}_1(w) = \frac{1}{r^{\frac{1}{2}}} \iint_{blue} \frac{1}{w(\lambda,s)} \log\left[2 + \frac{r\lambda}{(\lambda-\lambda_-)(\lambda_+ + \lambda)} H(t-s-r)\right] d\lambda ds, \tag{4.29}$$

$$\tilde{I}_5(w) = \frac{1}{r^{\frac{1}{2}}} \iint_{white} \frac{1}{w(\lambda,s)} \frac{r+\lambda}{\{(\lambda^2-\lambda_-^2)(\lambda_+^2-\lambda^2)\}^{\frac{1}{2}}} d\lambda ds. \tag{4.30}$$

In the following, we shall prove for $(r,t) \in \tilde{\Lambda}_i^c$

$$\tilde{I}_j(\xi) \le \frac{C}{(1+r)^{\frac{1}{2}}(1+t+r)^{\frac{1}{2}+\gamma}} \quad \text{for } j = 1, 5. \tag{4.31}$$

First we consider $\tilde{I}_1(\xi)$. It follows from (5.33) and (5.34) in [2] that for $0 \le s \le t$

$$\int_{\lambda_-}^{\lambda_-+\delta} \log\left[2 + \frac{r\lambda}{(\lambda-\lambda_-)(\lambda_+ + \lambda)} H(t-s-r)\right] d\lambda \le C\delta, \tag{4.32}$$

$$\int_{\lambda_+-\delta}^{\lambda_+} \log\left[2 + \frac{r\lambda}{(\lambda-\lambda_-)(\lambda_+ + \lambda)} H(t-s-r)\right] d\lambda \le C\delta^{\frac{1}{2}} \log[2+|t-r|]. \tag{4.33}$$

Therefore we have

$$\begin{aligned}\tilde{I}_1(\xi) &\le \frac{C\delta^{\frac{1}{2}}}{r^{\frac{1}{2}}} \log[2+|t-r|] \left\{\int_0^t \frac{1}{\xi(\lambda_-,s)} ds + \int_0^t \frac{1}{\xi(\lambda_+,s)} ds\right\} \\ &\le \frac{C\log[2+|t-r|]}{(1+r)^{\frac{1}{2}}(1+t+r)} \left\{\int_0^t \frac{1}{1+|2s-t+r|} ds + \int_0^t \frac{1}{1+|2s-t-r|} ds\right\}\end{aligned}$$

because $\delta r^{-1} \le C(1+r)^{-1}$ and $1+|t-r|$ is equivalent to $1+t+r$ for $(r,t) \in \tilde{\Lambda}_i^c$. Since $\gamma < 1/2$, we thus obtain (4.31) for $j=1$.

Next we consider $\tilde{I}_5(\xi)$. Notice that $\delta = 1/2$ if the domain *white* is not empty, hence r is equivalent to $1+r$. Moreover, since

$$\lambda \pm \lambda_- \ge \delta, \quad \lambda \pm \lambda_- \ge \delta \quad \text{for } (\lambda, s) \in \textit{ white},$$

we have

$$\tilde{I}_5(\xi) \le \frac{C}{(1+r)^{\frac{1}{2}}} \iint_{D'} \frac{1}{\xi(\lambda,s)} \times \frac{r+\lambda}{\{(\lambda-\lambda_-+1)(\lambda+\lambda_-+1)(\lambda_+-\lambda+1)(\lambda_++\lambda+1)\}^{\frac{1}{2}}} d\lambda ds.$$

Note that

$$\frac{r+\lambda}{\{(\lambda-\lambda_-+1)(\lambda+\lambda_-+1)(\lambda_+-\lambda+1)(\lambda_++\lambda+1)\}^{\frac{1}{2}}} \le \frac{2}{(\lambda-\lambda_-+1)^{\frac{1}{2}}} \left\{ \frac{1}{(\lambda+\lambda_-+1)^{\frac{1}{2}}} + \frac{1}{(\lambda_+-\lambda+1)^{\frac{1}{2}}} \right\},$$

which follows from

$$\lambda_+ + \lambda \ge \max\{r, \lambda\},$$
$$\lambda_+ - \lambda \ge \max\{r, \lambda\} \quad \text{for } \lambda \le \frac{\lambda_+ - \lambda_-}{2},$$
$$\lambda + \lambda_- \ge \max\{r, \lambda\} \quad \text{for } \lambda \ge \frac{\lambda_+ - \lambda_-}{2}.$$

Therefore we have

$$\begin{aligned}(1+r)^{\frac{1}{2}} \tilde{I}_5(\xi) \le C &\iint_{D'} \frac{1}{\xi(\lambda,s)} \frac{1}{\{(\lambda-t+s+r+1)(\lambda+t-s-r+1)\}^{\frac{1}{2}}} d\lambda ds \\ +C &\iint_{D'} \frac{1}{\xi(\lambda,s)} \frac{1}{\{(\lambda-t+s+r+1)(t-s+r-\lambda+1)\}^{\frac{1}{2}}} d\lambda ds \\ +C &\iint_{D'} \frac{1}{\xi(\lambda,s)} \frac{1}{\{(\lambda+t-s-r+1)(t-s+r-\lambda+1)\}^{\frac{1}{2}}} d\lambda ds.\end{aligned} \tag{4.34}$$

We shall show in the following that the right-hand side of (4.34) is dominated by $C(1+t+r)^{-\gamma-1/2}$. Since

$$1+s+\lambda \ge 1+|t-r| \quad \text{for } (\lambda,s) \in D', \tag{4.35}$$

the second term is dominated by

$$\begin{aligned}&\frac{C}{1+|t-r|} \iint_{D'} \frac{1}{1+|s-\lambda|} \left\{ \frac{1}{\lambda+s-t+r+1} + \frac{1}{t+r-s-\lambda+1} \right\} d\lambda ds \\ &\le \frac{C}{2(1+t+r)} \int_{|t-r|}^{t+r} \left\{ \frac{1}{\alpha-t+r+1} + \frac{1}{t+r-\alpha+1} \right\} d\alpha \int_{-\alpha}^{t-r} \frac{1}{1+|\beta|} d\beta,\end{aligned}$$

where we have changed the variables as

$$\alpha = s+\lambda, \quad \beta = s-\lambda. \tag{4.36}$$

Since the double integral is dominated by $C\{\log(1+t+r)\}^2$, we get the desired estimate.

To treat the first and third terms, we divide the domain D' into two parts:

$$D_- = \left\{(\lambda, s) \in D' : |\lambda - s| \leq \frac{1}{2}|t-r|\right\}, \quad D_-^c = white \setminus D_-. \tag{4.37}$$

Since D_- is empty if $0 < t \leq r$, we have

$$\lambda + t - s - r \geq \frac{1}{2}|t-r| \quad \text{for } (\lambda, s) \in D_-. \tag{4.38}$$

On the other hand, we have

$$1 + |s - \lambda| \geq \frac{1}{2}(1 + |t-r|) \quad \text{for } (\lambda, s) \in D_-^c. \tag{4.39}$$

Using these estimates together with (4.35) and changing the variables as (4.36), we find that the first term is majored by

$$\begin{aligned}
&\frac{C}{1+|t-r|}\left\{\iint_{D'} \frac{1}{(1+s+\lambda)^{\frac{1}{2}}(1+|s-\lambda|)(\lambda+s-t+r+1)^{\frac{1}{2}}} d\lambda ds\right.\\
&\left.+\iint_{D'} \frac{1}{(1+s+\lambda)}\left\{\frac{1}{\lambda-t+s+r+1}+\frac{1}{\lambda+t-s-r+1}\right\} d\lambda ds\right\}\\
\leq\ &\frac{C}{1+t+r}\left\{\int_{|t-r|}^{t+r}\left\{\frac{1}{1+\alpha}+\frac{1}{\alpha-t+r+1}\right\} d\alpha \int_{-\alpha}^{t-r}\frac{1}{1+|\beta|} d\beta\right.\\
&+\int_{|t-r|}^{t+r}\frac{1}{(1+\alpha)}\frac{1}{\alpha-t+r+1} d\alpha \int_{-\alpha}^{t-r} d\beta\\
&\left.+\int_{|t-r|}^{t+r}\frac{1}{(1+\alpha)} d\alpha \int_{-\alpha}^{t-r}\frac{1}{-\beta+t-r+1} d\beta\right\},
\end{aligned}$$

which yields the desired estimate. Since the third term is dealt with similarly, we omit the details. This completes the proof of (4.26).

Second, we shall show the statement (ii). By (4.21), we have for $|a| \leq 1$

$$|\Gamma^a F^i(y,s)| \leq \frac{M_{1,1}+M_{2,1}}{\lambda^{\frac{1}{2}}\tilde{w}(\lambda,s)}.$$

Therefore, by Lemma 4.2, we get for $(x,t) \in \mathbf{R}^2 \times (0,\infty)$

$$\begin{aligned}
w_i(r,t)I_j(F^i)(x,t) &\leq C(M_{1,1}+M_{2,1}) \quad \text{for } j=3,4,\\
w_i(r,t)J_j(F^i)(x,t) &\leq C(M_{1,1}+M_{2,1}) \quad \text{for } j=1,\dots,4.
\end{aligned}$$

Moreover, similarly to (4.26), we get for $(|x|,t) \in \tilde{\Lambda}_i^c$

$$w_i(r,t)I_j(F^i)(x,t) \leq C(M_{1,1}+M_{2,1}) \quad \text{for } j=1,2,5.$$

Thus it suffices to prove

$$w_i(r,t)I_j(F^i)(x,t) \leq C(M_{1,1}+M_{2,1}+M_{3,1}) \quad \text{for } j=1,2,5, \tag{4.40}$$

provided (4.21) and $(r,t) \in \tilde{\Lambda}_i$.

Having (3.1) in mind, we introduce a characteristic function of Λ_i denoted by $\chi(\lambda, s)$. Then we may write

$$\Gamma^a F^i = \Gamma^a(R^i + G^i) + (1-\chi)\Gamma^a N^i + \chi \Gamma^a N^i$$

and find from (4.28) and the definition of $M_{i,1}$ given in (4.22) that

$$|\Gamma^a F^i(y,s)| \le C(M_{1,1}+M_{2,1}+M_{3,1})\lambda^{-\frac{1}{2}} \left(\frac{1}{\bar{w}_i(\lambda,s)} + \frac{1-\chi(\lambda,s)}{\xi(\lambda,s)} + \frac{\chi(\lambda,s)}{\xi(\lambda,s)(1+s)^{\frac{1}{2}}} \right)$$

for $|a| \le 1$. Therefore, using (4.24), we have for $j = 1, 5$

$$I_j(F^i)(r,t) \le C(M_{1,1}+M_{2,1}+M_{3,1})(\{w_i(r,t)\}^{-1} + \tilde{I}_j(\tilde{\xi})), \tag{4.41}$$

where $\tilde{I}_j$ is defined in (4.29), (4.30) and we have set

$$\frac{1}{\tilde{\xi}(\lambda,s)} = \frac{1-\chi(\lambda,s)}{\xi(\lambda,s)} + \frac{\chi(\lambda,s)}{\xi(\lambda,s)(1+s)^{\frac{1}{2}}}.$$

In the following, we shall show for $(r,t) \in \tilde{\Lambda}_i$ and $j = 1, 5$

$$\tilde{I}_j(\tilde{\xi}) \le \frac{C}{(1+r)^{\frac{1}{2}}(1+|t-r|)^{\frac{1}{2}}} \tag{4.42}$$

because $1+r$ is equivalent to $1+t+r$ for $(r,t) \in \tilde{\Lambda}_i$.

First we consider $\tilde{I}_1(\tilde{\xi})$. Using (4.32) and (4.33), we have

$$\tilde{I}_1(\tilde{\xi}) \le \frac{C\log[2+|t-r|]}{(1+r)^{\frac{1}{2}}} \left\{ \int_0^t \frac{1}{\tilde{\xi}(\lambda_-,s)} ds + \int_0^t \frac{1}{\tilde{\xi}(\lambda_+,s)} ds \right\},$$

because $\delta r^{-1} \le C(1+r)^{-1}$. Since

$$(1+|s-\lambda|)^{\frac{1}{4}} \ge (1+s)^{\frac{1}{8}} \quad \text{for } (\lambda,s) \in \operatorname{supp}\{1-\chi\}, \tag{4.43}$$

we have from (4.27)

$$\frac{1}{\tilde{\xi}(\lambda,s)} \le \frac{2}{(1+s+\lambda)^{\frac{3}{4}}(1+|s-\lambda|)^{\frac{3}{4}}(1+s)^{\frac{3}{8}}}.$$

Therefore, we get

$$\begin{aligned}(1+r)^{\frac{1}{2}}\tilde{I}_1(\tilde{\xi}) \le{}& \frac{C\log[2+|t-r|]}{(1+|t-r|)^{\frac{3}{4}}} \\ &\times \left\{ \int_0^t \frac{1}{(1+|2s-t+r|)^{\frac{3}{4}}(1+s)^{\frac{3}{8}}} ds + \int_0^t \frac{1}{(1+|2s-t-r|)^{\frac{3}{4}}(1+s)^{\frac{3}{8}}} ds \right\},\end{aligned}$$

which yields (4.42) for $j = 1$.

Next we consider $\tilde{I}_5(\tilde{\xi})$. From (4.34), we have

$$\begin{aligned}(1+r)^{\frac{1}{2}}\tilde{I}_5(\tilde{\xi}) \le{}& C\iint_{D'} \frac{1}{\tilde{\xi}(\lambda,s)} \frac{1}{\{(\lambda-t+s+r+1)(\lambda+t-s-r+1)\}^{\frac{1}{2}}} d\lambda ds \\ &+C\iint_{D'} \frac{1}{\tilde{\xi}(\lambda,s)} \frac{1}{\{(\lambda-t+s+r+1)(t-s+r-\lambda+1)\}^{\frac{1}{2}}} d\lambda ds \\ &+C\iint_{D'} \frac{1}{\tilde{\xi}(\lambda,s)} \frac{1}{\{(\lambda+t-s-r+1)(t-s+r-\lambda+1)\}^{\frac{1}{2}}} d\lambda ds.\end{aligned} \tag{4.44}$$

We shall show that the right-hand side is dominated by $C(1+|t-r|)^{-1/2}$. Using (4.27) and (4.35), we have

$$\frac{1}{\tilde{\xi}(\lambda,s)} \le \frac{2}{\xi(\lambda,s)} \le \frac{2}{(1+|t-r|)^{\frac12}(1+s+\lambda)^{\frac14}(1+|s-\lambda|)^{\frac54}}.$$

Therefore, the second term is majored by $C(1+|t-r|)^{-1/2}$ times

$$\iint_{D'} \frac{1}{(1+s+\lambda)^{\frac14}(1+|s-\lambda|)^{\frac54}} \left\{ \frac{1}{\lambda+s-t+r+1} + \frac{1}{t+r-s-\lambda+1} \right\} d\lambda ds$$
$$\le C \int_{|t-r|}^{t+r} \left\{ \frac{1}{(1+\alpha)^{\frac54}} + \frac{1}{(\alpha-t+r+1)^{\frac54}} + \frac{1}{(t+r-\alpha+1)^{\frac54}} \right\} d\alpha \int_{-\alpha}^{t-r} \frac{1}{(1+|\beta|)^{\frac54}} d\beta,$$

which yields the desired estimate.

Next we deal with the first term, by dividing the domain D' as in (4.37). Using (4.38) and (4.39), we find that the first term is majored by $C(1+|t-r|)^{-1/2}$ times

$$\iint_{D'} \frac{1}{\xi(\lambda,s)} \frac{1}{(\lambda+s-t+r+1)^{\frac12}} d\lambda ds + \iint_{D'} \frac{(1+|s-\lambda|)^{\frac12}}{\xi(\lambda,s)} \frac{1}{\lambda+s-t+r+1} d\lambda ds + \iint_{D'} \frac{(1+|s-\lambda|)^{\frac12}}{\tilde{\xi}(\lambda,s)} \frac{1}{\lambda-s+t-r+1} d\lambda ds.$$

Analogous to the above calculation, we see that the first and second terms are bounded by some constant. Since we have by (4.43)

$$\frac{(1+|s-\lambda|)^{\frac12}}{\tilde{\xi}(\lambda,s)} \le \frac{2}{(1+s)^{\frac98}(1+|s-\lambda|)^{\frac14}},$$

the third term is dominated by

$$\int_0^\infty \frac{ds}{(1+s)^{\frac98}} \int_{-\infty}^\infty \left\{ \frac{1}{(1+|s-\lambda|)^{\frac54}} + \frac{1}{(\lambda-s+t-r+1)^{\frac54}} \right\} d\lambda \le C;$$

hence we obtain the desired estimate of the first term in the right-hand side of (4.44). Since the third term in the right-hand side of (4.44) is dealt with similarly, we omit the details. This completes the proof of the proposition.

In our analysis, we need an upper bound of not only ∂u^i but also u^i itself.

PROPOSITION 4.4. *Let u_1^i be the solution to $(\partial_t^2-\Delta)u_1^i = F^i(\partial u, \partial^2 u)$ with the zero initial data. Here u is a solution to* (1.1) *and* (1.2). *Let $0 \le \mu < 1/2$. Assume that $w(r,t)$ satisfies* (4.19). *Then we have for $(|x|,t) \in \tilde{\Lambda}_i$ with $t < T$*

$$(1+t+r)^\mu |u_1^i(x,t)| \le CM_{0,0}, \tag{4.45}$$

where $M_{0,0}$ is defined in (4.20).

Proof. It follows from (4.8) with $b = 0$, (4.12), (4.15), and (4.19) that

$$|u_1^i(x,t)| \le CM_{0,0}(P_1 + P_2), \tag{4.46}$$

where we have set

$$P_1 = \frac{1}{r^{\frac{1}{2}}} \iint_{D'} \frac{1}{\tilde{w}(\lambda,s)} \log\left[2 + \frac{r\lambda}{(\lambda-\lambda_-)(\lambda_+ + \lambda)} H(t-s-r)\right] d\lambda ds$$

and

$$P_2 = H(t-r) \iint_{D''} \frac{\lambda^{\frac{1}{2}}}{\tilde{w}(\lambda,s)\{(\lambda+\lambda_-)(\lambda_+ - \lambda)\}^{\frac{1}{2}}} \log\left[2 + \frac{r\lambda}{(\lambda_- - \lambda)(\lambda_+ + \lambda)}\right] d\lambda ds.$$

Since t is equivalent to r for $(r,t) \in \tilde{\Lambda}_i$, it suffices to show

$$P_j \le C(1+r)^{-\mu} \quad \text{for } j = 1, 2. \tag{4.47}$$

First, we treat P_1. We split the domain D' into *blue* and *white* defined by (4.10). According to this decomposition, we shall write $P_1 = P_{1,blue} + P_{1,white}$. By (4.32) and (4.33), we have

$$\begin{aligned} P_{1,blue} &\le \frac{C\delta^{\frac{1}{2}}}{r^{\frac{1}{2}}} \log[2 + |t-r|] \left\{ \int_0^t \frac{1}{\tilde{w}(\lambda_-,s)} ds + \int_0^t \frac{1}{\tilde{w}(\lambda_+,s)} ds \right\} \\ &\le \frac{C\log[2+|t-r|]}{(1+r)^{\frac{1}{2}}} \int_0^t \frac{1}{1+s} ds \\ &\le \frac{C}{(1+r)^{\mu}} \end{aligned}$$

for $0 \le \mu < 1/2$ because $1 + r$ is equivalent to $1 + t + r$ for $(r,t) \in \tilde{\Lambda}_i$.

On the other hand, we have for $(\lambda, s) \in$ *white* with $0 \le s \le t - r$

$$\frac{r\lambda}{(\lambda-\lambda_-)(\lambda_+ + \lambda)} \le \frac{\lambda}{\lambda - \lambda_-} \le 1 + 2\lambda_- \le 1 + 2(t-r)$$

because $\delta = 1/2$, if *white* is not empty. Therefore we have

$$\begin{aligned} (1+r)^{\frac{1}{2}} P_{1,white} &\le C\log[2+|t-r|] \iint_{D'} \frac{1}{\tilde{w}(\lambda,s)} d\lambda ds \\ &\le C\log[2+|t-r|] \int_0^t \frac{ds}{1+s} \int_0^{t+r} \left\{ \frac{1}{1+\lambda} + \sum_{j=1}^m \frac{1}{1+|c_j s - \lambda|} \right\} d\lambda; \end{aligned}$$

hence $P_{1,blue} \le C(1+r)^{-\mu}$ for $0 \le \mu < 1/2$. We thus get (4.47) for $j = 1$.

Second, we deal with P_2. Notice that $\tilde{w}(\lambda,s)$ is equivalent to $\tilde{w}(\lambda_-,s)$ for $\lambda_- - 1 \le \lambda \le \lambda_-$ and that $(\lambda_+ - \lambda)^{1/2} \le (1+r)^{1/2}$ for $0 < \lambda \le \lambda_- - 1$ and $0 \le s \le t - r$. Moreover, we have for $0 < \lambda \le \lambda_- - 1$ and $0 \le s \le t - r$

$$\frac{r\lambda}{(\lambda_- - \lambda)(\lambda_+ + \lambda)} \le \frac{\lambda}{\lambda_- - \lambda} \le -1 + \lambda_- \le t - r.$$

Splitting the integral into two parts, we have

$$P_2 \le CH(t-r-1)(1+r)^{-\frac{1}{2}} \log[2+|t-r|] \iint_{D''} \frac{1}{\tilde{w}(\lambda,s)} d\lambda ds$$
$$+CH(t-r)\int_0^{t-r} \frac{1}{\tilde{w}(\lambda_-,s)} ds \int_{(\lambda_- -1)_+}^{\lambda_-} \frac{\lambda^{\frac{1}{2}}}{\{(\lambda+\lambda_-)(\lambda_+ -\lambda)\}^{\frac{1}{2}}}$$
$$\times \log\left[2+\frac{r\lambda}{(\lambda_- -\lambda)(\lambda_+ +\lambda)}\right] d\lambda.$$

Notice that

$$\int_{(\lambda_- -1)_+}^{\lambda_-} \frac{\lambda^{\frac{1}{2}}}{\{(\lambda+\lambda_-)(\lambda_+ -\lambda)\}^{\frac{1}{2}}} \log\left[2+\frac{r\lambda}{(\lambda_- -\lambda)(\lambda_+ +\lambda)}\right] d\lambda \le C\frac{\log[2+|t-r|]}{(1+r)^{\frac{1}{2}}}.$$

(For the proof, see (5.73) in [2].) Therefore we have

$$(4.48) \quad (1+r)^{\frac{1}{2}} P_2 \le C\log[2+|t-r|] \left\{ \iint_{D''} \frac{1}{\tilde{w}(\lambda,s)} d\lambda ds + \int_0^{t-r} \frac{ds}{1+s} \right\},$$

which implies (4.47) for $j=2$. This completes the proof. □

LEMMA 4.3. *Let F^i satisfy* (1.12) *and u be smooth function satisfying* (4.1) *with* $k=[(N+1)/2]$. *If we set*

$$(4.49) \qquad \frac{1}{w(\lambda,s)} = \sum_{j,k=1}^{m} \frac{1}{(w_j w_k)(\lambda,s)} \qquad \text{for } \lambda>0, s>0,$$

then we have

$$(4.50) \qquad M_{0,N} \le C_N [\partial u]^2_{[\frac{N+1}{2}],t} \|\partial u\|_{N+3,t}.$$

Moreover, if we set

$$(4.51) \qquad \frac{1}{w(\lambda,s)} = \sum_{j,k,l=1}^{m} \frac{\lambda^{\frac{1}{2}}}{(w_j w_k w_l)(\lambda,s)} \text{ for } \lambda>0, s>0,$$

then we have

$$(4.52) \qquad M_{0,[\frac{N+1}{2}]} \le C_N [\partial u]^3_{[\frac{N+1}{2}]+1,t}.$$

Here $M_{0,N}$ is defined in (4.20).

Proof. First, we shall show (4.52). Since (4.1) with $k=[(N+1)/2]$ implies

$$(4.53) \quad \sum_{j=1}^{m} \sum_{|a|\le[(N+1)/2]} |\Gamma^a \partial u^j(y,s)| \le [\partial u]_{[\frac{N+1}{2}],T} < 1 \quad \text{for } 0\le s\le t<T, y\in \mathbf{R}^2,$$

by (1.12) we have for $|a|\le[(N+1)/2]$

$$|\Gamma^a F^i(y,s)| \le C \sum_{j,k,l=1}^{m} \frac{1}{(w_j w_k w_l)(\lambda,s)} [\partial u(s)]^3_{[\frac{N+1}{2}]+1}$$

with $\lambda = |y|$. By (4.51), we therefore get (4.52).

Second, we shall prove (4.50). It follows that for $|a| \le N$

$$|\Gamma^a F^i(y,s)| \le C \sum_{j,k,l=1}^{m} \frac{1}{(w_j w_k)(\lambda,s)} [\partial u(s)]^2_{[\frac{N+1}{2}]} \sum_{|b|\le|a|+1} |\Gamma^b \partial u^l(y,s)|.$$

We now use an imbedding theorem concerning the invariant norm

$$|x|^{\frac{1}{2}} |f(x)| \le \sum_{|a|\le 2} \|\Gamma^a f\|_{L^2} \quad \text{for } x \in \mathbf{R}^2. \tag{4.54}$$

(For the proof, see, e.g., Lemma 6 in [19].) Applying this and using (4.49), we obtain (4.50). The proof is complete. □

COROLLARY 4.1. *Let $u = (u^1, \ldots, u^m)$ be the solution of* (1.1) *and* (1.2) *and let F^i satisfy* (1.12). *Let $0 \le \mu < 1/2$. Then we have for $(|x|,t) \in \Lambda_i$ with $t < T$*

$$(1+t+r)^\mu |\Gamma^a u^i(x,t)| \le C_N \left(\varepsilon + [\partial u]^2_{[\frac{N+1}{2}],t} \|\partial u\|_{N+3,t}\right) \quad \textit{for } |a| \le N, \tag{4.55}$$

$$(1+t+r)^\mu |\Gamma^a u^i(x,t)| \le C_N \left(\varepsilon + [\partial u]^3_{[\frac{N+1}{2}],t}\right) \quad \textit{for } |a| \le [(N+1)/2], \tag{4.56}$$

provided (4.1) *with $k = [(N+1)/2]$ holds.*

Proof. Using the decomposition (4.5) with $b = 0$ and the estimates (4.6) and (4.45), we have

$$(1+t+r)^\mu |\Gamma^a u^i(x,t)| \le M_N \varepsilon + C_N M_{0,|a|}, \tag{4.57}$$

where $M_{0,|a|}$ is defined in (4.20), if $w(r,t)$ satisfies (4.19). Note that both (4.51) and (4.49) satisfy (4.19). Applying Lemma 4.3, we obtain (4.55) and (4.56). This completes the proof. □

End of the proof of Proposition 4.1. First we shall show (4.2). Using the decomposition (4.5) with $|b| = 1$ and the estimates (4.6) and (4.20), we have

$$w_i(r,t) |\Gamma^a \partial u^i(x,t)| \le M_N \varepsilon + C_N M_{0,|a|+1} \quad \text{for } (|x|,t) \in \tilde{\Lambda}_i^c \text{ with } t < T \tag{4.58}$$

if $w(r,t)$ satisfies (4.19). Using (4.50) with N replaced by $N+1$, we obtain (4.2).

Next we shall show (4.3). Similarly, it follows from (4.6) and (4.22) that for $|a| \le N$

$$w_i(r,t) |\Gamma^a \partial u^i(x,t)| \le M_N \varepsilon + C_N (M_{1,N+1} + M_{2,N+1} + M_{3,N+1}) \tag{4.59}$$

if $\eta_i(r,t)$ $(i = 1,2)$ satisfies (4.21).

First, we shall show

$$M_{1,N+1} \le C \left(\varepsilon + [\partial u]^2_{[\frac{N+4}{2}],t} \|\partial u\|_{N+6,t}\right). \tag{4.60}$$

If we set

$$\frac{1}{\eta_1(\lambda,s)} = \sum_{j,k,l=1}^{m} \frac{1}{(w_j w_k w_l)(\lambda,s)} + \sum_{(j,k)\ne(i,i)} \frac{1}{(w_j w_k)(\lambda,s)} + \frac{1-\tilde{\chi}(\lambda,s)}{\{w_i(\lambda,s)\}^2}, \tag{4.61}$$

then $\eta_1(r,t)$ satisfies the first condition in (4.21). Here $\tilde{\chi}$ is the characteristic function of $\tilde{\Lambda}_i$. In what follows, we always assume $|a| \le N+1$. By (1.12) and (4.1) with $k = [(N+2)/2]$, we have

$$(4.62)\quad |\Gamma^a G^i(y,s)| \le C \sum_{j,k,l=1}^{m} \frac{1}{(w_j w_k w_l)(\lambda,s)} [\partial u(s)]^3_{\left[\frac{|a|+1}{2}\right]} \sum_{|b|\le |a|+1} |\Gamma^b \partial u^l(y,s)|.$$

Using (4.54), we get

$$(4.63)\qquad |\lambda^{\frac{1}{2}} \eta_1(\lambda,s) \Gamma^a G^i(y,s)| \le C[\partial u(s)]^2_{\left[\frac{N+2}{2}\right]} \|\partial u(s)\|_{N+4}.$$

As for the resonance-form R^i, we find from (1.13) that there is at least one index among j, k, and l which does not coincide with i. Therefore, by (1.12) we have

$$(4.64)\qquad \begin{aligned} |\Gamma^a R^i(y,s)| \le\ & C \sum_{(j,k)\ne(i,i)} \sum_{l=1}^{m} \frac{1}{(w_j w_k)(\lambda,s)} [\partial u(s)]^2_{\left[\frac{|a|+1}{2}\right]} \sum_{|b|\le|a|+1} |\Gamma^b \partial u^l(y,s)| \\ & + C \sum_{\substack{(j,k)=(i,i)\\ l\ne i}} \frac{1-\tilde{\chi}(\lambda,s)}{\{w_i(\lambda,s)\}^2} [\partial u(s)]^2_{\left[\frac{|a|+1}{2}\right]} \sum_{|b|\le|a|+1} |\Gamma^b \partial u^l(y,s)| \\ & + C \sum_{\substack{(j,k)=(i,i)\\ l\ne i}} \frac{1}{(w_i w_l)(\lambda,s)} \frac{[\partial u(s)]^2_{\left[\frac{|a|+1}{2}\right]}}{w_i(\lambda,s)} \sum_{|b|\le|a|+1} |(\tilde{\chi} w_l)(\lambda,s) \Gamma^b \partial u^l(y,s)|. \end{aligned}$$

By (4.61) and (4.54), we find that the first and second terms are dominated by

$$(4.65)\qquad C\lambda^{-\frac{1}{2}} \{\eta_1(\lambda,s)\}^{-1} [\partial u(s)]^2_{\left[\frac{N+2}{2}\right]} \|\partial u(s)\|_{N+4}.$$

On the other hand, by (4.61), (4.1) with $k = [(N+2)/2]$, and $w_i(\lambda,s) \ge \lambda^{1/2}$, the third term is dominated by

$$(4.66)\qquad C\lambda^{-\frac{1}{2}} \{\eta_1(\lambda,s)\}^{-1} \sum_{|b|\le N+2} |(\tilde{\chi} w_l)(\lambda,s) \Gamma^b \partial u^l(y,s)|.$$

Moreover, since $\tilde{\Lambda}_i \subset \tilde{\Lambda}_l^c$ by (2.4), we get from (4.2),

$$|(\tilde{\chi} w_l)(\lambda,s) \Gamma^b \partial u^l(y,s)| \le C_N \left(\varepsilon + [\partial u]^2_{\left[\frac{N+4}{2}\right],t} \|\partial u\|_{N+6,t} \right)$$

for $|b| \le N+2$. We thus find that the third term is dominated by

$$(4.67)\qquad C\lambda^{-\frac{1}{2}} \{\eta_1(\lambda,s)\}^{-1} \left(\varepsilon + [\partial u]^2_{\left[\frac{N+4}{2}\right],t} \|\partial u\|_{N+6,t} \right);$$

hence, together with (4.65), we get

$$(4.68)\qquad |\lambda^{\frac{1}{2}} \eta_1(\lambda,s) \Gamma^a R^i(y,s)| \le C \left(\varepsilon + [\partial u]^2_{\left[\frac{N+4}{2}\right],t} \|\partial u\|_{N+6,t} \right).$$

Combining (4.63) and (4.68), we finally get (4.60).

Second, we consider $M_{2,N+1}$. Taking $\eta_2(r,t)=w_i(r,t)^2$, we easily see that $\eta_2(r,t)$ satisfies the second condition of (4.21) and that

$$M_{2,N+1} \le C[\partial u]^2_{[\frac{N+2}{2}],t}\|\partial u\|_{N+4,t}. \tag{4.69}$$

Third, we consider $M_{3,N+1}$ by taking $\eta_2(r,t)=w_i(r,t)^2$. By (3.1) we have

$$(1+s)^{\frac12}|\Gamma^a N^i(y,s)| \le C(\Phi^i_a + (1+s)^{-\frac12}\Theta^i_a)$$

for $(|y|,s)\in\Lambda_i$. Therefore, we obtain

$$\begin{aligned}|\lambda^{\frac12}\eta_2(\lambda,s)(1+s)^{\frac12}\Gamma^a N^i(y,s)| &\le C\sum_{|b+c+d|\le|a|+1}|\lambda^{\frac12}\eta_2(\lambda,s)|\partial\Gamma^b u^i\|\partial\Gamma^c u^i\|\partial\Gamma^d u^i\| \\ &+C\sum_{\substack{|b+c+d|\le|a|+2\\|b|,|c|,|d|\le|a|+1}}|\lambda^{\frac12}\eta_2(\lambda,s)(1+s)^{-\frac12}|\Gamma^b u^i\|\partial\Gamma^c u^i\|\partial\Gamma^d u^i\|.\end{aligned} \tag{4.70}$$

We easily see that the first term is dominated by $C[\partial u(s)]^2_{[(N+2)/2]}\|\partial u(s)\|_{N+4}$.

To treat the second term, we divide the argument into two cases. First we assume $|b|\ge[(N+2)/2]$. Since $1+\lambda$ is equivalent to $1+s$ for $(\lambda,s)\in\Lambda_i$ by (2.9), we have

$$\begin{aligned}&|\lambda^{\frac12}\eta_2(\lambda,s)(1+s)^{-\frac12}|\Gamma^b u^i\|\partial\Gamma^c u^i\|\partial\Gamma^d u^i\| \\ &\le C[\partial u(s)]^2_{[\frac{N+2}{2}]}|\Gamma^b u^i(y,s)| \\ &\le C\left(M_N\varepsilon + [\partial u]^2_{[\frac{N+4}{2}],s}\|\partial u\|_{N+6,s}\right),\end{aligned}$$

where we have used (4.1) with $k=[(N+4)/2]$ and (4.55) with $\mu=0$ and N replaced by $N+3$.

Next we assume $|b|\le[(N+2)/2]$. In this case, we have

$$\begin{aligned}&|\lambda^{\frac12}\eta_2(\lambda,s)(1+s)^{-\frac12}|\Gamma^b u^i\|\partial\Gamma^c u^i\|\partial\Gamma^d u^i\| \\ &\le C\|\partial u(s)\|_{N+4}[\partial u(s)]_{[\frac{N+2}{2}]}|w_i(\lambda,s)(1+s)^{-\frac12}|\Gamma^b u^i(\lambda,s)\| \\ &\le C(1+s)^{\frac14-\mu}\|\partial u(s)\|_{N+4}\left(\varepsilon+[\partial u]^3_{[\frac{N+2}{2}]+1,s}\right),\end{aligned}$$

where we have used (4.54), (2.8), (4.1) with $k=[(N+4)/2]$, and (4.56). Taking μ such that $\mu>1/4$, we obtain

$$M_{3,N+1}\le C\left((1+\|\partial u\|_{N+4,t})\varepsilon+[\partial u]^2_{[\frac{N+4}{2}],t}\|\partial u\|_{N+6,t}\right). \tag{4.71}$$

Combining (4.60), (4.69), and (4.71) with (4.59), we obtain (4.3). This completes the proof. ∎

5. Proof of Theorem 1.1. By the existence and the uniqueness of the local smooth solution of (1.1) and (1.2) (see, e.g., S. Klainerman [14]), it is enough to establish a uniform a priori estimate of $[\partial u(t)]_N$ for some large integer N. To deal with the L^2-norm in the right-hand side of (4.3), we need the following.

PROPOSITION 5.1. *Let $u^i\in C^\infty(\mathbf{R}^2\times[0,T))$ be a solution of* (1.1) *and* (1.2). *Suppose that* (1.5) *holds. Then there exists a sufficiently small $\delta_1>0$ independent* *of T and a constant $C_N > 0$ independent of T and δ_1 such that the following energy estimate holds for $0 \le t < T$:*

$$\|\partial u(t)\|_N \le C_N \|\partial u(0)\|_N (1+t)^{C_N[\partial u]^2_{[\frac{N+1}{2}],t}}, \tag{5.1}$$

provided (4.1) *with $k = [(N+1)/2]$ holds.*

PROPOSITION 5.2. *Let $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$ be a solution of* (1.1) *and* (1.2). *Also let $0 < \delta_1 < 1$ in* (4.1). *Suppose that* (1.11) *holds. Then there exists a constant $C_N > 0$ independent of T and δ_1 such that the following energy estimate holds for $0 \le t < T$:*

$$\begin{aligned}\|\partial u(t)\|_N^2 \le C_N^2 \Big\{ & \|\partial u(0)\|_N^2 \\ & + \int_0^t (1+s)^{-\frac{5}{4}} ([\partial u(s)]^2_{N+1} + \langle u(s)\rangle^2_{N+1}) \|\partial u(s)\|^2_{N+1} ds \Big\},\end{aligned} \tag{5.2}$$

provided (4.1) *with $k = [(N+1)/2]$ holds. Here we have set*

$$\langle u(s)\rangle_k = \sum_{i=1}^m \sum_{|a|\le k} \sup_{\{x \in \mathbf{R}^2 : (x,s) \in \Lambda_i\}} |\Gamma^a u^i(x,s)|.$$

Proof of Proposition 5.1. If we set

$$L_i v = \Box_i v^i - \sum_{l=1}^m \sum_{\gamma,\delta=0}^2 H_{il}^{\gamma\delta}(\partial u) \partial_\gamma \partial_\delta v^l - K_i(\partial u) \quad \text{for} \quad v = (v^1, \dots, v^m), \tag{5.3}$$

we have an identity

$$\begin{aligned}\frac{d}{dt} \int_{\mathbf{R}^2} \Big\{ & (\partial_t v^i)^2 + c_i^2 |\nabla v^i|^2 - \sum_{l=1}^m H_{il}^{00}(\partial u) \partial_t v^i \partial_t v^l \\ & + \sum_{p,q=1}^2 H_{il}^{pq}(\partial u) \partial_p v^i \partial_q v^q \Big\} dx = \int_{\mathbf{R}^2} J_i(v) dx,\end{aligned} \tag{5.4}$$

where

$$\begin{aligned} J_i(v) = 2L_i v \partial_t v^i & - \sum_{l=1}^m (\partial_t H_{il}^{00}(\partial u)) \partial_t v^i \partial_t v^l + 2 \sum_{l=1}^m \sum_{p=1}^2 (\partial_p H_{il}^{p0}(\partial u)) \partial_t v^i \partial_t v^l \\ & -2 \sum_{l=1}^m \sum_{p,q=1}^2 (\partial_p H_{il}^{pq}(\partial u)) \partial_q v^i \partial_t v^l \\ & + \sum_{l=1}^m \sum_{p,q=1}^2 (\partial_t H_{il}^{pq}(\partial u)) \partial_p v^i \partial_q v^l + 2K_i(\partial u) \partial_t v^i. \end{aligned}$$

Here we have used (1.5) and the divergence theorem. By (1.12), we have

$$|H_{il}^{\gamma\delta}(\partial u)| < \frac{1}{2m} \min\{1, c_m^2\}$$

if we take δ_1 in (4.1) to be sufficiently small. Therefore, (5.4) yields

$$\|\partial v(t)\|_0^2 \le C\left(\|\partial v(0)\|_0^2 + \sum_{i=1}^m \int_0^t ds \int_{\mathbf{R}^2} |J_i(v)|dx\right). \tag{5.5}$$

Hence, if we take $v = \Gamma^a u (|a| \le N)$ in (5.5), we have

$$\|\partial u(t)\|_N^2 \le C\left(\|\partial u(0)\|_N^2 + \sum_{i=1}^m \sum_{|a|\le N} \int_0^t ds \int_{\mathbf{R}^2} |J_i(\Gamma^a u)|dx\right). \tag{5.6}$$

Furthermore, it follows from (1.12), (4.1), and the Leibniz rule that

$$\int_{\mathbf{R}^2} |J_i(\Gamma^a u)|dx \le C|\partial u(s)|^2_{[\frac{N+1}{2}]}\|\partial u(s)\|_N^2. \tag{5.7}$$

Thus, combining (5.6) and (5.7) and using Gronwall's inequality, we have

$$\|\partial u(t)\|_N \le C_N\|\partial u(0)\|_N \exp\left(\int_0^t C_N|\partial u(s)|^2_{[\frac{N+1}{2}]}ds\right), \tag{5.8}$$

which yields (5.1), due to (2.5). This completes the proof. □

Proof of Proposition 5.2. Multiplying $\partial_t\Gamma^a u^i$ by (3.9) and integrating it over $\mathbf{R}^2 \times [0,t]$, we have

$$\|\partial u^i(t)\|_N^2 \le \|\partial u^i(0)\|_N^2 + C_N \sum_{|b|\le|a|\le N} \int_0^t \int_{\mathbf{R}^2} |\Gamma^b(F^i(\partial u, \partial^2 u))\partial_t\Gamma^a u^i|dxds. \tag{5.9}$$

We divide the function F^i into three parts: G^i, R^i, and N^i as in (1.12).

First, we derive the estimate for the higher-order term G^i. Using (2.5) and (4.1) with $k = [(N+1)/2]$, we have

$$|\Gamma^b G^i(x,s)| \le C_N(1+s)^{-\frac{3}{2}}[\partial u(s)]^3_{[\frac{N+1}{2}]} \sum_{j=1}^m \sum_{|c|\le|b|+1} |\partial\Gamma^c u^j(x,s)|, \tag{5.10}$$

which yields

$$\int_{\mathbf{R}^2} |\Gamma^b(G^i(x,s))\partial_t\Gamma^a u^i|dx \le C_N(1+s)^{-\frac{3}{2}}[\partial u(s)]^3_{[\frac{N+1}{2}]}\|\partial u(s)\|^2_{N+1}. \tag{5.11}$$

Second, we consider the resonance-form R^i. Without loss of generality, we may assume $l \ne i$ by (1.13). We now use the "resonance" property by the aid of (2.5), (2.6), and (2.4), namely,

$$\frac{1}{(w_l w_i)(|x|,s)} \le \frac{C}{(1+s)^{\frac{5}{4}}}. \tag{5.12}$$

Using this estimate, we get

$$\begin{aligned}|\Gamma^b(R^i(x,s))\partial_t\Gamma^a u^i| &\le C_N \sum_{j,k,=1}^m \sum_{l\ne i} \sum_{|c+d+e|\le|b|+1} |\Gamma^c(\partial u^j)\Gamma^d(\partial u^k)\Gamma^e(\partial u^l)\partial_t\Gamma^a u^i| \\ &\le C_N \sum_{j,k=1}^m \sum_{|c+d|\le|b|+1} (1+s)^{-\frac{5}{4}}[\partial u(s)]^2_{N+1}|\Gamma^c(\partial u^j)\Gamma^d(\partial u^k)|,\end{aligned}$$

which yields

$$\int_{\mathbf{R}^2} |\Gamma^b(R^i(x,s))\partial_t\Gamma^a u^i|dx \le C_N(1+s)^{-\frac{5}{4}}[\partial u(s)]_{N+1}^2\|\partial u(s)\|_{N+1}^2. \tag{5.13}$$

Finally, we treat the null-form N^i. When $(x,s)\in\Lambda_i^c$, we find from (2.7) that

$$|\Gamma^b(N^i(x,s))| \le C_N(1+s)^{-\frac{3}{2}}[\partial u(s)]_{[\frac{N+1}{2}]}^2 \sum_{|c|\le|b|+1} |\partial\Gamma^c u^i(x,s)|.$$

When $(x,s)\in\Lambda_i$, it follows from Proposition 3.1 and (2.5) that

$$\begin{aligned}|\Gamma^b(N^i(x,s))| &\le C_N((1+s)^{-\frac{1}{2}}\Phi_b^i + (1+s)^{-1}\Theta_b^i)\\ &\le C_N(1+s)^{-\frac{3}{2}}([\partial u(s)]_{N+1}^2 + [\partial u(s)]_{N+1}\langle u(s)\rangle_{N+1}) \sum_{|c|\le|b|+1}|\partial\Gamma^c u^i(x,s)|.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\int_{\mathbf{R}^2} |\Gamma^b(N^i(x,s))\partial_t\Gamma^a u^i|dx\\ &\le \|\Gamma^b(N^i(s))\|_0\|\partial u(s)\|_{N+1}\\ &\le C_N(1+s)^{-\frac{3}{2}}([\partial u(s)]_{N+1}^2 + \langle u(s)\rangle_{N+1}^2)\|\partial u(s)\|_{N+1}^2.\end{aligned} \tag{5.14}$$

Combining (5.11), (5.13), and (5.14) with (5.9), we obtain (5.2). The proof is complete. ∎

COROLLARY 5.1. *Let $u^i \in C^\infty(\mathbf{R}^2\times[0,T))$ be a solution of* (1.1) *and* (1.2). *Suppose that* (1.5) *and* (1.11) *hold. Then there exist a sufficiently small $\delta_1 > 0$ independent of T and a constant $C_N > 0$ independent of T and δ_1 such that the following holds for $0\le t<T$:*

$$\|\partial u(t)\|_{N+6}^2 \le C_N^2\varepsilon^2\left\{1+\int_0^t (1+s)^{-\frac{5}{4}+4C_N[\partial u]^2_{[\frac{N+14}{2}],s}}ds\right\}, \tag{5.15}$$

provided (4.1) *with $k=[(N+14)/2]$ holds and $0<\varepsilon\le 1$.*

Proof. It follows from (4.3) and (5.1) that for $0\le s\le t$

$$[\partial u(s)]_{N+7} \le C_N(\varepsilon + (\varepsilon+\delta_1^2)\|\partial u\|_{N+13,s}) \tag{5.16}$$

and

$$\|\partial u\|_{N+13,s} \le C_N\varepsilon(1+s)^{C_N[\partial u]^2_{[\frac{N+14}{2}],s}} \tag{5.17}$$

because $\|\partial u(0)\|_{N+13}\le C_N\varepsilon$ for sufficiently small δ_1. Therefore, we have

$$[\partial u(s)]_{N+7} \le C_N(1+\varepsilon+\delta_1^2)\varepsilon(1+s)^{C_N[\partial u]^2_{[\frac{N+14}{2}],s}}. \tag{5.18}$$

Moreover, $\langle u(s)\rangle_{N+7}$ has the same estimate as $[\partial u(s)]_{N+7}$, because of (4.55). Now (5.15) follows from (5.2) and (5.18) together with (5.17). The proof is complete. ∎

End of the proof of Theorem 1.1. As we stated at the beginning of the present section, what we need to prove Theorem 1.1 is an a priori estimate for $[\partial u(t)]_N$. We fix an integer N satisfying $N\ge 13$, which guarantees $[(N+14)/2]\le N$. We take a positive constant B_N such that $B_N \geq 2\tilde{C}_N$ and $B_N \geq M_N$, where M_N is the constant in (4.6) and $\tilde{C}_N$ is the constant larger than C_N appearing in (4.3) and (5.15). We also take ε_1 such that

$$0 < \varepsilon_1 \leq 1 \quad \text{and} \quad 3B_N\varepsilon_1 \leq \delta_1, \tag{5.19}$$

where δ_1 is the smallest one taken in Proposition 4.1 and Corollary 5.1. Moreover, set

$$\begin{aligned} T_\varepsilon = \sup\{T > 0:\ &(1.1) \text{ and } (1.2) \text{ have a solution } u^i \text{ in } C^\infty(\mathbf{R}^2 \times [0, T)) \\ &\text{and } [\partial u]_{N,T} \leq 3B_N\varepsilon \text{ holds}\}. \end{aligned} \tag{5.20}$$

We can see that $T_\varepsilon > 0$, because of the existence of a local solution, the continuity of $[\partial u]_{N,t}$, and (4.5). Then, for each ε satisfying $0 < \varepsilon \leq \varepsilon_1$, we have $u^i \in C^\infty(\mathbf{R}^2 \times [0, T_\varepsilon))$ and

$$[\partial u]_{[\frac{N+14}{2}],T_\varepsilon} \leq [\partial u]_{N,T_\varepsilon} \leq \delta_1,$$

which imply that (4.3) and (5.15) hold. In particular, we have for $0 \leq t < T_\varepsilon$

$$\|\partial u(t)\|_{N+6} \leq \tilde{C}_N\varepsilon \left\{1 + \int_0^t (1+s)^{-\frac{5}{4}+4\tilde{C}_N[\partial u]_{[\frac{N+14}{2}],s}} ds\right\}^{\frac{1}{2}}. \tag{5.21}$$

Now, we take ε_0 to be

$$0 < \varepsilon_0 \leq \varepsilon_1, \quad 3\tilde{C}_N\varepsilon_0 \leq 1, \quad \text{and} \quad 12\tilde{C}_N B_N\varepsilon_0 \leq \frac{1}{8}, \tag{5.22}$$

and fix an ε in $[0, \varepsilon_0)$ in the following. Then, by (5.21), (5.20), and (5.22), we have for $0 \leq t < T_\varepsilon$

$$\begin{aligned} \|\partial u(t)\|_{N+6} &\leq \tilde{C}_N\varepsilon \left(1 + \int_0^t (1+s)^{-\frac{9}{8}} ds\right)^{\frac{1}{2}} \\ &\leq 1. \end{aligned}$$

Substituting this into (4.3) and using (5.20), we have

$$[\partial u]_{N,T_\varepsilon} \leq \tilde{C}_N \left(2\varepsilon + 3B_N\varepsilon[\partial u]_{[\frac{N+4}{2}],T_\varepsilon}\right).$$

Hence, by $B_N \geq 2\tilde{C}_N$ and (5.22), we have

$$[\partial u]_{N,T_\varepsilon} \leq 2B_N\varepsilon. \tag{5.23}$$

By the blowup criterion (see, e.g., [22, Theorem 2.2, p. 31]), we see that if $T_\varepsilon < +\infty$, we must have $\lim_{t\to T_\varepsilon - 0}[\partial u]_{N,T} = 3B_N\varepsilon$, which contradicts (5.23). Therefore, we have $T_\varepsilon = +\infty$. This completes the proof of Theorem 1.1. ∎

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