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The null condition for quasilinear wave equations in two space dimensions I

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Introduction

In this paper, we prove quasi-global existence (resp. global existence) for quasilinear wave equations in two space dimensions satisfying the null condition (resp. both null conditions). In contrast with the case of three space dimensions where the result, due to D. Christodoulou and S. Klainerman, has been known since 1986, this problem has remained open until now, except for the special cases of cubic terms or rotationally invariant equations. Our proof relies on the construction of an approximate solution, combined with an energy integral method which displays the null condition(s).

1. Main results

We consider a quasilinear wave equation in $\mathbf{R}_+ \times \mathbf{R}^2$

$$(1.1) \qquad \left(\partial_t^2 - \Delta_x\right)u + \Sigma_{0\le i,j\le 2} g_{ij}(\partial u)\partial_{ij}^2 u = 0.$$

Here

$$\Sigma g_{ij}\partial_{ij}^2 = g_{00}\partial_t^2 + g_{01}\partial_t\partial_1 + g_{02}\partial_t\partial_2 + g_{11}\partial_1^2 + g_{12}\partial_{12}^2 + g_{22}\partial_2^2,$$
$$x_0 = t, x = (x_1, x_2), \partial u = (\partial_1 u, \partial_2 u, \partial_t u).$$

The coefficients g_{ij} are smooth real functions vanishing at the origin, and, more precisely,

$$g_{ij}(\xi) = \Sigma g_{ij}^k \xi_k + \Sigma h_{ij}^{kl}\xi_k\xi_l + r_{ij}(\xi), r_{ij}(\xi) = O(|\xi|^3).$$

The initial conditions for u are

(1.2)
$$u(x,0) = \epsilon u_1^0(x) + \epsilon^2 u_2^0(x) + \dots, \partial_t u(x,0) = \epsilon u_1^1(x) + \epsilon^2 u_2^1(x) + \dots,$$

where the real functions u_i^j are smooth and supported in $|x| \le M$.

As usual, we define

$$(1.3)_a \qquad g(\omega) = \Sigma g_{ij}^k \omega_i \omega_j \omega_k,$$

$$(1.3)_b \qquad h(\omega) = \Sigma h_{ij}^{kl} \omega_i \omega_j \omega_k \omega_l,$$

where $\omega_0 = -1$, $\omega_1 = \cos\omega$, $\omega_2 = \sin\omega$. We assume throughout this paper that the wave equation (1.1) satisfies the *null condition*, that is $g(\omega) \equiv 0$.

In three space dimensions, it has been shown by Christodoulou [5] and Klainerman [9] that the null condition implies global existence of smooth solutions. The proof of Christodoulou uses the conformal method; the null condition implies then that the nonlinear terms of the equations transform into smooth terms, and the problem is reduced to a local problem with small data. The proof of Klainerman uses a special energy inequality for the wave equation, which is obtained by multiplying by an appropriate vector field with quadratic coefficients (see [7] for an account of both aspects).

In the present case of two space dimensions, the problem is more delicate, as we explain now:

a. First, methods of nonlinear geometrical optics show that the solution *formally* looks like

$$u(x,t) = \frac{\epsilon}{r^{1/2}} S(r-t, \omega, \tau),$$

where

$$r = \sqrt{x_1^2 + x_2^2}, x_1 = r\cos\omega, x_2 = r\sin\omega$$

are the usual polar coordinates in the plane, while $\tau = \epsilon^2 \log(1+t)$ is a *slow time* taking into account the effects of the nonlinear terms. We use also $\sigma = r - t$.

b. Second, as a first approximation, we have $u \sim \epsilon u_1$, where u_1 is the solution of the linearized problem on $u = 0$

$$(1.4) \qquad \left(\partial_t^2 - \Delta\right) u_1 = 0, u_1(x,0) = u_1^0(x), \partial_t u_1(x,0) = u_1^1(x).$$

Defining

$$R_1(\sigma,\omega) = \frac{1}{2\sqrt{2\pi}} \chi_-^{-1/2} * \left[R\left(u_1^0\right) - \partial_s R\left(u_1^1\right)\right],$$

where

$$\chi_-^\lambda(t) = \chi_+^\lambda(-t), \chi_+^\lambda(t) = \frac{t_+^\lambda}{\Gamma(\lambda+1)}$$

and R is the Radon transform, we have for $r \geq t/2$

$$(1.5) \qquad u_1(x,t) \sim r^{-1/2} R_1(r-t,\omega).$$

We assume from now on that not both u_1^0 and u_1^1 are identically zero, which implies that R_1 is not identically zero either. We call "case I" the situation when h is not identically zero, "case II" the situation when $h \equiv 0$.

In case I, the function $h(\partial_\sigma R_1)(\partial_\sigma^2 R_1)$ has a positive maximum in (σ, ω), and we define

$$\bar{\tau} = \left[\max h(\omega)(\partial_\sigma R_1)\left(\partial_\sigma^2 R_1\right)(\sigma, \omega)\right]^{-1}. \tag{1.6}$$

The number $\bar{\tau}$ appears as the principal term in the expression of the lifespan of S. Hence the following two conjectures have been made in this case:

$$\liminf \epsilon^2 \log \bar{T}_\epsilon \geq \bar{\tau}, \tag{$\underline{C}$}$$

$$\limsup \epsilon^2 \log \bar{T}_\epsilon \leq \bar{\tau}. \tag{$\bar{C}$}$$

Here, $\bar{T}_\epsilon$ is the lifespan of the smooth solution u of our problem (1.1), (1.2). In case II (we say that the "second null condition" is satisfied), it has been conjectured that

$$\bar{T}_\epsilon = +\infty. \tag{C_∞}$$

This approach is very similar to what has been done for wave equations in two or three space dimensions (without null condition), where Hörmander [7] has conjectured the precise expression of the first term of an asymptotic expansion of $\bar{T}_\epsilon$ (see also [4]).

In the semilinear case (with lower order terms satisfying the null condition), Godin [6] has proved ($\underline{C}$) and C_∞. In the present quasilinear case, as far as we know, the conjectures (C) have only been proved in special cases:

i) If all $g_{ij}^k = 0$ (the "cubic case"), energy integral methods show easily that indeed ($\underline{C}$) and (C_∞) hold. One can see for instance Hoshiga [8] or Li Ta-tsien [11].
ii) For rotationally invariant equations and data, Ladhari [10] proved all three conjectures.

The methods used for $n = 3$ are not available, since

i) The energy integral method used by Klainerman [9] works only for $n \geq 3$,
ii) The conformal method of Christodoulou [5] yields, for $n = 2$, singular nonlinear terms, making the problem after transformation look as complicated as the original one.

In this paper, we prove ($\underline{C}$) and (C_∞) in the general case. The conjecture ($\bar{C}$) is proved in [1].

Theorem 1. *In case I ($h \not\equiv 0$), the lifespan $\bar{T}_\epsilon$ of the solution of (1.1), (1.2) satisfies*

$$\liminf \epsilon^2 \log \bar{T}_\epsilon \geq \bar{\tau}.$$

Theorem 1′. *In case II ($h \equiv 0$), the smooth solution u of (1.1), (1.2) exists globally.*

The method of proof is the following:

a) First, we construct an approximate solution u_a, using nonlinear geometrical optics techniques (see [4] for instance for an account of these techniques).
b) Second, splitting $u = u_a + \dot{u}$, we prove by induction on time estimates on

$$|Z^\alpha \partial \dot{u}(., t)|_{L^\infty}.$$

Part a) is straightforward.

Part b) uses the Z-fields method of Klainerman, along with the corresponding weighted Sobolev inequality. Recall that Z is one of the fields

(1.7) $$x_1\partial_2 - x_2\partial_1,\ t\partial_j + x_j\partial_t,\ x\partial_x + t\partial_t,\ \partial_k,\ j = 1, 2,\ k = 0, 1, 2,$$

and that

$$|v(x, t)| \leq C(1 + t + r)^{-1/2}(1 + |r - |t||)^{-1/2}\Sigma_{|\alpha|\leq 2}|Z^\alpha v(., t)|_{L^2}.$$

The key point is to obtain an energy inequality for the linearized operator P on u

$$P \equiv \partial_t^2 - \Delta + \Sigma g_{ij}(\partial u)\partial_{ij}^2 + \Sigma g_{ij}^k\left(\partial_{ij}^2 u\right)\partial_k.$$

It turns out that, in case I, for $\tau \leq \tau_0 < \bar{\tau}$ and ϵ small enough, we have for P exactly the same inequality as for the ordinary wave equation

(1.8) $$|\partial v(., t)|_{L^2} \leq C(|\partial v(., 0)|_{L^2} + \int_0^t |Pv(., s)|_{L^2} ds).$$

In case II, we cannot obtain a similar global inequality, but we get an inequality with some not too big amplification factor

(1.8)′ $$|\partial v(., t|_{L^2} \leq C(1 + t)^{C\epsilon^2}(|\partial v(., 0)|_{L^2} + \int_0^t |Pv(., s)|_{L^2} ds).$$

This is enough to finally obtain global existence.

To obtain (1.8) or (1.8)′, we just compute as usual

$$\int e^p Pv\partial_t v dx dt,$$

for some "ghost weight" p, that is, an appropriate *bounded* weight, which thus disappears from the final inequality. The technical aspects of this energy integral method are explained in some details in Sect. 3.2. It turns out that, for a careful choice of p, we obtain a better control (in $L^2_{x,t}$) of the "tangential" derivatives $\partial_i v + \omega_i \partial_t v$. This allows us to make the null condition(s) appear in the quadratic form on ∂v obtained by integrations by parts. We hope that this method will be also useful elsewhere.

The plan of the paper is as follows: first, we describe briefly the approximate solution u_a in cases I and II (Sect. 2). Section 3 is devoted to the proof of the energy inequalities (1.8) and (1.8)′ for the linearized operator (Theorems 2 and 2′ and Theorem 3). In Sect. 4 (resp. Sect. 5), we show how to control commutator terms, and finish the proof of Theorem 1 (resp. of Theorem 1′). In fact, more precise Theorems 4 and 4′ are stated and proved, giving estimates of the error $\dot{u}$. Finally, to illustrate how our energy method works, we give the proof of Theorem 3 in Sect. 6.

2. Construction of an approximate solution

We use here the simplest approximate solution, starting with ϵu_1, blowing up at time $\bar{T}_a$ such that

$$\epsilon^2 \log(1+\bar{T}_a) = \bar{\tau}.$$

For similar constructions, we refer to [7] or [8]. In Ladhari [10], a more precise approximate solution is obtained, but we will not use it here.

2.1. Description of the construction (case I)

We fix $0 < \tau_0 < \bar{\tau}$ and distinguish two periods of time: the period I is $0 \le t \le 2\epsilon^{-p}$, the period II is $\epsilon^{-p} \le t \le (\exp \tau_0/\epsilon^2) - 1$, and the transition period is $\epsilon^{-p} \le t \le 2\epsilon^{-p}$.

The approximate solution u_a will be constructed separately in each period as u_a^I and u_a^{II}, and we set

$$u_a(x,t) = \chi_1(t\epsilon^p)u_a^I + (1-\chi_1(t\epsilon^p))u_a^{II}$$

for a fix smooth function $\chi_1(s)$ being 1 for $s \le 1$ and 0 for $s \ge 2$.

a. In period I, we simply take $u_a = \epsilon u_1$. The function u_1 has already been introduced and studied in **1.** Recall that also

$$\left|Z^\alpha u_1\right| \le C(1+t)^{-\frac{1}{2}}(1+|\sigma|)^{-\frac{1}{2}}.$$

Introducing the slow time $\tau = \epsilon^2 \log(1+t)$, it is natural to look for u_a in period II as

$$u_a^{II} = u_a = \frac{\epsilon}{r^{1/2}}\chi_2 S(\sigma,\omega,\tau) + (1-\chi_2)u_a^I,$$

where $\chi_2 = \chi_2(r/(1+t))$, the smooth function $\chi_2(s)$ being 0 for $s \le 1/2$ and 1 for $s \ge 2/3$.

b. We have the identity

$$\big(\partial_t^2 - \Delta\big)(S/r^{1/2}) = - r^{-5/2}\big(S/4 + \partial_\omega^2 S\big) + \frac{\epsilon^2}{t\sqrt{r}}\big[- 2\partial_{\sigma\tau}^2 S - t^{-1}\partial_\tau S + \epsilon^2 t^{-1}\partial_\tau^2 S\big].$$

On the other hand, for $v = \epsilon S/r^{1/2}$,

$$\Sigma g_{ij}(\partial v)\partial_{ij}^2 v = \Sigma g_{ij}^k \partial_k v \partial_{ij}^2 v + \Sigma h_{ij}^{kl}\partial_k v \partial_l v \partial_{ij}^2 v + \Sigma O((\partial v)^3)\partial_{ij}^2 v = = \frac{\epsilon^3}{r^{3/2}} h(\omega)(\partial_\sigma S)^2(\partial_\sigma^2 S) + O(\epsilon^2 r^{-2}).$$

This suggests to take for S the solution of the equation

$$\partial_{\sigma\tau}^2 S - 1/2 h(\omega)(\partial_\sigma S)^2\big(\partial_\sigma^2 S\big) = 0,\ S(\sigma, \omega, 0) = R_1(\sigma, \omega).$$

The number $\bar\tau$ already defined in (1.6) is just the lifespan of this function S.

c. We fix $p = 4$ in the construction above. We denote by J_a^I, J_a^{II}, J_a the value of the left-hand side of the equation evaluated on u_a^I, u_a^{II}, u_a respectively. We have the following estimates, for $\tau = \epsilon^2 \log(1 + t) \le \tau_0 < \bar\tau$:

(2.1.4) $$\big|Z^\alpha u_a^I\big| \le C\epsilon(1 + t)^{-1/2}(1 + |\sigma|)^{-1/2},$$

(2.1.5) $$\big|Z^\alpha J_a^I\big| \le C\frac{\epsilon^2}{(1 + t)^{3/2}}(1 + |\sigma|)^{-3},$$

(2.1.6) $$\big|Z^\alpha u_a^{II}\big| \le C\epsilon(1 + t)^{-1/2}(1 + |\sigma|)^{-1/2},$$

(2.1.7) $$\big|Z^\alpha J_a^{II}\big| \le C\epsilon(1 + t)^{-2}(1 + |\sigma|)^{-1},$$

and, in the transition period $\epsilon^{-p} \le t \le \epsilon^{-2p}$,

(2.1.8) $$\big|Z^\alpha\big(u_a^I - u_a^{II}\big)\big| \le C\epsilon^2(1 + t)^{-1/2}(1 + |\sigma|)^{-3},$$

(2.1.9) $$\big|Z^\alpha J_a\big| \le C\epsilon^2(1 + t)^{-3/2}(1 + |\sigma|)^{-5/4}.$$

In particular, we have for $\tau \le \tau_0 < \bar\tau$,

(2.1.10) $$\int_0^t \big|Z^\alpha J_a(., s)\big|_{L^2} ds \le C\epsilon^2 |\log\epsilon|.$$

2.2. Description of the construction (case II)

This case is even simpler than the preceeding one. We just take $u_a = \epsilon u_1$ for all t. There is no slow time in this case. We have the estimates

(2.2.1) $$\big|Z^\alpha u_a\big| \le C\epsilon(1 + t)^{-1/2}(1 + |\sigma|)^{-1/2},$$

(2.2.2) $$\big|Z^\alpha J_a\big| \le C\epsilon^2(1 + t)^{-2}(1 + |\sigma|)^{-2}.$$

In particular,

$$\int_0^{+\infty} \left|Z^\alpha J_a(.,s)\right|_{L^2} ds \le C\epsilon^2. \tag{2.2.3}$$

2.3. Justification of the results

The proofs of the above estimates use two results:

i) The properties of u_1 and R_1 defined above,
ii) An analysis of the improved behavior of the nonlinear terms of the equation resulting from the null condition. This analysis is contained in the following two lemmas.

Lemma 2.3.1 (Hörmander [7], Lemma 6.6.4).

i) If the g_{ij}^k satisfy the null condition

$$\Sigma g_{ij}^k \omega_i \omega_j \omega_k = 0,$$

for functions u, v supported in $|x| \le M + t$,

$$\left|\Sigma g_{ij}^k \partial_k u \partial_{ij}^2 v\right| \le C(1+t)^{-1}(|Zu||\partial^2 v| + |\partial u||Z\partial v|). \tag{2.3.1}$$

ii) If the h_{ij}^{kl} satisfy the null condition

$$\Sigma h_{ij}^{kl} \omega_i \omega_j \omega_k \omega_l = 0,$$

for functions u, v, w supported in $|x| \le M + t$,

$$\begin{aligned}\left|\Sigma h_{ij}^{kl} \partial_k u \partial_l v \partial_{ij}^2 w\right| \le C(1+t)^{-1}(&|Zu||\partial v||\partial^2 w| \\ &+ |\partial u||Zv||\partial^2 w| + |\partial u||\partial v||Z\partial w|).\end{aligned} \tag{2.3.2}$$

Lemma 2.3.2 (Hörmander [7], Lemma 6.6.5). *Let*

$$G = G\big(u_1^{(k_1)}, u_2^{(k_2)}, \ldots, u_p^{(k_p)}\big)$$

be a multilinear form on $\mathbf{R}^3$ satisfying the null condition and Z one of the usual Klainerman vector fields. Then

$$\begin{aligned}ZG &= G\big((Zu_1)^{(k_1)}, u_2^{(k_2)}, \ldots, u_p^{(k_p)}\big) + \ldots \\ &+ G\big(u_1^{(k_1)}, \ldots, (Zu_p)^{(k_p)}\big) + G_1\big(u_1^{(k_1)}, \ldots, u_p^{(k_p)}\big),\end{aligned} \tag{2.3.3}$$

where G_1 also satisfies the null condition.

Finally, we list here some easy properties of u_a which will be useful in the proofs of the theorems.

Lemma 2.3.3. *For the approximate solution u_a (case I or case II), we have, for the relevant values of t in each case,*

i) $|\partial u_a| + |Z\partial u_a| \leq C\epsilon(1+t)^{-1/2}(1+|\sigma|)^{-3/2}$,
ii) $|\partial^2 u_a| \leq C\epsilon(1+t)^{-1/2}(1+|\sigma|)^{-5/2}$.

Proof. Because of the identity

$$\partial_k = (t^2 - r^2)^{-1}\Sigma c_k^{ij} x_i Z_j, \tag{2.3.4}$$

where the c's are constants, we have for any v

$$|\partial v| \leq C(1+|\sigma|)^{-1}|Zv|. \tag{2.3.5}$$

This implies immediately i) and ii) in view of (2.1.4)–(2.1.6) or (2.2.1). ◇

3. Energy inequalities for the linearized operator

3.1 Let u_a be the approximate solution from Sect. 2, and set $u = u_a + \dot{u}$. In this section, we prove the following energy inequalities, separating case I and case II for clarity.

Theorem 2 (case I). *For any fixed* $0 < \tau_0 < \bar{\tau}$, $s_0 \geq 3$, *there exist* $\epsilon_0 > 0$ *and* C_0 *such that: if* $\epsilon \leq \epsilon_0$, *if the solution u of (1.1), (1.2) exists and is smooth for* $t < T$ *with* $\epsilon^2 \log(1+T) \leq \tau_0$ *and if*

$$\Sigma_{|\alpha|\leq s_0}|Z^\alpha \partial\dot{u}(.,t)|_{L^\infty} \leq \epsilon^{3/2}(1+t)^{-1/2}(1+|\sigma|)^{-1/2} \tag{3.1.1}$$

there, then we have the inequality

$$|\partial v(.,t)|_{L^2} \leq C_0(|\partial v(.,0)|_{L^2} + \int_0^t |Lv(.,s)|_{L^2} ds), \tag{3.1.2}$$

where L is either one of the two operators

$$P_a = \partial_t^2 - \Delta + \Sigma g_{ij}(\partial u)\partial_{ij}^2 + \Sigma c_a^k \partial_k,\ c_a^k = \Sigma g_{ij}^k \partial_{ij}^2 u_a$$

or

$$P = \partial_t^2 - \Delta + \Sigma g_{ij}(\partial u)\partial_{ij}^2 + \Sigma c_k \partial_k,\ c_k = \Sigma g_{ij}^k \partial_{ij}^2 u.$$

Theorem 2′ (case II). *For any fixed* $s_0 \geq 3$, *there exist* $\epsilon_0 > 0$ *and* C_0 *such that: if* $\epsilon \leq \epsilon_0$, *if the solution u of (1.1), (1.2) exists and is smooth for* $t < T$ *with*

$$\Sigma_{|\alpha|\leq s_0}|Z^\alpha \partial\dot{u}(.,t)|_{L^\infty} \leq \epsilon^{3/2}(1+t)^{-1/2}(1+|\sigma|)^{-1/2}$$

there, then we have the inequality

$$|\partial v(.,t)|_{L^2} \leq C_0(1+t)^{C_0\epsilon^2}(|\partial v(.,0)|_{L^2} + \int_0^t |Lv(.,s)|_{L^2} ds) \tag{3.1.3}$$

where L is either P or P_a defined in Theorem 2.

Remark. Under the assumptions of Theorem 2′, we are not able to obtain a global energy inequality *without the amplification factor* $(1+t)^{C_0\epsilon^2}$, except in special cases (see the remark after the proof of Theorem 2 and 2′): the difficulty seems to come from the lower order terms of the linearized operator. We do not know whether such an inequality exists or not.

To illustrate how our energy technique can works in this direction, we also give the following "abstract" Theorem, closely related to Theorems 2 and 2′.

Theorem 3. *Let w be a smooth function for $t \geq 0$, supported in $|x| \leq M+t$. Define the linear operator P by*

$$P = \left(\partial_t^2 - \Delta\right) + \Sigma g_{ij}(\partial w)\partial_{ij}^2,$$

where

$$g_{ij}(\xi) = \Sigma g_{ij}^k \xi_k + \Sigma h_{ij}^{kl}\xi_k\xi_l$$

satisfies both null conditions $g = h = 0$. Assume that, for some $\eta > 1/3$,

$$|\partial w(.,t)|_{L^\infty} + \Sigma|Z\partial w(.,t)|_{L^\infty} \leq C_1(1+t)^{-\eta}.$$

Then, for C_1 small enough, the following standard energy inequality holds for some C_0

$$|\partial v(.,t)|_{L^2} \leq C_0(|\partial v(.,0)|_{L^2} + \int_0^t |Pv(.,s)|_{L^2} ds).$$

3.2. Idea of the method

The proof uses the method of "ghost weighted" energy inequalities. This means that we proceed as in the usual energy inequality by computing in a strip

$$\int (\exp p)\, Pv\partial_t v dx dt,$$

and determine a *bounded* p such that the quadratic form Q_1 in ∂v obtained by integration by parts is non negative (up to easily handled terms). Since p is bounded, it will be eventually eliminated from the inequality, yielding exactly the same inequality as for the unperturbed wave equation.

More precisely, the coefficients of Q_1 are sums of terms either linear in ∂p, or linear in $\partial^2 u$, or bilinear in $\partial u \partial^2 u$; moreover,

i) All second order derivatives of u can be expressed in terms of $\partial_t^2 u$ modulo integrable terms,
ii) $\partial_t^2 u$ can be bounded by $C\epsilon(1+t)^{-1/2}(1+|\sigma|)^{-3/2}$.

The second point suggests to use the simple weight

$$p = b(\sigma) - \theta(t), b' > 0, \theta' > 0.$$

On the other hand, the effect of a weight such as $b(\sigma)$ is to make appear in Q_1 the expression

$$b'\left[(\partial_1 v + \omega_1 \partial_t v)^2 + (\partial_2 v + \omega_2 \partial_t v)^2\right].$$

This means a (rather weak) control of the terms

$$Z_j v, Z_j = t\partial_j + x_j \partial_t, \ j = 1, 2.$$

In other words, it displays the fact that free solutions behave essentially like functions of $r-t$. This fact had already been observed and used in [2]. What is hidden here is that, thanks to the null condition(s), $r-t$ is a very good approximate phase function for the linearized operator on u.

The first point allows us to collect the terms of Q_1 linear in $\partial^2 u$ (resp. bilinear in $\partial u \partial^2 u$) as a single term containing $\partial_t^2 u$ as a factor (resp. a single term containing $\partial_t u \partial_t^2 u$ as a factor). Rearranging Q_1 as a quadratic form in

$$\partial_t v, \partial_1 v + \omega_1 \partial_t v, \partial_2 v + \omega_2 \partial_t v,$$

one sees *both* null conditions appear in the computation, causing some coefficients in Q_1 to cancel. It turns out that, thanks to this null condition, one can take for θ' either zero (Theorems 2 and 2′) or some integrable function of t (Theorem 3). The decay of $\partial_t^2 u$ towards the interior of the light cone makes it possible to take b' integrable, hence b is bounded and so is p.

3.3. Proof of Theorems 2 and 2′

To avoid heavy notations, we write Z generically for one of the fields (1.7), and, in inequalities, $|Zu|$ stands for $\Sigma|Zu|$ over all such fields, $|Z^\alpha u|$ stands for

$$\Sigma_{|\beta|\le|\alpha|}|Z^\beta u|,$$

and so on. The important thing is to distinguish for instance $Z^\alpha u$ from $Z^{\alpha-1}\partial u$, since the first term may not contain any "true" derivative $\partial_j u$.

Remark that if we replace u_a by u, the operator P_a becomes P; hence we give the proof for P_a, and observe that it works identically if we replace u_a by u. In case I, it is understood that (according to the hypothesis in Theorem 2) the induction hypothesis is true for $0 \le t < T$, for some $T > 0$, $\epsilon^2 \log(1+T) \le \tau_0$. In case II, no such limitation occurs.

Step 1: Perform the usual integrations by parts

We have

$$\begin{aligned}2e^p Pv\partial_t v = \partial_t\big\{e^p[(1+g_{00})(\partial_t v)^2 + (1-g_{11})(\partial_1 v)^2 \\ + (1-g_{22})(\partial_2 v)^2 - g_{12}(\partial_1 v)(\partial_2 v)]\big\} \\ + \partial_1\big\{e^p[g_{01}(\partial_t v)^2 - 2(\partial_t v)(\partial_1 v)(1-g_{11}) + g_{12}(\partial_t v)(\partial_2 v)]\big\} \\ + \partial_2\big\{e^p[g_{o2}(\partial_t v)^2 + g_{12}(\partial_t v)(\partial_1 v) - 2(\partial_t v)(\partial_2 v)(1-g_{22})]\big\} \\ + Q_1,\end{aligned}$$

where Q_1 denotes the quadratic terms in ∂v,

$$\begin{aligned}Q_1 = K_0(\partial_t v)^2 + K_1(\partial_1 v)^2 + K_2(\partial_2 v)^2 + 2H_1(\partial_t v)(\partial_1 v) \\ + 2H_2(\partial_t v)(\partial_2 v) + 2H_0(\partial_1 v)(\partial_2 v).\end{aligned}$$

Here,

$$\begin{aligned}K_0 &= -\partial_t\big(e^p(1+g_{00})\big) - \partial_1\big(e^p g_{01}\big) - \partial_2\big(e^p g_{02}\big) + 2e^p c_a^0,\\ K_1 &= -\partial_t\big(e^p(1-g_{11})\big),\ K_2 = -\partial_t\big(e^p(1-g_{22})\big),\\ H_1 &= -1/2\partial_2\big(e^p g_{12}\big) + \partial_1\big(e^p(1-g_{11})\big) + e^p c_a^1,\\ H_2 &= -1/2\partial_1\big(e^p g_{12}\big) + \partial_2\big(e^p(1-g_{22})\big) + e^p c_a^2,\\ H_0 &= 1/2\partial_t\big(e^p g_{12}\big).\end{aligned}$$

Integrating in the strip $0 \le t \le T_1 < T$, the divergence terms yield control of the usual energy at time T_1.

Step 2: Simplify terms modulo $Z\partial u$, $Z\partial u_a$

We set $\bar{g}^k = \Sigma g_{ij}^k \omega_i \omega_j$. Using for clarity the notation D_k to denote the derivatives of the functions g_{ij} with respect to $\partial_k u$, we write

$$\begin{aligned}e^{-p}K_0 = &-(1+g_{00})\partial_t p - g_{01}\partial_1 p - g_{02}\partial_2 p\\ &+ \big(\partial_t^2 u\big)[\Sigma D_k g_{00}\omega_k - \Sigma D_k g_{01}\omega_1\omega_k - \Sigma D_k g_{02}\omega_2\omega_k]\\ &+ 2\big(\partial_t^2 u_a\big)\bar{g}^0 - \Sigma D_k g_{00}\big(\partial_t\partial_k u + \omega_k\partial_t^2 u\big)\\ &- \Sigma D_k g_{01}\big(\partial_1\partial_k u - \omega_1\omega_k\partial_t^2 u\big)\\ &- \Sigma D_k g_{02}\big(\partial_2\partial_k u - \omega_2\omega_k\partial_t^2 u\big) + 2\Sigma g_{ij}^0\big(\partial_{ij}^2 u_a - \omega_i\omega_j\partial_t^2 u_a\big),\\ e^{-p}K_1 = &-\partial_t p(1-g_{11}) - \big(\partial_t^2 u\big)\Sigma D_k g_{11}\omega_k + \Sigma D_k g_{11}\big(\partial_t\partial_k u + \omega_k\partial_t^2 u\big),\\ e^{-p}K_2 = &-\partial_t p(1-g_{22}) - \big(\partial_t^2 u\big)\Sigma D_k g_{22}\omega_k + \Sigma D_k g_{22}\big(\partial_t\partial_k u + \omega_k\partial_t^2 u\big),\\ e^{-p}H_1 = &-\frac{1}{2}g_{12}\partial_2 p + (1-g_{11})\partial_1 p\\ &- \big(\partial_t^2 u\big)[1/2\Sigma D_k g_{12}\omega_2\omega_k + \Sigma D_k g_{11}\omega_1\omega_k] + \bar{g}^1\big(\partial_t^2 u_a\big)\\ &- 1/2\Sigma D_k g_{12}\big(\partial_k\partial_2 u - \omega_2\omega_k\partial_t^2 u\big) - \Sigma D_k g_{11}\big(\partial_k\partial_1 u - \omega_1\omega_k\partial_t^2 u\big)\\ &+ \Sigma g_{ij}^1\big(\partial_{ij}^2 u_a - \omega_i\omega_j\partial_t^2 u_a\big),\end{aligned}$$

$$
\begin{aligned}
e^{-p}H_2 = &-\frac{1}{2}g_{12}\partial_1 p + (1-g_{22})\partial_2 p\\
&-\left(\partial_t^2 u\right)[1/2\Sigma D_k g_{12}\omega_1\omega_k + \Sigma D_k g_{22}\omega_2\omega_k] + \bar{g}^2\left(\partial_t^2 u_a\right)\\
&-1/2\Sigma D_k g_{12}\left(\partial_k\partial_1 u - \omega_1\omega_k\partial_t^2 u\right) - \Sigma D_k g_{22}\left(\partial_k\partial_2 u - \omega_k\omega_2\partial_t^2 u\right)\\
&+\Sigma g_{ij}^2\left(\partial_{ij}^2 u_a - \omega_i\omega_j\partial_t^2 u_a\right),\\
e^{-p}H_0 = &\; 1/2 g_{12}\partial_t p - 1/2\left(\partial_t^2 u\right)\Sigma D_k g_{12}\omega_k + 1/2\Sigma D_k g_{12}\left(\partial_k\partial_t u + \omega_k\partial_t^2 u\right).
\end{aligned}
$$

This allows us to simplify the terms modulo terms easily handled.

a. First, $D_k g_{ij} = g_{ij}^k + O(\partial u)$. Since, according to the induction hypothesis,

$$|\partial u|\left|\partial_t^2 u\right| \le C\epsilon^2(1+t)^{-1},$$

we can replace everywhere $D_k g_{ij}$ by g_{ij}^k in the terms containing $(\partial_t^2 u)$ as a factor, modulo terms which are easily handled by Gronwall inequality, since

$$\int_0^{T_1} \epsilon^2(1+t)^{-1}dt \le \epsilon^2 \log(1+T_1).$$

b. Second, we can write, for $k = 1, 2$

$$\partial_k w + \omega_k\partial_t w = t^{-1}Z_k w - \omega_k t^{-1}\sigma\partial_t w,\; Z_k = t\partial_k + x_k\partial_t.$$

Applying this with $w = \partial u$, we obtain

$$\partial_k\partial_t u + \omega_k\partial_t^2 u = t^{-1}Z_k\partial_t u - \omega_k t^{-1}\sigma\partial_t^2 u,$$
$$\partial_k\partial_1 u - \omega_k\omega_1\partial_t^2 u = t^{-1}(Z_k\partial_1 u - \omega_k Z_1\partial_t u) - \omega_k t^{-1}\sigma\left(\partial_t\partial_1 u + \omega_1\partial_t^2 u\right),$$

and similarly for $\partial_k\partial_2 u$. From (2.3.4) and the induction hypothesis, we get

$$|\partial^2 u| \le C(1+|\sigma|)^{-1}|Z\partial u| \le C\epsilon(1+t)^{-1/2}(1+|\sigma|)^{-1},$$

hence

$$\left|\partial_k\partial_t u + \omega_k\partial_t^2 u\right| + \left|\partial_k\partial_1 u - \omega_k\omega_1\partial_t^2 u\right| \le C\epsilon(1+t)^{-3/2}.$$

We proceed similarly with the second order derivatives of u_a. Hence the corresponding terms are integrable.

We now write $Q_1 = Q_2 + \ldots$, where the dots denote the form with the coefficients handled in a. and b. For simplicity, we set

$$
\begin{aligned}
A_0 &= \Sigma\left(g_{00}^k\omega_k - g_{01}^k\omega_1\omega_k - g_{02}^k\omega_2\omega_k\right),\\
A_1 &= \Sigma\left(1/2 g_{12}^k\omega_2\omega_k + g_{11}^k\omega_1\omega_k\right),\\
A_2 &= \Sigma\left(1/2 g_{12}^k\omega_1\omega_k + g_{22}^k\omega_2\omega_k\right).
\end{aligned}
$$

Step 3: Write Q_2 in terms of $\partial_t v$, $\partial_1 v + \omega_1\partial_t v$, $\partial_2 v + \omega_2\partial_t v$

By definition of Q_2 and the choice $p = b(\sigma) - \theta(t)$, we have

$$
\begin{aligned}
e^{-p}Q_2 = {} & b'\big[(\partial_t v)^2 + (\partial_1 v)^2 + (\partial_2 v)^2 + 2\omega_1(\partial_1 v)(\partial_t v) + 2\omega_2(\partial_2 v)(\partial_t v)\big] \\
& + (\partial_t v)^2\big[b'(g_{00} - \omega_1 g_{01} - \omega_2 g_{02}) + \theta'(1 + g_{00}) \\
& \qquad + A_0\partial_t^2 u + 2\bar{g}^0\partial_t^2 u_a\big] \\
& + (\partial_1 v)^2\big[-b'g_{11} + \theta'(1 - g_{11}) - \big(\partial_t^2 u\big)\Sigma g_{11}^k\omega_k\big] \\
& + (\partial_2 v)^2\big[-b'g_{22} + \theta'(1 - g_{22}) + \big(\partial_t^2 u\big)\Sigma g_{22}^k\omega_k\big] \\
& + 2(\partial_1 v)(\partial_t v)\big[-b'(1/2\omega_2 g_{12} + \omega_1 g_{11}) - A_1\partial_t^2 u + \bar{g}^1\partial_t^2 u_a\big] \\
& + 2(\partial_2 v)(\partial_t v)\big[-b'(1/2\omega_1 g_{12} + \omega_2 g_{22}) - A_2\partial_t^2 u + \bar{g}^2\partial_t^2 u_a\big] \\
& + 2(\partial_1 v)(\partial_2 v)\big[-1/2(b' + \theta')g_{12} - 1/2\big(\partial_t^2 u\big)\Sigma g_{12}^k\omega_k\big].
\end{aligned}
$$

The coefficient of b' in the first term is just $X_1^2 + X_2^2$, where

$$X_j = \partial_j v + \omega_j\partial_t v,\ j = 1, 2.$$

This is the effect of a weight function of $r - t$ for the wave equation.

With

$$\partial_j v = X_j - \omega_j\partial_t v,$$

we rewrite now Q_2:

$$
\begin{aligned}
e^{-p}Q_2 = {} & X_1^2\big[(b' + \theta')(1 - g_{11}) - \big(\partial_t^2 u\big)\Sigma g_{11}^k\omega_k\big] + X_2^2\big[(b' + \theta')(1 - g_{22}) \\
& - \big(\partial_t^2 u\big)\Sigma g_{22}^k\omega_k\big] + 2X_1(\partial_t v)\big[-\omega_1\theta'(1 - g_{11}) + 1/2\omega_2\theta' g_{12} \\
& + \big(\partial_t^2 u\big)\big(-A_1 + \omega_1\Sigma g_{11}^k\omega_k + 1/2\omega_2\Sigma g_{12}^k\omega_k\big) + \bar{g}^1\partial_t^2 u_a\big] \\
& + 2X_2(\partial_t v)\big[-\omega_2\theta'(1 - g_{22}) + 1/2\omega_1\theta' g_{12} \\
& + \big(\partial_t^2 u\big)\big(-A_2 + \omega_2\Sigma g_{22}^k\omega_k + 1/2\omega_1\Sigma g_{12}^k\omega_k\big) + \bar{g}^2\partial_t^2 u_a\big] \\
& + \bar{K}(\partial_t v)^2 + 2X_1X_2\big[-1/2(b' + \theta')g_{12} - 1/2\big(\partial_t^2 u\big)\Sigma g_{12}^k\omega_k\big].
\end{aligned}
$$

The coefficient $\bar{K}$ of $(\partial_t v)^2$ deserves special attention:

$$
\begin{aligned}
\bar{K} = {} & \theta'(2 + A_3) + b'\big[g_{00} - \omega_1 g_{01} - \omega_2 g_{02} + \omega_1^2 g_{11} + \omega_2^2 g_{22} + \omega_1\omega_2 g_{12}\big] \\
& + \big(\partial_t^2 u\big)\big[A_0 + 2\omega_1 A_1 + 2\omega_2 A_2 - \omega_1^2\Sigma g_{11}^k\omega_k - \omega_2^2\Sigma g_{22}^k\omega_k \\
& - \omega_1\omega_2\Sigma g_{12}^k\omega_k\big] - 2\big(\partial_t^2 u_a\big)\big(-\bar{g}^0 + \omega_1\bar{g}^1 + \omega_2\bar{g}^2\big),
\end{aligned}
$$

with

$$A_3 = g_{00} - \omega_1^2 g_{11} - \omega_2^2 g_{22} - \omega_1\omega_2 g_{12}.$$

First, we remark that the coefficient of $\partial_t^2 u$ in $\bar{K}$ is just $g(\omega) \equiv 0$. Second, the coefficient of $-2\partial_t^2 u_a$ is also $g(\omega) \equiv 0$. Third, in the coefficient of b', we write

$$g_{ij}(\partial u) = \Sigma g_{ij}^k\partial_k u + O((\partial u)^2)$$

and use again $\partial_k u+\omega_k\partial_t u = t^{-1}Z_k u-\omega_k t^{-1}\sigma\partial_t u$. The induction hypothesis implies now

$$|(\partial u)^2| \le C\epsilon^2(1+t)^{-1},$$

and, by integration,

$$|Z\dot{u}| \le C\epsilon^{3/2}(1+t)^{-1/2}(1+|\sigma|)^{1/2}.$$

Since we will eventually choose $b' \le C(1+|\sigma|)^{-1}$, we get

$$b' g_{ij}(\partial u) = -(\partial_t u) b' \Sigma g_{ij}^k \omega_k + r$$

with

$$\begin{aligned}|r| &\le b' O(|\partial u|^2) + b'(1+t)^{-1}(|Zu| + (1+|\sigma|)|\partial u|)\\ &\le C\epsilon(1+t)^{-3/2} + C\epsilon^2(1+t)^{-1}.\end{aligned}$$

Hence the corresponding terms in $(\partial_t v)^2$ are handled by Gronwall's Lemma. The remaining part in the coefficient of b' is just $-(\partial_t u)g(\omega) \equiv 0$.

For the coefficient of $X_j\partial_t v$, we remark that the terms containing $\partial_t^2 u$ cancel. Finally, we have obtained

$$e^{-p}Q_2 = e^{-p}Q_3 + \dots,$$

where the dots denote the terms we have handled wia Gronwall's Lemma and

$$\begin{aligned}e^{-p}Q_3 = {} & X_1^2\big[(b'+\theta')(1-g_{11}) - \big(\partial_t^2 u\big)\Sigma g_{11}^k\omega_k\big] + X_2^2\big[(b'+\theta')(1-g_{22})\\ & - \big(\partial_t^2 u\big)\Sigma g_{22}^k\omega_k\big] + 2X_1(\partial_t v)\big[\theta'(-\omega_1(1-g_{11}) + 1/2\omega_2 g_{12})\\ & + \bar{g}^1\partial_t^2 u_a\big] + 2X_2(\partial_t v)\big[\theta'(-\omega_2(1-g_{22}) + 1/2\omega_1 g_{12}) + \bar{g}^2\partial_t^2 u_a\big]\\ & + 2X_1X_2\big[-1/2(b'+\theta')g_{12} - 1/2\big(\partial_t^2 u\big)\Sigma g_{12}^k\omega_k\big]\\ & + \theta'(2+A_3)(\partial_t v)^2.\end{aligned}$$

Step 4: Show the non negativity of Q_3 for suitable p

This is a straightforward step. For the proofs of Theorems 2 and 2′, we can take $\theta' = 0$. We have then

$$\begin{aligned}e^{-p}Q_3 = {} & b'\big[X_1^2(1-g_{11}) + X_2^2(1-g_{22}) - g_{12}X_1X_2\big] + \big(\partial_t^2 u\big)q\\ & + 2\big(\partial_t^2 u_a\big)\partial_t v\big(\bar{g}^1 X_1 + \bar{g}^2 X_2\big),\end{aligned}$$

where $q = q(X)$ is a quadratic form with bounded coefficients. Using

$$\big|\partial_t^2 u\big| + \big|\partial_t^2 u_a\big| \le C\epsilon(1+t)^{-1/2}(1+|\sigma|)^{-3/2},$$

we can bound the last two terms in $e^{-p}Q_3$ by

$$C\epsilon^2(1+t)^{-1}\big(X_1^2 + X_2^2 + (\partial_t v)^2\big) + (1+|\sigma|)^{-3}\big(X_1^2+X_2^2\big).$$

Hence it is enough to take $b' = 2(1+|\sigma|)^{-3}$. ◇

Remark. In case $\bar{g}^1 = \bar{g}^2 = 0$ (for instance, in the cubic case), we write

$$\begin{aligned} e^{-p} Q_3 = b'\big[X_1^2(1-g_{11}) + X_2^2(1-g_{22}) - g_{12}X_1X_2\big] + \big(\partial_t^2 u\big)q(X) \\ + \theta'\big[X_1^2(1-g_{11}) + X_2^2(1-g_{22}) - g_{12}X_1X_2 - 2\alpha_1 X_1 v_t \\ - 2\alpha_2 X_2 v_t + v_t^2(2+A_3)\big], \end{aligned}$$

with

$$\alpha_1 = \omega_1(1-g_{11}) - \frac{1}{2} g_{12}\omega_2$$

and similarly for α_2. Noting that $\alpha_1^2 + \alpha_2^2 = 1 + o(1)$, and writing for instance

$$\big|\partial_t^2 u\big| \le C(1+t)^{-1/2}(1+|\sigma|)^{-3/2} \le C\epsilon^{5/2}(1+t)^{-5/4} + C(1+|\sigma|)^{-5/2},$$

we obtain $Q_3 \gg 0$ for

$$b' = B(1+|\sigma|)^{-5/2}, \theta' = B(1+t)^{-5/4}$$

with B big enough. If we are in case II, we then obtain a global energy inequality without amplification factor, as in Theorem 3.

4. Proof of Theorem 1 (case I)

4.1. Proof of Theorem 1

a. A localisation Lemma

Before proceeding, we need to recall a simple Lemma (see [3], Lemma 2.2, p. 637).

Lemma 4.1.1. *Let $w(x)$ be a C^1 function supported in $|x| \le R$. Then, for $m \ne 1$,*

$$|w(1+|R-r|)^{-m}|_{L^2} \le C|\partial_r w|_{L^2} R^{(1-m)_+}.$$

b. Proof of Theorem 1

We write the equation on u in the form

$$\big(\partial_t^2 - \Delta\big)\dot{u} + \Sigma g_{ij}(\partial u)\partial_{ij}^2 \dot{u} + \Sigma\big(\partial_{ij}^2 u_a\big)[g_{ij}(\partial u_a + \partial \dot{u}) - g_{ij}(\partial u_a)] = -J_a,$$

where, as in Sect. 2,

$$J_a = \big(\partial_t^2 - \Delta\big)u_a + \Sigma g_{ij}(\partial u_a)\partial_{ij}^2 u_a.$$

Thus, to control $\partial\dot{u}$, it is enough to use the energy inequality for P_a. When we apply Z^α $(0 < |\alpha| \leq 2s_0 - 1)$ to the sum $\Sigma g_{ij}^k \partial_k u \partial_{ij}^2 \dot{u}$, we have to separate the two extreme terms:

$$Z^\alpha \Sigma g_{ij}^k \partial_k u \partial_{ij}^2 \dot{u} = \Sigma g_{ij}^k \partial_k u \partial_{ij}^2 Z^\alpha \dot{u} + \Sigma g_{ij}^k \partial_k Z^\alpha u \partial_{ij}^2 \dot{u} + \Sigma_{|\beta|,|\gamma| \leq |\alpha|-1} H(Z^\beta u, Z^\gamma \dot{u}),$$

where here and in the sequence $H(v, w)$ denotes various bilinear forms analogous to $\Sigma g_{ij}^k \partial_k v \partial_{ij}^2 w$, satisfying the null condition, according to Lemma 2.3.2. We see that we have to incorporate the first two terms of the right-hand side to the operator, hence use the inequality for P. We obtain, with $g_{ij}(\xi) = \Sigma g_{ij}^k \xi_k + \tilde{g}_{ij}(\xi)$,

$$(4.1.9) \quad \Sigma_{|\beta| \leq |\alpha|} * Z^\beta J_a = P(Z^\alpha \dot{u}) + \Sigma g_{ij}^k \partial_k Z^\alpha u_a \partial_{ij}^2 \dot{u} + \Sigma_{|\beta|,|\gamma| \leq |\alpha|-1} H(Z^\beta u, Z^\gamma \dot{u}) + \Sigma_{|\beta| \leq |\alpha|-1} H\big(Z^\beta \dot{u}, Z^\gamma u_a\big) + \Sigma_{|\gamma| \leq |\alpha|-1} * \big(Z^\beta \tilde{g}_{ij}(\partial u)\big) Z^\gamma \partial^2 \dot{u} + \Sigma * Z^\beta \partial_{ij}^2 u_a Z^\gamma [\tilde{g}_{ij}(\partial u) - \tilde{g}_{ij}(\partial u_a)].$$

Here the $*$ simply denote irrelevant numerical coefficients.

a. The terms containing $\tilde{g}_{ij}$ are easily handled. In fact,

$$\tilde{g}_{ij}(\partial u) - \tilde{g}_{ij}(\partial u_a) = [\int_0^1 \tilde{g}_{ij}'(s\partial u + (1-s)\partial u_a) ds] \partial\dot{u}.$$

When applying Z^α to this difference, we obtain a sum of products in which at most one term contains $Z^\beta \dot{u}$ for $|\beta| \geq s_0$. All other terms are bounded by $C\epsilon(1+t)^{-1/2}$. Hence the last term in the right-hand side is bounded by

$$C\epsilon^2(1+t)^{-1} \Sigma_{|\beta| \leq |\alpha|} |Z^\beta \partial\dot{u}|.$$

Writing

$$\tilde{g}_{ij}(\partial u) = \tilde{g}_{ij}(\partial u) - \tilde{g}_{ij}(\partial u_a) + \tilde{g}_{ij}(\partial u_a),$$

we get the same estimate for the previous term too.

b. From Lemma 2.3.1, we get

$$\left|H\big(Z^\alpha u_a, \dot{u}\big)\right| \leq C(1+t)^{-1}\left|Z^{\alpha+1} u_a\right| |Z\partial\dot{u}| \leq C\epsilon(1+t)^{-3/2} |Z\partial\dot{u}|.$$

For a term of the second sum in the right-hand side of (4.1.2), we get

$$|H(Z^\beta u, Z^\gamma \dot{u})| \leq C(1+t)^{-1}(|Z^{\beta+1} u||\partial^2 Z^\gamma \dot{u}| + |Z^\beta \partial u||Z^{\gamma+1} \partial\dot{u}|).$$

If $|\beta| \leq s_0 - 1$,

$$|Z^{\beta+1} \partial\dot{u}| \leq C\epsilon^{3/2}(1+t)^{-1/2}(1+|\sigma|)^{-1/2},$$

hence

$$|Z^{\beta+1}\dot{u}| \leq C\epsilon^{3/2}(1+t)^{-1/2}(1+|\sigma|)^{1/2},$$

while $|Z^{\beta+1}u_a| \leq C\epsilon(1+t)^{-1/2}$. In this case,

$$|Z^{\beta+1}u||\partial^2 Z^\gamma \dot{u}| \leq C\epsilon(1+t)^{-1/2}|Z^{\gamma+1}\partial\dot{u}|,$$
$$|H|_{L^2} \leq C\epsilon(1+t)^{-3/2}|Z^\alpha \partial\dot{u}|_{L^2}.$$

If $|\gamma| \leq s_0 - 1$,

$$|Z^{\gamma+1}\partial\dot{u}| \leq C\epsilon^{3/2}(1+t)^{-1/2}(1+|\sigma|)^{-1/2},$$
$$|\partial^2 Z^\gamma \dot{u}| \leq C\epsilon^{3/2}(1+t)^{-1/2}(1+|\sigma|)^{-3/2},$$

hence, by Lemma 4.1.1,

$$\begin{aligned}|(Z^{\beta+1}\dot{u})(\partial^2 Z^\gamma \partial\dot{u})|_{L^2} &\leq C\epsilon^{3/2}(1+t)^{-1/2}|(1+|\sigma|)^{-3/2}Z^{\beta+1}\dot{u}|_{L^2}\\ &\leq C\epsilon^{3/2}(1+t)^{-1/2}|Z^{\beta+1}\partial\dot{u}|_{L^2}.\end{aligned}$$

In both cases, the terms $|Z^\beta \partial u||Z^{\gamma+1}\partial\dot{u}|$ can be handled similarly (and more easily). Thus, all the terms of the second sum can be handled by Gronwall's Lemma.

Applying Lemma 2.3.1 to a term of the third sum, we get

$$\left|H\left(Z^\beta \dot{u}, Z^\gamma u_a\right)\right| \leq C(1+t)^{-1}|Z^{\beta+1}\dot{u}|\left|Z^{\gamma+1}\partial u_a\right|.$$

Since, for instance, $|Z^{\gamma+1}\partial u_a| \leq C\epsilon(1+t)^{-1/2}(1+|\sigma|)^{-5/4}$, we have, by Lemma 4.1.1 with $m = 5/4$,

$$\left|(Z^{\beta+1}\dot{u})\left(Z^{\gamma+1}\partial u_a\right)\right|_{L^2} \leq C\epsilon(1+t)^{-1/2}|Z^{\beta+1}\partial\dot{u}|_{L^2},$$

and the term H is handled via Gronwall's Lemma.

c. Since

$$\dot{u}(x,0) = O(\epsilon^2),\ \partial_t \dot{u}(x,0) = O(\epsilon^2)$$

and $J_a^I = O(\epsilon^2)$, the inequality (3.1.1) is certainly true for small ϵ and finite $T > 0$. We use now (3.1.2) for P_a to get a control of $\partial\dot{u}$. For $0 < |\alpha| \leq s_0$, we use (3.1.2) for P and the properties of J_a to get finally

$$\Sigma_{|\alpha|\leq 2s_0-1}|Z^\alpha \partial\dot{u}(.,t)|_{L^2} \leq C\epsilon^2|\log\epsilon|.$$

This implies, if $s_0 \geq 3$,

$$\Sigma_{|\gamma|\leq s_0}|Z^\gamma \partial\dot{u}(.,t)|_{L^\infty} \leq C\epsilon^2|\log\epsilon|(1+t)^{-1/2}(1+|\sigma|)^{-1/2},$$

which implies (3.1.1) with $\frac{1}{2}\epsilon^{3/2}$ instead of $\epsilon^{3/2}$ if ϵ is small enough. ◇

4.2. Theorem 4 (case I)

We have in fact proved above a quasi-global approximation theorem about the approximate solution u_a. We state it here for completeness.

Theorem 4.2.2. *For any fixed* $0 < \tau_0 < \bar{\tau}$, $s_0 \geq 3$, *let* u_a *be the approximate solution constructed in Sect. 2. Then there exists* $\epsilon_0 > 0$ *such that for* $\epsilon \leq \epsilon_0$, *the solution* u *exists for* $\epsilon^2 \log(1+t) \leq \tau_0$ *and*

$$\Sigma_{|\alpha| \leq 2s_0 - 1} |Z^\alpha \partial \dot{u}(., t)|_{L^2} \leq C\epsilon^2 |\log \epsilon|.$$

Remark. If one uses the better approximate solution u_a constructed in [8], one obtains exactly the same theorem with the corresponding improved approximation, showing that u_a is relevant.

5. Proof of Theorem 1′ (case II)

5.1. Proof of Theorem 1′

According to Theorem 2′, exactly the same proof as in 4.1 yields

$$\Sigma_{|\alpha| \leq 2s_0 - 1} |Z^\alpha \partial \dot{u}(., t)|_{L^2} \leq C\epsilon^2 (1+t)^{C\epsilon^2}. \tag{5.1.1}$$

We want to apply now Z^β to (4.1.1), for $|\beta| \leq 2s_1 - 1$, $s_1 \leq s_0$. In order to get a precise estimate, we have to make explicit the cubic terms. With

$$g_{ij}(\xi) = g_{ij}^k \xi_k + h_{ij}^{kl} \xi_k \xi_l + r_{ij}(\xi), \quad r_{ij}(\xi) = O(|\xi|^3),$$

we rewrite (4.1.1) in the form

$$\begin{aligned} -J_a = \left(\partial_t^2 - \Delta\right)\dot{u} &+ \Sigma\left[g_{ij}^k \partial_k u + h_{ij}^{kl} \partial_k u \partial_l u + r_{ij}(\partial u)\right]\partial_{ij}^2 \dot{u} \\ &+ \Sigma\left(\partial_{ij}^2 u_a\right)\left[g_{ij}^k \partial_k \dot{u} + h_{ij}^{kl}(\partial_k u \partial_l \dot{u} + \partial_k \dot{u} \partial_l u_a)\right. \\ &\left. + r_{ij}(\partial u_a + \partial \dot{u}) - r_{ij}(\partial u_a)\right]. \end{aligned} \tag{5.1.2}$$

Applying repeatedly Lemma 2.3.2, we denote by G various trilinear forms analogous to

$$G(v, w, z) = \Sigma h_{ij}^{kl} \partial_k v \partial_l w \partial_{ij}^2 z,$$

satisfying the null condition. We obtain from (5.1.2), with

$$|\gamma| + |\delta| \leq |\beta|, |\gamma| + |\delta| + |\nu| \leq |\beta|,$$

$$\begin{aligned} \left(\partial_t^2 - \Delta\right)Z^\beta \dot{u} &+ \Sigma H(Z^\gamma u, Z^\delta \dot{u}) + \Sigma H(Z^\gamma \dot{u}, Z^\delta u_a) \\ &+ \Sigma G(Z^\gamma u, Z^\delta u, Z^\nu \dot{u}) + G\left(Z^\gamma u, Z^\delta \dot{u}, Z^\nu u_a\right) \\ &+ \Sigma G\left(Z^\gamma \dot{u}, Z^\delta u_a, Z^\nu u_a\right) + \Sigma * Z^\gamma r_{ij}(\partial u) Z^\delta \partial_{ij}^2 \dot{u} \\ &+ \Sigma * Z^\gamma \partial_{ij}^2 u_a Z^\delta [r_{ij}(\partial u) - r_{ij}(\partial u_a)] = -Z^\beta J_a. \end{aligned} \tag{5.1.3}$$

a. Since $|Z^{\gamma'}u_a| \le C\epsilon(1+t)^{-1/2}$, we get

$$\left|H\left(Z^\gamma u_a, Z^\delta \dot{u}\right)\right|_{L^2} \le C\epsilon^3(1+t)^{-3/2+C\epsilon^2}$$

if $s_1 \le s_0 - 1$.

b. Consider a term $H(Z^\gamma \dot{u}, Z^\delta \dot{u})$ from the first sum of the left-hand side of (5.1.3). It is bounded according to (2.3.1). If $|\gamma| \le s_0 - 1$,

$$|Z^{\gamma+1}\dot{u}| \le C\epsilon^{3/2}(1+t)^{-1/2}(1+|\sigma|)^{1/2},$$

while

$$|\partial^2 Z^\delta \dot{u}| \le C(1+|\sigma|)^{-1}|Z^{\delta+1}\partial\dot{u}|. \tag{5.1.4}$$

Provided $s_1 \le s_0 - 1$, the L^2 norm of the product is bounded by

$$C\epsilon^{7/2}(1+t)^{-1/2+C\epsilon^2}.$$

The other type of terms in the estimate (2.3.1) of H is easily bounded by the same number, hence

$$|H|_{L^2} \le C\epsilon^{7/2}(1+t)^{-3/2+C\epsilon^2}.$$

If $|\delta| \le s_0 - 1$, terms bounded by $(1+t)^{-1}|\partial Z^\gamma \dot{u}||Z^{\delta+1}\partial\dot{u}|$ are easily handled as before. Finally, using (5.1.1) again and Lemma (4.1.1), we obtain

$$\begin{aligned} |Z^{\gamma+1}\dot{u}||\partial^2 Z^\delta \dot{u}|_{L^2} &\le C\epsilon^{3/2}(1+t)^{-1/2}|(1+|\sigma|)^{-3/2}Z^{\gamma+1}\dot{u}|_{L^2} \\ &\le C\epsilon^{7/2}(1+t)^{-1/2+C\epsilon^2}. \end{aligned}$$

In conclusion,

$$|H|_{L^2} \le C\epsilon^{7/2}(1+t)^{-3/2+C\epsilon^2}.$$

c. The terms of the second sum are handled exactly as in 4.1.

d. We use now Lemma 2.3.1 to handle the trilinear terms. A typical term would be for instance one bounded by

$$(1+t)^{-1}|Z^\gamma \partial\dot{u}||Z^{\delta+1}\dot{u}||\partial^2 Z^\nu \dot{u}|.$$

Proceeding as before, we can bound the L^2 norm by

$$C\epsilon^5(1+t)^{-2+C\epsilon^2}.$$

All similar terms are at least as easy to handle, $\dot{u}$ being replaced by u_a in some factors.

e. Finally, the terms involving r_{ij} are handled exactly as the cubic terms in 4.1. They are all bounded in L^2 norm by

$$C\epsilon^5(1+t)^{-3/2+C\epsilon^2}.$$

f. Applying the usual energy inequality for the wave equation to (5.1.3), we get, for $|\beta| \leq 2s_0 - 3$ and small ϵ,

$$|Z^\beta \partial \dot{u}(., t)|_{L^2} \leq C\epsilon^2.$$

If $s_0 \geq 5$, this implies (3.1.1) with $\frac{1}{2}\epsilon^{3/2}$ instead of $\epsilon^{3/2}$ for ϵ small enough and completes the proof of Theorem 1′. ◇

5.2. Theorem 4′

We have in fact proved above a global approximation theorem. We state it here for completeness.

Theorem 4′ (case II). *For any fixed $s_0 \geq 5$, let u_a be the approximate solution contructed in Sect. 2. Then there exists $\epsilon_0 > 0$ such that for $\epsilon \leq \epsilon_0$, the solution u exists globally and*

$$\Sigma_{|\alpha| \leq 2s_0 - 3} |Z^\alpha \partial \dot{u}(., t)|_{L^2} \leq C\epsilon^2.$$

Remark. As for Theorem 4, one can prove that the better approximation u_a contructed in [8] is relevant, with corresponding estimates of the remainder $\dot{u}$.

6. Proof of Theorem 3

We go back to the proof in Sect. 3.3, where formally we take $u_a \equiv 0$, and briefly discuss the minor changes to make. We use the notations

$$\bar{g}_{ij} = \Sigma g_{ij}^k \omega_k, \ \bar{g}^k = \Sigma g_{ij}^k \omega_i \omega_j,$$

and so on, the bar indicating "contraction" with respect to ω of the missing indexes.

The first Step 1 is identical. In Step 2, we have to improve the estimate on $D_k g_{ij}$. We write

$$(D_k g_{ij})(\partial w) = g_{ij}^k - 2(\partial_t w)\big(\Sigma h_{ij}^{kl} \omega_l\big) + 2\Sigma h_{ij}^{kl}(\partial_l w + \omega_l \partial_t w) + O((\partial w)^2).$$

Since

$$|\partial_k w + \omega_k \partial_t w| \leq C(1+t)^{-1}(|Zw| + (1+|\sigma|)|\partial w|),$$
$$|Zw| \leq C(1+t)^{-\eta}(1+|\sigma|),$$
$$|\partial^2 w| \leq C(1+t)^{-\eta}(1+|\sigma|)^{-1},$$

we have

$$\big(\partial_t^2 w\big) D_k g_{ij} = \big(\partial_t^2 w\big) g_{ij}^k - 2\big(\partial_t^2 w\big)(\partial_t w)\bar{h}_{ij}^k + r,$$

with

$$|r| \le C(1+t)^{-3\eta} + C(1+t)^{-1-2\eta}.$$

On the other hand,

$$\left|\partial_{ij}^2 w - \omega_i\omega_j\partial_t^2 w\right| \le C(1+t)^{-1-\eta}.$$

Hence we can simplify the coefficients of Q_1 up to integrable terms. We introduce the modified coefficients

$$\begin{aligned}
A_0 &= \bar g_{00} - \omega_1\bar g_{01} - \omega_2\bar g_{02} - 2(\partial_t w)(\bar h_{00} - \omega_1\bar h_{01} - \omega_2\bar h_{02}),\\
A_1 &= \frac{1}{2}\omega_2\bar g_{12} + \omega_1\bar g_{11} - 2(\partial_t w)\left(\frac{1}{2}\omega_2\bar h_{12} + \omega_1\bar h_{11}\right),
\end{aligned}$$

and similarly for A_2.

The computation in Step 3 is the same, and we obtain

$$\begin{aligned}
e^{-p}Q_2 = {} & X_1^2\big[(b'+\theta')(1-g_{11}) - \big(\partial_t^2 w\big)(\bar g_{11} - 2w_t\bar h_{11})\big]\\
& + X_2^2\big[(b'+\theta')(1-g_{22}) - \big(\partial_t^2 w\big)(\bar g_{22} - 2w_t\bar h_{22})\big]\\
& - 2\alpha_1\theta' X_1(\partial_t v) - 2\alpha_2\theta' X_2(\partial_t v)\\
& + 2X_1X_2\left[-\frac{1}{2}(b'+\theta')g_{12} - \frac{1}{2}\big(\partial_t^2 w\big)(\bar g_{12} - 2w_t\bar h_{12})\right] + \bar K(\partial_t v)^2,
\end{aligned}$$

with

$$\begin{aligned}
\alpha_1 &= \omega_1(1-g_{11}) - \frac{\omega_2}{2}g_{12},\ \alpha_2 = \omega_2(1-g_{22}) - \frac{\omega_1}{2}g_{12},\\
\bar K &= b'\big[g_{00} - \omega_1 g_{01} - \omega_2 g_{02} + \omega_1^2 g_{11} + \omega_2^2 g_{22} + \omega_1\omega_2 g_{12}\big]\\
&\quad + \big(\partial_t^2 w\big)[g(\omega) - 2(\partial_t w)h(\omega)] + (2 + A_3)\theta'.
\end{aligned}$$

Just as before, we see that we can replace, in the coefficient of b', $g_{ij}(\partial w)$ by

$$-(\partial_t w)\Sigma g_{ij}^k\omega_k + (\partial_t w)^2\Sigma h_{ij}^{kl}\omega_k\omega_l,$$

modulo integrable terms. Hence the assumption that g satisfies both null conditions yields $Q_2 = Q_3 + \ldots$, with $\bar K = \theta'(2 + A_3)$ in $e^{-p}Q_3$.

In Step 4, we write

$$\begin{aligned}
e^{-p}Q_3 = {} & \big(\partial_t^2 w\big)q(X) + b'\big[X_1^2(1-g_{11}) + X_2^2(1-g_{22}) - g_{12}X_1X_2\big]\\
& + \theta'\big[X_1^2(1-g_{11}) + X_2^2(1-g_{22}) - g_{12}X_1X_2\\
& - 2\alpha_1X_1w_t - 2\alpha_2X_2w_t + w_t^2(2+A_3)\big],
\end{aligned}$$

where q is a quadratic form with bounded coefficients. Since

$$\alpha_1^2 + \alpha_2^2 = 1 + O(|\partial w|),$$

the coefficient of θ' is a positive definite quadratic form for small ∂w. Using now

$$\left|\partial_t^2 w\right| \leq C(1+t)^{-\eta}(1+|\sigma|)^{-1} \leq C(1+t)^{-p\eta} + C(1+|\sigma|)^{-q}$$

for conjugate indexes p, q, we can pick $p < \infty$ big enough to ensure $p\eta > 1$. Choosing

$$b' = B(1+|\sigma|)^{-q}, \theta' = B(1+t)^{-p\eta}$$

for B big enough turns Q_3 into a positive quantity, and this completes the proof. ◇

References

1. Alinhac S., The null condition for quasilinear wave equations in two space dimensions II, to appear, Amer. J. Math. 2001
2. Alinhac S., Temps de vie et comportement explosif des solutions d'équations d'ondes quasi-linéaires en dimension deux I, Ann. Sci. Ecole Norm. Sup. **28**, 1995, 225–251
3. Alinhac S., Temps de vie des solutions des équations d'Euler compressibles axisymétriques en dimension deux, Invent. math. **111**, 1993, 627–670
4. Alinhac S., Blowup for nonlinear hyperbolic equations, Prog. Nonlinear Diff. Equations and Appl., Birkhäuser, Boston, 1995
5. Christodoulou D., Global solutions of nonlinear hyperbolic equations for small initial data, Comm. Pure Appl. Math. **39**, 1986, 267–282
6. Godin P., Lifespan of solutions of semilinear wave equations in two space dimensions, CPDE **18** (5–6), 1993, 895–916
7. Hörmander L., Lectures on nonlinear hyperbolic differential equations, Math. Appl. 26, Springer Verlag, 1997
8. Hoshiga A., The initial value problems for quasilinear wave equations in two space dimensions with small data, Adv. Math. Sci. Appl. **5**, 1995, 67–89
9. Klainerman S., The null condition and global existence to nonlinear wave equations, Lect. Appl. Math. **23**, 1986, 293–326
10. Ladhari R., Petites solutions d'équations d'ondes quasi-linéaires en dimension deux d'espace, Thèse de Doctorat, Université Paris-Sud, Orsay, 1999
11. Li Ta-tsien, Global existence for systems of nonlinear wave equations in two space dimensions, Publ. RIMS **231**, 1995, 645–665