Journal of Differential Equations **177**, 146–208 (2001)
doi:10.1006/jdeq.2000.3983, available online at http://www.idealibrary.com on IDEAL®

Weighted Decay Estimates for the Wave Equation[1]

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Received February 8, 2000; revised October 12, 2000

In this work we study weighted Sobolev spaces in $\mathbf{R}^n$ generated by the Lie algebra of vector fields $(1+|x|^2)^{1/2}\,\partial_{x_j}$, $j=1, ..., n$. Interpolation properties and Sobolev embeddings are obtained on the basis of a suitable localization in $\mathbf{R}^n$. As an application we derive weighted L^q estimates for the solution of the homogeneous wave equation. For the inhomogeneous wave equation we generalize the weighted Strichartz estimate established by V. Georgiev (1997, *Amer. J. Math.* **119**, 1291–1319) and establish global existence results for the supercritical semilinear wave equation with non-compact small initial data in these weighted Sobolev spaces.

Key Words: wave equation; decay estimates; semilinear equation; global solution; weighted Sobolev spaces; supercritical.

[1] The first and second authors were partially supported by the Programma Nazionale M.U.R.S.T. "Problemi e Metodi nella Teoria delle Equazioni Iperboliche." The third author was partially supported by Grant-in-Aid for Science Research (No. 09304012), Ministry of Education, Science and Culture, Japan.

0022-0396/01 $35.00

1. INTRODUCTION

In this work we study the decay properties of the wave equation

$$\square u \equiv \partial_t^2 u - \Delta u = 0, \tag{1.1}$$

$$u(0, x) = u_0(x), \qquad \partial_t u(0, x) = u_1(x). \tag{1.2}$$

Among the most important a priori estimates for this classical equation we mention the standard energy estimate, the estimate of von Wahl [28], and the Strichartz type estimates [23].

The energy estimate gives a control of derivatives of L^2-norms of the solution:

$$\|\nabla_{t,x} u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leqslant C(\|\nabla_x u_0\|_{L^2(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}).$$

The estimate of von Wahl controls the L^∞ norm of the solution:

$$(1+t+|x|)^{\frac{n-1}{2}} |u(t, x)| \leqslant C(\|u_0\|_{W^{[n/2]+1,1}} + \|u_1\|_{W^{[n/2],1}}).$$

Strichartz estimates give an estimate of the $L^q(\mathbb{R}^{n+1}_+)$ norm of the solution in terms of the L^p norm of the data, for suitable values of p, q.

Our goal is to obtain unified decay estimates of the solution in terms of the norm of the data in suitable weighted Sobolev spaces. These spaces are natural extensions of the weighted Sobolev spaces studied by Choquet-Bruhat and Christodoulou [3]. They defined, for any integer $s \geqslant 0$ and real δ,

$$\|u\|_{H^{s,\delta}} = \sum_{|\alpha| \leqslant s} \|\langle x \rangle^{\delta+|\alpha|} D_x^\alpha u\|_{L^2(\mathbb{R}^n)}$$

(where $\langle x \rangle = (1+|x|^2)^{1/2}$). Here we extend their definition to the L^p case and more generally to any real order s (see Sections 2 and 3). This is essential to handle initial data of minimal regularity for Problem (1.1), (1.2). To this end, we consider a dyadic partition of unity in $\mathbf{R}^n$, i.e., a sequence of functions $\phi_j \in C_c^\infty(\mathbb{R}^n)$ such that $\phi_j \geqslant 0$, $\sum \phi_j = 1$, and

$$\operatorname{supp} \phi_0 \subseteq \{|x| \leqslant 2\}, \qquad \operatorname{supp} \phi_j \subseteq \{2^{j-1} \leqslant |x| \leqslant 2^{j+1}\}, \quad j \geqslant 1.$$

Moreover, we define the pseudodifferential operators Λ_j^s as

$$\Lambda_j^s \text{ has symbol } \langle 2^j \xi \rangle^s = (1+2^{2j}|\xi|^2)^{s/2}. \tag{1.3}$$

Then the norm of the space $H^{s,\delta}_p$ is defined as

$$\|u\|^p_{H^{s,\delta}_p} = \sum_{j\geqslant 0} \|\Lambda^s_j(\langle x\rangle^\delta \phi_j u)\|^p_{L^p} \sim \sum_{j\geqslant 0} 2^{j\delta p} \|\Lambda^s_j(\phi_j u)\|^p_{L^p}.$$

Notice that the dyadic decomposition used is in the x-variables and not in the dual ξ-variables as usual. When $p=2$, we write simply $H^{s,\delta}$ instead of $H^{s,\delta}_p$.

In order to permit a unified treatment of several concrete cases, we give in Section 2 an abstract framework for handling such situations; a fairly complete theory of the spaces $H^{s,\delta}_p$ is developed in Section 3, with special attention to interpolation, duality and embedding properties. Section 4 is devoted to technical lemmas concerning $H^{s,\delta}$ spaces.

In Sections 5, 6, and 7 we shall prove the following decay estimates:

THEOREM 1.1. *Let $n\geqslant 2$. For $d\in[0,(n-1)/2]$, the solution $u(t,x)$ of* (1.1), (1.2) *satisfies for $t\geqslant 0$ the estimate*

$$(1.4)\qquad (1+t+|x|)^{(n-1)/2}(1+|t-|x||)^d\,|u(t,x)|\leqslant C(\|u_0\|_{H^{s_0,\delta_0}}+\|u_1\|_{H^{s_1,\delta_1}})$$

provided

$$s_0>\frac{n}{2},\qquad \delta_0>-\frac{1}{2}+d,\qquad s_1>\frac{n}{2}-1,\qquad \delta_1>\frac{1}{2}+d,$$

with a constant $C=C(d,\delta_0,\delta_1,s_0,s_1,n)>0$ independent of t,x,u_0,u_1.

THEOREM 1.2. *Let $n\geqslant 3$. For any real $a<-1/2$, $b\in]-1/2,0]$ the solution $u(t,x)$ of* (1.1),(1.2) *satisfies the estimate*

$$(1.5)\qquad \|(1+t+|x|)^a(1+|t-|x||)^b\,u\|_{L^2(\mathbb{R}^{n+1}_+)}\leqslant C(\|u_0\|_{H^{-b,b}}+\|u_1\|_{H^{-b-1,b+1}})$$

with a constant $C=C(a,b,n)>0$ independent of u_0,u_1.

The estimate is also true for $n=2$, provided $b<0$ strictly.

Moreover, interpolating between Theorems 1.1 and 1.2 we prove the following

THEOREM 1.3. *Let $n\geqslant 3$, $q\in[2,\infty]$. For any*

$$\rho<\frac{n-1}{2}-\frac{n}{q},\qquad 0\leqslant\sigma\leqslant\frac{n-1}{2}-\frac{n-1}{q}$$

the solution $u(t,x)$ of (1.1),(1.2) *satisfies the estimate*

$$\|(1+t+|x|)^{\rho}\,(1+|t-|x||)^{\sigma}\,u\|_{L^q(\mathbb{R}^{n+1}_+)} \leqslant C(\|u_0\|_{H^{s_0,\delta_0}}+\|u_1\|_{H^{s_1,\delta_1}}) \tag{1.6}$$

provided

$$s_0>\frac{n}{2}-\frac{n}{q},\qquad \delta_0>\frac{1}{q}-\frac{1}{2}+\sigma,\qquad s_1>\frac{n}{2}-\frac{n}{q}-1,\qquad \delta_1>\frac{1}{q}+\frac{1}{2}+\sigma,$$

with a constant $C=C(\sigma,\rho,\delta_0,\delta_1,s_0,s_1,n)>0$ independent of u_0, u_1.

Moreover, (1.6) *is also true for any $\rho<(n-1)/2-n/q$, $-1/q<\sigma\leqslant 0$ provided $s_0>n/2-n/q-\sigma$, $\delta_0>1/q-1/2+\sigma$, $s_1>n/2-n/q-\sigma-1$, $\delta_1>1/q+1/2+\sigma$.*

The above estimates hold also for $n=2$, provided $\sigma<(n-1)/2-(n-1)/q$ strictly.

Notice in particular that choosing $\rho=\sigma$ we obtain the estimate

$$\|(1+t+|x|)^{\sigma}\,(1+|t-|x||)^{\sigma}\,u\|_{L^q(\mathbb{R}^{n+1}_+)} \leqslant C(\|u_0\|_{H^{s_0,\delta_0}}+\|u_1\|_{H^{s_1,\delta_1}}) \tag{1.7}$$

which is valid for

$$2+\frac{2}{n-1}\leqslant q\leqslant\infty,\qquad 0\leqslant\sigma<\frac{n-1}{2}-\frac{n}{q}$$

and

$$s_0>\frac{n}{2}-\frac{n}{q},\qquad \delta_0>\frac{1}{q}-\frac{1}{2}+\sigma,\qquad s_1>\frac{n}{2}-\frac{n}{q}-1,\qquad \delta_1>\frac{1}{q}+\frac{1}{2}+\sigma.$$

Finally, Section 8 is dedicated to the initial value problem with small data for the semilinear wave equations of the form

$$\Box u=F(u)\qquad \text{in } \mathbb{R}^{n+1}_+, \tag{1.8}$$

$$u(0,x)=u_0(x),\qquad \partial_t u(0,x)=u_1(x)\qquad \text{for}\quad x\in\mathbb{R}^n, \tag{1.9}$$

where $n\geqslant 2$. We shall assume that $F\in C^1(\mathbb{R})$ satisfies

$$F(0)=0,\qquad |F'(u)|\leqslant C\,|u|^{\lambda-1}, \tag{1.10}$$

where $C>0$ and $\lambda>1$. Typical examples are $F=|u|^{\lambda}$ and $F=|u|^{\lambda-1}\,u$.

Equation (1.8) has a long history, starting with Strauss' paper [20]; see the survey paper [21] for earlier references. In 1979 John [17] proved that

(1.8), (1.9) has global solution for $n=3$, provided the initial data are smooth and small enough, and $\lambda > 1+\sqrt{2}$; he also proved that for λ below this value in general solutions blow up in a finite time even with small data. This agreed with Walter Strauss' conjecture [22] that for $n \geqslant 2$ and λ greater than the positive root $\lambda_0(n)$ of the equation

$$\lambda\left(\frac{n-1}{2}\lambda - \frac{n+1}{2}\right) = 1. \tag{1.11}$$

Problem (1.8), (1.9) has a global solution. The conjecture was proved true for $n=2$ by Glassey [10], who also proved the blow up below $\lambda_0(2)$ [11]. The critical case $\lambda = \lambda_0$ was considered by Schaeffer [18] who proved blow up for $n=2, 3$. Sideris [19] completely solved the subcritical case, showing that one has always blow up in general for $\lambda < \lambda_0(n)$, $n \geqslant 2$. On the other hand, the supercritical case has been treated by many authors (see, e.g., [2, 4, 13, 14, 25, 29] and the references cited therein). The gobal existence result is established in [9] for any $\lambda > \lambda_0(n)$ (see also [7, 24]).

Our aim is to extend the result of [9], in two directions: on one hand, we relax the regularity assumptions on the initial data; on the other hand, we remove the assumption that the initial data are compactly supported. This result is obtained combining estimate (1.7) with a suitable extension of the weighted Strichartz type estimate established in [9, 24] (see Lemma 8.1). In conclusion we obtain

THEOREM 1.4. *Assume $n \geqslant 2$, $F(u) \in C^1(\mathbb{R})$ satisfies (1.10) with*

$$\lambda_0(n) < \lambda \leqslant \frac{n+3}{n-1} \tag{1.12}$$

and that the initial data (1.9) satisfy $u_0 \in H^{s_0, \delta_0}$, $u_1 \in H^{s_1, \delta_1}$ with

$$s_0 > \frac{\lambda-1}{\lambda+1}\cdot\frac{n}{2}, \qquad \delta_0 > \frac{1}{\lambda} - \frac{1}{2}, \qquad s_1 > \frac{\lambda-1}{\lambda+1}\cdot\frac{n}{2} - 1, \qquad \delta_1 > \frac{1}{\lambda} + \frac{1}{2}. \tag{1.13}$$

Then there exists $\varepsilon > 0$ such that, for all data with $\|u_0\|_{H^{s_0,\delta_0}} + \|u_1\|_{H^{s_1,\delta_1}} < \varepsilon$, Problem (1.8), (1.9) has a unique weak global solution

$$u(t, x) \in L^{\lambda+1}(\mathbb{R}^{n+1}_+). \tag{1.14}$$

Actually, we have $(1+|t-|x||)^a (1+t+|x|)^a u \in L^{\lambda+1}(\mathbb{R}^{n+1}_+)$ for any $a < (n-1)/2 - n/(\lambda+1)$.

By weak solution we mean as usual a solution of the integral equation corresponding to (1.8), (1.9). For instance, in $n=4$ space dimensions, and for λ close to the critical value $\lambda_0(4)=2$, Theorem 1.4 implies global existence for any small initial data $u_0 \in H^1$, $u_1 \in L^2$ such that $\langle x \rangle \nabla u_0$ and $\langle x \rangle u_1$ are in L^2; actually the regularity can be even lower, indeed (1.13) give for $\lambda = 2$

$$s_0 > \tfrac{2}{3}, \quad s_1 > -\tfrac{1}{3}.$$

We refer to Sections 3 and 4 for the precise definition and properties of the spaces $H^{s,\delta}$.

2. ABSTRACT LOCALIZED NORMS

DEFINITION 2.1. Let A be a Banach space with norm $\|\cdot\|_A$. A *Paley–Littlewood partition of identity* (*PL partition* for short) is a sequence $\pi = \{\pi_j\}_{j\geqslant 0}$ of bounded operators on A such that: the series $\sum_{j\geqslant 0} \pi_j$ converges strongly (i.e., pointwise) to the identity operator on A, and in addition there exists an integer $N \geqslant 1$ such that

$$\pi_j \pi_k = 0 \qquad \text{for} \quad |j-k| \geqslant N. \tag{2.1}$$

Remark 2.1. We shall frequently encounter the following situation: we have two functions $\phi(x)$ and $\psi(x)$ defined on some vector space A, in general norms or norms raised to a fixed power, and there exists a constant $C>0$ such that for all $x \in A$

$$C^{-1}\phi(x) \leqslant \psi(x) \leqslant C\phi(x).$$

In such cases we shall say that *ϕ and ψ are equivalent on A*, and we shall write

$$\phi(x) \sim \psi(x) \tag{2.2}$$

for $x \in A$.

EXAMPLE 2.1. Let $\{\phi_j\}_{j\geqslant 0}$ be a Paley–Littlewood partition of unity on $\mathbb{R}^n$, i.e., a sequence $\phi_j \in C_c^\infty(\mathbb{R}^n)$ such that $\phi_j \geqslant 0$, $\sum \phi_j = 1$, and

$$\operatorname{supp} \phi_0 \subseteq \{|x| \leqslant 2\}, \qquad \operatorname{supp} \phi_j \subseteq \{2^{j-1} \leqslant |x| \leqslant 2^{j+1}\} \quad j \geqslant 1. \tag{2.3}$$

More precisely, fix an arbitrary nonnegative $\psi \in C_c^\infty(\mathbb{R}^n)$, $0 \leqslant \psi \leqslant 1$, equal to 1 on the ball $B(0, 1/2)$ and vanishing outside $B(0, 1)$, and define

$$\phi(x) = \psi(x/2) - \psi(x), \quad \phi_0(x) = \psi(x/2), \quad \phi_j(x) = \phi(2^{-j}x), \quad j \geqslant 1. \tag{2.4}$$

This gives a partition of unity satisfying (2.3), and we shall call it a *(standard) Paley–Littlewood partition of unity* (*PL partition* for short).

We remark that if we choose $A = L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, and define $\pi_j: A \to A$ as the multiplication operator by ϕ_j then $\pi = \{\pi_j\}$ is a PL partition of identity in the sense of Definition 2.1. Moreover, it enjoys the following important property, which will be used several times in the sequel: for any $1 \leqslant p < \infty$

$$\|u\|_{L^p(\mathbb{R}^n)}^p \sim \sum_{j \geqslant 0} \|\phi_j u\|_{L^p(\mathbb{R}^n)}^p \tag{2.5}$$

and similarly

$$\|u\|_{L^\infty(\mathbb{R}^n)} \sim \sup_{j \geqslant 0} \|\phi_j u\|_{L^\infty(\mathbb{R}^n)}.$$

The second relation is obvious. On the other hand, for $p < \infty$ we have

$$\frac{1}{2^{p-1}} \leqslant \sum_{j \geqslant 0} \phi_j(x)^p \leqslant 1$$

since, at each $x \in \mathbb{R}^n$, at most 2 of the functions ϕ_j do not vanish. This implies

$$\frac{1}{2^{p-1}} \int |u|^p \, dx \leqslant \int \sum |\phi_j u|^p \, dx \leqslant \int |u|^p \, dx$$

and noticing that

$$\int \sum |\phi_j u|^p \, dx = \sum \int |\phi_j u|^p \, dx$$

by monotone convergence, we obtain (2.5).

In the sequel we shall need the following technical

LEMMA 2.4. *Assume $\{\lambda_j\}_{j=-\infty}^{+\infty}$ is a two sided sequence of nonnegative real numbers, and let $\{\Lambda_j\}_{j \geqslant 0}$ be a sequence of positive real numbers such that for some $C_0 > 0$ and all $j, k \geqslant 0$*

$$\sum_{h \geqslant 0} \Lambda_h \lambda_{k-h} \leqslant C_0 \Lambda_k, \qquad \sum_{h \geqslant 0} \frac{\lambda_{h-j}}{\Lambda_h} \leqslant \frac{C_0}{\Lambda_j}. \tag{2.6}$$

Then for all $q \in [1, \infty[$*, for any sequence* $\{a_j\}_{j \geqslant 0}$ *of complex numbers,*

$$\sum_{j \geqslant 0} \Lambda_j^q \left| \sum_{k \geqslant 0} \lambda_{k-j} a_k \right|^q \leqslant C_0^q \sum_{j \geqslant 0} \Lambda_j^q \, |a_j|^q \tag{2.7}$$

and also ("$q = \infty$*")*

$$\sup_{j \geqslant 0} \Lambda_j \left| \sum_{k \geqslant 0} \lambda_{k-j} a_k \right| \leqslant C_0 \sup_{j \geqslant 0} \Lambda_j \, |a_j|. \tag{2.8}$$

Proof. Let T be the operator acting on sequences of $\mathbb{C}$

$$T(\{a_k\}) = \{b_j\}, \qquad b_j = \Lambda_j \sum_{k \geqslant 0} \frac{\lambda_{k-j}}{\Lambda_k} a_k, \quad j \geqslant 0. \tag{2.9}$$

The operator T is easily seen to be bounded on ℓ^∞; indeed, by the second property in (2.6),

$$\begin{aligned} \|T(\{a_k\})\|_{\ell^\infty} &\equiv \sup_{j \geqslant 0} \Lambda_j \left| \sum_{k \geqslant 0} \frac{\lambda_{k-j}}{\Lambda_k} a_k \right| \\ &\leqslant \sup_{k \geqslant 0} |a_k| \cdot \sup_{j \geqslant 0} \Lambda_j \sum_{k \geqslant 0} \frac{\lambda_{k-j}}{\Lambda_k} \leqslant C_0 \sup_{k \geqslant 0} |a_k|. \end{aligned} \tag{2.10}$$

Notice that, when applied to the sequence $\Lambda_k a_k$, this proves (2.8). Moreover, T is bounded on ℓ^1; indeed, using the first property in (2.6) we have

$$\|T(\{a_k\})\|_{\ell^1} \equiv \sum_{j \geqslant 0} \Lambda_j \left| \sum_{k \geqslant 0} \frac{\lambda_{k-j}}{\Lambda_k} a_k \right| \leqslant \sum_{k \geqslant 0} \frac{|a_k|}{\Lambda_k} \sum_{j \geqslant 0} \Lambda_j \lambda_{k-j} \leqslant C_0 \sum_{k \geqslant 0} |a_k|. \tag{2.11}$$

Now, by the Riesz-Thorin interpolation theorem (see e.g., [1]), we see that T is a bounded operator on ℓ^q for all q $(1 \leqslant q < \infty)$, with norm not greater than C_0; this gives the inequality

$$\|T(\{a_k\})\|_{\ell^q}^q \equiv \sum_{j \geqslant 0} \Lambda_j^q \left| \sum_{k \geqslant 0} \frac{\lambda_{k-j}}{\Lambda_k} a_k \right|^q \leqslant C_0^q \sum_{k \geqslant 0} |a_k|^q. \tag{2.12}$$

If we apply (2.12) to the sequence $\Lambda_k a_k$ we obtain (2.7). ∎

We are now ready to prove an abstract localization lemma, which in the next section will be applied to produce several equivalent norms on weighted Sobolev spaces.

LEMMA 2.2 (Localization Lemma). *Let A, B be Banach spaces with norms $\|\cdot\|_A$, $\|\cdot\|_B$, endowed with PL partitions of identity $\{\pi_j\}$ and $\{p_k\}$ respectively, with the same integer N from Definition 2.1, and assume $F\colon A \to B$ is an invertible isometry. Let $\{\lambda_j\}_{j=-\infty}^{+\infty}$, $\{\Lambda_j\}_{j\geqslant 0}$ be two nonnegative sequences satisfying (2.6) and the following additional property: for some $C_1 > 0$*

$$\lambda_j \leqslant C_1 \lambda_k \qquad \text{for} \quad |j-k| \leqslant N. \tag{2.13}$$

Finally, assume that for some $C_2 > 0$ and all j, k

$$\|F\pi_j F^{-1} p_k\|_{\mathscr{L}(B)} + \|p_j F\pi_k F^{-1}\|_{\mathscr{L}(B)} \leqslant C_2 \lambda_{k-j}. \tag{2.14}$$

Then the following equivalences of norms hold on A:

$$\left(\sum_{k\geqslant 0} \Lambda_k^q \|p_k F u\|_B^q\right)^{1/q} \sim \left(\sum_{j\geqslant 0} \Lambda_j^q \|F\pi_j u\|_B^q\right)^{1/q}, \qquad q \in [1, \infty[, \tag{2.15}$$

$$\sup_{k\geqslant 0} \|p_k F u\|_B \sim \sup_{j\geqslant 0} \|F\pi_j u\|_B. \tag{2.16}$$

Proof. Using $\sum p_k = I$ and the property (2.1) we can write

$$F\pi_j u = \sum_k F\pi_j F^{-1} p_k F u = \sum_k \sum_{|\ell-k|\leqslant N} F\pi_j F^{-1} p_\ell p_k F u. \tag{2.17}$$

Hence, by (2.14) and (2.13),

$$\|F\pi_j u\|_B \leqslant C_2 \sum_k \sum_{|\ell-k|\leqslant N} \lambda_{\ell-j} \|p_k F u\|_B \leqslant (2N+1)\, C_1 C_2 \sum_k \lambda_{k-j} \|p_k F u\|_B.$$

Thus we can apply Lemma 2.1 to the sequence $a_k = \|p_k F u\|_B$, and we obtain easily

$$\sum \Lambda_j^q \|F\pi_j u\|_B^q \leqslant C \sum \Lambda_k^q \|p_k F u\|_B^q, \tag{2.18}$$

which is the first inequality to prove (the case $q = \infty$ is analogous). The reverse inequality is proved in a similar way, writing

$$p_k F u = \sum_j \sum_{|j-\ell|\leqslant N} p_k F\pi_j F^{-1} F\pi_\ell u. \quad \blacksquare$$

3. LOCALIZED SOBOLEV NORMS AND WEIGHTED SPACES

Notation. In the following we shall frequently use the operators $\Lambda^s = (1-\Delta)^{s/2}$, $s \in \mathbb{R}$, defined as

$$\Lambda^s u = \mathscr{F}^{-1} \langle \xi \rangle^s \mathscr{F} u,$$

where $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$ and $\mathscr{F}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the Fourier transform.

Remark 3.1. In the sequel we shall exclusively use the complex interpolation in the sense of Chapter 4 of [1]. We recall briefly the definition. Given a couple $A = (A_0, A_1)$ of Banach spaces embedded continuously in a common Hausdorff topological vector space, let Ω be the complex strip $0 < \operatorname{Re} z < 1$, and denote by $F(A)$ the space of functions bounded and continuous on $\bar{\Omega}$ and holomorphic on Ω, with values in $A_0 + A_1$, such that $\|F(iy)\|_{A_0}$ and $\|F(1+iy)\|_{A_1}$ are bounded for $y \in \mathbb{R}$. $F(A)$ is a Banach space with the norm

$$\|f\|_F = \sup_y \, [\|F(iy)\|_{A_0} + \|F(1+iy)\|_{A_1}].$$

Then $A_\theta = (A_0, A_1)_\theta$, $0 < \theta < 1$, is defined as the Banach space of values $\{f(\theta)\}$ with $f \in F(A)$, endowed with the norm

$$\|u\|_{A_\theta} = \inf\{\|f\|_F : f \in F, f(\theta) = u\}.$$

3.1. *The Generalized Sobolev Spaces.* To give a first example of localized norms we shall consider the spaces

$$H^s_p = H^s_p(\mathbb{R}^n), \qquad s \in \mathbb{R}, \quad p \in [1, \infty],$$

also denoted by $\mathscr{L}^s_p(\mathbb{R}^n)$, whose norm is defined as

$$\|u\|_{H^s_p} = \|\Lambda^s u\|_{L^p}. \tag{3.1}$$

As usual H^s_p is defined as the space of all tempered distributions u such that $\Lambda^s u \in L^p$ and the above norm is finite. These spaces are well studied; see e.g., [1], [26], [27]. We list a few properties of these spaces, whose proofs can be found in the given references:

(1) If $s \geqslant 0$ is an integer and $1 < p < \infty$, then H^s_p coincides with the usual Sobolev space $W^{s,p}(\mathbb{R}^n)$.

(2) Λ^s is an isomorphism of H^σ_p onto $H^{\sigma-s}_p$, $s, \sigma \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$.

(3) We have the Sobolev type continuous embeddings

(3.2) $$H^s_p \subseteq C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \qquad s > \frac{n}{p}, \quad 1 < p < \infty;$$

(3.3) $$H^s_p \subseteq L^q(\mathbb{R}^n), \qquad s \geqslant \frac{n}{p} - \frac{n}{q}, \quad 1 < p \leqslant q < \infty;$$

(3.4) $$H^s_p \subseteq H^\sigma_p, \qquad s \geqslant \sigma, \quad 1 \leqslant p \leqslant \infty.$$

(4) If $s \in \mathbb{R}$ and $1 \leqslant p < \infty$, then

$$(H^s_p)' = H^{-s}_q, \qquad \frac{1}{p} + \frac{1}{q} = 1;$$

moreover, $C^\infty_c(\mathbb{R}^n)$ and $\mathscr{S}$ are dense in H^s_p.

(5) Probably the most useful property of these spaces is their behaviour with respect to interpolation: for all real $s_0 \neq s_1$ and all $p_0, p_1 \in]1, \infty[$ we have

(3.5) $$(H^{s_0}_{p_0}, H^{s_1}_{p_1})_\theta = H^s_p,$$

where

$$0 < \theta < 1,$$
$$s = (1-\theta)\, s_0 + \theta s_1,$$
$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Remark 3.2. The following property will be used frequently in the sequel. Let $\phi(x)$ be a smooth function such that

(3.6) $$\|\partial^\alpha_x \phi\|_{L^\infty} \leqslant C_N \qquad \text{for} \quad |\alpha| \leqslant N.$$

Then the multiplication operator by ϕ is a bounded operator on H^s_p, for all $s \in \mathbb{R}$ with $|s| \leqslant N$ and $1 < p < \infty$. This is trivial when $s \geqslant 0$ is an integer (Leibnitz' rule), hence is true for real $s \geqslant 0$ by interpolation property (3.5), and follows easily by duality for negative s.

We show now how it is possible to localize the H^s_p norm. A first localization is trivial, and follows immediately by the equivalence (2.5),

(3.7) $$\|u\|^p_{H^s_p} \sim \sum_{j \geqslant 0} \|\phi_j \Lambda^s u\|^p_{L^p},$$

where $\phi_j(x)$ is a PL partition of unity as in Example 2.1. The following result is more subtle:

LEMMA 3.1. *For $s \in \mathbb{R}$, $1 < p < \infty$, we have*

$$\|u\|^p_{H^s_p} \sim \sum_{j \geqslant 0} \|\phi_j \Lambda^s u\|^p_{L^p} \sim \sum_{j \geqslant 0} \|\Lambda^s(\phi_j u)\|^p_{L^p}. \tag{3.8}$$

Proof. Taking into account (3.7), we need only to prove the equivalence of the last two quantities. We shall apply Lemma 2.2 with the choices $A = H^s_p$, $B = L^p$, while the partitions of identity p_j, π_j are both defined as multiplication by ϕ_j as in Example 2.1; we can take $N = 2$. Moreover we choose

$$\Lambda_j = 1, \qquad \lambda_j = 2^{-m|j|}, \tag{3.9}$$

where $m > 1$ will be precised in the following. It is trivial to verify that assumptions (2.6), (2.13) are satisfied. Finally we take $F = \Lambda^s$ which is an invertible isometry of A onto B. With these choices, (3.8) is exactly (2.15) (with $q = p$), thus the result will follow as soon as we verify that (2.14) is satisfied. Hence we must prove that for some C independent of $u \in H^s_p$

$$\|\Lambda^s \phi_j \Lambda^{-s} \phi_k u\|_{L^p} \leqslant \frac{C}{2^{|j-k|\, m}} \|u\|_{L^p}, \tag{3.10}$$

$$\|\phi_k \Lambda^s \phi_j \Lambda^{-s} u\|_{L^p} \leqslant \frac{C}{2^{|j-k|\, m}} \|u\|_{L^p}. \tag{3.11}$$

Actually, it is possible to choose any $m > 1$, as it will be clear at the end of the proof. Notice that (3.11) is a consequence of (3.10), since the operator $\Lambda^s \phi_j \Lambda^{-s} \phi_k$ is dual to $\phi_k \Lambda^{-s} \phi_j \Lambda^s$ in the pairing $\langle L^p, L^{p'} \rangle$ (with s arbitrary real and $1 < p < \infty$). To prove (3.10), we begin by remarking that

$$\|\Lambda^s \phi_j \Lambda^{-s} \phi_k\|_{\mathscr{L}(L^p)} \leqslant C \tag{3.12}$$

with C independent of j, k; this follows from Remark 3.2:

$$\|\Lambda^s \phi_j \Lambda^{-s} \phi_k u\|_{L^p} = \|\phi_j \Lambda^{-s} \phi_k u\|_{H^s_p} \leqslant C \, \|\Lambda^{-s} \phi_k u\|_{H^s_p} \leqslant C \, \|\phi_k u\|_{L^p} \leqslant C \, \|u\|_{L^p}$$

(we have used the fact that $\Lambda^{-s}\colon H^s_p \to L^p$ is an isometry and that $|\partial^\alpha \phi_j| \leqslant C_\alpha$ with C_α independent of j). Thus it is sufficient to prove (3.10) for $|j-k| \geqslant 3$, i.e., when the supports of ϕ_j and ϕ_k are disjoint.

Let $u \in C_0^\infty(\mathbb{R}^n)$. By the standard computation ($đ\xi = (2\pi)^{-n}\, d\xi$, $D = \partial/i$)

$$\iint e^{i(x-y)\xi} \langle \xi \rangle^s\, u(y)(x-y)^\alpha\, dy\, đ\xi = \int D_\xi^\alpha \left(\int e^{i(x-y)\,\xi} u(y)\, dy \right) \langle \xi \rangle^s\, đ\xi$$

$$= \iint e^{i(x-y)\,\xi} u(y)(-D_\xi)^\alpha \, \langle \xi \rangle^s\, dy\, đ\xi,$$

we see that the kernel $K_s(x-y)$ of the operator Λ^s, defined by

$$\Lambda^s u(x) = \langle K_s(x - \cdot\,), u(\,\cdot\,) \rangle,$$

satisfies for any α

$$(x-y)^\alpha\, K_s(x-y) = \int e^{i(x-y)\,\xi}\, (-D_\xi)^\alpha\, \langle \xi \rangle^s\, đ\xi$$

which is an ordinary (not oscillatory) integral as soon as $|\alpha| > s+n$. So $K_s(z)$ is smooth for $z \neq 0$ and we have

$$|z^\alpha K_s(z)| \leqslant C(\alpha, s) \qquad \text{for any} \quad |\alpha| > s+n.$$

In a similar way,

$$|D_z^\beta z^\alpha K_s(z)| \leqslant C(\alpha, \beta, s) \qquad \text{for any} \quad |\alpha| - |\beta| > s+n.$$

Consequently,

$$|z^\alpha D_z^\beta K_s(z)| \leqslant C(\alpha, \beta, s) \qquad \text{for any} \quad |\alpha| - |\beta| > s+n.$$

So we arrive at

$$|D_z^\beta K_s(z)| \leqslant \frac{C(\alpha, s, M)}{|z|^M} \qquad \text{for any} \quad M - |\beta| > s+n. \tag{3.13}$$

Since $|j-k| \geqslant 3$ the supports of ϕ_j, ϕ_k are disjoint and more precisely

$$x \in \operatorname{supp} \phi_j,\ y \in \operatorname{supp} \phi_k \Rightarrow |x-y| \geqslant \tfrac{1}{4}\, 2^{|j-k|} \tag{3.14}$$

as it is readily seen. Thus the operator $\phi_j \Lambda^s \phi_k$ has the kernel

$$K_{ij}(x, y) = \phi_j(x)\, K_s(x-y)\, \phi_k(y) \tag{3.15}$$

which is a smooth function. Since

$$D_x^\alpha(\phi_j \Lambda^s \phi_k u) = D_x^\alpha \int \phi_j(x)\, K_s(x-y)\, \phi_k(y)\, u(y)\, dy,$$

by Leibnitz' rule and using (3.13), (3.14), we obtain

$$|D_x^\alpha(\phi_j \Lambda^s \phi_k u)| \leqslant C \sum_{\beta \leqslant \alpha} |D^\beta \phi_j(x)| \cdot \int \phi_k(y)\, |u(y)|\, dy \cdot 2^{-|j-k|\, M}.$$

This implies easily for $p=1$ or $p=\infty$

$$\|D^\alpha(\phi_j \Lambda^s \phi_k u)\|_{L^p} \leqslant \frac{C(\alpha, s, M)}{2^{|j-k|\, M}} \|u\|_{L^p} \tag{3.16}$$

and hence for any $p \in [1, \infty]$ by interpolation. In particular we have proved that

$$\|\Lambda^{2\ell} \phi_j \Lambda^{-s} \phi_k u\|_{\mathscr{L}(L^p)} \leqslant C(\ell, s, M) \cdot 2^{-|j-k|\, M} \tag{3.17}$$

for any nonnegative integer ℓ, $1 \leqslant p \leqslant \infty$, and any $M, j, k \geqslant 0$. From this, (3.10) follows easily, for $1 < p < \infty$, by the well known L^p boundedness of the operator $\Lambda^{s-2\ell}$ for $s \leqslant 2\ell$ (and in fact of any operator in $OPS^0_{1,0}$). ∎

3.2. *The Weighted Sobolev Spaces* $H^s_p(\rho)$.

DEFINITION 3.1. Let $\chi(x) \in C^\infty(\mathbb{R}^n)$ be a smooth, strictly positive, radial function $\chi(x) = \rho(|x|)$. We shall say that $\chi(x)$ (or $\rho(R)$) is a *weight function*, or simply a *weight*, if for all $k \geqslant 0$

$$|\rho^{(k)}(R)| \leqslant C_k \rho(R), \tag{3.18}$$

and for any $\delta > 0$ there exists $C = C(\delta) > 0$ such that

$$\begin{aligned} C^{-1}\rho(R_1)\, \rho(R_2) &\leqslant \rho(R_1 R_2) \\ &\leqslant C\rho(R_1)\, \rho(R_2) \qquad \text{for any} \quad R_1, R_2 > \delta. \end{aligned} \tag{3.19}$$

In some cases it is useful to require the stronger property

$$|\rho^{(k)}(R)| \leqslant C_k \langle R \rangle^{-k} \rho(R); \tag{3.20}$$

we shall call such a ρ a *strong weight*.

The most typical example of a weight corresponds to the choice

$$\rho(R) = \langle R \rangle^s \tag{3.21}$$

for any $s \in \mathbb{R}$; notice this is also a strong weight.

We notice two consequences of this definition. There exists $C > 0$ independent of j such that

$$2^{j-1} \leqslant |x| \leqslant 2^{j+1} \Rightarrow C^{-1}\rho(2^j) \leqslant \rho(|x|) \leqslant C\rho(2^j) \tag{3.22}$$

(trivial proof). Moreover, the reciprocal of a weight is still a weight, and in particular

$$\left| \left(\frac{1}{\rho}\right)^{(k)} \right| \leqslant C_k \frac{1}{\rho}. \tag{3.23}$$

In a similar way, the reciprocal of a strong weight is a strong weight. This is easily proved using the formula

$$\left(\frac{1}{\rho}\right)^{(k)} = \sum_{\nu=1}^{k} \sum_{j_1 + \cdots + j_\nu = k} \binom{k}{j_1 \cdots j_\nu} (-1)^\nu \frac{\rho^{(j_1)} \cdots \rho^{(j_\nu)}}{\nu \rho^{\nu+1}}$$

together with (3.18).

We are now ready to introduce the weighted Sobolev space $H^s_p(\rho)$, whose norm is defined, for any $s \in \mathbb{R}$ and $1 < p < \infty$, by

$$\|u\|_{H^s_p(\rho)} = \|\Lambda^s[\rho(|x|)\, u(x)]\|_{L^p(\mathbb{R}^n)} = \|\Lambda^s(\rho u)\|_{L^p} = \|\rho u\|_{H^s_p}. \tag{3.24}$$

The properties of the spaces H^s_p can easily be extended to the case of weighted spaces $H^s_p(\rho)$. In particular we have the complex interpolation property: for all real $s_0 \neq s_1$ and all $p_0, p_1 \in]1, \infty[$

$$(H^{s_0}_{p_0}(\rho_0), H^{s_1}_{p_1}(\rho_1))_\theta = H^s_p(\rho), \tag{3.25}$$

where

$$\begin{aligned} &0 < \theta < 1, \\ &\rho = \rho_0^{1-\theta} \rho_1^\theta \\ &s = (1-\theta)\, s_0 + \theta s_1, \\ &\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \end{aligned}$$

This is an immediate consequence of the corresponding property for the spaces H^s_p. Indeed, the operator

$$\phi(z) \mapsto \rho_0^{1-z} \rho_1^z \phi(z)$$

is evidently an isomorphism of $F(H^{s_0}_{p_0}(\rho_0), H^{s_1}_{p_1}(\rho_1))$ onto $F(H^{s_0}_{p_0}, H^{s_1}_{p_1})$ (see Remark 3.1 for notations).

Moreover, we have for any $s \in \mathbb{R}$ and $1 < p < \infty$

$$(H^s_p(\rho))' = H^{-s}_q(1/\rho), \qquad \frac{1}{q} + \frac{1}{p} = 1$$

(trivial consequence of the duality property of H^s_p).

Finally, it is easy to obtain from (3.2)–(3.4) corresponding embedding properties for weighted spaces. In particular we notice

$$\|\rho u\|_{L^\infty} \leqslant C(s, n, \rho)\, \|u\|_{H^s_2(\rho)} \tag{3.26}$$

valid for any real $s > n/2$.

We give now several equivalent localizations of the weighted norm.

LEMMA 3.2. *Let $\{\phi_j\}$ be a PL partition of unity, $1 < p < \infty$, $s \in \mathbb{R}$ and $\rho(|x|)$ be a weight. Then the following norms raised to power p are equivalent on $H^s_p(\rho)$:*

$$I = \|u\|^p_{H^s_p(\rho)} \equiv \|\Lambda^s(\rho u)\|^p_{L^p}, \tag{3.27}$$

$$II = \sum_{j \geqslant 0} \|\phi_j \Lambda^s(\rho u)\|^p_{L^p}, \tag{3.28}$$

$$III = \sum_{j \geqslant 0} \|\Lambda^s(\phi_j \rho u)\|^p_{L^p}, \tag{3.29}$$

$$IV = \sum_{j \geqslant 0} \rho(2^j)^p\, \|\Lambda^s(\phi_j u)\|^p_{L^p}, \tag{3.30}$$

$$V = \|\rho \Lambda^s u\|^p_{L^p}. \tag{3.31}$$

Proof. $I \sim II$ is a simple consequence of (2.5). $II \sim III$ follows by (3.8) of Lemma 3.1 applied to the function $\rho(|x|)\, u(x)$. To prove $III \sim IV$ we remark that, by properties (3.18) and (3.22) (resp. (3.23) and (3.22)), the functions

$$\psi_j(x) = (\phi_{j-1} + \phi_j + \phi_{j+1}) \frac{\rho(|x|)}{\rho(2^j)}, \qquad \chi_j(x) = (\phi_{j-1} + \phi_j + \phi_{j+1}) \frac{\rho(2^j)}{\rho(|x|)},$$

(set $\phi_{-1} \equiv 0$) satisfy for all α

$$|\partial_x^\alpha \psi_j(x)| + |\partial_x^\alpha \chi_j(x)| \leqslant C_\alpha \tag{3.32}$$

with constants C_α independent of j. Hence by Remark 3.2 multiplication by ψ_j or χ_j is a bounded operator on H_p^s, $s \in \mathbb{R}$, $1 < p < \infty$, with norm uniformly bounded in j; equivalently,

$$\|\Lambda^s \psi_j \Lambda^{-s}\|_{\mathscr{L}(L^p)} + \|\Lambda^s \chi_j \Lambda^{-s}\|_{\mathscr{L}(L^p)} \leqslant C(s)$$

with $C(s)$ independent of j. Notice that $\phi_j \rho = \rho(2^j)\, \psi_j \phi_j$, because $\phi_{j-1} + \phi_j + \phi_{j+1} \equiv 1$ on the support of ϕ_j. Thus writing

$$\|\Lambda^s(\phi_j \rho u)\|_{L^p} = \rho(2^j)\, \|(\Lambda^s \psi_j \Lambda^{-s})\, \Lambda^s(\phi_j u)\|_{L^p} \leqslant C\rho(2^j)\, \|\Lambda^s(\phi_j u)\|_{L^p},$$

we get $III \leqslant C \cdot IV$, and similarly for the reverse inequality writing $\rho(2^j)\, \Lambda^s(\phi_j u) = \Lambda^s \chi_j \Lambda^{-s} \Lambda^s \rho \phi_j u$.

Finally, $IV \sim V$ is a consequence of Lemma 2.2. Indeed, we choose A, B, p_j, π_j, λ_j exactly as in the proof of Lemma 3.1 (recall in particular (3.10), (3.11) already proved there), the only difference consisting in the choice

$$\Lambda_j = \rho(2^j);$$

assumption (2.6) is readily verified. Indeed, by property (3.18) it follows easily that, for a suitable $C > 1$,

$$C^{-j} \rho(1) \leqslant \rho(2^j) \leqslant C^j \rho(1);$$

hence it is clear that, choosing m large enough in (3.10), (3.11), we obtain (2.6). Thus by Lemma 2.2 we get

$$IV \sim \sum_{j \geqslant 0} \rho(2^j)^p\, \|\phi_j \Lambda^s u\|_{L^p}^p$$

and the last quantity is clearly equivalent to V by (3.22) and (2.5). ∎

Remark 3.3. When s is a nonnegative integer, $1 < p < \infty$, we may use the identity $H_p^s \equiv W^{s,p}$ (classical Sobolev spaces) in connection with Lemma 3.2 to give further equivalent representations for the $H_p^s(\rho)$ norm (on power p):

$$\|u\|^p_{H^s_p(\rho)}, \qquad \sum_{|\alpha| \leqslant s} \|\rho\, \partial^\alpha u\|^p_{L^p},$$

$$\sum_{j \geqslant 0,\, |\alpha| \leqslant s} \|\phi_j\, \partial^\alpha(\rho u)\|^p_{L^p}, \qquad \sum_{j \geqslant 0,\, |\alpha| \leqslant s} \|\partial^\alpha(\phi_j \rho u)\|^p_{L^p},$$

$$\sum_{j \geqslant 0,\, |\alpha| \leqslant s} \rho(2^j)^p\, \|\partial^\alpha \phi_j u\|^p_{L^p}, \qquad \sum_{j \geqslant 0,\, |\alpha| \leqslant s} \rho(2^j)^p\, \|\phi_j\, \partial^\alpha u\|^p_{L^p}.$$

3.3. *Sobolev Spaces Associated to Lie Algebras.* Let Z be an N-tuple of smooth vector fields on $\mathbb{R}^n$

$$Z = (Z_1, ..., Z_N),$$

such that their commutators satisfy

$$[Z_j, Z_k] = \sum_{m=1}^{N} c^m_{jk}(x)\, Z_m \tag{3.33}$$

for suitable $c^m_{jk} \in C^\infty(\mathbb{R}^n)$. It is convenient to require also that

$$|Z^\alpha c^m_{jk}(x)| \leqslant C_\alpha \tag{3.34}$$

for all x, α. Moreover, let $\rho(|x|)$ be a weight function. Then one can define, for any integer $s \geqslant 0$ and $1 \leqslant p \leqslant \infty$, the Sobolev spaces generated by Z, written $H^s_p(\rho, Z)$ through the norm

$$\|u\|^p_{H^s_p(\rho, Z)} = \sum_{|\alpha| \leqslant s} \|\rho Z^\alpha u\|^p_{L^p}. \tag{3.35}$$

In the following we shall consider only the following choice of Z:

$$Z = \langle x \rangle\, \partial_x = (\langle x \rangle\, \partial_1, ..., \langle x \rangle\, \partial_n).$$

Of special importance are the weight functions

$$\rho(|x|) = \langle x \rangle^\delta, \qquad \delta \in \mathbb{R}; \tag{3.36}$$

we shall denote the corresponding spaces by $H^{s,\delta}_p$,

$$H^{s,\delta}_p \equiv H^s_p(\langle x \rangle^\delta, \langle x \rangle\, \partial_x), \tag{3.37}$$

and in particular we shall omit p when $p=2$:

$$H^{s,\delta} \equiv H_2^{s,\delta} \equiv H_2^s(\langle x \rangle^\delta, \langle x \rangle \partial_x).$$

The $H^{s,\delta}$ spaces where introduced in [3] for integer s, in connection with elliptic systems. These spaces are especially well suited to estimate solutions of the wave equation; in order to obtain optimal results, it will be necessary to extend the definition to any real s. The simplest way would be to use interpolation and duality arguments, but the abstract spaces thus obtained are not easy to handle. Instead, we prefer to give explicit representations of the norms as in the following definition, and to recover *a posteriori* the interpolation and duality properties.

To motivate our definition, let us first rephrase the definition in the integer case in a suitable way:

Remark 3.4. Let $s \geqslant 0$ be a positive integer, $1 \leqslant p < \infty$, $\delta \in \mathbb{R}$. According to Definition (3.37), the norm of the space $H_p^{s,\delta}$, which we shall denote by X for short, has (on power p)

$$\|u\|_X^p \equiv \sum_{|\alpha| \leqslant s} \|(\langle x \rangle D)^\alpha (\langle x \rangle^\delta u)\|_{L^p}^p. \tag{3.38}$$

Noticing that

$$(\langle x \rangle D)^\alpha = \sum_{\beta \leqslant \alpha} \psi_\beta(x) D^\beta \qquad \text{with} \quad |\psi_\beta(x)| \leqslant C_\beta \langle x \rangle^{|\beta|},$$

and an identical property for $D^\alpha \langle x \rangle^{|\alpha|}$, $(D\langle x \rangle)^\alpha$, it is clear that the following equivalences hold:

$$\|u\|_X^p \sim \sum_{|\alpha| \leqslant s} \|\langle x \rangle^{|\alpha|} D^\alpha \langle x \rangle^\delta u\|_{L^p}^p \sim \sum_{|\alpha| \leqslant s} \|D^\alpha(\langle x \rangle^{|\alpha|+\delta} u)\|_{L^p}^p \tag{3.39}$$
$$\sim \sum_{|\alpha| \leqslant s} \|(D\langle x \rangle)^\alpha \langle x \rangle^\delta u\|_{L^p}^p.$$

We use now a PL partition of identity (recall (2.5)) to obtain

$$\|u\|_X^p \sim \sum_{\substack{j \geqslant 0 \\ |\alpha| \leqslant s}} \|\langle x \rangle^{|\alpha|} D^\alpha (\phi_j \langle x \rangle^\delta u)\|_{L^p}^p, \tag{3.40}$$

and by (2.3) we get

$$\|u\|_X^p \sim \sum_{\substack{j\geqslant 0\\|\alpha|\leqslant s}} \|(2^j D)^\alpha (\phi_j \langle x\rangle^\delta u)\|_{L^p}^p. \tag{3.41}$$

We introduce now the *dilation operators* S_λ, $\lambda > 0$, defined by

$$(S_\lambda u)(x) = u(\lambda x),$$

and we notice the following properties:

$$\|S_\lambda u\|_{L^p} = \lambda^{-n/p} \|u\|_{L^p}, \tag{3.42}$$

$$D^\alpha S_\lambda u = \lambda^{|\alpha|} S_\lambda D^\alpha u = S_\lambda ((\lambda D)^\alpha u), \tag{3.43}$$

$$S_{1/\lambda} D^\alpha S_\lambda u = (\lambda D)^\alpha u, \tag{3.44}$$

$$\mathscr{F} S_\lambda u = \lambda^{-n} S_{1/\lambda} \hat{u}. \tag{3.45}$$

Thus using (3.44) we may write for any even integer $s \geqslant 0$

$$\begin{aligned} \|u\|_{H_p^{s,\delta}} &\sim \sum_{\substack{j\geqslant 0\\|\alpha|\leqslant s}} \|S_{2^{-j}} D^\alpha S_{2^j} (\phi_j \langle x\rangle^\delta u)\|_{L^p}^p \\ &\sim \sum_{j\geqslant 0} \|S_{2^{-j}} (1-\Delta)^{s/2} S_{2^j} (\phi_j \langle x\rangle^\delta u)\|_{L^p}^p. \end{aligned} \tag{3.46}$$

This suggests the following definition.

DEFINITION 3.2. Let $s \in \mathbb{R}$, $1 < p < \infty$, let $\{\phi_j\}$ be a PL partition of unity, and let $\rho(|x|)$ be a strong weight (see (3.20)). The $H_p^s(\rho, \langle x\rangle\, \partial)$ norm raised to power p is defined as

$$\|u\|_{H_p^s(\rho,\langle x\rangle\,\partial)}^p = \sum_{j\geqslant 0} \|S_{2^{-j}} \Lambda^s S_{2^j} (\rho\phi_j u)\|_{L^p}^p, \tag{3.47}$$

and $H_p^s(\rho, \langle x\rangle\, \partial)$ is the Banach space of all tempered distributions such that the above norm is (defined and) finite. We shall also write

$$\Lambda_j^s = S_{2^{-j}} \Lambda^s S_{2^j}; \tag{3.48}$$

it is trivial to verify that Λ_j^s is a pseudodifferential operator, and more precisely

$$\Lambda_j^s \text{ has symbol } \langle 2^j \xi\rangle^s = (1 + 2^{2j} |\xi|^2)^{s/2}. \tag{3.49}$$

Thus we may write also

$$\|u\|^p_{H^s_p(\rho,\,\langle x\rangle\,\partial)}=\sum_{j\geqslant 0}\|\Lambda^s_j(\rho\phi_j u)\|^p_{L^p}. \tag{3.50}$$

For integer $s\geqslant 0$, this is equivalent to the norms (3.39) and (3.38).

The next lemma gives an equivalent form of the norm:

LEMMA 3.3. *For any* $1<p<\infty$, $s\in\mathbb{R}$ *and* $\rho(|x|)$ *strong weight, we have the equivalence*

$$\|u\|^p_{H^s_p(\rho,\,\langle x\rangle\,\partial)}=\sum_{j\geqslant 0}\|\Lambda^s_j(\rho\phi_j u)\|^p_{L^p}\sim\sum_{j\geqslant 0}\rho(2^j)^p\,\|\Lambda^s_j(\phi_j u)\|^p_{L^p}. \tag{3.51}$$

Proof. The equivalence of the terms with $j=0$ is obvious; for $j\geqslant 1$ we shall prove that

$$\|\Lambda^s_j(\rho\phi_j u)\|_{L^p}\leqslant C\rho(2^j)\,\|\Lambda^s_j(\phi_j u)\|_{L^p} \tag{3.52}$$

with a constant independent of j, and a similar reverse inequality, from which (3.51) follows immediately.

We recall that, for $j\geqslant 1$, $\phi_j(x)=\phi(2^{-j}x)$ (see (2.4)). Now, let $\psi\in C^\infty_c(\mathbb{R}^n)$ be equal to 1 on supp $\phi\subset\{1/2\leqslant|x|\leqslant 2\}$, and set

$$\psi_j(x)=\frac{\rho(2^j\,|x|)}{\rho(2^j)}\,\psi(x).$$

Then it is trivial to verify that

$$\Lambda^s_j(\rho\phi_j u)=S_{2^{-j}}\Lambda^s\psi_j\Lambda^{-s}S_{2^j}\Lambda^s_j(\phi_j u)\cdot\rho(2^j),$$

and in order to prove (3.52) it is sufficient to prove that the operators

$$S_{2^{-j}}\Lambda^s\psi_j\Lambda^{-s}S_{2^j}$$

are bounded on L^p uniformly in j. Since S_{2^j}, $S_{2^{-j}}$ are isomorphisms of L^p onto itself, with norms $2^{-jn/p}$, $2^{jn/p}$, respectively (see (3.42)), it is sufficient to prove that

$$\|\Lambda^s\psi_j\Lambda^{-s}\|_{\mathscr{L}(L^p)}\leqslant C,$$

with C independent of j, or equivalently that multiplication by ψ_j is bounded on H^s_p, with uniform bound in j. Thus we may use again Remark 3.2, and we are reduced to prove that

$$|D^\alpha \psi_j(x)| \leqslant C_\alpha \qquad \text{independent of } j;$$

but this is a simple consequence of property (3.20) of the strong weight ρ.

The proof of the reverse inequality is similar, using a function of the form

$$\chi_j(x) = \frac{\rho(2^j)}{\rho(2^j x)} \psi(x)$$

and recalling that $1/\rho$ is also a strong weight. ∎

Remark 3.5. An equivalent characterization of the $H^s_p(\rho, \langle x \rangle \partial)$ spaces can be given using the selfadjoint operator

$$A = D(\langle x \rangle^2 D).$$

In fact, A is a selfadjoint due to Firmani [6]. Indeed, we have

$$\|u\|^p_{H^s_p(\rho, \langle x \rangle \partial)} \sim \|A^s(\rho u)\|_{L^p}$$

(compare (3.24)). Here we shall not use this equivalent norm.

In the sequel we shall restrict ourselves to the spaces $H^{s,\delta}_p$ defined in (3.37), with norm on power p

$$\|u\|^p_{H^{s,\delta}_p} = \sum_{j \geqslant 0} \|\Lambda^s_j(\langle x \rangle^\delta \phi_j u)\|^p_{L^p} \sim \sum_{j \geqslant 0} 2^{j\delta p} \|\Lambda^s_j(\phi_j u)\|^p_{L^p}.$$

The following lemma collects a few properties of these spaces:

LEMMA 3.4. *Let $p, p_0, p_1 \in]1, \infty[$, $a, s, s_0, s_1, \delta, \delta_0, \delta_1 \in \mathbb{R}$.*

(1) *The following duality relation holds:*

$$(H^{s,\delta}_p)' = H^{-s,-\delta}_q, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.53}$$

Moreover, the complex interpolation property holds,

$$(H^{s_0,\delta_0}_{p_0}, H^{s_1,\delta_1}_{p_1})_\theta = H^{s,\delta}_p, \tag{3.54}$$

where

$$0<\theta<1,$$
$$\delta=(1-\theta)\,\delta_0+\theta\delta_1,$$
$$s=(1-\theta)\,s_0+\theta s_1,$$
$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}.$$

(2) *The following Sobolev type embeddings hold: for any* $1<p<\infty$, $\delta\in\mathbb{R}$, $s>n/p$,

$$\langle x\rangle^{\delta+n/p}\,|u(x)|\leqslant C\,\|u\|_{H_p^{s,\delta}} \tag{3.55}$$

with $C=C(p,s,\delta,n)$ *independent of* $u\in H_p^{s,\delta}$; *and for any* $1<p\leqslant q<\infty$, $\delta\in\mathbb{R}$, $s\geqslant n/p-n/q$,

$$\|\langle x\rangle^{\delta+n/p-n/q}\,u\|_{L^q}\leqslant C\,\|u\|_{H_p^{s,\delta}} \tag{3.56}$$

with $C=C(p,q,s,\delta,n)$ *independent of* $u\in H_p^{s,\delta}$. *Moreover, if* $s_0\geqslant s_1$ *and* $\delta_0\geqslant\delta_1$,

$$H_p^{s_0,\delta_0}\subseteq H_p^{s_1,\delta_1}.$$

(3) *Multiplication by a function* $\psi\in C_c^\infty(\mathbb{R}^n)$ *is a bounded operator on* $H_p^{s,\delta}$. *More generally, let* $\psi\in C^N(\mathbb{R}^n)$ *be a function constant outside a compact set, such that*

$$|D^\alpha\psi|\leqslant C_\alpha\qquad \text{for}\quad |\alpha|\leqslant N.$$

Then multiplication by ψ *is a bounded operator on* $H_p^{s,\delta}$ *provided* $|s|\leqslant N$,

$$\|\psi u\|_{H_p^{s,\delta}}\leqslant C\,\|u\|_{H_p^{s,\delta}} \tag{3.57}$$

with C *depending only on* s, δ, p *and on* C_α *for* $|\alpha|\leqslant N$.

(4) *The multiplication operator by* $\langle x\rangle^a$ *is an isometry of* $H_p^{s,\delta}$ *onto* $H_p^{s,\delta-a}$; *moreover, for any multi-index* α,

$$x^\alpha\colon H_p^{s,\delta}\to H_p^{s,\delta-|\alpha|},\qquad D^\alpha\colon H_p^{s,\delta}\to H_p^{s-|\alpha|,\delta+|\alpha|}, \tag{3.58}$$

are bounded operators. Thus in particular

$$\langle x\rangle^{|\alpha|}\,D^\alpha,\ x^\alpha D^\alpha\colon H_p^{s,\delta}\to H_p^{s-|\alpha|,\delta} \tag{3.59}$$

are bounded.

Proof. We begin by introducing the auxiliary spaces $A_p^{s,\delta}$, defined as follows: $A_p^{s,\delta}$ is the space of all sequences $\{u_j\}_{j\geqslant 0}$ with $u_j \in H_p^s$, such that the norm on power p

$$\|\{u_j\}\|^p_{A_p^{s,\delta}} = \sum_{j\geqslant 0} 2^{pj\delta} \|\Lambda_j^s u_j\|^p_{L^p} \tag{3.60}$$

is finite. Notice that

$$\|u\|_{H_p^{s,\delta}} = \|\{\phi_j u\}\|_{A_p^{s,\delta}}; \tag{3.61}$$

we shall return on this below. The space $A_p^{s,\delta}$ can be regarded as a space of type $\ell^p(A_j)$ of ℓ^p sequences with values in a sequence of Banach spaces; indeed, it is sufficient to define A_j as the Banach space of $u \in H_p^s$ with norm

$$\|u\|_{A_j} = 2^{j\delta} \|\Lambda_j^s u\|_{L^p}$$

and then

$$\|\{u_j\}\|^p_{A_p^{s,\delta}} \equiv \sum_{j\geqslant 0} \|u_j\|^p_{A_j},$$

as required.

(1) To prove the duality property, we remark that

$$(A_p^{s,\delta})' \sim A_q^{-s,-\delta}, \qquad \frac{1}{p}+\frac{1}{q}=1, \tag{3.62}$$

meaning that a $T \in (A_p^{s,\delta})'$ can be identified to a sequence $\{v_j\} \in A_q^{-s,-\delta}$ through the identity

$$T(\{u_j\}) \equiv \sum_{j\geqslant 0} v_j(u_j) \qquad \forall \{u_j\} \in A_p^{s,\delta},$$

(and of course $v_j(u_j) = \langle v_j, u_j\rangle$ is the usual duality pairing $\langle \mathscr{S}', \mathscr{S}\rangle$). The proof of (3.62) is standard. Now, let $T \in (H_p^{s,\delta})'$ and define an element $T_1 \in (A_p^{s,\delta})'$ according to the rule

$$T_1(\{u_j\}) = T\left(\sum_{j\geqslant 0} \tilde{\phi}_j u_j\right),$$

where

$$\tilde{\phi}_j = \phi_{j-1} + \phi_j + \phi_{j+1}, \qquad \phi_{-1} \equiv 0; \tag{3.63}$$

notice that $\tilde{\phi}_j \equiv 1$ on the support of ϕ_j. We know T_1 can be identified with a sequence $\{v_j\} \in A_q^{-s,-\delta}$, and

$$\sum_{j \geqslant 0} v_j(u_j) = T_1(\{u_j\}) = T\left(\sum_{j \geqslant 0} \tilde{\phi}_j u_j\right)$$

for any $\{u_j\} \in A_p^{s,\delta}$. Thus, in particular, for a fixed $u \in H_p^{s,\delta}$ we can write

$$T(u) = T\left(\sum \tilde{\phi}_j \phi_j u\right) = \sum v_j(\phi_j u) = v(u),$$

where

$$v = \sum \phi_j v_j;$$

notice the last sum is locally finite, and gives an element $v \in H_q^{-s,-\delta}$. This proves the embedding $(H_p^{s,\delta})' \subseteq H_q^{-s,-\delta}$; the reverse embedding is trivial.

To prove (3.54), we start from the interpolation property

$$(A_{p_0}^{s_0,\delta_0}, A_{p_1}^{s_1,\delta_1})_\theta = A_p^{s,\delta}, \tag{3.64}$$

with indices as for $H_p^{s,\delta}$ spaces above (for a proof, see, e.g., Section 1.18.1 of [27]).

We notice now that $H_p^{s,\delta}$ can be regarded as a *retract* of $A_p^{s,\delta}$, meaning that there exist two bounded maps

$$R: A_p^{s,\delta} \to H_p^{s,\delta}, \qquad S: H_p^{s,\delta} \to A_p^{s,\delta}$$

with the property

$$RS = I \qquad \text{on } H_p^{s,\delta};$$

R and S are called *retraction* and *coretraction* respectively (*belonging* to each other). Notice that S is an isomorphism of B with a subspace of A.

We recall the following general property of complex (and real) interpolation with respect to retractions. Assume A_j, B_j, $j = 0, 1$ are Banach spaces, embedded in some common Hausdorff vector topological space. Moreover, let R be a bounded operator from $A_0 + A_1$ to $B_0 + B_1$ whose restriction is bounded from A_j to B_j, $j = 0, 1$; similarly, let S be bounded from $B_0 + B_1$ to $A_0 + A_1$ and from B_j to A_j, $j = 0, 1$. Finally, let R be a retraction of A_j on B_j, $j = 0, 1$, with coretraction S. Then S is an isomorphism of the complex interpolation space $(B_0, B_1)_\theta$, $0 < \theta < 1$, onto a complemented subspace of $(A_0, A_1)_\theta$; this subspace is exactly the range of SR restricted to $(A_0, A_1)_\theta$, and SR is a projection onto it. For a proof see Section 1.2.4 of [27]; see also Section 6.4 of [1].

In the present case, we can define

$$R(\{u_j\}) = \sum \tilde{\phi}_j u_j, \qquad S(u) = \{\phi_j u\}. \tag{3.65}$$

It is trivial to prove that $S\colon H_p^{s,\delta} \to A_p^{s,\delta}$ is bounded, actually it is an isometry onto its image, in view of (3.61). To prove that $R\colon A_p^{s,\delta} \to H_p^{s,\delta}$ is bounded, we notice that

$$\|R(\{u_j\})\|_{H_p^{s,\delta}}^p = \sum_{j\geqslant 0} 2^{j\delta p} \left\| \Lambda_j^s \phi_j \sum_{k\geqslant 0} \tilde{\phi}_k u_k \right\|_{L^p}^p;$$

now the products $\tilde{\phi}_k \phi_j$ are different from zero only for $|j-k| \leqslant 2$, so that it is sufficient to estimate the sums

$$\Sigma_\varepsilon = \sum_{j\geqslant 0} 2^{j\delta p} \|\Lambda_j^s \phi_j \tilde{\phi}_{j+\varepsilon} u_{j+\varepsilon}\|_{L^p}^p,$$

with $\varepsilon = \pm 2, \pm 1, 0$. We show, e.g., how to estimate Σ_{-1}, the others are identical. Let $\psi_j = \tilde{\phi}_{j-1}\phi_j$; we have, for $j \geqslant 2$, $\psi_j = S_{2^j}\psi$ for a fixed function ψ with compact support (the terms for $j = 0, 1, 2$ are treated by a similar argument). Then

$$\|\Lambda_j^s \psi_j u_{j-1}\|_{L^p} = 2^{jn/p} \|\Lambda^s \psi S_{2^j} u_{j-1}\|_{L^p},$$

and noticing that multiplication by ψ is a bounded operator on H_p^s we obtain

$$\begin{aligned}\|\Lambda_j^s \psi_j u_{j-1}\|_{L^p} &\leqslant C 2^{jn/p} \|\Lambda^s S_{2^j} u_{j-1}\|_{L^p} \\ &= C\, \|\Lambda_j^s u_{j-1}\|_{L^p} = C\, \|\Lambda_j^s \Lambda_{j-1}^{-s} \Lambda_{j-1}^s u_{j-1}\|_{L^p}.\end{aligned}$$

If we can prove that $\Lambda_j^s \Lambda_{j-1}^{-s}$ is bounded on L^p with norm independent of j, we obtain

$$\Sigma_{-1} \leqslant \sum_{j\geqslant 0} 2^{j\delta p} C\, \|\Lambda_{j-1}^s u_{j-1}\|_{L^p} \leqslant C\, \|\{u_j\}\|_{A_p^{s,\delta}},$$

i.e., the thesis. Now, $\Lambda_j^s \Lambda_{j-1}^{-s}$ has symbol

$$\left(\frac{1+2^{2j}\,|\xi|^2}{1+2^{2(j-1)}\,|\xi|^2}\right)^{s/2} = \chi(2^j\xi)^{s/2}, \qquad \chi(\xi) = \frac{1+|\xi|^2}{1+|\xi|^2/4}.$$

To prove the L^p-boundedness we can use the Mikhlin theorem (see, e.g., [26]), and we need only to verify that

$$|\xi|^{|\alpha|}\,|D^\alpha_\xi \chi(2^j\xi)| \leqslant C_\alpha \tag{3.66}$$

for $|\alpha| \leqslant [n/2]+1$, with C_α independent of j; but (3.66) follows easily from the condition

$$|\xi|^{|\alpha|}\,|D^\alpha_\xi \chi(\xi)| \leqslant C_\alpha \tag{3.67}$$

and (3.67) is obvious by the definition of $\chi(\xi)$.

Now, let H be the interpolation space $(H^{s_0,\delta_0}_{p_0}, H^{s_1,\delta_1}_{p_1})_\theta$; by the above general result S is an isomorphism of H onto a subspace of $A^{s,\delta}_p$, which can be characterized as the range of SR restricted to $A^{s,\delta}_p$. Thus given $u \in H$ we know that $S(u) = \{\phi_j u\} \in A^{s,\delta}_p$, and this implies $u \in H^{s,\delta}_p$ at once by the definition; conversely, if $u \in H^{s,\delta}_p$ then it is easy to see that $\{\tilde{\phi}_j u\} \in A^{s,\delta}_p$, hence $R(\{\tilde{\phi}_j u\}) \in H$, but $R(\{\tilde{\phi}_j u\}) = \sum \phi_j \tilde{\phi}_j u = u$ and this concludes the proof.

(2) Recalling (3.2) we have, for $s > n/p$, $1 < p < \infty$,

$$\|v\|_{L^\infty} \leqslant C\,\|\Lambda^s v\|_{L^p}$$

with $C = C(s, n, p)$ independent of v. By (3.42) we get

$$\|v\|_{L^\infty} \leqslant C 2^{-jn/p}\,\|S_{2^{-j}}(\Lambda^s v)\|_{L^p}$$

and if we apply this to $v = S_{2^j}(\phi_j u)$ we obtain

$$\|\phi_j u\|_{L^\infty} \equiv \|S_{2^j}(\phi_j u)\|_{L^\infty} \leqslant C 2^{-jn/p}\,\|\Lambda^s_j(\phi_j u)\|_{L^p}$$

with a constant independent of j. This implies

$$\|\phi_j u\|^p_{L^\infty} \leqslant C \sum 2^{-jn}\,\|\Lambda^s_j(\phi_j u)\|^p_{L^p} \equiv C\,\|u\|^p_{H^{s,-n/p}_p}$$

with C independent of j, and using the fact that

$$\|u\|_{L^\infty} \leqslant \sup_{j \geqslant 0}\,\|\phi_j u\|_{L^\infty}$$

we obtain

$$\|u\|_{L^\infty} \leqslant C\,\|u\|_{H^{s,-n/p}_p}.$$

This gives (3.55) at once, using the definition of the $H^{s,\delta}_p$ norm.

The other properties are proved in a similar way, starting from the corresponding properties of the H^s_p spaces.

(3) The property is trivial for $s \geqslant 0$ integer and follows from Leibnitz rule (recall (3.38)). Thus it can be extended to $s \geqslant 0$ real using the interpolation property (3.54). Finally, it holds also for $s \leqslant 0$ using an easy duality argument and (3.53).

(4) The first property is an immediate consequence of the definition of the $H^{s,\delta}_p$ norm. Properties (3.58) are trivial for s integer and nonnegative, extend to real s by interpolation, and to negative values of s by duality. The last property (3.59) is a consequence of (3.58). ∎

4. CALCULUS IN $H^{s,\delta}$ SPACES

The following technical lemmas will be essential in order to apply the theory of Section 3.3 to solutions of the wave equation.

LEMMA 4.1. *Let $s, \delta \in \mathbb{R}$, $R \geqslant 1$. If $u \in H^{s,\delta}$ vanishes on the ball $B(0, R)$, then for all $a \geqslant 0$ we have*

$$\| |x|^{-a} u \|_{H^{s,\delta}} \leqslant CR^{-a} \|u\|_{H^{s,\delta}} \tag{4.1}$$

and

$$R^a \|u\|_{H^{s,\delta}} \leqslant C \|u\|_{H^{s,\delta+a}} \tag{4.2}$$

with $C = C(s, \delta, a)$ independent of R and u.

Proof. To prove the first inequality, we define for $|x| \geqslant 1/2$

$$\rho(|x|) = \langle x \rangle^{\delta} |x|^{-a}$$

and extend $\rho > 0$ smoothly for $|x| \leqslant 1/2$. It is easy to verify that $\rho(|x|)$ is a strong weight; taking into account that u vanishes for $|x| \leqslant 1$ (at least) we may write

$$\| |x|^{-a} u\|^2_{H^{s,\delta}} = \sum_{j \geqslant 0} \|\Lambda^s_j(\rho \phi_j u)\|^2_{L^2} = \|u\|^2_{H(\rho, \langle x \rangle \partial)} \sim \sum_{j \geqslant 0} 2^{2j(\delta - a)} \|\Lambda^s_j(\phi_j u)\|^2_{L^2},$$

where we have used Lemma 3.3 and the explicit form of the weight. Since $\phi_j u \equiv 0$ for $2^{j+1} \leqslant R$, the last sum contains only terms for which $2^{j+1} > R$ and is less than

$$R^{-2a} \sum_{2^{j+1} > R} 2^{2j\delta} \|\Lambda^s_j(\phi_j u)\|^2_{L^2} \leqslant CR^{-2a} \|u\|^2_{H^{s,\delta}}.$$

To prove the second inequality, just remark that

$$R^{2a}\,\|u\|^2_{H^{s,\delta}} \leqslant C \sum_{2^{j+1}>R} 2^{2j(\delta+a)}\,\|\Lambda^s_j(\phi_j u)\|^2_{L^2} \leqslant C\,\|u\|^2_{H^{s,\delta+a}}. \quad \blacksquare$$

LEMMA 4.2. *Let $n \geqslant 2$, $s > 1/2$, $C_0 > 1$, $R \geqslant 1$. If $u \in H^{s,-1/2}$ has support contained in the annulus*

$$\text{supp } u \subseteq \{y: C_0^{-1}R \leqslant |y| \leqslant C_0 R\},$$

then for any $x \in \mathbb{R}^n$ with $|x| \leqslant \frac{R}{2C_0}$ we have

$$\int_{|\xi|=R} |u(x+\xi)|^2\, dH^{n-1}_\xi \leqslant C\,\|u\|^2_{H^{s,-1/2}}, \tag{4.3}$$

with $C = C(C_0, s, n)$ independent of x, R and u (here dH^{n-1}_ξ denotes the $n-1$ dimensional surface (i.e., Hausdorff) measure).

Proof. For $|x| \leqslant R/(2C_0)$ and $y \in \text{supp } u$ we have

$$\frac{|y|}{2C_0^2} \leqslant \frac{R}{2C_0} \leqslant |y|-|x| \leqslant |y-x| \leqslant |y|+|x| \leqslant \left(C_0+\frac{1}{2C_0}\right)R \leqslant \left(C_0^2+\frac{1}{2}\right)|y|,$$

whence

$$C_1^{-1}\langle y\rangle \leqslant \langle y-x\rangle \leqslant C_1\langle y\rangle, \qquad C_1 = 2C_0^2.$$

Thus for any integer $s \geqslant 0$ and any $\delta \in \mathbb{R}$ we have

$$\begin{aligned} C_2^{-1} \sum_{|\alpha|\leqslant s} \langle y\rangle^{|\alpha|+\delta}\,|D^\alpha_y u(y)| &\leqslant \sum_{|\alpha|\leqslant s} \langle y-x\rangle^{|\alpha|+\delta}\,|D^\alpha_y u(y)| \\ &\leqslant C_2 \sum_{|\alpha|\leqslant s} \langle y\rangle^{|\alpha|+\delta}\,|D^\alpha_y u(y)| \end{aligned}$$

with $C_2 = C(C_0, s, \delta)$, and this gives

$$C_3^{-1}\,\|u\|_{H^{s,\delta}} \leqslant \|u(x+\cdot)\|_{H^{s,\delta}} \leqslant C_3\,\|u\|_{H^{s,\delta}}$$

for some $C_3 = C(C_0, s)$. By interpolation, this inequality is true for all real $s \geqslant 0$.

Thus we see that, in order to prove (4.3), it is sufficient to prove the inequality

$$\int_{|\xi|=R} |u(\xi)|^2 \, dH_\xi^{n-1} \leqslant C \, \|u\|^2_{H^{s,-1/2}}.$$

We can write now

$$\int_{|\xi|=R} |u(\xi)|^2 \, dH \leqslant \sum_{j\geqslant 0} \int_{|\xi|=R} |\phi_j u|^2 \, dH$$

$$= \sum_{j\geqslant 0} 2^{j(n-1)} \int_{|\xi|=R/2^j} |S_{2^j}(\phi_j u)|^2 \, dH;$$

notice that the sum is finite, indeed, $\phi_j u \equiv 0$ unless

(4.4) $$2C_0 \geqslant \frac{R}{2^j} \geqslant \frac{1}{2C_0}$$

by the assumption on supp u. We apply now the trace theorem on the ball $\Omega = B(0, R/2^j) \subseteq \mathbb{R}^n$, $n \geqslant 2$: for any $s > 1/2$

$$\|u\|_{L^2(\partial\Omega)} \leqslant C(s, \Omega) \, \|u\|_{H^s(\Omega)} \leqslant C(s, \Omega) \, \|u\|_{H^s(\mathbb{R}^n)};$$

notice that the constant $C(s, \Omega) = C(s)$ may be taken independent of Ω thanks to (4.4). Thus

$$\int_{|\xi|=R} |u(\xi)|^2 \, dH \leqslant C(s) \sum_{j\geqslant 0} 2^{j(n-1)} \, \|\Lambda^s S_{2^j}(\phi_j u)\, \|^2_{L^2}$$

$$= C(s) \sum_{j\geqslant 0} 2^{-j} \, \|S_{2^{-j}} \Lambda^s S_{2^j}(\phi_j u)\|^2_{L^2}$$

$$= C(s) \, \|u\|^2_{H^{s,-1/2}}$$

(recall property (3.42)). ∎

Of special interest are the spaces $H^{s,-s}$, whose norm on power 2 is equivalent to

$$\|u\|^2_{H^{s,-s}} \sim \sum_{j\geqslant 0} 2^{-2js} \, \|\Lambda^s_j(\phi_j u)\|^2_{L^2}.$$

LEMMA 4.3. *The spaces $H^{s,-s}$ have the following properties.*

(1) *For any $s \geqslant 0$, we have the equivalence on $H^{s,-s}$*

$$\|u\|_{H^{s,-s}} \sim \|\langle x\rangle^{-s} u\|_{L^2} + \||\xi|^s \hat{u}\|_{L^2}. \tag{4.5}$$

If in addition $0 \leqslant s < n/2$, we have the equivalence

$$\|u\|_{H^{s,-s}} \sim \||\xi|^s \hat{u}\|_{L^2}. \tag{4.6}$$

(2) *For any $\lambda > 0$, $0 \leqslant s < n/2$, we have*

$$C^{-1} \|u\|_{H^{s,-s}} \leqslant \lambda^{n/2-s} \|S_\lambda u\|_{H^{s,-s}} \leqslant C \|u\|_{H^{s,-s}} \tag{4.7}$$

with $C = C(s, n)$ independent of λ and $u \in H^{s,-s}$.

(3) *For any $s \geqslant 0$ we have*

$$\|u\|_{H^{-s,s}} \leqslant C \|\langle x\rangle^s u\|_{L^2} \tag{4.8}$$

with $C = C(s, n)$ independent of $u \in H^{-s,s}$.

(4) *For any $s > -n/2$ we have*

$$\||\xi|^s \hat{u}\|_{L^2} \leqslant C \|u\|_{H^{s,-s}} \tag{4.9}$$

with $C = C(s, n)$ independent of $u \in H^{s,-s}$.

Proof. (1) Thanks to the interpolation property (3.54), it is sufficient to prove (4.5) only when $s \geqslant 0$ is an even integer. Notice that for integer s

$$\||\xi|^s \hat{u}\|_{L^2}^2 \sim \sum_{|\alpha|=s} \|D^\alpha u\|_{L^2}^2.$$

We begin by showing

$$\Sigma_{1 \leqslant |\alpha| \leqslant m-1} \|D^\alpha u\|_{L^2(U)} \leqslant C(\|u\|_{L^2(U)} + \Sigma_{|\alpha|=m} \|D^\alpha u\|_{L^2(U)}), \tag{4.10}$$

for any positive integer m and for any open set $U \subset \mathbb{R}^n$ with smooth boundary. For simplicity, we write

$$\|u\|_{\dot{H}^j(U)} = \Sigma_{|\alpha|=j} \|D^\alpha u\|_{L^2(U)}, \qquad \|u\|_{H^m(U)} = \Sigma_{0 \leqslant j \leqslant m} \|u\|_{\dot{H}^j(U)}.$$

By Theorem 9.6 in [15] we have

$$\begin{aligned} \|u\|_{\dot{H}^j(U)} &\leqslant C_j \|u\|_{L^2(U)}^{1-j/m} \|u\|_{H^m(U)}^{j/m} \\ &\leqslant C_j \varepsilon \|u\|_{H^m(U)} + C_{j,\varepsilon} \|u\|_{L^2(U)} \end{aligned}$$

for any $\varepsilon > 0$. Setting $S = \Sigma_{1 \leqslant j \leqslant m-1} \|u\|_{\dot H^j(U)}$, we get

$$S \leqslant C_m \varepsilon S + C_m \varepsilon \|u\|_{\dot H^m(U)} + C_{m,\varepsilon} \|u\|_{L^2(U)},$$

because $\|u\|_{H^m(U)} \leqslant \|u\|_{L^2(U)} + S + \|u\|_{\dot H^m(U)}$. If we choose ε sufficiently small, we obtain (4.10).

Now recall that, for $j \geqslant 1$, $\phi_j = \phi(2^{-j}x)$ with ϕ defined in (2.4), and let $\tilde\phi(\xi) = \phi(\xi) + \phi(2\xi) + \phi(\xi/2) + \phi(\xi/4)$. By analogy we write for $j \geqslant 2$

$$\tilde\phi_j = \tilde\phi(2^{-j}x) = \phi_{j-1}(x) + \phi_j(x) + \phi_{j+1}(x) + \phi_{j+2}(x),$$

and we define

$$\tilde\phi_1 = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x), \qquad \tilde\phi_0 = \phi_0(x) + \phi_1(x) + \phi_2(x).$$

Notice that $\tilde\phi(\xi) = 1$ on $U = \{1/2 \leqslant |\xi| \leqslant 4\}$ and that $\operatorname{supp} \phi \subset \{1/2 \leqslant |\xi| \leqslant 2\}$. Then we can write (recall s is an even integer)

$$\begin{aligned}
\|\Lambda^s(\phi w)\|_{L^2}^2 &\leqslant c(s) \sum_{|\alpha| \leqslant s} \|D^\alpha(\phi w)\|_{L^2}^2 \\
&\leqslant c(s) \sum_{|\alpha| \leqslant s} \|D^\alpha w\|_{L^2(U)}^2 \\
&\leqslant c(s) \left(\|w\|_{L^2(U)}^2 + \sum_{|\alpha| = s} \|D^\alpha w\|_{L^2(U)}^2 \right) \\
&\leqslant c(s) \left(\|\tilde\phi w\|_{L^2}^2 + \sum_{|\alpha| = s} \|\tilde\phi D^\alpha w\|_{L^2}^2 \right).
\end{aligned}$$

Hence for $j \geqslant 2$, using property (3.42) and (3.43),

$$\begin{aligned}
\|\Lambda_j^s(\phi_j u)\|_{L^2}^2 &= 2^{nj} \|\Lambda^s \phi S_{2^j} u\|_{L^2}^2 \\
&\leqslant C 2^{nj} \left(\|\tilde\phi S_{2^j} u\|_{L^2}^2 + \sum_{|\alpha| = s} \|\tilde\phi D^\alpha S_{2^j} u\|_{L^2}^2 \right) \\
&= C \left(\|\tilde\phi_j u\|_{L^2}^2 + 2^{2sj} \sum_{|\alpha| = s} \|\tilde\phi_j D^\alpha u\|_{L^2}^2 \right);
\end{aligned}$$

the same estimate holds true for $j = 0, 1$ by an almost identical proof. Since

$$\|u\|_{H^{s,-s}}^2 \sim \sum_{j \geqslant 0} 2^{-2js} \|\Lambda_j^s(\phi_j u)\|_{L^2}^2,$$

we obtain

$$\|u\|^2_{H^{s,-s}} \leqslant C \sum_{j\geqslant 0} 2^{-2sj} \|\tilde{\phi}_j u\|^2_{L^2} + C \sum_{\substack{|\alpha|=s \\ j\geqslant 0}} \|\tilde{\phi}_j D^\alpha u\|^2_{L^2}$$

$$\leqslant C \|\langle x\rangle^{-s} u\|^2_{L^2} + C \sum_{|\alpha|=s} \|D^\alpha u\|^2_{L^2}.$$

Conversely, we have (for $s \geqslant 0$)

$$\|\Lambda^s_j w\|^2_{L^2} = 2^{nj} \|\Lambda^s S_{2^j} w\|^2_{L^2}$$

$$\sim 2^{nj} \left(\|S_{2^j} w\|^2_{L^2} + \sum_{|\alpha|=s} \|D^\alpha S_{2^j} w\|^2_{L^2} \right)$$

$$= \|w\|^2_{L^2} + 2^{2sj} \sum_{|\alpha|=s} \|D^\alpha w\|^2_{L^2},$$

hence

$$\|u\|^2_{H^{s,-s}} \sim \sum_{j\geqslant 0} 2^{-2js} \|\phi_j u\|^2_{L^2} + \sum_{\substack{|\alpha|=s \\ j\geqslant 0}} \|D^\alpha(\phi_j u)\|^2_{L^2}.$$

The first term is equivalent to $\|\langle x\rangle^{-s} u\|^2_{L^2}$, and to handle the second it is sufficient to write

$$\sum_{|\alpha|=s} \left\| \sum_{j\geqslant 0} D^\alpha(\phi_j u) \right\|^2_{L^2} \leqslant 2 \sum_{|\alpha|=s} \sum_{j\geqslant 0} \|D^\alpha(\phi_j u)\|^2_{L^2}$$

since $\phi_j \phi_k \equiv 0$ for $|j-k| \geqslant 2$. This give the second inequality need to prove (4.5).

To prove (4.6), it is sufficient to show the inequality

$$\|u\|_{H^{s,-s}} \leqslant C \||\xi|^s \hat{u}\|_{L^2},$$

in view of (4.5). Indeed, for nonnegative s we have

$$\|\langle x\rangle^{-s} u\|_{L^2} \leqslant \||x|^{-s} u\|_{L^2} \leqslant C \||\xi|^s \hat{u}\|_{L^2}$$

where the last inequality is true for $s < n/2$, thanks to the extended Hardy inequality (9.5). By (4.5), we conclude the proof.

(2) By (3.45) and (3.42), we have

$$\||\xi|^s \mathscr{F} S_\lambda u\|_{L^2} = \lambda^{-n} \||\xi|^s S_{1/\lambda} \hat{u}\|_{L^2} = \lambda^{s-n} \|S_{1/\lambda} |\xi|^s \hat{u}\|_{L^2} = \lambda^{s-n/2} \||\xi|^s \hat{u}\|_{L^2}.$$

Recalling (4.6), we obtain

$$\|S_\lambda u\|_{H^{s,-s}} \sim \lambda^{s-n/2} \|u\|_{H^{s,-s}}.$$

(3) Recall that

$$\|u\|^2_{H^{-s,s}} \sim \sum_{j\geqslant 0} 2^{2js} \|\Lambda_j^{-s}(\phi_j u)\|^2_{L^2}.$$

For $s \geqslant 0$ we have $\|\Lambda^{-s} v\|_{L^2} \leqslant \|v\|_{L^2}$, so that

$$\|u\|^2_{H^{-s,s}} \leqslant C \sum_{j\geqslant 0} 2^{2js} \|\phi_j u\|^2_{L^2} \sim \sum_{j\geqslant 0} \|\phi_j \langle x\rangle^s u\|^2_{L^2} \sim \|\langle x\rangle^s u\|^2_{L^2}.$$

(4) For $s \geqslant 0$ the inequality is a consequence of (4.5). Assume now $s < 0$, and define the Hilbert spaces

$$A = L^2(\mathbb{R}^n_x, \langle x\rangle^s dx),$$
$$B_1 = L^2(\mathbb{R}^n_\xi, |\xi|^{-s} d\xi).$$

If $s > -n/2$, we have

$$B_1 \subseteq L^1_{loc} \subseteq \mathscr{S}',$$

since on any compact set K

$$\int_K |v| \, d\xi = \int_K |v| \, |\xi|^s \, |\xi|^{-s} \, d\xi \leqslant \|v\|_{B_1} \left(\int_K |\xi|^{2s} \, d\xi \right)^{1/2}$$

and the last integral is finite for $s > -n/2$. Thus we can define the Hilbert space

$$B = \mathscr{F}^{-1}(B_1).$$

Formula (4.5) with s replaced by $-s$ (since now $-s \geqslant 0$) can be written

$$\|u\|_{H^{-s,s}} \sim \|u\|_A + \|u\|_B,$$

i.e., we have the isomorphism of Hilbert spaces

$$H^{-s,s} \sim A \cap B.$$

Hence

$$H^{s,-s} = (H^{-s,s})' \sim (A \cap B)' \sim A' + B'$$

and by general properties of Hilbert spaces we can write

$$\|u\|_{A'+B'} \sim \inf_{u=u_1+u_2} (\|u_1\|_A + \|u_2\|_B),$$

where the infimum is taken over all decompositions $u = u_1 + u_2$ with $u_1 \in A'$ and $u_2 \in B'$. This means, for $0 \geqslant s > -n/2$,

$$\|u\|_{H^{s,-s}} \sim \inf_{u=u_1+u_2} (\|\langle x\rangle^{-s} u_1\|_{L^2} + \||\xi|^s \hat{u}_2\|_{L^2}). \tag{4.11}$$

Now take $u \in H^{s,-s}$; for any decomposition $u = u_1 + u_2$, by the extended Hardy inequality (9.5) proved in the Appendix,

$$\begin{aligned}\||\xi|^s \hat{u}\|_{L^2} &\leqslant \||\xi|^s \hat{u}_1\|_{L^2} + \||\xi|^s \hat{u}_2\|_{L^2} \\ &\leqslant C \,\||x|^{-s} u_1\|_{L^2} + \||\xi|^s \hat{u}_2\|_{L^2} \\ &\leqslant C \,\|\langle x\rangle^{-s} u_1\|_{L^2} + \||\xi|^s \hat{u}_2\|_{L^2},\end{aligned}$$

and, by (4.11), this implies (4.9). ∎

THEOREM 4.4 (Special Hardy Inequality). *Let $s \in [0, 1/2[$, $\lambda \geqslant 0$. Then*

$$\left\|\frac{u}{||x|-\lambda|^s}\right\|_{L^2} \leqslant C \,\|u\|_{H^{s,-s}} \tag{4.12}$$

with $C = C(s, n)$ independent of $u \in H^{s,-s}$, λ.

Proof. When $\lambda = 0$, (4.11) is a consequence of the extended Hardy inequality (Theorem 9.2) and of property (4.9). Thus we shall consider $\lambda > 0$.

Assume first $\lambda = 1$. Let $\psi_0, \psi_1, ..., \psi_{2n+1}$ be C^∞ functions on $\mathbb{R}^n$ such that $\sum_{j=0}^{2n+1} \psi_j = 1$, the support of ψ_0 is the closed ball $\overline{B(0, 1/2)}$, the support of ψ_{2n+1} is $\mathbb{R}^n \setminus B(0, 2)$, and the supports of ψ_j for $1 \leqslant j \leqslant 2n$ are compact and contained in one of the open half spaces $\pm x_j > 0$. We can write $u = \sum_{j=0}^{2n+1} u_j$, $u_j = \psi_j u$. We have trivially

$$\left\|\frac{u_0}{||x|-1|^s}\right\|_{L^2} + \left\|\frac{u_{2n+1}}{||x|-1|^s}\right\|_{L^2} \leqslant C(s) \,\|\langle x\rangle^{-s} u\|_{L^2} \leqslant C(s) \,\|u\|_{H^{s,-s}} \tag{4.13}$$

by property (4.5).

Now, consider the u_j for $j = 1, ..., 2n$. We can assume, e.g., that supp $u_j = K$ is contained in $x_n > 0$. Consider the map $x = \Phi(y)$ defined by

$$x_1 = y_1, ..., x_{n-1} = y_{n-1},$$

and

$$x_n = [(1+y_n)^2 - (y_1^2 + \cdots + y_{n-1}^2)]^{1/2}.$$

Writing $K' = \Phi^{-1}(K)$, it is clear that Φ is a diffeomorphism of a neighbourhood of K' onto a neighbourhood of K; notice that Φ maps $K' \cap \{y_n = 0\}$ onto $K \cap S^{n-1}$. We can modify Φ outside K' in such a way that $\Phi = I$ (the identity map of $\mathbb{R}^n$) outside a compact set, Φ is C^∞ and globally invertible on $\mathbb{R}^n$. Hence, writing

$$v = u_j \circ \Phi,$$

we have

$$\left\| \frac{u_j}{||x|-1|^s} \right\|_{L^2} \leqslant C(\Phi) \left\| \frac{v}{|y_n|^s} \right\|_{L^2}.$$

Since $s < 1/2$, we can apply the extended Hardy inequality (Theorem 9.2 in the Appendix) with respect to the variable y_n, Denoting by $\mathscr{F}_n$ the partial Fourier transform with respect to y_n, we have

$$\left\| \frac{v}{|y_n|^s} \right\|_{L^2} \leqslant C \, \||\xi_n|^s \, \mathscr{F}_n v\|_{L^2} = C \, \||\xi_n|^s \, \hat{v}\|_{L^2}$$

by Plancherel's identity (with respect to $y_1, ..., y_{n-1}$). We thus obtain

$$\left\| \frac{u_j}{||x|-1|^s} \right\|_{L^2} \leqslant C\||\xi_n|^s \, \hat{v}\|_{L^2} \leqslant C\|\langle\xi\rangle^s \hat{v}\|_{L^2} = C\|v\|_{H^s}.$$

Remark now that the linear operator

$$T\colon H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

defined as

$$T(g) = g \circ \Phi$$

is bounded for all real $s \geqslant 0$. Indeed, for integer s this follows by standard differentiation of composite functions, and for real s by interpolation. This implies

$$\|v\|_{H^s} = \|T(u_j)\|_{H^s} \leqslant C(s, \Phi) \, \|u_j\|_{H^s}.$$

Since ψ_j has compact support (independent of u) we have

$$\|u_j\|_{H^s} \leqslant C(\|u_j\|_{L^2} + \||\xi|^s \hat{u}_j\|_{L^2}) \leqslant C(\|\langle x\rangle^{-s} u_j\|_{L^2} + \||\xi|^s \hat{u}_j\|_{L^2})$$

and by property (4.5) we obtain

$$\|u_j\|_{H^s} \leqslant C \|u_j\|_{H^{s,-s}}.$$

Summing up, we have proved that

$$\left\| \frac{u_j}{||x|-1|^s} \right\|_{L^2} \leqslant C \|u_j\|_{H^{s,-s}},$$

and using the fact that multiplication by ψ_j is a bounded operator on $H^{s,\delta}$ (Lemma 3.11), we obtain

$$\left\| \frac{u_j}{||x|-1|^s} \right\|_{L^2} \leqslant C \|u\|_{H^{s,-s}}.$$

Together with (4.13), this proves the thesis for $\lambda = 1$.

For general $\lambda > 0$, we can write

$$\left\| \frac{u}{||x|-\lambda|^s} \right\|_{L^2} = \left\| S_{1/\lambda} S_\lambda \frac{u}{||x|-\lambda|^s} \right\|_{L^2} = \left\| S_{1/\lambda} \frac{S_\lambda u}{\lambda^s ||x|-1|^s} \right\|_{L^2}$$

and by property (3.42) we obtain

$$\left\| \frac{u}{||x|-\lambda|^s} \right\|_{L^2} = \lambda^{n/2-s} \left\| \frac{S_\lambda u}{||x|-1|^s} \right\|_{L^2} \leqslant C\lambda^{n/2-s} \|S_\lambda u\|_{H^{s,-s}}$$

using the thesis for $\lambda = 1$ already proved. Recalling property (4.7), we conclude the proof. ∎

Lemma 4.5. *For any real $b > -n/2$, $s \geqslant 0$, and any $\hat{g} \in H^{s+b,-b}$*

$$N_s(g) = \int_0^\infty \rho^{n-1+2b} \|g(\rho\,\cdot)\|^2_{H^s(S^{n-1})}\, d\rho \leqslant C \|\hat{g}\|^2_{H^{s+b,-b}}, \tag{4.14}$$

with $C = C(n, s, b)$ independent of g.

Proof. Denote by Ω_{jk}, $1 \leqslant j < k \leqslant n$, the operators

$$\Omega_{jk} = x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}.$$

As it is well known, the family $\Omega = \{\Omega_{jk}\}$ generates the Sobolev spaces $H^k(S^{n-1})$ on S^{n-1}, the unit sphere in $\mathbb{R}^n$. In other words

$$\|u\|^2_{H^k(S^{n-1})} = \sum_{|\alpha| \leqslant k} \|\Omega^\alpha u\|^2_{L^2(S^{n-1})},$$

where we used the customary multi-index notation $\Omega^\alpha = \Omega_{12}^{\alpha_{12}} \cdots \Omega_{n-1,n}^{\alpha_{n-1,n}}$. This implies, for all integer $k \geqslant 0$,

$$\|g(\rho\,\cdot)\|^2_{H^k(S^{n-1})} \leqslant C(k) \sum_{|\alpha| \leqslant k} \rho^{2|\alpha|} \|D^\alpha g(\rho\,\cdot)\|^2_{L^2(S^{n-1})}$$

and hence

$$\begin{aligned} N_k(g) &\leqslant C(k) \int_0^\infty \rho^{n-1+2b} \sum_{|\alpha| \leqslant k} \rho^{2|\alpha|} \|D^\alpha g(\rho\,\cdot)\|^2_{L^2(S^{n-1})} \\ &= C(k) \sum_{|\alpha| \leqslant k} \||x|^{b+|\alpha|} D^\alpha g(x)\|^2_{L^2(\mathbb{R}^n)} \end{aligned}$$

(recall (3.39)). Since $b+|\alpha| > -n/2$, we can apply (4.9), and we obtain

$$\begin{aligned} \||x|^{b+|\alpha|} D^\alpha g\|^2_{L^2} &\leqslant C \|\mathcal{F}(D^\alpha g)\|^2_{H^{b+|\alpha|,\,-b-|\alpha|}} \\ &\equiv C \|\xi^\alpha \hat{g}\|_{H^{b+|\alpha|,\,-b-|\alpha|}} \leqslant C \|\hat{g}\|^2_{H^{b+|\alpha|,\,-b}}, \end{aligned}$$

where in the last step we have used property (3.58). Thanks to the continuous embedding $H^{b+|\alpha|,\,-b} \subseteq H^{b+k,\,-b}$ if $|\alpha| \leqslant k$, we see that we have proved (4.14) for $s = k$ integer.

The proof for real $s \geqslant 0$ follows by complex interpolation. Indeed, define the Hilbert spaces X^s, $s \in \mathbb{R}$, as

$$X^s = L^2(\mathbb{R}^+, \rho^{2b} d\rho; H^s(S^{n-1}));$$

notice that their norm is exactly

$$\|u\|^2_{X^s} = N_s(u).$$

These space interpolate by standard results (see, e.g., [1]), and we have $(X^{s_0}, X^{s_1})_\theta = X^s$ with $s = (1-\theta)\, s_0 + \theta s_1$. In the first part of the proof we have showed that the embedding

$$X^k \subseteq H^{k+b,\,-b}$$

is bounded for any integer $k \geqslant 0$. Recalling (3.54), the proof is concluded. ∎

Our final technical lemma concerns the operators $I^{\pm}_a$, $a \in \mathbb{R}$, defined as

$$I^{\pm}_a(v)(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi \pm t|\xi|)} \frac{\hat{v}}{|\xi|^a}\, d\xi.$$

LEMMA 4.6. *For any $a < (n+1)/2$, $s > n/2 - a$, $\delta > a - 1/2$ and all $v \in H^{s,\delta}$ we have for $x \neq 0$*

$$|I^{\pm}_a(v)(t,x)| \leqslant C\,|x|^{-\frac{n-1}{2}}\,\|v\|_{H^{s,\delta}}, \tag{4.15}$$

with $C = C(n, s, \delta, a)$ independent of v, t, x.

Proof. We shall consider $I_a = I^+_a$ only, the proof for I^-_a being identical. We have in polar coordinates

$$I_a(v) = \int_0^\infty e^{it\rho} \rho^{n-1-a} J(x,\rho),$$

where

$$J(x,\rho) = \int_{|\xi|=1} e^{ix\cdot\xi\rho}\hat{v}(\rho\xi)\, dH^{n-1}_\xi.$$

By the stationary phase theorem (9.4) (see the Appendix) we have, for any $s > (n-1)/2$,

$$|J(x,\rho)| \leqslant C(s)(\rho\,|x|)^{-\frac{n-1}{2}}\,\|\hat{v}(\rho\,\cdot)\|_{H^s(S^{n-1})}$$

so that

$$|I_a| \leqslant C(s)\int_0^\infty \rho^{\frac{n-1}{2}-a}\,\|\hat{v}(\rho\,\cdot)\|_{H^s}\, d\rho.$$

We now split the last integral on the intervals $[0,1]$ and $[1,\infty[$. Thus for any $\varepsilon > 0$ we have, by the Cauchy inequality,

$$\int_0^1 \rho^{-\frac{1}{2}+\varepsilon}\rho^{\frac{n}{2}-a-\varepsilon}\,\|\hat{v}(\rho\,\cdot)\|_{H^s}\, d\rho \leqslant C(\varepsilon)\left(\int_0^\infty \rho^{n-2(a+\varepsilon)}\,\|\hat{v}(\rho\,\cdot)\|^2_{H^s}\, d\rho\right)^{1/2},$$

and similarly

$$\int_1^\infty \rho^{-\frac{1}{2}-\varepsilon}\rho^{\frac{n}{2}-a+\varepsilon}\,\|\hat{v}(\rho\,\cdot)\|_{H^s}\, d\rho \leqslant C(\varepsilon)\left(\int_0^\infty \rho^{n-2(a-\varepsilon)}\,\|\hat{v}(\rho\,\cdot)\|^2_{H^s}\, d\rho\right)^{1/2}.$$

We are now in position to apply Lemma 4.5, provided

$$\frac{1}{2}-(a+\varepsilon)>-\frac{n}{2}, \qquad \frac{1}{2}-(a-\varepsilon)>-\frac{n}{2},$$

i.e.,

$$a<\frac{n+1}{2}.$$

Thus we obtain

$$|I_a(v)| \leqslant C\,|x|^{-\frac{n-1}{2}}\,(\|v\|_{H^{s+\frac{1}{2}-(a+\varepsilon),\,-\frac{1}{2}+a+\varepsilon}}+\|v\|_{H^{s+\frac{1}{2}-(a-\varepsilon),\,-\frac{1}{2}+a-\varepsilon}}).$$

Recalling that $s>(n-1)/2$ is arbitrary, we conclude the proof. ∎

5. PROOF OF THEOREM 1.1 FOR $d=0$

In the course of the proof we shall make use of three different classical representations of the solution $u(t, x)$. The first one is the Fourier representation

$$u(t,\xi)=C\int e^{ix\cdot\xi}\cos(t\,|\xi|)\,\hat{u}_0(\xi)\,d\xi+C\int e^{ix\cdot\xi}\,\frac{\sin(t\,|\xi|)}{|\xi|}\,\hat{u}_1(\xi)\,d\xi. \tag{5.1}$$

The second one is the representation by spherical means (see, e.g., [5]): for $n\geqslant 3$ odd,

$$u(t,x)=\sum_{\nu=0}^{(n-1)/2} b_\nu t^\nu\,\partial_t^\nu M(u_0)+\sum_{\nu=0}^{(n-3)/2} a_\nu t^{\nu+1}\,\partial_t^\nu M(u_1), \tag{5.2}$$

and for $n\geqslant 2$ even

$$u(t,x)=\sum_{\nu=0}^{n/2} b'_\nu t^\nu\,\partial_t^\nu N(u_0)+\sum_{\nu=0}^{(n-2)/2} a'_\nu t^{\nu+1}\,\partial_t^\nu N(u_1), \tag{5.3}$$

where

$$M(g)(t,x)=\int_{|\xi|=1} g(x+t\xi)\,dH_\xi^{n-1}, \tag{5.4}$$

$$N(g)(t,x)=\int_{|y|\leqslant 1} g(x+ty)(1-|y|^2)^{-1/2}\,dy. \tag{5.5}$$

The third one is the distributional representation (see e.g., [12])

$$u(t, x) = \partial_t E(t, \cdot) * u_0(\cdot) + E(t, \cdot) * u_1(\cdot), \tag{5.6}$$

where $E(t, x)$ is the distribution equal to 0 for $t \leqslant 0$ and

$$E(t, x) = c_n \chi_{\frac{n-1}{2}}(t^2 - |x|^2) \tag{5.7}$$

for $t > 0$. It will not be necessary to give here the complete construction of $\chi_s(t^2 - |x|^2)$, which can be found in [12]; all we need to know here is that the singular support of $E(t, x)$ is the forward light cone $C_+ = \{t = |x|, t > 0\}$, and, outside C_+, $E(t, x)$ is a smooth function. On the exterior of this cone, i.e., for $t < |x|$, $E(t, x)$ vanishes identically, for any space dimension n. On the other hand, the behaviour on the interior of the cone depends on the space dimension. Indeed, when $n \geqslant 2$ is even or for $n = 1$ E coincides with the smooth function

$$E(t, x) = c_n (t^2 - |x|^2)^{-\frac{n-1}{2}} \tag{5.8}$$

for $t > |x|$; while for $n \geqslant 3$ odd, $E(t, x) \equiv 0$ (Huygens' principle).

We have to prove the estimate

$$(1 + t + |x|)^{(n-1)/2} \, |u(t, x)| \leqslant C(\|u_0\|_{H^{s_0, \delta_0}} + \|u_1\|_{H^{s_1, \delta_1}}) \tag{5.9}$$

for

$$s_0 > \frac{n}{2}, \quad \delta_0 > -\frac{1}{2}, \quad s_1 > \frac{n}{2} - 1, \quad \delta_1 > \frac{1}{2}.$$

Fix $(t, x) \in \mathbb{R}^{n+1}_+$ and consider the following possibilities: $t + |x| \leqslant 1$; $t + |x| \geqslant 1$ and $|x| \leqslant t/16$; $|x| \geqslant t/16$.

5.1. *Case* A.$t + |x| \leqslant 1$. By finite speed of propagation, we can freely replace the initial data u_0, u_1 by $\phi_0 u_0, \phi_0 u_1$ with ϕ_0 in (2.4). Recalling the Fourier representation (5.1), we can write (we write for simplicity again u_j instead of $\phi_0 u_j$ in the following formulas)

$$|u(t, x)| \leqslant C \int |\hat{u}_0| \, d\xi + C \int_{|\xi| \leqslant 1} \frac{|\hat{u}_1|}{|\xi|} \, d\xi + C \int_{|\xi| \geqslant 1} \frac{|\hat{u}_1|}{|\xi|} \, d\xi.$$

We introduce in the first integral $\langle \xi \rangle^{s_0} \langle \xi \rangle^{-s_0}$ with $s_0 > n/2$, in the third one $\langle \xi \rangle^{s_1} \langle \xi \rangle^{-s_1}$ with $s_1 > n/2 - 1$, while in the second one we split $|\xi|^{-1}$ as $|\xi|^{-1/2-\varepsilon} \cdot |\xi|^{-1/2+\varepsilon}$ with $\varepsilon > 0$. Thus by Cauchy inequality we get

$$|u(t, x)| \leqslant C \, [\|\langle \xi \rangle^{s_0} \, \hat{u}_0\|_{L^2} + \|\hat{u}_1 \, |\xi|^{-1/2-\varepsilon}\|_{L^2} + \|\langle \xi \rangle^{s_1} \, \hat{u}_1\|_{L^2}]$$

and by the extended Hardy inequality (9.5) in the Appendix, we obtain

$$|u(t,x)| \leqslant C[\|u_0\|_{H^{s_0}} + \|\langle x\rangle^{1/2+\varepsilon} u_1\|_{L^2} + \|u_1\|_{H^{s_1}}].$$

Recall now that we have modified the data by multiplication by ϕ_0. Having in mind (3.47), we see that

$$\|\phi_0 u_0\|_{H^{s_0}} \leqslant C\, \|u_0\|_{H^{s_0,\delta_0}}, \qquad \|\phi_0 u_1\|_{H^{s_1}} \leqslant C\, \|u_1\|_{H^{s_1,\delta_1}}.$$

And recalling that $H^{0,\delta} \subseteq H^{s,\delta}$ for all $s \geqslant 0$, we obtain (5.9).

5.2. *Case* B. $t+|x| \geqslant 1$ *and* $|x| \leqslant t/16$. Fix $\psi \in C_c^\infty(\mathbb{R}^n)$ with support contained in the ball $B(0, 2)$ and equal to one in the ball $B(0, 17/16)$. Since $1 \leqslant t+|x| \leqslant 17t/16$, we have

$$|D_y^\alpha \psi(y/t)| \leqslant C_\alpha$$

for all α, with C_α independent of t. Thus, by (3.57), multiplication by $\psi(y/t)$ is a bounded operator on $H^{s,\delta}$, with norm independent of t. By the finite speed of propagation, if we multiply the data u_0, u_1 by $\psi(y/t)$ the value of u at (t, x) is unchanged. In conclusion, we see that we can assume that

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subseteq B(0, 2t).$$

With a similar argument we see that we can split the data as

$$u_j = u_j^I + u_j^{II}$$

in such a way that

$$\text{(5.10)} \qquad \operatorname{supp} u_j^I \subseteq B(0, t/4), \qquad \operatorname{supp} u_j^{II} \subseteq \{y\colon t/8 \leqslant |y| \leqslant 2t\}$$

and it is sufficient to prove (5.9) for the solutions u^I, u^{II} corresponding to the initial data u_j^I, u_j^{II} respectively.

In order to estimate u^{II} we use the representations (5.2), (5.3). We can write, for $g = u_j^{II}$ $(j=0, 1)$ and $k=0, 1$,

$$t^{\nu+k}\, \partial_t^\nu M(g) = \int_{|\xi|=1} t^k((t\xi\cdot\partial)^\nu g)(x+t\xi)\, dH_\xi^{n-1}$$
$$\leqslant C \sum_{|\alpha|=\nu} t^{1-n} \int_{|\eta|=t} \langle\eta\rangle^{\nu+k}\, |(D^\alpha g)(x+\eta)|\, dH_\eta^{n-1}$$

and by Cauchy inequality, recalling that the measure of $\{|\xi|=t\}$ is $c_n t^{n-1}$, we obtain

$$|t^{\nu+k}\,\partial_t^\nu M(g)| \leqslant C t^{\frac{1-n}{2}} \sum_{|\alpha|=\nu} \left(\int_{|\xi|=t} |\langle\xi\rangle^{|\alpha|+k}\,(D^\alpha g)(x+\xi)|^2\, dH_\xi^{n-1} \right)^{1/2}.$$

Notice that, in the last integral, $\langle\xi\rangle=\langle t\rangle$, while $\langle x+\xi\rangle \geqslant \langle t/8\rangle$ by the definition of u_j^{II}, hence $\langle\xi\rangle \leqslant 8\langle x+\xi\rangle$ and we can estimate the right hand member by replacing $\langle\xi\rangle^{|\alpha|+k}$ with $\langle x+\xi\rangle^{|\alpha|+k}$. Now we can apply Lemma 4.2 with $R=t$, $C_0=8$ (recall (5.10) and notice that $|x| \leqslant t/16 = t/(2C_0)$); we obtain for all $s>1/2$ and some $C=C(s,n)$ independent of t, x, g

$$|t^{\nu+k}\,\partial_t^\nu M(g)| \leqslant C t^{\frac{1-n}{2}} \sum_{|\alpha|=\nu} \|\langle\xi\rangle^{|\alpha|+k}\, D^\alpha g\|_{H^{s,-1/2}} \leqslant C t^{\frac{1-n}{2}} \|g\|_{H^{s+\nu,-1/2+k}},$$

where we have used property (3.59). In conclusion, by formula (5.2) we get

$$|u^{II}| \leqslant C t^{\frac{1-n}{2}} (\|u_0\|_{H^{s_0,-1/2}} + \|u_1\|_{H^{s_1,1/2}})$$

for any $s_0>n/2$, $s_1>(n-2)/2$, whence (5.9) for u^{II} follows, since $1+t+|x| \leqslant 2(t+|x|) \leqslant 17t/8$.

The computation for n even is similar. We must estimate, for $g=u_j^{II}$ $(j=0,1)$,

$$\begin{aligned} t^{\nu+k}\,\partial_t^\nu N(g) &= \int_{|y|\leqslant 1} t^k((ty\cdot\partial)^\nu g)(x+ty)(1-|y|^2)^{-1/2}\,dy \\ &= t^{k-n+1} \int_{|y|\leqslant t} ((y\cdot\partial)^\nu g)(x+y)(t^2-|y|^2)^{-1/2}\,dy. \end{aligned}$$

Now for $x+y \in \operatorname{supp} g$, and $|x| \leqslant t/16$,

$$|y| = |x+y-x| \geqslant \frac{t}{8} - \frac{t}{16} = \frac{t}{16},$$

so that

$$(t^2-|y|^2)^{-1/2} \leqslant (t^2 - t^2/256)^{-1/2} = \frac{c}{t}$$

and also

$$|y| = |x+y-x| \leqslant 2t + \frac{t}{16} \leqslant \frac{33}{2}\,|x+y|, \tag{5.11}$$

(i.e., $|y| \sim t \sim |x+y|$), so that, for any real ℓ,

$$\begin{aligned}|t^{\nu+k}\,\partial_t^\nu N(g)| &\leqslant Ct^{k-n}\sum_{|\alpha|=\nu}\int_{|y|\leqslant t}|y|^\nu\,|D^\alpha g(x+y)|\,dy\\ &\leqslant Ct^{k-n-\ell}\sum_{|\alpha|=\nu}\int_{|y|\leqslant t}|y|^{\nu+\ell}\,|D^\alpha g(x+y)|\,dy\\ &\leqslant Ct^{k-\frac{n}{2}-\ell}\sum_{|\alpha|=\nu}\left(\int_{|y|\leqslant t}(|y|^{\nu+\ell}\,|D^\alpha g(x+y)|)^2\,dy\right)^{1/2}\\ &\leqslant Ct^{k-\frac{n}{2}-\ell}\sum_{|\alpha|=\nu}\|\langle y\rangle^{\nu+\ell}\,D^\alpha g\|_{L^2}\end{aligned}$$

by Cauchy inequality and by (5.11). In conclusion,

$$|t^{\nu+k}\,\partial_t^\nu N(g)| \leqslant Ct^{k-\frac{n}{2}-\ell}\,\|g\|_{H^{\nu,\ell}}$$

and choosing $k=0$, $\ell=-1/2$ or $k=1$, $\ell=1/2$ and plugging these estimates into (5.3) we obtain the thesis (since $1+t+|x|\leqslant ct$).

To estimate u^I we use the representation (5.6). We notice that in the convolution we apply $E(t,\,\cdot)$ to the functions $u_j^I(x-\cdot)$; for $x-y\in \operatorname{supp} u_j^I$ and $|x|\leqslant t/16$ we have

$$|y| = |x-y-x| \leqslant \frac{t}{4}+\frac{t}{16}\leqslant \frac{t}{2},$$

thus $E(t,y)$ vanishes identically for $n\geqslant 3$ odd, and for n even $E(t,y)$ coincides with an ordinary function

$$E(t,y) = c_n(t^2-|y|^2)^{-\frac{n-1}{2}}$$

for such values of x, y, t. The case $n\geqslant 3$ odd is trivial, and we shall consider only the second case. Hence we have also

$$\partial_t E(t,y) = c_n'\,t\,(t^2-|y|^2)^{-\frac{n+1}{2}}.$$

Now we can write directly

$$\begin{aligned}|u^I| &\leqslant C\left(t\int (t^2-|y|^2)^{-\frac{n+1}{2}}\,|u_0^I(x-y)|\,dy+\int (t^2-|y|^2)^{-\frac{n-1}{2}}\,|u_1^I(x-y)|\,dy\right)\\ &\leqslant C\left(t^{-n}\int |u_0^I|\,dy+t^{1-n}\int |u_1^I|\,dy\right).\end{aligned}$$

By Cauchy inequality, since supp $u_j^I \subseteq B(0, t/4)$,

$$\int |u_0^I| \, dy = \int \langle y \rangle^{1/2} \langle y \rangle^{-1/2} |u_0^I| \, dy$$

$$\leqslant \left(\int_{|y| \leqslant t/4} \langle y \rangle \right)^{1/2} \|\langle y \rangle^{-1/2} u_0\|_{L^2} \leqslant t^{\frac{1}{2}+\frac{n}{2}} \|\langle y \rangle^{-1/2} u_0\|_{L^2},$$

and also

$$\int |u_1^I| \, dy = \int \langle y \rangle^{-1/2} \langle y \rangle^{1/2} |u_1^I| \, dy \leqslant t^{\frac{n}{2}-\frac{1}{2}} \|\langle y \rangle^{1/2} u_1\|_{L^2}.$$

In conclusion

$$|u^I| \leqslant C t^{-\frac{n-1}{2}} (\|\langle y \rangle^{-1/2} u_0\|_{L^2} + \|\langle y \rangle^{1/2} u_1\|_{L^2})$$

which implies (5.9) for u^I.

5.3. *Case* C. $t+|x| \geqslant 1$ *and* $|x| \geqslant t/16$. We use again the representation (5.1); recalling the operators I_a introduced in Lemma 4.6, we see that

$$|u(t, x)| \leqslant C(|I_0^+(u_0)| + |I_0^-(u_0)| + |I_1^+(u_1)| + |I_1^-(u_1)|).$$

Now, estimate (4.15) gives

$$|u(t, x)| \leqslant C \, |x|^{-\frac{n-1}{2}} (\|u_0\|_{H^{s_0, \delta_0}} + \|u_1\|_{H^{s_1, \delta_1}})$$

with s_0, δ_0, s_1, δ_1 exactly as in the thesis. Since

$$|x| \geqslant \tfrac{1}{2} |x| + \tfrac{1}{32} t \geqslant C(1+t+|x|)$$

this concludes the proof.

6. PROOF OF THEOREM 1.1 FOR GENERAL d

Thanks to the interpolation properties of the spaces $H^{s,\delta}$, it is sufficient to prove Theorem 1.1 only in the two extreme cases $d=0$, $d=(n-1)/2$. We already considered the case $d=0$; thus from now on we shall assume that $d=(n-1)/2$, and we have to prove the estimate

$$(6.1) \quad (1+t+|x|)^{(n-1)/2} (1+|t-|x||)^{(n-1)/2} |u(t, x)| \leqslant C(\|u_0\|_{H^{s_0, \delta_0}} + \|u_1\|_{H^{s_1, \delta_1}})$$

for

$$s_0 > \frac{n}{2}, \quad \delta_0 > \frac{n}{2} - 1, \quad s_1 > \frac{n}{2} - 1, \quad \delta_1 > \frac{n}{2}.$$

We consider two cases.

6.1. *Case* A. $t + |x| \leqslant 1$ *or* $|t - |x|| \leqslant 1$. In both cases we have

$$(1 + t + |x|)^{\frac{n-1}{2}} (1 + |t - |x||)^{\frac{n-1}{2}} \leqslant 2^{\frac{n-1}{2}} (1 + t + |x|)^{\frac{n-1}{2}},$$

hence using estimate (5.9) already proved, we have for all $s_0 > n/2$, $s_1 > n/2 - 1$, $\delta_0 > -1/2$, $\delta_1 > 1/2$,

$$(1 + t + |x|)^{\frac{n-1}{2}} (1 + |t - |x||)^{\frac{n-1}{2}} |u(t, x)| \leqslant C(\|u_0\|_{H^{s_0, \delta_0}} + \|u_1\|_{H^{s_1, \delta_1}}).$$

Since for any $a \geqslant 0$

$$\|v\|_{H^{s, \delta}} \leqslant C \|v\|_{H^{s, \delta + a}},$$

the estimate (6.1) follows.

6.2. *Case* B. $t + |x| \geqslant 1$ *and* $|t - |x|| \geqslant 1$. As above, we split the data

$$u_j = u_j^I + u_j^{II}$$

in such a way that

$$\text{(6.2)} \qquad \operatorname{supp} u_j^I \subseteq \left\{ y \colon \frac{|t - |x||}{8} \leqslant |y| \right\}, \quad \operatorname{supp} u_j^{II} \subseteq \left\{ y \colon |y| \leqslant \frac{|t - |x||}{4} \right\}$$

and it is sufficient to prove (1.4) for the solutions u^I, u^{II} corresponding to the initial data u_j^I, u_j^{II} respectively.

To estimate u^I we begin by applying estimate (5.9) already proved, and we obtain

$$[(1 + t + |x|)(1 + |t - |x||)]^{\frac{n-1}{2}} |u^I(t, x)| \leqslant C\, |t - |x||^{\frac{n-1}{2}} (\|u_0^I\|_{H^{s_0, \delta_0}} + \|u_1^I\|_{H^{s_1, \delta_1}}).$$

Now we use property (4.2) of Lemma 4.1 with $R = |t - |x||$, $a = (n-1)/2$, recalling (6.2), and we obtain

$$|t - |x||^{(n-1)/2} \|u_j^I\|_{H^{s, \delta}} \leqslant C \|u_j^I\|_{H^{s, \delta + (n-1)/2}}$$

whence (6.1) for u^I follows easily.

We now estimate u^{II}. If $t<|x|$ then $u^{II}=0$; indeed, by finite speed of propagation u^{II} vanishes outside the cone

$$K=\{(s,y)\colon |y|-s\leqslant |t-|x||/4\}$$

and if $|x|-t>0$ clearly the point (t,x) lies outside K. Thus we may assume $t\geqslant |x|$, and we have the chain of inequalities

$$t-|x-y|\geqslant t-|x|-|y|\geqslant \tfrac{3}{4}|t-|x||\geqslant \tfrac{3}{4}. \tag{6.3}$$

We are in position to use again the representation (5.6) of the solution. Indeed, in the convolution $E(t,\cdot)*u_j^{II}(\cdot)$ the distribution E is computed only at $(t,x-y)$ with $y\in\operatorname{supp} u_j^{II}$, and by (6.3) this means E, $\partial_t E$ coincide with the ordinary functions

$$E(t,x-y)=c_n(t^2-|x-y|^2)^{-(n-1)/2}, \tag{6.4}$$

$$\partial_t E(t,x-y)=c'_n\, t\,(t^2-|x-y|^2)^{-(n+1)/2}, \tag{6.5}$$

at least when $n=1$ or $n\geqslant 2$ is even. As above, the case $n\geqslant 3$ odd is trivial since $E(t,x-y)$ vanishes identically for t,x,y as in (6.3). Notice that, by (6.3), if $y\in\operatorname{supp} u_j^{II}$ we have also

$$t-|x-y|\geqslant \tfrac{3}{4}|t-|x|| \tag{6.6}$$

and

$$t+|x-y|\geqslant t+|x|-|y|\geqslant t+|x|-\tfrac{1}{4}|t-|x||\geqslant \tfrac{3}{4}(t+|x|), \tag{6.7}$$

whence, for all $y\in\operatorname{supp} u_0^{II}\cup\operatorname{supp} u_1^{II}$

$$t^2-|x-y|^2\geqslant \tfrac{9}{16}(t-|x|)(t+|x|)\geqslant \tfrac{1}{4}(1+t+|x|)(1+|t-|x||) \tag{6.8}$$

(we have repeatedly used the assumptions $t+|x|\geqslant 1$, $t\geqslant |x|$, $t-|x|\geqslant 1$). Thus we may write, using (6.5),

(6.9)

$$\begin{aligned}|\partial_t E*u_0^{II}|&\leqslant C\cdot t\cdot\int\frac{|u_0^{II}(y)|}{(t^2-|x-y|^2)^{(n+1)/2}}\,dy\\ &\leqslant Ct\cdot(1+t+|x|)^{-(n+1)/2}\,(1+|t-|x||)^{-(n-1)/2}\int\frac{|u_0^{II}(y)|}{(t-|x-y|)}\,dy;\end{aligned}$$

since

$$t-|x-y|\geqslant \tfrac{3}{4}|t-|x||\geqslant \tfrac{3}{8}(1+|t-|x||)\geqslant \tfrac{3}{8}(1+4|y|)\geqslant \tfrac{3}{8}\langle y\rangle,$$

and obviously $1+t+|x| \geqslant t$, (6.9) implies

$$(6.10)\quad |\partial_t E * u_0^{II}| \leqslant C[(1+t+|x|)(1+|t-|x||)]^{-(n-1)/2} \int \frac{|u_0^{II}(y)|}{\langle y \rangle}\, dy.$$

In a similar way,

$$(6.11)\quad |E * u_1^{II}| \leqslant C \int \frac{|u_1^{II}(y)|}{(t^2-|x-y|^2)^{(n-1)/2}}$$

$$\leqslant C[(1+t+|x|)(1+|t-|x||)]^{-(n-1)/2} \int |u_1^{II}(y)|\, dy.$$

By (6.10) and (6.11), we have proved

$$[(1+t+|x|)(1+|t-|x||)]^{-(n-1)/2}\, |u^{II}(t,x)| \leqslant C \int \frac{|u_0^{II}(y)|}{\langle y \rangle}\, dy + C \int |u_1^{II}(y)|\, dy,$$

with $C = C(n)$, and by Cauchy inequality, for any $\varepsilon > 0$,

$$(6.12)\quad \leqslant C(n, \varepsilon)[\|\langle y \rangle^{\frac{n}{2}-1+\varepsilon}\, u_0^{II}(y)\|_{L^2} + \|\langle y \rangle^{\frac{n}{2}+\varepsilon}\, u_1^{II}(y)\|_{L^2}].$$

The last sum can be written

$$\|u_0^{II}(y)\|_{H^{0,\delta_0}} + \|u_1^{II}(y)\|_{H^{0,\delta_1}}$$

with $\delta_0 > -1/2+(n-1)/2$, $\delta_1 > 1/2+(n-1)/2$, and recalling that $H^{0,\delta} \subseteq H^{s,\delta}$ for all $s \geqslant 0$, this concludes the proof.

7. PROOF OF THEOREMS 1.2, 1.3

7.1. *Proof of Theorem* 1.2. Since $a \leqslant 0$, $b \leqslant 0$, we have

$$(1+t+|x|)^{2a}\,(1+|t-|x||)^{2b}\,|u(t,x)|^2 \leqslant (1+t)^{2a}\,|t-|x||^{2b}\,|u(t,x)|^2.$$

Thus, recalling that $a < -1/2$, we get

$$N(u) = \|(1+t+|x|)^a\,(1+|t-|x||)^b\,u(t,x)\|_{L^2(\mathbb{R}^{n+1}_+)}$$

$$\leqslant C(a) \sup_{t \geqslant 0} \||t-|x||^b\,u(t,\cdot)\|_{L^2(\mathbb{R}^n)}.$$

Since $b > -1/2$, we can now apply the special Hardy inequality (4.12), and we obtain

$$N(u) \leqslant C \sup_{t \geqslant 0} \||\xi|^{-b} \hat{u}(t, \cdot)\|_{L^2}.$$

Recalling the Fourier representation

$$\hat{u}(t, \xi) = \cos(t\,|\xi|)\,\hat{u}_0 + |\xi|^{-1} \sin(t\,|\xi|)\,\hat{u}_1,$$

we obtain easily

$$N(u) \leqslant C\,\||\xi|^{-b}\,\hat{u}_0\|_{L^2} + C\,\||\xi|^{-b-1}\,\hat{u}_1\|_{L^2}.$$

Property (4.5) of Lemma 4.3 gives (notice that $-b \geqslant 0$)

$$\||\xi|^{-b}\,\hat{u}_0\|_{L^2} \leqslant C\,\|u_0\|_{H^{-b,b}},$$

while property (4.9) gives

$$\||\xi|^{-b-1}\,\hat{u}_1\|_{L^2} \leqslant C\,\|u_1\|_{H^{-b-1,b+1}}$$

provided $b+1 < n/2$; notice that this condition holds for any $b \leqslant 0$ when $n \geqslant 3$, and for $b < 0$ when $n = 2$. This concludes the proof.

7.2. *Proof of Theorem* 1.3. Consider two Lebesgue spaces $X_0 = L^{p_0}(\mathbb{R}^n, d\mu_0)$ and $X_1 = L^{p_1}(\mathbb{R}^n, d\mu_1)$ with measures $\mu_j = w_j(x)\,dx$, w_j being a strictly positive function on $\mathbb{R}^n$ while dx is the standard Lebesgue measure. Then the complex interpolation space $X_\theta = (X_0, X_1)_\theta$ has the same form, that is, $X_\theta = L^p(\mathbb{R}^n, d\mu)$ with $d\mu = w_0^{1-\theta} w_1^\theta\,dx$, $1/p = (1-\theta)/p_0 + \theta/p_1$, as it is well known [1]. Recall also the interpolation property (3.54).

Assume now $n \geqslant 3$. Interpolating between (1.4) with $d = (n-1)/2$ and (1.5) with $b = 0$, we obtain (1.6) for any $\rho < (n-1)/2 - n/q$ and $\sigma = (n-1)/2 - (n-1)/q$. Similarly, interpolating between (1.4) with $d = 0$ and (1.5) with $b = 0$, obtain (1.6) for any $\rho < (n-1)/2 - n/q$ and $\sigma = 0$. Finally, interpolating between the two estimates thus obtained, we obtain the first part of the theorem for any $0 \leqslant \sigma \leqslant (n-1)/2 - (n-1)/q$.

The second part of the theorem is proved in a similar way, interpolating between (1.4) with $d = 0$ and (1.5) with $b \in]-1/2, 0]$ arbitrary.

Finally, in the case $n = 2$ we can proceed similarly; only, (1.5) does not hold for $b = 0$ and we can use $b = -\varepsilon$ arbitrarily small instead.

8. APPLICATION TO SEMILINEAR WAVE EQUATIONS

In order to apply the foregoing theory to the semilinear wave equation (1.8), we need first a suitable weighted Strichartz type estimate for the linear initial value problem

(8.1) $$\Box u = F(t, x) \qquad \text{in } \mathbb{R}^{n+1}_+,$$

(8.2) $$u(0, x) = 0, \qquad \partial_t u(0, x) = 0 \qquad \text{for} \quad x \in \mathbb{R}^n.$$

In the following, given a function $F \in L^q(\mathbb{R}^{n+1}_+)$ for some q, we denote with $S(F)$ the solution on $\mathbb{R}^{n+1}_+$ to the linear problem (8.1), (8.2) (more correctly, the solution to the integral equation equivalent to the Cauchy problem). We shall omit the reference to $\mathbb{R}^{n+1}_+$ and write

$$L^p = L^p(\mathbb{R}^{n+1}_+).$$

Then we shall prove the following estimate, adapted from the estimate of [9] (see also [8, 24]):

LEMMA 8.1. *Assume that*

(8.3) $$\frac{n-1}{2(n+1)} \leqslant \frac{1}{q} \leqslant \frac{1}{2}, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

(8.4) $$a < \frac{n-1}{2} - \frac{n}{q}, \qquad b > \frac{1}{q}, \qquad \delta > 0.$$

Then for any $F \in L^p(\mathbb{R}^{n+1}_+)$ *we have*

(8.5) $$\|(1+|t-|x||)^a \, (1+t+|x|)^a \, S(F)\|_{L^q} \leqslant C(\delta, a, b, p, n) \, \|(1+|t-|x||)^{b+\delta} \, (1+t+|x|)^b \, F\|_{L^p}.$$

Remark 8.1. We shall prove estimate (8.5) as a consequence of the estimate

(8.6) $$\|(t^2-|x|^2)^a \, S(F)\|_{L^q} \leqslant C \, \|(t^2-|x|^2)^b \, F\|_{L^p},$$

which holds only for F with

$$\operatorname{supp} F(s, y) \subseteq \{(s, y) \colon |y| + 1 \leqslant s\}$$

with indices satisfying (8.3), (8.4). For a proof we refer to [9] (see also [24]). Notice that, thanks to the assumption on the support of F and the finite speed of propagation, (8.6) is equivalent to

(8.7)
$$\|(1+|t-|x||)^a (1+t+|x|)^a S(F)\|_{L^q} \leqslant C \|(1+|t-|x||)^b (1+t+|x|)^b F\|_{L^p}.$$

Proof. We need here a different type of Littlewood decomposition from the rest of the paper. Starting from the usual decomposition $\{\phi_j\}_{j\geqslant 0}$ introduced in Example 2.3, we define the following partition of unity on $\mathbb{R}$:

$$\psi_j(s) = \begin{cases} \phi_j(s) & \text{if } \ j\geqslant 0,\, s\geqslant 0 \\ \phi_{-j}(-s) & \text{if } \ j\leqslant 0,\, s\leqslant 0 \end{cases} \tag{8.8}$$

and $\psi_j(s)=0$ otherwise. Notice that we obtain a partition of unity subordinated to the covering of $\mathbb{R}$

$$\{[2^{|j|-1}, 2^{|j|+1}]\}_{j\in\mathbb{N}} \cup \{[-2, 2]\} \cup \{[-2^{|j|+1}, -2^{|j|-1}]\}_{j\in\mathbb{N}}.$$

We decompose the function F as

$$F=\sum_{j\in\mathbb{Z}} F_j, \qquad F_j(t, x)=\psi_j(t-|x|)\, F(t, x) \tag{8.9}$$

and accordingly, recalling that $u=S(F)$, we denote by $u_j=S(F_j)$ the solution to (8.1), (8.2) with right hand member F_j (notice $F_j=0$ for $t\leqslant 0$). Clearly

$$u(t, x)=\sum_{j\in\mathbb{Z}} u_j(t, x),$$

where the sum is finite for each fixed (t, x), thanks to the finite speed of propagation.

We begin by proving the estimate

$$\|(1+|t-|x||)^a (1+t+|x|)^a u_j\|_{L^q} \leqslant C2^{b|j|} \|(1+t+|x|)^b F_j\|_{L^p} \tag{8.10}$$

with a constant independent of $j\in\mathbb{Z}$. When $j\geqslant 1$ we have

$$\operatorname{supp} F_j(s, y) \subseteq \{(s, y)\colon |y|+1\leqslant s\},$$

hence (8.10) follows immediately from (8.7). Consider now the case $j \leqslant 0$. Define

$$v_j(t, x) = u_j(2^{|j|}(t-3), 2^{|j|}x), \qquad G_j(t, x) = 2^{2|j|}F_j(2^{|j|}(t-3), 2^{|j|}x), \tag{8.11}$$

so that $v_j = S(G_j)$, and notice that

$$\operatorname{supp} G_j \subseteq \{(s, y) \colon 1 \leqslant s - |y| \leqslant 5\}. \tag{8.12}$$

Thus we can apply again (8.7), and we obtain (using (8.12))

$$\|(1+|t-|x||)^a (1+t+|x|)^a v_j\|_{L^q} \leqslant C \|(1+t+|x|)^b G_j\|_{L^p}.$$

Rescaling back the variables as

$$s = 2^{|j|}(t-3), \qquad y = 2^{|j|}x$$

we obtain, writing for brevity $\kappa = 2^{|j|}$,

$$\begin{aligned}\|(\kappa+|t+3\kappa-|x||)^a (4\kappa+t+|x|)^a u_j\|_{L^q} \\ \leqslant C\kappa^{2a+\frac{n+1}{q}+2-b-\frac{n+1}{p}} \|(4\kappa+t+|x|)^b F_j\|_{L^p}.\end{aligned} \tag{8.13}$$

From (8.4) it follows that

$$2a+\frac{n+1}{q}+2-b-\frac{n+1}{p} \leqslant b;$$

moreover, we have $\kappa/2 \leqslant |x|-t \leqslant 2\kappa$ on the support of F_j $(j \leqslant -1)$, whence $t+|x| \geqslant \kappa/2$, and this implies

$$4\kappa+t+|x| \leqslant 9(1+t+|x|)$$

on the support of F_j $(j \leqslant 0)$; finally, we have $t-|x| \geqslant -2\kappa$ on the support of F_j $(j \leqslant -1)$ and hence also on the support of u_j by a domain of dependence argument, thus

$$3(t+3\kappa-|x|) \geqslant \kappa+|t-|x||$$

on the support of u_j $(j \leqslant 0)$. In conclusion, from (8.13) we deduce (8.10).

It remains now to deduce (8.5) from (8.10). By Fatou's Lemma, since $u = \sum_{j=-\infty}^{+\infty} u_j$, we can write

$$\begin{aligned}&\|(1+|t-|x||)^a (1+t+|x|)^a u\|_{L^q} \\ &\quad \leqslant \liminf_{N\to\infty} \sum_{j=-N}^{N} \|(1+|t-|x||)^a (1+t+|x|)^a u_j\|_{L^q}.\end{aligned}$$

Thus, recalling (8.10), in order to prove (8.5) it is sufficient to prove that

(8.14)

$$\liminf_{N\to\infty} \sum_{j=-N}^{N} 2^{b|j|} \|(1+t+|x|)^b F_j\|_{L^p} \leqslant C(\delta) \|(1+|t-|x||)^{b+\delta} (1+t+|x|)^b F\|_{L^p}$$

for $\delta > 0$. Noticing that $2^{|j|} \leqslant C(1+|t-|x||)$ on $\operatorname{supp} F_j$, with a constant independent of j, we have by Hölder's inequality

(8.15)

$$\sum_{j=-N}^{N} 2^{b|j|} \|(1+t+|x|)^b F_j\|_{L^p} \leqslant \left(\sum_{j=-\infty}^{\infty} 2^{-\delta|j|q}\right)^{1/q} \left(\sum_{j=-N}^{N} \|(1+|t-|x||)^{b+\delta} (1+t+|x|)^b F_j\|_{L^p}^p\right)^{1/p}.$$

The first factor is a constant depending only on δ and q. As to the second, we can write, recalling (8.9),

$$\sum_{j=-N}^{N} \|(1+|t-|x||)^{b+\delta} (1+t+|x|)^b F_j\|_{L^p}^p = \int_{\mathbb{R}_+^{n+1}} (1+|t-|x||)^{(b+\delta)p} (1+t+|x|)^{bp} |F(t,x)|^p \sum_{j=-N}^{N} |\psi_j(t-|x|)|^p.$$

By definition of the ψ_j, at each point (t, x) at most two of the functions $\psi_j(t-|x|)$ are different from 0; this implies

$$\sum_{j=-N}^{N} |\psi_j(t-|x|)|^p \leqslant 2$$

and the proof is concluded. ∎

Thanks to estimate (8.5), we can achieve the proof of Theorem 1.4 by a standard application of the contraction mapping principle. More precisely, we introduce the function space X defined as

$$X = \{u \in L^{\lambda+1}(\mathbb{R}_+^{n+1}) : |||u||| < +\infty\}, \tag{8.16}$$

where λ is the power of the nonlinear term $F(u)$ (see (1.10)), with norm

$$|||u||| = \|(1+|t-|x||)^a (1+t+|x|)^a u\|_{L^{\lambda+1}(\mathbb{R}_+^{n+1})}.$$

Then we have

LEMMA 8.2. *Assume* $n \geqslant 2$,

$$\lambda_0(n) < \lambda \leqslant \frac{n+3}{n-1},$$

and $a = a(\lambda)$ *is chosen such that*

$$\frac{1}{\lambda} < (\lambda+1)\, a < \frac{n-1}{2}\lambda - \frac{n+1}{2}. \tag{8.17}$$

Then we have for all $u \in X$

$$|||S(|u|^{\lambda})||| \leqslant C\, |||u|||^{\lambda} \tag{8.18}$$

with C *independent of* u.

Proof. Recalling the definition (1.11) of $\lambda_0(n)$, it is clear that we can choose $a(\lambda)$ such that (8.17) holds. Moreover, the upper restriction on a in (8.17) ensures that

$$a < \frac{n-1}{2} - \frac{n}{\lambda+1},$$

while we have

$$q = \lambda + 1, \qquad p = \frac{\lambda+1}{\lambda};$$

since $1 < \lambda \leqslant (n+3)/(n-1)$, q satisfies (8.3). Finally, thanks to (8.17), we can choose positive numbers b, δ such that $b > 1/(\lambda+1)$ and $b + \delta \leqslant \lambda a$. Thus we have

$$\|(1+|t-|x||)^{b+\delta}\,(1+t+|x|)^{b}\,|u|^{\lambda}\|_{L^p(\mathbb{R}^{n+1}_+)} \leqslant C\, |||u|||^{\lambda}.$$

Combining this with Lemma 8.1, we obtain (8.18). ∎

LEMMA 8.3. *Assume* F *satisfies* (1.10) *and*

$$\lambda_0(n) < \lambda \leqslant \frac{n+3}{n-1}.$$

If $a = a(\lambda)$ is chosen so that (8.17) *holds, then we have, for all $u, v \in X$,*

(8.19) $$|||S(F(u))||| \leqslant C_1 \; |||u|||^{\lambda},$$

(8.20) $$|||S(F(u)) - S(F(v))||| \leqslant C_1 \; |||u-v||| \cdot (|||u||| + |||v|||)^{\lambda-1},$$

with a constant C_1 independent of u, v.

Proof. By (1.10) we know that

$$|F(u)| \leqslant C \; |u|^{\lambda}$$

and that

$$|F(u) - F(v)| \leqslant C[|u-v|^{\frac{1}{\lambda}} \cdot (|u| + |v|)^{1-\frac{1}{\lambda}}]^{\lambda}.$$

Noticing that

$$|||\,|v_1|^{\theta_1} \; |v_2|^{\theta_2}||| \leqslant |||v_1|||^{\theta_1} \cdot |||v_2|||^{\theta_2}$$

for all $v_1, v_2 \in X$, provided $\theta_1, \theta_2 > 0$ satisfy $\theta_1 + \theta_2 = 1$, from Lemma 8.3 we obtain (8.19), (8.20) immediately. ∎

To prove Theorem 1.4, we define a sequence $u_{(k)}(t, x)$ in X as follows: $u_{(0)}$ is the solution of the homogeneous problem (1.1), (1.2), and for $k \geqslant 0$

(8.21) $$u_{(k+1)} = v + S(F(u_{(k)})).$$

Choose $a(\lambda)$ such that (8.17) holds, so that in particular

$$a < \frac{n-1}{2} - \frac{n}{\lambda+1}.$$

Then from (1.7) we have

(8.22) $$|||u_{(0)}||| \leqslant C_0(\|u_0\|_{H^{s_0, \delta_0}} + \|u_1\|_{H^{s_1, \delta_1}})$$

provided

(8.23) $$s_0 > \frac{n}{2} - \frac{n}{q}, \quad \delta_0 > \frac{1}{q} - \frac{1}{2} + a, \quad s_1 > \frac{n}{2} - \frac{n}{q} - 1, \quad \delta_1 > \frac{1}{q} + \frac{1}{2} + a.$$

We now take $\varepsilon_0 > 0$ such that

(8.24) $$2C_1(4C_0)^{\lambda} \, \varepsilon_0^{\lambda-1} \leqslant 1$$

where C_0, C_1 are the constants in (8.22), (8.19), and (8.20). If we have

$$\|u_0\|_{H^{s_0,\delta_0}} + \|u_1\|_{H^{s_1,\delta_1}} \leqslant \varepsilon, \qquad \text{for} \quad 0 < \varepsilon \leqslant \varepsilon_0, \tag{8.25}$$

with indices satisfying (8.23), then, inductively, we have from (8.21), (8.22) and (8.19)

$$|||u_{(k)}||| \leqslant 2C_0\varepsilon \qquad \text{for} \quad 0 < \varepsilon \leqslant \varepsilon_0, \quad k \geqslant 0. \tag{8.26}$$

In a similar way we have

$$|||u_{(k+1)} - u_{(k)}||| \leqslant \tfrac{1}{2}\, |||u_{(k)} - u_{(k-1)}||| \qquad \text{for} \quad 0 < \varepsilon \leqslant \varepsilon_0, \quad k \geqslant 1. \tag{8.27}$$

Thus $\{u_{(k)}\}$ is a Cauchy sequence in X, whose limit is the weak solution to (1.8), (1.9).

9. APPENDIX

We collect here two technical results used several times in the course of the paper; they are of independent interest.

The first one is a classical result; however we need a more refined version of the stationary phase method, showing the precise dependence on the fractional Sobolev norms of the functions. Note that usually estimates (9.1)–(9.4) are proved for s integer only.

We state the theorems for smooth functions; it is trivial to extend the estimates to functions belonging to appropriate Sobolev spaces.

THEOREM 9.1 (Stationary Phase). *Let $\phi, v \in C^\infty(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^n)$, and consider the integrals*

$$I(R) = \int_{\mathbb{R}^n} e^{iR\phi(x)} g(x)\, dx, \qquad R \in \mathbb{R}$$

and

$$J(x) = \int_{|\xi|=1} e^{ix\cdot\xi} v(\xi)\, dH_\xi^{n-1}, \qquad x \in \mathbb{R}^n.$$

(1) *If $D\phi$ does not vanish on supp g, then for all real $s \geqslant 0$ and $R \neq 0$*

$$|I(R)| \leqslant C\, \|g\|_{H^s(\mathbb{R}^n)}\, |R|^{-s} \tag{9.1}$$

with C depending only on n, s, ϕ and the diameter of the support of g.

(2) *If* $D\phi = 0$ *only at a finite number of points of* supp g, *and in these points* $\det(D^2\phi)$ *does not vanish, then for all* $R \neq 0$

$$|I(R)| \leqslant C \, \|\hat{g}\|_{L^1(\mathbb{R}^n)} \, |R|^{-\frac{n}{2}}, \tag{9.2}$$

with C *depending only on* n, s, ϕ *and the diameter of the support of* g. *This implies, for any real* $s > n/2$, $R \neq 0$,

$$|I(R)| \leqslant C \, \|g\|_{H^s(\mathbb{R}^n)} \, |R|^{-\frac{n}{2}}. \tag{9.3}$$

(3) *For all real* $s > (n-1)/2$, $x \neq 0$,

$$|J(x)| \leqslant C(n, s) \, \|v\|_{H^s(S^{n-1})} \, |x|^{-\frac{n-1}{2}}. \tag{9.4}$$

Proof. (1) It is sufficient to consider the case $R > 0$. By a partition of unity (depending only on ϕ) we are reduced to the case $\partial_j \phi \neq 0$ on supp g for some $j = 1, ..., n$. Then the operator

$$L = \frac{1}{i \, \partial_j \phi} \frac{\partial}{\partial x_j}$$

has the property

$$L^k e^{iR\phi(x)} = R^k e^{iR\phi(x)}$$

for all integer k. Hence, integration by parts gives, for any integer $k \geqslant 0$,

$$|R^k I(R)| = \left| \int e^{iR\phi(x)} ({}^tL)^k \, g(x) \, dx \right| \leqslant C(n, k, \phi) \, \|g\|_{W^{k,1}} \leqslant C d^{n/2} \, \|g\|_{H^k(B)},$$

where B is a ball containing the support of g and d is its diameter. By interpolation, we obtain (9.1) for any real $s \geqslant 0$.

(2) By a partition of unity depending only on ϕ, we may split $I(R)$ as

$$I(R) = \sum_{j=1}^{v} \int e^{iR\phi(x)} g_j(x) \, dx,$$

in such a way that either supp g_j does not contain critical points $D\phi = 0$, in which case (9.1) can be applied; or supp g_j contains exactly one such point, and supp g_j is so small that the Morse lemma can be applied to ϕ on it. Since multiplication by an element of C_c^∞ is a bounded operator on H^s, it is sufficient to prove (9.2) for each g_j, and this introduces only a constant depending on ϕ (restricted to supp g) in the final estimate.

Thus we are reduced to the simpler case

$$I(R) = \int e^{iR\langle Ax, x\rangle} g(x)\, dx,$$

where A is a diagonal matrix with entries ± 1 on the diagonal. We now apply the well know formula for the Fourier transform of a Gaussian function

$$\mathscr{F}(e^{iR\langle Ax, x\rangle}) = \pi^{n/2} e^{\pm \pi i/4} e^{i\langle A^{-1}\xi, \xi\rangle/(4R)} \cdot R^{-n/2}$$

(see, e.g., [12, Theorem 7.6.1]), where the $\pm$ is the opposite of the signature of A, in this case $-\det A$. Thus by Plancherel formula we have

$$I(R) = c(n)\, R^{-n/2} \int e^{i\langle A^{-1}\xi, \xi\rangle/(4R)} \hat{g}(\xi)\, d\xi$$

whence

$$|I(R)| \leqslant c(n)\, R^{-n/2} \int |\hat{g}|\, d\xi.$$

This proves (9.2), and (9.3) follows immediately by Cauchy inequality, writing

$$\int |\hat{g}|\, d\xi = \int \langle \xi \rangle^{-s} \langle \xi \rangle^{s} |\hat{g}|\, d\xi.$$

(3) After a rotation, we see it is sufficient to consider the case $x = (0, ..., 0, |x|)$, and we are reduced to estimate

$$J(x) = \int_{|\xi| = 1} e^{i\,|x|\,\xi_n} g(\xi)\, dH^{n-1}_{\xi}.$$

Using a partition of unity depending only on n, composed of $2n$ elements, we can assume $\operatorname{supp} g$ is contained in one of the half spaces $\xi_j > 0$ (or $\xi_j < 0$) for some j. Then we write

$$\xi' = (\xi_1, ..., \check{\xi}_j, ..., \xi_n) \in \mathbb{R}^{n-1},$$

with jth coordinate suppressed, and use ξ' as a coordinate on $S^{n-1} \cap \operatorname{supp} g$.

If $j \neq n$, we have

$$J(x) = \int_{\mathbb{R}^{n-1}} e^{i\,|x|\,\xi_n} g(\xi) \, \frac{d\xi'}{\sqrt{1-|\xi'|^2}}$$

since $\xi_j = \sqrt{1-|\xi'|^2}$ represents S^{n-1} in this coordinate system. To this integral (9.1) can be applied and we get (9.4), actually with stronger decay $|x|^{-s}$ for any s.

If $j = n$, we have

$$J(x) = \int_{\mathbb{R}^{n-1}} e^{i\,|x|\,\sqrt{1-|\xi'|^2}} g(\xi) \, \frac{d\xi'}{\sqrt{1-|\xi'|^2}}.$$

To this integral we can apply (9.3), and we obtain (9.4) again.

As above, we have used the fact that, if M is a smooth manifold and $\phi \in C_c^\infty(M)$, then

$$\|\phi g\|_{H^s(M)} \leqslant C(s, \phi) \, \|g\|_{H^s(M)}$$

(a trivial fact that can be proved, e.g., by interpolation). ∎

THEOREM 9.2 (Extended Hardy Inequality). *For any real $a \in [0, n/2[$ and any $f \in C_c^\infty(\mathbb{R}^n)$, we have*

$$\left\| \frac{\hat{f}(\xi)}{|\xi|^a} \right\|_{L^2} \leqslant C \, \||x|^a f\|_{L^2}, \tag{9.5}$$

with $C = C(n, a)$ independent of f.

Proof. Inequality (9.5) is a special case of a result of Muckenhoupt (see Theorem 1 in [16]). For sake of completeness, we give here a proof since it is particularly simple in this case.

We must prove that

$$\int \left| \int u(x) \, e^{-ix\cdot\xi} \, dx \right|^2 |\xi|^{-2a} \, d\xi \leqslant C \int |u(x)|^2 \, |x|^{2a} \, dx.$$

Split the first integral as $I + II$, with

$$I = \sum_{j \in \mathbb{Z}} \int_{2^j < |\xi|^{-a} \leqslant 2^{j+1}} \left| \int_{|x|^a > 2^j} u(x) \, e^{-ix\xi} \, dx \right|^2 |\xi|^{-2a} \, d\xi$$

and

$$II = \sum_{j \in \mathbb{Z}} \int_{2^j < |\xi|^{-a} \leqslant 2^{j+1}} \left| \int_{|x|^a \leqslant 2^j} u(x)\, e^{-ix\xi}\, dx \right|^2 |\xi|^{-2a}\, d\xi$$

We can write

$$I = \sum_{j \in \mathbb{Z}} 2^{2j+2} \, \|\mathcal{F}(u \cdot \chi_{\{|x|^a > 2^j\}})\|_{L^2}^2,$$

where $\mathcal{F}$ is the Fourier transform and χ_A is the characteristic function of the set A. Thus, by Plancherel's theorem,

$$I \leqslant \sum_{j \in \mathbb{Z}} 2^{2j+2} \, \|u \cdot \chi_{\{|x|^a > 2^j\}}\|_{L^2}^2 \leqslant \|u \cdot h\|_{L^2}^2,$$

where the function $h(x)$ is defined by

$$h(x) = \sum_{j \in \mathbb{Z}} \chi_{\{|x|^a > 2^j\}} 2^{j+1}$$

and hence satisfies

$$h(x) \leqslant 2\, |x|^a.$$

This concludes the estimate for I.

To estimate II, we begin by noticing that

$$II \leqslant \int \left(\int_{|x| \leqslant 1/|\xi|} |u(x)|\, dx \right)^2 |\xi|^{-2a}\, d\xi.$$

Now, consider the lowest integer $J \in \mathbb{Z}$ such that $2^J \geqslant \|u\|_{L^1}$; then define $r_J = \infty$ and, for $j < J$, choose any nondecreasing sequence of positive numbers r_j such that

$$\int_{|x|^a \leqslant r_j} |u|\, dx = 2^j;$$

finally, define the sets for $-\infty < j \leqslant J$

$$A_j = \{x \colon r_{j-2} < |x|^a \leqslant r_{j-1}\}, \qquad B_j = \{x \colon r_{j-1} < |\xi|^{-a} \leqslant r_j\}.$$

We notice the following property:

$$\int_{|x|^a \leqslant r_j} |u|\,dx \leqslant 2^j = 4\int_{|x|^a \leqslant r_{j-2}} |u|\,dx = 4\int_{A_j} |u|\,dx;$$

hence, for $\xi \in B_j$,

$$\begin{aligned}\left(\int_{|x|\leqslant 1/|\xi|} |u(x)|\,dx\right)^2 &\leqslant \left(\int_{|x|^a \leqslant r_j} |u|\,dx\right)^2\\ &\leqslant \left(4\int_{A_j} |u|\,dx\right)^2 \leqslant 16\int_{A_j} |u|^2\,|x|^{2a}\,dx \cdot \int_{A_j} |x|^{-2a}\,dx\\ &\leqslant c_n r_{j-1}^{-2+n/a}\int_{A_j} |u|^2\,|x|^{2a}\,dx\end{aligned}$$

by Cauchy–Schwarz inequality and the explicit computation (valid for $a < n/2$)

$$\int_{A_j} |x|^{-2a}\,dx \leqslant c_n r_{j-1}^{-2+n/a}.$$

Thus we can write

$$\begin{aligned}II &= \sum_{j=-\infty}^{J} \int_{B_j} \left(\int_{|x|\leqslant 1/|\xi|} |u(x)|\,dx\right)^2 |\xi|^{-2a}\,d\xi\\ &\leqslant c_n \sum_{j=-\infty}^{J} r_{j-1}^{-2+n/a} \int_{A_j} |u|^2\,|x|^{2a}\,dx \int_{B_j} |\xi|^{-2a}\,d\xi\end{aligned}$$

whence

$$II \leqslant c_n \sum_{j=-\infty}^{J} \int_{A_j} |u|^2\,|x|^{2a}\,dx$$

by the explicit computation

$$\int_{B_j} |\xi|^{-2a}\,d\xi \leqslant c_n r_{j-1}^{-n/a+2}.$$

Since the A_j are disjoint sets, the proof is concluded. ∎

ACKNOWLEDGMENT

We thank Dr. Sandra Lucente for the helpful remarks about the subject of this paper.

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