Journal of Functional Analysis 281 (2021) 109232

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Global solution to the wave and Klein-Gordon system under null condition in dimension two

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A R T I C L E   I N F O

Article history:
Received 28 September 2020
Accepted 25 August 2021
Available online 14 September 2021
Communicated by Camil Muscalu

MSC:
35L05
35L52

Keywords:
Semilinear wave and Klein-Gordon equations
Null condition
Global-in-time solutions
Ghost weight method

A B S T R A C T

We are interested in studying the coupled wave and Klein-Gordon equations with null quadratic nonlinearities in $\mathbb{R}^{2+1}$. We aim to establish the small data global existence result, and in addition, we also illustrate the sharp pointwise asymptotic behaviour of the solution to the coupled system. The initial data are not required to have compact support, and this is achieved by applying Alinhac's ghost weight method to both the wave and the Klein-Gordon equations.

1. Introduction

Model of interest We are interested in the following coupled wave and Klein-Gordon equations

$$-\Box u = P_1^{\alpha\beta} Q_{\alpha\beta}(u, v),$$

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https://doi.org/10.1016/j.jfa.2021.109232

$$-\Box v + v = P_2^{\alpha\beta} Q_{\alpha\beta}(u,v), \tag{1.1}$$

where

$$Q_{\alpha\beta}(u,v) = \partial_\alpha u \partial_\beta v - \partial_\alpha v \partial_\beta u, \quad Q_0(u,v) = \partial_\alpha u \partial^\alpha v \text{ (to be used later)}$$

represent the classical null forms, and $P_1^{\alpha\beta}, P_2^{\alpha\beta}$ are constants.

The prescribed initial data are denoted by

$$\big(u, \partial_t u\big)(t_0, \cdot) = (u_0, u_1), \quad \big(v, \partial_t v\big)(t_0, \cdot) = (v_0, v_1). \tag{1.2}$$

Our goal is to show the small data global existence result for the system (1.1) (without compactness assumption on the initial data), and to demonstrate the sharp pointwise asymptotic behaviour of the solution (u, v) in $\mathbb{R}^{2+1}$. This extends the study by Georgiev [22] in $\mathbb{R}^{3+1}$.

Throughout of the paper, we use $A \lesssim B$ to denote $A \leq CB$ with C a generic constant, and use the notation $\langle s \rangle = \sqrt{1 + |s|^2}$. The spacetime indices are represented by $\alpha, \beta, \gamma \in \{0, 1, 2\}$, while the space indices are denoted by $a, b, c \in \{1, 2\}$, and the Einstein summation convention is adopted unless otherwise specified. As usual, we use $L^p, W^{k,p}$ (with abbreviation $H^k = W^{k,2}$) to denote the standard Sobolev spaces, and we might use the notation $\| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^2)}$ for simple illustration.

Brief history After the seminal work on nonlinear wave and nonlinear Klein-Gordon equations in $\mathbb{R}^{3+1}$, by Klainerman [32] and Christodoulou [6], and by Klainerman [33] and Shatah [48], various exciting results on nonlinear wave equations, nonlinear Klein-Gordon equations, and their coupled systems come out. In [4], Bachelot considered a Dirac-wave-Klein-Gordon system in $\mathbb{R}^{3+1}$, and then in [22], Georgiev studied the coupled wave and Klein-Gordon equations (1.1) with strong null nonlinearities (i.e. nonlinearities of type $Q_{\alpha\beta}$) in dimension $\mathbb{R}^{3+1}$, where the initial data are assumed to be compactly supported. The study in [4,22] was generalised by [15,30,31,36] and many others in $\mathbb{R}^{3+1}$, where more general nonlinearities were studied.

On the other hand, the study of the coupled wave and Klein-Gordon equations is motivated by some important models from mathematical physics, for example the Einstein-Klein-Gordon model in [28,29,37,38,55], the electroweak standard model as well as their simplified models in [18,21,35,46,47,53,54], the Klein-Gordon-Zakharov equations in [13,20,45], and many others.

Due to the fact that the wave components and the Klein-Gordon components decay slower in lower dimensions, the study of coupled wave and Klein-Gordon systems has crucial difficulties in $\mathbb{R}^{2+1}$. Recently, Ma [43,44] has initialised, as far as we know, the study of coupled (quasilinear) wave and Klein-Gordon systems in $\mathbb{R}^{2+1}$ using the hyperboloidal foliation method [36,37], and obtained global existence results (under the compactness assumption on the initial data) for such systems, and then extended the study to more types of (semilinear) nonlinearities, including null forms, in [40–42]. The hyperboloidal foliation method, dating back to Klainerman [33] and Hormander [25] (see also the pioneering work using the hyperbolic space by Tataru [52]), turns out to be very powerful in studying coupled wave and Klein-Gordon systems in $\mathbb{R}^{2+1}$. On the other hand, the microlocal analysis method has also been used to investigate the quasilinear wave and Klein-Gordon systems under the null condition, and Stingo obtained a global existence result (first such result without compactness assumption) in [51] for a quasilinear system. The almost global existence result in [27] by Ifrim and Stingo is also relevant to our study. We also recall a very recent work [13] by the author on the Klein-Gordon-Zakharov equations in $\mathbb{R}^{2+1}$, in which a class of coupled wave and Klein-Gordon equations violating the null condition was studied, see also [19]. Worth to mention, there also exist many results on nonlinear wave equations as well as nonlinear Klein-Gordon equations in $\mathbb{R}^{2+1}$, see for instance [1,2,5,11,24,26,56], see also some other related works [7–10].

Motivated by the existing results on coupled wave and Klein-Gordon systems, our prime goal is to show the global existence result for the semilinear wave and Klein-Gordon equations under strong null conditions in dimension $\mathbb{R}^{2+1}$, where the decay of wave and Klein-Gordon components is slower, and which will be considered to be more difficult to handle, and by relying on new techniques we do not need the compactness assumption on the initial data. Our argument is also expected to have other applications to the coupled wave and Klein-Gordon systems, which for instance appears in the later work [12].

Major difficulties and key ideas We now revisit the major difficulties arising in studying coupled wave and Klein-Gordon equations in $\mathbb{R}^{2+1}$ using Klainerman's vector field method with non-compactly supported initial data. Besides the slow decay nature of linear wave and Klein-Gordon components as well as the non-commutation of the scaling vector field and the Klein-Gordon operator, the difficulties also include: 1) obtaining good estimates of the undifferentiated wave components and the null forms; 2) obtaining $\langle t - r\rangle$ decay for the differentiated wave components; 3) closing the bootstrap for the high order energy.

First, the L^2–type norm of the undifferentiated wave components is required when estimating the null forms, see Lemma 2.2 and the explanations below. We recall that the L^2–type norm of the wave components cannot be bounded by the natural wave energy, and the following Hardy–type inequality

$$\big\|u/|t-|x||\big\|_{L^2(\mathbb{R}^2)} \lesssim \big\|\partial u\big\|_{L^2(\mathbb{R}^2)}$$

can be used to bound the L^2–type norm for the wave components, see for instance [39], but the compactness assumption on the solution is required to use this type of Hardy inequality. So in order to obtain the L^2–type norm bounds for the wave components without the compactness assumption, we will rely on the hidden divergence form structure of the nonlinearities $Q_{\alpha\beta}(u, v)$ (see [30]), i.e.

$$Q_{\alpha\beta}(u,v) = \partial_\beta\big(\partial_\alpha u v\big) - \partial_\alpha\big(\partial_\beta u v\big),$$

by which the L^2–type norm estimates are decomposed into estimates for linear waves as well as differentiated waves. Combining this L^2–type norm bounds and the Klainerman-Sobolev inequality in [34] (see Proposition 2.3), we find that one can obtain the pointwise decay for the undifferentiated wave components. However, due to the use of the Klainerman-Sobolev inequality in Proposition 2.3, an iteration procedure is expected.

Second, when treating the coupled wave and Klein-Gordon systems, the scaling vector field $L_0 = S = t\partial_t + x_a\partial^a$ is in general avoided to use, which is due to the fact that the scaling vector field does not commute with the Klein-Gordon operator. However, we find that the conformal energy (together with other observations) of the wave component allows us to bound the L^2 norm of Su, which further allows us to treat the (hidden) null form $Q_0(u,v)$, see (2.4). Worth to mention, combined with the Klainerman-Sobolev inequality in Proposition 2.3 we are able to get the L^∞ norm of Su. To be more precise, in order to estimate the (hidden) null form $Q_0(u,v)$, which appears in (4.15), we rely on the estimates in Lemma 2.2 to have

$$\langle t+|x|\rangle |Q_0(u,v)| \lesssim \big(|\Gamma u| + |L_0 u|\big) \sum_a \big(|L_a v| + |\partial v|\big),$$

in which $\Gamma \in \{L_a = t\partial_a + x_a\partial_t, \Omega_{ab} = x_a\partial_b - x_b\partial_a, \partial_\alpha\}$. The important thing here is that we can avoid acting the scaling vector field L_0 on the Klein-Gordon component, which was a key observation in [15] when treating a coupled wave and Klein-Gordon system in $\mathbb{R}^{3+1}$ using the hyperboloidal foliation method and is now adapted to the $\mathbb{R}^{2+1}$ case. However, the new difficulty then lies in estimating (the L^2 norm of) the term Su. Recall that the conformal energy for wave component in $\mathbb{R}^{2+1}$ is of the form

$$E_{con}(t,u) = \|Su + u\|^2_{L^2(\mathbb{R}^2)} + \sum_{a<b} \|\Omega_{ab} u\|^2_{L^2(\mathbb{R}^2)} + \sum_a \|L_a u\|^2_{L^2(\mathbb{R}^2)},$$

which is not yet an upper bound of the term Su. But thanks to the hidden divergence form structure of the null forms $Q_{\alpha\beta}$ explained before, we can first obtain the L^2 norm estimate of u, and then use the simple triangle inequality to get the L^2 norm estimate on Su from the conformal energy estimate $E_{con}(t,u)^{1/2}$.

Another difficulty lies in that when wave equations are coupled with the Klein-Gordon equations we might lose the $\langle t-|x|\rangle$ decay for the wave component (see the Klainerman-Sobolev inequality in Proposition 2.3). However, we surprisingly find that the $\langle t-|x|\rangle$ decay can be retained by first obtaining the pointwise bound for $L_0 u$, and then by relying on the fact that

$$\langle t-|x|\rangle |\partial u| \lesssim \big|L_0 u\big| + \big|\Gamma u\big|, \qquad \Gamma = \partial_\alpha, L_a, \Omega_{ab}.$$

Thus, we are allowed to gain the $\langle t-|x|\rangle$ decay for the wave components with partial derivatives ∂u. The reason why we need to retain the $\langle t-|x|\rangle$ decay is that this is necessary when applying Alinhac's ghost weight method, which is explained right below.

In addition, it is not clear how to bound the highest order energy, which is because we cannot rely on the null estimates in Lemma 2.2 in the highest order cases (due to the presence of the Klein-Gordon component v). Fortunately, we find that the null forms can be alternatively bounded by

$$\big|Q_0(u,v)\big| + \big|Q_{\alpha\beta}(u,v)\big| \lesssim \sum_a \big|\partial_a u + x_a \partial_t u/|x|\big|\big|\partial v\big| + \sum_a \big|\partial_a v + x_a \partial_t v/|x|\big|\big|\partial u\big|,$$

and we also observe that Alinhac's ghost weight method [1] can be applied to Klein-Gordon equations (see also an earlier application to quasilinear Klein-Gordon equations in [27]), which is one key novelty of the paper, and hence we are able to close the bootstrap for the highest order energy by the aid of the ghost weight energy estimates adapted to Klein-Gordon equations. However one more problem arises: the application of the ghost weight method demands the estimate

$$\big|\partial u\big| \lesssim \big(1+\big|t-|x|\big|\big)^{-1/2-\delta_1}(1+t)^{-1/2}, \qquad \delta_1>0$$

to be true. In order to achieve this, on one hand, we rely on the hidden divergence form structure of the null forms $Q_{\alpha\beta}(u,v)$ again and the estimates of type $\partial\partial u$ (see Lemma 3.4 or [43]) within the region $\{(t,x) : |x| \leq 2t\}$, which roughly tells us that within this region $\partial\partial u$ enjoys an extra $\langle t-r\rangle^{-1}$ decay compared with ∂u. On the other hand we rely on the extra $\langle t-|x|\rangle$ decay in the region $\{(t,x) : |x| \geq 2t\}$, which is equivalent to $\langle t\rangle$ decay in this region, from the pointwise decay of $L_0 u, \Gamma u$ as explained before. The way we obtain the $\langle t-|x|\rangle$ decay is another novelty of the paper. More details are demonstrated in the analysis in Section 4. Worth to mention, we find that the ghost weight method on the Klein-Gordon equations (see also [27]) has other applications [12].

Main theorem We are now ready to state the main result.

Theorem 1.1. *[Global existence result for the coupled wave and Klein-Gordon equations] Consider the coupled wave and Klein-Gordon system* (1.1)*, and let $N \geq 14$ be an integer. There exits an $\epsilon_0>0$, such that for all $\epsilon<\epsilon_0$, and all initial data satisfying the smallness condition*

$$\begin{aligned}&\sum_{k\leq N+1}\Big(\big\|\langle|x|\rangle^k\nabla^k u_0\big\|_{L^1\bigcap L^2} + \big\|\langle|x|\rangle^{k+1}\nabla^k v_0\big\|_{L^2}\Big)\\ &+\sum_{k\leq N}\Big(\big\|\langle|x|\rangle^{k+1}\nabla^k u_1\big\|_{L^1\bigcap L^2} + \big\|\langle|x|\rangle^{k+2}\nabla^k v_1\big\|_{L^2}\Big) \leq \epsilon,\end{aligned} \tag{1.3}$$

with $\nabla=(\partial_a)$ and $\bigcap$ the intersection notation, the Cauchy problem (1.1)–(1.2) *admits a global-in-time solution (u,v), which satisfies the following sharp pointwise decay results*

$$|v(t,x)| \lesssim \langle t\rangle^{-1}, \quad |u(t,x)| \lesssim \langle t\rangle^{-1/2}, \quad |\partial u(t,x)| \lesssim \langle t-|x|\rangle^{-3/4}\langle t\rangle^{-1/2}. \tag{1.4}$$

We note that the regularity required $N \geq 14$ in Theorem 1.1 is quite high, but more or less the same regularity assumptions have also appeared in [26,27], where no compactness assumptions on the initial data are needed. The main reason why we make high regularity assumptions on the initial data is that this is needed in our analysis to obtain sufficient $\langle t-|x|\rangle$ decay for the wave component, see Lemma 4.11. As a comparison, in a recent work [19] on a class of two dimensional wave and Klein-Gordon equations the regularity needed is quite low ($N \geq 3$ suffices), but the compactness assumptions are required. We conclude that, using the current method, the price to pay for removing the compactness assumptions on the initial data is making higher regularity assumptions on the initial data. To reduce the regularity required on the (non-compactly supported) initial data will be considered in our future work.

It is worth to mention that in Theorem 1.1 we need the initial data to decay sufficiently fast at infinity, which is also the case of [26] on pure wave equations. However, in the global existence result of [51] and the almost global existence result of [27], they only need mild decay on the initial data.

In the current paper, which is mainly motivated by the pioneering study of Georgiev [22] in $\mathbb{R}^{3+1}$, we only consider nonlinearities of the type $Q_{\alpha\beta}(u,v)$ in (1.1). We note that $v-v$ type interactions appearing in the u (wave) equation have been studied in [13,19,27,40,51], for instance. Instead of making an exhaustive discussion for other types of nonlinearities, we lead one to the discussion on this direction of [27,41]. One important goal in our future study is to treat more general quadratic nonlinearities for the coupled equations.

Nevertheless the slow decay nature of the wave and the Klein-Gordon components in $\mathbb{R}^{2+1}$, we can still get the global-in-time solution, as well as pointwise decay results of the solution, to the system (1.1) without compactness restrictions on the initial data, and this is the first such result. Together with the theorem in [30], we know the global existence result to the system (1.1) (with no compactness assumptions) is valid in all $\mathbb{R}^{n+1}$, with $n \geq 2$. To the best of our knowledge, whether such result to the system (1.1) holds in $\mathbb{R}^{1+1}$ is still unknown. But since, as far as we know, there does not exist any (nontrivial) blow-up result on the coupled wave and Klein-Gordon systems in any dimensions, we believe the answer to the $\mathbb{R}^{1+1}$ question is also positive.

Organisation The rest of the paper is planned as follows. In Section 2, we revisit some notations and some basic results on the wave and Klein-Gordon equations. Then in Section 3, we prepare some key results on estimating the L^2 and the L^∞ estimates for the linear wave equations. Finally, we provide the proof for Theorem 1.1 by relying on the fixed point theorem in Section 4.

2. Preliminaries

2.1. Basic notations

In the $(2+1)$ dimensional spacetime, we adopt the signature $(-,+,+)$. We denote a point in $\mathbb{R}^{2+1}$ by $(x^0, x^1, x^2) = (t, x^1, x^2)$, and denote its spacial radius by $r = \sqrt{x_1^2 + x_2^2}$.

In order to apply Klainerman's vector field method, we first introduce the vector fields:

- Translations: $\partial_\alpha = \partial_{x^\alpha}$, $\alpha = 0, 1, 2$.
- Rotations: $\Omega_{ab} = x_a \partial_b - x_b \partial_a$, $a, b = 1, 2$.
- Lorentz boosts: $L_a = x_a \partial_t + t \partial_a$, $a = 1, 2$.
- Scaling vector field: $L_0 = S = t\partial_t + r\partial_r$.

We will use Γ to denote a general vector field (not the scaling vector field L_0) in

$$V := \{\partial_\alpha, \Omega_{ab}, L_a\}.$$

In addition, we also introduce the notation of (the ghost derivative)

$$G_a := r^{-1}\big(x_a \partial_t + r\partial_a\big),$$

which appears in Alinhac's ghost weight method.

Given a sufficiently nice function $w = w(t, x)$, we define its energy on the constant time slice $t = \textit{constant}$ by

$$E_m(t, w) := \int_{\mathbb{R}^2} \Big(|\partial_t w|^2 + \sum_a |\partial_a w|^2 + m^2 |w|^2\Big)\, dx. \tag{2.1}$$

For abbreviation, we use the notation

$$E(t, w) = E_0(t, w).$$

2.2. Estimates for commutators and null forms

The following results for commutators will be frequently used, see [50].

Lemma 2.1. *For any $\Gamma', \Gamma'' \in V$ we have*

$$[\Box, \Gamma'] = 0, \qquad \big|[\Gamma', \Gamma'']w\big| \lesssim \big|\Gamma w\big|, \qquad \big|[\Gamma, \partial]w\big| + \big|[L_0, \partial]w\big| \lesssim \big|\partial w\big|, \tag{2.2}$$

in which w is sufficiently nice function. In addition, if we act the vector field Γ on the null forms, we further have

$$
\begin{aligned}
\big|\Gamma Q_0(u,v) - Q_0(\Gamma u, v) - Q_0(u,\Gamma v)\big| &= 0,\\
\big|\Gamma Q_{\alpha\beta}(u,v) - Q_{\alpha\beta}(\Gamma u, v) - Q_{\alpha\beta}(u,\Gamma v)\big| &\leq \sum_{\alpha',\beta'} \big|Q_{\alpha'\beta'}(u,v)\big|.
\end{aligned}
\tag{2.3}
$$

In order to estimate null forms, we need the following lemma which gives very detailed estimates on the null forms and can be found in [26,50] for example.

Lemma 2.2. *It holds that*

$$
\begin{aligned}
|Q_0(u,v)| &\lesssim \langle t+|x|\rangle^{-1}\big(\big|L_0 u \Gamma v\big| + \big|\Gamma u \Gamma v\big|\big),\\
|Q_{\alpha\beta}(u,v)| &\lesssim \langle t+|x|\rangle^{-1}\big(\big|\Gamma v \partial u\big| + \big|\Gamma u \partial v\big|\big).\\
|Q_0(u,v)| + |Q_{\alpha\beta}(u,v)| &\lesssim \sum_a \big(\big|G_a u\big||\partial v| + \big|G_a v\big||\partial u|\big).
\end{aligned}
\tag{2.4}
$$

2.3. Sobolev–type inequalities

Now, in order to obtain the pointwise wave decay or Klein-Gordon decay estimates from the weighted energy bounds we recall the following inequalities. We note that the importance of the inequalities below to coupled wave and Klein-Gordon equations is that we do not need to rely on the scaling vector field $L_0 = t\partial_t + x^a\partial_a$.

We first revisit one special version of the Klainerman-Sobolev inequality in [34], see the inequalities (4), (5'), and (6) therein. We note that it is not required to use the scaling vector field L_0 in the right hand side L^2–type norms, so this version is very well adapted to the study of the coupled wave and Klein-Gordon systems. However, in the inequality (2.5), we need the future information till time $2t$ when deriving the pointwise bounds for the function at time t, and thus we rely on the fixed point iteration method to prove Theorem 1.1.

Proposition 2.3. *Let $u = u(t,x)$ be a sufficiently smooth function which decays sufficiently fast at space infinity for each fixed $t \geq 0$. Then for any $t \geq 0$, $x \in \mathbb{R}^2$, we have*

$$
|u(t,x)| \lesssim \langle t\rangle^{-1/2} \sup_{0\leq s\leq 2t, |I|\leq 3} \big\|\Gamma^I u(s)\big\|_{L^2}, \qquad \Gamma \in V = \{L_a, \partial_\alpha, \Omega_{ab} = x^a\partial_b - x^b\partial_a\}.
\tag{2.5}
$$

Before recalling the following inequality, which was proved by Georgiev in [23], we first introduce some notations. Denote $\{p_j\}_0^\infty$ a usual Paley-Littlewood partition of the unity

$$
1 = \sum_{j\geq 0} p_j(s), \qquad s \geq 0,
$$

which also satisfies

$$0 \le p_j \le 1, \qquad p_j \in C_0^\infty(\mathbb{R}), \qquad \text{for all } j \ge 0,$$

as well as

$$\text{supp } p_0 \subset (-\infty, 2], \qquad \text{supp } p_j \subset [2^{j-1}, 2^{j+1}], \qquad \text{for all } j \ge 1.$$

Proposition 2.4. *Let w solve the Klein-Gordon equation*

$$-\Box w + w = f,$$

with $f = f(t,x)$ a sufficiently nice function. Then for all $t \ge 0$, it holds

$$\begin{aligned} &\langle t + |x|\rangle |w(t,x)| \\ \lesssim &\sum_{j\ge 0,\, |I|\le 4} \sup_{0\le s\le t} p_j(s) \big\| \langle s + |x|\rangle \Gamma^I f(s,x)\big\|_{L^2} \\ &+ \sum_{j\ge 0,\, |I|\le 4} \big\| \langle |x|\rangle p_j(|x|) \Gamma^I w(0,x)\big\|_{L^2} \end{aligned} \tag{2.6}$$

As a consequence, we have the following simplified version of Proposition 2.4.

Proposition 2.5. *With the same settings as Proposition 2.4, let $\delta' > 0$ and assume*

$$\sum_{|I|\le 4} \big\| \langle s + |x|\rangle \Gamma^I f(s,x)\big\|_{L^2} \le C_f \langle s\rangle^{-\delta'},$$

then we have

$$\langle t + |x|\rangle |w(t,x)| \lesssim C_f + \sum_{|I|\le 4} \big\| \langle |x|\rangle \Gamma^I w(0,x)\big\|_{L^2}. \tag{2.7}$$

2.4. Energy estimates for wave and Klein-Gordon equations

We first recall the conformal energy estimates for wave equations in $\mathbb{R}^{2+1}$, which is rarely used but will play an important role in our analysis later. For its proof, one refers to [3].

Proposition 2.6. *Let w be the solution to*

$$-\Box w = f, \qquad \big(w, \partial_t w\big)(0) = (w_0, w_1),$$

then it holds

$$E_{con}(t,w)^{1/2} \lesssim E_{con}(0,w)^{1/2} + \int_0^t \big\| \langle t' + |x|\rangle f \big\|\, dt', \tag{2.8}$$

in which

$$E_{con}(t,w) = \|Sw + w\|^2 + \sum_{a<b} \|\Omega_{ab} w\|^2 + \sum_a \|L_a w\|^2. \tag{2.9}$$

We now extend a little bit Alinhac's ghost weight method for wave equations, so that it can also be applied to Klein-Gordon equations (see also [27]). The following ghost weight energy estimates will be frequently used, which are valid for both wave and Klein-Gordon equations.

Proposition 2.7. *Assume w is the solution to*

$$-\Box w + m^2 w = f,$$

then we have

$$E_{gst1,m}(t,w) \le \int_{\mathbb{R}^2} e^q \big(|\partial_t w|^2 + \sum_a |\partial_a w|^2 + m^2 w^2\big)\, dx(0) + 2 \int_0^t \int_{\mathbb{R}^2} \big| f \partial_t w e^q \big|\, dx dt', \tag{2.10}$$

in which

$$q = \int_{-\infty}^{r-t} \langle s \rangle^{-3/2}\, ds,$$

and

$$\begin{aligned} & E_{gst1,m}(t,w) \\ = & \int_{\mathbb{R}^2} e^q \big(|\partial_t w|^2 + \sum_a |\partial_a w|^2 + m^2 w^2\big)\, dx(t) + m^2 \int_0^t \int_{\mathbb{R}^2} \frac{e^q}{\langle r - t' \rangle^{3/2}} w^2\, dx dt' \\ & + \sum_a \int_0^t \int_{\mathbb{R}^2} \frac{e^q}{\langle r - t' \rangle^{3/2}} \big| G_a w \big|^2\, dx dt'. \end{aligned} \tag{2.11}$$

Proof. The proof is almost the same as the proof for the case of $m = 0$.

We multiply on both sides of the w equation with $e^q \partial_t w$ to get

$$\begin{aligned} & \frac{1}{2} \partial_t \big(e^q (\partial w)^2 + m^2 e^q w^2\big) - \partial_a \big(e^q \partial^a w \partial_t w\big) + \frac{1}{2} \frac{e^q}{\langle t - r \rangle^{3/2}} \sum_a \big(G_a w\big)^2 \\ & + \frac{m^2}{2} \frac{e^q}{\langle t - r \rangle^{3/2}} w^2 = e^q f \partial_t w. \end{aligned}$$

Integrating over the region $[0,t]\times\mathbb{R}^2$ to arrive at the desired energy estimates. Hence the proof is done. □

Since $-\pi/2\le q\le\pi/2$, we thus have the following version of the ghost weight energy estimates

$$E_{gst,m}(t,w)\lesssim E_m(0,w)+\int_0^t\int_{\mathbb{R}^2}|f\partial_t w|\,dxdt', \tag{2.12}$$

in which

$$\begin{aligned}E_{gst,m}(t,w)=&E_m(t,w)+m^2\int_0^t\int_{\mathbb{R}^2}\frac{w^2}{\langle r-t'\rangle^{3/2}}\,dxdt'\\&+\sum_a\int_0^t\int_{\mathbb{R}^2}\frac{|G_a w|^2}{\langle r-t'\rangle^{3/2}}\,dxdt'.\end{aligned} \tag{2.13}$$

We note that the ghost weight energy estimates imply the usual energy estimates

$$E_m(t,w)^{1/2}\lesssim E_m(0,w)^{1/2}+\int_0^t\|f\|\,dt'. \tag{2.14}$$

Besides, we also have the following type of ghost weight energy estimates.

Proposition 2.8. *With the same assumptions as in Proposition 2.7, we get*

$$\begin{aligned}&m^2\int_0^t\langle t'\rangle^{-\delta}\int_{\mathbb{R}^2}\frac{w^2}{\langle r-t'\rangle^{3/2}}\,dxdt'+\sum_a\int_0^t\langle t'\rangle^{-\delta}\int_{\mathbb{R}^2}\frac{\left|G_a w\right|^2}{\langle r-t'\rangle^{3/2}}\,dxdt'\\&\lesssim E_m(0,w)+\int_0^t\int_{\mathbb{R}^2}\langle t'\rangle^{-\delta}|f\partial_t w|\,dxdt'.\end{aligned} \tag{2.15}$$

Proof. We multiply on both sides of the w equation with $\langle t\rangle^{-\delta}e^q\partial_t w$ to get

$$\begin{aligned}&\frac12\partial_t\big(\langle t\rangle^{-\delta}\big(e^q(\partial w)^2+m^2e^qw^2\big)\big)\\&+\frac{\delta}{2}t\langle t\rangle^{-2-\delta}\big(e^q(\partial w)^2+m^2e^qw^2\big)-\partial_a\big(\langle t\rangle^{-\delta}e^q\partial^a w\partial_t w\big)\\&+\frac12\frac{\langle t\rangle^{-\delta}e^q}{\langle t-r\rangle^{3/2}}\sum_a\big(G_a w\big)^2+\frac{m^2}{2}\frac{\langle t\rangle^{-\delta}e^q}{\langle t-r\rangle^{3/2}}w^2=\langle t\rangle^{-\delta}e^q f\partial_t w.\end{aligned}$$

We integrate over the region $[0,t]\times\mathbb{R}^2$, and the facts $t\geq 0$, $1\lesssim e^q\lesssim 1$ imply the desired energy estimates. We thus complete the proof. □

3. L^2 and L^∞ estimates for wave equations

3.1. L^2 estimates for homogeneous wave equations

We have the following lemmas which help bound the L^2 norm of the solution (with no derivatives in front) to wave equations, which was used in [14,16,17].

Lemma 3.1. *Let w be the solution to the linear wave equation*

$$\begin{aligned} -\Box w &= 0, \\ w(0,\cdot) = w_0, &\quad \partial_t w(0,\cdot) = w_1. \end{aligned} \tag{3.1}$$

We assume that ($\bigcap$ is the notation for the intersection of two sets)

$$\|w_0\|_{L^2} + \|w_1\|_{L^2\bigcap L^1} < +\infty. \tag{3.2}$$

Then the following L^2 norm bound is valid

$$\|w\|_{L^2} \lesssim \|w_0\|_{L^2} + \langle t\rangle^\delta \|w_1\|_{L^2\bigcap L^1} \tag{3.3}$$

for $0<\delta\ll 1$.

Proof. Recall that the Fourier transform is defined by

$$\widehat{w}(t,\xi) = \int_{\mathbb{R}^2} w(t,x)e^{-ix_a\xi^a}\,dx.$$

We express the equation of w in the Fourier space

$$\begin{aligned} \partial_{tt}\widehat{w}(t,\xi) + |\xi|^2\widehat{w}(t,\xi) = 0, \\ \widehat{w}(0,\cdot) = \widehat{w}_0, \qquad \partial_t\widehat{w}(0,\cdot) = \widehat{w}_1. \end{aligned}$$

Then we obtain the solution w in Fourier space by solving the above ordinary differential equation

$$\widehat{w}(t,\xi) = \cos(t|\xi|)\widehat{w}_0 + \frac{\sin(t|\xi|)}{|\xi|}\widehat{w}_1.$$

Thus we can bound the L^2 norm of w as (recall the Plancherel's identity)

$$\|w\|_{L^2} \lesssim \|w_0\|_{L^2} + \Big\| \frac{\sin(t|\xi|)}{|\xi|} \widehat{w}_1 \Big\|_{L^2}. \tag{3.4}$$

We proceed by

$$\Big\| \frac{\sin(t|\xi|)}{|\xi|} \widehat{w}_1 \Big\|_{L^2} \lesssim t^\delta \Big\| \frac{\widehat{w}_1}{|\xi|^{1-\delta}} \Big\|_{L^2} \lesssim t^\delta \Big\| \frac{w_1}{\Lambda^{1-\delta}} \Big\|_{L^2},$$

in which $\Lambda = \sqrt{-\partial_a \partial^a}$ and we used the simple relations $|\sin(t|\xi|)| \leq t|\xi|, |\sin(t|\xi|)| \leq 1$. The Sobolev embedding (δ_1 below needs not be an integer)

$$\Big\| \frac{f}{\Lambda^{\delta_1}} \Big\|_{L^q} \lesssim \|f\|_{L^p}, \qquad \delta_1 = \frac{2}{p} - \frac{2}{q}, \quad 1 < p < q < +\infty$$

further implies

$$\begin{aligned} \Big\| \frac{\sin(t|\xi|)}{|\xi|} \widehat{w}_1 \Big\|_{L^2} &\lesssim t^\delta \Big\| \frac{w_1}{\Lambda^{1-\delta}} \Big\|_{L^2} \\ &\lesssim t^\delta \|w_1\|_{L^{2/(2-\delta)}} \lesssim t^\delta \|w_1\|_{L^1 \bigcap L^2}, \end{aligned}$$

in which we used the fact $\delta \ll 1$. Gathering the estimates finishes the proof. □

3.2. L^∞ estimates for wave equations

Recall that we do not have any $\langle t - |x| \rangle$ decay when applying the Klainerman-Sobolev inequality of version (2.5). But the following result helps gain $\langle t - |x| \rangle^{-1}$ decay for ∂u components, which is of vital importance when using the ghost weight energy estimates (2.12). Its proof can be found in [26,50].

Lemma 3.2. *We have*

$$\big|\partial u\big| \lesssim \langle t - |x| \rangle^{-1} \big(|L_0 u| + |\Gamma u| \big), \qquad \big|G_a u\big| \lesssim \langle t + |x| \rangle^{-1} \big(|L_0 u| + |\Gamma u| \big). \tag{3.5}$$

Next, we recall the pointwise estimates for homogeneous waves, see for instance [26,49]. We note that the regularity required for the initial data is much weaker in [49], where the Besov spaces are used, but due to some regularity loss in other places we will use the following version of estimates with proof.

Lemma 3.3. *Let w be the solution to*

$$\begin{aligned} -\Box w &= 0, \\ w(0,\cdot) &= w_0, \quad \partial_t w(0,\cdot) = w_1, \end{aligned} \tag{3.6}$$

then we have

$$|w| \lesssim \langle t \rangle^{-1/2} \big(\|w_0\|_{W^{2,1} \bigcap H^3} + \|w_1\|_{W^{1,1} \bigcap H^2} \big). \tag{3.7}$$

Proof. We revisit the proof given in [26], and since the result we need is weaker, the analysis is simpler.

The solution can be represented by $w = w^0 + w^1$, with

$$w^0(t,x) = \frac{1}{2\pi}\partial_t \int\limits_{|x-y|\le t} \frac{w_0(y)\,dy}{\sqrt{t^2-|x-y|^2}}, \qquad w^1(t,x) = \frac{1}{2\pi} \int\limits_{|x-y|\le t} \frac{w_1(y)\,dy}{\sqrt{t^2-|x-y|^2}}.$$

We will only provide the proof for the estimate of $w^1(t,x)$ when $t \ge 2$, since other cases are either similar or simpler. We note that (with $p = y - x$)

$$\begin{aligned} \big|w^1(t,x)\big| \lesssim & \Big| \int\limits_{|p|\le t} \frac{w_1(x+p)\,dp}{\sqrt{t^2-|p|^2}} \Big| \\ =& \Big| \int\limits_{|p|\le t} \frac{w_1(x+p)\,dp}{\sqrt{t-|p|}\sqrt{t+|p|}} \Big|, \end{aligned}$$

and the fact $\langle t\rangle^{1/2} \lesssim \sqrt{t+|p|}$ for $t \ge 2$ further implies

$$\begin{aligned} \big|w^1(t,x)\big| \lesssim & \langle t\rangle^{-1/2} \int\limits_{|p|\le t-1} \frac{|w_1(x+p)|\,dp}{\sqrt{t-|p|}} + \Big| \int\limits_{t-1\le|p|\le t} \frac{w_1(x+p)\,dp}{\sqrt{t+|p|}\sqrt{t-|p|}} \Big| \\ \lesssim & \langle t\rangle^{-1/2} \big\|w_1\big\|_{L^1} + \Big| \int\limits_{t-1\le|p|\le t} \frac{w_1(x+p)\,dp}{\sqrt{t+|p|}\sqrt{t-|p|}} \Big|. \end{aligned}$$

We observe that

$$\begin{aligned} & \Big| \int\limits_{t-1\le|p|\le t} \frac{w_1(x+p)\,dp}{\sqrt{t+|p|}\sqrt{t-|p|}} \Big| \\ \lesssim & \Big| \int\limits_{S^1} \int\limits_{t-1}^{t} \frac{w_1(x+\omega|p|)|p|}{\sqrt{t+|p|}}\, d\sqrt{t-|p|} d\omega \Big| \\ \lesssim & \langle t\rangle^{-1/2} \Big| \int\limits_{S^1} w_1(x+\omega(t-1))(t-1)\,d\omega \Big| \\ & + \langle t\rangle^{-1/2} \int\limits_{S^1} \int\limits_{t-1}^{t} \big(|w_1(x+\omega|p|)| + |\partial w_1(x+\omega|p|)||p|\big)\,d|p|d\omega, \end{aligned}$$

in which we used integration by part in the last step. Recall that $t \ge 2$, we thus have

$$\begin{aligned}&\int\limits_{S^1}\int\limits_{t-1}^{t}\big(|w_1(x+\omega|p|)|+|\partial w_1(x+\omega|p|)||p|\big)\,d|p|d\omega\\ \lesssim&\int\limits_{S^1}\int\limits_{t-1}^{t}\big(|w_1(x+\omega|p|)||p|+|\partial w_1(x+\omega|p|)||p|\big)\,d|p|d\omega\\ \lesssim&\|w_1\|_{L^1}+\|\partial w_1\|_{L^1}.\end{aligned}$$

To proceed, we get

$$\begin{aligned}&\langle t\rangle^{1/2}\Big|\int\limits_{t-1\le|p|\le t}\frac{w_1(x+p)\,dp}{\sqrt{t+|p|}\sqrt{t-|p|}}\Big|\\ \lesssim&\Big|\int\limits_{S^1}w_1(x+\omega(t-1))(t-1)\,d\omega\Big|+\|w_1\|_{L^1}+\|\partial w_1\|_{L^1}\\ \lesssim&\int\limits_{S^1}\int\limits_{0}^{t-1}\big(|\partial w_1(x+\omega|p|)||p|+|w_1(x+\omega|p|)|\big)\,d|p|d\omega+\|w_1\|_{L^1}+\|\partial w_1\|_{L^1}\\ \lesssim&\int\limits_{S^1}\int\limits_{0}^{1}|w_1(x+\omega|p|)|\,d|p|d\omega+\int\limits_{S^1}\int\limits_{1}^{t-1}|w_1(x+\omega|p|)||p|\,d|p|d\omega+\|w_1\|_{L^1}+\|\partial w_1\|_{L^1\bigcap L^2}\\ \lesssim&\|w_1\|_{L^\infty}+\|w_1\|_{L^1}+\|\partial w_1\|_{L^1}\lesssim\|w_1\|_{H^2}+\|w_1\|_{L^1}+\|\partial w_1\|_{L^1},\end{aligned}$$

in which we used the fundamental theorem of calculus in the second step. We thus obtain

$$\big|w^1(t,x)\big|\lesssim\langle t\rangle^{-1/2}\big(\|w_1\|_{H^2}+\|w_1\|_{L^1}+\|\partial w_1\|_{L^1}\big),\qquad t\ge 2.\quad\Box$$

Besides, the following key observation, see for instance [42,43], claims that $\partial\partial u$ has extra $\langle t-|x|\rangle^{-1}$ decay than ∂u in the spacetime region $\{(t,x):|x|\le 2t\}$, and this can be used to get the L^∞ bound for ∂u thanks to the divergence form structure in the u equation.

Lemma 3.4. *Let w solve*

$$-\Box w=f,$$

and we further assume

$$\big|\partial w\big|+\big|\partial\Gamma w\big|\lesssim C_w\langle t\rangle^{-1/2},\qquad |f|\lesssim C_f\langle t\rangle^{-3/2},\tag{3.8}$$

with C_w, C_f constants, then we have

$$\big|\partial\partial w\big|\lesssim\big(C_w+C_f\big)\langle t-|x|\rangle^{-1}\langle t\rangle^{-1/2},\qquad \textit{in }\{(t,x):|x|\le 2t\}.\tag{3.9}$$

Proof. For completeness we revisit the proof in [42,43]. Since it is easily seen that the results hold for $t \leq 1$, so we will only consider the case $t \geq 1$.

We first express the wave operator $-\Box$ by ∂_t, L_a to get

$$-\Box = \frac{(t-|x|)(t+|x|)}{t^2}\partial_{tt} + 2\frac{x^a}{t^2}\partial_t L_a - \frac{1}{t^2}L^a L_a + \frac{2}{t}\partial_t - \frac{x^a}{t^2}\partial_a. \tag{3.10}$$

Then we find that

$$\frac{1+|t-|x||}{t}|\partial_{tt}w| \lesssim \frac{1}{t}\big(|\partial\Gamma w| + |\partial w|\big) + |f|,$$

in which we used the relation $|x| \leq 2t$, and thus we are led to

$$|\partial_{tt}w| \lesssim \big(C_w + C_f\big)\frac{1}{\langle t-|x|\rangle\langle t\rangle^{1/2}}.$$

On the other hand, we note that the following relations hold true

$$\partial_a\partial_t = -\frac{x_a}{t}\partial_t\partial_t + \frac{1}{t}\partial_t L_a + \frac{x_a}{t^2}\partial_t - \frac{1}{t^2}L_a,$$
$$\partial_a\partial_b = \frac{x_a x_b}{t^2}\partial_t\partial_t - \frac{x_a}{t^2}\partial_t L_b + \frac{1}{t}\partial_b L_a - \frac{\delta_{ab}}{t}\partial_t + \frac{x_a}{t^2}\partial_b,$$

which, using again $|x| \leq 2t$, means

$$\big|\partial_\alpha\partial_\beta w\big| \lesssim \big|\partial_t\partial_t w\big| + \frac{1}{t}\big(|\partial\Gamma w| + |\partial w|\big) \lesssim \big|\partial_t\partial_t w\big| + \frac{1}{\langle t-|x|\rangle}\big(|\partial\Gamma w| + |\partial w|\big).$$

We thus complete the proof. $\Box$

4. Proof of the main theorem

4.1. Initialisation of the iteration method

As we explained in the introduction part that the utilisation of the Klainerman-Sobolev inequality (2.5) requires us to rely on an iteration procedure in order to show the global existence result for the system (1.1), we thus first provide the basics for the fixed point iteration method.

We now introduce the solution space which is denoted by X, recalling $N \geq 14$ in Theorem 1.1.

Definition 4.1. Let $\phi = \phi(t,x), \psi = \psi(t,x)$ be sufficiently regular functions, and we say (ϕ, ψ) belongs to the function space X if

- It satisfies

$$\big(\phi, \partial_t\phi, \psi, \partial_t\psi\big)(0,\cdot) = \big(u_0, u_1, v_0, v_1\big). \tag{4.1}$$

- It satisfies

$$\big\|(\phi,\psi)\big\|_X \le C_1\epsilon, \tag{4.2}$$

in which $C_1 \gg 1$ is a large constant to be determined, the size of the initial data $\epsilon \ll 1$ is sufficiently small such that $C_1\epsilon \ll \delta$, and the $\|\cdot\|_X$ norm is defined by

$$\begin{aligned}
\big\|(u,v)\big\|_X := &\sup_{t\ge 0,\,|I|\le N} \langle t\rangle^{-\delta}\big(\big\|\Gamma^I u\big\| + E_{gst}(\Gamma^I u,t)^{1/2} + E_{gst,1}(\Gamma^I v,t)^{1/2}\big)\\
&+ \sup_{t\ge 0,\,|I|\le N} \langle t\rangle^{-\delta/2}\Big(\int_0^t \langle t'\rangle^{-\delta}\Big(\Big\|\frac{\Gamma^I v}{\langle t'-|x|\rangle^{3/4}}\Big\|^2 + \Big\|\frac{G_a\Gamma^I v}{\langle t'-|x|\rangle^{3/4}}\Big\|^2\Big)\,dt'\Big)^{1/2}\\
&+ \sup_{t\ge 0,\,|I|\le N-1} \big(E_{gst}(\Gamma^I u,t)^{1/2} + E_{gst,1}(\Gamma^I v,t)^{1/2}\big)\\
&+ \sup_{t\ge 0,|I|\le N-1} \langle t\rangle\big(\big\|\Box\Gamma^I u\big\| + \big\|-\Box\Gamma^I v + \Gamma^I v\big\|\big)\\
&+ \sup_{t\ge 0,\,|I|\le N-2} \langle t\rangle^{-1/2-\delta}\big\|L_0\Gamma^I u\big\|\\
&+ \sup_{t\ge 0,\,|I|\le N-6} \langle t\rangle^{-\delta}\big\|L_0\Gamma^I u\big\| + \sup_{x,\,t\ge 0,\,|I|\le N-5} \langle t+|x|\rangle\big|\Gamma^I v\big|\\
&+ \sup_{x,\,t\ge 0,|I|\le N-6} \langle t\rangle^2\big(\big|\Box\Gamma^I u\big| + \big|-\Box\Gamma^I v + \Gamma^I v\big|\big)\\
&+ \sup_{x,\,t\ge 0,\,|I|\le N-9} \langle t-|x|\rangle^{3/4}\langle t\rangle^{1/2}\big|\partial\Gamma^I u\big|.
\end{aligned} \tag{4.3}$$

We note that the function space X is complete with respect to the metric induced from the $\|\cdot\|_X$ norm.

4.2. The solution mapping

Definition 4.2. Given a pair of functions $(m,n) \in X$, we define

$$T(m,n) := (\phi,\psi), \tag{4.4}$$

in which (ϕ,ψ) is the solution to the following (linear) system

$$\begin{aligned}
-\Box\phi &= P_1^{\alpha\beta}Q_{\alpha\beta}(m,n),\\
-\Box\psi + \psi &= P_2^{\alpha\beta}Q_{\alpha\beta}(m,n),\\
\big(\phi, \partial_t\phi, \psi, \partial_t\psi\big)(0,\cdot) &= \big(u_0, u_1, v_0, v_1\big).
\end{aligned} \tag{4.5}$$

To track the components ϕ, ψ, m, n easily, we remind one that ϕ, m are wave components while ψ, n represent Klein-Gordon components.

We have the following proposition about the solution mapping T.

Proposition 4.3. *The images of the solution mapping T lie in X.*

We need the following results to prove Proposition 4.3.

Lemma 4.4. *We have*

$$\begin{aligned} \big|L_0\Gamma^I m\big| &\lesssim C_1\epsilon\langle t\rangle^{-1/2+\delta}, & |I|&\le N-9,\\ \big|\Gamma^I m\big| &\lesssim C_1\epsilon\langle t\rangle^{-1/2+\delta}, & |I|&\le N-4,\\ \big|\partial\Gamma^I m\big| &\lesssim C_1\epsilon\langle t\rangle^{-1/2}, & |I|&\le N-4,\\ \big|\partial\Gamma^I m\big| &\lesssim C_1\epsilon\langle t-|x|\rangle^{-3/4}\langle t\rangle^{-1/2}, & |I|&\le N-9. \end{aligned} \tag{4.6}$$

Proof. The first three estimates follow from the Klainerman-Sobolev inequality (2.5) and the commutator estimates, and the last one is from the definition of the function space X. □

Lemma 4.5. *We have*

$$\begin{aligned} &E_{gst}(\Gamma^I\psi,t)^{1/2} \lesssim \epsilon + (C_1\epsilon)^{3/2}, \qquad |I|\le N-1,\\ &E_{gst}(\Gamma^I\psi,t)^{1/2} \lesssim \epsilon + (C_1\epsilon)^{3/2}\langle t\rangle^{\delta}, \qquad |I|\le N,\\ &\Big(\int_0^t \langle t'\rangle^{-\delta}\Big(\Big\|\frac{\Gamma^I\psi}{\langle t'-|x|\rangle^{3/4}}\Big\|^2 + \Big\|\frac{G_a\Gamma^I\psi}{\langle t'-|x|\rangle^{3/4}}\Big\|^2\Big)\,dt'\Big)^{1/2} \lesssim \epsilon + (C_1\epsilon)^{3/2}\langle t\rangle^{\delta/2},\\ &\qquad |I|\le N. \end{aligned} \tag{4.7}$$

Proof. We act the vector field Γ^I on both sides of ψ equation in (4.5) to get

$$-\Box\Gamma^I\psi + \Gamma^I\psi = P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n).$$

The usual energy estimates (2.14) give

$$E_1(t,\Gamma^I\psi)^{1/2} \lesssim E_1(0,\Gamma^I\psi)^{1/2} + \int_0^t \big\|P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big\|\,dt'.$$

For the case $|I|\le N-1$, we have (recall $N\ge 14$)

$$\big\|P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big\| \lesssim \sum_{\alpha,\beta,|I_1|+|I_2|\le|I|} \big\|Q_{\alpha\beta}(\Gamma^{I_1}m,\Gamma^{I_2}n)\big\|$$

$$
\begin{aligned}
&\lesssim \sum_{|I_1|+|I_2|\le|I|} \big\|\langle t'\rangle^{-1}\Gamma\Gamma^{I_1}m\Gamma\Gamma^{I_2}n\big\| \\
&\lesssim \sum_{\substack{|I_1|+|I_2|\le|I|\\ |I_2|\le N-6}} \langle t'\rangle^{-1}\big\|\Gamma\Gamma^{I_1}m\big\|\big\|\Gamma\Gamma^{I_2}n\big\|_{L^\infty} \\
&\quad + \sum_{\substack{|I_1|+|I_2|\le|I|\\ |I_1|\le N-5}} \langle t'\rangle^{-1}\big\|\Gamma\Gamma^{I_1}m\big\|_{L^\infty}\big\|\Gamma\Gamma^{I_2}n\big\| \\
&\lesssim (C_1\epsilon)^2\langle t'\rangle^{-3/2+2\delta}.
\end{aligned}
$$

So we are led to

$$
E_1(t,\Gamma^I\psi)^{1/2} \lesssim \epsilon + (C_1\epsilon)^2\int_0^t \langle t'\rangle^{-3/2+2\delta}\,dt' \lesssim \epsilon + (C_1\epsilon)^2.
$$

Then, we apply the ghost weight energy estimates (2.12) to obtain

$$
E_{gst,1}(t,\Gamma^I\psi) \lesssim E_{gst,1}(0,\Gamma^I\psi) + \int_0^t \big\|P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\partial_t\Gamma^I v\big\|_{L^1}\,dt'.
$$

Similarly, we get

$$
\begin{aligned}
E_{gst,1}(t,\Gamma^I\psi) &\lesssim \epsilon^2 + \int_0^t \big\|P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big\|\big\|\partial_t\Gamma^I v\big\|\,dt' \\
&\lesssim \epsilon^2 + (C_1\epsilon)^3\int_0^t \langle t'\rangle^{-3/2+2\delta}\,dt' \lesssim \epsilon^2 + (C_1\epsilon)^3.
\end{aligned}
$$

Next, we turn to the case of $|I|\le N$, and we start with estimating (recall $N\ge 14$)

$$
\begin{aligned}
&\big\|P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big\| \\
\lesssim& \sum_{\alpha,\beta,|I_1|+|I_2|\le|I|} \big\|Q_{\alpha\beta}(\Gamma^{I_1}m,\Gamma^{I_2}n)\big\| \\
\lesssim& \sum_{a,|I_1|+|I_2|\le|I|} \Big(\big\|G_a\Gamma\Gamma^{I_1}m\partial\Gamma^{I_2}n\big\| + \big\|\partial\Gamma\Gamma^{I_1}mG_a\Gamma^{I_2}n\big\|\Big) \\
\lesssim& \sum_{\substack{|I_1|+|I_2|\le|I|\\ |I_2|\le N-9}} \Big(\Big\|\frac{G_a\Gamma^{I_1}n}{\langle t'-|x|\rangle^{3/4}}\Big\|\big\|\langle t'-|x|\rangle^{3/4}\partial\Gamma^{I_2}m\big\|_{L^\infty} + \big\|\partial\Gamma^{I_1}n\big\|\big\|G_a\Gamma^{I_2}m\big\|_{L^\infty}\Big)
\end{aligned}
$$

$$+\sum_{\substack{|I_1|+|I_2|\le|I|\\|I_1|\le N-6}}\big\|\partial\Gamma^{I_1}m\big\|\big\|\partial\Gamma^{I_2}n\big\|_{L^\infty}.$$

Recall the relation

$$\big|G_a m\big|\lesssim\langle t+|x|\rangle^{-1}\big(\big|L_0 m\big|+\big|\Gamma m\big|\big),$$

as well as the bounds

$$\big|\langle t-|x|\rangle^{3/4}\partial\Gamma^{I_2}m\big|\lesssim C_1\epsilon\langle t\rangle^{-1/2},$$

we thus arrive at

$$\big\|P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big\|\lesssim C_1\epsilon\langle t\rangle^{-1/2}\sum_{|I|\le N}\Big\|\frac{G_a\Gamma^I n}{\langle t-|x|\rangle^{3/4}}\Big\|+(C_1\epsilon)^2\langle t\rangle^{-1+\delta}.$$

Then the energy estimates (2.14) yield

$$\begin{aligned}E_1(t,\Gamma^I)^{1/2}\lesssim&\epsilon+\int_0^t\big\|P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big\|\,dt'\\ \lesssim&\epsilon+\int_0^t\Big((C_1\epsilon)^2\langle t'\rangle^{-1+\delta}+C_1\epsilon\langle t'\rangle^{-1/2}\sum_{|I|\le N}\Big\|\frac{G_a\Gamma^I n}{\langle t'-|x|\rangle^{3/4}}\Big\|\Big)\,dt'\\ \lesssim&\epsilon+(C_1\epsilon)^2t^\delta\\ &+C_1\epsilon\sum_{|I|\le N}\Big(\int_0^t\langle t'\rangle^{-1+\delta}\,dt'\Big)^{1/2}\Big(\int_0^t\langle t'\rangle^{-\delta/2}\Big\|\frac{G_a\Gamma^I n}{\langle t'-|x|\rangle^{3/4}}\Big\|^2\Big)\,dt'\Big)^{1/2},\end{aligned}$$

which leads us to

$$E_1(t,\Gamma^I)^{1/2}\lesssim\epsilon+(C_1\epsilon)^2\langle t\rangle^\delta.$$

In succession, we apply the ghost weight energy estimates (2.12) to get

$$\begin{aligned}&E_{gst,1}(t,\Gamma^I\psi)\\ \lesssim&E_{gst,1}(0,\Gamma^I\psi)+\int_0^t\big\|P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big\|\big\|\partial_t\Gamma^I v\big\|\,dt'\\ \lesssim&\epsilon^2+\int_0^t\Big((C_1\epsilon)^3\langle t'\rangle^{-1+2\delta}+(C_1\epsilon)^2\langle t'\rangle^{-1/2+\delta}\sum_{|I|\le N}\Big\|\frac{G_a\Gamma^I n}{\langle t'-|x|\rangle^{3/4}}\Big\|\Big)\,dt'\\ \lesssim&\epsilon^2+(C_1\epsilon)^3t^{2\delta}.\end{aligned}$$

Finally, we use the ghost weight energy estimates (2.15) to proceed

$$\sum_a \int_0^t \langle t'\rangle^{-\delta}\Big(\Big\|\frac{\Gamma^I\psi}{\langle t'-|x|\rangle^{3/2}}\Big\|^2+\Big\|\frac{G_a\Gamma^I\psi}{\langle t'-|x|\rangle^{3/4}}\Big\|^2\Big)\,dt'$$

$$\lesssim E_m(0,\Gamma^I\psi)+\int_0^t\int_{\mathbb{R}^2}\langle t'\rangle^{-\delta}|P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\partial_t\Gamma^I\psi|\,dxdt'$$

$$\lesssim \epsilon^2+\int_0^t\Big((C_1\epsilon)^3\langle t'\rangle^{-1+\delta}+(C_1\epsilon)^2\langle t'\rangle^{-1/2}\sum_{|I|\le N}\Big\|\frac{G_a\Gamma^I n}{\langle t'-|x|\rangle^{3/4}}\Big\|\Big)\,dt'\lesssim \epsilon^2+(C_1\epsilon)^3 t^\delta.$$

The proof is complete now. □

Lemma 4.6. *We have*

$$\begin{aligned} E_{gst}(\Gamma^I\phi,t)^{1/2}&\lesssim \epsilon+(C_1\epsilon)^{3/2}, && |I|\le N-1,\\ E_{gst}(\Gamma^I\phi,t)^{1/2}&\lesssim \epsilon+(C_1\epsilon)^{3/2}\langle t\rangle^\delta, && |I|\le N. \end{aligned} \tag{4.8}$$

Proof. The same proof in Lemma 4.5 also applies here, so we omit the proof. □

Lemma 4.7. *We have*

$$\begin{aligned} \big\|\Box\Gamma^I\phi\big\|+\big\|(-\Box+1)\Gamma^I\psi\big\|&\lesssim (C_1\epsilon)^2\langle t\rangle^{-1}, && |I|\le N-1,\\ \big|\Box\Gamma^I\phi\big|+\big|(-\Box+1)\Gamma^I\psi\big|&\lesssim (C_1\epsilon)^2\langle t\rangle^{-2}, && |I|\le N-6. \end{aligned} \tag{4.9}$$

Proof. The proof of the L^2–type norm estimates was covered in the proof of Lemma 4.5.

As for the sup-norm estimates for $|I|\le N-6$, we observe that it suffices to show

$$\big|P_a^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big|\lesssim (C_1\epsilon)^2\langle t\rangle^{-2},\qquad |I|\le N-6.$$

We indeed have for $|I|\le N-6$ that

$$\big|P_a^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big|\lesssim \frac{1}{\langle t\rangle}\sum_{|I_1|,|I_2|\le N-5}\big|\Gamma^{I_1}m\big|\big|\Gamma^{I_2}n\big|\lesssim (C_1\epsilon)^2\langle t\rangle^{-5/2+\delta}\lesssim (C_1\epsilon)^2\langle t\rangle^{-2}.$$

Hence we complete the proof. □

Lemma 4.8. *We have*

$$\big|\Gamma^I\psi\big|\lesssim \epsilon+(C_1\epsilon)^{3/2}\langle t+|x|\rangle^{-1},\qquad |I|\le N-5. \tag{4.10}$$

Proof. According to the result in Proposition 2.5, it suffices to show

$$\big\|\langle t+|x|\rangle P_2^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big\| \lesssim (C_1\epsilon)^2\langle t\rangle^{-\delta}, \qquad |I|\le N-1.$$

But this was done (not exactly the same but very similar) in the proof of Lemma 4.5.

The proof is done. □

Before we proceed further, we now decompose the wave component ϕ as

$$\phi = \phi^5 + \partial_\gamma \phi^\gamma, \tag{4.11}$$

in which ϕ^5, ϕ^γ are solutions to the following (linear) equations:

$$-\Box\phi^5 = 0, \qquad \big(\phi^5, \partial_t\phi^5\big)(0) = (u_0, u_1), \tag{4.12}$$

as well as

$$-\Box\phi^\gamma = P_1^{\alpha\gamma} n\partial_\alpha m - P_1^{\gamma\beta} n\partial_\beta m, \qquad \big(\phi^\gamma, \partial_t\phi^\gamma\big)(0) = (0,0). \tag{4.13}$$

In addition, we reveal the hidden null structure in the equation of (4.13) with the new variables

$$\Phi^\gamma := \phi^\gamma + P_1^{\alpha\gamma} n\partial_\alpha m - P_1^{\gamma\beta} n\partial_\beta m, \tag{4.14}$$

which is the solution to

$$\begin{aligned}-\Box\Phi^\gamma =& P_1^{\alpha\gamma}(-\Box+1)n\partial_\alpha m + P_1^{\alpha\gamma} n(-\Box\partial_\alpha m) - P_1^{\alpha\gamma} Q_0(n,\partial_\alpha m)\\ &- P_1^{\gamma\beta}(-\Box+1)n\partial_\beta m - P_1^{\gamma\beta} n(-\Box\partial_\beta m) + P_1^{\gamma\beta} Q_0(n,\partial_\beta m),\end{aligned} \tag{4.15}$$

and this is obtained by the product rule for derivatives. We observe that the nonlinearities decay very fast, and this decomposition will be used in the proof of Lemma 4.11 to obtain sharp time decay result of ϕ.

Lemma 4.9. *We have*

$$\big\|\Gamma^I\phi\big\|_{L^2} \lesssim \Big(\epsilon + (C_1\epsilon)^{3/2}\Big)\langle t\rangle^\delta, \qquad |I|\le N. \tag{4.16}$$

Proof. First, the result in Lemma 3.1 implies

$$\|\Gamma^I\phi^5\| \lesssim \epsilon\langle t\rangle^\delta, \qquad |I|\le N. \tag{4.17}$$

Taking into account the relation (4.11), it suffices to show

$$\sum_\gamma E(t,\Gamma^I\phi^\gamma)^{1/2} \lesssim \epsilon + (C_1\epsilon)^2\langle t\rangle^\delta, \qquad |I|\le N. \tag{4.18}$$

Applying the usual energy estimates for the ϕ^γ equation, we get

$$E(t,\Gamma^I\phi^\gamma)^{1/2}\lesssim E(0,\Gamma^I\phi^\gamma)^{1/2}+\int_0^t\big\|\Gamma^I\big(P_1^{\alpha\gamma}n\partial_\alpha m-P_1^{\gamma\beta}n\partial_\beta m\big)\big\|\,dt'.$$

Successively, we have (recall that $N\geq 14$)

$$\begin{aligned}
&\big\|\Gamma^I\big(P_1^{\alpha\gamma}n\partial_\alpha m-P_1^{\gamma\beta}n\partial_\beta m\big)\big\|\\
\lesssim&\sum_{|I_1|+|I_2|\leq N}\big\|\Gamma^{I_1}n\partial\Gamma^{I_2}m\big\|\\
\lesssim&\sum_{\substack{|I_1|\leq N-5\\|I_2|\leq N}}\big\|\Gamma^{I_1}n\big\|_{L^\infty}\big\|\partial\Gamma^{I_2}m\big\|\\
&+\sum_{\substack{|I_1|\leq N\\|I_2|\leq N-9}}\Big\|\frac{\Gamma^{I_1}n}{\langle t\rangle^{\delta/2}\langle t-|x|\rangle^{3/4}}\Big\|\big\|\langle t\rangle^{\delta/2}\langle t-|x|\rangle^{3/4}\partial\Gamma^{I_2}m\big\|_{L^\infty}\\
\lesssim&(C_1\epsilon)^2\langle t\rangle^{-1+\delta}+C_1\epsilon\langle t\rangle^{-1/2+\delta/2}\sum_{|I_1|\leq N}\Big\|\frac{\Gamma^{I_1}n}{\langle t\rangle^{\delta/2}\langle t-|x|\rangle^{3/4}}\Big\|.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&E(t,\Gamma^I\phi^\gamma)^{1/2}\\
\lesssim&\epsilon+(C_1\epsilon)^2\langle t\rangle^\delta+\sum_{|I_1|\leq N}\Big(\int_0^t\Big\|\frac{\Gamma^{I_1}n}{\langle t'\rangle^{\delta/2}\langle t'-|x|\rangle^{3/4}}\Big\|^2dt'\Big)^{1/2}\Big(\int_0^t\langle t'\rangle^{-1+\delta}\,dt'\Big)^{1/2}\\
\lesssim&\epsilon+(C_1\epsilon)^2\langle t\rangle^\delta.\quad\square
\end{aligned}$$

Lemma 4.10. *We have*

$$\begin{aligned}
\big\|L_0\Gamma^I\phi\big\|_{L^2}&\lesssim\epsilon+(C_1\epsilon)^{3/2}t^\delta, &&|I|\leq N-6,\\
\big\|L_0\Gamma^I\phi\big\|_{L^2}&\lesssim\epsilon+(C_1\epsilon)^{3/2}t^{1/2+\delta}, &&|I|\leq N-2.
\end{aligned}\tag{4.19}$$

Proof. We only provide the proof for the case $|I|\leq N-6$, and the case of $|I|\leq N-2$ can be shown in the similar way.

We apply the conformal energy estimates (2.8) on the equation

$$-\Box\Gamma^I\phi=P_1^{\alpha\beta}\Gamma^IQ_{\alpha\beta}(m,n),$$

with $|I|\leq N-6$, to get

$$E_{con}(t,\Gamma^I\phi)^{1/2} \lesssim E_{con}(0,\Gamma^I)^{1/2} + \int_0^t \big\|\langle t'+|x|\rangle P_1^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big\|\, dt'.$$

We note that

$$\big\|\langle t'+|x|\rangle P_1^{\alpha\beta}\Gamma^I Q_{\alpha\beta}(m,n)\big\| \lesssim \sum_{|I|\le N-6} \big\|\Gamma\Gamma^I m\big\| \sum_{|I|\le N-6} \big\|\Gamma\Gamma^I n\big\|_{L^\infty} \lesssim (C_1\epsilon)^2\langle t'\rangle^{-1+\delta},$$

which further yields

$$E_{con}(t,\Gamma^I\phi)^{1/2} \lesssim \epsilon + (C_1\epsilon)^2\langle t\rangle^{\delta}.$$

Thus the proof is done after recalling the estimates in Lemma 4.9 as well as the triangle inequality. □

Lemma 4.11. *We have*

$$\big|\partial\Gamma^I\phi\big| \lesssim \big(\epsilon+(C_1\epsilon)^{3/2}\big)\langle t-|x|\rangle^{-3/4}\langle t\rangle^{-1/2}, \qquad |I|\le N-9. \tag{4.20}$$

Proof. It suffices to show the following two types of estimates

$$\big|\partial\Gamma^I\phi\big| \lesssim \big(\epsilon+(C_1\epsilon)^{3/2}\big)\langle t-|x|\rangle^{-1}\langle t\rangle^{-1/2+\delta}, \qquad |I|\le N-9, \tag{4.21}$$

which implies that

$$\big|\partial\Gamma^I\phi\big| \lesssim \big(\epsilon+(C_1\epsilon)^{3/2}\big)\langle t-|x|\rangle^{-3/4}\langle t\rangle^{-1/2}, \qquad \text{in } \{(t,x): |x|\ge 2t\}, \qquad |I|\le N-9,$$

and

$$\big|\Gamma^I\phi\big| \lesssim \big(\epsilon+(C_1\epsilon)^{3/2}\big)\langle t-|x|\rangle^{-1}\langle t\rangle^{-1/2}, \qquad \text{in } \{(t,x): |x|\le 2t\}, \qquad |I|\le N-9. \tag{4.22}$$

For the first estimate (4.21), the estimates (4.16), (4.19), and the commutator estimates imply

$$\sum_{|I_1|\le 3, |I_2|\le N-9} \big(\big\|\Gamma^{I_1}L_0\Gamma^{I_2}\phi\big\| + \big\|\Gamma^{I_1}\Gamma\Gamma^{I_2}\phi\big\|\big) \lesssim \big(\epsilon+(C_1\epsilon)^{3/2}\big)\langle t\rangle^{\delta}.$$

Then we apply the Klainerman-Sobolev inequality (2.5) to get

$$\big|L_0\Gamma^I\phi\big| + \big|\Gamma\Gamma^I\phi\big| \lesssim \big(\epsilon+(C_1\epsilon)^{3/2}\big)\langle t\rangle^{-1/2+\delta}, \qquad |I|\le N-9.$$

In junction with the fact

$$|\partial\Gamma^I\phi| \lesssim \langle t-|x|\rangle^{-1}\big(\big|L_0\Gamma^I\phi\big| + \big|\Gamma\Gamma^I\phi\big|\big),$$

we arrive at (4.21).

Next, we derive (4.22), and we only need to consider the case $|x| \leq 2t$. Recall the decomposition (4.11) (and the commutator estimates), and we observe that it suffices to show

$$\left|\partial\Gamma^I \phi^5\right| + \sum_\gamma \left|\partial\partial\Gamma^I \phi^\gamma\right| \lesssim \left(\epsilon + (C_1\epsilon)^{3/2}\right)\langle t-|x|\rangle^{-1}\langle t\rangle^{-1/2}, \qquad |I| \leq N-9.$$

Thanks to Lemma 3.3, we get

$$\left|L_0\Gamma^I\phi^5\right| + \left|\Gamma\Gamma^I\phi^5\right| \lesssim \epsilon\langle t\rangle^{-1/2},$$

and hence

$$\left|\partial\Gamma^I\phi^5\right| \lesssim \epsilon\langle t-|x|\rangle^{-1}\langle t\rangle^{-1/2}, \qquad |I| \leq N-9.$$

On the other hand, consider the definition of Φ^γ and the equation (4.15)

$$\begin{aligned}
\Phi^\gamma =& \phi^\gamma + P_1^{\alpha\gamma} n\partial_\alpha m - P_1^{\gamma\beta} n\partial_\beta m,\\
-\Box\Phi^\gamma =& P_1^{\alpha\gamma}(-\Box+1)n\partial_\alpha m + P_1^{\alpha\gamma} n(-\Box\partial_\alpha m) - P_1^{\alpha\gamma} Q_0(n,\partial_\alpha m)\\
&- P_1^{\gamma\beta}(-\Box+1)n\partial_\beta m - P_1^{\gamma\beta} n(-\Box\partial_\beta m) + P_1^{\gamma\beta} Q_0(n,\partial_\beta m).
\end{aligned}$$

We recall the estimates for null forms in Lemmae 2.2 that

$$|Q_0(n,\partial_\alpha m)| \lesssim \langle t+|x|\rangle^{-1}\left(\left|L_0\partial_\alpha m\Gamma n\right| + \left|\Gamma\partial_\alpha m\Gamma n\right|\right),$$

in which we take advantage of that the scaling vector field L_0 only acts on the wave component m. Then the usual energy estimates easily give

$$\sum_\gamma E(t,\Gamma^J\Phi^\gamma)^{1/2} \lesssim \epsilon + (C_1\epsilon)^{3/2}, \qquad |J| \leq N-4,$$

which further yields

$$\sum_\gamma \left\|\partial\Gamma^J\phi^\gamma\right\| \lesssim \epsilon + (C_1\epsilon)^{3/2}, \qquad |J| \leq N-4.$$

Again, we apply the Klainerman-Sobolev inequality (2.5) (and the commutator estimates) to obtain the sup-norm bounds

$$\sum_\gamma \left|\partial\Gamma^J\phi^\gamma\right| \lesssim \left(\epsilon + (C_1\epsilon)^{3/2}\right)\langle t\rangle^{-1/2}, \qquad |J| \leq N-7. \tag{4.23}$$

Finally, Lemma 3.4 implies

$$\sum_{\gamma} \left|\partial\partial\Gamma^I \phi^\gamma\right| \lesssim \left(\epsilon + (C_1\epsilon)^{3/2}\right)\langle t-|x|\rangle\langle t\rangle^{-1/2}, \qquad |I| \le N-9.$$

Till now, the proof is complete. □

As a consequence of (4.23) and Lemma 3.3, together with the relation (4.11)

$$\phi = \phi^5 + \partial_\gamma \phi^\gamma,$$

we obtain the sharp time decay result of ϕ, which reads

$$|\phi| \lesssim \langle t\rangle^{-1/2}, \tag{4.24}$$

which proves the second inequality in (1.4) in Theorem 1.1.

We are now ready to show Proposition 4.3.

Proof of Proposition 4.3. By carefully choosing $C_1 \gg 1$ large enough and $\epsilon \ll 1$ sufficiently small, we get from Lemmas 4.5–4.11 that

$$\|(\phi,\psi)\|_X \le \frac{1}{2} C_1 \epsilon, \tag{4.25}$$

and hence $(\phi,\psi) \in X$. □

4.3. Contraction mapping and the global existence result

We now want to show that the solution mapping T is also a contraction mapping.

Proposition 4.12. *T is a contraction mapping from X to itself, i.e.*

$$\|(\phi-\phi', \psi-\psi')\|_X \le \frac{1}{2}\|(m-m', n-n')\|_X, \tag{4.26}$$

in which $(m,n), (m',n') \in X$, and $(\phi,\psi) = T(m,n), (\phi',\psi') = T(m',n')$.

Proof. The proof for Proposition 4.3 can also be applied here (we might further shrink the size of the initial data ϵ if needed), so we omit it. □

Proof of Theorem 1.1. By the Banach fixed point theorem, we know the mapping T has a unique fixed point, which is the solution to the system (1.1) As for the pointwise decay estimates (1.4), they can be obtained from the Definition 4.1, the inequality (4.24), and Lemma 4.4. □

Acknowledgments

The author is deeply grateful to the anonymous referees for carefully reading the manuscript. Their valuable comments help greatly improve the presentation as well as the precision of the paper. This project was partially supported by the China Postdoctoral Science Foundation, with grant number 2021M690702.

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