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Global small data smooth solutions of 2-D null-form wave equations with non-compactly supported initial data ☆

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Abstract

For the 2-D nonlinear wave equations $\Box u = F(\partial u, \partial^2 u)$ with initial data $(u(0,x), \partial_t u(0,x)) = (\varepsilon u_0(x), \varepsilon u_1(x))$, where $x = (x_1, x_2)$, $x_0 = t$, $\partial = (\partial_0, \partial_1, \partial_2)$, $\varepsilon > 0$ is small enough, $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^2)$, and the smooth nonlinearity $F(\partial u, \partial^2 u) = O(|\partial u|^2 + |\partial^2 u|^2)$, when $F(\partial u, \partial^2 u)$ satisfies the null conditions, S. Alinhac in [2] shows that the smooth solution u exists globally. The proof relies on the compactness of the support of $(u_0(x), u_1(x))$. Recently, for a class of quasilinear wave equations $\Box u = N^{\alpha\beta\mu\nu}\partial^2_{\alpha\beta}u\partial^2_{\mu\nu}u$ or $\Box u = A^\alpha \partial_\alpha(|\partial_t u|^2 - |\nabla u|^2)$ with small and non-compactly supported initial data, where $N^{\alpha\beta\mu\nu}$ and A^α are constants $(0 \le \alpha, \beta, \mu, \nu \le 2)$, when the related null condition hold, the authors in [6] prove the global existence of solution u. In this paper, we will prove the global existence for the general 2-D null-form wave equations $\Box u = F(\partial u, \partial^2 u)$ with non-compactly supported initial data. The new key ingredient is to establish a class of weighted L^∞-L^∞ estimates of solution w to the 2-D linear wave equation $\Box w = f(t,x)$ instead of the usual L^∞-L^2 estimates used in [2], [6] and so on. From this, we also get a better time-decay rate for the "good derivatives" of solution u of nonlinear wave equations.

MSC: 35L05; 35L70

Keywords: Global existence; Null condition; Null-form; Non-compactly supported initial data; Poisson formula; Weighted L^∞-L^∞ estimate

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1. Introduction

In this paper, we are concerned with the second order nonlinear wave equation in $[0,\infty)\times\mathbb{R}^2$:

$$\begin{cases}\Box u = F(\partial u, \partial^2 u),\\ u(0,x)=u_0(x),\ \partial_t u(0,x)=u_1(x),\end{cases} \tag{1.1}$$

where $x=(x_1,x_2)$, $x_0=t$, $\partial=(\partial_0,\partial_1,\partial_2)$, $(u_0(x),u_1(x))\in C^\infty(\mathbb{R}^2)$, and the smooth real nonlinearity $F(\partial u,\partial^2 u)=O(|\partial u|^2+|\partial^2 u|^2)$. Without loss of generality, one can write $F(\partial u,\partial^2 u)$ as the following form:

$$F(\partial u,\partial^2 u)=A_1^{\alpha\beta}\partial_\alpha u\partial_\beta u + A_2^{\alpha\beta\gamma}\partial_\alpha u\partial_\beta u\partial_\gamma u + F^{\alpha\beta}(\partial u,\partial^2 u)\partial^2_{\alpha\beta}u + O(|\partial u|^4+|\partial^2 u|^4),$$

where $0\le\alpha,\beta,\gamma\le 2$, $A_1^{\alpha\beta}$ and $A_2^{\alpha\beta\gamma}$ are constants, here and throughout the whole paper, Einstein's summation convention is used. In addition, the smooth functions $F^{\alpha\beta}(\partial u,\partial^2 u)$ are

$$\begin{aligned}&F^{\alpha\beta}(\partial u,\partial^2 u)=F^{\beta\alpha}(\partial u,\partial^2 u)\\ &=A_3^{\alpha\beta\gamma}\partial_\gamma u + A_4^{\alpha\beta\mu\nu}\partial^2_{\mu\nu}u + A_5^{\alpha\beta\mu\nu}\partial_\mu u\partial_\nu u + A_6^{\alpha\beta\gamma\mu\nu}\partial_\gamma u\partial^2_{\mu\nu}u + A_7^{\alpha\beta\gamma\delta\mu\nu}\partial^2_{\gamma\delta}u\partial^2_{\mu\nu}u,\end{aligned}$$

where $0\le\alpha,\beta,\gamma,\mu,\nu,\delta\le 2$, and $A_l^{\alpha\beta\cdots}$ are constants for $l=3,4,\cdots,7$.

We call the nonlinearity $F(\partial u,\partial^2 u)$ fulfills the null-form, which means that $A_l(\widetilde{\omega})\equiv 0$ for all $l=1,2,\cdots,7$, where $\widetilde{\omega}=(\omega_0,\omega)=(\omega_0,\omega_1,\omega_2)=(-1,\frac{x_1}{|x|},\frac{x_2}{|x|})$ with $|x|=\sqrt{x_1^2+x_2^2}$, and

$$\begin{aligned}&A_1(\widetilde{\omega}):=A_1^{\alpha\beta}\omega_\alpha\omega_\beta,\ A_2(\widetilde{\omega}):=A_2^{\alpha\beta\gamma}\omega_\alpha\omega_\beta\omega_\gamma,\ A_3(\widetilde{\omega}):=A_3^{\alpha\beta\gamma}\omega_\alpha\omega_\beta\omega_\gamma,\\ &A_4(\widetilde{\omega}):=A_4^{\alpha\beta\mu\nu}\omega_\alpha\omega_\beta\omega_\mu\omega_\nu,\ A_5(\widetilde{\omega}):=A_5^{\alpha\beta\mu\nu}\omega_\alpha\omega_\beta\omega_\mu\omega_\nu,\ A_6(\widetilde{\omega}):=A_6^{\alpha\beta\gamma\mu\nu}\omega_\alpha\omega_\beta\omega_\gamma\omega_\mu\omega_\nu,\\ &A_7(\widetilde{\omega}):=A_7^{\alpha\beta\gamma\delta\mu\nu}\omega_\alpha\omega_\beta\omega_\gamma\omega_\delta\omega_\mu\omega_\nu.\end{aligned} \tag{1.2}$$

The main result in this paper is:

Theorem 1.1. *Suppose that the nonlinearity $F(\partial u,\partial^2 u)$ in (1.1) admits the null-form. There exists a small constant $\varepsilon_0>0$ such that for $\varepsilon\le\varepsilon_0$, and for the initial data $(u_0(x),u_1(x))$ satisfying*

$$\begin{aligned}&\sum_{0\le k\le N}\|\langle|x|\rangle^k\nabla^k u_0(x)\|_{L^2}+\sum_{0\le k\le N-1}\|\langle|x|\rangle^{k+1}\nabla^k u_1(x)\|_{L^2}\\ &+\sum_{0\le k\le N-4}\Big[\|\langle|x|\rangle^{k+1}\nabla^k u_0(x)\|_{W^{2,1}}+\|\langle|x|\rangle^{k+1}\nabla^k u_1(x)\|_{W^{1,1}}\Big]\le\varepsilon,\end{aligned} \tag{1.3}$$

where $\langle|x|\rangle=\sqrt{1+|x|^2}$, $\nabla=(\partial_1,\partial_2)$, $N\ge 14$ is a fixed constant, and $\|\cdot\|_{W^{k,p}}$ stands for the standard Sobolev norms. Then problem (1.1) admits a global smooth solution u.

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Remark 1.1. Consider the 3-D nonlinear wave equation

$$\begin{cases} \Box u = F(\partial u, \partial^2 u), \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.4}$$

where $F(\partial u, \partial^2 u) = O(|\partial u|^2 + |\partial^2 u|^2)$, and

$$F(\partial u, \partial^2 u) = A_1^{\alpha\beta} \partial_\alpha u \partial_\beta u + A_2^{\alpha\beta\gamma} \partial_\alpha u \partial^2_{\beta\gamma} u + A_3^{\alpha\beta\mu\nu} \partial^2_{\alpha\beta} u \partial^2_{\mu\nu} u + O(|\partial u|^3 + |\partial^2 u|^3).$$

When $F(\partial u, \partial^2 u)$ fulfills the null-form, that is, $A_1^{\alpha\beta}\omega_\alpha\omega_\beta = A_2^{\alpha\beta\gamma}\omega_\alpha\omega_\beta\omega_\gamma = A_3^{\alpha\beta\mu\nu}\omega_\alpha\omega_\beta\omega_\mu\omega_\nu \equiv 0$ hold for $(\omega_0, \omega_1, \omega_2, \omega_3) = (-1, \frac{x_1}{|x|}, \frac{x_2}{|x|}, \frac{x_3}{|x|})$ with $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, then under condition (1.3) with $N \geq 16$, analogously we can prove that problem (1.4) has a global smooth small data solution u.

Remark 1.2. For the n-dimensional ($n \geq 4$) nonlinear wave equation

$$\begin{cases} \Box u = F(\partial u, \partial^2 u), \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.5}$$

where $x = (x_1, x_2, ..., x_n)$, and $F(\partial u, \partial^2 u) = O(|\partial u|^2 + |\partial^2 u|^2)$, when $\|u_0\|_{H^{2[\frac{n}{2}]+4}} + \|u_1\|_{H^{2[\frac{n}{2}]+3}}$ is small enough, it follows from Theorem 6.5.2 of [12] that problem (1.5) has a global smooth small data solution u.

Remark 1.3. The typical nonlinear null-form wave equation in (1.1) comes from the irrotational potential equation of 2-D Chaplygin gases. Indeed, the 2D isentropic Euler equations of Chaplygin gases is

$$\begin{cases} \partial_t \rho + div(\rho u) = 0 & \text{(Conservation of mass)}, \\ \partial_t(\rho u) + div(\rho u \otimes u) + \nabla P(\rho) = 0 & \text{(Conservation of momentum)}, \\ P(\rho) = P_0 - \dfrac{B}{\rho} & \text{(State equation)}, \\ \rho(0,x) = \bar{\rho} + \varepsilon\rho_0(x), u(0,x) = \varepsilon u_0(x), \end{cases} \tag{1.6}$$

where $t \geq 0$, $x = (x_1, x_2) \in \mathbb{R}^2$, P_0, B and $\bar{\rho}$ are positive constants, $\varepsilon > 0$ is small enough, $(\rho_0(x), u_0(x)) \in C_0^\infty(\mathbb{R}^2)$, and $u = (u_1, u_2)$, ρ, P stand for the velocity, density, pressure of gases respectively. For simplicity, we assume the sound speed $c(\bar{\rho}) = \sqrt{P'(\bar{\rho})} = 1$. When $rot u_0(x) \equiv 0$, one can introduce the potential function $\varphi(t,x)$ such that $u = \nabla\varphi$. In this case, it follows from the Bernoulli's law that

$$\partial_t \varphi + \frac{1}{2}|\nabla\varphi|^2 + h(\rho) = 0,$$

where $h(\rho) = \frac{1}{2} - \frac{B}{2\rho^2}$ is the enthalpy of the gases and $\rho = h^{-1}(-\partial_t\varphi - \frac{1}{2}|\nabla\varphi|^2)$. Substituting this into the conservation of mass yields

$$\Box\varphi = Q(\partial\varphi, \partial^2\varphi), \tag{1.7}$$

where

$$Q(\partial\varphi, \partial^2\varphi) = Q^{\alpha\beta}(\partial\varphi)\partial^2_{\alpha\beta}\varphi = -2\sum_{i=1}^{2}\partial_i\varphi\partial_t\partial_i\varphi - \sum_{i,j=1}^{2}\partial_i\varphi\partial_j\varphi\partial^2_{ij}\varphi + (2\partial_t\varphi + |\nabla\varphi|^2)\Delta\varphi.$$

It is easy to know that $Q(\partial\varphi, \partial^2\varphi)$ in (1.7) admits a null-form.

We now recall some fundamental results closely related to our works. Investigate the second order quasilinear wave equation in $[0, \infty) \times \mathbb{R}^n$ $(n \geq 3)$

$$\begin{cases} g^{\alpha\beta}(u, \partial u)\partial^2_{\alpha\beta}u = 0, \\ (u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x)), \end{cases} \tag{1.8}$$

where $x_0 = t$, $x = (x_1, ..., x_n)$, $\partial = (\partial_0, \partial_1, ..., \partial_n)$, $\varepsilon > 0$ is a sufficiently small constant, $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^n)$, $g^{\alpha\beta}(u, \partial u) = g^{\beta\alpha}(u, \partial u)$ $(0 \leq \alpha, \beta \leq n)$ are smooth functions in their arguments. Let

$$g^{\alpha\beta}(u, \partial u) = c^{\alpha\beta} + d^{\alpha\beta}u + e^{\alpha\beta}_\gamma\partial_\gamma u + O(|u|^2 + |\partial u|^2),$$

where $c^{\alpha\beta}, d^{\alpha\beta}$ and $e^{\alpha\beta}_\gamma$ $(0 \leq \alpha, \beta, \gamma \leq n)$ are constants, $c^{\alpha\beta}\partial^2_{\alpha\beta} = \Box$. By the results in [16] and [19], one knows that (1.8) has a global smooth small data solution for $n \geq 4$. If $n = 3$ and $d^{\alpha\beta} = 0$ for all $0 \leq \alpha, \beta \leq 3$, then (1.8) has a global smooth solution when the null condition holds (namely, $e^{\alpha\beta}_\gamma\omega_\alpha\omega_\beta\omega_\gamma \equiv 0$ holds for $\omega_0 = -1$ and $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2$), otherwise, the smooth solution of (1.8) blows up in finite time (see [1], [7] and [11–15]). If $n = 3$ and $d^{\alpha\beta} \neq 0$ for some (α, β), but $e^{\alpha\beta}_\gamma = 0$ for all $0 \leq \alpha, \beta, \gamma \leq 3$, then it follows from the results in [4], [20] and [21] that (1.8) has a global smooth solution u. If $n = 3$, $d^{\alpha\beta} \neq 0$ for some (α, β) and $e^{\alpha\beta}_\gamma \neq 0$ for some (α, β, γ), when $e^{\alpha\beta}_\gamma\omega_\alpha\omega_\beta\omega_\gamma \not\equiv 0$, the authors in [9] have established the blowup result of solution u in finite time as long as $(u_0(x), u_1(x)) \not\equiv 0$. In addition, if $n = 3$, $d^{\alpha\beta} \neq 0$ for some (α, β), and $e^{\alpha\beta}_\gamma \neq 0$ for some (α, β, γ), when $e^{\alpha\beta}_\gamma\omega_\alpha\omega_\beta\omega_\gamma \equiv 0$, the authors in [8] prove that the smooth solution u exists globally.

Investigate the second order quasilinear wave equation in $[0, \infty) \times \mathbb{R}^2$

$$\begin{cases} g^{\alpha\beta}(\partial u)\partial^2_{\alpha\beta}u = 0, \\ (u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x)), \end{cases} \tag{1.9}$$

where $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^2)$, $g^{\alpha\beta}(\partial u) = g^{\beta\alpha}(\partial u)$ $(0 \leq \alpha, \beta \leq 2)$ can be written as

$$g^{\alpha\beta}(\partial u) = c^{\alpha\beta} + d^{\alpha\beta}_\gamma\partial_\gamma u + e^{\alpha\beta}_{\gamma\delta}\partial_\gamma u\partial_\delta u + O(|\partial u|^3),$$

here $c^{\alpha\beta}, d^{\alpha\beta}_\gamma$ and $e^{\alpha\beta}_{\gamma\delta}$ $(0 \leq \alpha, \beta, \gamma, \delta \leq 2)$ are constants, $c^{\alpha\beta}\partial^2_{\alpha\beta} = \Box$. By the results in [2] and [3], one knows that (1.9) has a global smooth solution as long as $d^{\alpha\beta}_\gamma\omega_\alpha\omega_\beta\omega_\gamma \equiv 0$ and

$e_{\gamma\delta}^{\alpha\beta}\omega_\alpha\omega_\beta\omega_\gamma\omega_\delta \equiv 0$ for $\omega_0 = -1$ and $\omega = (\omega_1, \omega_2) \in \mathbb{S}^1$, otherwise, the solution u blows up in finite time.

When the compact support property of $(u_0(x), u_1(x))$ in (1.9) is removed, the authors in [6] consider the following two classes of 2-D nonlinear wave equations

$$\begin{cases} \Box v = N^{\alpha\beta\mu\nu}\partial_{\alpha\beta}^2 v\partial_{\mu\nu}^2 v, \\ v(0,x) = v_0(x), \partial_t v(0,x) = v_1(x) \end{cases} \tag{1.10}$$

and

$$\begin{cases} \Box v = A^\alpha \partial_\alpha(|\partial_t v|^2 - |\nabla v|^2), \\ v(0,x) = v_0(x), \partial_t v(0,x) = v_1(x), \end{cases} \tag{1.11}$$

where the corresponding null conditions $N^{\alpha\beta\mu\nu}\omega_\alpha\omega_\beta\omega_\mu\omega_\nu \equiv 0$ for $\omega_0 = -1$ and $\omega = (\omega_1, \omega_2) \in \mathbb{S}^1$ hold. The authors in [6] make full use of the nonlinear structure $N^{\alpha\beta\mu\nu}\partial_{\alpha\beta}^2 v\partial_{\mu\nu}^2 v$ to prove that the smooth small data solution v of problem (1.10) exists globally for the non-compactly supported $(v_0(x), v_1(x))$. Meanwhile, by Section 5 of [6] one knows that problem (1.11) can be transformed into the fully nonlinear wave equation problem (1.10) due to its special structure. Obviously, such a quasilinear wave equation $\Box v = \partial_1(\partial_1 v\partial_2 v) - \partial_2(|\partial_1 v|^2) = \partial_2 v\partial_{11}^2 v - \partial_1 v\partial_{12}^2 v$ does not belong to the class of (1.11), however its nonlinearity actually fulfills the null condition. On the other hand, we point out that problem (1.7) does not satisfy the null-forms in (1.10) or (1.11). In the present paper, we will prove the global existence of small data smooth solution u for all the 2-D null-form wave equations $\Box u = F(\partial u, \partial^2 u)$ with non-compactly supported initial data.

Next we give some comments on the proof of Theorem 1.1. As interpreted in [2] or [6], in order to prove the global existence of small data solution u of (1.1), such a crucial estimate $\|\partial(\partial_t + \partial_r)Z^\alpha u\|_{L^2} \le C_\alpha(1+t)^{-1}$ is required, where $Z \in \{\partial, S = t\partial_t + \sum_{i=1}^{2} x_i\partial_i, \Omega = x_1\partial_2 - x_2\partial_1, H_i = x_i\partial_t + t\partial_i, i = 1, 2\}$. In this process, the compact support of $(u_0(x), u_1(x))$ should be posed for utilizing the basic Hardy-type inequality. To remove the restriction of compact support for $(v_0(x), v_1(x))$ in (1.10), the authors in [6] use the form of fully nonlinear $N^{\alpha\beta\mu\nu}\partial_{\alpha\beta}^2 v\partial_{\mu\nu}^2 v$ to let v admit one more derivative than the second order quasilinear nonlinearity $N^{\alpha\beta\mu}\partial_{\alpha\beta}^2 v\partial_\mu v$. Starting from this key point, an extra $(1+t)^{-1}$ decay can be derived for the lower-order energy estimate of v in [6] and further establish the global existence of v. However, the method in [6] can not be applied to treat the general null-form nonlinearity $F(\partial u, \partial^2 u)$ in (1.1) due to the appearances of such terms $\partial_\alpha u\partial_{\beta\gamma}^2 u$, $\partial_\alpha u\partial_\beta u\partial_\gamma u$ and so on. We have to use new ideas for proving Theorem 1.1. At first, motivated by the technique in [10] for treating the second order semilinear wave equation, we establish a kind of weighted L^∞-L^∞ estimate for the solution w of 2-D linear wave equation $\Box w = F(t,x)$ with $(w(0,x), \partial_t w(0,x)) = (w_0(x), w_1(x))$. Then based on this, instead of the L^∞-L^2 estimates used in [2] and [6], we can derive better time-decay rates for the "good derivatives" $T_i u$ (see (2.11) and (3.21) below), where $T_i = \partial_i + \omega_i\partial_t$ with $\omega_i = \frac{x_i}{|x|}$ for $i = 1, 2$. By this, we can complete the proof of Theorem 1.1.

The paper is organized as follows: In Section 2, some estimates with null-form structures are given. In addition, several basic inequalities are listed or derived. In Section 3, the commutations between the vector fields and 2-D linear wave operator are shown. Moreover,

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the pointwise estimates of the "good derivatives" for the linear equation $\Box w = F(t, x)$ with $(w(0, x), \partial_t w(0, x)) = (w_0(x), w_1(x))$ are established by the Poisson formula. As in [2], by suitable choice of ghost weight and by integration by parts, the related energy estimates for the smooth solution u of problem (1.1) are established in Section 4. In Section 5, Theorem 1.1 is proved by the continuous induction method together with the local existence of solution u to problem (1.1).

Through the whole paper, we will use the following notation:

$$Z \in \{\partial, S = t\partial_t + \sum_{i=1}^{2} x_i\partial_i, \Omega = x_1\partial_2 - x_2\partial_1, H_i = x_i\partial_t + t\partial_i, i = 1, 2\}.$$

The "good derivatives" are

$$T_i = \partial_i + \omega_i\partial_t \text{ with } \omega_i = \frac{x_i}{|x|} \text{ and } i = 1, 2.$$

Set $\widetilde{\omega} = (\omega_0, \omega) = (\omega_0, \omega_1, \omega_2) = (-1, \frac{x_1}{|x|}, \frac{x_2}{|x|})$.

Define the energies

$$\tilde{E}_1[u](t) = \|\partial u\|, \quad \tilde{E}_k[u](t) = \sum_{|a|\le k-1} \tilde{E}_1[Z^a u](t),$$

where $k \in \mathbb{N}$, $\|\cdot\| = \|\cdot\|_{L^2_x}$, and $a \in \mathbb{N}_0^7$ denotes the multi-indices. In addition, set

$$E_k[u](t) = \sup_{0\le\tau\le t} \tilde{E}_k[u](\tau).$$

For the positive quantities A and B, $A \lesssim B$ means $A \le CB$ with generic positive constant C which is independent of t and ε.

2. Preliminaries

In this section, we will establish some estimates for the null-form structures and list or derive some basic inequalities.

Lemma 2.1. *The null condition*

$$A_1(\widetilde{\omega}) = A_1^{\alpha\beta}\omega_\alpha\omega_\beta \equiv 0 \tag{2.1}$$

holds if and only if

$$A_1^{\alpha\beta}\partial_\alpha u\partial_\beta u = A_1^{00}(|\partial_t u|^2 - |\nabla u|^2),$$

where $A_1^{\alpha\beta}$ $(0 \le \alpha, \beta \le 2)$ are constants.

Proof. Substituting $\omega_0 = -1$ into (2.1) yields that for $(\omega_1, \omega_2) \in \mathbb{S}^1$

$$A_1^{00} - 2(A_1^{01}\omega_1 + A_1^{02}\omega_2) + A_1^{11}\omega_1^2 + A_1^{22}\omega_2^2 + 2A_1^{12}\omega_1\omega_2 = 0. \tag{2.2}$$

Replacing (ω_1, ω_2) by $(-\omega_1, -\omega_2)$ in (2.2), we have

$$A_1^{01}\omega_1 + A_1^{02}\omega_2 = 0,$$
$$A_1^{00} + A_1^{11}\omega_1^2 + A_1^{22}\omega_2^2 + 2A_1^{12}\omega_1\omega_2 = 0.$$

Together with the choices of $(\omega_1, \omega_2) = (1, 0)$ and $(0, 1)$ respectively, this implies

$$A_1^{01} = A_1^{02} = 0, \quad A_1^{11} = A_1^{22} = -A_1^{00}. \tag{2.3}$$

Inserting (2.3) into (2.2) and taking $\omega_1 = \omega_2 = \sqrt{2}/2$, we arrive at $A_1^{12} = 0$. Thus, proof of Lemma 2.1 is completed. □

From Lemma 2.1, without loss of generality, we assume $A_1^{00} \neq 0$. In this case, let $v := e^{-A_1^{00}u} - 1$, we then obtain that from (1.1)

$$\begin{cases} \Box v = \dfrac{\tilde{A}_2^{\alpha\beta\gamma}}{(1+v)^2}\partial_\alpha v\partial_\beta v\partial_\gamma v + \tilde{\mathcal{F}}^{\alpha\beta}(\partial v, \partial^2 v)\partial^2_{\alpha\beta}v + (1+v)O\Big(\Big|\dfrac{\partial v}{1+v}\Big|^4 + \Big|\dfrac{\partial^2 v}{1+v}\Big|^4\Big), \\ v(0,x) = e^{-A_1^{00}u_0(x)} - 1, \partial_t v(0,x) = -A_1^{00}e^{-A_1^{00}u_0(x)}u_1(x), \end{cases} \tag{2.4}$$

where

$$\tilde{\mathcal{F}}^{\alpha\beta}(\partial v, \partial^2 v) = \tilde{\mathcal{F}}^{\beta\alpha}(\partial v, \partial^2 v) = \frac{1}{1+v}\Big[\tilde{A}_3^{\alpha\beta\gamma}\partial_\gamma v + \tilde{A}_4^{\alpha\beta\mu\nu}\partial^2_{\mu\nu}v\Big]$$
$$+ \frac{1}{(1+v)^2}\Big[\tilde{A}_5^{\alpha\beta\mu\nu}\partial_\mu v\partial_\nu v + \tilde{A}_6^{\alpha\beta\gamma\mu\nu}\partial_\gamma v\partial^2_{\mu\nu}v + \tilde{A}_7^{\alpha\beta\gamma\delta\mu\nu}\partial^2_{\gamma\delta}v\partial^2_{\mu\nu}v\Big].$$

The new constants $\tilde{A}_l^{\alpha\beta\gamma\cdots}$ with $l = 2, 3, \cdots, 7$ also fulfill (1.2). For notational convenience, we still denote $\tilde{A}_l^{\alpha\beta\gamma\cdots}$ by $A_l^{\alpha\beta\gamma\cdots}$. Here we point out that the nonlinear terms $(1+v)O\big(|\frac{\partial v}{1+v}|^4 + |\frac{\partial^2 v}{1+v}|^4\big)$ in the right hand side of (2.4) can be neglected since there is no any influence on the proof of global existence of small data solution v to (2.4). In addition, it is convenient to suppose that all the coefficients in (2.4) are symmetric

$$\begin{aligned} &A_2^{\alpha\beta\gamma} = A_2^{\beta\alpha\gamma} = A_2^{\gamma\beta\alpha}, \quad A_4^{\alpha\beta\mu\nu} = A_4^{\mu\nu\alpha\beta}, \quad A_5^{\alpha\beta\mu\nu} = A_5^{\alpha\beta\nu\mu}, \\ &A_6^{\alpha\beta\gamma\mu\nu} = A_6^{\mu\nu\gamma\alpha\beta}, \quad A_7^{\alpha\beta\gamma\delta\mu\nu} = A_7^{\gamma\delta\alpha\beta\mu\nu} = A_7^{\mu\nu\gamma\delta\alpha\beta}. \end{aligned} \tag{2.5}$$

Lemma 2.2. *If $A_l(\tilde{\omega}) \equiv 0$ for $1 \le l \le 7$ (see (1.2) for definitions), then for any smooth functions u, v, w and ψ, there exists a positive constant c which only depends on the constants $A_l^{\alpha\beta\gamma\cdots}$, such that the following inequalities hold*

$$c|A_l^{\alpha\beta\gamma}\partial^2_{\alpha\beta}u\partial_\gamma v| \le |T\partial u||\partial v| + |\partial^2 u||Tv|,$$
$$c|A_l^{\alpha\beta\gamma}\partial_\alpha u\partial_\beta v\partial_\gamma w| \le |Tu||\partial v||\partial w| + |\partial u||Tv||\partial w| + |\partial u||\partial v||Tw|,$$
$$c|A_l^{\alpha\beta\mu\nu}\partial^3_{\alpha\beta\mu}u\partial_\nu v| \le |T\partial^2 u||\partial v| + |\partial^3 u||Tv|,$$

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$$
\begin{aligned}
&c|A_l^{\alpha\beta\mu\nu}\partial^2_{\alpha\beta}u\partial^2_{\mu\nu}v| \le |T\partial u||\partial^2 v| + |\partial^2 u||T\partial v|,\\
&c|A_l^{\alpha\beta\mu\nu}\partial^2_{\alpha\beta}u\partial_\mu v\partial_\nu w| \le |T\partial u||\partial v||\partial w| + |\partial^2 u||Tv||\partial w| + |\partial^2 u||\partial v||Tw|,\\
&c|A_l^{\alpha\beta\mu\nu}\partial_\alpha u\partial_\beta v\partial_\mu w\partial_\nu \psi| \le |Tu||\partial v||\partial w||\partial\psi| + |\partial u||Tv||\partial w||\partial\psi|\\
&\qquad\qquad + |\partial u||\partial v||Tw||\partial\psi| + |\partial u||\partial v||\partial w||T\psi|,\\
&c|A_l^{\alpha\beta\gamma\mu\nu}\partial^2_{\alpha\beta}u\partial^2_{\gamma\mu}v\partial_\nu w| \le |T\partial u||\partial^2 v||\partial w| + |\partial^2 u||T\partial v||\partial w| + |\partial^2 u||\partial^2 v||Tw|,\\
&c|A_l^{\alpha\beta\gamma\mu\nu}\partial^3_{\alpha\beta\gamma}u\partial_\mu v\partial_\nu w| \le |T\partial^2 u||\partial v||\partial w| + |\partial^3 u||Tv||\partial w| + |\partial^3 u||\partial v||Tw|,\\
&c|A_l^{\alpha\beta\gamma\mu\nu}\partial^2_{\alpha\beta}u\partial_\gamma v\partial_\mu w\partial_\nu\psi| \le |T\partial u||\partial v||\partial w||\partial\psi| + |\partial^2 u||Tv||\partial w||\partial\psi|\\
&\qquad\qquad + |\partial^2 u||\partial v||Tw||\partial\psi| + |\partial^2 u||\partial v||\partial w||T\psi|,\\
&c|A_l^{\alpha\beta\gamma\delta\mu\nu}\partial^2_{\alpha\beta}u\partial^2_{\gamma\delta}v\partial^2_{\mu\nu}w| \le |T\partial u||\partial^2 v||\partial^2 w| + |\partial^2 u||T\partial v||\partial^2 w| + |\partial^2 u||\partial^2 v||T\partial w|,\\
&c|A_l^{\alpha\beta\gamma\delta\mu\nu}\partial^3_{\alpha\beta\mu}u\partial^2_{\gamma\delta}v\partial_\nu w| \le |T\partial^2 u||\partial^2 v||\partial w| + |\partial^3 u||T\partial v||\partial w| + |\partial^3 u||\partial^2 v||Tw|,\\
&c|A_l^{\alpha\beta\gamma\delta\mu\nu}\partial^2_{\alpha\beta}u\partial^2_{\gamma\delta}v\partial_\mu w\partial_\nu\psi| \le |T\partial u||\partial^2 v||\partial w||\partial\psi| + |\partial^2 u||T\partial v||\partial w||\partial\psi|\\
&\qquad\qquad + |\partial^2 u||\partial^2 v||Tw||\partial\psi| + |\partial^2 u||\partial^2 v||\partial w||T\psi|,
\end{aligned}
\tag{2.6}
$$

where $|Tg| = |T_1 g| + |T_2 g|$.

Proof. Let $T_0 = 0$, as in Section 9.1 of [5], one has

$$
\begin{aligned}
&\partial_\alpha = T_\alpha - \omega_\alpha\partial_t, \quad \partial^2_{\alpha\beta} = T_\alpha\partial_\beta - \omega_\alpha\partial_\beta\partial_t = T_\alpha\partial_\beta - \omega_\alpha T_\beta\partial_t + \omega_\alpha\omega_\beta\partial_t^2,\\
&\partial^3_{\alpha\beta\gamma} = T_\alpha\partial^2_{\beta\gamma} - \omega_\alpha T_\beta\partial_t\partial_\gamma + \omega_\alpha\omega_\beta\partial_t^2\partial_\gamma = T_\alpha\partial^2_{\beta\gamma} - \omega_\alpha T_\beta\partial_t\partial_\gamma + \omega_\alpha\omega_\beta T_\gamma\partial_t^2 - \omega_\alpha\omega_\beta\omega_\gamma\partial_t^3.
\end{aligned}
$$

This, together with the definition (1.2) of null conditions and direct algebraic computation, yields (2.6). □

In the whole paper, we will always make the assumptions

$$
\begin{aligned}
&E_N[v](t) \le M\varepsilon(1+t)^{M'\varepsilon}, \quad M'\varepsilon_0 \le \frac{1}{16},\\
&E_{N-4}[v](t) \le M\varepsilon, \qquad M\varepsilon \le 1,
\end{aligned}
\tag{2.7}
$$

where M and M' are positive constants which will be chosen later. In subsequent Section 5, we will show that M in (2.7) can be replaced by $\frac{1}{2}M$.

Next we cite such a Klainerman-Sobolev's embedding inequality.

Lemma 2.3 *(Proposition 6.5.1 in [12]). For the smooth function* $u(t,x)$*, then*

$$
\langle |x|+t\rangle^{\frac12}\langle |x|-t\rangle^{\frac12}|u(t,x)| \lesssim \sum_{|a|\le 2}\|Z^a u(t,x)\|. \tag{2.8}
$$

In addition, for smooth function $u(t,x)$, we have

Lemma 2.4. *The following inequalities hold for $i = 1, 2$,*

$$\langle |x| - t\rangle |\partial u(t,x)| \lesssim \sum_{|a|=1} |Z^a u(t,x)|, \tag{2.9}$$

$$\langle |x| + t\rangle |T_i u(t,x)| \lesssim \sum_{|a|=1} |Z^a u(t,x)|, \tag{2.10}$$

$$\langle t\rangle \|T_i \partial u(t,x)\| \lesssim \sum_{|a|=1} \|Z^a \partial u(t,x)\|, \tag{2.11}$$

$$\langle |x| + t\rangle |T_i u(t,x)| \lesssim E_3[u](t), \tag{2.12}$$

$$|Zu(t,x)| \lesssim \sum_{j=1,2} \langle |x| + t\rangle |T_j u(t,x)| + \langle |x| - t\rangle |\partial u(t,x)|, \tag{2.13}$$

$$|u(t,x)| \lesssim \sup_y |u(0,y)| + E_3[u](t). \tag{2.14}$$

Proof. (2.9) comes from a direct computation, here we omit it. Next, we prove (2.10). For $i = 1, 2$,

$$\begin{aligned} tT_i u(t,x) &= (t - |x|)\partial_i u + \sum_{j=1}^{2} \omega_j x_j \partial_i u + \omega_i Su - \sum_{j=1}^{2} \omega_i x_j \partial_j u \\ &= (t - |x|)\partial_i u + \omega_i Su + \sum_{j=1}^{2} \omega_j (x_j \partial_i - x_i \partial_j) u. \end{aligned}$$

This together with (2.9) implies (2.10). In addition, taking the L^2-norms of (2.10) directly yields (2.11).

Now, we focus on the proof of (2.12). Divide $\langle |x| + t\rangle^2 |T_i u(t,x)|^2$ into such two parts:

$$\begin{aligned} &\langle |x| + t\rangle^2 |T_i u(t,x)|^2 \\ &= \langle |x| + t\rangle^2 |T_i u(t,x)|^2 \Big[1 - \chi\Big(\frac{|x|}{\langle t\rangle}\Big)\Big] + \langle |x| + t\rangle^2 |T_i u(t,x)|^2 \chi\Big(\frac{|x|}{\langle t\rangle}\Big) \\ &= J_1 + J_2, \end{aligned} \tag{2.15}$$

where χ is the smooth cutoff function satisfying

$$0 \le \chi(s) \le 1, \qquad \chi(s) = \begin{cases} 1, & \frac{1}{2} \le s \le \frac{5}{4}, \\ 0, & s \le \frac{1}{4} \ \text{ or } \ s \ge \frac{3}{2}. \end{cases} \tag{2.16}$$

For term J_1, it is easy to get

$$J_1 \lesssim \langle |x| + t\rangle \langle |x| - t\rangle |T_i u(t,x)|^2 \Big[1 - \chi\Big(\frac{|x|}{\langle t\rangle}\Big)\Big] \lesssim E_3^2[u](t). \tag{2.17}$$

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For term J_2, from the Sobolev embedding theorem on the unit sphere, we achieve

$$J_2 \lesssim \sum_{k=0}^{1} \int_{\mathbb{S}^1} \langle t\rangle^2 |\Omega^k T_i u(t,x)|^2 \chi\Big(\frac{|x|}{\langle t\rangle}\Big) d\omega. \tag{2.18}$$

Therefore,

$$\begin{aligned} J_2 &\lesssim \sum_{k=0}^{1} \left| \int_{|x|}^{\infty} \int_{\mathbb{S}^1} \langle t\rangle^2 \frac{d}{dr}\Big[|\Omega^k T_i u(t,r\omega)|^2 \chi\Big(\frac{r}{\langle t\rangle}\Big)\Big] d\omega dr \right| \\ &\lesssim \sum_{k=0}^{1} \int_{|x|}^{\infty} \int_{\mathbb{S}^1} |\Omega^k T_i u(t,r\omega)|^2 r dr d\omega + \langle t\rangle^2 \sum_{k=0}^{1} \int_{|x|}^{\infty} \int_{\mathbb{S}^1} |\chi \partial_r \Omega^k T_i u(t,r\omega)|^2 r dr d\omega. \end{aligned} \tag{2.19}$$

Substituting $\Omega T_1 = T_1\Omega - T_2$ and $\Omega T_2 = T_2\Omega + T_1$ into (2.19) and collecting (2.15)–(2.18), (2.12) is then proved.

Inequality (2.13) comes from the following direct computations:

$$\begin{aligned} &(t\partial_t + x_1\partial_1 + x_2\partial_2)u = (x_1T_1 + x_2T_2)u + (t-|x|)\partial_t u, \\ &(x_1\partial_2 - x_2\partial_1)u = (x_1T_2 - x_2T_1)u, \end{aligned}$$

and for $i = 1, 2$

$$(t\partial_i + x_i\partial_t)u = tT_i u + \omega_i(|x| - t)\partial_t u.$$

Finally, we prove (2.14). For any $(t,x) \in \mathbb{R}^{1+2}$ with $|x| \neq 0$, set $\omega = \frac{x}{|x|}$, we then arrive at

$$\begin{aligned} |u(t,x) - u(0,(|x|+t)\omega)| &= \left| \int_0^t \frac{d}{d\tau} u(\tau,(|x|+t-\tau)\omega) d\tau \right| \\ &\lesssim \int_0^t \langle |x|+t\rangle^{-\frac12} \langle |x|+t-2\tau\rangle^{-\frac12} d\tau\, E_3[u](t) \lesssim E_3[u](t). \end{aligned} \tag{2.20}$$

For $|x| = 0$, we can choose $\omega = (1, 0)$ and the same inequality as (2.20) holds.

Consequently, the proof of Lemma 2.4 is finished. □

Remark 2.1. Inequalities (2.13) and (2.14) will be used to control the factor $\frac{1}{1+v}$ and $\frac{1}{(1+v)^2}$ in (2.4). For examples, one can see (4.16) (for application of (2.13)), and (3.25) or (4.15) (for application of (2.14)).

3. Weighted L^∞-L^∞ estimates by the Poisson formula

At first, the commutations between the vector fields and $\Box$ are given as follows:

Lemma 3.1. *Let v be the solution of (2.4). Then for any multi-index $a \in \mathbb{N}_0^7$, we have the following equation for $Z^a v$:*

$$\begin{aligned}\Box Z^a v = &\sum_{b+c+d+e\le a} A^{\alpha\beta\gamma}_{2,abcde}\partial_\alpha Z^b v\partial_\beta Z^c v\partial_\gamma Z^d v Z^e\Big(\frac{1}{(1+v)^2}\Big)\\ &+\sum_{b+c+d\le a}\partial^2_{\alpha\beta}Z^b v Z^c\Big(\frac{1}{1+v}\Big)\Big[A^{\alpha\beta\gamma}_{3,abcd}\partial_\gamma Z^d v + A^{\alpha\beta\mu\nu}_{4,abcd}\partial^2_{\mu\nu}Z^d v\Big]\\ &+\sum_{b+c+d+e\le a}\partial^2_{\alpha\beta}Z^b v Z^c\Big(\frac{1}{(1+v)^2}\Big)\Big[A^{\alpha\beta\mu\nu}_{5,abcde}\partial_\mu Z^d v\partial_\nu Z^e v\\ &\qquad + A^{\alpha\beta\gamma\mu\nu}_{6,abcde}\partial_\gamma Z^d v\partial^2_{\mu\nu}Z^e v + A^{\alpha\beta\gamma\delta\mu\nu}_{7,abcde}\partial^2_{\gamma\delta}Z^d v\partial^2_{\mu\nu}Z^e v\Big],\end{aligned} \tag{3.1}$$

where $A^{\alpha\beta\gamma\cdots}_{l,abcd\cdots}$ are real constants satisfying $A_{l,abcd\cdots}(\widetilde{\omega}) \equiv 0$ with $l = 2, 3, \cdots, 7$. In addition, similar to (2.5), the coefficients satisfy

$$\begin{aligned}&A^{\alpha\beta\gamma}_{2,abcde} = A^{\beta\alpha\gamma}_{2,abcde} = A^{\gamma\beta\alpha}_{2,abcde} = A^{\alpha\beta\gamma}_{2,acbde} = A^{\alpha\beta\gamma}_{2,adcbe}, \quad A^{\alpha\beta\mu\nu}_{4,abcd} = A^{\mu\nu\alpha\beta}_{4,abcd} = A^{\alpha\beta\mu\nu}_{4,adcb},\\ &A^{\alpha\beta\mu\nu}_{5,abcde} = A^{\alpha\beta\nu\mu}_{5,abcde} = A^{\alpha\beta\mu\nu}_{5,abced}, \quad A^{\alpha\beta\gamma\mu\nu}_{6,abcde} = A^{\mu\nu\gamma\alpha\beta}_{6,abcde} = A^{\alpha\beta\gamma\mu\nu}_{6,aecdb},\\ &A^{\alpha\beta\gamma\delta\mu\nu}_{7,abcde} = A^{\gamma\delta\alpha\beta\mu\nu}_{7,abcde} = A^{\mu\nu\gamma\delta\alpha\beta}_{7,abcde} = A^{\alpha\beta\gamma\delta\mu\nu}_{7,adcbe} = A^{\alpha\beta\gamma\delta\mu\nu}_{7,aecdb}.\end{aligned}$$

Proof. Applying Lemma 6.6.5 in [5] to equation (2.4), we then complete the proof of Lemma 3.1. □

We now consider the following linear problem

$$\begin{cases}\Box u = F(t,x),\\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x).\end{cases} \tag{3.2}$$

By the Poisson formula, we have

$$u(t,x) = u_{hom}(t,x) + u_{inh}(t,x), \tag{3.3}$$

where

$$\begin{aligned}u_{hom}(t,x) &= u^0_{hom}(t,x) + u^1_{hom}(t,x)\\ &= \frac{1}{2\pi}\frac{\partial}{\partial t}\int_{|x-y|\le t}\frac{u_0(y)dy}{\sqrt{t^2-|x-y|^2}} + \frac{1}{2\pi}\int_{|x-y|\le t}\frac{u_1(y)dy}{\sqrt{t^2-|x-y|^2}}\end{aligned} \tag{3.4}$$

and

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$$u_{inh}(t,x) = \frac{1}{2\pi}\int_0^t \int_{|x-y|\le t-t'} \frac{F(t',y)}{\sqrt{(t-t')^2-|x-y|^2}} dydt'. \tag{3.5}$$

Lemma 3.2. *Let $u_{hom}(t,x)$ be defined in (3.4), then we have*

$$\langle|x|+t\rangle^{\frac{1}{2}}\langle|x|-t\rangle^{\frac{1}{2}}|u_{hom}(t,x)| \lesssim \|\tilde{u}_0(y)\|_{W^{2,1}} + \|\tilde{u}_1(y)\|_{W^{1,1}}, \tag{3.6}$$

where $\tilde{u}_l(y) = \langle|y|\rangle u_l(y)$ with $l = 0, 1$.

Proof. At first, we treat the case of $t \ge 2$. In this case, it is then to obtain

$$2(t+|x-y|)(1+|y|) \ge (1+t+|x-y|)(1+|y|) \ge 1+t+|x-y|+|y| \ge 1+t+|x|. \tag{3.7}$$

On the other hand, similar treatment also works for the situation of $t-|x-y| \ge 1$. Indeed, when $|x| \le t$, we see that

$$2(t-|x-y|)(1+|y|) \ge (1+t-|x-y|)(1+|y|) \ge 1+t-|x-y|+|y| \ge 1+t-|x|; \tag{3.8}$$

while, for $|x| \ge t$, we arrive at

$$1+|x|-t \le 1+|x-y|+|y|-t \le 1+|y| \le (1+|y|)(t-|x-y|), \tag{3.9}$$

here we have used the fact of $|x-y| \le t$.

Consequently, applying (3.7)–(3.9) to u^1_{hom}, we achieve

$$\langle|x|+t\rangle^{\frac{1}{2}}\langle|x|-t\rangle^{\frac{1}{2}}|u^1_{hom}(t,x)| \lesssim \|\langle y\rangle u_1(y)\|_{L^1} + (1+|t^2-|x|^2|)^{\frac{1}{2}}\mathcal{J}, \tag{3.10}$$

where

$$\mathcal{J} = \int_{t-1\le|x-y|\le t} \frac{|u_1(y)|dy}{\sqrt{t^2-|x-y|^2}}. \tag{3.11}$$

We next deal with the integration $\mathcal{J}$. By $t-1 \le |x-y| \le t$, one has

$$t-|x| \le t-|x-y|+|y| \le 1+|y| \tag{3.12}$$

or

$$|x|-t \le |x-y|+|y|-t \le |y|. \tag{3.13}$$

Collecting (3.7) and (3.11)–(3.13), we arrive at

$$(1+|t^2-|x|^2|)^{\frac{1}{2}}\mathcal{J} \lesssim \int_{t-1\le|x-y|\le t} \frac{|\tilde{u}_1(y)|dy}{\sqrt{t-|x-y|}}. \tag{3.14}$$

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On the other hand,

$$\begin{aligned}
&\int\limits_{t-1\le|x-y|\le t}\frac{|\tilde{u}_1(y)|dy}{\sqrt{t-|x-y|}}=\int\limits_{t-1}^{t}\int\limits_{|\omega|=1}\frac{|\tilde{u}_1(x+\rho\omega)|}{\sqrt{t-\rho}}\rho d\omega d\rho\\
&=\int\limits_{t-1}^{t}\int\limits_{|\omega|=1}|\tilde{u}_1(x+\rho\omega)|\rho d\omega d(-2\sqrt{t-\rho})\\
&\lesssim\int\limits_{|\omega|=1}|\tilde{u}_1(x+(t-1)\omega)|(t-1)d\omega+\left|\int\limits_{t-1}^{t}\int\limits_{|\omega|=1}\frac{d}{d\rho}\Big(\tilde{u}_1(x+\rho\omega)\rho\Big)d\omega d\rho\right|\\
&\lesssim\int\limits_{t-1}^{\infty}\int\limits_{|\omega|=1}\Big(|\nabla\tilde{u}_1(x+\rho\omega)|\rho+|\tilde{u}_1(x+\rho\omega)|\Big)d\omega d\rho\lesssim\|\langle|y|\rangle u_1(y)\|_{W^{1,1}}.
\end{aligned}\tag{3.15}$$

Substituting (3.14)–(3.15) into (3.10) yields (3.6) for u^1_{hom} when $t\ge 2$.

Next we focus on the estimate of u^0_{hom} when $t\ge 2$. From the proof of Lemma 1 in [10], we have

$$2\pi u^0_{hom}(t,x)=\frac{1}{t}\int\limits_{|x-y|\le t}\frac{u_0(y)dy}{\sqrt{t^2-|x-y|^2}}+\frac{1}{t}\int\limits_{|x-y|\le t}\frac{|x-y|}{\sqrt{t^2-|x-y|^2}}\Big(\frac{y-x}{t}\Big)\cdot\nabla u_0(y)dy.$$

By the same argument as for u^1_{hom} with $t\ge 2$, one has

$$\langle|x|+t\rangle^{\frac12}\langle|x|-t\rangle^{\frac12}|u^0_{hom}(t,x)|\lesssim\|\langle y\rangle u_0(y)\|_{W^{1,1}}+\|\langle y\rangle\nabla u_0(y)\|_{W^{1,1}}\lesssim\|\langle y\rangle u_0(y)\|_{W^{2,1}}.\tag{3.16}$$

Finally, we turn our attention to the case of $t\le 2$. As in [14], we deduce that

$$\begin{aligned}
&\langle|x|\rangle|u^1_{hom}(t,x)|\lesssim\int\limits_{|x-y|\le t}\frac{\langle|y|\rangle|u_1(y)|dy}{\sqrt{t^2-|x-y|^2}}=\iint\limits_{0}^{t}\frac{|\tilde{u}_1(x+\rho\omega)|\rho d\rho d\omega}{\sqrt{t^2-\rho^2}}\\
&\lesssim\frac{1}{\sqrt{t}}\int\limits_{0}^{t}\frac{d\rho}{\sqrt{t-\rho}}\sup_{0\le\rho\le t}\Big[\rho\int|\tilde{u}_1(x+\rho\omega)|d\omega\Big]\\
&\lesssim\sup_{0\le\rho\le t}\Big[\rho\int\limits_{\rho}^{\infty}\int\Big|\frac{d}{ds}\tilde{u}_1(x+s\omega)\Big|d\omega ds\Big]\\
&\lesssim\sup_{0\le\rho\le t}\int\limits_{\rho}^{\infty}\int|\nabla\tilde{u}_1(x+s\omega)|sd\omega ds\lesssim\|\nabla\tilde{u}_1(y)\|_{L^1}.
\end{aligned}\tag{3.17}$$

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Analogously,

$$\begin{aligned}
\langle|x|\rangle|u^0_{hom}(t,x)| &\lesssim \frac{1}{t}\int_{|x-y|\le t}\frac{\langle|y|\rangle|u_0(y)|dy}{\sqrt{t^2-|x-y|^2}}+\int_{|x-y|\le t}\frac{\langle|y|\rangle|\nabla u_0(y)|dy}{\sqrt{t^2-|x-y|^2}}\\
&\lesssim \frac{1}{t\sqrt{t}}\int_0^t\frac{d\rho}{\sqrt{t-\rho}}\sup_{0\le\rho\le t}\Big[\rho\int|\tilde u_0(x+\rho\omega)|d\omega\Big]+\|\nabla(\langle|y|\rangle\nabla u_0(y))\|_{L^1}\\
&\lesssim \sup_{0\le\rho\le t}\int_\rho^\infty\int|\nabla\tilde u_0(x+s\omega)|d\omega ds+\|\tilde u_0(y)\|_{W^{2,1}}\\
&\lesssim \sup_{0\le\rho\le t}\Big[\rho\int|\nabla\tilde u_0(x+\rho\omega)|d\omega+\int_\rho^\infty\int|\nabla^2\tilde u_0(x+s\omega)|d\omega s ds\Big]+\|\tilde u_0(y)\|_{W^{2,1}}\\
&\lesssim \|\tilde u_0(y)\|_{W^{2,1}}.
\end{aligned}\tag{3.18}$$

Collecting (3.16)–(3.18) completes the proof of Lemma 3.2. □

Remark 3.1. Different from Lemma 1 in [10], the proof of our Lemma 3.2 does not depend on the compactness of $\operatorname{supp}(u_0(x),u_1(x))$.

Remark 3.2. Here we point out that the control norms $\|\tilde u_0(y)\|_{W^{2,1}}+\|\tilde u_1(y)\|_{W^{1,1}}$ on the right hand side of (3.6) are not optimal, and thus the control norms of the initial data in (1.3) can be improved (one can be referred to Proposition 2.1 of [18]).

Next, we will use a modified version of Theorem 1.1 from [17] to estimate (3.5).

Lemma 3.3. *Let $u_{inh}(t,x)$ be defined in (3.5). For any $\ell_1\in[0,\frac12)$, $\ell_2\in(0,\frac12)$ and $\kappa>0$, then the following weighted L^∞-L^∞ estimate holds*

$$\langle|x|+t\rangle^{\frac12-\ell_1}\langle|x|-t\rangle^{\ell_2}|u_{inh}(t,x)|\lesssim\sup_{(t',y)\in\mathbb{R}^{1+2}}\{|y|^{\frac12}\langle|y|+t'\rangle^{1-\ell_1+\ell_2}\langle|y|-t'\rangle^{1+\kappa}|F(t',y)|\}.\tag{3.19}$$

Proof. For the case of $\ell_1=0$ in (3.19), one sees Theorem 1.1 of [17]. Next, we assume $\ell_1\in(0,\frac12)$. It is easy to know that

$$\begin{aligned}
\langle|x|+t\rangle^{-\ell_1}|u_{inh}(t,x)| &\lesssim\int_0^t\int_{|x-y|\le t-t'}\frac{\langle|x|+t\rangle^{-\ell_1}|F(t',y)|}{\sqrt{(t-t')^2-|x-y|^2}}dydt'\\
&\lesssim\int_0^t\int_{|x-y|\le t-t'}\frac{\langle|y|+t'\rangle^{-\ell_1}|F(t',y)|}{\sqrt{(t-t')^2-|x-y|^2}}dydt',
\end{aligned}\tag{3.20}$$

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where we have used the fact of $|y|+t' \le |x|+|x-y|+t' \le |x|+t$ in the last inequality above. Applying (3.19) with $\ell_1=0$ to (3.20) yields

$$\begin{aligned}&\langle |x|+t\rangle^{-\ell_1}|u_{inh}(t,x)|\\ &\lesssim \langle |x|+t\rangle^{-\frac{1}{2}}\langle |x|-t\rangle^{-\ell_2}\sup_{(t',y)\in\mathbb{R}^{1+2}}\{|y|^{\frac{1}{2}}\langle |y|+t'\rangle^{1-\ell_1+\ell_2}\langle |y|-t'\rangle^{1+\kappa}|F(t',y)|\}.\end{aligned}$$

This completes the proof of Lemma 3.3. □

Remark 3.3. For $p\in(\frac{3+\sqrt{17}}{2},4)$, R. Glassey [10] has proved (3.19) with $\ell_1=0$, $\ell_2=\frac{p-3}{2}$ and $\kappa=\frac{p^2-3p-2}{2}>0$.

Based on Lemma 3.2–3.3, the better decay rates of $T_i v$ than (2.12) can be derived as follows:

Lemma 3.4. *Let v be the solution of (2.4) and (1.3) holds. For $i=1,2$ and $a\in\mathbb{N}_0^7$ with $|a|\le N-5$, there exists a positive constant C which is independent of t and ε, such that the following inequality holds*

$$\langle |x|+t\rangle^{\frac{5}{4}}\langle |x|-t\rangle^{\frac{1}{8}}|T_iZ^a v(t,x)|\le \varepsilon+CM^2\varepsilon^2. \tag{3.21}$$

Proof. In fact, in order to prove (3.21), by (2.9) and (2.10) we only need to prove

$$\langle |x|+t\rangle^{\frac{1}{4}}\langle |x|-t\rangle^{\frac{1}{8}}|ZZ^a v(t,x)|\le \varepsilon+CM^2\varepsilon^2. \tag{3.22}$$

Denoting $Z^{a'}=ZZ^a$ and applying Lemma 3.2–3.3 to $Z^{a'}v$, then

$$\begin{aligned}\langle |x|+t\rangle^{\frac{1}{4}}\langle |x|-t\rangle^{\frac{1}{8}}|Z^{a'}v(t,x)|&\lesssim \|\langle y\rangle Z^{a'}v(0,y)\|_{W^{2,1}}+\|\langle y\rangle\partial_t Z^{a'}v(0,y)\|_{W^{1,1}}\\ &+\sup_{(t',y)\in\mathbb{R}^{1+2}}\{|y|^{\frac{1}{2}}\langle |y|+t'\rangle^{\frac{7}{8}}\langle |y|-t'\rangle^{\frac{5}{4}}|\Box Z^{a'}v(t',y)|\}.\end{aligned} \tag{3.23}$$

Next we treat the term $\Box Z^{a'}v(t',y)$ on the right hand side of (3.23). To this end, applying Lemma 2.2 to (3.1) yields that

$$\begin{aligned}|\Box Z^{a'}v(t',y)|\lesssim &\sum_{b+c+d+e\le a',k\le 1}|T\partial^k Z^b v||\partial\partial^k Z^c v||\partial\partial^k Z^d v|\Big|Z^e\Big(\frac{1}{(1+v)^2}\Big)\Big|\\ &+\sum_{b+c+d\le a',k\le 1}\Big|Z^c\Big(\frac{1}{1+v}\Big)\Big|\Big\{|\partial^2 Z^b v||T\partial^k Z^d v|+|T\partial Z^b v||\partial\partial^k Z^d v|\Big\}.\end{aligned} \tag{3.24}$$

By using Lemma 2.3–2.4 to (3.24), we further obtain

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$$
\begin{aligned}
&|y|^{\frac12}\langle |y|+t'\rangle\langle |y|-t'\rangle^{\frac54}|\Box Z^{a'}v(t',y)|\\
&\lesssim \langle |y|+t'\rangle^{-\frac12}\langle |y|-t'\rangle^{\frac14}\sum_{b+c+d+e\le a'}E_{|b|+4}[v](t')E_{|c|+4}[v](t')E_{|d|+4}[v](t')\\
&\qquad\times\Big\{\sup_y\Big|Z^e\Big(\frac{1}{(1+v)^2}\Big)(0,y)\Big|+E_{|e|+3}\Big[\frac{1}{(1+v)^2}\Big](t')\Big\}\\
&\quad+\sum_{b+c+d\le a'}E_{|b|+4}[v](t')E_{|d|+4}[v](t')\Big\{\sup_y\Big|Z^c\Big(\frac{1}{1+v}\Big)(0,y)\Big|+E_{|c|+3}\Big[\frac{1}{1+v}\Big](t')\Big\}\\
&\lesssim E_{|a'|+4}[v](t')E_{[\frac{|a'|}{2}]+4}[v](t')\Big\{1+E_{[\frac{|a'|}{2}]+4}[v](t')+E^2_{[\frac{|a'|}{2}]+4}[v](t')\Big\},
\end{aligned}
\tag{3.25}
$$

where $[s]:=\sup\{k\in\mathbb{N}:k\le s\}$. Note that $|a'|=1+|a|\le N-4$ and $N\ge 14$ in (1.3). Then substituting assumption (2.7) into (3.25) derives

$$
\begin{aligned}
&\sup_{(t',y)\in\mathbb{R}^{1+2}}\{|y|^{\frac12}\langle |y|+t'\rangle^{\frac78}\langle |y|-t'\rangle^{\frac54}|\Box Z^{a'}v(t',y)|\}\\
&\le C\sup_{t'}\Big\{\langle t'\rangle^{-\frac18}E_N[v](t')E_{N-4}[v](t')\Big(1+E_{N-4}[v](t')+E^2_{N-4}[v](t')\Big)\Big\} \qquad (3.26)\\
&\le CM^2\varepsilon^2\sup_{t'}(1+t')^{M'\varepsilon-\frac18}\le CM^2\varepsilon^2.
\end{aligned}
$$

Thus, combining (3.23)–(3.26) with (1.3) yields (3.22) and then Lemma 3.4 is proved. □

Remark 3.4. In fact, the weight $\langle |x|+t\rangle^{\frac54}\langle |x|-t\rangle^{\frac18}$ on the left hand side of (3.21) can be replaced by $\langle |x|+t\rangle^{\frac32-\ell}$ for any $\ell\in(0,\frac12)$.

4. Energy estimates

Lemma 4.1. *Let v be the solution of (2.4) and (1.3) holds. For $N\ge 14$, there exists a positive constant C which is independent of ε and t, such that the following inequalities hold*

$$
E_N^2[v](t')\le C\varepsilon^2+C\int_0^{t'}\langle t\rangle^{-1}E_{N-4}[v](t)E_N^2[v](t)dt, \tag{4.1}
$$

$$
E_{N-4}^2[v](t')\le C\varepsilon^2+C\int_0^{t'}\langle t\rangle^{-\frac54}E_{N-4}[v](t)E_{N-3}[v](t)\Big\{\varepsilon+E_N[v](t)\Big\}dt. \tag{4.2}
$$

Proof. For $a\in\mathbb{N}_0^7$ with $|a|\le N-1$, by integrating $2e^{q(|x|-t)}\partial_tZ^av\Box Z^av$ over $[0,t']\times\mathbb{R}^2$ with the ghost weight $q(|x|-t)=\arctan(|x|-t)$, we directly get

$$\|e^{q/2}\partial_t Z^a v(t',x)\|^2+\|e^{q/2}\nabla Z^a v(t',x)\|^2+\sum_{i=1,2}\int_0^{t'}\Big\|\frac{e^{q/2}T_i Z^a v}{\langle |x|-t\rangle}\Big\|^2 dt$$
$$=\|e^{q(|x|)/2}\partial_t Z^a v(0,x)\|^2+\|e^{q(|x|)/2}\nabla Z^a v(0,x)\|^2+\int_0^{t'}\int 2e^q\partial_t Z^a v\Box Z^a v dxdt. \tag{4.3}$$

Here we emphasize that the top-order derivative terms in the expression of $\Box Z^a v$ should be paid more attention. By the symmetry assumptions of coefficients (2.5), the identity (3.1) can be rewritten as

$$\begin{aligned}\Box Z^a v&=\partial^2_{\alpha\beta}Z^a v\mathcal{A}^{\alpha\beta}_{top}(v)\\&+\sum_{b+c+d+e\le a}A^{\alpha\beta\gamma}_{2,abcde}\partial_\alpha Z^b v\partial_\beta Z^c v\partial_\gamma Z^d v Z^e\Big(\frac{1}{(1+v)^2}\Big)\\&+\sum_{b+c+d\le a,b<a}A^{\alpha\beta\gamma}_{3,abcd}\partial^2_{\alpha\beta}Z^b v Z^c\Big(\frac{1}{1+v}\Big)\partial_\gamma Z^d v\\&+\sum_{b+c+d\le a,b+d<a}A^{\alpha\beta\mu\nu}_{4,abcd}\partial^2_{\alpha\beta}Z^b v Z^c\Big(\frac{1}{1+v}\Big)\partial^2_{\mu\nu}Z^d v\\&+\sum_{b+c+d+e\le a,b<a}A^{\alpha\beta\mu\nu}_{5,abcde}\partial^2_{\alpha\beta}Z^b v Z^c\Big(\frac{1}{(1+v)^2}\Big)\partial_\mu Z^d v\partial_\nu Z^e v\\&+\sum_{b+c+d+e\le a,b+e<a}A^{\alpha\beta\gamma\mu\nu}_{6,abcde}\partial^2_{\alpha\beta}Z^b v Z^c\Big(\frac{1}{(1+v)^2}\Big)\partial_\gamma Z^d v\partial^2_{\mu\nu}Z^e v\\&+\sum_{b+c+d+e\le a,b+d+e<a}A^{\alpha\beta\gamma\delta\mu\nu}_{7,abcde}\partial^2_{\alpha\beta}Z^b v Z^c\Big(\frac{1}{(1+v)^2}\Big)\partial^2_{\gamma\delta}Z^d v\partial^2_{\mu\nu}Z^e v,\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}\mathcal{A}^{\alpha\beta}_{top}(v)&=\frac{1}{1+v}\Big(A_3^{\alpha\beta\gamma}\partial_\gamma v+2A_4^{\alpha\beta\mu\nu}\partial^2_{\mu\nu}v\Big)\\&+\frac{1}{(1+v)^2}\Big(A_5^{\alpha\beta\mu\nu}\partial_\mu v\partial_\nu v+2A_6^{\alpha\beta\gamma\mu\nu}\partial_\gamma v\partial^2_{\mu\nu}v+3A_7^{\alpha\beta\gamma\delta\mu\nu}\partial^2_{\gamma\delta}v\partial^2_{\mu\nu}v\Big).\end{aligned} \tag{4.5}$$

For the term $\partial^2_{\alpha\beta}Z^a v\mathcal{A}^{\alpha\beta}_{top}(v)$ in the first line of (4.4), direct computation yields

$$\begin{aligned}&2e^q\partial_t Z^a v\partial^2_{\alpha\beta}Z^a v\mathcal{A}^{\alpha\beta}_{top}(v)\\&=2\partial_\alpha[e^q\partial_t Z^a v\partial_\beta Z^a v\mathcal{A}^{\alpha\beta}_{top}(v)]-\partial_t[e^q\partial_\alpha Z^a v\partial_\beta Z^a v\mathcal{A}^{\alpha\beta}_{top}(v)]\\&\quad-2(\partial_\alpha q)e^q\partial_t Z^a v\partial_\beta Z^a v\mathcal{A}^{\alpha\beta}_{top}(v)-2e^q\partial_t Z^a v\partial_\beta Z^a v\partial_\alpha\mathcal{A}^{\alpha\beta}_{top}(v)\\&\quad+(\partial_t q)e^q\partial_\alpha Z^a v\partial_\beta Z^a v\mathcal{A}^{\alpha\beta}_{top}(v)+e^q\partial_\alpha Z^a v\partial_\beta Z^a v\partial_t\mathcal{A}^{\alpha\beta}_{top}(v),\end{aligned} \tag{4.6}$$

where for $\lambda = 0, 1, 2$,

$$\begin{aligned}\partial_\lambda \mathcal{A}^{\alpha\beta}_{top}(v) &= \frac{1}{1+v}\Big(A_3^{\alpha\beta\gamma}\partial^2_{\lambda\gamma}v + 2A_4^{\alpha\beta\mu\nu}\partial^3_{\lambda\mu\nu}v\Big) - \frac{\partial_\lambda v}{(1+v)^2}\Big(A_3^{\alpha\beta\gamma}\partial_\gamma v + 2A_4^{\alpha\beta\mu\nu}\partial^2_{\mu\nu}v\Big)\\ &+ \frac{1}{(1+v)^2}\Big[2A_5^{\alpha\beta\mu\nu}\partial_\mu v\partial^2_{\lambda\nu}v + 2A_6^{\alpha\beta\gamma\mu\nu}(\partial^2_{\lambda\gamma}v\partial^2_{\mu\nu}v + \partial_\gamma v\partial^3_{\lambda\mu\nu}v) + 6A_7^{\alpha\beta\gamma\delta\mu\nu}\partial^2_{\gamma\delta}v\partial^3_{\lambda\mu\nu}v\Big]\\ &- \frac{2\partial_\lambda v}{(1+v)^3}\Big(A_5^{\alpha\beta\mu\nu}\partial_\mu v\partial_\nu v + 2A_6^{\alpha\beta\gamma\mu\nu}\partial_\gamma v\partial^2_{\mu\nu}v + 3A_7^{\alpha\beta\gamma\delta\mu\nu}\partial^2_{\gamma\delta}v\partial^2_{\mu\nu}v\Big).\end{aligned} \tag{4.7}$$

Therefore, applying the fact that $T_i q(|x|-t) \equiv 0$ and (2.6) to (4.4)–(4.7), it follows from direct computation that

$$\begin{aligned}&\left|\int_0^{t'}\int 2e^q\partial_t Z^a v \square Z^a v dxdt\right|\\ &\lesssim \sum_{\substack{b+c+d\le a,\\ b<a}}(I_1^{abcd}+I_2^{abcd}) + \sum_{\substack{b+c+d\le a,\\ b+d<a}} I_3^{abcd} + \sum_{b+c+d\le a} I_4^{abcd} + \mathcal{I}^a_{l.o.t.} + \mathcal{I}^a_{top},\end{aligned} \tag{4.8}$$

where

$$I_1^{abcd} := \int_0^{t'}\int |\partial Z^a v||\partial^2 Z^b v|\Big|Z^c\Big(\frac{1}{1+v}\Big)\Big||TZ^d v|dxdt, \tag{4.9}$$

$$I_2^{abcd} := \int_0^{t'}\int |\partial Z^a v||T\partial Z^b v|\Big|Z^c\Big(\frac{1}{1+v}\Big)\Big||\partial Z^d v|dxdt, \tag{4.10}$$

$$I_3^{abcd} := \int_0^{t'}\int |\partial Z^a v||T\partial Z^b v|\Big|Z^c\Big(\frac{1}{1+v}\Big)\Big||\partial^2 Z^d v|dxdt, \tag{4.11}$$

$$I_4^{abcd} := \int_0^{t'}\int \langle |x|-t\rangle^{-2}|\partial Z^a v||TZ^b v|\Big|Z^c\Big(\frac{1}{1+v}\Big)\Big||\partial Z^d v|dxdt, \tag{4.12}$$

and

$$\begin{aligned}\mathcal{I}^a_{l.o.t.} &:= \sum_{b+c+d+e\le a}\int_0^{t'}\int |\partial Z^a v||TZ^b v||\partial Z^c v||\partial Z^d v|\Big|Z^e\Big(\frac{1}{(1+v)^2}\Big)\Big|dxdt\\ &+ \sum_{b+c+d+e\le a, b<a}\int_0^{t'}\int |\partial Z^a v||\partial Z^d v|\Big|Z^c\Big(\frac{1}{(1+v)^2}\Big)\Big|\end{aligned}$$

$$
\begin{aligned}
&\times \Big[|T\partial Z^b v||\partial Z^e v| + |\partial^2 Z^b v||TZ^e v|\Big]dxdt \\
&+ \sum_{b+c+d+e\le a, b+d<a} \int_0^{t'}\int |\partial Z^a v||\partial^2 Z^d v|\Big|Z^c\Big(\frac{1}{(1+v)^2}\Big)\Big| \\
&\quad \times \Big[|T\partial Z^b v||\partial Z^e v| + |\partial^2 Z^b v||TZ^e v|\Big]dxdt \\
&+ \sum_{b+c+d+e\le a, b+d+e<a} \int_0^{t'}\int |\partial Z^a v||T\partial Z^b v||\partial^2 Z^d v||\partial^2 Z^e v|\Big|Z^c\Big(\frac{1}{(1+v)^2}\Big)\Big|dxdt,
\end{aligned}
\tag{4.13}
$$

and

$$
\begin{aligned}
\mathcal{I}^a_{top} :=& \left|\int_0^{t'}\int 2e^q \partial_t Z^a v \partial^2_{\alpha\beta} Z^a v \mathcal{A}^{\alpha\beta}_{top}(v)dxdt\right| \\
\lesssim& \sum_{k\le 2}\int_0^{t'}\int |\partial Z^a v|(1+|\partial\partial^k v| + |\partial\partial^k v|^2)\Big\{|\partial Z^a v||T\partial^k v| \\
&+ |TZ^a v|\Big[|\partial^2 v| + |\partial^3 v| + \langle |x|-t\rangle^{-2}|\partial\partial^k v|\Big]\Big\}dxdt + \varepsilon^3 + E_4[v](t')E^2_{|a|+1}[v](t') \\
\lesssim& \sum_{i=1,2}\int_0^{t'}\Big\|\frac{e^{q/2}T_i Z^a v}{\langle |x|-t\rangle}\Big\|^2 E_5[v](t)dt + \int_0^{t'}\langle t\rangle^{-1}E_5[v](t)E^2_{|a|+1}[v](t)dt \\
&+ \varepsilon^3 + E_4[v](t')E^2_{|a|+1}[v](t').
\end{aligned}
\tag{4.14}
$$

At first, we focus on the treatments of the terms I_l^{abcd} with $l = 1, 2, 3, 4$.

$\underline{I_1^{abcd}}$: If $|c| \le N-7$, by Lemma 2.3–2.4 we have

$$
\begin{aligned}
\sum_{\substack{b+c+d\le a,\\ b<a, |c|\le N-7}} I_1^{abcd} \lesssim& \sum_{b+d\le a, b<a}\int_0^{t'}\int |\partial Z^a v||\partial^2 Z^b v||TZ^d v|dxdt \\
\lesssim& \sum_{d\le a, i=1,2}\int_0^{t'}\Big\|\frac{e^{q/2}T_i Z^d v}{\langle |x|-t\rangle}\Big\|^2 E_{[\frac{|a|}{2}]+4}[v](t)dt + \int_0^{t'}\langle t\rangle^{-1}E_{[\frac{|a|}{2}]+4}[v](t)E^2_{|a|+1}[v](t)dt.
\end{aligned}
\tag{4.15}
$$

If $|c| \ge N-6 \ge 1$, denote $Z^c = ZZ^{c'}$. Then applying (2.13) to I_1^{abcd} yields

$$\begin{aligned}
&\sum_{\substack{b+c+d\le a,\\ b<a,|c|\ge N-6}} I_1^{abcd}\\
&\lesssim \sum_{\substack{b+c+d\le a,\\ b<a,|c|\ge N-6}} \int_0^{t'}\int |\partial Z^a v||\partial^2 Z^b v||TZ^d v|\\
&\quad\times\Big[\Big|\langle |x|+t\rangle TZ^{c'}\Big(\frac{1}{1+v}\Big)\Big|+\Big|\langle |x|-t\rangle \partial Z^{c'}\Big(\frac{1}{1+v}\Big)\Big|\Big]dxdt\\
&\lesssim \sum_{c\le a, i=1,2}\int_0^{t'}\Big\|\frac{e^{q/2}T_i Z^c v}{\langle |x|-t\rangle}\Big\|^2 E_9[v](t)dt+\int_0^{t'}\langle t\rangle^{-1}E_9[v](t)E^2_{|a|+1}[v](t)dt.
\end{aligned} \tag{4.16}$$

I_2^{abcd} and I_3^{abcd}: We conclude from (2.8), (2.10) and (2.13) that

$$\begin{aligned}
&\sum_{b+c+d\le a, b<a} I_2^{abcd}+\sum_{b+c+d\le a, b+d<a} I_3^{abcd}\\
&\lesssim \sum_{b+c+d\le a, b<a}\int_0^{t'}\int \langle t\rangle^{-1}|\partial Z^a v||Z\partial Z^b v|\Big|Z^c\Big(\frac{1}{1+v}\Big)\Big||\partial Z^d v|dxdt\\
&\quad+\sum_{b+c+d\le a, b+d<a}\int_0^{t'}\int \langle t\rangle^{-1}|\partial Z^a v||Z\partial Z^b v|\Big|Z^c\Big(\frac{1}{1+v}\Big)\Big||\partial^2 Z^d v|dxdt\\
&\lesssim \sum_{c\le a, i=1,2}\int_0^{t'}\Big\|\frac{e^{q/2}T_i Z^c v}{\langle |x|-t\rangle}\Big\|^2 E_9[v](t)dt+\int_0^{t'}\langle t\rangle^{-\frac{3}{2}}E_{\max\{[\frac{|a|}{2}]+4,9\}}[v](t)E^2_{|a|+1}[v](t)dt.
\end{aligned} \tag{4.17}$$

I_4^{abcd}: Analogously to the treatment on I_2^{abcd}, we achieve

$$\begin{aligned}
\sum_{b+c+d\le a} I_4^{abcd} &\lesssim \int_0^{t'}\langle t\rangle^{-1}E_{\max\{[\frac{|a|}{2}]+4,9\}}[v](t)E^2_{|a|+1}[v](t)dt\\
&\quad+\sum_{b\le a, i=1,2}\int_0^{t'}\Big\|\frac{T_i Z^b v}{\langle |x|-t\rangle}\Big\|^2 E_{\max\{[\frac{|a|}{2}]+4,9\}}[v](t)dt.
\end{aligned} \tag{4.18}$$

In addition, it is easy to check that $\mathcal{I}^a_{l.o.t.}$ verifies

$$\mathcal{I}^a_{l.o.t.}\lesssim \int_0^{t'}\langle t\rangle^{-\frac{3}{2}}E_{N-4}[v](t)E^2_{|a|+1}[v](t)dt+\sum_{c\le a, i=1,2}\int_0^{t'}\Big\|\frac{e^{q/2}T_i Z^c v}{\langle |x|-t\rangle}\Big\|^2 E_{N-4}[v](t)dt. \tag{4.19}$$

Therefore, inserting (4.8)–(4.19) into (4.3) and according to the smallness of ε_0 and the fact of $N \geq 14$, we derive (4.1).

Finally, we start to prove (4.2). For $|a| \leq N-5$, applying (2.6) to (4.4) directly, we get

$$\begin{aligned}&\left|\int_0^{t'}\int 2e^q \partial_t Z^a v \Box Z^a v dx dt\right| \\ &\lesssim \sum_{b+c+d\leq a}(I_1^{abcd}+I_2^{abcd}+I_3^{abcd}+I_4^{abcd})+\mathcal{J}^a_{l.o.t.},\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\mathcal{J}^a_{l.o.t.} &:= \sum_{\substack{b+c+d+e\leq a,\\ k\leq 1}}\int_0^{t'}\int |\partial Z^a v||T\partial^k Z^b v|\left|Z^c\Big(\frac{1}{(1+v)^2}\Big)\right||\partial\partial^k Z^d v||\partial\partial^k Z^e v|dxdt \\ &= \sum_{\substack{b+c+d+e\leq a,\\ d+e=a}}\cdots+\sum_{\substack{b+c+d+e\leq a,\\ d+e<a}}\cdots \\ &\lesssim \int_0^{t'}\langle t\rangle^{-\frac{3}{2}}E_{|a|+1}[v](t)E_{[\frac{|a|}{2}]+4}[v](t)\Big\{E_{|a|+2}[v](t)E_4[v](t)+E_{|a|+1}[v](t)E_{|a|+4}[v](t)\Big\}dt \\ &\quad+\sum_{c\leq a, i=1,2}\int_0^{t'}\left\|\frac{e^{q/2}T_i Z^c v}{\langle |x|-t\rangle}\right\|^2 E_{N-4}[v](t)dt.\end{aligned} \tag{4.21}$$

Applying (3.21) to I_1^{abcd} and I_4^{abcd} yields

$$\begin{aligned}&\sum_{b+c+d\leq a}(I_1^{abcd}+I_4^{abcd}) \\ &\lesssim \sum_{b+c+d\leq a, b<a}I_1^{abcd}+\sum_{b+c+d\leq a}I_4^{abcd}+\int_0^{t'}\int|\partial Z^a v||\partial^2 Z^a v|\left|\frac{1}{1+v}\right||Tv|dxdt \\ &\lesssim \varepsilon(1+M^2\varepsilon)\int_0^{t'}\langle t\rangle^{-\frac{5}{4}}E_{|a|+1}[v](t)E_{|a|+2}[v](t)dt.\end{aligned} \tag{4.22}$$

In addition, it follows from direct computation that

$$\sum_{b+c+d\leq a, b=a} I_2^{abcd}+\sum_{b+c+d\leq a, b+d=a} I_3^{abcd}$$

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$$\lesssim \int_0^{t'} \int \langle t\rangle^{-1} |\partial Z^a v||Z\partial Z^a v|\left|\frac{1}{1+v}\right||\partial v|dxdt$$
$$+\sum_{b+d=a} \int_0^{t'} \int \langle t\rangle^{-1} |\partial Z^a v||Z\partial Z^b v|\left|\frac{1}{1+v}\right||\partial^2 Z^d v|dxdt \tag{4.23}$$
$$\lesssim \int_0^{t'} \langle t\rangle^{-\frac{3}{2}} E_{|a|+1}[v](t) E_{|a|+2}[v](t) E_{[\frac{|a|}{2}]+4}[v](t)dt.$$

Therefore, substituting (4.9)–(4.12), (4.17) and (4.20)–(4.23) into (4.3), we achieve (4.2). □

5. Proof of Theorem 1.1

Proof of Theorem 1.1. Applying Gronwall's inequality to (4.1) and substituting (2.7) into (4.2), we achieve

$$E_N^2[v](t) \le C_1\varepsilon^2(1+t)^{C_2 M\varepsilon},$$
$$E_{N-4}^2[v](t) \le C\varepsilon^2 + CM^2\varepsilon^3(1+M)\int_0^t (1+\tau)^{2M'\varepsilon-\frac{5}{4}}d\tau$$
$$\le C\varepsilon^2 + CM^2\varepsilon^3(1+M)$$
$$\le C_3\varepsilon^2[1+M^2\varepsilon(1+M)],$$

where C_1, C_2, C_3 are positive constants. By choosing $M = \max\{1, 2\sqrt{C_1}, 2\sqrt{2C_3}\}$, $M' = 2C_2 M$ and $\varepsilon_0 = \min\{\frac{1}{16M'}, \frac{1}{M^2(1+M)}\}$, we then have

$$E_N[v](t) \le \frac{1}{2}M\varepsilon(1+t)^{M'\varepsilon},$$
$$E_{N-4}[v](t) \le \frac{1}{2}M\varepsilon.$$

Thus, by the continuous induction method and the local well-posedness of problem (2.4) (see [12]), then problem (1.1) admits a global smooth solution u. □

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