Advances in Differential Equations Volume 7, Number 4, April 2002, Pages 441–468

SCATTERING FOR SYSTEMS OF SEMILINEAR WAVE EQUATIONS WITH DIFFERENT SPEEDS OF PROPAGATION

Hideo Kubo*
Department of Applied Mathematics, Faculty of Engineering, Shizuoka University
Hamamatsu 432-8561, Japan

Kôji Kubota
Department of Mathematics, Hokkaido University of Education
Hakodate, Hakodate 040-8567, Japan

(Submitted by: Sergiu Klainerman)

Abstract. We give an extension of the a priori estimate, obtained in [8], for a solution of the inhomogeneous wave equation in $\mathbf{R}^n \times \mathbf{R}$, where $n = 2$ or $n = 3$. As an application, we study the asymptotic behavior as $t \to \pm\infty$ of solutions to systems of semilinear wave equations. The discrepancy of the speeds of propagation may make a significant difference from the case of common propagation speeds. (See also Theorem 3.3 and 3.4). Whether such a phenomenon occurs or not depends on the type of the interaction determined by the nonlinearities.

1. Introduction

In a previous paper [8], we have derived an a priori estimate for a solution of the inhomogeneous wave equation in $\mathbf{R}^n \times \mathbf{R}$ with $n = 2$ or $n = 3$, and studied the asymptotic behavior of solutions to the system

$$\partial_t^2 u - \Delta u = |v|^p \quad \text{in } \mathbf{R}^n \times \mathbf{R},$$
$$\partial_t^2 v - \Delta v = |u|^q \quad \text{in } \mathbf{R}^n \times \mathbf{R},$$

where $\partial_t = \partial/\partial t$ and $1 < p \leq q$. The aim of this paper is to extend the a priori estimate so that one can treat more general systems of semilinear wave equations:

$$\partial_t^2 u^i - c_i^2 \Delta u^i = F^i(u) \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{1.1}$$

Accepted for publication: August 2001.

*Partially supported by Grant-in-Aid for Science Research (No.09304012), Ministry of Education, Science and Culture, Japan.

AMS Subject Classifications: 35L70, 35B40, 35B45.

where $1 \leq i \leq m$, $u(x,t) = (u^1(x,t), \cdots, u^m(x,t))$ is an unknown $\mathbf{R}^m$-valued function, c_i are positive numbers, and either $n = 2$ or $n = 3$. Besides, we assume that $F^i \in C^1(\mathbf{R}^m)$ satisfies

$$F^i(0) = \frac{\partial F^i}{\partial u^j}(0) = 0 \quad \text{for } 1 \leq i, j \leq m.$$

We will require some other specific conditions on F^i later on.

To show the global existence of solutions of the Cauchy problem for (1.1), we need to evaluate an operator, associated with the inhomogeneous wave equation, defined by

$$L_c^+(F)(x,t) = \frac{1}{2\pi}\int_0^t ds \int_0^{t-s} \frac{\rho d\rho}{\sqrt{(t-s)^2 - \rho^2}} \int_{|\omega|=1} F(x + c\rho\omega, s) dS_\omega \quad (1.2)$$

for $(x,t) \in \mathbf{R}^2 \times [0,\infty)$, and by

$$L_c^+(F)(x,t) = \frac{1}{4\pi}\int_0^t (t-s) ds \int_{|\omega|=1} F(x + c(t-s)\omega, s) dS_\omega \quad (1.3)$$

for $(x,t) \in \mathbf{R}^3 \times [0,\infty)$, where $F \in C(\mathbf{R}^n \times [0,\infty))$ and $c > 0$. In fact, $L_c^+(F)(x,t)$ satisfies the inhomogeneous wave equation

$$\partial_t^2 u - c^2 \Delta u = F \quad \text{in } \mathbf{R}^n \times (0,\infty), \quad (1.4)$$

together with the zero initial data, if $\partial_x^\alpha F \in C(\mathbf{R}^n \times [0,\infty))$ for $|\alpha| \leq 2$. For the operator, we obtain the following basic estimate.

Theorem 1.1. *Let $n = 2$ or $n = 3$. Let $F \in C(\mathbf{R}^n \times [0,\infty))$ and let c, a, ν and μ be positive numbers. Then we have*

$$|L_c^+(F)(x,t)| \leq C M_{\nu,\mu}^+(F,a)(1 + r + t)^{-\frac{n-1}{2}} \Phi_n(r, ct; \nu) \quad (1.5)$$

for $(x,t) \in \mathbf{R}^n \times [0,\infty)$ with $r = |x|$, where C is a constant depending only on c, a, ν and μ. In addition,

$$M_{\nu,\mu}^+(F,a) = \sup_{(y,s) \in \mathbf{R}^n \times [0,\infty)} \{|y|^{\frac{n-1}{2}}(1 + |y| + s)^{1+\nu}(1 + ||y| - as|)^{1+\mu}|F(y,s)|\}.$$

Moreover, $\Phi_n(r,t;\nu)$ is defined by

$$\Phi_3(r,t;\nu) = (1 + |r - t|)^{-\nu} \quad (1.6)$$

and

$$\Phi_2(r,t;\nu) = \begin{cases} (1 + |r-t|)^{-\nu} & \text{if} \quad -\infty < t \leq r \\ (1 + t - r)^{-\frac{1}{2}}(1 + t - r)^{[\frac{1}{2} - \nu]_+} & \text{if} \quad r < t \end{cases}.$$

Here we have set $[b]_+ = \max\{b, 0\}$ for $b \in \mathbf{R}$ with $b \neq 0$,

$$A^{[0]_+} = 1 + \log A \quad \text{for } A \geq 1.$$

Remark. If the system (1.1) has the common propagation speeds, i.e., $c_1 = \cdots = c_m$, it suffices to show (1.5) for $a = c$. However, in general, we need to derive the estimate for arbitrary a.

Next we turn our attention to the asymptotic behavior of solutions of (1.1). In this paper, we call the following fact "small data nonlinear scattering": Let $u_-(x,t) = (u_-^1(x,t), \cdots, u_-^m(x,t))$ be the classical solution of the Cauchy problem for the homogeneous wave equation

$$\partial_t^2 u^i - c_i^2 \Delta u^i = 0 \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{1.7}$$

satisfying

$$|\partial_{x,t}^\alpha u_-^i(x,t)| \le \varepsilon(1+|x|+|t|)^{-\frac{n-1}{2}} \Phi_n(r, c_i|t|; \nu_i) \tag{1.8}$$

for some $\varepsilon > 0$ and $\nu_i > 0$, where $|\alpha| \le 1$. Then for sufficiently small ε, there is uniquely a C^1-solution $u(x,t)$ of (1.1) such that

$$|||(u^i - u_-^i)(t)|||_{e,i} \le C\varepsilon(1+|t|)^{-\rho_i} \quad \text{for } t \le 0 \tag{1.9}$$

for some $\rho_i > 0$ and a positive constant C. Here $|||\cdot|||_{e,i}$ stands for the energy norm, namely

$$|||w(t)|||_{e,i}^2 = \frac{1}{2}\int_{\mathbf{R}^n} (|\partial_t w(x,t)|^2 + c_i^2 |\partial_x w(x,t)|^2) dx \quad \text{for } 1 \le i \le m. \tag{1.10}$$

Furthermore, there exists uniquely a C^1-solution $u_+(x,t) = (u_+^1(x,t), \ldots, u_+^m(x,t))$ of the homogeneous wave equations (1.7) satisfying

$$|||(u^i - u_+^i)(t)|||_{e,i} \le C\varepsilon(1+t)^{-\rho_i} \quad \text{for } t \ge 0. \tag{1.11}$$

In other words, the scattering operator for the system (1.1):

$$(u_-(0), \partial_t u_-(0)) \longmapsto (u_+(0), \partial_t u_+(0)) \tag{1.12}$$

is defined on a dense set of a neighborhood of 0 in the energy space.

To prove that "small data nonlinear scattering" happens for the system (1.1) under sutable assumptions on F^i, it is not enough to study $L_c^+(F)(x,t)$. We need to modify the operator as follows:

$$L_c(F)(x,t) = \frac{1}{2\pi}\int_{-\infty}^t ds \int_0^{t-s} \frac{\rho d\rho}{\sqrt{(t-s)^2-\rho^2}} \int_{|\omega|=1} F(x+c\rho\omega, s) dS_\omega \tag{1.13}$$

for $(x,t) \in \mathbf{R}^2 \times \mathbf{R}$, and

$$L_c(F)(x,t) = \frac{1}{4\pi}\int_{-\infty}^t (t-s) ds \int_{|\omega|=1} F(x+c(t-s)\omega, s) dS_\omega \tag{1.14}$$

for $(x,t) \in \mathbf{R}^3 \times \mathbf{R}$.

Since $L_c(F)(x,t)$ includes integral over unbounded region, we are forced to require that the solutions decay fast as $|x|+|t|\to\infty$. More precisely, we have to consider such solutions behaves as in (1.8) with $\nu_i>0$. But it is not the case for the Cauchy problem. (See e.g. [2]).

When $n=3$, we can evaluate $L_c(F)(x,t)$ by almost the same way as $L_c^+(F)(x,t)$. On the other hand, when $n=2$, we need to add an essentially extra estimate (for J_2 defined in (2.24) below) which is not appeared in the Cauchy problem. This is due to the lack of the strong Huygens' principle. (See also the introductions in [12] and [15]).

The following a priori estimate is an extension of [8], Theorem 1.1.

Theorem 1.2. *Let $n=2$ or $n=3$. Let $F\in C(\mathbf{R}^n\times\mathbf{R})$ and let c, a and ν be positive numbers. Then we have*

$$|L_c(F)(x,t)|\le CM_{\nu,\mu}(F,a)(1+r+|t|)^{-\frac{n-1}{2}}\Phi_n(r,ct;\nu)(1+|r-ct|)^{[-\mu]_+} \tag{1.15}$$

for $(x,t)\in\mathbf{R}^n\times\mathbf{R}$ with $r=|x|$, provided either $\mu>0$ or $t\le 0$ and $\mu>-(n-1)/2$, where C is a constant depending only on c, a, ν and μ. Here

$$M_{\nu,\mu}(F,a)=\sup_{(y,s)\in\mathbf{R}^n\times\mathbf{R}}\{|y|^{\frac{n-1}{2}}(1+|y|+|s|)^{1+\nu}(1+||y|-a|s||)^{1+\mu}|F(y,s)|\}. \tag{1.16}$$

Moreover, $\Phi_n(r,t;\nu)$ is defined by (1.6)*.*

Remark. When $c=a=1$ and $\mu>0$, the estimate (1.15) coincides with (1.4) in [8], since $[-\mu]_+=0$ for $\mu>0$.

The paper is organaized as follows. In Section 2 we prove Theorems 1.1 and 1.2 by strengthening a little the approach of [8]. In Section 3 we state our results for the system of semilinear wave equations (1.1). Fianlly, in Section 4 we prove theorems given in the previous section by making use of Theorems 1.1 and 1.2.

The authors would like to thank the referee for stimulative comments. The first author is greatfull to Dipartimento di Matematica Pura ed Applicata, Università degli Studi dell'Aquila, where he was a visitor during the period of this work was prepared.

2. Basic estimates

In this section we shall prove Theorems 1.1 and 1.2. We begin by showing the following identity concerning the spherical mean.

Lemma 2.1. *Let $b(\lambda)$ be a continuous function of $\lambda \in [0,\infty)$. Let $n \geq 2$. Then we have*

$$\int_{|\omega|=1} \frac{b(|x+\rho\omega|)}{|x+\rho\omega|^\gamma} dS_\omega = \frac{2^{3-n}\omega_{n-1}}{(r\rho)^{n-2}} \int_{|\rho-r|}^{\rho+r} \lambda^{1-\gamma} b(\lambda) h(\lambda,\rho,r)\, d\lambda \tag{2.1}$$

for $0 \leq \gamma \leq 1$, $\rho > 0$ and $x \in \mathbf{R}^n$ with $r = |x| > 0$, where $\omega_k = 2\pi^{k/2}/\Gamma(k/2)$, $\Gamma(s)$ being the Gamma function, and $h(\lambda,\rho,r)$ is defined by

$$\begin{aligned} h(\lambda,\rho,r) &= (\lambda^2 - (\rho-r)^2)^{\frac{n-3}{2}} ((\rho+r)^2 - \lambda^2)^{\frac{n-3}{2}} \\ &= (\rho^2 - (\lambda-r)^2)^{\frac{n-3}{2}} ((\lambda+r)^2 - \rho^2)^{\frac{n-3}{2}}. \end{aligned} \tag{2.2}$$

Proof. It is well known that (2.1) holds when $\gamma = 0$. For the proof see [14] (or [11], Lemma 2.3). Therefore, we have for any positive integer k

$$\int_{|\omega|=1} \frac{b(|x+\rho\omega|)}{((1/k)+|x+\rho\omega|)^\gamma} dS_\omega = \frac{2^{3-n}\omega_{n-1}}{(r\rho)^{n-2}} \int_{|\rho-r|}^{\rho+r} \frac{\lambda b(\lambda) h(\lambda,\rho,r)}{((1/k)+\lambda)^\gamma} d\lambda. \tag{2.3}$$

Notice that the right hand side of (2.3) is bounded with respct to k, so is the left hand side. Now, applying Beppo Levi's theorem, we obtain (2.1) and the proof is complete. □

We are now in a position to estimate the operator $L_c(F)(x,t)$ given by (1.13) and (1.14). A scaling argument shows that it suffices to consider it for $c = 1$. Indeed, since $L_c(F)(x,t) = L_1(F_c)(x,ct)$ with $F_c(x,t) = c^{-2}F(x,t/c)$, (1.15) with $c = 1$ will give us

$$|L_c(F)(x,t)| \leq CM_{\nu,\mu}(F_c, a/c)(1+r+|ct|)^{-\frac{n-1}{2}} \Phi_n(r,ct;\nu)(1+|r-ct|)^{[-\mu]_+}, \tag{2.4}$$

hence, (1.15) holds for any $c > 0$. If we set

$$z_{\nu,\mu}(\lambda,s;a) = (1+|s|+\lambda)^{1+\nu}(1+|\lambda-a|s||)^{1+\mu} \tag{2.5}$$

for $a > 0$, $\nu > 0$ and $\mu > -(n-1)/2$, we have from (1.16)

$$M_{\nu,\mu}(F,a) = \sup_{(y,s)\in\mathbf{R}^n\times\mathbf{R}} \{|y|^{\frac{n-1}{2}} z_{\nu,\mu}(|y|,s;a)|F(y,s)|\},$$

hence,

$$\int_{|\omega|=1} |F(x+\rho\omega,s)| dS_\omega \leq M_{\nu,\mu}(F,a) \int_{|\omega|=1} \frac{dS_\omega}{\lambda^{\frac{n-1}{2}} z_{\nu,\mu}(\lambda,s;a)} \tag{2.6}$$

with $\lambda = |x+\rho\omega|$. Applying Lemma 2.1 with $\gamma = (n-1)/2$ to the integral on the right hand side, we get

$$\int_{|\omega|=1} \frac{dS_\omega}{\lambda^{\frac{n-1}{2}} z_{\nu,\mu}(\lambda,s;a)} = 2^{3-n}\omega_{n-1}(r\rho)^{2-n} \int_{|\rho-r|}^{\rho+r} \frac{\lambda^{\frac{3-n}{2}} h(\lambda,\rho,r)}{z_{\nu,\mu}(\lambda,s;a)} d\lambda. \tag{2.7}$$

Case 1: n = 2. If we set

$$I(r,t)=\frac{2}{\pi}\int_{-\infty}^{t}ds\int_0^{t-s}\frac{\rho}{\sqrt{(t-s)^2-\rho^2}}d\rho\int_{|\rho-r|}^{\rho+r}\frac{\lambda^{1/2}h(\lambda,\rho,r)}{z_{\nu,\mu}(\lambda,s;a)}d\lambda, \tag{2.8}$$

it follows from (1.13), (2.6) and (2.7) with $n=2$ that

$$|L_1(F)(x,t)|\le M_{\nu,\mu}(F,a)\times I(r,t). \tag{2.9}$$

Changing the order of the integrals, we get

$$I(r,t)=I_1(r,t)+I_2(r,t), \tag{2.10}$$

where we have set

$$I_1(r,t)=\int_{-\infty}^{t}ds\int_{|t-s-r|}^{t-s+r}\frac{1}{z_{\nu,\mu}(\lambda,s;a)}K_1(\lambda,r,t-s)d\lambda, \tag{2.11}$$

$$I_2(r,t)=\int_{-\infty}^{t-r}ds\int_0^{t-s-r}\frac{1}{z_{\nu,\mu}(\lambda,s;a)}K_2(\lambda,r,t-s)d\lambda. \tag{2.12}$$

Here

$$K_1(\lambda,r,t)=\frac{2\sqrt{\lambda}}{\pi}\int_{|\lambda-r|}^{t}\frac{\rho h(\lambda,\rho,r)}{\sqrt{t^2-\rho^2}}d\rho\quad\text{for } |t-r|<\lambda<t+r, \tag{2.13}$$

$$K_2(\lambda,r,t)=\frac{2\sqrt{\lambda}}{\pi}\int_{|\lambda-r|}^{\lambda+r}\frac{\rho h(\lambda,\rho,r)}{\sqrt{t^2-\rho^2}}d\rho\quad\text{for } 0<\lambda<t-r. \tag{2.14}$$

We introduce new variables by

$$\alpha=\lambda+s\quad\text{and}\quad\beta=\lambda-s. \tag{2.15}$$

Then we have from (2.5)

$$z_{\nu,\mu}(\lambda,s;a)=(1+\alpha)^{1+\nu}(1+|\Psi_+(\alpha,\beta)|)^{1+\mu}\quad\text{for } s\ge 0,$$
$$z_{\nu,\mu}(\lambda,s;a)=(1+\beta)^{1+\nu}(1+|\Psi_-(\alpha,\beta)|)^{1+\mu}\quad\text{for } s\le 0,$$

where $\Psi_\pm(\alpha,\beta)$ are defined by

$$\Psi_+(\alpha,\beta)=\frac{1-a}{2}\alpha+\frac{1+a}{2}\beta \tag{2.16}$$

$$\Psi_-(\alpha,\beta)=\frac{1+a}{2}\alpha+\frac{1-a}{2}\beta. \tag{2.17}$$

Therefore, if we denote by $I_1^\pm$ and $I_2^\pm$ the integrals over $\pm s\ge 0$ of I_1 and I_2, respectively, then we have

$$I_1^+(r,t)=\tfrac{1}{2}\chi(t)\int_{|t-r|}^{t+r}(1+\alpha)^{-1-\nu}d\alpha\int_{r-t}^{\alpha}(1+|\Psi_+(\alpha,\beta)|)^{-1-\mu}K_1d\beta, \tag{2.18}$$

$$I_1^-(r,t) = \tfrac{1}{2}\int_{t-r}^{t+r} d\alpha \int_{\alpha\vee|r-t|}^{\infty} (1+\beta)^{-1-\nu}(1+|\Psi_-(\alpha,\beta)|)^{-1-\mu} K_1 d\beta, \quad (2.19)$$

$$I_2^+(r,t) = \tfrac{1}{2}\chi(t-r)\int_0^{t-r} (1+\alpha)^{-1-\nu} d\alpha \int_{-\alpha}^{\alpha} (1+|\Psi_+(\alpha,\beta)|)^{-1-\mu} K_2 d\beta, \quad (2.20)$$

$$I_2^-(r,t) = \tfrac{1}{2}\int_{-\infty}^{t-r} d\alpha \int_{|\alpha|}^{\infty} (1+\beta)^{-1-\nu}(1+|\Psi_-(\alpha,\beta)|)^{-1-\mu} K_2 d\beta, \quad (2.21)$$

where $\chi(t) = 1$ for $t > 0$, $\chi(t) = 0$ for $t \le 0$, and we will use the following notations in what follows repeatedly:

$$a \vee b = \max\{a,b\}, \quad [c]_+ = c \vee 0, \quad A^{[0]+} = 1 + \log A, \quad (2.22)$$

for a, b, $c \in \mathbf{R}$ with $c \ne 0$ and $A \ge 1$. Moreover we divide I_2^- into two integrals as follows. $I_2^- = J_1 + J_2$, where

$$J_1(r,t) = \tfrac{1}{2}\chi(t-r)\int_0^{t-r} (1+\beta)^{-1-\nu} d\beta \int_{-\beta}^{\beta} (1+|\Psi_-(\alpha,\beta)|)^{-1-\mu} K_2 d\alpha \quad (2.23)$$

$$J_2(r,t) = \tfrac{1}{2}\int_{|r-t|}^{\infty} (1+\beta)^{-1-\nu} d\beta \int_{-\beta}^{t-r} (1+|\Psi_-(\alpha,\beta)|)^{-1-\mu} K_2 d\alpha. \quad (2.24)$$

Then we have

$$I_1 = I_1^+ + I_1^-, \quad I_2 = I_2^+ + J_1 + J_2. \quad (2.25)$$

The following estimates for K_1 and K_2 are part of Lemma 4.1 in [8].

Lemma 2.2. *Let $t - r < \alpha < t + r$ and $\beta > r - t$ with* (2.15)*. Then it holds that*

$$K_1(\lambda, r, t-s) \le \frac{\sqrt{\alpha}}{\sqrt{\beta+r+t}\sqrt{\alpha+r-t}} \quad \text{for } \alpha \ge \beta, \quad (2.26)$$

$$K_1(\lambda, r, t-s) \le \frac{\sqrt{\beta}}{\sqrt{\beta+r+t}\sqrt{\alpha+r-t}} \quad \text{for } \alpha \le \beta \quad (2.27)$$

$$K_1(\lambda, r, t-s) \le \frac{1}{\sqrt{\alpha+r-t}}. \quad (2.28)$$

Let $-\beta < \alpha < t - r$. Then it holds that

$$K_2(\lambda, r, t-s) \le \frac{\sqrt{\alpha}}{\sqrt{t-r-\alpha}\sqrt{t+r+\beta}} \quad \text{for } \alpha \ge \beta, \quad (2.29)$$

$$K_2(\lambda, r, t-s) \le \frac{\sqrt{\beta}}{\sqrt{t-r-\alpha}\sqrt{t+r+\beta}} \quad \text{for } \alpha \le \beta \quad (2.30)$$

$$K_2(\lambda, r, t-s) \leq \frac{1}{\sqrt{t-r-\alpha}}. \tag{2.31}$$

The following elementary inequalities are the same ones as in Lemma 3.3 of [8].

Lemma 2.3. *Let κ and γ be real numbers such that $\kappa > 0$, $0 \leq \gamma < 1$ and $\kappa + \gamma > 1$. Then we have*

$$\int_{|b|}^{\infty} (1+\sigma)^{-\kappa}(d+\sigma)^{-\gamma} d\sigma \leq C(1+|b|)^{-\kappa-\gamma+1} \quad \textit{for } d \geq -|b|. \tag{2.32}$$

Moreover, we have

$$\int_{b}^{\infty} (1+|\sigma|)^{-\kappa}(d+\sigma)^{-\gamma} d\sigma \leq C(1+|d|)^{-\gamma}(1+|d|)^{[1-\kappa]_+} \quad \textit{for } d \geq -b, \tag{2.33}$$

or equivalently,

$$\int_{-\infty}^{b} (1+|\sigma|)^{-\kappa}(d-\sigma)^{-\gamma} d\sigma \leq C(1+|d|)^{-\gamma}(1+|d|)^{[1-\kappa]_+} \quad \textit{for } d \geq b, \tag{2.34}$$

Here C are constants depending only on κ and γ.

From (2.33) and (2.34) we easily have the following.

Corollary 2.1. *Let $\mu > -1/2$. Then we have*

$$\begin{aligned} &\int_{t-r}^{\infty} (1+|\Psi_-(\alpha,\beta)|)^{-1-\mu}(\alpha+r-t)^{-\frac{1}{2}} d\alpha \\ &\leq C(1+|\Psi_-(t-r,\beta)|)^{-\frac{1}{2}}(1+|\Psi_-(t-r,\beta)|)^{[-\mu]_+} \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} &\int_{-\infty}^{t-r} (1+|\Psi_-(\alpha,\beta)|)^{-1-\mu}(t-r-\alpha)^{-\frac{1}{2}} d\alpha \\ &\leq C(1+|\Psi_-(t-r,\beta)|)^{-\frac{1}{2}}(1+|\Psi_-(t-r,\beta)|)^{[-\mu]_+} \end{aligned} \tag{2.36}$$

for $r > 0$, $t \in \mathbf{R}$ and $\beta \geq 0$. Moreover,

$$\int_{-b}^{\infty} (1+|\Psi_+(\alpha,\beta)|)^{-1-\mu}(t+r+\beta)^{-\frac{1}{2}} d\beta \leq C(1+r+t)^{-\frac{1}{2}} \tag{2.37}$$

for $r > 0$, $t > 0$, $0 \leq \alpha \leq r+t$ and $b \leq r+t$, provided $\mu > 0$. Here C are constants depending only on μ and a. Besides, $\Psi_\pm$ are given by (2.16) and (2.17).

Proof. First consider (2.35). By I we denote the integral. Changing the variable α by $\sigma = \Psi_-(\alpha, \beta)$, we have

$$I = \left(\frac{2}{1+a}\right)^{\frac{1}{2}} \int_{\Psi_-(t-r,\beta)}^{\infty} (1+|\sigma|)^{-1-\mu}(\sigma - \Psi_-(t-r,\beta))^{-\frac{1}{2}} d\sigma.$$

Therefore, we get (2.35), making use of (2.33). Analogously we obtain (2.36) by (2.34) with $b = d = \Psi_-(t-r,\beta)$.

Next consider (2.37). By I we denote the integral. Then we have as above

$$I = \left(\frac{2}{1+a}\right)^{\frac{1}{2}} \int_{\Psi_+(\alpha,-b)}^{\infty} (1+|\sigma|)^{-1-\mu}(\sigma + \Psi_+(-\alpha,t+r))^{-\frac{1}{2}} d\sigma.$$

Since

$$\Psi_+(\alpha,-b) + \Psi_+(-\alpha,t+r) = \frac{1+a}{2}(t+r-b) \geq 0 \quad \text{for } b \leq t+r,$$

one can apply (2.33) to the integral on the right hand side. Hence, we get

$$I \leq C(1+\Psi_+(-\alpha,t+r))^{-\frac{1}{2}}, \quad \text{if } \mu > 0.$$

Moreover,

$$\Psi_+(-\alpha,r+t) \geq (\min\{1,a\})(r+t) \quad \text{for } 0 \leq \alpha \leq r+t.$$

Therefore, we obtain (2.37). □

We will also make use of another elementary iniquality.

Lemma 2.4. *Let κ and γ be positive numbers with $\kappa+\gamma > 1$. Then we have*

$$\int_{|b|}^{\infty} (1+\sigma)^{-\kappa}(1+|\sigma-d|)^{-\gamma} d\sigma \leq C(1+|b|)^{-\kappa}(1+|b|+|d|)^{[1-\gamma]_+} \quad (2.38)$$

for b, $d \in \mathbf{R}$. Moreover, if $\mu \leq 0$ and $\kappa+\gamma+\mu > 1$, we have

$$\begin{aligned} &\int_{|b|}^{\infty} (1+\sigma)^{-\kappa}(1+\sigma)^{[-\mu]_+}(1+|\sigma-d|)^{-\gamma} d\sigma \qquad (2.39) \\ \leq\ & C(1+|b|)^{-\kappa}(1+|b|+|d|)^{[-\mu]_+}(1+|b|+|d|)^{[1-\gamma]_+} \end{aligned}$$

for b, $d \in \mathbf{R}$. Here C are constants depending only on κ, γ and μ.

Proof. First we shall show (2.39). If we set

$$I_1 = \int_{|b|}^{\infty} \chi(|\sigma-d| - \frac{\sigma}{2})(1+\sigma)^{-\kappa}(1+\sigma)^{[-\mu]_+}(1+|\sigma-d|)^{-\gamma} d\sigma,$$

$$I_2 = \int_{|b|}^{\infty} (1-\chi(|\sigma-d| - \frac{\sigma}{2}))(1+\sigma)^{-\kappa}(1+\sigma)^{[-\mu]_+}(1+|\sigma-d|)^{-\gamma} d\sigma,$$

we see that the left hand side is estimated by $I_1 + I_2$, where $\chi(t) = 1$ for $t > 0$, $\chi(t) = 0$ for $t \leq 0$. When $\mu < 0$, it is easy to see that

$$I_1 \leq C \int_{|b|}^{\infty} (1+\sigma)^{-\kappa-\gamma-\mu} d\sigma \leq C(1+|b|)^{-\kappa-\gamma+1}(1+|b|)^{[-\mu]_+}. \tag{2.40}$$

If $\mu = 0$, then I_1 is estimated by

$$C \int_{|b|}^{\infty} (1+\sigma)^{-\kappa-\gamma}(1+\log(1+\sigma))d\sigma \leq C(1+|b|)^{-\kappa-\gamma+1}(1+\log(1+|b|)), \tag{2.41}$$

by integration by parts. Since $|\sigma - d| \leq \sigma/2$ is equivalent to $2d/3 \leq \sigma \leq 2d$, we see that $I_2 = 0$ if $d \leq 0$ and that for $d > 0$

$$\begin{aligned} I_2 &\leq C(1+|b|)^{-\kappa}(1+|d|)^{[-\mu]_+} \int_{\frac{2}{3}d}^{2d} (1+|\sigma-d|)^{-\gamma} d\sigma \\ &\leq C(1+|b|)^{-\kappa}(1+|d|)^{[-\mu]_+}(1+|d|)^{[1-\gamma]_+}, \end{aligned}$$

which gives (2.39) together with (2.40) and (2.41).

Analogously to the proof of (2.39) for the case when $\mu < 0$, we can prove (2.38). The proof is complete. □

We are now in a position to estimate the integrals in (2.25). The basic estimate (1.15) for $t \leq 0$ is implied by the following.

Proposition 2.1. *Let $\nu > 0$ and $\mu > -1/2$. Then we have*

$$I_1^-(r,t) + J_2(r,t) \leq C(1+|r-t|)^{-\frac{1}{2}-\nu}(1+|r-t|)^{[-\mu]_+} \tag{2.42}$$

for $r > 0$ and $t \in \mathbf{R}$.

Proof. We first claim that

$$I_1^- + J_2 \leq C \int_{|r-t|}^{\infty} (1+\beta)^{-1-\nu}(1+|\Psi_-(t-r,\beta)|)^{-\frac{1}{2}}(1+|\Psi_-(t-r,\beta)|)^{[-\mu]_+} d\beta, \tag{2.43}$$

where $\Psi_-(\alpha,\beta)$ is given by (2.17). Indeed, it follows from (2.19) and (2.28) that

$$I_1^- \leq \frac{1}{2} \int_{|r-t|}^{\infty} (1+\beta)^{-1-\nu} d\beta \int_{t-r}^{\infty} (1+|\Psi_-(\alpha,\beta)|)^{-1-\mu}(\alpha+r-t)^{-\frac{1}{2}} d\alpha.$$

Hence, we have (2.43) for I_1^-, making use of (2.35). Analogously we get the estimate for J_2 from (2.24) and (2.31), making use of (2.36) instead of (2.35).

If $a = 1$, we have easily (2.42) from (2.43), since (2.17) implies $\Psi_-(t-r,\beta) = t - r$. On the contrary, suppose that $a \neq 1$. Then we set for convenience

$$d(r,t) = \frac{1+a}{1-a}(r-t), \tag{2.44}$$

so that

$$|\Psi_-(t-r,\beta)| = \frac{|1-a|}{2}|\beta - d(r,t)|. \tag{2.45}$$

Since $d(r,t)$ is equivalent to $|r-t|$, we get by (2.43)

$$I_1^- + J_2 \le C\int_{|r-t|}^{\infty}(1+\beta)^{-1-\nu}(1+\beta)^{[-\mu]_+}(1+|\beta-d(r,t)|)^{-\frac{1}{2}}d\beta.$$

Since $\mu > -1/2$, making use of Lemma 2.4 with $b = r-t$ and $d = d(r,t)$, we arraive at (2.42). This completes the proof. □

In what follows we shall deal with the case where

$$\nu > 0, \quad \mu > 0 \quad \text{and} \quad t > 0. \tag{2.46}$$

First we shall prove the following.

Proposition 2.2. *Suppose that* (2.46) *is satisfied. Then we have*

$$I_1^+(r,t) + I_1^-(r,t) + J_2(r,t) \le C(1+r+t)^{-\frac{1}{2}}(1+|r-t|)^{-\nu}. \tag{2.47}$$

Proof. First consider I_1^+. It follows from (2.18) and (2.26) that

$$I_1^+ \le \tfrac{1}{2}\int_{|t-r|}^{t+r}(1+\alpha)^{-\frac{1}{2}-\nu}(\alpha+r-t)^{-\frac{1}{2}}d\alpha\int_{r-t}^{\infty}(1+|\Psi_+(\alpha,\beta)|)^{-1-\mu}(r+t+\beta)^{-\frac{1}{2}}d\beta.$$

Making use of (2.37) with $b = t-r$, we have

$$(1+r+t)^{\frac{1}{2}}I_1^+ \le C\int_{|t-r|}^{\infty}(1+\alpha)^{-\frac{1}{2}-\nu}(\alpha+r-t)^{-\frac{1}{2}}d\alpha.$$

Therefore, we get (2.47) for I_1^+ by (2.32) with $b = d = r-t$.

Next we shall deal with I_1^- and J_2. If either $0 < r \le 1$ or $t \ge 2r$, then it follows from Proposition 2.1 with $\mu > 0$ that

$$I_1^- + J_2 \le C(1+r+t)^{-\frac{1}{2}-\nu}, \tag{2.48}$$

because $1+r+t \le C(1+|r-t|)$ for such (r,t). Hence, we assume in the following that

$$r \ge 1 \quad \text{and} \quad 0 < t \le 2r. \tag{2.49}$$

If $a = 1$, then (2.47) with $I_1^+ = 0$ follows from [8], Lemma 4.3. Therefore, we also suppose that $a \ne 1$. First we shall show that

$$I_1^- + J_2 \le C\int_{|r-t|}^{\infty}(1+\beta)^{-\frac{1}{2}-\nu}(\beta+r+t)^{-\frac{1}{2}}(1+|\Psi_-(t-r,\beta)|)^{-\frac{1}{2}}d\beta. \tag{2.50}$$

It follows from (2.19) and (2.27) that

$$I_1^- \le \tfrac{1}{2}\int_{|r-t|}^{\infty}(1+\beta)^{-\frac{1}{2}-\nu}(\beta+r+t)^{-\frac{1}{2}}d\beta\int_{t-r}^{\infty}(1+|\Psi_-(\alpha,\beta)|)^{-1-\mu}(\alpha+r-t)^{-\frac{1}{2}}d\alpha.$$

Since $\mu>0$, we have (2.50) for I_1^-, making use of (2.35). Analogusly we get the estimate for J_2 from (2.24) and (2.30), making use of (2.36) instead of (2.35). By virtue of (2.45) and (2.49), we see from (2.50) that

$$I_1^- + J_2 \le C(1+r+t)^{-\frac{1}{2}}\int_{|r-t|}^{\infty}(1+\beta)^{-\frac{1}{2}-\nu}(1+|\beta-d(r,t)|)^{-\frac{1}{2}}d\beta.$$

Making use of (2.38), we obtain (2.47) with $I_1^+=0$. The proof is complete. □

Finally we deal with I_2^+ and J_1 which are involved to integrals over regions near the origin.

Proposition 2.3. *Suppose that $\mu>0$, $\nu>0$ and $t>r$. Then we have*

$$I_2^+(r,t)+J_1(r,t) \le C(1+r+t)^{-\frac{1}{2}}(1+t-r)^{-\frac{1}{2}}(1+t-r)^{[\frac{1}{2}-\nu]_+}. \tag{2.51}$$

Proof. First consider I_2^+. It follows from (2.20) and (2.29) that

$$I_2^+ \le \tfrac{1}{2}\int_0^{t-r}(1+\alpha)^{-\frac{1}{2}-\nu}(t-r-\alpha)^{-\frac{1}{2}}d\alpha\int_{-\alpha}^{\infty}(1+|\Psi_+(\alpha,\beta)|)^{-1-\mu}(t+r+\beta)^{-\frac{1}{2}}d\beta.$$

Hence, by (2.37) we get

$$I_2^+ \le C(1+r+t)^{-\frac{1}{2}}\int_{-\infty}^{t-r}(1+|\alpha|)^{-\frac{1}{2}-\nu}(t-r-\alpha)^{-\frac{1}{2}}d\alpha.$$

Therefore, we obtain (2.51) for I_2^+, making use of (2.34).

Next consider J_1. It follows from (2.23) and (2.30) that

$$J_1 \le \tfrac{1}{2}\int_0^{t-r}(1+|\beta|)^{-\frac{1}{2}-\nu}(t+r+\beta)^{-\frac{1}{2}}d\beta\int_{-\beta}^{\beta}(1+|\Psi_-(\alpha,\beta)|)^{-1-\mu}(t-r-\alpha)^{-\frac{1}{2}}d\alpha.$$

Since

$$\int_{-\infty}^{\infty}(1+|\Psi_-(\alpha,\beta)|)^{-1-\mu}d\alpha = \frac{4}{(1+a)\mu}, \tag{2.52}$$

we have

$$J_1 \le C\int_0^{t-r}(1+|\beta|)^{-\frac{1}{2}-\nu}(t+r+\beta)^{-\frac{1}{2}}(t-r-\beta)^{-\frac{1}{2}}d\beta.$$

If $t+r\le 1$, then

$$J_1 \le C\int_0^{t-r}\beta^{-\frac{1}{2}}(t-r-\beta)^{-\frac{1}{2}}d\beta = C\pi,$$

which yields (2.51) for J_1. If $t+r \geq 1$, we have

$$(1+r+t)^{\frac{1}{2}} J_1 \leq C \int_{-\infty}^{t-r} (1+|\beta|)^{-\frac{1}{2}-\nu} (t-r-\beta)^{-\frac{1}{2}} d\beta.$$

Therefore, we obtain (2.51) for J_1, making use of (2.35). The proof is complete. $\square$

Case 2: $\mathbf{n=3}$. If we set

$$I(r,t) = \frac{1}{2r} \int_{-\infty}^{t} ds \int_{|t-s-r|}^{t-s+r} \frac{1}{z_{\nu,\mu}(\lambda, s; a)} d\lambda, \tag{2.53}$$

it follows from (1.14), (2.6) and (2.7) with $n=3$ and $\rho = t-s$ that

$$|L_1(F)(x,t)| \leq M_{\nu,\mu}(F,a) \times I(r,t). \tag{2.54}$$

By $I^{\pm}(r,t)$ we denote the integrals over $\pm s \geq 0$ of $I(r,t)$, respectively, so that $I(r,t) = I^+(r,t) + I^-(r,t)$. Then we have

$$I^+(r,t) = \frac{1}{4r} \chi(t) \int_{|t-r|}^{t+r} (1+\alpha)^{-1-\nu} d\alpha \int_{r-t}^{\alpha} (1+|\Psi_+(\alpha,\beta)|)^{-1-\mu} d\beta \tag{2.55}$$

$$I^-(r,t) = \frac{1}{4r} \int_{t-r}^{t+r} d\alpha \int_{\alpha \vee |r-t|}^{\infty} (1+\beta)^{-1-\nu} (1+|\Psi_-(\alpha,\beta)|)^{-1-\mu} d\beta, \tag{2.56}$$

where $\chi(t) = 1$ for $t > 0$, $\chi(t) = 0$ for $t \leq 0$ and $\Psi_{\pm}$ are given by (2.16) and (2.17).

The estimate (1.15) for $t \leq 0$ is implied by the following.

Proposition 2.4. *Let $\nu > 0$ and $\mu > -1$. Then we have*

$$I^+(r,t) + I^-(r,t) \leq C(1+|r-t|)^{-1-\nu}(1+|r-t|)^{[-\mu]_+} \tag{2.57}$$

for $r > 0$ and $t \in \mathbf{R}$, provided either $\mu > 0$ or $t \leq 0$.

Proof. First consider I_+. We may suppose $\mu > 0$ and $t > 0$. Then it follows from (2.16) that

$$\int_{-\infty}^{\infty} (1+|\Psi_+(\alpha,\beta)|)^{-1-\mu} d\beta = \frac{4}{(1+a)\mu}. \tag{2.58}$$

Hence, by (2.55) we get

$$I^+ \leq C(1+|r-t|)^{-1-\nu} \frac{1}{r} \int_{t-r}^{t+r} d\alpha,$$

which yields (2.57) for I^+.

Next consider I^-. If $a = 1$, since $\Psi_-(\alpha, \beta) = \alpha$, it follows from (2.56) that

$$I^- \leq \frac{1}{4r}\int_{t-r}^{t+r}(1+|\alpha|)^{-1-\mu}d\alpha\int_{|r-t|}^{\infty}(1+\beta)^{-1-\nu}d\beta.$$

Moreover, we see from the proof of [8], Lemma 4.4 that

$$\frac{1}{r}\int_{t-r}^{t+r}(1+|\alpha|)^{-1-\mu}d\alpha \leq C(1+r+|t|)^{-1}(1+r+|t|)^{[-\mu]_+} \tag{2.59}$$

for $r > 0$, $t \in \mathbf{R}$ and $\mu > -1$. Therefore, we obtain (2.57) for I^-.

In what follows we suppose that $a \neq 1$. Set $d(\alpha) = \frac{a+1}{a-1}\alpha$, so that

$$|\Psi_-(\alpha, \beta)| = \frac{|1-a|}{2}|\beta - d(\alpha)|. \tag{2.60}$$

Then it follows from (2.56) that

$$I^- \leq C\frac{1}{r}\int_{t-r}^{t+r}d\alpha\int_{|r-t|}^{\infty}(1+\beta)^{-1-\nu}(1+|\beta-d(\alpha)|)^{-1-\mu}d\beta. \tag{2.61}$$

If $\mu > 0$, we easily obtain (2.57) for I^-.

Suppose that $-1 < \mu \leq 0$ and $t \leq 0$, so that $|r - t| = r + |t|$. Then we have $|d(\alpha)| \leq C|r-t|$ for $t - r \leq \alpha \leq t + r$. Therefore, for such α, making use of (2.38), we get

$$\int_{|r-t|}^{\infty}(1+\beta)^{-1-\nu}(1+|\beta-d(\alpha)|)^{-1-\mu}d\beta \leq C(1+|r-t|)^{-1-\nu}(1+|r-t|)^{[-\mu]_+}$$

Therefore, by (2.61) we obtain (2.57) for I^-. The proof is complete. $\square$

Finally we shall prove the following.

Proposition 2.5. *Let $\nu > 0$ and $\mu > 0$. Suppsoe that* (2.49) *is satisfied. Then we have*

$$I^+(r,t) + I^-(r,t) \leq C(1+r+t)^{-1}(1+|r-t|)^{-\nu}. \tag{2.62}$$

Proof. First consider I^+. Let $t > 0$. Then it follows from (2.55) and (2.58) that

$$I^+ \leq C\frac{1}{r}\int_{|r-t|}^{\infty}(1+\alpha)^{-1-\nu}d\alpha.$$

Hence, by (2.49) we easily obtain (2.62) for I^+.

Next consider I^-. It follows from (2.56) and (2.52) that

$$I^- \leq C\frac{1}{r}\int_{|r-t|}^{\infty}(1+\beta)^{-1-\nu}d\beta,$$

which gives (2.62) for I^- as above. The proof is complete. $\square$

End of proof of Theorem 1.2. Case 1: $\mathbf{n=2}$. If $t \le 0$ and $\mu > -1/2$, we have (1.15) with $c = 1$ from (2.9), (2.10), (2.25) and Proposition 2.1. Let (2.46) hold. Then we obtain the estimate from Propositions 2.2 and 2.3.

Case 2: $\mathbf{n=3}$. If $t \le 0$ and $\mu > -1$, we have (1.15) with $c = 1$ from (2.54) and Proposition 2.4. If (2.46) holds, we obtain the estimate from Propositions 2.4 and 2.5, because the former implies (1.15) with $c = 1$ when $0 < r \le 1$ or $t \ge 2r$. Thus we have proved Theorem 1.2. □

End of proof of Theorem 1.1. Since $L_c^+(x,t)$ is part of $L_c(x,t)$, we easily see that Theorem 1.1 holds. □

3. An Application

As we have mentioned in section 1, we shall consider the system (1.1) as an application of Theorems 1.1 and 1.2. Since one can deal with the general case analogously to the case of $m = 2$, we restrict ourselves to the following system:

$$\partial_t^2 u - c_1^2 \Delta u = F(u,v) \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{3.1}$$

$$\partial_t^2 v - c_2^2 \Delta v = G(u,v) \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{3.2}$$

where $n = 2$ or $n = 3$ and c_1, c_2 are positive constants. We divied the argument into three cases, according to the nonlinearity $F(u,v)$ and $G(u,v)$. But in each case, we have to asume that F and G have at least thier first order derivatives.

Case 1 (Weakly coupled case): Let the function $F(u,v)$ (resp. $G(u,v)$) depends only on v (resp. u). Namely, we consider

$$\partial_t^2 u - c_1^2 \Delta u = F(v) \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{3.3}$$

$$\partial_t^2 v - c_2^2 \Delta v = G(u) \quad \text{in } \mathbf{R}^n \times \mathbf{R}. \tag{3.4}$$

We suppose that $F(v) \in C^2(\mathbf{R})$ and $G(u) \in C^2(\mathbf{R})$ satisfy

$$F(0) = F^{'}(0) = F^{''}(0) = 0, \quad G(0) = G^{'}(0) = G^{''}(0) = 0, \tag{3.5}$$

and that there are $p > 2$, $q > 2$ and $A > 0$ such that for $|u_i| \le 2$, $|v_i| \le 2$ $(i = 1, 2)$

$$|F^{''}(v_1) - F^{''}(v_2)| \le \begin{cases} Ap(p-1)|v_1 - v_2|^{p-2} & \text{if} \quad 2 < p \le 3, \\ Ap(p-1)|v_1 - v_2|(|v_1| + |v_2|)^{p-3} & \text{if} \quad p > 3, \end{cases} \tag{3.6}$$

$$|G^{''}(u_1) - G^{''}(u_2)| \le \begin{cases} Aq(q-1)|u_1 - u_2|^{q-2} & \text{if} \quad 2 < q \le 3, \\ Aq(q-1)|u_1 - u_2|(|u_1| + |u_2|)^{q-3} & \text{if} \quad q > 3. \end{cases} \tag{3.7}$$

Remark. Typical examples of F and G are

$$F(v)=|v|^{p-1}v \quad \text{or} \quad F(v)=|v|^p, \tag{3.8}$$

$$G(u)=|u|^{q-1}u \quad \text{or} \quad G(u)=|u|^q. \tag{3.9}$$

For convenience, we set

$$p^*=\frac{n-1}{2}p-\frac{n+1}{2}, \quad q^*=\frac{n-1}{2}q-\frac{n+1}{2} \tag{3.10}$$

and

$$\alpha=pq^*-1, \quad \beta=qp^*-1, \quad \Gamma=\alpha+p\beta. \tag{3.11}$$

When $c_1=c_2$, D. Del Santo, V. Georgiev and E. Mitidieri [4] proved, among other things, that there is a global solution of the Cauchy problem (3.3)–(3.4) in $\mathbf{R}^n\times(0,\infty)$ with the data given at $t=0$ as

$$u(x,0)=f_1(x), \quad \partial_t u(x,0)=g_1(x), \tag{3.12}$$

$$v(x,0)=f_2(x), \quad \partial_t v(x,0)=g_2(x), \tag{3.13}$$

provided that the data f_j, g_j are sufficiently small in a suitable sense and are compactly supported, and that

$$0<p^*\le q^*, \quad \text{i.e.,} \quad (n+1)/(n-1)<p\le q, \tag{3.14}$$

$$\Gamma=\Gamma(p,q,n)>0. \tag{3.15}$$

However, in order to study the asymptotic behavior, we need to solve the Cauchy problrm with the data given at $t=-\infty$. Therefore, the assumption that the data have compact support is not adequet for the purpouse. Having this in mind, we assume that $f_j\in C^3(\mathbf{R}^n)$ and $g_j\in C^2(\mathbf{R}^n)$ $(j=1,2)$ satisfy

$$|||(f_1,g_1)|||_\nu+|||(f_2,g_2)|||_\kappa\le\varepsilon \tag{3.16}$$

for positive numbers ν, κ and ε. Here we have set

$$\begin{aligned}|||(f,g)|||_\nu=\|\langle\cdot\rangle^{\frac{n-1}{2}+\nu}f\|_{L^\infty(\mathbf{R}^n)}&+\sum_{1\le|\alpha|\le3}\|\langle\cdot\rangle^{\frac{n+1}{2}+\nu}\partial_x^\alpha f\|_{L^\infty(\mathbf{R}^n)} \\ &+\sum_{0\le|\alpha|\le2}\|\langle\cdot\rangle^{\frac{n+1}{2}+\nu}\partial_x^\alpha g\|_{L^\infty(\mathbf{R}^n)}.\end{aligned} \tag{3.17}$$

For the initial data satisfying (3.16), the existence theorem is still true even for the case where c_1 and c_2 are arbitrary positive constants. Morever, we are able to show that "small data nonlinear scattering" happens. The latter is an extention of the results in the previous paper [8], where we assumed $c_1=c_2=1$.

In order to state the result more precisely, we shall introduce some notations. For positive numbers ν and κ we set

$$X_{\nu,\kappa} = \{(u,v) \in C(\mathbf{R}^n \times \mathbf{R}) \times C(\mathbf{R}^n \times \mathbf{R}) : \ \|u\|_{\nu,1} + \|v\|_{\kappa,2} < +\infty\}, \tag{3.18}$$

where the norms $\|w\|_{\nu,i}$ $(i = 1, 2)$ are defined by

$$\|w\|_{\nu,i} = \sup_{(x,t)\in\mathbf{R}^n\times\mathbf{R}} \{|w(x,t)|(1 + r + |t|)^{\frac{n-1}{2}}/\Phi_n(r, c_i|t|;\nu)\} \tag{3.19}$$

with $r = |x|$, where $\Phi_n(r,t;\nu)$ is defined by (1.6). In addition, we set for $\delta > 0$

$$X_{\nu,\kappa}(\delta) = \{(u,v) \in X_{\nu,\kappa} : \ \|u\|_{\nu,1} + \|v\|_{\kappa,2} \le \delta\}. \tag{3.20}$$

In the same manner, we define $X^+_{\nu,\kappa}$, $\|w\|^+_{\nu,i}$ and $X^+_{\nu,\kappa}(\delta)$ by replacing $\mathbf{R}^n \times \mathbf{R}$ by $\mathbf{R}^n \times [0,\infty)$, which are used for the Cauchy problem.

Moreover, let us denote by $u^-(x,t)$ and $v^-(x,t)$, respectively, the solutions of the homogeneous wave equations

$$\partial_t^2 u - c_1^2 \Delta u = 0 \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{3.21}$$

$$\partial_t^2 v - c_2^2 \Delta v = 0 \quad \text{in } \mathbf{R}^n \times \mathbf{R} \tag{3.22}$$

satisfying initial conditions (3.12), (3.13).

If $f_j \in C^3(\mathbf{R}^n)$ and $g_j \in C^2(\mathbf{R}^n)$ $(j = 1, 2)$ satisfy (3.16), then there is a positive constant $C_0 = C_0(c_1, c_2, \nu, \kappa, n)$ such that

$$(u^-, v^-) \in X_{\nu,\kappa}(C_0\varepsilon), \quad (\partial_x^\alpha u^-, \partial_x^\alpha v^-) \in X_{\nu,\kappa} \quad \text{for } |\alpha| \le 2. \tag{3.23}$$

For the proof, see Lemma 2 in [13] for 3-dimensional case and Proposition 1.1 in [12] (or Proposition 2.1 in [11]) for 2-dimensional case. (See also those proofs).

To apply the basic estimate (1.15), we take ν and κ as follows. (See also Lemma 4.1 below).

Lemma 3.1. *If* (3.14) *and* (3.15) *hold, then there are ν and κ satisfying*

$$0 < \nu \le p^*, \tag{3.24}$$

$$0 < \kappa \le q^*, \tag{3.25}$$

$$p^* - \nu + p\kappa > 1, \quad q^* - \kappa + q\nu > 1. \tag{3.26}$$

Moreover, when $n = 2$, we can choose them so that

$$\nu < 1/2, \quad \kappa < 1/2. \tag{3.27}$$

Proof. To find κ satisfying (3.25) and (3.26) for some ν, we need to assure that

$$(1-p^*+\nu)/p < q^*+q\nu-1, \quad (1-p^*+\nu)/p < q^*, \quad 0 < q^*+q\nu-1.$$

Equivalently,

$$\nu > (1-p^*-pq^*+p)/(pq-1), \quad \nu < pq^*+p^*-1, \quad \nu > (1-q^*)/q. \tag{3.28}$$

Note that $(1-q^*)/q \leq 0$ if $q^* \geq 1$, and

$$\frac{1-q^*}{q} - \frac{1-p^*-pq^*+p}{pq-1} = \frac{q^*-1+q(p^*-1)}{q(pq-1)} \leq 0$$

if $q^* \leq 1$. Therefore, to take ν satisfying (3.28) together with (3.24), it suffices to assure

$$\begin{aligned}&(1-p^*-pq^*+p)/(pq-1) < pq^*+p^*-1,\\ &(1-p^*-pq^*+p)/(pq-1) < p^*, \quad 0 < pq^*+p^*-1.\end{aligned}$$

Simplifying the above relations with the aid of (3.11), we get

$$\beta+q\alpha > 0, \quad \alpha+p\beta > 0, \quad \alpha+p^* > 0. \tag{3.29}$$

Notice that

$$\beta \leq \frac{\Gamma}{p+1} \leq \alpha \quad \text{for } 1 < p \leq q. \tag{3.30}$$

So $\beta+q\alpha \geq \alpha+p\beta = \Gamma$ for $1 < p \leq q$. Therefore, (3.15) implies (3.29).

In addition, to choose ν and κ so that (3.27) holds, we need

$$(1-p^*+\nu)/p < 1/2, \quad \text{i.e.,} \quad \nu < p^*+(p-2)/2,$$

hence,

$$(1-p^*-pq^*+p)/(pq-1) \;<\; p^*+(p-2)/2, \tag{3.31}$$

$$(1-p^*-pq^*+p)/(pq-1) \;<\; 1/2. \tag{3.32}$$

Since $(1-p^*-pq^*+p)/(pq-1) = -\Gamma/(pq-1)+p^*$, we easily have (3.31). While (3.32) is equivalent to $p^*+3q^*+2p^*q^* > 0$ when $n=2$, which follows from (3.14). Thus we finish the proof. □

We are now in a position to state the main results in this case.

Theorem 3.1. *Let $n=2$ or $n=3$. Suppose that (3.5), (3.6), (3.7), (3.14), (3.15) and (3.16) hold. Let ν and κ satisfy (3.24) through (3.26), and also (3.27) when $n=2$. Then there is a positive constant ε_0 (depending only on c_1, c_2, A, p, q, ν and κ) such that for any ε with $0 < \varepsilon \leq \varepsilon_0$, there exists uniquely a classical solution $(u,v) \in X^+_{\nu,\kappa}(2C_0\varepsilon)$ of the Cauchy problem (3.3)–(3.4) in $\mathbf{R}^n \times [0,\infty)$ with (3.12) and (3.13). Besides, we have $(\partial_x^\alpha u, \partial_x^\alpha v) \in X^+_{\nu,\kappa}$ $(|\alpha| = 1,2)$.*

Remark. The condition (3.15) seems to be optimal in the sense that there is a solution of the problem which blows up in finite time, even if the initial data are small enough, when $c_1 = c_2$ and (3.15) does not hold. (See [4], [3], [6], [5], [1], [9]). Moreover, when $c_1 \neq c_2$ and $n = 3$, if (3.15) does not hold, the above blow-up occurs. (See [10]).

Theorem 3.2. *We suppose the same assumptions as in Theorem* 3.1.

(A) *There is a positive number* $\varepsilon_0 = \varepsilon_0(c_1, c_2, A, p, q, \nu, \kappa)$ *such that for any* ε *with* $0 < \varepsilon \leq \varepsilon_0$*, there exists uniquely a classical solution* $(u, v) \in X_{\nu,\kappa}(2C_0\varepsilon)$ *of* (3.3)–(3.4) *verifying* $(\partial_x^\alpha u, \partial_x^\alpha v) \in X_{\nu,\kappa} \quad (|\alpha| = 1, 2)$,

$$|\!|\!|(u - u^-)(t)|\!|\!|_{e,1} \leq C\|v\|_{\kappa,2}^p (1 + |t|)^{-p^*} \left\{(1 + |t|)^{[1-2p\kappa]_+}\right\}^{\frac{1}{2}} \text{ for } t \leq 0 \tag{3.33}$$

$$|\!|\!|(v - v^-)(t)|\!|\!|_{e,2} \leq C\|u\|_{\nu,1}^q (1 + |t|)^{-q^*} \left\{(1 + |t|)^{[1-2q\nu]_+}\right\}^{\frac{1}{2}} \text{ for } t \leq 0, \tag{3.34}$$

where u^- *and* v^- *are respectively the solutions of the homogeneous wave equations* (3.21) *and* (3.22) *satisfying* (3.12) *and* (3.13)*. Moreover, for* $(x, t) \in \mathbf{R}^n \times \mathbf{R}$ *with* $r = |x|$ *and* $|\alpha| \leq 2$*, we have*

$$|\partial_x^\alpha (u(x, t) - u^-(x, t))| \leq C[v]_{\kappa,2}^p (1 + r + |t|)^{-\frac{n-1}{2}} \Phi_n(r, c_1 t; \nu) \tag{3.35}$$

$$|\partial_x^\alpha (v(x, t) - v^-(x, t))| \leq C[u]_{\nu,1}^q (1 + r + |t|)^{-\frac{n-1}{2}} \Phi_n(r, c_2 t; \kappa). \tag{3.36}$$

In particular, if $t \leq 0$*, we have*

$$|\partial_x^\alpha (u(x, t) - u^-(x, t))| \leq C[v]_{\kappa,2}^p (1 + r + |t|)^{-\frac{n-1}{2} - p^*} (1 + r + |t|)^{[1-p\kappa]_+} \tag{3.37}$$

$$|\partial_x^\alpha (v(x, t) - v^-(x, t))| \leq C[u]_{\nu,1}^q (1 + r + |t|)^{-\frac{n-1}{2} - q^*} (1 + r + |t|)^{[1-q\nu]_+} \tag{3.38}$$

for $|\alpha| \leq 2$*. Here we have set for* $i = 1, 2$

$$[u]_{\nu,i} = \max_{|\gamma| \leq 2} \|\partial_x^\gamma u\|_{\nu,i}, \tag{3.39}$$

and C *are constants depending only on* c_1, c_2, ν, κ, p, q *and* A.

(B) *Let* (u, v) *be as in the part* (*A*) *of the theorem. Then there exists uniquely a classical solution* $(u^+, v^+) \in X_{\nu,\kappa}$ *to the system of homogeneous wave equations* (3.21)–(3.22) *such that* $(\partial_x^\alpha u^+, \partial_x^\alpha v^+) \in X_{\nu,\kappa} \quad (|\alpha| = 1, 2)$,

$$|\!|\!|(u - u^+)(t)|\!|\!|_{e,1} \leq C\|v\|_{\kappa,2}^p (1 + t)^{-p^*} \left\{(1 + t)^{[1-2p\kappa]_+}\right\}^{\frac{1}{2}} \quad \text{for } t \geq 0 \tag{3.40}$$

$$|\!|\!|(v - v^+)(t)|\!|\!|_{e,2} \leq C\|u\|_{\nu,1}^q (1 + t)^{-q^*} \left\{(1 + t)^{[1-2q\nu]_+}\right\}^{\frac{1}{2}} \quad \text{for } t \geq 0. \tag{3.41}$$

Moreover, we have

$$|\partial_x^\alpha (u(x, t) - u^+(x, t))| \leq C[v]_{\kappa,2}^p (1 + r + |t|)^{-\frac{n-1}{2}} \Phi_n(r, -c_1 t; \nu) \tag{3.42}$$

$$|\partial_x^\alpha(v(x,t)-v^+(x,t))| \le C[u]_{\nu,1}^q(1+r+|t|)^{-\frac{n-1}{2}}\Phi_n(r,-c_2t;\kappa) \tag{3.43}$$

for $(x,t)\in\mathbf{R}^n\times\mathbf{R}$ *with* $r=|x|$ *and* $|\alpha|\le 2$. *In particular, if* $t\ge 0$, *we have*

$$|\partial_x^\alpha(u(x,t)-u^+(x,t))| \le C[v]_{\kappa,2}^p(1+r+t)^{-\frac{n-1}{2}-p^*}(1+r+t)^{[1-p\kappa]_+} \tag{3.44}$$

$$|\partial_x^\alpha(v(x,t)-v^+(x,t))| \le C[u]_{\nu,1}^q(1+r+t)^{-\frac{n-1}{2}-q^*}(1+r+t)^{[1-q\nu]_+}. \tag{3.45}$$

Remarks. 1) The existence of such ν and κ as in the theorem is guaranteed by Lemma 3.1. In particular, if we take ν as $\nu=p^*$, we have $\kappa>1/p$ from (3.26). Hence, we can drop the factors $\{(1+|t|)^{[1-2p\kappa]_+}\}^{\frac{1}{2}}$ and $(1+r+|t|)^{[1-p\kappa]_+}$ in (3.33), (3.37), (3.40) and (3.44).

2) If p and q satisfy

$$\beta=qp^*-1>0, \tag{3.46}$$

then $\alpha=pq^*-1>0$ by (3.30), hence, $p^*>1/q$, $q^*>1/p$. We see that for any ν and κ satisfying

$$1/q<\nu\le p^*, \quad 1/p<\kappa\le q^*, \tag{3.47}$$

we have (3.24) through (3.26). Therefore, choosing ν and κ veryfing (3.47), we can drop not only the same factors as in the item **1)** but also the factors $\{(1+|t|)^{[1-2q\nu]_+}\}^{\frac{1}{2}}$ and $(1+r+|t|)^{[1-q\nu]_+}$ in (3.34), (3.38), (3.41) and (3.45).

3) When $p\kappa\le 1$ or $q\nu\le 1$, the decay rate in (3.37) or (3.38) is better than the one in (3.35) or (3.36) respectively, thanks to (3.26).

4) Even in the case where $2p\kappa\le 1$ or $2q\nu\le 1$, the right hand side of (3.33) or (3.34) tends to zero as $t\to-\infty$, since (3.26) implies

$$-p^*+\tfrac{1}{2}-p\kappa<-\tfrac{1}{2}-\nu, \quad -q^*+\tfrac{1}{2}-q\nu<-\tfrac{1}{2}-\kappa. \tag{3.48}$$

Case 2 (Strogly coupled case): First we take the functions $F(u,v)$ and $G(u,v)$ as

$$F(u,v)=|v|^{p-1}u, \quad G(u,v)=|u|^{q-1}v. \tag{3.49}$$

Then we shall see that the discrepancy of the propagtion speeds c_1 and c_2 makes a difference from the case of common propagation speeds. Indeed, if $c_1\ne c_2$, then it suffices to assume (3.14) on p and q. On the other hand, when $c_1=c_2$, we need to assume additionally

$$qp^*-1+q^*-p^*>0, \quad \text{i.e.,} \quad \Gamma>pp^*-1 \tag{3.50}$$

so that we can choose ν and κ satisfying (3.24), (3.25), (3.27) and

$$p^*+(p-1)\kappa>1, \quad q^*+(q-1)\nu>1. \tag{3.51}$$

Note that it can be realized if (3.14), (3.50) and

$$pq^* - 1 + p^* - q^* > 0 \tag{3.52}$$

hold. Since (3.52) follows from (3.50), we can do this.

Theorem 3.3. *Let $n = 2$ or $n = 3$. If $c_1 \neq c_2$, then we assume that* (3.14) *and* (3.16) *hold, and we take positive numbers ν and κ such that*

$$\nu = \kappa, \quad p^* + (p-1)\nu > 1. \tag{3.53}$$

In addition, we choose ν satisfying also $\nu < 1/2$ when $n = 2$.

If $c_1 = c_2$, then we assume that (3.14), (3.50) *and* (3.16) *hold, and we take ν and κ to satisfy* (3.24), (3.25), (3.51), *and also* (3.27) *when $n = 2$.*

(A) *Then there is a positive constant ε_0 (depending only on c_1, c_2, p, q, ν and κ) such that for any ε with $0 < \varepsilon \leq \varepsilon_0$, there exists uniquely a C^1-solution $(u, v) \in X^+_{\nu,\kappa}(2C_0\varepsilon)$ of the Cauchy problem* (3.1)–(3.2) *with* (3.49) *in $\mathbf{R}^n \times [0, \infty)$ with* (3.12) *and* (3.13). *Besides, we have $(\partial_x u, \partial_x v) \in X^+_{\nu,\kappa}$.*

(B) *There is a positive number $\varepsilon_0 = \varepsilon_0(c_1, c_2, p, q, \nu, \kappa)$ such that for any ε with $0 < \varepsilon \leq \varepsilon_0$, there exists uniquely a C^1-solution $(u, v) \in X_{\nu,\kappa}(2C_0\varepsilon)$ of* (3.1)–(3.2) *with* (3.49) *verifying $(\partial_x u, \partial_x v) \in X_{\nu,\kappa}$, and* (3.33), (3.34) *with $\|v\|^p_{\kappa,2}$, $\|u\|^q_{\nu,1}$ replaced by*

$$\|u\|^{p_1}_{\nu,1}\|v\|^{p_2}_{\kappa,2}, \quad \|u\|^{q_1}_{\nu,1}\|v\|^{q_2}_{\kappa,2},$$

and $[1-2p\kappa]_+$, $[1-2q\nu]_+$ replaced by $[1-2p_1\nu-2p_2\kappa]_+$, $[1-2q_1\nu-2q_2\kappa]_+$. Here $p_1 = 1$, $p_2 = p-1$, $q_1 = q-1$ and $q_2 = 1$.

Moreover, we have (3.35), (3.36) *with $[v]^p_{\kappa,2}$, $[u]^q_{\nu,1}$ replaced by*

$$\max_{|\gamma|\leq 1}\left\{\|\partial_x^\gamma u\|^p_{\nu,1} + \|\partial_x^\gamma v\|^p_{\kappa,2}\right\}, \quad \max_{|\gamma|\leq 1}\left\{\|\partial_x^\gamma u\|^q_{\nu,1} + \|\partial_x^\gamma v\|^q_{\kappa,2}\right\}.$$

In particular, if $t \leq 0$, we have (3.37), (3.38) *with $[v]^p_{\kappa,2}$, $[u]^q_{\nu,1}$ replaced as in the above, and also $[1-p\kappa]_+$, $[1-q\nu]_+$ replaced by $[1-p_1\nu-p_2\kappa]_+$, $[1-q_1\nu-q_2\kappa]_+$.*

(C) *Let (u, v) be as in the part (B) of the theorem. Then there exists uniquely a C^1 solution $(u^+, v^+) \in X_{\nu,\kappa}$ to the system of homogeneous wave equations* (3.21)–(3.22) *such that $(\partial_x u^+, \partial_x v^+) \in X_{\nu,\kappa}$. Moreover, we have* (3.40) *through* (3.45) *with the same modification as in the part (B).*

Secondly we take the functions $F(u, v)$ and $G(u, v)$ as

$$F(u, v) = |u|^{p-1}v, \quad G(u, v) = |v|^{q-1}u, \tag{3.54}$$

where p and q satisfy (3.14).

Lemma 3.2. *Suppose that* (3.14) *and*

$$pp^* - 1 + q^* - p^* > 0 \tag{3.55}$$

hold. Besides, when $n = 2$, we further assume

$$pp^* + (p-2)p^* > 1, \quad i.e., \quad p > 2 + \sqrt{2}. \tag{3.56}$$

Then we can choose ν and κ satisfying (3.24), (3.25) *and*

$$p^* + (p-2)\nu + \kappa > 1, \quad q^* + (q-2)\kappa + \nu > 1. \tag{3.57}$$

Moreover, when $n = 2$, we can choose them so that (3.27) *holds.*

Proof. We consider only the case of $n = 2$. First suppose $q^* \geq 1/2$. Then to find κ satisfying (3.25), (3.57) and (3.27), we need

$$1 - p^* - (p-2)\nu < 1/2, \quad (1 - q^* - \nu)/(q-2) < 1/2.$$

Since the conditions can be reduced to the former inequality, in order to take ν satisfying it together with (3.24) and (3.27), we need to have

$$(1/2 - p^*)/(p-2) < p^*, \quad (1/2 - p^*)/(p-2) < 1/2.$$

Equivalently, (3.56) and $p^* > 0$. Hence, we have done.

Next suppose $q^* < 1/2$. Then to find κ satisfying (3.25), (3.57) and (3.27), we need

$$1 - p^* - (p-2)\nu < q^*, \quad (1 - q^* - \nu)/(q-2) < q^*.$$

Since $p^* \leq q^* < 1/2$, to take ν satisfying them together with (3.24) and (3.27), it suffices to assure

$$(1 - p^* - q^*)/(p-2) < p^*, \quad 1 - q^* - (q-2)q^* < p^*.$$

Equivalently, (3.55) and $qq^* - 1 + p^* - q^* > 0$. But the foremer yields the latter, since $q \geq p$. This completes the proof. □

Theorem 3.4. *Let $n = 2$ or $n = 3$. If $c_1 \neq c_2$, then we assume that* (3.14) *and* (3.16) *hold, and we take positive numbers ν and κ such that* (3.53). *In addition, we choose ν satisfying also $\nu < 1/2$ when $n = 2$.*

If $c_1 = c_2$, then we assume that (3.14), (3.55) *and* (3.16) *hold. In addition, we assume* (3.56) *when $n = 2$. As for ν and κ, we take them so that they satisfy* (3.24), (3.25), (3.57), *and also* (3.27) *when $n = 2$.*

Then we have the same conclusion for (3.1)–(3.2) *with* (3.54) *as in Theorem* 3.3 *with $p_1 = p - 1$, $p_2 = 1$, $q_1 = 1$ and $q_2 = q - 1$.*

Case 3 (Intermediate case): We take the functions $F(u, v)$ and $G(u, v)$ as

$$F(u, v) = |v|^{p-1}u, \quad G(u) = |u|^q, \tag{3.58}$$

where we assume that p and q satisfy (3.14).

Lemma 3.3. *If we suppose that* (3.14) *and* (3.52) *hold, then we can choose* ν *satisfying* (3.25) *and*

$$p^* + (p-1)\nu > 1, \quad q^* + (q-1)\nu > 1. \tag{3.59}$$

Moreover, when $n = 2$*, we can choose it so that* $\nu < 1/2$ *holds.*

If we suppose that (3.14) *and*

$$\alpha + (p-1)\beta + p^* - q^* > 0 \tag{3.60}$$

hold, then we can choose ν *and* κ *satisfying* (3.24), (3.25) *and*

$$p^* + (p-1)\kappa > 1, \quad q^* - \kappa + q\nu > 1. \tag{3.61}$$

Moreover, when $n = 2$*, we can choose them so that* (3.27) *holds.*

Proof. We shall prove the second assertion for the case of $n = 2$. To find κ satisfying (3.25), (3.61) and (3.27), we need (3.60), (3.14) and

$$\nu > (1 - p^* - (p-1)(q^* - 1))/(pq - q), \quad \nu > (1 - q^*)/q. \tag{3.62}$$

Note that $(1 - q^*)/q \le 0$ if $q^* \ge 1$, and

$$\frac{1-q^*}{q} - \frac{1 - p^* - (p-1)(q^*-1)}{q(p-1)} = \frac{p^*-1}{q(p-1)} \le 0$$

if $q^* \le 1$. Therefore, to take ν satisfying (3.62) together with (3.24) and (3.27), we need

$$(1-p^*-(p-1)(q^*-1))/(pq-q) < p^*, \quad (1-p^*-(p-1)(q^*-1))/(pq-q) < 1/2. \tag{3.63}$$

Equivalently, $\alpha+(p-1)\beta+p^*-q^* > 0$, $p^*+2q^*+2p^*q^* > 0$. Since the former follows from (3.60) and the latter from (3.14), we finish the proof. □

Theorem 3.5. *Let* $n = 2$ *or* $n = 3$*. If* $c_1 \ne c_2$*, then we assume that* (3.14), (3.52) *and* (3.16) *hold, and we take positive numbers* ν *and* κ *such that* $\nu = \kappa$, (3.25) *and* (3.59)*. In addition, we choose* ν *satisfying also* $\nu < 1/2$ *when* $n = 2$*. If* $c_1 = c_2$*, then we assume that* (3.14), (3.60) *and* (3.16) *hold. As for* ν *and* κ*, we take them so that they satisfy* (3.24), (3.25), (3.61)*, and also* (3.27) *when* $n = 2$*. Then we have the same conclusion for* (3.1)–(3.2) *with* (3.58) *as in Theorem 3.3 with* $p_1 = 1$, $p_2 = p - 1$, $q_1 = q$ *and* $q_2 = 0$.

Remark. If $\beta \ge 0$, then (3.60) follows from (3.52). Therefore, the effect of the difference of the propagation speeds might appear only for the case of $\beta < 0$.

4. Proof of theorems in section 3

First we prove Theorem 3.2. As is well known, a solution (u, v) of the system (3.3)–(3.4) having the asymptotic behavior (3.33) and (3.34) is obtained by solving the following system of integral equations:

$$u(x,t) = u^-(x,t) + L_{c_1}(F(v))(x,t) \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{4.1}$$

$$v(x,t) = v^-(x,t) + L_{c_2}(G(u))(x,t) \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{4.2}$$

where $L_c(F)(x,t)$ is defined by (1.13) and (1.14).

In order to prove Theorem 3.2, the following lemma is crucial.

Lemma 4.1. *Assume that* (3.14) *and* (3.15) *hold. Let* $(u, v) \in X_{\nu,\kappa}$. *If* ν *and* κ *satisfy* (3.24), (3.25) *and the first inequality in* (3.26), *and also* (3.27) *when* $n = 2$, *then we have*

$$\|L_{c_1}(|v|^p)\|_{\nu,1} \le K_0 \|v\|_{\kappa,2}^p, \tag{4.3}$$

where K_0 *is a constant depending only on* c_1, c_2, p, q, ν *and* κ. *Moreober, if* ν *and* κ *satisfy* (3.24), (3.25) *and the second inequality in* (3.26), *and also* (3.27) *when* $n = 2$, *then we have*

$$\|L_{c_2}(|u|^q)\|_{\kappa,2} \le K_0 \|u\|_{\nu,1}^q. \tag{4.4}$$

Proof. We shall prove only (4.3), since the other can be analogously handled. We make use of Theorem 1.2 by taking $c = c_1$, $a = c_2$, $\mu > 0$ and $F = |v|^p$. Then we get

$$\|L_{c_1}(|v|^p)\|_{\nu,1} \le C M_{\nu,\mu}(|v|^p, c_2), \tag{4.5}$$

since $\Phi_n(r,t;\nu) \le \Phi_n(r,|t|;\nu)$, where $M_{\nu,\mu}(F,a)$ is given by (1.16) (recalling also (3.19)). It follows from (3.10) and (3.24) that

$$\begin{aligned} |v(y,s)|^p &\le \|v\|_{\kappa,2}^p (1+\lambda+|s|)^{-\frac{n+1}{2}-p^*}(1+|\lambda-c_2|s||)^{-p\kappa} \\ &\le C\|v\|_{\kappa,2}^p (1+\lambda+|s|)^{-\frac{n+1}{2}-\nu}(1+|\lambda-c_2|s||)^{-p^*+\nu-p\kappa} \end{aligned}$$

for $(y,s) \in \mathbf{R}^n \times \mathbf{R}$ with $\lambda = |y|$, where C is a constant depending only on c_2 and $\nu - p^*$. Here we have used the fact that $\kappa < 1/2$ in (3.27), when $n = 2$. Furthermore we see from the first inequality of (3.26) that one can take a positive number μ satisfying $1 + \mu - p^* + \nu - p\kappa \le 0$. Therefore, we obtain

$$M_{\nu,\mu}(|v|^p, c_2) \le C\|v\|_{\kappa,2}^p \tag{4.6}$$

for such μ. Now the desired estimate (4.3) follows immediately from (4.5) and (4.6), and the proof is complete. □

By Lemma 4.1 one can prove the part (A) of the theorem except the estimates (3.37) and (3.38), analogously to [8], Theorem 5.1, if we take the positive number ε_0 in such a way that $2C_0\varepsilon_0 \le 1$ and

$$ApK_0(4C_0\varepsilon_0)^{p-1} \le \tfrac{1}{2}, \quad AqK_0(4C_0\varepsilon_0)^{q-1} \le \tfrac{1}{2}.$$

Here C_0 and K_0 are the constants in (3.23) and Lemma 4.1 respectively. In particular, we can get a unique solution $(u, v) \in X_{\nu,\kappa}(2C_0\varepsilon)$ with $0 < \varepsilon \le \varepsilon_0$ of the system (4.1)–(4.2) such that $[u]_{\nu,1} + [v]_{\kappa,2} < \infty$. For completeness, we remark that

$$|u(x, t)| + |v(x, t)| \le 2 \quad \text{for } (x, t) \in \mathbf{R}^n \times \mathbf{R},$$

if $(u, v) \in X_{\nu,\kappa}(1)$. In fact, it follows from (1.6) that

$$\Phi_2(|x|, c_i|t|; 1/2) \le 2/\sqrt{e} \quad \text{for } (x, t) \in \mathbf{R}^2 \times \mathbf{R},\ c_i > 0,$$

hence, we see from (1.6) and (3.19) that

$$|u(x, t)| \le 2\|u\|_{\nu,i} \quad \text{for } (x, t) \in \mathbf{R}^n \times \mathbf{R},\ \nu > 0,\ c_i > 0,\ n = 2, 3.$$

Since $\|u\|_{\nu,1} + \|v\|_{\kappa,2} \le 1$, we thus obtain the desired estimate.

It also follows from the proof of Theorem 5.1 in [8] that a solution $(u, v) \in X_{\nu,\kappa}(1)$ of (3.3)–(3.4) verifying (3.33), (3.34) and $[u]_{\nu,1} + [v]_{\kappa,2} < \infty$ satisfies the system of integral equations (4.1)–(4.2). Therefore, we have only to show that (u, v) considered in the above satisfies the estimates (3.37) and (3.38).

First we deal with (3.37). It follows from (4.1) and Theorem 1.2 with $c = c_1$, $a = c_2$ and $F = A|v|^p$ that

$$|u(x, t) - u^-(x, t)| \le CAM_{\delta,\mu}(|v|^p, c_2)(1+r+|t|)^{-\frac{n-1}{2}-\delta}(1+r+|t|)^{[-\mu]_+} \quad (4.7)$$

for $t \le 0$ and $x \in \mathbf{R}^n$ with $r = |x|$, provided $\delta > 0$ and $\mu > -(n-1)/2$, since (3.5) and (3.6) yield

$$|F(v)| \le A|v|^p \quad \text{for } |v| \le 2.$$

If $n = 3$, we take $\delta = p^*$ and $\mu = p\kappa - 1$. Then analogously to the proof of (4.6) we have

$$M_{p^*,\ p\kappa-1}(|v|^p, c_2) \le C\|v\|^p_{\kappa,2}, \quad (4.8)$$

hence, we obtain (3.37) with $|\alpha| = 0$ by (4.7). If $n = 2$ and $p\kappa > 1/2$, we also get as above the same estimate.

Next suppose that $n = 2$ and $p\kappa \le 1/2$. Then we take μ in such a way as $-1/2 < \mu < 0$ and set $\delta = p^* + p\kappa - 1 - \mu$. Then we have $0 < \delta < p^*$. Indeed, $\delta - p^* = p\kappa - 1 - \mu < 0$ and $\delta > \nu - \mu > 0$, by $p\kappa \le 1/2$, $-1/2 < \mu < 0$ and (3.26). Therefore, we get similarly to (4.6)

$$M_{p^*+p\kappa-1-\mu,\ \mu}(|v|^p, c_2) \le C\|v\|^p_{\kappa,2}.$$

Now (4.7) gives

$$|u(x,t) - u^-(x,t)| \leq CA\|v\|^p_{\kappa,2}(1 + r + |t|)^{-\frac{1}{2}-p^*+1-p\kappa},$$

hence, (3.37) for $|\alpha| = 0$ holds. The case where $1 \leq |\alpha| \leq 2$ is also similar, because $\partial_x^\alpha L_c(F) = L_c(\partial_x^\alpha F)$ according to (1.13) and (1.14). Hence, (3.37) follows.

The proof of (3.38) is analogous to that of (3.37), if we use (3.7) instead of (3.6). So we omit the details. This completes the proof of the part (A) of Theorem 3.2.

Let (u, v) be the solution of the system (3.4)–(3.3) which is obtained by the part (A). Set

$$u^+(x,t) = u(x,t) - \widetilde{L}_{c_1}(F(v))(x,t) \quad \text{in } \mathbf{R}^n \times \mathbf{R},$$
$$v^+(x,t) = v(x,t) - \widetilde{L}_{c_2}(G(u))(x,t) \quad \text{in } \mathbf{R}^n \times \mathbf{R},$$

where $\widetilde{L}_c(F)(x,t)$ $(c > 0)$ is defined by

$$\widetilde{L}_c(F)(x,t) = \frac{1}{2\pi}\int_t^\infty ds \int_0^{s-t} \frac{\rho d\rho}{\sqrt{(s-t)^2 - \rho^2}} \int_{|\omega|=1} F(x + c\rho\omega, s)dS_\omega$$

for $(x,t) \in \mathbf{R}^2 \times \mathbf{R}$, and by

$$\widetilde{L}_c(F)(x,t) = \frac{1}{4\pi}\int_t^\infty (s-t)ds \int_{|\omega|=1} F(x + c(s-t)\omega, s)dS_\omega \tag{4.9}$$

for $(x,t) \in \mathbf{R}^3 \times \mathbf{R}$. Then, since it follows from (1.13) and (1.14) that

$$\widetilde{L}_c(F)(x,t) = L_c(\hat{F})(x,-t), \quad \text{with} \quad \hat{F}(x,t) = F(x,-t),$$

one can prove the part (B) of the theorem, by repeating exactly the same procedure as in the proof the part (A). In particular, we see that (u^+, v^+) is a solution to the system of homogeneous wave equations (3.21)–(3.22). Thus we have proved Theorem 3.2. $\square$

Next we prove Theorem 3.1.

The global solution of the Cauchy probelm is obtained from the integral equations (4.1)–(4.2) with $L_{c_1}(F(v))(x,t)$, $L_{c_2}(G(u))(x,t)$ replaced by $L^+_{c_1}(F(v))(x,t)$, $L^+_{c_2}(G(u))(x,t)$. By Theorem 1.1, one can derive a variant of Lemma 4.1 for $L^+_{c_1}(|v|^p)(x,t)$ and $L^+_{c_2}(|u|^q)(x,t)$. Then from the stanard argument we see that Theorem 3.1 holds. (See e.g. [7]).

The proof of Theorems 3.3 and 3.4 can be done by using the following lemma analogously to that of Theorems 3.1 and 3.2.

Lemma 4.2. *Let $(u, v) \in X_{\nu,\kappa}$ and let p_1, $p_2 \geq 1$. Put $p = p_1 + p_2$. If $c_1 \neq c_2$, then we assume that*

$$p^* > 0, \quad \nu = \kappa, \quad p^* + (p-1)\nu > 1. \tag{4.10}$$

In addition, we suppose that ν satisfies also $\nu < 1/2$ when $n = 2$.

If $c_1 = c_2$, then we assume that

$$0 < \nu \leq p^*, \quad p^* + (p_1 - 1)\nu + p_2\kappa > 1. \tag{4.11}$$

In addition, we suppose that ν and κ satisfy also $\nu < 1/2$, $\kappa < 1/2$ when $n = 2$. Then we have

$$\|L_{c_1}(|u|^{p_1}|v|^{p_2})\|_{\nu,1} \leq K_1 \|u\|_{\nu,1}^{p_1} \|v\|_{\kappa,2}^{p_2}, \tag{4.12}$$

where K_1 is a constant depending only on c_1, c_2, p_1, p_2, ν and κ.

Proof. We use Theorem 1.2 with $c = c_1$, $a = c_2$, $\mu > 0$ and $F = |u|^{p_1}|v|^{p_2}$. Then

$$\|L_{c_1}(|u|^{p_1}|v|^{p_2})\|_{\nu,1} \leq C M_{\nu,\mu}(|u|^{p_1}|v|^{p_2}, c_2). \tag{4.13}$$

When $c_1 = c_2$, analogously to the proof of (4.6) we obtain

$$M_{\nu,\mu}(|u|^{p_1}|v|^{p_2}, c_2) \leq C \|u\|_{\nu,1}^{p_1} \|v\|_{\kappa,2}^{p_2}, \tag{4.14}$$

by taking a positive number μ so that $1 + \mu - p^* - (p_1 - 1)\nu - p_2\kappa \leq 0$, hence, (4.12) holds.

Next suppose $c_1 \neq c_2$. We take a positive number μ satisfying $1 + \mu - p^* - (p-1)\nu \leq 0$ and set

$$F_i(y, s) = (1 + |y| + |s|)^{-\frac{n+1}{2} - \nu} (1 + ||y| - c_i|s||)^{-1-\mu}$$

for $i = 1, 2$. Then we have

$$|u(y, s)|^{p_1} |v(y, s)|^{p_2} \leq C \|u\|_{\nu,1}^{p_1} \|v\|_{\kappa,2}^{p_2} \, (F_1(y, s) + F_2(y, s))$$

for $(y, s) \in \mathbf{R}^n \times \mathbf{R}$, since $\nu = \kappa$ and p_1, $p_2 \geq 1$. Therefore,

$$\|L_{c_1}(|u|^{p_1}|v|^{p_2})\|_{\nu,1} \leq C \|u\|_{\nu,1}^{p_1} \|v\|_{\kappa,2}^{p_2} \, (\|L_{c_1}(F_1)\|_{\nu,1} + \|L_{c_1}(F_2)\|_{\nu,1}).$$

Since $M_{\nu,\mu}(F_i, c_i)$ is bounded, using Theorem 1.2 with $c = c_1$, $a = c_i$, $\mu > 0$ and $F = F_i$, we obtain (4.12) also in this case. The proof is complete. $\square$

Finally we prove Theorem 3.5. Using both Lemmas 4.1 and 4.2, we can carry out the proof of it, analogously to that of Theorems 3.1 and 3.2. This completes the proof.

REFERENCES

[1] R. Agemi, Y. Kurokawa and H. Takamura, Critical curve for p-q systems of nonlinear wave equations in three space dimensions, *J. Differential Equations.* **167** (2000), 87–133.

[2] F. Asakura, Existence of a global solution to a semi-linear wave equation with slowly decreasing initial data in three space dimensions, *Comm. Partial Differential Equations* **11** (1986), 1459–1487.

[3] D. Del Santo, Global existence and blow-up for a hyperbolic system in three space dimensions, *Rend. Istit. Math. Univ. Trieste* **26** (1997), 115-140.

[4] D. Del Santo, V. Georgiev and E. Mitidieri, Global existence of the solutions and formation of singularities for a class of hyperbolic systems, *in* "Geometric Optics and Related topics" (F. Colombini and N. Lerner ed.), Progress in Nonlinear Differential Equations and Their Applications, Vol. 32, 117–140, Birkhäuser, Boston, 1997.

[5] D. Del Santo and E. Mitidieri, Blow-up of solutions of a hyperbolic system: the critical case, *Differential Equations* **34(9)** (1998), 1157–1163.

[6] K. Deng, Nonexistence of global solutions of a nonlinear hyperbolic system, *Trans. Amer. Math. Soc.* **349** (1997), 1685–1696.

[7] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.* **28** (1979), 235–268.

[8] H. Kubo and K. Kubota, Asymptotic behavior of classical solutions to a system of semilinier wave equations in low space dimensions, to appear in *J. Math. Soc. Japan.*

[9] H. Kubo and M. Ohta, Critical blowup for systems of semilinear wave equations in low space dimensions, *J. Math. Anal. Appl.* **240** (1999), 340–360.

[10] H. Kubo and M. Ohta, Small data blowup for systems of semilinear wave equations with different propagation speeds in three space dimensions, *J. Differential Equations* **163** (2000), 457–492.

[11] K. Kubota, Existence of a global solutions to a semi-linear wave equation with initial data of non-compact support in low space dimensions, *Hokkaido Math. J.* **22** (1993), 123–180.

[12] K. Kubota and K. Mochizuki, On small data scattering for 2-dimensional semilinear wave equations, *Hokkaido Math. J.* **22** (1993), 79–97.

[13] H. Pecher, Scattering for semilinear wave equations with small data in three space dimensions, *Math. Z.* **198** (1988), 277–289.

[14] J. Schaeffer, Wave equation with positive nonlinearities, Ph. D. thesis, Indiana Univ. (1983)

[15] K. Tsutaya, Scattering theory for semilinear wave equations with small data in two space dimensions, *Trans. A.M.S.* **342** (1994), 595–618.