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Effective Maximum Principles for Spectral Methods

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Abstract. Many physical problems such as Allen-Cahn flows have natural maximum principles which yield strong point-wise control of the physical solutions in terms of the boundary data, the initial conditions and the operator coefficients. Sharp/strict maximum principles insomuch of fundamental importance for the continuous problem often do not persist under numerical discretization. A lot of past research concentrates on designing fine numerical schemes which preserves the sharp maximum principles especially for nonlinear problems. However these sharp principles not only sometimes introduce unwanted stringent conditions on the numerical schemes but also completely leaves many powerful frequency-based methods unattended and rarely analyzed directly in the sharp maximum norm topology. A prominent example is the spectral methods in the family of weighted residual methods.

In this work we introduce and develop a new framework of almost sharp maximum principles which allow the numerical solutions to deviate from the sharp bound by a controllable discretization error: we call them effective maximum principles. We showcase the analysis for the classical Fourier spectral methods including Fourier Galerkin and Fourier collocation in space with forward Euler in time or second order Strang splitting. The model equations include the Allen-Cahn equations with double well potential, the Burgers equation and the Navier-Stokes equations. We give a comprehensive proof of the effective maximum principles under very general parametric conditions.

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1 Introduction

In solving physical problems such as Allen-Cahn flows in interfacial dynamics, the maximum principle plays an important role since it gives strong point-wise control of the physical solutions in terms of the boundary data, the initial conditions and the operator coefficients. For practical numerical simulations, it is often the case that sharp/strict maximum principles for the continuous problem often do not persist under numerical discretization. A lot of past research is centered on designing fine numerical schemes which preserves the maximum in a sharp way especially for nonlinear problems. For linear parabolic equations, it is well known that central finite difference in space with backward Euler time stepping can preserve the sharp maximum principle (cf. Chapter 9 of [5] for a textbook analysis of 1D homogeneous heat equation). This is also the case if one employs lumped mass linear finite element in space using acute simplicial triangulation. Although preserving the sharp maximum principle is highly desirable for numerical simulations, these often introduce unwanted stringent conditions on the numerical schemes. Moreover it completely leaves out many powerful L^2-based methods unattended and rarely analyzed directly in the sharp maximum norm topology. In this respect a prominent example is the spectral methods in the family of weighted residual methods. In this work we introduce and develop a new framework of almost sharp maximum principles which allow the numerical solutions to deviate from the sharp bound by a controllable discretization error: we call them effective maximum principles. Our main models are Allen-Cahn equations in physical dimensions $d\leq 3$, but we also discuss related models such as Burgers equations, Navier-Stokes equations. All these will be discussed in this introduction.

We begin by considering the Allen-Cahn equation in physical dimensions $d=1,2,3$:

$$\begin{cases} \partial_t u=\nu^2\Delta u-f(u), & t>0,\\ u\big|_{t=0}=u_0, & \\ \text{periodic boundary conditions}, & \end{cases} \tag{1.1}$$

where u is a scalar function which typically represents the concentration of one of the two metallic components of the alloy. For simplicity we consider the periodic boundary condition and assume the function to have period 1 in each spatial coordinate axis. The parameter $\nu>0$ controls the interfacial width which is small compared with the system size under study. The nonlinear term has the usual double well

form:

$$f(u)=F'(u)=u^3-u,\quad F(u)=\frac{1}{4}(u^2-1)^2.$$

Introduce the energy functional

$$\mathcal{E}(u)=\int\Big(\frac{1}{2}\nu^2|\nabla u|^2+F(u)\Big)dx.$$

The system (1.1) can be regarded as a gradient flow of $\mathcal{E}(\cdot)$ in the L^2 metric. On the other hand if one changes the topology to H^{-1} then we obtain the usual Cahn-Hilliard system. Due to the gradient flow structure

$$u_t=-\frac{\delta\mathcal{E}}{\delta u},$$

one has the energy law

$$\frac{d}{dt}\mathcal{E}(u(t))=-\|\partial_t u\|_2^2=-\int|\nu^2\Delta u-f(u)|^2dx.$$

Thus for smooth solutions, we have the energy decay

$$\mathcal{E}(u(t))\leq\mathcal{E}(u(s)),\quad\forall 0\leq s\leq t<\infty.$$

Similar energy laws also exists for other phase field models such as the Cahn-Hilliard system. However for the Allen-Cahn system due to its particular structure one has an additional maximum principle which asserts that the L^∞ norm of the smooth solution is bounded by 1 if the initial data is bounded by 1. Verification of these two fundamental conservation laws are of pivotal role in designing robust and stable numerical schemes for the Allen-Cahn system.

We begin with an implicit-explicit Fourier Galerkin discretization of (1.1). For simplicity consider the system in one dimension, i.e., the torus $\mathbb{T}=[0,1)$. For a periodic function f with Fourier coefficients $\widehat{f}(k)$, we define its projection to the first N-modes ($N\geq 2$ is an integer) as:

$$(\Pi_N f)(x)=\sum_{|k|\leq N}\widehat{f}(k)e^{2\pi ik\cdot x}.$$

In yet other words Π_N is the projection to the space

$$S_N=\text{span}\{e^{2\pi ik\cdot x}:\ k\in\mathbb{Z}\text{ and }|k|\leq N\}.$$

A prototypical implicit-explicit Fourier spectral scheme has the form:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau}=\nu^2\partial_{xx}u^{n+1}-\Pi_N((u^n)^3-u^n), \quad n\geq 0, \\ u^0=\Pi_N u_0, \end{cases} \tag{1.2}$$

where $\tau>0$ is the size of the time step, and u^n denotes the numerical solution at the time step $t=n\tau$. Note that the linear part is treated implicitly whereas the nonlinear part is explicit which makes the practical computation very convenient. Thanks to the frequency projection the Fourier modes of u^n are all trapped in the window $[-N,N]$. In later sections we also consider the full collocation case and aliasing errors. On the other hand, the system (1.2) in some sense captures the essential difficulties of the numerical analysis for the Fourier Galerkin method.

As was already mentioned we are concerned with the L^∞ maximum principle for the approximation system (1.2). For this purpose it is convenient to recast it as

$$\begin{cases} u^{n+1}=T_{N,\tau\nu^2}\Big(f_\tau(u^n)\Big), \quad n\geq 0, \\ u^0=\Pi_N u_0, \end{cases} \tag{1.3}$$

where $T_{N,\tau\nu^2}=(\mathrm{Id}-\tau\nu^2\partial_{xx})^{-1}\Pi_N$, and

$$f_\tau(z)=(1+\tau)z-\tau z^3.$$

Similar reformulation can also be written down for other discretization methods such as the finite difference scheme. One immediate problem which makes the analysis of (1.3) nontrivial in the L^∞ setting is the lack of L^∞ preservation due to the frequency truncation Π_N. Indeed, even for $n=0$ and generic initial data u_0 with $\|u_0\|_\infty\leq 1$, one can have

$$\|u^0\|_\infty>1,$$

which is caused by the lack of positivity of the Dirichlet kernel. For this and similar other technical obstructions (cf. page 219 of [8]), there has been no discussion of L^∞ maximum principle for the Fourier spectral method in the literature prior to this work. The very purpose of this work is to settle this issue and develop a new framework for spectral methods.

Our first result is concerned with a detailed analysis of the operator $T_{N,\tau\nu^2}$. Albeit classical the main novelty here is the quantification of the parameters and the sharpness of the involved constants. To allow some generality we denote for $N\geq 1$, $\beta>0$, $T_{N,\beta}=(\mathrm{Id}-\beta\partial_{xx})^{-1}\Pi_N$. Also denote

$$\Pi_{>N}=\mathrm{Id}-\Pi_N \quad \text{and} \quad K_{>N,\beta}=(\mathrm{Id}-\beta\partial_{xx})^{-1}\Pi_{>N}.$$

Theorem 1.1 (Maximum principle for $T_{N,\beta}$, 1D)**.** *Let $\beta>0$ and $N\geq 1$.*

- *Strict maximum principle: If*

$$N\geq N_0(\beta)=\frac{1}{2\pi^2\sqrt{\beta}}e^{\frac{1}{2\sqrt{\beta}}},$$

then we have the strict maximum principle:

$$\|T_{N,\beta}f\|_{L^\infty(\mathbb{T})}\leq\|f\|_{L^\infty(\mathbb{T})}.$$

On the other hand if $N<N_0$ there are counterexamples.

- *Effective maximum principle: For any $N\geq 1$, we have*

$$\|T_{N,\beta}f\|_{L^\infty(\mathbb{T})}\leq\Big(1+\frac{\alpha_1}{1+\beta N^2}\log(N+2)\Big)\|f\|_{L^\infty(\mathbb{T})},$$

where $\alpha_1>0$ is an absolute constant.

- *Sharp bounds on $K_{>N,\beta}$: there are absolute constants $c_1>0$, $c_2>0$, such that*

$$\frac{c_1}{1+\beta N^2}\log(N+2)\leq\|K_{>N,\beta}\|_{L^1(\mathbb{T})}\leq\frac{c_2}{1+\beta N^2}\log(N+2).$$

These results are essentially sharp with respect to the dependence on the parameters. See Theorem 2.2 in Section 2 for some results concerning lower bounds and more definite and precise statements.

Remark 1.1. For the strict maximum principle of $T_{N,\beta}$ to hold we needed $N\geq N_0(\beta)=\mathcal{O}(e^{C/\sqrt{\beta}})$ which is large in the regime $0<\beta\ll 1$. The sharpness of such bounds is established in Proposition 2.8. But heuristically there are several ways to see why this is needed. One way is as follows. Denote the kernel of $T_{N,\beta}$ as $K_N=D_N*g_\beta$, where

$$g_\beta(x)=\sum_{n\in\mathbb{Z}}g_\beta^{(0)}(x+n),\quad g_\beta^{(0)}(x)=\frac{1}{2\sqrt{\beta}}\exp\Big(-\frac{|x|}{\sqrt{\beta}}\Big),$$

and $*$ is the usual convolution on the torus. Take the fundamental domain for the torus as $[-\frac{1}{2},\frac{1}{2}]$ and consider

$$K_N(x_0)=\int_{|y|\leq\frac{1}{2}}D_N(y)g_\beta(x_0-y)dy$$

with x_0 near $\frac{1}{2}-\frac{0.5}{2N+1}$. Note that the Dirichlet kernel is sharply peaked at $y=0$ with value $(2N+1)$ followed by a minimum at $y=\frac{3}{2}\cdot\frac{1}{2N+1}$ with value approximately

$$-\frac{2}{3\pi}(2N+1)\approx -0.212*(2N+1)$$

and so on. Clearly for $y=\mathcal{O}(\frac{1}{2N+1})$ the contribution is approximately $N\cdot e^{-\frac{1}{2\sqrt{\beta}}}$ and this has to beat the $\mathcal{O}(1)$ fluctuation near x_0. Thus we need $N\gtrsim\exp(c/\sqrt{\beta})$. Another way to see it is to observe that $g_\beta(y)$ has minimum value around $\frac{1}{2\sqrt{\beta}}\exp(-c/\sqrt{\beta})$. Now we take the Dirichlet kernel and convolve it with g_β around this minimum. Clearly we need $N\gg\exp(c/\sqrt{\beta})$ in order for the first peak of D_N to beat the first negative trough.

Theorem 1.1 is a special case of Theorem 2.2 in Section 2. In Section 2 we give two proofs of the sharp characterization that

$$\|K_{>N,\beta}\|_{L^1(\mathbb{T})}\sim\frac{1}{1+N^2\beta}\log(N+2).$$

The first proof (Proposition 2.3) is essentially based on the Poisson summation whereas the second proof (Proposition 2.6) uses the Fejer kernel which exploits the convexity of the Fourier coefficients in certain regimes. The proof therein naturally generalizes to certain convex trigonometric series.

Our next result generalizes Theorem 1.1 to dimensions $d\geq 1$. In numerical computations, we usually work with the projection operators

$$\Pi_N=\Pi_N^{(d)}=\Pi_{N,k_1}\cdots\Pi_{N,k_d},$$

where Π_{N,k_j} refers to projection into each frequency coordinate k_j. In yet other words $\Pi_N^{(d)}$ is the frequency truncation to the domain $\{k:|k|_\infty\leq N\}$, i.e.,

$$\widehat{\Pi_N^{(d)}}(k)=1_{|k|_\infty\leq N},\quad k=(k_1,\cdots,k_d)\in\mathbb{Z}^d.$$

For $\beta>0$, $N\geq 2$, we define

$$T_{>N,\beta}=(\mathrm{Id}-\beta\Delta)^{-1}(\mathrm{Id}-\Pi_N^{(d)}),\qquad \widehat{T_{>N,\beta}}(k)=\frac{1}{1+\beta(2\pi|k|)^2}\cdot 1_{|k|_\infty>N},$$

$$T_{N,\beta}=(\mathrm{Id}-\beta\Delta)^{-1}\Pi_N^{(d)},\qquad \widehat{T_{N,\beta}}(k)=\frac{1}{1+\beta(2\pi|k|)^2}\cdot 1_{|k|_\infty\leq N}.$$

Theorem 1.2 (Maximum principle for $T_{N,\beta}$ and $T_{>N,\beta}$, $d\geq 1$)**.** *Let $d\geq 1$, $N\geq 2$, $\beta>0$. Then the following hold for*

$$T_{>N,\beta}=(\mathrm{Id}-\beta\Delta)^{-1}(\mathrm{Id}-\Pi_N^{(d)})$$

with the kernel $K_{>N,\beta}$, and

$$T_{N,\beta}=(\mathrm{Id}-\beta\Delta)^{-1}\Pi_N^{(d)}.$$

1. *Sharp L_x^1 bound.*

$$\frac{c_1}{1+\beta N^2}(\log(N+2))^d\leq\|K_{>N,\beta}\|_{L_x^1(\mathbb{T}^d)}\leq\frac{c_2}{1+\beta N^2}(\log(N+2))^d,$$

where $c_1>0$, $c_2>0$ depend only on the dimension d.

2. *Effective maximum principle. For all $N\geq 2$, we have*

$$\|T_{N,\beta}f\|_{L_x^\infty(\mathbb{T}^d)}\leq\Big(1+\frac{c_3}{1+\beta N^2}(\log(N+2))^d\Big)\|f\|_{L_x^\infty(\mathbb{T}^d)},$$

where $c_3>0$ depends only on the dimension d.

3. *Sharp maximum principle. Let $d=1,2$. Denote the kernel of $T_{N,\beta}$ as $K_{N,\beta}$. For all $N\geq N_0=N_0(\beta,d)$ (i.e., N_0 is constant depending only on β and d), it holds that $K_{N,\beta}$ is a strictly positive function on $\mathbb{T}^d$ with unit L_x^1 mass, and consequently*

$$\|T_{N,\beta}f\|_{L_x^\infty(\mathbb{T}^d)}\leq\|f\|_{L_x^\infty(\mathbb{T}^d)}.$$

Theorem 1.2 is a restatement of Theorem 2.4 in Section 2. The corresponding analysis and proof can be found therein. From a practical point of view, the statement (2) in Theorem 1.2 is most useful and effective in the regime $0<\beta\ll 1$ since we only need to take N moderately large in order to obtain an almost sharp maximum principle. The $\log^d(N+2)$ factor in the upper bound reflects the fact that the corresponding kernel is a tensor product of one-dimensional Dirichlet kernels.

A natural generalization of the operator $(\mathrm{Id}-\beta\Delta)^{-1}\Pi_N$ is the truncated Bessel type operators $(\mathrm{Id}-\beta\Delta)^{-s}\Pi_N$, where $s>0$. We denote the kernel function corresponding to $(\mathrm{Id}-\beta\Delta)^{-s}\Pi_N$ as $F_{N,s}$ and note that $F_{N,s}\in C^\infty(\mathbb{T}^d)$ for each finite N. Let $s_*\in[0,1]$ be the unique solution to the equation

$$\int_0^{\frac{3}{2}\pi}x^{1-s}\sin x dx=0,\quad 0\leq s\leq 1.$$

In Lemma 2.12 of Section 2, we show that $s_*\in(0.308443,0.308444)$. The following group of results is proved in Section 2. Perhaps a bit surprisingly, for the sharp maximum principle to hold, the transition threshold occurs at $s=s_*<1$.

Theorem 1.3 (Maximum principle for general Bessel case)**.** *Let $d\geq 1$, $N\geq 2$, $\beta>0$ and $s>0$. Then for all $N\geq 2$, we have the effective maximum principle:*

$$\|F_{N,s}\|_{L^1_x(\mathbb{T}^d)}\leq 1+\frac{\alpha_{s,d}}{(1+\beta N^2)^{\frac{s}{2}}}(\log(N+2))^d,$$

where $\alpha_{s,d}>0$ depend only on $(d,\ s)$.

Concerning sharp maximum principles, we have the following

1. *If $s>d$ and*

$$N\geq c_{s,d}(\beta^{\frac{d-1}{2}}e^{\frac{2}{\sqrt{\beta}}})^{\frac{1}{s-d}}\quad (c_{s,d}>0\ \text{ depends only on }\ (s,d)),$$

   then $F_{N,s}$ is a positive function and $\|F_{N,s}\|_{L^1(\mathbb{T}^d)}=1$.

2. *Let $d=1$ and $s=1$. If*

$$N\geq \alpha\beta^{-\frac{1}{2}}e^{\frac{1}{\sqrt{\beta}}}\quad (\alpha>0 \text{ is an absolute constant}),$$

   then $F_{N,1}$ is positive and has unit L^1 mass.

Let $d=1$, $\beta>0$ and $0<s<1$. We identify $\mathbb{T}=[-\frac{1}{2},\frac{1}{2})$. For some $s_*\in(0.308443,0.308444)$, the following hold.

1. *Lack of positivity for $0<s<s_*$. There are positive constants $\gamma_2(s)>\gamma_1(s)>0$ and $\gamma(s)>0$, such that for any $y\in[\gamma_1(s),\gamma_2(s)]$ and*

$$N\gg_s 1+\beta^{-\frac{1}{2(1-s)}},$$

   we have

$$\frac{1}{N^{1-s}}F_{N,s}(\frac{y}{N})=-\gamma(s)+\Big(1+\frac{1}{\sqrt{\beta}}\Big)\cdot O_s(N^{-(1-s)})<-\frac{1}{2}\gamma(s),$$

   In particular this shows that for $0<s<s_*$ the kernel function $F_{N,s}$ must be negative on an interval of length $O_s(N^{-1})$ for all large N.

2. *Positivity for $s_*<s<1$. For any $s_*<s<1$ there are constants $c_1(s)>0$ and $c_2(s)>0$ such that if*

$$N\geq \max\big\{c_1(s)\beta^{-\frac{1}{2}}e^{\frac{1}{\sqrt{\beta}}},c_2(s)\cdot(1+\beta^{-\frac{1}{2(1-s)}})\big\},$$

   then $F_N(x)$ is positive and hence has unit L^1_x mass.

3. *Lack of positivity for $s=s_*$ and $0<\beta<\beta_*$. Let $s=s_*$ and $y=\frac{3}{4}$. Let $0<\beta<\beta_*$ where β_* is the same absolute constant as in Lemma 2.14. Then for $N\geq\frac{1}{\sqrt{\beta}}$, we have*

$$F_{N,s}\Big(\frac{y}{N}\Big)=\frac{1}{2\pi\sqrt{\beta}}A_\beta(s)+O_{s,\beta}(N^{-s}),$$

where $A_\beta(s)$ is given by the expression

$$A_\beta(s)=\int_{\mathbb{R}}(\langle x\rangle^{-s}-|x|^{-s})dx-s2\pi\sqrt{\beta}\int_{\mathbb{R}}\langle x\rangle^{-s-2}x\Big\{\frac{x}{2\pi\sqrt{\beta}}\Big\}dx,$$

where $\langle x\rangle=(1+x^2)^{\frac{1}{2}}$, and for $y\in\mathbb{R}$, $\{y\}=y-[y]$ with $[y]$ being the smallest integer less than or equal to y. Furthermore for $0<\beta<\beta_*$, we have

$$0<\alpha_1<-A_\beta(s)<\alpha_2<\infty,$$

where $\alpha_1>0$, $\alpha_2>0$ are absolute constants.

Remark 1.2. For the sharp maximum principle, further interesting cases are

1. $d\geq 2$, $0<s\leq d$;
2. $d=1$, $s=s_*$, $\beta>\beta_*$.

Some partial results are available but we will not dwell on this issue here and will investigate it elsewhere.

We now return to (1.3). For $\beta=\nu^2\tau>0$ Theorem 1.1 shows that in general we can only expect the effective maximum principle:

$$\|T_{N,\nu^2\tau}f\|_{L^\infty(\mathbb{T})}\leq\Big(1+\frac{\text{const}}{1+N^2\nu^2\tau}\log(N+2)\Big)\|f\|_{L^\infty(\mathbb{T})},\quad\forall N\geq 2.$$

Moreover by Theorem 2.2, this upper bound is optimal for $0<\beta=\nu^2\tau\ll 1$ (which is typically the case in practical numerical simulations) and moderately large N, i.e.,

$$\frac{\text{const}}{\nu\sqrt{\tau}}\leq N\leq\frac{\text{const}}{\nu\sqrt{\tau}}\exp\Big(\frac{1}{6\nu^2\tau}\Big).$$

The sharp maximum does hold for large $N\gtrsim\exp(\frac{\text{const}}{\nu^2\tau})$ but it is computationally unfeasible especially when $0<\nu^2\tau\ll 1$. Thus in order for (1.3) to admit any sort of maximum principle in some reasonable generality, we must have some strong contractive estimates on the nonlinear map $u^n\to f_\tau(u^n)$ which will compensate for the loss in the linear estimate. As it turns out, this is indeed possible under some mild constraints on the time step. The heart of the matter is encapsulated in the following elementary iterative system.

Proposition 1.1. *Let $\tau>0$ and consider the cubic polynomial $f_\tau(x)=(1+\tau)x-\tau x^3$ for $x\in\mathbb{R}$. Then the following hold:*

- *If $0<\tau\leq\frac{1}{2}$, then*

$$\max_{|x|\leq 1}|f_\tau(x)|=1.$$

Actually for any $1\leq\alpha\leq\sqrt{1+\frac{2}{\tau}}$, we have

$$\max_{|x|\leq\alpha}|f_\tau(x)|\leq\alpha.$$

Furthermore for any $1<\alpha<\sqrt{1+\frac{2}{\tau}}$, we have the strict inequality

$$\max_{|x|\leq\alpha}|f_\tau(x)|<\alpha.$$

- *If $\frac{1}{2}\leq\tau\leq 2$, then for any*

$$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}\leq\alpha\leq\sqrt{\frac{2+\tau}{\tau}},$$

we have

$$\max_{|x|\leq\alpha}|f_\tau(x)|\leq\alpha.$$

Furthermore if

$$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}<\alpha<\sqrt{\frac{2+\tau}{\tau}},$$

then we have the strict inequality

$$\max_{|x|\leq\alpha}|f_\tau(x)|<\alpha.$$

- *For $\tau>2$, define*

$$x_0=\sqrt{\frac{1+\tau}{3\tau}}\in(0,1)\quad\text{ and }\quad x_{n+1}=f_\tau(x_n).$$

Then $|x_n|\to\infty$ as $n\to\infty$.

Remark 1.3. We have

$$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}=\begin{cases}1, & \tau=\dfrac{1}{2},\\ \sqrt{2}, & \tau=2,\end{cases}\qquad \sqrt{\frac{2+\tau}{\tau}}=\begin{cases}\sqrt{5}, & \tau=\dfrac{1}{2},\\ \sqrt{2}, & \tau=2.\end{cases}$$

The most useful result in Proposition 1.1 is the strict inequality for the maximum. In particular for $0<\tau\leq\frac{1}{2}$ and $\alpha=1+\delta_0$ with $\delta_0>0$ suitably small, we have

$$\frac{1}{\alpha}\max_{|x|\leq\alpha}|f_\tau(x)|\leq\theta_0(\alpha,\tau)<1, \tag{1.4}$$

where $\theta_0(\alpha,\tau)$ depends only on $(\alpha,\,\tau)$.

Now we return to (1.3). Denote $\alpha_n=\|u^n\|_\infty$. Assume $\alpha_n>1$ and α_n-1 is suitably small. By Theorem 1.1 and (1.4), for $0<\tau\leq\frac{1}{2}$ we deduce

$$\alpha_{n+1}\leq\Big(1+\mathcal{O}\Big(\frac{1}{1+N^2\nu^2\tau}\log(N+2)\Big)\Big)\cdot\theta_0(\alpha_n,\tau)\cdot\alpha_n.$$

Before we proceed further, we should point out one subtle technical difficulty with the above simplified system in the regime $0<\tau\ll1$. Note that $\theta_0(\alpha,\tau)\to1$ as $\tau\to0$. Since the the damping factor contains $N^2\tau$, it follows that the threshold N must be taken τ-dependent in order to obtain contractive estimates which is hardly desirable in practice. In order to retain stability for $\tau\to0$ and build a stability analysis for N moderately large and *independent of* τ, a different line of argument is needed and indeed we develop a refined analysis in Section 3 (and later sections) to cover the regime $0<\tau\ll1$.

To simplify the discussion, we now consider the case $\tau_*\leq\tau\leq\frac{1}{2}$ where $\tau_*>0$ is fixed. In this case, it is not difficult to see that for $N\geq N_0(\tau_*,\nu)$ sufficiently large, we have

$$\alpha_{n+1}\leq\max_{|x|\leq\alpha_n}|f_\tau(x)|+\eta_n,$$

where η_n accounts for the spectral error and can be made sufficiently small. The next proposition quantifies the desired strong stability.

Proposition 1.2 (Strong stability of the prototype iterative system)**.** *Let* $\tau>0$ *and* $f_\tau(x)=(1+\tau)x-\tau x^3$ *for* $x\in\mathbb{R}$. *Consider the recurrent relation*

$$\alpha_{n+1}=\max_{|x|\leq\alpha_n}|f_\tau(x)|+\eta,\quad n\geq0,$$

where $\eta>0$.

1. *Case* $0<\tau\leq\frac{1}{2}$. *Let* $\alpha_0=2$. *There exists an absolute constant* $\eta_0>0$ *sufficiently small, such that for all* $0\leq\eta\leq\eta_0$, *we have* $1\leq\alpha_n\leq 2$ *for all* n. *Furthermore for all* $n\geq 1$,

$$1+\eta\leq\alpha_n\leq 1+\theta^n+\frac{1-\theta^n}{1-\theta}\eta,$$

where $\theta=1-2\tau$.

2. *Case* $\frac{1}{2}\leq\tau\leq 2-\epsilon_0$ *where* $0<\epsilon_0\leq 1$. *Let*

$$\alpha_0=\frac{1}{2}\Big(\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\sqrt{\frac{2+\tau}{\tau}}\Big).$$

Then there exists a constant $\eta_0>0$ *depending only on* ϵ_0, *such that if* $0\leq\eta\leq\eta_0$, *then for all* $n\geq 1$, *we have*

$$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\eta\leq\alpha_n\leq\alpha_0.$$

Both Proposition 1.1 and Proposition 1.2 are proved in Section 3 (see in particular Lemmas 3.2 and 3.3 therein). For the one-dimensional system (1.3), a complete theory of L^∞-stability and instability is worked out in Section 3 for all $0<\tau<\infty$ and $N\geq 2$. We shall not reproduce all the details here and turn now to the general theory for dimensions $1\leq d\leq 3$ developed in Section 4. Consider

$$\begin{cases}\dfrac{u^{n+1}-u^n}{\tau}=\nu^2\Delta u^{n+1}-\Pi_N((u^n)^3-u^n), \quad n\geq 0, \quad (t,x)\in(0,\infty)\times\mathbb{T}^d,\\ u^0=\Pi_N u_0.\end{cases}\tag{1.5}$$

Concerning (1.5), the following group of stability results is proved in Section 4. For the first time we are able to establish effective maximum principles for the Fourier spectral methods applied on the nonlinear system.

Theorem 1.4 (Effective maximum principles for (1.5))**.** *Consider* (1.5) *on* $\mathbb{T}^d=[0,1)^d$ *with* $1\leq d\leq 3$ *and* $\nu>0$. *Then the following hold:*

1. *Energy stability for* $0<\tau\leq 0.86$. *Let* $0<\tau\leq 0.86$. *Assume* $u_0\in H^s(\mathbb{T}^d)\cap H^1(\mathbb{T}^d)$, $s>\frac{d}{2}$ *and*

$$\|u^0\|_\infty\leq\sqrt{\frac{5}{3}}\quad\Big(\textit{note that}\ \sqrt{\frac{5}{3}}\approx 1.29099\Big).$$

Then we have energy stability for any $N \geq N_0 = N_0(s,d,\nu,u_0)$*:*

$$E(u^{n+1}) \leq E(u^n), \quad \forall n \geq 0,$$

where

$$E(u) = \int_{\mathbb{T}^d} \Big(\frac{1}{2}\nu^2 |\nabla u|^2 + \frac{1}{4}(u^2-1)^2 \Big) dx.$$

Furthermore,

$$\sup_{n\geq 0} \|u^n\|_{H^s} \leq U_1 < \infty,$$

where $U_1 > 0$ *is a constant depending only on* $(s,\ d,\ \nu,\ u_0)$*.*

2. *Effective maximum principle for* $0 < \tau \leq \frac{1}{2}$*. Let* $0 < \tau \leq \frac{1}{2}$*. Assume* $u_0 \in H^s(\mathbb{T}^d) \cap H^1(\mathbb{T}^d)$*,* $s > \frac{d}{2}$ *and* $\|u^0\|_\infty = \|\Pi_N u_0\|_\infty \leq 1 + \alpha_0$ *for some* $0 \leq \alpha_0 \leq \sqrt{\frac{5}{3}} - 1$ *(note that* $\sqrt{\frac{5}{3}} - 1 \approx 0.29099$*). Then for* $N \geq N_1 = N_1(s,d,\nu,u_0)$*, we have*

$$\|u^n\|_\infty \leq 1 + (1-2\tau)^n \alpha_0 + N^{-(s-\frac{d}{2})} C_{s,d,\nu,u_0}, \quad \forall n \geq 1,$$

where $C_{s,d,\nu,u_0} > 0$ *depends only on* $(s,\ d,\ \nu,\ u_0)$*.*

3. *L^∞-stability for* $\frac{1}{2} < \tau < 2$*. Assume* $\frac{1}{2} < \tau < 2$*. Then the following hold:*

   (a) *Let* $\frac{1}{2} < \tau < 2 - \epsilon_0$ *for some* $0 < \epsilon_0 \leq 1$*. Denote*

   $$M_0 = \frac{1}{2}\Big(\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}} \cdot \frac{2}{3} + \sqrt{\frac{2+\tau}{\tau}} \Big).$$

   If $\|u^0\|_\infty \leq M_0$ *and*

   $$N \geq N_2 = N_2(\nu) = C(\epsilon_0) \cdot \nu^{-1} (|\log \nu|)^{\frac{d}{2}},$$

   when $0 < \nu \ll 1$ *($C(\epsilon_0)$ is a constant depending only on ϵ_0), then*

   $$\sup_{n\geq 0} \|u^n\|_\infty \leq M_0.$$

   (b) *Let* $d = 1,2$*. For any*

   $$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}} \cdot \frac{2}{3} \leq \alpha \leq \sqrt{\frac{2+\tau}{\tau}},$$

if $\|u^0\|_\infty \le \alpha$ *and*

$$N \ge N_3 = N_3(\nu) = \mathcal{O}(e^{\frac{\text{const}}{\nu}}),$$

then

$$\sup_{n\ge 0} \|u^n\|_\infty \le \alpha.$$

Remark 1.4. In Statement (1), our stability region is much wider than the result obtained by Tang and Yang [6]. For the finite difference case the analysis therein requires $0<\tau\le\frac{1}{2}$. Our analysis here covers both the finite difference case and the spectral case, and produces energy stability for $0<\tau\le 0.86$ under much less stringent assumptions on the initial data. Also we should point out that for $0<\nu\ll 1$, the dependence of N_0 is only power like which is a very mild constraint in practice.

Remark 1.5. For $\tau=2$ and N not large, there are counterexamples as shown in Proposition 3.5.

To understand the effect of pure spectral truncation on L^∞-stability, we now consider the following model system

$$\begin{cases} \partial_t u = \nu^2 \Delta u - \Pi_N(u^3-u), \quad (t,x)\in(0,\infty)\times\mathbb{T}^d, \\ u\big|_{t=0} = \Pi_N u_0, \end{cases} \tag{1.6}$$

where $\nu>0$ and $d\ge 1$. The following results are proved in Section 4.

Theorem 1.5 (Maximum principle for the continuous in time system with spectral truncation)**.** *Consider* (1.6) *with* $\nu>0$, $d\ge 1$. *Then the following hold*

1. *Effective maximum principle for regular initial data. Assume* $\|u_0\|_\infty\le 1$ *and* $u_0\in H^s(\mathbb{T}^d)$, $s>\frac{d}{2}$. *Then for* $N\ge N_0=N_0(s,d,\nu,u_0)$, *we have*

$$\sup_{0\le t<\infty} \|u(t,\cdot)\|_\infty \le 1 + N^{-\gamma_0} C_1,$$

where $\gamma_0=\min\{s-\frac{d}{2},1\}$, *and* $C_1>0$ *depends only on* (s, d, ν, u_0).

2. *Effective maximum principle for* L^∞ *initial data. Assume* $\|u_0\|_\infty\le 1$. *Then for* $N\ge N_1=N_1(d,\nu,u_0)$, *we have*

$$\sup_{0\le t<\infty} \|u(t,\cdot)\|_\infty \le \max\{\|\Pi_N u_0\|_\infty, 1\} + N^{-\frac{1}{2}} C_2,$$

where $C_2>0$ *depends only on* (d, ν, u_0).

We now turn to the Fourier collocation method which is widely used in practical numerical simulations. To illustrate the theory we consider the 1D Allen-Cahn on the periodic torus $\mathbb{T}=[0,1)$ using discrete Fourier transform in space. We discretize the domain $[0,1)$ using $x_j=\frac{j}{N}$, $j=0, 1, \cdots, N-1$, where N is usually taken to be an even number. In typical FFT simulations, N is usually taken to be a dyadic number. We use $u^n=(u_0^n,\cdots,u_{N-1}^n)^T$ to denote the approximation of $u(t,x_j)=u(t,\frac{j}{N})$, $t=n\tau$, $0\leq j\leq N-1$. We shall adopt the following convention for discrete Fourier transform:

$$\tilde{U}_k=\frac{1}{N}\sum_{j=0}^{N-1}U_j e^{-2\pi ik\cdot\frac{j}{N}},$$

$$U_j=\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\tilde{U}_k e^{2\pi ik\cdot\frac{j}{N}},$$

where $\tilde{U}=(\tilde{U}_0,\cdots,\tilde{U}_{N-1})^T$ is the (approximate) Fourier coefficient vector of the input data $U=(U_0,\cdots,U_{N-1})^T$. Note that $\tilde{U}_{k\pm N}=\tilde{U}_k$ for any $k\in\mathbb{Z}$. The discrete Laplacian operator Δ_h corresponds to the Fourier multiplier $-(2\pi k)^2$ (when k is restricted to $-\frac{N}{2}<k\leq\frac{N}{2}$).

To understand how Fourier collocation affects L^∞-stability, we consider the following ODE system which is continuous-in-time with Fourier-collation-in-space discretization of Allen-Cahn. Here $U(t)=(U(t)_0,\cdots,U(t)_{N-1})^T$ is the numerical approximation of the exact node values $u(t,x_j)_{j=0}^{N-1}$.

$$\begin{cases}\dfrac{d}{dt}U=\nu^2\Delta_h U+U-U^{\cdot 3},\\ U\big|_{t=0}=U^0\in\mathbb{R}^N,\end{cases}\tag{1.7}$$

where $U^{\cdot 3}=((U_0)^3,\cdots,(U_{N-1})^3)^T$.

Concerning (1.7), the following results are proved in Section 5.

Theorem 1.6. *Consider* (1.7) *with* $\nu>0$. *Then the following hold:*

1. *Effective maximum principle for regular initial data. Suppose* $u_{\text{init}}:\mathbb{T}\to\mathbb{R}$ *satisfies* $u_{\text{init}}\in H^{\frac{3}{2}}(\mathbb{T})$ *and* $\|u_{\text{init}}\|_\infty\leq 1$. *Take the initial data* U^0 *such that* $U_j^0=u_{\text{init}}(\frac{j}{N})$ *for* $0\leq j\leq N-1$. *Then for* $N\geq N_1=N_1(u_{\text{init}},\nu)>0$, *we have*

$$\|U(t)\|_\infty\leq 1+C_1\cdot N^{-\frac{1}{2}},$$

where $C_1>0$ *depends only on* $(u_{\text{init}},\ \nu)$.

2. *Effective maximum principle: version 2. Suppose* $u_{\text{init}}:\mathbb{T}\to\mathbb{R}$ *has Fourier support in* $\{|k|\le\frac{N}{2}\}$ *with*

$$\widehat{u_{\text{init}}}\Big(\frac{N}{2}\Big)=\widehat{u_{\text{init}}}\Big(-\frac{N}{2}\Big)\quad\text{and}\quad\|u_{\text{init}}\|_\infty\le 1.$$

Take the initial data U^0 *such that*

$$U_j^0=u_{\text{init}}\Big(\frac{j}{N}\Big)\quad\text{for}\quad 0\le j\le N-1.$$

Then for $N\ge N_2=N_2(u_{\text{init}},\nu)>0$, *we have*

$$\|U(t)\|_\infty\le 1+C_2\cdot N^{-\frac{1}{4}},$$

where $C_2>0$ *depends only on* $(u_{\text{init}},\ \nu)$.

3. *Effective maximum principle for rough initial data. Suppose* $U^0\in\mathbb{R}^N$ *satisfies* $\|U^0\|_\infty\le 1$. *If* $N\ge N_3(\nu)>0$, *it holds that*

$$\sup_{0\le t<\infty}\|U(t)\|_\infty\le 1+\epsilon_1,$$

where $0<\epsilon_1<10^{-2}$ *is an absolute constant. More precisely the following hold. (Below we shall write* $X=O(Y)$ *if* $|X|\le CY$ *where the constant* C *only depends on* ν.*)*

(a) *For* $t\ge T_0=T_0(\nu)>0$,

$$\|U(t)\|_\infty\le 1+\mathcal{O}(N^{-\frac{1}{8}}(\log N)^2),\quad E(U(t))\le\mathcal{O}(1),$$

where $E(U)$ *was defined in* (5.8).

(b) *For* $C_1=C_1(\nu)>0$ *and* $C_1\cdot N^{-\frac{3}{4}}\le t\le T_0$, *it holds that*

$$\|U(t)\|_\infty\le 1+\mathcal{O}(N^{-\frac{1}{8}}(\log N)^2).$$

(c) *For* $C_2=C_2(\nu)>0$ *and* $C_2\cdot N^{-2}\log N\le t\le C_1\cdot N^{-\frac{3}{4}}$,

$$\|U(t)\|_\infty\le 1+\mathcal{O}(N^{-\frac{3}{16}}).$$

(d) *For* $0<t\le C_2N^{-2}\log N$, *we have*

$$\|U(t)\|_\infty\le\Big(1+\frac{1}{2}\epsilon_1\Big)\|U^0\|_\infty+\mathcal{O}(N^{-\frac{1}{2}}(\log N)^{\frac{1}{4}}).$$

(e) If $\|U^0\|_\infty \le \frac{1}{1+\frac{1}{2}\epsilon_1}$ *(note that* $\frac{1}{1+\frac{1}{2}\epsilon_1} > 0.995$*), then for all* $t \ge 0$*, we have*

$$\|U(t)\|_\infty \le 1 + \mathcal{O}(N^{-\frac{1}{8}}(\log N)^2).$$

(f) Suppose $f : \mathbb{T} \to \mathbb{R}$ *is continuous and* $\|f\|_\infty \le 1$*. If* $(U^0)_l = f(\frac{l}{N})$ *for all* $0 \le l \le N-1$*, then*

$$\sup_{t\ge 0} \|U(t)\|_\infty \le 1 + \mathcal{O}(\omega_f(N^{-\frac{2}{3}})) + \mathcal{O}(N^{-c}),$$

where $c > 0$ *is an absolute constant, and* ω_f *is defined in* (B.5)*.*

(g) Suppose $f : \mathbb{T} \to \mathbb{R}$ *is* C^α*-continuous* (*see* (B.6)) *for some* $0 < \alpha < 1$ *and* $\|f\|_\infty \le 1$*. If* $(U^0)_l = f(\frac{l}{N})$ *for all* $0 \le l \le N-1$*, then*

$$\sup_{t\ge 0} \|U(t)\|_\infty \le 1 + \mathcal{O}(N^{-c_1}),$$

where $c_1 > 0$ *is a constant depending only on* α*.*

Moreover we have the following result which shows the sharpness of our estimates above. There exists a function f*:* $\mathbb{T} \to \mathbb{R}$*, continuous at all of* $\mathbb{T}\setminus\{x_*\}$ *for some* $x_* \in \mathbb{T}$ *(i.e., continuous at all* $x \ne x_*$*) and has the bound* $\|f\|_\infty \le 1$ *such that the following hold: for a sequence of even numbers* $N_m \to \infty$*,* $t_m = \nu^{-2} N_m^{-2}$ *and* $x_m = j_m/N_m$ *with* $0 \le j_m \le N_m - 1$*, if* $\tilde{U}_m(t) \in \mathbb{R}^{N_m}$ *solves* (1.7) *with*

$$\tilde{U}_m(0) = v_m \in R^{N_m} \quad \text{satisfying} \quad (v_m)_j = f\Big(\frac{j}{N_m}\Big)$$

for all $0 \le j \le N_m - 1$*. Then*

$$|\tilde{U}_m(t_m, x_m)| \ge 1 + \eta_*, \quad \forall m \ge 1,$$

where $\eta_* > 0.001$ *is an absolute constant.*

Remark 1.6. For $0 < \nu \ll 1$ the dependence of $N_i(\nu)$, $i = 1,2,3$ is only power like which is a mild constraint in practice.

We now consider the fully discrete system.

$$\frac{U^{n+1} - U^n}{\tau} = \nu^2 \Delta_h U^{n+1} + U^n - (U^n)^{.3}, \tag{1.8}$$

where $\nu > 0$, $\tau > 0$ and $(U^n)^{.3} = ((U_0^n)^3, \cdots, (U_{N-1}^n)^3)^T$. This is a first order IMEX method applied to Allen-Cahn with Fourier collocation in space. The following results are proved in Section 5.

Theorem 1.7 (Effective maximum principles for (1.8))**.** *Consider* (1.8) *with* $\nu>0$, $\tau>0$. *Then the following hold*

1. *Sharp maximum principle for* $0<\tau\le\frac{1}{2}$ *and very large* N. *Assume* $0<\tau\le\frac{1}{2}$ *and* $\|U^0\|_\infty\le 1$. *If*
$$N\ge 4+\frac{1}{\pi^2\nu\sqrt{\tau}}e^{\frac{1}{2\nu\sqrt{\tau}}},$$
then
$$\|U^n\|_\infty\le 1,\quad \forall n\ge 1.$$

2. L^∞ *stability for* $\frac{1}{2}\le\tau\le 1.99$. *Assume* $\frac{1}{2}\le\tau\le 1.99$ *and* $\|U^0\|_\infty\le 1$. *If* $N\ge N_4(\nu)>0$ *(the dependence of* N_4 *on* ν *is only power-like), then*
$$\sup_{n\ge 0}\|U^n\|_\infty\le M_a=\frac{1}{2}\Big(\frac{2}{3}\cdot\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}+\sqrt{\frac{2+\tau}{\tau}}\Big).$$
Moreover if $\frac{1}{2}\le\tau\le 0.86$, *then*
$$\sup_{n\ge 0}\|U^n\|_\infty\le M_b=\frac{2}{3}\cdot\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}+\eta_0,\quad \eta_0=10^{-5},$$
$$E(U^{n+1})\le E(U^n),\qquad \forall n\ge 0, \tag{1.9a}$$
$$E(U^n)\le E(U^1)\le\mathcal{E}_1,\quad \forall n\ge 1, \tag{1.9b}$$
where $\mathcal{E}_1>0$ *depends only on* ν.

3. *Effective maximum principle for* $N^{-0.3}\le\tau\le\frac{1}{2}$. *Suppose* $N^{-0.3}\le\tau\le\frac{1}{2}$ *and* $\|U^0\|_\infty\le 1$. *If* $N\ge N_5(\nu)>0$ *(the dependence of* N_5 *on* ν *is only power-like), then*
$$\begin{aligned}&\|U^n\|_\infty\le 1+\mathcal{O}(N^{-\frac{1}{3}}),&&\forall n\ge 0,\\ &E(U^{n+1})\le E(U^n),&&\forall n\ge 0,\\ &E(U^n)\le\mathcal{E}_2,&&\forall n\ge\frac{1}{\tau},\end{aligned}$$
where $\mathcal{E}_2>0$ *depends only on* ν.

4. *Effective maximum principle for* $0<\tau\le N^{-0.3}$. *Assume* $0<\tau\le N^{-0.3}$ *and* $\|U^0\|_\infty\le 1$. *If* $N\ge N_6(\nu)>0$ *(the dependence of* N_6 *on* ν *is only power-like), then it holds that*
$$\sup_{n\ge 0}\|U^n\|_\infty\le 1+\epsilon_1,$$

where $0<\epsilon_1<10^{-2}$ is an absolute constant. More precisely the following hold. (Below we shall write $X=O(Y)$ if $|X|\le CY$ and the constant C depends only on ν.)

(a) $E(U^{n+1})\le E(U^n)$ *for all* $n\ge 0$.

(b) For all $n\ge 2N^{-0.3}/\tau$, *we have*

$$\|U^n\|_\infty \le 1+\mathcal{O}(N^{-\frac{3}{40}}).$$

(c) For $1\le n\le 2N^{-0.3}/\tau$, *we have*

$$\|U^n\|_\infty \le \Big(1+\frac{1}{2}\epsilon_1\Big)\|U^0\|_\infty+\mathcal{O}(N^{-\frac{3}{40}}).$$

(d) If $\|U^0\|_\infty \le \frac{1}{1+\frac{1}{2}\epsilon_1}$ *(note that* $1/(1+\frac{1}{2}\epsilon_1)>0.995$*), then*

$$\sup_{n\ge 0}\|U^n\|_\infty \le 1+\mathcal{O}(N^{-\frac{3}{40}}).$$

(e) Suppose $f:\mathbb{T}\to\mathbb{R}$ *is continuous and* $\|f\|_\infty\le 1$. *If* $(U^0)_l=f(\frac{l}{N})$ *for all* $0\le l\le N-1$, *then*

$$\sup_{n\ge 0}\|U^n\|_\infty \le 1+\mathcal{O}(\omega_f(N^{-\frac{2}{3}}))+\mathcal{O}(N^{-c}),$$

where $c>0$ *is an absolute constant, and* ω_f *is defined in* (B.5).

(f) Suppose $f:\mathbb{T}\to\mathbb{R}$ *is* C^α*-continuous for some* $0<\alpha<1$ *and* $\|f\|_\infty\le 1$. *If* $(U^0)_l=f(\frac{l}{N})$ *for all* $0\le l\le N-1$, *then*

$$\sup_{n\ge 0}\|U^n\|_\infty \le 1+\mathcal{O}(N^{-c_1}),$$

where $c_1>0$ *is a constant depending only on* α.

Moreover we have the following result which shows the sharpness of our estimates above. There exists a function $f:\ \mathbb{T}\to\mathbb{R}$, *continuous at all of* $\mathbb{T}\backslash\{x_*\}$ *for some* $x_*\in\mathbb{T}$ (*i.e., continuous at all* $x\ne x_*$) *and has the bound* $\|f\|_\infty\le 1$ *such that the following hold: for a sequence of even numbers* $N_m\to\infty$,

$$\tau_m=\frac{1}{4}\nu^{-2}N_m^{-2}\quad \textit{and}\quad x_m=j_m/N_m$$

with $0\le j_m\le N_m-1$, *if* $\tilde{U}_m\in\mathbb{R}^{N_m}$ *solves*

$$(\mathrm{Id}-\tau\nu^2\Delta_h)\tilde{U}_m=V_m+\tau_m f(V_m),$$

where $(V_m)_l = f(\frac{l}{N_m})$ *for all* $0 \le l \le N_m - 1$. *Then*

$$|\tilde{U}_m(x_m)| \ge 1 + \eta_*, \quad \forall m \ge 1,$$

where $\eta_* > 0.001$ *is an absolute constant.*

Remark 1.7. In Statement (1) the cut-off $\tau = 1.99$ is for convenience only. One can replace it by any number less than 2 but N_4 will have to be adjusted correspondingly.

Concerning time discretization, all the numerical methods we discussed so far are only first order in time. With further work our effective maximum principles can be generalized to higher order in time methods. To showcase the theory, we consider Strang splitting with Fourier collocation for Allen-Cahn on $\mathbb{T}$. The exact model equation is

$$\partial_t u = \nu^2 \partial_{xx} u + f(u), \quad f(u) = u - u^3, \quad (t,x) \in (0,\infty) \times \mathbb{T}.$$

We slightly abuse the notation and denote for $U \in \mathbb{R}^N$

$$f(U) = U - U^{\cdot 3}.$$

We consider the time splitting as follows. Let $\tau > 0$ be the time step. Consider the ODE

$$\begin{cases} \dfrac{d}{dt} U = f(U), \\ U\big|_{t=0} = a \in \mathbb{R}^N. \end{cases}$$

We define the solution operator $\mathcal{N}_\tau : \mathbb{R}^N \to \mathbb{R}^N$ as the map $a \to U(\tau)$. Thanks to the explicit form of $f(U)$, we have ($a = (a_0, \cdots, a_{N-1})^T$ and with no loss we shall assume that $N \ge 2$)

$$(\mathcal{N}_\tau a)_j = U(\tau)_j = \frac{a_j}{\sqrt{e^{-2\tau} + (1 - e^{-2\tau})(a_j)^2}}, \quad j = 0, 1, \cdots, N-1.$$

Define $\mathcal{S}_\tau = e^{\frac{\tau}{2}\nu^2 \Delta_h}$. Then U^{n+1}, $U^n \in \mathbb{R}^N$ are related via the relation:

$$U^{n+1} = \mathcal{S}_\tau \mathcal{N}_\tau \mathcal{S}_\tau U^n, \quad n \ge 0. \tag{1.10}$$

It is not difficult to check that this particular Strang splitting method is second order in time. The following results are established in Section 6.

Theorem 1.8 (Effective maximum principle for Strang splitting of Allen-Cahn with Fourier Collocation)**.** *Consider* (1.10) *with* $\nu > 0$, $\tau > 0$. *Then the following hold*

1. *Effective maximum principle for $N^{-0.4}\leq\tau<\infty$. Assume $N^{-0.4}\leq\tau<\infty$ and $\|U^0\|_\infty\leq 1$. If $N\geq N_1(\nu)>0$ (the dependence of N_1 on ν is only power-like), then for all $n\geq 0$, we have*

$$\|U^n\|_\infty\leq 1+\mathcal{O}(N^{-0.5}).$$

2. *Effective maximum principle for $0<\tau<N^{-0.4}$. Assume $0<\tau<N^{-0.4}$ and $\|U^0\|_\infty\leq 1$. If $N\geq N_2(\nu)>0$ (the dependence of N_2 on ν is only power-like), then*

$$\sup_{n\geq 0}\|U^n\|_\infty\leq 1+\epsilon_1,$$

where $0<\epsilon_1<10^{-2}$ is an absolute constant. More precisely the following hold.

   (a) *For some $T_0=\mathcal{O}(N^{-0.3})$ sufficiently small and for all $n\geq T_0/\tau$, we have*

$$\|U^n\|_\infty\leq 1+\mathcal{O}(N^{-0.29}),$$

   (b) *For $1\leq n\leq T_0/\tau$, we have*

$$\begin{aligned}&\|U^n-\mathcal{S}_{2n\tau}U^0\|_\infty\leq\mathcal{O}(N^{-0.3}),\\&\|\mathcal{S}_{2n\tau}U^0\|_\infty\leq\Big(1+\frac{1}{2}\epsilon_1\Big)\|U^0\|_\infty,\\&\|U^n\|_\infty\leq\Big(1+\frac{1}{2}\epsilon_1\Big)\|U^0\|_\infty+\mathcal{O}(N^{-0.3}).\end{aligned}$$

   (c) *If $\|U^0\|_\infty\leq\frac{1}{1+\frac{1}{2}\epsilon_1}$ (note that $1/(1+\frac{1}{2}\epsilon_1)>0.995$), then*

$$\sup_{n\geq 0}\|U^n\|_\infty\leq 1+\mathcal{O}(N^{-0.3}).$$

   (d) *Suppose $f:\mathbb{T}\to\mathbb{R}$ is continuous and $\|f\|_\infty\leq 1$. If $(U^0)_l=f(\frac{l}{N})$ for all $0\leq l\leq N-1$, then*

$$\sup_{n\geq 0}\|U^n\|_\infty\leq 1+\mathcal{O}(\omega_f(N^{-\frac{2}{3}}))+\mathcal{O}(N^{-c}),$$

   where $c>0$ is an absolute constant, and ω_f is defined in (B.5).

   (e) *Suppose $f:\mathbb{T}\to\mathbb{R}$ is C^α-continuous for some $0<\alpha<1$ and $\|f\|_\infty\leq 1$. If $(U^0)_l=f(\frac{l}{N})$ for all $0\leq l\leq N-1$, then*

$$\sup_{n\geq 0}\|U^n\|_\infty\leq 1+\mathcal{O}(N^{-c_1}),$$

   where $c_1>0$ is a constant depending only on α.

Moreover we have the following result which shows the sharpness of our estimates above. There exists a function $f:\mathbb{T}\to\mathbb{R}$, continuous at all of $\mathbb{T}\backslash\{x_*\}$ for some $x_*\in\mathbb{T}$ (i.e., continuous at all $x\neq x_*$) and has the bound $\|f\|_\infty\leq 1$ such that the following hold: for a sequence of even numbers $N_m\to\infty$, $\tau_m=\frac{1}{4}\nu^{-2}N_m^{-2}$, $n_m\geq 1$, and $x_m=j_m/N_m$ with $0\leq j_m\leq N_m-1$, if $\tilde{U}_m\in\mathbb{R}^{N_m}$ satisfies

$$\tilde{U}_m=\underbrace{e^{\frac{1}{2}\tau_m\Delta_h}\mathcal{N}_{\tau_m}e^{\frac{1}{2}\tau_m\Delta_h}\cdots e^{\frac{1}{2}\tau_m\Delta_h}\mathcal{N}_{\tau_m}e^{\frac{1}{2}\tau_m\Delta_h}}_{\text{iterate } n_m \text{ times}}a_m,$$

where $a_m\in\mathbb{R}^{N_m}$ satisfies $(a_m)_l=f(\frac{l}{N_m})$ for all $0\leq l\leq N_m-1$. Then

$$|\tilde{U}_m(x_m)|\geq 1+\eta_*,\quad \forall m\geq 1,$$

where $\eta_*>0.001$ is an absolute constant.

Our analysis is certainly not restricted to the Allen-Cahn equation and be generalized and developed in many other directions. In Sections 7 and 8 we introduce effective maximum principles for the 1D Burgers equation and the 2D Navier-Stokes equation with periodic boundary conditions. Quite interestingly, these almost sharp maximum principles exhibit similar strong stability properties much as the Allen-Cahn case, although they are derived using slightly different mechanisms. In the following we give a short summary of the results obtained in Sections 7 and 8.

Consider the 1D Burgers equation:

$$\begin{cases}\partial_t u+\Pi_N(uu_x)=\nu^2u_{xx},\quad (t,x)\in(0,\infty)\times\mathbb{T},\\ u\big|_{t=0}=u^0=\Pi_N f,\end{cases}\tag{1.11}$$

where $f\in L^\infty(\mathbb{T})$.

Theorem 1.9 (Effective maximum principle for continuous in time Burgers with spectral truncation)**.** *Let $\nu>0$ and u be the solution to* (1.11)*. We have for all $N\geq 2$,*

$$\sup_{t\geq 0}\|u(t)\|_\infty\leq\|u^0\|_\infty+\alpha\cdot N^{-c},$$

where $c>0$ is an absolute constant, and $\alpha>0$ depends only on $(\|f\|_\infty,\nu)$.

Remark 1.8. The dependence of α on $1/\nu$ is at most power-like for $0<\nu\lesssim 1$. Also if one work with $u^0=\tilde{P}_N f$, where $\tilde{P}_N$ is a nice Fourier projector such as the Fejer's kernel which satisfies $\|\tilde{P}_N f\|_\infty\leq\|f\|_\infty$ and $\mathrm{supp}(\tilde{P}_N f)\subset\{k:-\frac{N}{2}<k\leq\frac{N}{2}\}$, then we obtain in this case

$$\sup_{t\geq 0}\|u(t)\|_\infty\leq\|f\|_\infty+\mathcal{O}(N^{-c}).$$

Such pre-processing of initial data is quite easy to implement in practice.

Consider the following time-discretized version of (1.11).

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau}+\Pi_N(u^n\partial_x u^n)=\nu^2\partial_{xx}u^{n+1}, \quad (t,x)\in(0,\infty)\times\mathbb{T}, \\ u\big|_{t=0}=u^0=\Pi_N f, \end{cases} \tag{1.12}$$

where $f\in L^\infty(\mathbb{T})$.

Theorem 1.10 (Effective maximum principle, Burgers with forward Euler and spectral truncation)**.** *Let $\nu>0$ and u^n be the solution to* (1.12)*. We have for all $0<\tau<\tau_0=\tau_0(\nu,\|f\|_\infty)$ and $N\geq 2$,*

$$\sup_{n\geq 0}\|u^n\|_\infty\leq\|u^0\|_\infty+\alpha_1\cdot N^{-\frac{1}{2}}+\alpha_2\tau^{0.49},$$

where $\alpha_1,\alpha_2>0$ depend only on ($\|f\|_\infty$, ν).

Remark 1.9. One should observe that the above L^∞ bound has a very nice pattern:

$$\sup_{n\geq 0}\|u^n\|_\infty\leq \text{“}L^\infty \text{ of initial data”}+\text{“Spectral error”}+\text{“Time discretization error”}.$$

The decay rate in N and τ is probably not optimal but we will not dwell on it here.

We now consider the Fourier collocation method applied to the 1D Burgers. Let $U(t)\in\mathbb{R}^N$, $N\geq 2$ solve

$$\begin{cases} \dfrac{d}{dt}U+\dfrac{1}{2}\partial_h(U^{\cdot 2})=\Delta_h U, \quad t>0, \\ U\big|_{t=0}=U^0\in\mathbb{R}^N, \end{cases} \tag{1.13}$$

where for convenience we have set the viscosity coefficient $\nu=1$, and the operators ∂_h, Δ_h correspond to the Fourier multipliers $2\pi ik$, $-4\pi^2k^2$ for $-\frac{N}{2}<k\leq\frac{N}{2}$ respectively in the DFT formula.

Theorem 1.11 (Effective maximum principle, 1D Burgers with Fourier collocation and regular data)**.** *Consider* (1.13) *with $N\geq 2$. Suppose f is a real-valued function on $\mathbb{T}$ having $\hat{f}$ supported in $\{|k|\leq\frac{N}{2}\}$ with $\hat{f}(\frac{N}{2})=\hat{f}(-\frac{N}{2})$ (so that $f=Q_Nf$, where Q_N is defined in the beginning part of Section 5). Let $(U^0)_j=f(\frac{j}{N})$ for all $0\leq j\leq N-1$. Then for all $N\geq N_0=N_0(\|f\|_2)$, we have*

$$\sup_{t\geq 0}\|U(t)\|_\infty\leq\|f\|_\infty+\gamma_1\cdot N^{-\frac{1}{2}}\cdot(1+\|\partial_x f\|_2^2),$$

where $\gamma_1>0$ depends only on $\|f\|_2$.

Remark 1.10. We stress that the condition on the initial function f can certainly be removed, see Theorem 7.4 for a much more general and technical result for rough initial data. Our results give a clear explanation and justification of almost sharp L^∞ bounds observed in practical numerical simulations. By using the machinery developed in this work, it is also possible to give a comprehensive analysis of the fully discrete scheme such as forward Euler in time with implicit treatment of the dissipation term, and Fourier collocation in space. All these will be addressed elsewhere.

Consider on the torus $\mathbb{T}^2=[0,1)^2$ the two dimensional Navier-Stokes system expressed in the vorticity form:

$$\begin{cases}\partial_t\omega+\Pi_N(u\cdot\nabla\omega)=\Delta\omega,\\ \omega\big|_{t=0}=\omega^0=\Pi_N f,\end{cases}\tag{1.14}$$

where $f\in L^\infty(\mathbb{T}^2)$ and has mean zero. We shall work with ω having zero mean which is clearly preserved in time. The velocity u is connected to the vorticity ω through the Biot-Savart law: $u=\nabla^\perp\Delta^{-1}\omega$.

Theorem 1.12 (Effective maximum principle for 2D Navier-Stokes with spectral Galerkin truncation)**.** *Let ω be the solution to* (1.14)*. We have for all $N\geq 2$,*

$$\sup_{t\geq 0}\|\omega(t)\|_\infty\leq\|\omega^0\|_\infty+\alpha\cdot N^{-\frac{1}{2}},$$

where $\alpha>0$ depends only on $\|f\|_\infty$.

For the time-discretized case, consider

$$\begin{cases}\dfrac{\omega^{n+1}-\omega^n}{\tau}+\Pi_N(u^n\cdot\nabla\omega^n)=\Delta\omega^{n+1},\quad (t,x)\in(0,\infty)\times\mathbb{T}^2,\\ \omega\big|_{t=0}=\omega^0=\Pi_N f,\end{cases}\tag{1.15}$$

where $f\in L^\infty(\mathbb{T}^2)$ and has mean zero.

Theorem 1.13 (Effective maximum principle for 2D Navier-Stokes, Forward Euler with spectral truncation)**.** *Let ω^n be the solution to* (1.15)*. We have for all $0<\tau<\tau_0=\tau_0(\|f\|_\infty)$ and $N\geq 2$,*

$$\sup_{n\geq 0}\|\omega^n\|_\infty\leq\|\omega^0\|_\infty+\alpha_1\cdot N^{-\frac{1}{2}}+\alpha_2\tau^{0.4},$$

where $\alpha_1,\alpha_2>0$ depend only on $\|f\|_\infty$.

Notation and preliminaries

For any $k=(k_1,\cdots,k_d)\in\mathbb{R}^d$, we denote

$$|k|=|k|_2=\sqrt{k_1^2+\cdots+k_d^2},\quad |k|_\infty=\max_{1\le j\le d}|k_j|,$$
$$\langle k\rangle=(1+|k|^2)^{\frac{1}{2}}.$$

For any two positive quantities X and Y, we shall write $X\lesssim Y$ or $Y\gtrsim X$ if $X\le CY$ for some constant $C>0$ whose precise value is unimportant. We shall write $X\sim Y$ if both $X\lesssim Y$ and $Y\lesssim X$ hold. We write $X\lesssim_{\alpha_1,\cdots,\alpha_k} Y$ if the constant C depends on some parameters $\alpha_1,\cdots,\alpha_k$. We shall write $X=O(Y)$ if $|X|\lesssim Y$ and $X=O_{\alpha_1,\cdots,\alpha_k}(Y)$ if $|X|\lesssim_{\alpha_1,\cdots,\alpha_k} Y$.

We shall denote $X\ll Y$ if $X\le cY$ for some sufficiently small constant c. The smallness of the constant c is usually clear from the context. The notation $X\gg Y$ is similarly defined. Note that our use of $\ll$ and $\gg$ here is *different* from the usual Vinogradov notation in number theory or asymptotic analysis.

We adopt the following convention for Fourier transforms. Denote for $f:\mathbb{R}^d\to\mathbb{C}$, $g:\mathbb{R}^d\to\mathbb{C}$,

$$(\mathcal{F}f)(\xi)=\int_{\mathbb{R}^d} f(x)e^{-2\pi ix\cdot\xi}dx,$$
$$(\mathcal{F}^{-1}g)(x)=\int_{\mathbb{R}^d} g(\xi)e^{2\pi i\xi\cdot x}d\xi.$$

For any function f defined on the periodic torus $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$ which we identify as $[0,1)^d$, denote

$$\widehat{f}(k)=\int_{[0,1)^d} f(x)e^{-2\pi ik\cdot x}dx.$$

Occasionally we identify $\mathbb{T}^d=[-\frac{1}{2},\frac{1}{2})^d$ in order to isolate the singularity near $x=0$.

For $0<\gamma<1$ and $f:\mathbb{T}^d\to\mathbb{R}$, we define

$$\|f\|_{\dot{C}^\gamma(\mathbb{T}^d)}=\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\gamma},$$
$$\|f\|_{C^\gamma(\mathbb{T}^d)}=\|f\|_{L^\infty(\mathbb{T}^d)}+\|f\|_{\dot{C}^\gamma(\mathbb{T}^d)}.$$

Recall the Dirichlet kernel and unscaled Fejer kernel:

$$D_j(x)=\frac{\sin(2j+1)\pi x}{\sin\pi x}=\sum_{|l|\leq j}e^{2\pi ilx}, \qquad j\geq 0,$$
$$\tilde{F}_k=(\frac{\sin(k+1)\pi x}{\sin\pi x})^2=\sum_{j=0}^{k}D_j(x), \qquad k\geq 0.$$

We also define $\tilde{F}_{-1}=0$, $\tilde{F}_{-2}=1$ and note that

$$2\cos 2\pi kx=\tilde{F}_k-2\tilde{F}_{k-1}+\tilde{F}_{k-2}, \qquad \forall k\geq 0.$$

Now we recall the usual Poisson summation formula which will be used sometimes without explicit mentioning.

Lemma 1.1 (Poisson summation)**.** *Let δ be the usual Dirac distribution. Then*

$$\sum_{k\in\mathbb{Z}^d}e^{-2\pi ik\cdot x}=\sum_{n\in\mathbb{Z}^d}\delta(x-n).$$

For $f\in L^1(\mathbb{R}^d)$ with

$$|f(x)|+|(\mathcal{F}f)(x)|\lesssim(1+|x|)^{-d-\epsilon}$$

for some $\epsilon>0$, we have

$$\sum_{k\in\mathbb{Z}^d}(\mathcal{F}f)(k)=\sum_{n\in\mathbb{Z}^d}f(n),$$
$$\sum_{k\in\mathbb{Z}^d}(\mathcal{F}f)(k)e^{2\pi ik\cdot y}=\sum_{n\in\mathbb{Z}^d}f(y+n), \quad \forall y\in\mathbb{R}^d.$$

Remark 1.11. This is just saying that under suitable decay assumptions the natural periodization of the original function inherits its Fourier coefficients. An immediate useful estimate is: if

$$|f(x)|+|(\mathcal{F}f)(x)|\lesssim(1+|x|)^{-d-\epsilon}$$

for some $\epsilon>0$, then

$$\Big\|\sum_{k\in\mathbb{Z}^d}(\mathcal{F}f)(k)e^{2\pi ik\cdot y}\Big\|_{L^1_y(\mathbb{T}^d)}\leq\|f\|_{L^1(\mathbb{R}^d)},$$

where equality holds when f has a definite sign.

Remark 1.12. Another useful corollary is as follows. Suppose $f\in L^1(\mathbb{T}^d)$ and has absolutely converging Fourier series expansion with Fourier coefficients $\hat{f}(k)$. Then

$$\Big\|\sum_k \hat{f}(k)(\mathcal{F}g)(k)e^{2\pi ik\cdot x}\Big\|_{L^1_x(\mathbb{T}^d)}\leq \|f\|_{L^1_x(\mathbb{T}^d)}\|g\|_{L^1_x(\mathbb{R}^d)},$$

where g and $\mathcal{F}g$ is assumed to have sufficient decay.

Remark 1.13. Observe that for $w\in\mathbb{R}$, one has

$$e^{-|w|}=\frac{1}{\pi}\int_{\mathbb{R}}\frac{1}{1+y^2}e^{iw\cdot y}dy=\int_0^\infty\frac{1}{\sqrt{\pi s}}e^{-s}e^{-\frac{|w|^2}{4s}}ds.$$

Then for $x\in\mathbb{R}^d$, $\beta\geq 0$,

$$e^{-\beta|x|}=\int_0^\infty\frac{1}{\sqrt{\pi s}}e^{-s}e^{-\beta^2\frac{|x|^2}{4s}}ds.$$

Thus

$$\int_{\mathbb{R}^d}e^{-2\pi\beta|x|}e^{-2\pi i\xi\cdot x}dx=\pi^{-\frac{d+1}{2}}\cdot\beta^{-d}\cdot\Big(1+\frac{|\xi|^2}{\beta^2}\Big)^{-\frac{d+1}{2}}\Gamma\Big(\frac{d+1}{2}\Big),$$

where Γ is the usual Gamma function. It follows that for $\beta>0$, $y\in\mathbb{R}^d$,

$$\sum_{k\in\mathbb{Z}^d}\Big(1+\frac{|k|^2}{\beta^2}\Big)^{-\frac{d+1}{2}}e^{2\pi ik\cdot y}=\pi^{\frac{d+1}{2}}\cdot\beta^d\cdot\frac{1}{\Gamma(\frac{d+1}{2})}\sum_{n\in\mathbb{Z}^d}e^{-2\pi\beta|y+n|}.$$

Interestingly, for dimension $d=1$, one can take the limit $\beta\to 0$ and derive

$$\sum_{0\neq k\in\mathbb{Z}}\frac{1}{k^2}e^{2\pi ik|y|}=2\pi^2\Big(y^2-|y|+\frac{1}{6}\Big),\quad\forall y\in[-1,1].$$

Denote for $0\leq y<1$,

$$F_2(y)=\sum_{0\neq k\in\mathbb{Z}}\frac{1}{k^2}e^{2\pi iky}=2\pi^2\Big(y^2-y+\frac{1}{6}\Big),$$

and extend F_2 to the whole $\mathbb{R}$ via $F_2(y)=F_2(1+y)$. It is easy to check that F_2 coincides with $2\pi^2(y^2-|y|+\frac{1}{6})$ in the domain $[-1,1]$ and there is no inconsistency in the definition. One should recognize that the function $F_2(y)$ is the usual Bernoulli polynomial.

2 Linear setting: 1D torus continuous case

Consider the periodic 1D torus $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ which can be identified as $[0,1)$. For periodic function $f:\mathbb{T}\to\mathbb{C}$ and integer $N\geq 1$, recall

$$\Pi_N f=\sum_{|k|\leq N}\hat{f}(k)e^{2\pi ik\cdot x}.$$

Note that $\Pi_N f=D_N*f$, where D_N is the usual Dirichlet kernel given by

$$D_N(x)=\sum_{|k|\leq N}e^{2\pi ik\cdot x}=\frac{\sin(2N+1)\pi x}{\sin\pi x}.$$

Proposition 2.1. *Let $N\geq 1$. Then*

$$\|\Pi_N f\|_{L^\infty(\mathbb{T})}=\|D_N*f\|_{L^\infty(\mathbb{T})}\lesssim\log(N+2)\|f\|_{L^\infty(\mathbb{T})}.$$

The bound is sharp in the following sense. For each $N\geq 1$, there exists $f_N\in C^\infty(\mathbb{T})$, such that

$$|(D_N*f_N)(0)|>c_1\log(N+2)\cdot\|f_N\|_{L^\infty(\mathbb{T})},$$

where $c_1>0$ is an absolute constant.

Proof. This is rather standard. For the lower bound define

$$\tilde{f}_N=\begin{cases}\displaystyle\sum_{j=0}^{N}\chi_{[\frac{2j}{2N+1},\frac{2j+1}{2N+1})}-\sum_{j=0}^{N-1}\chi_{[\frac{2j+1}{2N+1},\frac{2j+2}{2N+1})}, & 0\leq x<1,\\ \text{periodic extension.}\end{cases}$$

Mollifying $\tilde{f}_N$ suitably then yields the lower bound. □

Proposition 2.2. *Let $N\geq 1$. Define*

$$\Omega_N^+=\{x\in[0,1]:\,D_N(x)>0\}=\bigcup_{j=0}^{N}\Big(\frac{2j}{2N+1},\frac{2j+1}{2N+1}\Big),$$

$$\Omega_N^-=\{x\in[0,1]:\,D_N(x)<0\}=\bigcup_{j=0}^{N-1}\Big(\frac{2j+1}{2N+1},\frac{2j+2}{2N+1}\Big).$$

Then

$$\mathrm{Leb}(\Omega_N^+)=\frac{N+1}{2N+1},\quad \mathrm{Leb}(\Omega_N^-)=\frac{N}{2N+1},$$

and

$$\int_{\Omega_N^+} D_N(x)dx \geq \frac{1}{\pi^2}\log(2N+3),$$
$$\int_{\Omega_N^-} (-D_N(x))dx \geq \frac{1}{\pi^2}\log(N+1).$$

Proof. Clearly

$$\begin{aligned}
&\sum_{j=0}^{N}\int_{\frac{2j}{2N+1}}^{\frac{2j+1}{2N+1}} \frac{\sin(2N+1)\pi x}{\sin\pi x}dx\\
=&\sum_{j=0}^{N}\int_{0}^{\frac{1}{2N+1}} \frac{\sin(2N+1)\pi y}{\sin(\frac{2j\pi}{2N+1}+\pi y)}dy\\
\geq&\sum_{j=0}^{N}\frac{1}{\frac{2j+1}{2N+1}\pi}\int_{0}^{\frac{1}{2N+1}}\sin(2N+1)\pi y dy\\
\geq&\frac{2}{\pi^2}\sum_{j=0}^{N}\frac{1}{2j+1}\geq\frac{2}{\pi^2}\int_0^{N+1}\frac{1}{2x+1}dx=\frac{1}{\pi^2}\log(2N+3).
\end{aligned}$$

Similarly,

$$-\sum_{j=0}^{N-1}\int_{\frac{2j+1}{2N+1}}^{\frac{2j+2}{2N+1}} \frac{\sin(2N+1)\pi x}{\sin\pi x}dx \geq \frac{1}{\pi^2}\sum_{j=0}^{N-1}\frac{1}{j+1}\geq\frac{1}{\pi^2}\log(N+1).$$

Thus, we complete the proof. □

Let $\nu>0$, $N\geq 1$, and consider

$$K_N(x)=\sum_{|k|\leq N}\frac{1}{1+4\pi^2\nu^2k^2}e^{2\pi ik\cdot x}, \qquad x\in\mathbb{T},$$
$$K_{>N}(x)=\sum_{|k|>N}\frac{1}{1+4\pi^2\nu^2k^2}e^{2\pi ik\cdot x}, \qquad x\in\mathbb{T}.$$

Note that

$$\|K_{>N}\|_1\leq\|K_{>N}\|_2=\Big(\sum_{|k|>N}\frac{1}{(1+4\pi^2\nu^2k^2)^2}\Big)^{\frac{1}{2}}\lesssim\nu^{-2}\cdot N^{-\frac{3}{2}}.$$

A better bound is available. See below.

Proposition 2.3. *Let $\nu>0$. We have*

$$\frac{c_1}{1+(\pi N\nu)^2}\log(N+2)\le\|K_{>N}\|_{L^1(\mathbb{T})}\le\frac{c_2}{1+(\pi N\nu)^2}\log(N+2),\quad\forall N\ge1,$$

where $c_1>0$, $c_2>0$ are absolute constants.

Proof. We first show the upper bound.

Choose $\phi\in C_c^\infty(\mathbb{R})$ such that $\phi(x)=1$ for $|x|\le1$ and $\phi(x)=0$ for $|x|\ge2$. Choose $\phi_1\in C_c^\infty(\mathbb{R})$ such that $\phi_1(x)=1$ for $1\le|x|\le2$ and $\phi_1(x)=0$ for $x\le0.9$ or $x\ge2.1$. Then

$$\begin{aligned}K_{>N}(x)&=\sum_{|k|>N}\frac{1}{1+4\pi^2\nu^2k^2}\phi(\frac{k}{N})e^{2\pi ik\cdot x}+\sum_{|k|>N}\frac{1}{1+4\pi^2\nu^2k^2}\Big(1-\phi\Big(\frac{k}{N}\Big)\Big)e^{2\pi ik\cdot x}\\&=\sum_{N<|k|\le2N}\frac{1}{1+4\pi^2\nu^2k^2}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}+\sum_{k\in\mathbb{Z}}\frac{1}{1+4\pi^2\nu^2k^2}\Big(1-\phi\Big(\frac{k}{N}\Big)\Big)e^{2\pi ik\cdot x}\\&=\sum_{N<|k|\le2N}\frac{1}{1+4\pi^2\nu^2k^2}\phi\Big(\frac{k}{N}\Big)\phi_1\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}+\sum_{k\in\mathbb{Z}}\frac{1}{1+4\pi^2\nu^2k^2}\Big(1-\phi\Big(\frac{k}{N}\Big)\Big)e^{2\pi ik\cdot x}\\&=:A(x)+B(x).\end{aligned}$$

For the first piece, denote

$$K_1(x)=\mathcal{F}^{-1}\Big(\frac{1}{1+(2\pi\nu\xi)^2}\phi\Big(\frac{\xi}{N}\Big)\phi_1\Big(\frac{\xi}{N}\Big)\Big)$$

and note that (this is where the localization ϕ_1 is needed)

$$\|K_1\|_{L^1(\mathbb{R})}\lesssim\frac{1}{1+(\pi\nu N)^2}.$$

Observe

$$\begin{aligned}A(x)&=\int_{\mathbb{R}}\Big(\frac{1}{1+(2\pi\nu\xi)^2}\phi\Big(\frac{\xi}{N}\Big)\phi_1\Big(\frac{\xi}{N}\Big)\Big)\cdot\Big(\sum_{N<|k|\le2N}\delta(\xi-k)\Big)e^{2\pi i\xi\cdot x}d\xi\\&=\sum_{n\in\mathbb{Z}}\int_0^1K_1(x-y-n)\tilde{D}_N(y)dy,\end{aligned}$$

where

$$\tilde{D}_N(y)=\sum_{N<|k|\le2N}e^{2\pi ik\cdot y}.$$

Thus

$$\|A\|_{L^1(\mathbb{T})}\lesssim\|K_1\|_{L^1(\mathbb{R})}\|\tilde{D}_N\|_{L^1(\mathbb{T})}\lesssim\frac{1}{1+(\pi\nu N)^2}\log N.$$

For the second piece we discuss two cases. If $N\leq\frac{1}{\pi\nu}$, then we simply have

$$\|B\|_{L^1(\mathbb{T})}\lesssim 1.$$

If $N>\frac{1}{\pi\nu}$, then one can make use of the identity

$$\begin{aligned}&\frac{1}{1+4\pi^2\nu^2\xi^2}\cdot\Big(1-\phi\Big(\frac{\xi}{N}\Big)\Big)\\=&\frac{1}{4\pi^2\nu^2N^2}\cdot\Big(1-\frac{1}{1+4\pi^2\nu^2\xi^2}\Big)\cdot\frac{1-\phi\big(\frac{\xi}{N}\big)}{\big(\frac{\xi}{N}\big)^2}.\end{aligned}$$

Thus in this case

$$\|B\|_{L^1(\mathbb{T})}\lesssim\frac{1}{\pi^2\nu^2N^2}\lesssim\frac{1}{1+(\pi\nu N)^2}.$$

Now we turn to the lower bound. We first consider the situation that N is sufficiently large (larger than an absolute constant). This amounts to showing

$$\|A\|_{L^1(\mathbb{T})}\gtrsim\frac{1}{1+(\pi\nu N)^2}\log N.$$

Denote (below ϕ and ϕ_1 are the same functions used in the definition of $A(x)$)

$$\begin{aligned}\tilde{A}(x)=&\sum_{N<|k|\leq 2N}\phi\Big(\frac{k}{N}\Big)\phi_1\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}\\=&\sum_{N<|k|\leq 2N}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}=\sum_{|k|>N}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}\\=&-\sum_{|k|\leq N}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}+\sum_{k\in\mathbb{Z}}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}\\=&-\sum_{|k|\leq N}e^{2\pi ik\cdot x}+\sum_{k\in\mathbb{Z}}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}\\=&-D_N(x)+\sum_{k\in\mathbb{Z}}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}.\end{aligned}$$

Clearly for N large,

$$\|\tilde{A}\|_{L^1(\mathbb{T})} \gtrsim \log N.$$

Now observe that on $\mathbb{T}$:

$$\tilde{A} = (1-\nu^2\partial_{xx})A.$$

Let $\psi \in C_c^\infty(\mathbb{R})$ be such that $\psi(\xi)=1$ for $0.5 \le |\xi| \le 3$ and $\psi(\xi)=0$ for $|\xi| \le 0.4$ or $|\xi|>4$. Let

$$m(x) = \int_{\mathbb{R}} \frac{1+(2\pi\nu\xi)^2}{1+(2\pi\nu N)^2} \psi\Big(\frac{\xi}{N}\Big) e^{2\pi i \xi\cdot x} d\xi.$$

It is easy to check that $\|m\|_{L^1(\mathbb{R})} \lesssim 1$. Now since

$$\tilde{A}(x) = (1+(2\pi\nu N)^2) \sum_{n\in\mathbb{Z}} \int_0^1 m(x-y)A(y)dy,$$

we clearly have

$$\|\tilde{A}\|_{L^1(\mathbb{T})} \lesssim (1+(2\pi\nu N)^2) \cdot \|A\|_{L^1(\mathbb{T})}.$$

Thus the lower bound for $\|A\|_{L^1(\mathbb{T})}$ is shown when N is large.

Now if N is of constant order, we can use the interpolation inequality

$$\|K_{>N}\|_2 \lesssim \|K_{>N}\|_1^{\frac{1}{2}} \|K_{>N}\|_\infty^{\frac{1}{2}}.$$

Note that for $N \sim 1$,

$$\|K_{>N}\|_2^2 \sim \sum_{k\gtrsim 1} \frac{1}{(1+k^2\nu^2)^2} \sim \begin{cases} \dfrac{1}{\nu}, & 0<\nu<1, \\ \nu^{-4}, & \nu \ge 1. \end{cases} \tag{2.1}$$

Also

$$\|K_{>N}\|_\infty \lesssim \sum_{k\gtrsim 1} \frac{1}{1+k^2\nu^2} \lesssim \begin{cases} \dfrac{1}{\nu}, & 0<\nu<1, \\ \nu^{-2}, & \nu \ge 1. \end{cases} \tag{2.2}$$

Thus for $N \sim 1$,

$$\|K_{>N}\|_1 \gtrsim \frac{1}{1+\nu^2}. \tag{2.3}$$

Thus, we complete the proof. □

In Proposition 2.3, the lower bound on $K_{>N}$ can also be obtained from a more general result, see Proposition 2.4 below.

We first need a simple lemma.

Lemma 2.1. *For any $\theta_0\in[0,1)$, there exits α_1 (α_1 may depend on θ_0) with $0<10^{-6}<\alpha_1<1-10^{-6}$, such that*

$$\mathrm{Im}(e^{2\pi i(\theta_0+x)})\geq 10^{-9},\quad \forall x\in[\alpha_1,\alpha_1+10^{-7}].$$

Proof. Obvious. One just need to discuss separately the cases θ_0 in different intervals and choose the corresponding shifts suitably. □

Let $\phi\in\mathcal{S}(\mathbb{R})$ and note that $\widehat{\phi}(1)$ may be complex-valued. If $\widehat{\phi}(1)\neq 0$, then by Lemma 2.1, we have for some $0<10^{-6}<\alpha_1<1-10^{-6}$,

$$\mathrm{Im}\Big(e^{2\pi iy}\frac{\widehat{\phi}(1)}{|\widehat{\phi}(1)|}\Big)\geq 10^{-9},\quad \forall y\in[\alpha_1,\alpha_1+10^{-7}].$$

Define

$$\begin{aligned}Q_N(D_N)(x)&=\int_{\mathbb{R}}N\phi(Ny)\frac{\sin((2N+1)\pi(x-y))}{\sin(\pi(x-y))}dy\\&=\int_{\mathbb{R}}\phi(y)\frac{\sin\big((2N+1)\pi\big(x-\frac{1}{N}y\big)\big)}{\sin\big(\pi\big(x-\frac{1}{N}y\big)\big)}dy.\end{aligned}$$

One typical case is that $\widehat{\phi}(\xi)$ is localized to $\{|\xi|\sim 1\}$ so that $Q_N(D_N)$ is the frequency projection of the Dirichlet kernel to the frequency annulus $\{|\xi|\sim N\}$. Note that if $x=\frac{2j_0+\epsilon}{2N+1}$, where j_0 is an integer, $N^{\frac{1}{10}}\leq j_0\leq N^{\frac{1}{2}}$, $\alpha_1\leq\frac{\epsilon}{2}\leq\alpha_1+10^{-7}$ with α_1 the same as in Lemma 2.1, then as $N\to\infty$,

$$\begin{aligned}Q_N(D_N)(x)\cdot x&=\int_{\mathbb{R}}\phi(y)\frac{\sin((2+\frac{1}{N})\pi(\frac{N}{2N+1}\epsilon-y))}{\sin(\pi(\frac{2j_0+\epsilon}{2N+1}-\frac{y}{N}))}\Big(\frac{2j_0+\epsilon}{2N+1}\Big)dy\\&\to\frac{1}{\pi}\int_{\mathbb{R}}\phi(y)\sin\Big(2\pi\Big(\frac{1}{2}\epsilon-y\Big)\Big)dy\\&=\frac{1}{\pi}\mathrm{Im}(e^{\pi i\epsilon}\widehat{\phi}(1))\gtrsim|\widehat{\phi}(1)|.\end{aligned}$$

This heuristic computation then leads to the following proposition.

Proposition 2.4. *If $\widehat{\phi}(1)=0$, then*

$$\lim_{N\to\infty}\frac{\|Q_N(D_N)\|_{L^1(\mathbb{T})}}{\log N}=0.$$

If $\widehat{\phi}(1)\neq 0$, then there exists some $N_0=N_0(\phi)$ such that if $N\geq N_0$, then

$$c_1|\widehat{\phi}(1)|\log N\leq\|Q_N(D_N)\|_{L^1(\mathbb{T})}\leq c_2|\widehat{\phi}(1)|\log N,$$

where $c_1>0$, $c_2>0$ are absolute constants.

Proof. Consider the case $\widehat{\phi}(1)\neq 0$. We first show the lower bound. Let $x_0=\frac{2j_0+\epsilon}{2N+1}$, where j_0 is an odd integer (this is to ensure that the intervals corresponding to different j_0 are non-overlapping), $N^{\frac{1}{10}}\leq j_0\leq N^{\frac{1}{2}}$, $\alpha_1\leq\frac{\epsilon}{2}\leq\alpha_1+10^{-7}$ with α_1 the same as in Lemma 2.1, then

$$\begin{aligned}Q_N(D_N)(x_0)=&\int_{\mathbb{R}}\phi(y)\cdot\chi_{|y|\leq N^{\frac{1}{20}}}\frac{\sin((2+\frac{1}{N})\pi(\frac{N}{2N+1}\epsilon-y))}{\sin(\pi(\frac{2j_0+\epsilon}{2N+1}-\frac{y}{N}))}dy\\&+\int_{\mathbb{R}}\phi(y)\cdot\chi_{|y|>N^{\frac{1}{20}}}\frac{\sin((2+\frac{1}{N})\pi(\frac{N}{2N+1}\epsilon-y))}{\sin(\pi(\frac{2j_0+\epsilon}{2N+1}-\frac{y}{N}))}dy.\end{aligned}$$

Since ϕ is a Schwartz function, the second piece above can be easily bounded by N^{-10} which is negligible as N tends to infinity. In the computation below we shall completely ignore this piece. Then

$$\begin{aligned}Q_N(D_N)(x_0)\cdot x_0\approx&\int_{\mathbb{R}}\phi(y)\chi_{|y|\leq N^{\frac{1}{20}}}\frac{\sin((2+\frac{1}{N})\pi(\frac{N}{2N+1}\epsilon-y))}{\sin(\pi(\frac{2j_0+\epsilon}{2N+1}-\frac{y}{N}))}\Big(\frac{2j_0+\epsilon}{2N+1}\Big)dy\\=&\int_{\mathbb{R}}\phi(y)\chi_{|y|\leq N^{\frac{1}{20}}}\frac{\sin((2+\frac{1}{N})\pi(\frac{N}{2N+1}\epsilon-y))}{\sin(\pi(\frac{2j_0+\epsilon}{2N+1}-\frac{y}{N}))}\Big(\frac{2j_0+\epsilon}{2N+1}-\frac{y}{N}\Big)dy\\&+\frac{1}{N}\int_{\mathbb{R}}y\phi(y)\chi_{|y|\leq N^{\frac{1}{20}}}\frac{\sin((2+\frac{1}{N})\pi(\frac{N}{2N+1}\epsilon-y))}{\sin(\pi(\frac{2j_0+\epsilon}{2N+1}-\frac{y}{N}))}dy.\end{aligned}$$

Note that the second piece is bounded by $\mathcal{O}(N^{-\frac{1}{10}})$ and it is negligible. For the first piece, one notes that $|\frac{x}{\sin x}-1|\lesssim x^2$ for $|x|\ll 1$. Thus

$$\begin{aligned}Q_N(D_N)(x_0)\cdot x_0=&\frac{1}{\pi}\int_{\mathbb{R}}\phi(y)\chi_{|y|\leq N^{\frac{1}{20}}}\sin\Big(\Big(2+\frac{1}{N}\Big)\pi\Big(\frac{N}{2N+1}\epsilon-y\Big)\Big)dy+\mathcal{O}\Big(N^{-\frac{1}{10}}\Big)\\=&\frac{1}{\pi}\underbrace{\int_{\mathbb{R}}\phi(y)\sin\Big(\Big(2+\frac{1}{N}\Big)\pi\Big(\frac{N}{2N+1}\epsilon-y\Big)\Big)dy}_{=:I_N}+\mathcal{O}\Big(N^{-\frac{1}{10}}\Big).\end{aligned}$$

Note that in I_N there is no dependence on j_0. As $N\to\infty$, we have

$$I_N\to\int_{\mathbb{R}}\phi(y)\sin\Big(2\pi\Big(\frac{1}{2}\epsilon-y\Big)\Big)dy=\mathrm{Im}(e^{\pi i\epsilon}\widehat{\phi}(1))\gtrsim|\widehat{\phi}(1)|,$$

where in the last step we used Lemma 2.1. It follows that

$$\sum_{\substack{N^{\frac{1}{10}}\leq j_0\leq N^{\frac{1}{2}}\\ j_0 \text{ is odd}}}\int_{\frac{2j_0+2\alpha_1}{2N+1}}^{\frac{2j_0+2\alpha_1+2\cdot 10^{-7}}{2N+1}} Q_N(D_N)(x)dx\gtrsim|\widehat{\phi}(1)|\sum_{\substack{N^{\frac{1}{10}}\leq j_0\leq N^{\frac{1}{2}}\\ j_0 \text{ is odd}}}\frac{1}{\frac{j_0}{N}}\cdot\frac{1}{N}\gtrsim(\log N)|\widehat{\phi}(1)|.$$

Now we turn to the upper bound. For $x=\frac{j+\epsilon}{2N+1}$ with $0\leq\epsilon\leq 1$ and j being an integer in $[0,2N]$, we have

$$\begin{aligned}Q_N(D_N)(x)=&\int_{\mathbb{R}}\phi(y)\frac{\sin((2N+1)\pi(x-\frac{1}{N}y))}{\sin(\pi(x-\frac{1}{N}y))}dy\\ =&(-1)^j\int_{\mathbb{R}}\phi(y)\frac{\sin((2N+1)\pi(\frac{\epsilon}{2N+1}-\frac{1}{N}y))}{\sin(\pi(\frac{j+\epsilon}{2N+1}-\frac{1}{N}y))}dy.\end{aligned}$$

For the regime $j\leq\alpha\log N$ where $\alpha=|\widehat{\phi}(1)|/\|\phi\|_{L^1(\mathbb{R})}$, we note that $\|Q_N(D_N)\|_\infty\lesssim\|\phi\|_{L^1(\mathbb{R})}\cdot N$, and

$$\|Q_N(D_N)\|_{L^1([\frac{j}{2N+1},\frac{j+1}{2N+1}])}\lesssim\|\phi\|_{L^1(\mathbb{R})}.$$

Thus

$$\sum_{j\leq\alpha\log N}\|Q_N(D_N)\|_{L^1([\frac{j}{2N+1},\frac{j+1}{2N+1}])}\lesssim\|\phi\|_{L^1(\mathbb{R})}\alpha\log N\lesssim|\widehat{\phi}(1)|\log N.$$

Next consider the regime $\frac{N}{\log N}\leq j\leq 2N$. Clearly we have

$$\begin{aligned}&\sum_{\frac{N}{\log N}\leq j\leq 2N}\int_{\frac{j}{2N+1}}^{\frac{j+1}{2N+1}}|Q_N(D_N)(x)|dx\\ \lesssim&\|\phi\|_{L^1(\mathbb{R})}\sum_{\frac{N}{\log N}\leq j\leq 2N}\frac{1}{j}\lesssim\|\phi\|_{L^1(\mathbb{R})}\log\log N\lesssim|\widehat{\phi}(1)|\log N,\end{aligned}$$

for N sufficiently large. Finally consider the regime $\alpha\log N\leq j\leq\frac{N}{\log N}$. For $x=\frac{j+\epsilon}{2N+1}$, we have

$$\begin{aligned}Q_N(D_N)(x)\cdot x=&(-1)^j\int_{\mathbb{R}}\phi(y)\frac{\sin\Big((2N+1)\pi\Big(\frac{\epsilon}{2N+1}-\frac{1}{N}y\Big)\Big)}{\sin\Big(\pi\Big(\frac{j+\epsilon}{2N+1}-\frac{1}{N}y\Big)\Big)}\Big(\frac{j+\epsilon}{2N+1}-\frac{1}{N}y\Big)dy\\ &+\frac{(-1)^j}{N}\int_{\mathbb{R}}y\phi(y)\frac{\sin\Big((2N+1)\pi\Big(\frac{\epsilon}{2N+1}-\frac{1}{N}y\Big)\Big)}{\sin\Big(\pi\Big(\frac{j+\epsilon}{2N+1}-\frac{1}{N}y\Big)\Big)}dy.\end{aligned}$$

Note that the second piece above can be easily bounded by $(\log N)^{-1}$. For the first piece one can use the cut-off $\chi_{|y|\leq\frac{1}{100}\alpha\log N}$ and the inequality $|\frac{x}{\sin x}-1|\ll x^2$ for $|x|\ll 1$ to extract the main order. It follows easily that

$$|Q_N(D_N)(x)|\cdot|x|\lesssim|\widehat{\phi}(1)|+\mathcal{O}((\log N)^{-1})\lesssim|\widehat{\phi}(1)|$$

for N sufficiently large. Collecting all estimates, we obtain

$$\|Q_N(D_N)\|_{L^1(\mathbb{T})}\lesssim|\widehat{\phi}(1)|\cdot\log N.$$

Finally we turn to the case $\widehat{\phi}(1)=0$. The analysis is similar. First the regime $\frac{N}{\log N}\leq j\leq 2N$ is acceptable since it gives at most $\log\log N$ growth. The other two regimes are

$$j\leq\sqrt{\log N}\quad\text{and}\quad\sqrt{\log N}\leq j\leq\frac{N}{\log N}.$$

We omit the details. □

In the next few propositions, we outline an alternative approach to obtain the L^1-norm bound on $K_{>N}$. The advantage is that it can be used on more general trigonometric series whose coefficients satisfy certain convexity properties.

Proposition 2.5. *Let $\nu>0$. Let $N_0=e^{3\pi^2}$. If $N\geq\max\{N_0,\frac{1}{2\sqrt{3}\pi\nu}\}$, then*

$$\frac{c_1}{(N\nu)^2}\log N\leq\|K_{>N}\|_{L^1(\mathbb{T})}\leq\frac{c_2}{(N\nu)^2}\log N,$$

where $c_1>0$, $c_2>0$ are absolute constants.

Remark 2.1. In the regime $N\ll\frac{1}{\nu}$ (for $0<\nu\ll 1$), the L^1 bound of $K_{>N}$ and K_N can also be quite large. For example, fix $N=N_0$ independent of ν and take $\nu\to 0$, then the main order of K_{N_0} is given by the Dirichlet kernel, and we have (below we use L^2 to estimate the error piece)

$$\begin{aligned}\|K_{N_0}\|_{L^1(\mathbb{T})}&\geq\Big\|\frac{\sin(2N_0+1)\pi x}{\sin\pi x}\Big\|_{L^1(\mathbb{T})}-\text{const}\cdot\nu^2\cdot N_0^{2.5}\\&\geq\text{const}\cdot\log N_0-\text{const}\cdot\nu^2\cdot N_0^{2.5}\\&\gtrsim\log N_0,\end{aligned}$$

where $\nu>0$ is sufficiently small. Noting that

$$\|K_{N_0}+K_{>N_0}\|_{L^1(\mathbb{T})}=1,$$

we obtain a similar lower bound for $K_{>N_0}$.

Proof. Denote

$$c_k=\frac{1}{1+4\pi^2\nu^2k^2},$$
$$\tilde{F}_k(x)=\sum_{j=0}^{k}D_j(x)=\left(\frac{\sin(k+1)\pi x}{\sin\pi x}\right)^2.$$

Note that $\|\tilde{F}_k\|_1=k+1$ and $\tilde{F}_k-\tilde{F}_{k-1}=D_k$. Then

$$\begin{aligned}K_{>N}(x)&=\sum_{|k|\geq N+1}\frac{1}{1+4\pi^2\nu^2k^2}e^{2\pi ik\cdot x}=\sum_{k\geq N+1}c_k(\tilde{F}_k-2\tilde{F}_{k-1}+\tilde{F}_{k-2})\\&=\sum_{k\geq N+1}c_k\tilde{F}_k-2\sum_{k\geq N}c_{k+1}\tilde{F}_k+\sum_{k\geq N-1}c_{k+2}\tilde{F}_k\\&=\sum_{k\geq N}(c_k-2c_{k+1}+c_{k+2})\tilde{F}_k-c_N\tilde{F}_N+c_{N+1}\tilde{F}_{N-1}\\&=\sum_{k\geq N}(c_k-2c_{k+1}+c_{k+2})\tilde{F}_k-(c_N-c_{N+1})\tilde{F}_{N-1}-c_ND_N.\end{aligned}$$

Note that if $12k^2\pi^2\nu^2-1\geq 0$, then

$$c_k-2c_{k+1}+c_{k+2}\geq 0.$$

It is easy to check that

$$(c_N-c_{N+1})\|\tilde{F}_{N-1}\|_{L^1(\mathbb{T})}=\frac{4\pi^2(2N+1)\nu^2N}{(1+4\pi^2N^2\nu^2)(1+4\pi^2(N+1)^2\nu^2)}\leq\frac{2}{1+4\pi^2N^2\nu^2}.$$

By Proposition 2.2, we have

$$\int_{\Omega_N^-}K_{>N}(x)dx\geq\frac{1}{1+4\pi^2N^2\nu^2}\frac{1}{\pi^2}\log(N+1)-\frac{2}{1+4\pi^2N^2\nu^2}\gtrsim\frac{1}{(N\nu)^2}\log N.$$

Thus

$$\begin{aligned}\|K_{>N}\|_{L^1(\mathbb{T})}\geq\|K_{>N}\|_{L^1(\Omega_N^-)}&\geq\int_{\Omega_N^-}K_{>N}(x)dx\\&\gtrsim\frac{1}{(N\nu)^2}\log N.\end{aligned}$$

For the upper bound, one notes that $\int_{\mathbb{T}}K_{>N}(x)dx=0$, and

$$\|K_{>N}\|_{L^1(\mathbb{T})}\leq 2N(c_N-c_{N+1})+2c_N\|D_N\|_{L^1(\mathbb{T})}\lesssim\frac{1}{(N\nu)^2}\log N.$$

So, we complete the proof. □

Proposition 2.6. *Let $\nu>0$. There exists an absolute constant $N_1>0$ such that if $N_1\leq N$ and $N\leq\frac{1}{2\sqrt{3}\pi\nu}$, then*

$$\|K_N\|_{L^1(\mathbb{T})}\sim\log N,$$
$$\|K_{>N}\|_{L^1(\mathbb{T})}\sim\log N.$$

Consequently by Proposition 2.5*, for all $N\geq\max\{N_1,e^{3\pi^2}\}$, we have*

$$\|K_{>N}\|_{L^1(\mathbb{T})}\sim\frac{1}{1+(N\nu)^2}\log N.$$

Proof. We only need to show that $\|K_N\|_{L^1(\mathbb{T})}\gtrsim\log N$. Denote again $c_k=\frac{1}{1+4\pi^2\nu^2k^2}$. Then

$$\begin{aligned}K_N(x)&=\sum_{|k|\leq4}c_ke^{2\pi ik\cdot x}+\sum_{4<k\leq N}c_k(\tilde{F}_k-2\tilde{F}_{k-1}+\tilde{F}_{k-2})\\&=\sum_{|k|\leq4}c_ke^{2\pi ik\cdot x}+\sum_{k=4}^{N-1}(c_k-2c_{k+1}+c_{k+2})\tilde{F}_k+c_ND_N\\&\quad+(c_N-c_{N+1})\tilde{F}_{N-1}+(c_5-c_4)\tilde{F}_3-c_4D_4.\end{aligned}$$

Note that for $k\leq N-1$, we have

$$c_k-2c_{k+1}+c_{k+2}=\frac{\beta}{1+\beta(k+1)^2}\cdot\frac{2\beta(3k^2+6k+2)-2}{(1+\beta k^2)(1+\beta(k+2)^2)}\leq0,$$

where $\beta=(2\pi\nu)^2$. Here we used the assumption $N\leq\frac{1}{2\sqrt{3}\pi\nu}$.

Recall $\Omega_N^-=\{x\in\mathbb{T}:D_N(x)<0\}$. Then

$$\|K_N\|_{L^1(\mathbb{T})}\geq\|K_N\|_{L^1(\Omega_N^-)}\geq c_N\|D_N\|_{L^1(\Omega_N^-)}-\text{const}-(c_N-c_{N+1})N\gtrsim\log N.$$

Thus, we complete the proof. □

Lemma 2.2. *Let*

$$F_N(x)=-\frac{\sin(2N+1)\pi x}{\sin\pi x}-\frac{2}{N}\cdot\left(\frac{\sin N\pi x}{\sin\pi x}\right)^2.$$

There exists an absolute constant $N_0>0$ sufficiently large, such that the following hold for any $N\geq N_0$, $\epsilon\in[\frac{1}{10},\frac{1}{6}]$:

If $x=\frac{j+1-\epsilon}{2N}$, $N^{\frac{1}{10}}<j<N^{\frac{1}{2}}$, and j is odd, then

$$F_N(x)\geq\beta_1\frac{N}{j},$$

where β_1 is an absolute constant.

Proof. Note that

$$-\sin(2N+1)\pi x=(-1)^{j+1}\sin\left(\epsilon\pi-\frac{j+1-\epsilon}{2N}\pi\right)=\sin\left(\epsilon\pi-\frac{j+1-\epsilon}{2N}\pi\right)\geq\frac{1}{2}\epsilon\pi,$$
$$2\sin^2 N\pi x=1-\cos 2N\pi x=1-(-1)^{j+1}\cos(\epsilon\pi)=\frac{1}{2}\epsilon^2\pi^2+\mathcal{O}(\epsilon^4),$$

where in the first inequality above we have used the assumptions on N and ϵ. Then

$$-\frac{\sin(2N+1)\pi x}{\sin\pi x}\geq\text{const}\cdot\epsilon\cdot\frac{N}{j},$$
$$\frac{2}{N}\cdot\left(\frac{\sin N\pi x}{\sin\pi x}\right)^2\leq\text{const}\cdot\frac{N}{j^2}\cdot\epsilon^2.$$

The desired result then clearly follows. □

Lemma 2.2 leads to an interesting point-wise lower bound on $K_{>N}$. It also yields another proof for $\|K_{>N}\|_{L^1(\mathbb{T})}$. We record it here for the sake of completeness.

Proposition 2.7. *Let $\nu>0$. If $N\geq\max\{N_0,\frac{1}{2\sqrt{3}\pi\nu}\}$ where N_0 is the same absolute constant as in Lemma 2.2, then for any $0\leq x<1$,*

$$K_{>N}(x)\geq\beta_1\sum_{\substack{N^{\frac{1}{10}}<j<N^{\frac{1}{2}}\\ j\ is\ odd}}\frac{1}{1+4\pi^2\nu^2N^2}\cdot\frac{N}{j}\cdot 1_{[\frac{j+\frac{5}{6}}{2N},\frac{j+\frac{9}{10}}{2N}]},$$

where β_1 is the same absolute constant as in Lemma 2.2. Consequently

$$\|K_{>N}\|_{L^1(\mathbb{T})}\gtrsim\frac{1}{1+4\pi^2\nu^2N^2}\log(N+2).$$

Proof. Clearly

$$\begin{aligned}K_{>N}(x)&=\sum_{|k|\geq N+1}\frac{1}{1+4\pi^2\nu^2k^2}e^{2\pi ik\cdot x}\\&=\sum_{k\geq N}(c_k-2c_{k+1}+c_{k+2})\tilde{F}_k-c_ND_N-(c_N-c_{N+1})\tilde{F}_{N-1},\end{aligned}$$

where we use again the kernel

$$\tilde{F}_k(x)=\sum_{j=0}^{k}D_j(x)=\left(\frac{\sin(k+1)\pi x}{\sin\pi x}\right)^2$$

and

$$c_k = \frac{1}{1+4\pi^2\nu^2k^2}.$$

Note that for $k \geq N \geq \frac{1}{2\sqrt{3}\pi\nu}$, we have

$$c_k - 2c_{k+1} + c_{k+2} \geq 0.$$

Noting that $c_N - c_{N+1} \leq c_N \cdot \frac{2}{N}$, we obtain

$$\sum_{|k|\geq N+1} \frac{1}{1+4\pi^2\nu^2k^2} e^{2\pi ik\cdot x} \geq \frac{1}{1+4\pi^2\nu^2N^2} \cdot \left[-\frac{\sin(2N+1)\pi x}{\sin \pi x} - \frac{2}{N} \cdot \left(\frac{\sin N\pi x}{\sin \pi x} \right)^2 \right].$$

The point-wise lower bound then easily follows from Lemma 2.2. The L^1 lower bound follows from direct integration. □

Proposition 2.8. *Let $\nu > 0$. If $N \geq \frac{1}{2\pi^2\nu} e^{\frac{1}{2\nu}}$, then $K_N(x) > 0$ for all $x \in \mathbb{T}$ and $\|K_N\|_{L^1(\mathbb{T})} = 1$.*

The dependence of N on ν is almost sharp in the following sense: for $0 < \nu \ll 1$, if

$$\frac{1}{2\sqrt{3}\pi\nu} \leq N \lesssim \frac{1}{\sqrt{\nu}} e^{\frac{1}{6\nu}},$$

then

$$K_N < -\frac{0.3}{1+4\pi^2\nu^2(N+1)^2}$$

on a set of measure $\mathcal{O}(1)$ and

$$\|K_N\|_{L^1(\mathbb{T})} \geq 1 + \frac{\gamma_1}{1+4\pi^2\nu^2(N+1)^2},$$

where γ_1 is an absolute constant.

Proof. For the first result, note that

$$\begin{aligned} K_N(x) &\geq \frac{1}{2\nu} \sum_{n\in\mathbb{Z}} e^{-\frac{1}{\nu}|x+n|} - \sum_{|k|>N} \frac{1}{1+4\pi^2\nu^2k^2} \\ &> \frac{1}{\nu} e^{-\frac{1}{2\nu}} - \frac{1}{4\pi^2\nu^2} \frac{2}{N} \geq 0, \end{aligned}$$

if $N \geq \frac{1}{2\pi^2\nu} e^{\frac{1}{2\nu}}$.

For the last result we recall the identity for $K_{>N}(x)$:

$$\sum_{|k|\geq N+1}\frac{1}{1+4\pi^2\nu^2k^2}e^{2\pi ik\cdot x}=\sum_{k\geq N}(c_k-2c_{k+1}+c_{k+2})\tilde{F}_k-c_N\tilde{F}_N+c_{N+1}\tilde{F}_{N-1}.$$

Note that for $k\geq N\geq\frac{1}{2\sqrt{3}\pi\nu}$, we have

$$c_k-2c_{k+1}+c_{k+2}\geq 0.$$

Now if we take $x_0=\frac{j_0}{N+1}$, where j_0 is an integer in $[1,N]$, then $\tilde{F}_N=0$ and

$$\tilde{F}_{N-1}=\left(\frac{\sin\left(N\pi\frac{j_0}{N+1}\right)}{\sin\left(\frac{j_0\pi}{N+1}\right)}\right)^2=1.$$

It follows that for $x_0=\frac{j_0}{N+1}$, $N\geq\frac{1}{2\sqrt{3}\pi\nu}$,

$$K_{>N}(x_0)\geq\frac{1}{1+4\pi^2\nu^2(N+1)^2}.$$

Thus

$$\begin{aligned}K_N(x_0)\leq&\frac{1}{2\nu}\sum_{n\in\mathbb{Z}}e^{-\frac{1}{\nu}|x_0+n|}-\frac{1}{1+4\pi^2\nu^2(N+1)^2}\\<&\frac{1}{\nu}\cdot 2\frac{e^{-\frac{1}{\nu}\min\{x_0,1-x_0\}}}{1-e^{-\frac{1}{\nu}}}-\frac{1}{1+4\pi^2\nu^2(N+1)^2}.\end{aligned}$$

Assume $j_0\in[\frac{1}{3}(N+1),\frac{2}{3}(N+1)]$ such that $x_0\in[\frac{1}{3},\frac{2}{3}]$, then for

$$\frac{1}{2\sqrt{3}\pi\nu}\leq N\lesssim\frac{1}{\sqrt{\nu}}e^{\frac{1}{6\nu}},$$

we have

$$K_N(x_0)<\frac{1}{\nu}\cdot\frac{2}{e^{\frac{1}{3\nu}}(1-e^{-\frac{1}{\nu}})}-\frac{1}{1+4\pi^2\nu^2(N+1)^2}<-\frac{0.5}{1+4\pi^2\nu^2(N+1)^2}.$$

One may then pick $\mathcal{O}(N)$ of j_0 from the interval $[\frac{1}{3}(N+1),\frac{2}{3}(N+1)]$, and choose $x_0=\frac{j_0+\epsilon}{N+1}$, with $\epsilon\in[0,\epsilon_*]$ (ϵ_* is a sufficiently small absolute constant), such that

$$K_N(x_0)<-\frac{0.3}{1+4\pi^2\nu^2(N+1)^2}.$$

Note that the total measure of such x_0 is $\mathcal{O}(1)$. Finally we should point out that since $\int_{\mathbb{T}} K_N dx = 1$, the above point-wise bound then yields the (slightly inferior) L^1 lower bound of K_N as:

$$\|K_N\|_{L^1(\mathbb{T})} \geq 1 + \frac{\gamma_1}{1+4\pi^2\nu^2(N+1)^2},$$

where γ_1 is an absolute constant. Note that we lose a logarithm here compared to the optimal bound. □

2.1 Generalization to convex sequences

The simple method used in the proof of Proposition 2.5 is quite robust and can be generalized. For example, consider a sequence of real numbers $(c_k)_{k\geq 0}$ such that $\sup_{k\geq 0}|c_k| < \infty$ and the following hold:

1. for some $k_0 \geq 0$,

$$c_k - 2c_{k+1} + c_{k+2} \geq 0, \quad \forall k \geq k_0.$$

2. $\lim_{k\to\infty}(c_{k+1} - c_k)k = 0$.

3. $\lim_{k\to\infty} c_k \log k = 0$.

Remark 2.2. Condition (1) and (2) implies that

$$\sum_{k=0}^{\infty} |c_k - 2c_{k+1} + c_{k+2}| \cdot (k+1) < \infty.$$

Remark 2.3. Condition (3) cannot be deduced from (1) and (2). For example, let

$$c_k = \frac{1}{\log\log k},$$

then condition (1) and (2) are satisfied, but not condition (3).

Remark 2.4. Condition (2) and (3) is convenient for extracting convergence rates. These conditions can be weakened further provided one works with other norms such as total variational norms and so on. However, we do not dwell on this issue here.

Now let

$$G_N(x) = \sum_{|k|\leq N} c_{|k|} e^{2\pi i k\cdot x} = c_0 + 2\sum_{k=1}^{N} c_k \cos(2\pi k \cdot x).$$

Theorem 2.1. *Suppose the sequence $(c_k)_{k\geq 0}$ satisfies the conditions* (1), (2) *and* (3). *Then G_N converges in L^1 to a function G_∞ as N tends to infinity. Furthermore the following upper and lower estimates hold:*

$$\|G_N-G_\infty\|_{L^1(\mathbb{T})}\leq 2|c_N-c_{N+1}|\cdot N+\alpha_1|c_N|\log N, \qquad \forall N\geq 4,$$
$$\|G_N-G_\infty\|_{L^1(\mathbb{T})}\geq c_N\frac{1}{\pi^2}\log(N+1)-|c_N-c_{N+1}|\cdot N, \qquad \forall N\geq 4,$$

where $\alpha_1>0$ is an absolute constant.

Remark 2.5. This theorem is quite handy in practical applications. For example, one can take $c_k=k^{-\alpha}$, $k\geq 1$, where $\alpha>0$ (note that we do not require $\alpha>1$!). It is easy to check that conditions (1), (2) and (3) are satisfied. Then corresponding to this sequence $(c_k)_{k\geq 0}$ we have

$$\|G_N-G_\infty\|_{L^1(\mathbb{T})}\sim N^{-\alpha}\log N$$

for all large N.

Proof. Note that for $4\leq N<M$,

$$\begin{aligned}
G_M-G_N&=\sum_{N<k\leq M}c_k(\tilde{F}_k-2\tilde{F}_{k-1}+\tilde{F}_{k-2})\\
&=\sum_{k=N+1}^{M}c_k\tilde{F}_k-2\sum_{k=N}^{M-1}c_{k+1}\tilde{F}_k+\sum_{k=N-1}^{M-2}c_{k+2}\tilde{F}_k\\
&=\sum_{k=N}^{M-2}(c_k-2c_{k+1}+c_{k+2})\tilde{F}_k+c_M\tilde{F}_M+c_{M-1}\tilde{F}_{M-1}-2c_M\tilde{F}_{M-1}\\
&\quad -c_N\tilde{F}_N+c_{N+1}\tilde{F}_{N-1}\\
&=\sum_{k=N}^{M-2}(c_k-2c_{k+1}+c_{k+2})\tilde{F}_k+c_MD_M+(c_{M-1}-c_M)\tilde{F}_{M-1}\\
&\quad -(c_N-c_{N+1})\tilde{F}_{N-1}-c_ND_N.
\end{aligned}$$

Since (by Remark 2.2)

$$\sum_k|c_k-2c_{k+1}+c_{k+2}|(k+1)<\infty,$$

it follows easily that G_N is Cauchy in L^1 and converges to an L^1 function as $N\to\infty$. The upper and lower bounds are similarly estimated as before. We omit the details. □

Remark 2.6. One can also change the metric and obtain more precise point-wise convergence results with suitable weights. For example, fix any $\delta_0>0$, one can show (say identify $\mathbb{T}=[-\frac{1}{2},\frac{1}{2})$)

$$\|G_N-G_M\|_{L^\infty_x(\delta_0<|x|<\frac{1}{2})}\to 0$$

as $N<M$ tends to infinity.

2.2 PDE results

Let $\nu>0$ and consider the following linear elliptic PDE posed on the 1D torus $\mathbb{T}=[0,1)$:

$$u-\nu^2\partial_{xx}u=\Pi_N f, \tag{2.4}$$

where we recall Π_N defined as

$$\Pi_N f=\sum_{|k|\le N}\hat{f}(k)e^{2\pi ik\cdot x}.$$

This equation is perhaps one of the simplest cases for the spectral Galerkin method. By using the results derived earlier, we have the following theorem.

Theorem 2.2. *Let $f\in L^\infty(\mathbb{T})$. Then the unique solution u to (2.4) satisfies the following:*

- *Strict maximum principle. If $N\ge\frac{1}{2\pi^2\nu}e^{\frac{1}{2\nu}}$, then*

$$\|u\|_{L^\infty(\mathbb{T})}\le\|f\|_{L^\infty(\mathbb{T})}.$$

- *Generalized maximum principle. For all $N\ge 1$,*

$$\|u\|_{L^\infty(\mathbb{T})}\le\left(1+\frac{\alpha_1}{1+(\pi\nu N)^2}\log(N+2)\right)\|f\|_{L^\infty(\mathbb{T})},$$

  where $\alpha_1>0$ is an absolute constant.

- *Lower bound. For each $N\ge 1$, there exists smooth $f^{(N)}$ with $\|f^{(N)}\|_{L^\infty(\mathbb{T})}=1$, such that the corresponding solution $u^{(N)}$ satisfies*

$$\|u^{(N)}\|_{L^\infty(\mathbb{T})}\ge\frac{\alpha_2}{1+(\pi N\nu)^2}\log(N+2)-1,$$

  where $\alpha_2>0$ is an absolute constant.

- *Lower bound for small ν. Assume $0<\nu\ll 1$. For each N satisfying*

$$\frac{1}{2\sqrt{3}\pi\nu}\leq N\lesssim\frac{1}{\sqrt{\nu}}e^{\frac{1}{6\nu}},$$

there exists smooth $f^{(N)}$ with $\|f^{(N)}\|_{L^\infty(\mathbb{T})}=1$, such that the corresponding solution $u^{(N)}$ satisfies

$$\|u^{(N)}\|_{L^\infty(\mathbb{T})}\geq 1+\frac{\alpha_3}{1+4\pi^2\nu^2(N+1)^2},$$

where $\alpha_3>0$ is an absolute constant.

Proof. The first and fourth result follow from Proposition 2.8. The second and third results follow from Proposition 2.3. □

2.3 The case $d\geq 2$

We now discuss some higher dimensional analogues of the previous results. In practical numerical computations, we usually use the projection

$$\Pi_N=\Pi_N^{(d)}=\Pi_{N,k_1}\cdots\Pi_{N,k_d},$$

where Π_{N,k_j} refers to projection into each frequency coordinate k_j. In yet other words $\Pi_N^{(d)}$ is the frequency truncation with $|k|_\infty\leq N$.

We begin with a technical lemma.

Lemma 2.3. *Let $d\geq 1$, $\beta>0$, and $N\geq 2$. Suppose $\phi\in C_c^\infty(\mathbb{R}^d)$ is such that $\phi(\xi)=1$ for $|\xi|\leq\frac{3}{2}\sqrt{d}$, and $\phi(\xi)=0$ for $|\xi|>2\sqrt{d}$. Then*

$$\Big\|\sum_{|k|_\infty>N}\frac{1}{1+4\pi^2\beta|k|^2}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}\Big\|_{L_x^1(\mathbb{T}^d)}\sim\frac{1}{1+\beta N^2}(\log(N+2))^d,$$

where the implied constants depend only on d and the function ϕ.

Proof. First we show the upper bound. Let $\phi_1\in C_c^\infty(\mathbb{R}^d)$ be such that $\phi_1(\xi)=1$ when $1\leq|\xi|\leq 2\sqrt{d}$, and $\phi_1(\xi)=0$ for $|\xi|<\frac{1}{2}$ or $|\xi|>3\sqrt{d}$. Then clearly we have

$$\begin{aligned}
&\Big\|\sum_{|k|_\infty>N}\frac{1}{1+4\pi^2\beta|k|^2}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}\Big\|_{L_x^1(\mathbb{T}^d)}\\
=&\Big\|\sum 1_{N<|k|_\infty\leq 2\sqrt{d}N}\cdot\frac{1}{1+4\pi^2\beta|k|^2}\phi\Big(\frac{k}{N}\Big)\phi_1\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}\Big\|_{L_x^1(\mathbb{T}^d)}\\
\leq&\|\tilde{D}_N\|_{L_x^1(\mathbb{T}^d)}\cdot\|H\|_{L_x^1(\mathbb{R}^d)}\lesssim(\log(N+2))^d\cdot\frac{1}{1+\beta N^2},
\end{aligned}$$

where $\tilde{D}_N$ is a Dirichlet kernel corresponding to the part

$$1_{N<|k|_\infty\leq 2\sqrt{d}N}=1_{|k|_\infty\leq 2\sqrt{d}N}-1_{|k|_\infty\leq N}$$

(Note that both summands naturally split as product of one-dimensional kernels) and

$$H=\mathcal{F}^{-1}\Big(\frac{1}{1+4\pi^2\beta|\xi|^2}\phi\Big(\frac{\xi}{N}\Big)\phi_1\Big(\frac{\xi}{N}\Big)\Big).$$

Next we show the lower bound. Note that

$$\tilde{A}(x)=\sum_{|k|_\infty>N}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}=-\sum_{|k|_\infty\leq N}e^{2\pi ik\cdot x}+\sum_{k\in\mathbb{Z}^d}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x},$$

which implies

$$\|\tilde{A}\|_{L^1_x(\mathbb{T}^d)}\gtrsim(\log N)^d\quad\text{for } N \text{ large}.$$

Observe

$$\tilde{A}(x)=(1+4\pi^2\beta N^2)\sum_{|k|_\infty>N}\frac{1}{1+4\pi^2\beta|k|^2}\phi\Big(\frac{k}{N}\Big)\phi\Big(\frac{k}{N}\Big)\frac{1+4\pi^2\beta|k|^2}{1+4\pi^2\beta N^2}e^{2\pi ik\cdot x}.$$

The desired lower bound (for N large) then follows since

$$\Big\|\mathcal{F}^{-1}\Big(\phi_1\Big(\frac{\xi}{N}\Big)\frac{1+4\pi^2\beta|\xi|^2}{1+4\pi^2\beta N^2}\Big)\Big\|_{L^1_x(\mathbb{R}^d)}\lesssim 1.$$

Finally for $N\sim 1$ one can note that

$$(x)=\sum_{|k|_\infty>N}\frac{1}{1+4\pi^2\beta|k|^2}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}$$

has the bounds:

$$\|A\|_{L^\infty_x(\mathbb{T}^d)}\lesssim\frac{1}{1+\beta},\quad \|A\|_{L^2_x(\mathbb{T}^d)}\sim\frac{1}{1+\beta}.$$

One can then use interpolation to get the lower bound. □

Recall the Japanese bracket notation $\langle x\rangle=(1+|x|^2)^{\frac{1}{2}}$ for $x\in\mathbb{R}^d$. Let $\epsilon>0$ and consider

$$g_N(x)=\sum_{|k|_\infty\leq N}\langle k\rangle^{-\epsilon}e^{2\pi ik\cdot x},$$

$$\tilde{g}_N(x)=\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-\epsilon}\psi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x},$$

where $\psi\in C^\infty_c(\mathbb{R}^d)$ satisfies $\psi(\xi)=0$ for $|\xi|\leq\sqrt{d}$, and $\psi(\xi)=0$ for $|\xi|>2\sqrt{d}$.

Lemma 2.4. *Both $(g_N)_{N\geq 2}$ and $(\tilde{g}_N)_{N\geq 2}$ are Cauchy in $L^1(\mathbb{T}^d)$, and*

$$\|g_N-\tilde{g}_N\|_{L^1_x(\mathbb{T}^d)}\lesssim N^{-\epsilon}\cdot(\log N)^d\to 0 \quad as \quad N\to\infty.$$

Proof. Due to the assumption on the support of ψ, we see that $\psi(\frac{k}{N})=1$ when $|k|_\infty\leq N$, and $\psi(\frac{k}{N})=0$ when $|k|_\infty>2N$. Choose $\psi_1\in C_c^\infty$ with support in $\{|\xi|\sim 1\}$ such that $\psi_1(\xi)=1$ on the support on ψ. Then

$$\begin{aligned}\|g_N-\tilde{g}_N\|_{L^1_x(\mathbb{T}^d)}=&\Big\|\sum_{N<|k|_\infty\leq 2N}\langle k\rangle^{-\epsilon}\psi\Big(\frac{k}{N}\Big)\psi_1\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}\Big\|_{L^1_x(\mathbb{T}^d)}\\ \lesssim&\|\tilde{D}_N\|_{L^1_x(\mathbb{T}^d)}\Big\|\int\langle\xi\rangle^{-\epsilon}\psi\Big(\frac{\xi}{N}\Big)\psi_1\Big(\frac{\xi}{N}\Big)e^{2\pi i\xi\cdot x}d\xi\Big\|_{L^1_x(\mathbb{R}^d)}\\ \lesssim&(\log N)^d\cdot N^{-\epsilon},\end{aligned}$$

where $\tilde{D}_N$ is the Dirichlet kernel corresponding to $1_{N<|k|_\infty\leq 2N}$.

It then suffices for us to show that $(\tilde{g}_N)_{N\geq 2}$ is Cauchy in $L^1_x(\mathbb{T}^d)$. Observe that for $M>N\geq 2$, we have

$$\|\tilde{g}_N-\tilde{g}_M\|_{L^1_x(\mathbb{T}^d)}\leq\Big\|\int_{\mathbb{R}^d}\langle\xi\rangle^{-\epsilon}\Big(\psi\Big(\frac{\xi}{N}\Big)-\psi\Big(\frac{\xi}{M}\Big)\Big)e^{2\pi i\xi\cdot x}d\xi\Big\|_{L^1_x(\mathbb{R}^d)}\to 0$$

as we take $M>N$ to infinity. □

Lemma 2.4 can be used to show the following folklore identity that for $\epsilon>0$,

$$\sum_{k\in\mathbb{Z}^d}\langle 2\pi k\rangle^{-\epsilon}e^{2\pi ik\cdot x}=\sum_{n\in\mathbb{Z}^d}f(x+n),$$

where $f=\mathcal{F}^{-1}(\langle 2\pi\xi\rangle^{-\epsilon})$ is the usual Bessel potential. Note that even though f has exponential decay, the decay of the Fourier coefficient is not enough to apply the classic version of the Poisson summation formula. To resolve this the natural idea is to truncate the LHS and in some sense Lemma 2.4 assures the equivalence of different two truncations. The convergence of the Fourier series on the LHS can then be understood in $L^1_x(\mathbb{T}^d)$ sense. Stronger point-wise asymptotics are also available but we will not dwell on this issue here.

Consider the operator

$$T_{>N,\beta}=(\mathrm{Id}-\beta\Delta)^{-1}(\mathrm{Id}-\Pi_N^{(d)}).$$

Note that it is spectrally localized to the sector $\{k:|k|_\infty>N\}$. Denote the corresponding kernel as $K_{>N,\beta}$.

Theorem 2.3. *Let $N\geq 2$. $K_{>N,\beta}$ is an L^1 function on $\mathbb{T}^d$ and*

$$\|K_{>N,\beta}\|_{L^1(\mathbb{T}^d)}\sim\frac{1}{1+\beta N^2}\log^d(N+2),$$

where the implied constants depend only on d.

Proof. Choose ϕ as in Lemma 2.3 and note that

$$\begin{aligned}K_{>N,\beta}(x)&=\sum_{|k|_\infty>N}\frac{1}{1+4\pi^2\beta|k|^2}\phi\Big(\frac{k}{N}\Big)e^{2\pi ik\cdot x}+\sum_{k\in\mathbb{Z}^d}\frac{1}{1+4\pi^2\beta|k|^2}\Big(1-\phi\Big(\frac{k}{N}\Big)\Big)e^{2\pi ik\cdot x}\\&=:A_1(x)+A_2(x).\end{aligned}$$

Here we used the fact that when $1-\phi(\frac{k}{N})=0$ we must have $|k|\geq\frac{3}{2}N\sqrt{d}$, which implies $|k|_\infty\geq\frac{3}{2}N$ (so that the condition $|k|_\infty>N$ can be dropped).

To bound A_2 we can use Lemma 2.4 and the discussion afterwards which gives

$$\|A_2\|_{L^1_x(\mathbb{T}^d)}\leq\|\mathcal{F}^{-1}\Big(\frac{1}{1+4\pi^2\beta|\xi|^2}\Big(1-\phi\Big(\frac{\xi}{N}\Big)\Big)\Big)\|_{L^1_x(\mathbb{R}^d)}\lesssim\frac{1}{1+\beta N^2}.$$

By Lemma 2.3 we have

$$\|A_1\|_{L^1_x(\mathbb{T}^d)}\sim\frac{1}{1+\beta N^2}(\log(N+2))^d,$$

and thus the desired bound for $K_{>N,\beta}$ follows when N is large.

Note that the desired upper bound for $K_{>N,\beta}$ holds for all $N\geq 2$ and we only need to show the lower bound when N is of order 1. To this end consider

$$\begin{aligned}B(x)&=\sum_{|k|_\infty>N}\frac{1}{|k|^{d+1}}\cdot\frac{1}{1+4\pi^2\beta|k|^2}e^{2\pi ik\cdot x},\\B_1(x)&=\sum_{|k|_\infty>N}\frac{1}{|k|^{d+1}}e^{2\pi ik\cdot x}.\end{aligned}$$

Easy to check that for $N\sim 1$,

$$\|B\|_{L^2_x(\mathbb{T}^d)}\sim\frac{1}{1+\beta},\quad\|B_1\|_{L^2_x(\mathbb{T}^d)}\sim 1.$$

Since $B=B_1*K_{>N,\beta}$ (here $*$ denotes the usual convolution on the torus), we then have

$$\frac{1}{1+\beta}\lesssim\|B\|_{L^2_x(\mathbb{T}^d)}\leq\|B_1\|_{L^2_x(\mathbb{T}^d)}\cdot\|K_{>N,\beta}\|_{L^1_x(\mathbb{T}^d)}\lesssim\|K_{>N,\beta}\|_{L^1_x(\mathbb{T}^d)}.$$

Thus the desired lower bound holds also when N is of order 1. □

Theorem 2.4. *Let $d\geq 1$, $N\geq 2$, $\beta>0$. Then the following hold for*

$$T_{>N,\beta}=(\mathrm{Id}-\beta\Delta)^{-1}(\mathrm{Id}-\Pi_N^{(d)})$$

with the kernel $K_{>N,\beta}$, and

$$T_{N,\beta}=(\mathrm{Id}-\beta\Delta)^{-1}\Pi_N^{(d)}.$$

1. *Sharp L_x^1 bound.*

$$\frac{c_1}{1+\beta N^2}(\log(N+2))^d\leq\|K_{>N,\beta}\|_{L_x^1(\mathbb{T}^d)}\leq\frac{c_2}{1+\beta N^2}(\log(N+2))^d,$$

   where $c_1>0$, $c_2>0$ depend only on the dimension d.

2. *Generalized maximum principle. For all $N\geq 2$, we have*

$$\|T_{N,\beta}f\|_{L_x^\infty(\mathbb{T}^d)}\leq\Big(1+\frac{c_3}{1+\beta N^2}(\log(N+2))^d\Big)\|f\|_{L_x^\infty(\mathbb{T}^d)},$$

   where $c_3>0$ depend only on the dimension d.

3. *Strict maximum principle. Let $d=1,2$. Denote the kernel of $T_{N,\beta}$ as $K_{N,\beta}$. For all $N\geq N_0=N_0(\beta,d)$ (i.e., N_0 is constant depending only on β and d), it holds that $K_{N,\beta}$ is a strictly positive function on $\mathbb{T}^d$ with unit L_x^1 mass, and consequently*

$$\|T_{N,\beta}f\|_{L_x^\infty(\mathbb{T}^d)}\leq\|f\|_{L_x^\infty(\mathbb{T}^d)}.$$

Remark 2.7. An interesting problem is to investigate the maximum principle for the spherical operator

$$\tilde{T}_{N,\beta}=(\mathrm{Id}-\beta\Delta)^{-1}\tilde{\Pi}_N,$$

where $\tilde{\Pi}_N$ corresponds to the Fourier cut-off $|k|\leq N$ (i.e., $|k|_\infty$ is replaced by the usual l^2 norm $|k|$).

Proof. The first two follow from Theorem 2.3. We only need to show (3). Note that the case $d=1$ is already settled before with explicit dependence of constants. The case $d=2$ is proved in the appendix. □

2.4 Truncation of general Bessel case

We first recall the usual Bessel potential on $\mathbb{R}^d$: for $s>0$,

$$G_s=(\mathrm{Id}-\Delta)^{-\frac{s}{2}}\delta_0,\quad \widehat{G_s}(\xi)=\langle 2\pi\xi\rangle^{-s}.$$

At the origin, we have for $|x|\to 0$,

$$G_s(x)=\begin{cases}\frac{\Gamma(\frac{d-s}{2})}{2^s\pi^{\frac{s}{2}}|x|^{d-s}}(1+o(1)), & 0<s<d,\\ \frac{1}{2^{d-1}\pi^{\frac{d}{2}}}\ln\frac{1}{|x|}(1+o(1)), & s=d,\\ \frac{\Gamma(\frac{s-d}{2})}{2^s\pi^{\frac{s}{2}}}(1+o(1)), & s>d.\end{cases}$$

When $|x|\to\infty$,

$$G_s(x)=\frac{e^{-|x|}}{2^{\frac{d+s-1}{2}}\pi^{\frac{d-1}{2}}\Gamma(\frac{s}{2})|x|^{\frac{d+1-s}{2}}}(1+o(1)).$$

It follows that for $s>d$, we have the rough bound

$$G_s(x)\gtrsim e^{-2|x|},\quad \forall x\in\mathbb{R}^d.$$

For $s=d$, we have

$$G_s(x)\gtrsim\begin{cases}e^{-2|x|}, & \forall|x|\geq\frac{1}{5},\\ -\ln|x|, & \forall 0<|x|<\frac{1}{5}.\end{cases}$$

For $0<s<d$, we have

$$G_s(x)\gtrsim\begin{cases}e^{-2|x|}, & \forall|x|\geq\frac{1}{5},\\ |x|^{-(d-s)}, & \forall 0<|x|<\frac{1}{5}.\end{cases}$$

Denote for $\beta>0$, $N\geq 2$,

$$F_{N,s}(x)=\sum_{|k|_\infty\leq N}\frac{1}{(1+4\pi^2\beta|k|^2)^{\frac{s}{2}}}e^{2\pi ik\cdot x},\quad x\in\mathbb{T}^d.$$

We suppress the dependence of $F_{N,s}$ on β since we focus only on studying other parameters while keeping $\beta>0$ fixed. It is easy to check that the L^1_x limit of $F_{N,s}$ is given by

$$F_{\infty,s}(x)=\beta^{-\frac{d}{2}}\sum_{n\in\mathbb{Z}^d}G_s\Big(\frac{x+n}{\sqrt{\beta}}\Big)\geq\beta^{-\frac{d}{2}}G_s\Big(\frac{x}{\sqrt{\beta}}\Big),\quad\forall x\in\mathbb{T}^d=\Big[-\frac{1}{2},\frac{1}{2}\Big)^d.$$

Lemma 2.5. *Let $d\geq1$ and $s>d$. There exists*

$$N_0=c_{s,d}\cdot(\beta^{\frac{d-1}{2}}e^{\frac{2}{\sqrt{\beta}}})^{\frac{1}{s-d}},$$

($c_{s,d}>0$ depends only on (s,d)) such that if $N\geq N_0$, then each $F_{N,s}$ is positive and hence has unit L^1_x mass.

Remark 2.8. Note that for $d=1$ and $s=2$, our $N_0\sim e^{\frac{2}{\sqrt{\beta}}}$ is slightly worse than the constant $\mathcal{O}(\frac{1}{\sqrt{\beta}}e^{\frac{1}{\sqrt{\beta}}})$ obtained before. This is due to the use of the rough lower bound on G_s.

Proof. There is no issue with point-wise convergence since $s>d$. Observe that

$$\sum_{|k|_\infty>N}\frac{1}{(1+4\pi^2\beta|k|^2)^{\frac{s}{2}}}\lesssim\beta^{-\frac{s}{2}}N^{-(s-d)}.$$

It follows that

$$F_{N,s}(x)\gtrsim\beta^{-\frac{d}{2}}e^{-2\frac{1}{\sqrt{\beta}}}-\beta^{-\frac{s}{2}}N^{-(s-d)}>0,$$

if

$$N\geq N_0\sim(\beta^{\frac{d-s}{2}}e^{\frac{2}{\sqrt{\beta}}})^{\frac{1}{s-d}}.$$

Thus, we complete the proof. □

The complementary regime $0<s\leq d$ requires more delicate analysis.

First we make an easy note. For all $s>0$, $d\geq1$, and $|x|\leq\frac{1}{6N\sqrt{d}}$, we have

$$2\pi|k\cdot x|\leq\frac{\pi}{3},\quad\forall k\in\mathbb{Z}^d\quad\text{with}\quad|k|_\infty\leq N.$$

Clearly then for all $|x|\leq\frac{1}{6N\sqrt{d}}$, we have

$$F_{N,s}(x)\geq\frac{1}{2}\sum_{|k|_\infty\leq N}\frac{1}{(1+4\pi^2\beta|k|^2)^{\frac{s}{2}}}>0.$$

Thus to analyze positivity we only need to focus on the regime $\{x\in[-\frac{1}{2},\frac{1}{2}]^d:|x|\geq\frac{1}{6N\sqrt{d}}\}$. In the rest of the analysis we shall only treat x in this regime.

2.4.1 Complete analysis for $1D$ case

We first investigate the 1D case in complete detail. For convenience throughout this subsection we shall identify

$$\mathbb{T}=\Big[-\frac{1}{2},\frac{1}{2}\Big).$$

This is to single out the possible singularity near 0.

To continue it is useful to denote (by a slight abuse of notation)

$$c_k=c(k)=\langle 2\pi\sqrt{\beta}k\rangle^{-s}=\frac{1}{(1+4\pi^2\beta|k|^2)^{\frac{s}{2}}}.$$

Note that

$$(\langle x\rangle^{-s})''=s\langle x\rangle^{-(s+4)}\cdot((s+1)x^2-1).$$

Thus $c(k)$ is convex whenever $k\geq\frac{1}{2\pi\sqrt{s+1}\sqrt{\beta}}$. We will use the following identity for any $4\leq N<M$:

$$\begin{aligned}F_{M,s}-F_{N,s}=&\sum_{k=N}^{M-2}(c_k-2c_{k+1}+c_{k+2})\tilde{F}_k+c_MD_M+(c_{M-1}-c_M)\tilde{F}_{M-1}\\&-(c_N-c_{N+1})\tilde{F}_{N-1}-c_ND_N,\end{aligned}\tag{2.5}$$

where

$$D_N(x)=\frac{\sin(2N+1)\pi x}{\sin\pi x}\quad\text{ and }\quad\tilde{F}_N(x)=\Big(\frac{\sin(N+1)\pi x}{\sin\pi x}\Big)^2.$$

From this it is not difficult to check that $(F_{N,s})_{N\geq4}$ converges uniformly on $\mathbb{T}_\delta=\{x\in[-\frac{1}{2},\frac{1}{2}):|x|>\delta\}$ for any fixed small $\delta>0$. Sending M to infinity and using convexity of c_k (for $k\geq N$) then yields for $N\geq\frac{1}{2\pi\sqrt{s+1}\sqrt{\beta}}$,

$$F_{N,s}(x)\geq F_{\infty,s}(x)-h_N(x)+c_ND_N(x)\quad\text{for any }\delta<|x|\leq\frac{1}{2},\tag{2.6}$$

where

$$h_N(x)=\sum_{k=N}^{\infty}(c_k-2c_{k+1}+c_{k+2})\tilde{F}_k.$$

Lemma 2.6. *Let $d=1$ and $0<s\leq1$. For all $0<s\leq1$ and $\frac{1}{5}\sqrt{\beta}\leq|x|\leq\frac{1}{2}$, we have*

$$F_{\infty,s}(x)\gtrsim\beta^{-\frac{1}{2}}e^{-\frac{1}{\sqrt{\beta}}}.$$

For all $0<|x|<\min\{\frac{1}{5}\sqrt{\beta},\frac{1}{2}\}$*, we have*

$$F_{\infty,s}(x)\gtrsim\begin{cases}\beta^{-\frac{1}{2}}\ln\Big|\dfrac{\sqrt{\beta}}{|x|}\Big|, & s=1,\\ \beta^{-\frac{1}{2}}\Big(\dfrac{\sqrt{\beta}}{|x|}\Big)^{1-s}, & 0<s<1.\end{cases}$$

For any $N\geq\frac{1}{2\pi\sqrt{\beta}}$ *and* $\frac{1}{6N}<|x|\leq\frac{1}{2}$*, we have*

$$|c_N|+|h_N(x)|\lesssim\beta^{-\frac{s}{2}}N^{-s}\frac{1}{|x|}.$$

Proof. Direct computation. Note that

$$\begin{aligned}h_N(x)&\lesssim|x|^{-2}\sum_{k=N}^{\infty}(c_k-2c_{k+1}+c_{k+2})=|x|^{-2}(c_N-c_{N+1})\\&\lesssim|x|^{-2}\beta^{-\frac{s}{2}}N^{-s-1}\lesssim|x|^{-1}\beta^{-\frac{s}{2}}N^{-s}.\end{aligned}$$

Thus, we complete the proof. □

Lemma 2.7. *Let* $d=1$ *and* $s=1$*. Then for any*

$$N\geq N_0(\beta)=\alpha\cdot\beta^{-\frac{1}{2}}e^{\frac{1}{\sqrt{\beta}}}$$

(*alpha*>0 *is some absolute constant*)*,* $F_{N,1}$ *is positive and has unit* L^1_x *mass.*

Proof. Consider first the regime $\frac{1}{5}\sqrt{\beta}\leq|x|\leq\frac{1}{2}$ (note that in this case for such x to exist we must have $0<\beta\leq(\frac{5}{2})^2$). We have

$$\begin{aligned}F_{N,s}(x)&\geq\alpha_1\beta^{-\frac{1}{2}}e^{-\frac{1}{\sqrt{\beta}}}-\alpha_2\beta^{-\frac{1}{2}}\frac{1}{N|x|}\\&\geq\alpha_1\beta^{-\frac{1}{2}}e^{-\frac{1}{\sqrt{\beta}}}-5\alpha_2\beta^{-\frac{1}{2}}\frac{1}{N\sqrt{\beta}},\end{aligned}$$

where $\alpha_1>0$, $\alpha_2>0$ are absolute constants. Clearly then

$$N_0\sim\beta^{-\frac{1}{2}}e^{\frac{1}{\sqrt{\beta}}}$$

suffices. Next consider the regime

$$\frac{1}{6N}\leq|x|\leq\min\Big\{\frac{1}{5}\sqrt{\beta},\frac{1}{2}\Big\}.$$

We have

$$F_{N,s}(x)\geq\alpha_3\beta^{-\frac{1}{2}}\ln\Big|\frac{\sqrt{\beta}}{|x|}\Big|-\alpha_2\beta^{-\frac{1}{2}}\cdot\frac{1}{N|x|},$$

where $\alpha_3>0$ is also an absolute constant. Now if

$$\ln\Big|\frac{\sqrt{\beta}}{|x|}\Big|\geq 2\frac{\alpha_2}{\alpha_3},$$

clearly we have positivity. On the other hand if

$$\ln\Big|\frac{\sqrt{\beta}}{|x|}\Big|<2\frac{\alpha_2}{\alpha_3},$$

then

$$F_{N,s}(x)\geq\alpha_3\beta^{-\frac{1}{2}}\ln 5-\alpha_2\beta^{-1}\cdot\frac{1}{N}\cdot e^{2\frac{\alpha_2}{\alpha_3}}>0,$$

if we take $N_0\gg\beta^{-\frac{1}{2}}$. □

We now focus on $0<s<1$.

Lemma 2.8. *Let $d=1$ and $0<s<1$. If*

$$\frac{1}{5}\sqrt{\beta}\leq|x|\leq\frac{1}{2}\quad\text{ and }\quad N\geq N_0(\beta,s)=\alpha(s)\cdot\beta^{-\frac{1}{2}}e^{\frac{1}{s\sqrt{\beta}}}$$

($\alpha(s)>0$ depends only on s), then $F_{N,s}(x)>0$.

Proof. Observe that

$$F_{N,s}(x)\geq\alpha_1(s)\cdot\beta^{-\frac{1}{2}}e^{-\frac{1}{\sqrt{\beta}}}-\alpha_2(s)\cdot\beta^{-\frac{s}{2}}\cdot N^{-s}\cdot\frac{1}{\sqrt{\beta}},$$

where $\alpha_1(s)>0$, $\alpha_2(s)>0$ depend only on s. □

Lemma 2.9. *Let $d=1$ and $0<s<1$. Assume $N\geq\frac{1}{2\pi\sqrt{\beta}}$. There exist a constant $\tilde{\gamma}_1(s)>0$ depending only on s such that if*

$$\frac{\tilde{\gamma}_1(s)}{N}\leq|x|\leq\min\Big\{\frac{1}{5}\sqrt{\beta},\frac{1}{2}\Big\},$$

then $F_{N,s}(x)>0$.

Proof. Observe that

$$\begin{aligned}F_{N,s}(x)\geq&\alpha_3(s)\beta^{-\frac{1}{2}}\Big(\frac{\sqrt{\beta}}{|x|}\Big)^{1-s}-\alpha_4(s)\cdot\beta^{-\frac{s}{2}}N^{-s}\cdot\frac{1}{|x|}\\=&\alpha_3(s)\beta^{-\frac{s}{2}}|x|^{-1+s}-\alpha_4(s)\cdot\beta^{-\frac{s}{2}}N^{-s}|x|^{-1},\end{aligned}$$

where $\alpha_3>0$, $\alpha_4>0$ depend only on s. Matching $|x|^{-1+s}$ with the term $N^{-s}|x|^{-1}$ then determines the constant $\tilde{\gamma}_1(s)$. □

Remark 2.9. Yet another way to obtain the positivity for $F_{N,s}(x)$ in the regime $\frac{1}{N}\ll|x|\leq\frac{1}{2}$ is as follows. For simplicity consider $\beta=\frac{1}{(4\pi)^2}$ and

$$K_{>N}(x)=\sum_{|k|>N}\langle k\rangle^{-s}e^{2\pi ik\cdot x}=\sum_{k\in\mathbb{Z}}\langle k\rangle^{-s}\eta_N(k)e^{2\pi ik\cdot x},$$

where η_N is a linear interpolation such that $\eta_N(k)=0$ for $|k|\leq N$ and $\eta_N(k)=1$ for $|k|\geq N+1$. In particular

$$\eta_N'(k)=\mathrm{sgn}(k)1_{|k|\in[N,N+1]}.$$

We only need to show

$$|K_{>N}(x)|\lesssim N^{-s}\frac{1}{|x|}.$$

It then suffices to consider

$$\begin{aligned}G(x)=&\int_{\mathbb{R}}\langle\xi\rangle^{-s}\eta_N(\xi)e^{2\pi i\xi\cdot x}d\xi\\=&-\frac{1}{2\pi ix}\int_{\mathbb{R}}(-s)\langle\xi\rangle^{-s-2}\xi\eta_N(\xi)e^{2\pi i\xi\cdot x}d\xi-\frac{1}{2\pi ix}\int_N^{N+1}\langle\xi\rangle^{-s}2i\sin2\pi\xi\cdot xd\xi.\end{aligned}$$

One can then obtain for $0<|x|\leq\frac{1}{2}$,

$$|G(x)|\lesssim|x|^{-1}\cdot N^{-s},$$

and for $|x|\geq\frac{1}{2}$,

$$|G(x)|\lesssim|x|^{-2}\cdot N^{-s}.$$

The desired estimate then follows. One should observe that this line of computation is in some sense analogous to (2.6) whereas the latter can be viewed as discrete integration by parts.

Remark 2.10. In the preceding remark, one may also take η_N to be a smooth function as

$$\eta_N(\xi)=\eta_1*\Big(1_{|\xi|>N+\frac{1}{2}}\Big),$$

where $*$ is the usual convolution on $\mathbb{R}$, and $\eta_1\in C_c^\infty(\mathbb{R})$ is an even function such that $\eta_1(x)=1$ for $|x|\le 0.01$ and $\eta_1(x)=0$ for $|x|>0.02$. This way one does not need to worry about boundary terms when doing integration by parts.

Since we have shown the positivity of $F_{N,s}$ for $|x|<\frac{1}{6N}$ and $\frac{1}{N}\ll|x|\le\frac{1}{2}$, it then suffices for us to study the positivity or non-positivity of $F_{N,s}(x)$ for $0<s<1$ and $|x|\in(\frac{1}{6N},\frac{\tilde{\gamma}_1(s)}{N})$. To this end let $x=\frac{y}{N}$ and assume that $|y|\in[\gamma_1,\gamma_2]$ for some $0<\gamma_1(s)<\gamma_2(s)$. Here we allow some flexibility and the values of γ_1 and γ_2 will become clear later. Consider the expression

$$H_{N,s}(y)=N^{-(1-s)}F_{N,s}\Big(\frac{y}{N}\Big)=\frac{1}{N^{1-s}}\sum_{|k|\le N}\frac{1}{(1+4\pi^2\beta|k|^2)^{\frac{s}{2}}}e^{2\pi i\frac{k}{N}\cdot y}$$

and note that

$$H_{\infty,s}(y)=(2\pi\sqrt{\beta})^{-s}\int_{-1}^{1}|t|^{-s}e^{2\pi it\cdot y}dt.$$

Lemma 2.10. *We have*

$$\max_{|y|\in[\gamma_1(s),\gamma_2(s)]}|H_{N,s}(y)-H_{\infty,s}(y)|\lesssim_s N^{-(1-s)}\Big(1+\frac{1}{\sqrt{\beta}}\Big).$$

Proof. Utterly standard and thus we only sketch the details (so as to check the constants). First note that the tail piece is OK:

$$N^{-(1-s)}\Big|\sum_{|k|\le\frac{1}{\sqrt{\beta}}}\frac{1}{\Big(1+4\pi^2\beta|k|^2\Big)^{\frac{s}{2}}}e^{2\pi i\frac{k}{N}\cdot y}\Big|\lesssim_s\Big(1+\frac{1}{\sqrt{\beta}}\Big)N^{-(1-s)}.$$

Here we have the factor $1+\frac{1}{\sqrt{\beta}}$ instead of purely $\frac{1}{\sqrt{\beta}}$ due to the $k=0$ term. This is needed especially when $\beta>1$.

Next observe that

$$\begin{aligned}&N^{-(1-s)}\Big|\sum_{\frac{1}{\sqrt{\beta}}<|k|\le N}\Big(\frac{1}{(1+4\pi^2\beta|k|^2)^{\frac{s}{2}}}-\frac{1}{(4\pi^2\beta|k|^2)^{\frac{s}{2}}}\Big)e^{2\pi i\frac{k}{N}\cdot y}\Big|\\ \lesssim_s&N^{-(1-s)}\sum_{\frac{1}{\sqrt{\beta}}<|k|\le N}\frac{1}{(1+4\pi^2\beta|k|^2)^{\frac{s}{2}+1}}\lesssim_s\Big(1+\frac{1}{\sqrt{\beta}}\Big)N^{-(1-s)}.\end{aligned}$$

Finally to bound the difference between the integral and the summation, it suffices to consider the function $g(\xi)=|\xi|^{-s}e^{2\pi i\xi\cdot y}$. Note that on the mesh $[\frac{k}{N},\frac{k+1}{N}]$ (say WLOG $k\geq 2$), we have

$$\Big|g(\xi)-g\Big(\frac{k}{N}\Big)\Big|\lesssim_s\Big(\frac{k}{N}\Big)^{-s-1}\cdot\frac{1}{N}=k^{-1-s}\cdot N^s.$$

Since the mesh size is $1/N$, the total error for the function $\beta^{-\frac{s}{2}}g(\xi)$ can then be estimated by

$$\beta^{-\frac{s}{2}}\sum_{|k|>\frac{1}{\sqrt{\beta}}}|k|^{-1-s}\cdot N^s\cdot\frac{1}{N}\lesssim N^{-(1-s)}\cdot\Big(1+\frac{1}{\sqrt{\beta}}\Big).$$

Thus, we complete the proof. □

Thanks to Lemma 2.10, we have for all $y\in[\gamma_1(s),\gamma_2(s)]$,

$$F_{N,s}(\frac{y}{N})=N^{1-s}H_{\infty,s}(y)+O_s(1)\cdot\Big(1+\frac{1}{\sqrt{\beta}}\Big),$$

where for any quantity X we denote $X=O_s(1)$ if $|X|\leq C_s$ for some constant $C_s>0$ depending only on s. Clearly

$$\begin{aligned}H_{\infty,s}(y)=&2(2\pi\sqrt{\beta})^{-s}\int_0^1 t^{-s}\cos(2\pi ty)dt\\=&2(2\pi\sqrt{\beta})^{-s}(2\pi y)^{-(1-s)}\int_0^{2\pi y}t^{-s}\cos t dt.\end{aligned}$$

Consider for $0<s<1$, and for $y\in[0,\infty)$,

$$h_s(y)=\int_0^y t^{-s}\cos t dt.$$

Lemma 2.11. *For any $0<\delta_1<\frac{1}{10}$, we have*

$$\inf_{\delta_1\leq y<\infty}h_s(y)=\min\Big\{h_s(\delta_1),h_s\Big(\frac{3\pi}{2}\Big)\Big\}.$$

Proof. Note that

$$h_s(\infty)=s(1+s)\int_0^\infty t^{-s-2}(1-\cos t)dt>0.$$

Clearly h_s only has interior minimums at $y=(n+\frac{1}{2})\pi$ with $n\geq 1$ being any odd integer. The result then follows from the fact that for odd $n\geq 1$,

$$\int_{(n+\frac{1}{2})\pi}^{(n+2+\frac{1}{2})\pi} t^{-s}\cos t dt=\int_0^{2\pi}\Big(\Big(n+\frac{1}{2}\Big)\pi+t\Big)^{-s}\sin t dt>0.$$

The lemma is proved. □

Now note that

$$h_s\Big(\frac{3}{2}\pi\Big)=\frac{1}{1-s}\int_0^{\frac{3}{2}\pi}\xi^{1-s}\sin\xi d\xi.$$

Lemma 2.12. *Consider for* $0\leq s\leq 1$

$$f_1(s)=\int_0^{\frac{3}{2}\pi}\xi^{1-s}\sin\xi d\xi.$$

Then $f_1(0)=-1$ *and* $f_1(1)=1$, *and*

$$\inf_{0<s<1} f_1'(s)\geq\frac{1}{2}.$$

For some $s_*\in(0.308443,0.308444)$, *we have* $f_1(s_*)=0$, *and* $f_1(s)<0$ *for* $0\leq s<s_*$, *and* $f_1(s)>0$ *for* $s_*<s\leq 1$.

Remark 2.11. One can check that $f_1''(s)<0$ and

$$\inf_{0<s<1} f_1'(s)=f_1'(1)\approx 0.77.$$

Remark 2.12. For $0\leq s\leq 1$, the function $f_1(s)$ is given by a hypergeometric expression:

$$\frac{1}{3-s}\cdot\Big(\frac{3\pi}{2}\Big)^{3-s}\text{HypergeometricPFQ}\Big[\frac{3}{2}-\frac{s}{2},\Big\{\frac{3}{2},\frac{5-s}{2}\Big\},-\frac{9\pi^2}{16}\Big].$$

Proof. We have

$$f_1'(s)=\int_0^{\frac{3}{2}\pi}\xi^{1-s}(-\log\xi)\sin\xi d\xi,\quad f_1''(s)=\int_0^{\frac{3}{2}\pi}\xi^{1-s}(\log\xi)^2\sin\xi d\xi.$$

Clearly we have uniform bound on $\|f_1''\|_{L_s^\infty([0,1])}$. A rigorous numerical computation together with error estimates then shows that the desired lower bound on f_1' indeed

holds true. Since f_1 is strictly monotonically increasing, the existence of s_* follows from the intermediate value theorem. The precise numerical value is found via rigorous numerical integration. In particular, we have

$$f_1(0.308443)\approx -1.92202\times 10^{-6}<0, \quad f_1(0.308444)\approx 5.43492\times 10^{-7}>0.$$

Thus, we complete the proof. □

The following theorem follows immediately from the preceding series of lemmas and computations.

Theorem 2.5 (Sharp characterization of $F_{N,s}$ when $0<s<1$)**.** *For some* $s_*\in(0.308443,0.308444)$*, the following hold*

1. *Lack of positivity for* $0<s<s_*$*. There are positive constants* $\gamma_2(s)>\gamma_1(s)>0$ *and* $\gamma(s)>0$*, such that for any* $y\in[\gamma_1(s),\gamma_2(s)]$ *and* $N\gg_s 1+\beta^{-\frac{1}{2(1-s)}}$*, we have*

$$\frac{1}{N^{1-s}}F_{N,s}\Big(\frac{y}{N}\Big)=-\gamma(s)+\Big(1+\frac{1}{\sqrt{\beta}}\Big)\cdot O_s(N^{-(1-s)})<-\frac{1}{2}\gamma(s).$$

   In particular this shows that for $0<s<s_*$ *the kernel function* $F_{N,s}$ *must be negative on an interval of length* $O_s(N^{-1})$ *for all large* N*.*

2. *Positivity for* $s_*<s<1$*. For any* $s_*<s<1$ *there are constants* $c_1(s)>0$ *and* $c_2(s)>0$ *such that if*

$$N\geq \max\Big\{c_1(s)\beta^{-\frac{1}{2}}e^{\frac{1}{\sqrt{\beta}}},c_2(s)\cdot(1+\beta^{-\frac{1}{2(1-s)}})\Big\},$$

   then $F_N(x)$ *is positive and hence has unit* L^1_x *mass.*

Remark 2.13. It is also possible to characterize the lack of positivity of $F_{N,s}$ via real-space representation. For example, for $0<s\ll 1$ we can consider (take $\beta=1$)

$$F_{N,s}(x)=\sum_{n\in\mathbb{Z}}\int_{|y|<\frac{1}{2}} g_0(x-y+n)\frac{\sin M\pi y}{\sin \pi y}dy,$$

where $M=2N+1$ and $g_0=\mathcal{F}^{-1}((1+4\pi^2\xi^2)^{-\frac{s}{2}})$. Note that

$$g_0(z)\sim |z|^{-(1-s)}(1+o(1)) \quad \text{as} \quad |z|\to 0$$

and has exponential decay as $|z|\to\infty$. By a discussion similar to that in the proof of Statement (3) of Theorem 2.4 in the appendix, we can consider only the main piece (below $f_0(y)=\frac{y}{\sin\pi y}$)

$$\begin{aligned}&\sum_{|n|\leq 3}\int_{|y|<\frac{M}{2}}\Big|x+n-\frac{y}{M}\Big|^{-(1-s)}f_0\Big(\frac{y}{M}\Big)\frac{\sin\pi y}{y}dy\\ =&M^{1-s}\Big(\int_{|y|<\frac{M}{2}}|Mx-y|^{-(1-s)}f_0\Big(\frac{y}{M}\Big)\frac{\sin\pi y}{y}dy\\ &+\sum_{1\leq|n|\leq 3}\int_{|y|<\frac{M}{2}}|Mx+Mn-y|^{-(1-s)}f_0\Big(\frac{y}{M}\Big)\frac{\sin\pi y}{y}dy\Big).\end{aligned}$$

Setting $x=\frac{3}{2M}$ and taking the limit $M\to\infty$ for the first term inside the bracket, we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\Big|y-\frac{3}{2}\Big|^{-(1-s)}\frac{\sin\pi y}{y}dy\\ =&\int_{|y-\frac{3}{2}|<0.01}\Big|y-\frac{3}{2}\Big|^{-(1-s)}\frac{\sin\pi y}{y}dy+\int_{|y-\frac{3}{2}|>0.01}\Big|y-\frac{3}{2}\Big|^{-(1-s)}\frac{\sin\pi y}{y}dy.\end{aligned}$$

Clearly the first term diverges as $-\frac{C_1}{s}$ ($C_1>0$ is a constant) as $s\to 0$. On the other hand, it is easy to check that

$$\Big|\sum_{1\leq|n|\leq 3}\int_{|y|<\frac{M}{2}}|Mx+Mn-y|^{-(1-s)}f_0\Big(\frac{y}{M}\Big)\frac{\sin\pi y}{y}dy\Big|\lesssim M^{-(1-s)}\log M.$$

One can then see that $F_{N,s}(\frac{3}{4N+2})$ must be negative for $0<s\ll 1$ and N sufficiently large.

Now we turn to the critical case $s=s_*$ which requires a more refined analysis since the main integral term vanishes in the limit.

Lemma 2.13.

$$\begin{aligned}\sum_{i=m}^{n}f(i)=&\int_m^n f(x)dx+\frac{f(m)+f(n)}{2}+\int_m^n f'(x)\Big(\{x\}-\frac{1}{2}\Big)dx\\ =&\int_m^n f(x)dx+f(m)+\int_m^n f'(x)\{x\}dx,\end{aligned}$$

where $\{x\}=x-[x]$ denotes the usual fractional part (i.e., $[x]$ denotes the largest integer less than or equal to x so that $\{x\}\in[0,1)$). For example $\{-3.1\}=0.9$ and $\{3.1\}=0.1$.

Proof. This is the usual Euler-Maclaurin formula. See for example [1]. □

Lemma 2.14. *Recall* $\langle x\rangle=(1+x^2)^{\frac{1}{2}}$. *For* $0<s<1$ *let*

$$A_\beta(s)=\int_{-\infty}^{\infty}(\langle x\rangle^{-s}-|x|^{-s})dx-s2\pi\sqrt{\beta}\int_{-\infty}^{\infty}\langle x\rangle^{-s-2}x\Big\{\frac{x}{2\pi\sqrt{\beta}}\Big\}dx.$$

Then for $s\in[0.29,0.31]$ *and* $0<\beta<\beta_*$ *(*β_* *is an absolute constant), we have*

$$0<\alpha_1<-A_\beta(s)<\alpha_2<\infty,$$

where a_1 *and* α_2 *are absolute constants.*

Remark 2.14. One can compute numerically that for $\beta=\frac{1}{4\pi^2}$ and $s=0.3$

$$-A_\beta(s)\in[1.13-0.53,1.13+0.53].$$

A further interesting question is to investigate the case $\beta\gtrsim 1$.

Proof. Observe that for $s\in[0.29,0.31]$,

$$s\Big|\int_{-\infty}^{\infty}\langle x\rangle^{-s-2}x\Big\{\frac{x}{2\pi\sqrt{\beta}}\Big\}dx\Big|\leq c_1\cdot\beta^{\frac{1}{2}},$$

where $c_1>0$ is an absolute constant.

On the other hand for $s\in[0.29,0.31]$, we have

$$\int_{-\infty}^{\infty}(|x|^{-s}-\langle x\rangle^{-s})dx=-\sqrt{\pi}\frac{\Gamma(\frac{-1+s}{2})}{\Gamma(\frac{s}{2})}\geq c_2>0,$$

where $\Gamma(\cdot)$ is the usual Gamma function and $c_2>0$ is another absolute constant. The desired result then follows easily. □

Theorem 2.6 (Lack of positivity for $s=s_*$ and $0<\beta<\beta_*$)**.** *Let* $s=s_*$ *and* $y=\frac{3}{4}$. *Let* $0<\beta<\beta_*$ *where* β_* *is the same constant as in Lemma* 2.14. *Then for* $N\geq\frac{1}{\sqrt{\beta}}$, *we have*

$$F_{N,s}\Big(\frac{y}{N}\Big)=\frac{1}{2\pi\sqrt{\beta}}A_\beta(s)+O_{s,\beta}(N^{-s}),$$

where $A_\beta(s)<0$ *is given by the expression*

$$A_\beta(s)=\int_{\mathbb{R}}(\langle x\rangle^{-s}-|x|^{-s})dx-s2\pi\sqrt{\beta}\int_{\mathbb{R}}\langle x\rangle^{-s-2}x\Big\{\frac{x}{2\pi\sqrt{\beta}}\Big\}dx.$$

Remark 2.15. A further interesting issue is to investigate the case $|y-\frac{3}{4}|\ll 1$. However we shall not dwell on it here.

Proof. Denote $\beta_1=4\pi^2\beta$ and $y_1=2\pi y$. Clearly for

$$f(k)=(1+\beta_1 k^2)^{-\frac{s}{2}}e^{i\frac{k}{N}y_1},$$

we have

$$\begin{aligned}F_{N,s}\Big(\frac{y}{N}\Big)&=\sum_{k=-N}^{N}f(k)\\&=\int_{-N}^{N}f(x)dx+f(-N)+\int_{-N}^{N}((1+\beta_1x^2)^{-\frac{s}{2}})'e^{i\frac{x}{N}y_1}\{x\}dx\\&\quad+\int_{-N}^{N}(1+\beta_1x^2)^{-\frac{s}{2}}\frac{iy_1}{N}e^{i\frac{x}{N}y_1}\{x\}dx.\end{aligned}$$

Note that for $s=s_*$, $y=\frac{3}{4}$, we have

$$\int_{-N}^{N}(\beta_1k^2)^{-\frac{s}{2}}e^{2\pi i\frac{k}{N}y}dk=0.$$

Also it is easy to check that (here and below we need $N\sqrt{\beta}\geq 1$):

$$|f(-N)|+\int_{-N}^{N}(1+\beta_1x^2)^{-\frac{s}{2}}\frac{1}{N}dx=O_s(\beta^{-\frac{s}{2}}N^{-s}).$$

Thus we obtain

$$\begin{aligned}F_{N,s}(\frac{y}{N})&=\int_{-N}^{N}((1+\beta_1x^2)^{-\frac{s}{2}}-(\beta_1x^2)^{-\frac{s}{2}})e^{i\frac{x}{N}y_1}dx\\&\quad-s\int_{-N}^{N}(1+\beta_1x^2)^{-\frac{s}{2}-1}\cdot\beta_1xe^{i\frac{x}{N}y_1}\{x\}dx+O_s(\beta^{-\frac{s}{2}}N^{-s})\\&=\frac{1}{\sqrt{\beta_1}}\Big(\int_{-N_1}^{N_1}((1+x^2)^{-\frac{s}{2}}-|x|^{-s})e^{i\frac{x}{N_1}y_1}dx\Big)\\&\quad-s\int_{-N_1}^{N_1}(1+x^2)^{-\frac{s}{2}-1}\cdot xe^{i\frac{x}{N_1}y_1}\Big\{\frac{x}{\sqrt{\beta_1}}\Big\}dx+O_s(\beta^{-\frac{s}{2}}N^{-s}),\end{aligned}$$

where we have denoted $N_1=N\sqrt{\beta_1}$. The result then follows from the error estimates:

$$\begin{aligned}
&\int_{|x|\leq N_1}|(1+x^2)^{-\frac{s}{2}}-|x|^{-s}|\cdot|1-e^{i\frac{x}{N_1}y_1}|dx=\mathcal{O}(N_1^{-1}),\\
&\int_{|x|>N_1}|(1+x^2)^{-\frac{s}{2}}-|x|^{-s}|dx=\mathcal{O}(N_1^{-1-s}),\\
&\int_{|x|>N_1}(1+x^2)^{-\frac{s}{2}-1}|x|dx=\mathcal{O}(N_1^{-s}),\\
&\int_{|x|\leq N_1}(1+x^2)^{-\frac{s}{2}-1}\cdot|x|\cdot|e^{i\frac{x}{N_1}y_1}-1|dx=\mathcal{O}(N_1^{-s}).
\end{aligned}$$

Here we used the fact that for $|x|\gg 1$,

$$(1+x^2)^{-\frac{s}{2}}=|x|^{-s}+\mathcal{O}(|x|^{-s-2}).$$

Thus, we complete the proof. □

Remark 2.16. Let $\beta>0$, $N\geq 2$. Let $d\geq 2$, $0<s\leq d$. One can consider the general Bessel cases:

$$\begin{aligned}
F^{\text{square}}_{N,s}(x)&=\sum_{|k|_\infty>N}(1+4\pi^2\beta|k|^2)^{-\frac{s}{2}}e^{2\pi ik\cdot x},\\
F^{\text{sphere}}_{>N,s}(x)&=\sum_{|k|>N}(1+4\pi^2\beta|k|^2)^{-\frac{s}{2}}e^{2\pi ik\cdot x}.
\end{aligned}$$

These will be investigated elsewhere.

We summarize the results obtained in this section as the following theorem.

Theorem 2.7 (Sharp maximum principle)**.** *Let $d\geq 1$ and $\beta>0$. Then the following hold*

1. *If $s>d$ and*

$$N\geq c_{s,d}(\beta^{\frac{d-1}{2}}e^{\frac{2}{\sqrt{\beta}}})^{\frac{1}{s-d}}$$

($c_{s,d}>0$ depends only on $(s,\ d)$), then $F_{N,s}$ is a positive function and

$$\|F_{N,s}\|_{L^1(\mathbb{T}^d)}=1.$$

2. *Let $d=1$ and $s=1$. If*

$$N\geq\alpha\beta^{-\frac{1}{2}}e^{\frac{1}{\sqrt{\beta}}}$$

($\alpha>0$ is an absolute constant), then $F_{N,1}$ is positive and has unit L^1 mass.

Let $d=1$, $\beta>0$ and $0<s<1$. We identify $\mathbb{T}=[-\frac{1}{2},\frac{1}{2})$. For some $s_*\in(0.308443,0.308444)$, the following hold

1. *Lack of positivity for $0<s<s_*$. There are positive constants $\gamma_2(s)>\gamma_1(s)>0$ and $\gamma(s)>0$, such that for any $y\in[\gamma_1(s),\gamma_2(s)]$ and $N\gg_s 1+\beta^{-\frac{1}{2(1-s)}}$, we have*

$$\frac{1}{N^{1-s}}F_{N,s}\Big(\frac{y}{N}\Big)=-\gamma(s)+\Big(1+\frac{1}{\sqrt{\beta}}\Big)\cdot O_s(N^{-(1-s)})<-\frac{1}{2}\gamma(s).$$

   In particular this shows that for $0<s<s_*$ the kernel function $F_{N,s}$ must be negative on an interval of length $O_s(N^{-1})$ for all large N.

2. *Positivity for $s_*<s<1$. For any $s_*<s<1$ there are constants $c_1(s)>0$ and $c_2(s)>0$ such that if*

$$N\geq\max\Big\{c_1(s)\beta^{-\frac{1}{2}}e^{\frac{1}{\sqrt{\beta}}},c_2(s)\cdot(1+\beta^{-\frac{1}{2(1-s)}})\Big\},$$

   then $F_N(x)$ is positive and hence has unit L^1_x mass.

3. *Lack of positivity for $s=s_*$ and $0<\beta<\beta_*$. Let $s=s_*$ and $y=\frac{3}{4}$. Let $0<\beta<\beta_*$ where β_* is the same constant as in Lemma 2.14. Then for $N\geq\frac{1}{\sqrt{\beta}}$, we have*

$$F_{N,s}(\frac{y}{N})=\frac{1}{2\pi\sqrt{\beta}}A_\beta(s)+O_{s,\beta}(N^{-s}),$$

   where $A_\beta(s)$ is given by the expression

$$A_\beta(s)=\int_{\mathbb{R}}(\langle x\rangle^{-s}-|x|^{-s})dx-s2\pi\sqrt{\beta}\int_{\mathbb{R}}\langle x\rangle^{-s-2}x\Big\{\frac{x}{2\pi\sqrt{\beta}}\Big\}dx.$$

   Furthermore for $0<\beta<\beta_*$, we have

$$0<\alpha_1<-A_\beta(s)<\alpha_2<\infty,$$

   where $\alpha_1>0$, $\alpha_2>0$ are absolute constants.

Theorem 2.8 (Effective maximum principle)**.** *Let $d\geq1$, $N\geq2$, $\beta>0$ and $s>0$. Then for all $N\geq2$, we have*

$$\|F_{N,s}\|_{L^1_x(\mathbb{T}^d)}\leq1+\frac{c_{s,d}}{(1+\beta N^2)^{\frac{s}{2}}}(\log(N+2))^d,$$

where $c_{s,d}>0$ depends only on $(d,\,s)$.

Proof of Theorem 2.8. The proof is a simple adaptation of the proof of Theorem 2.3. We omit the details. □

3 1D torus case for Allen-Cahn: semi-discretization

Consider the scheme for Allen-Cahn on the torus $\mathbb{T}=[0,1)$:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau}=\nu^2\partial_{xx}u^{n+1}-\Pi_N((u^n)^3-u^n), \quad n\geq 0, \\ u^0=\Pi_N u_0. \end{cases}$$

Here $\tau>0$ is the size of the time step.

The following lemma will be used later which we will often refer to as a standard discrete energy estimate.

Lemma 3.1 (Energy estimate)**.** *For any $n\geq 0$, we have*

$$\begin{aligned} &E(u^{n+1})-E(u^n)+\frac{1}{2}\nu^2\|\partial_x(u^{n+1}-u^n)\|_2^2+\left(\frac{1}{\tau}+\frac{1}{2}\right)\|u^{n+1}-u^n\|_2^2 \\ \leq&\frac{3}{2}\max\{\|u^n\|_\infty^2,\|u^{n+1}\|_\infty^2\}\cdot\|u^{n+1}-u^n\|_2^2, \end{aligned}$$

where $F(x)=\frac{1}{4}(x^2-1)^2$, and

$$E(u)=\frac{1}{2}\nu^2\|\partial_x u\|_{L^2(\mathbb{T})}^2+\int_{\mathbb{T}}F(u)dx.$$

Proof. Denote by $(,)$ the usual L^2 pairing of functions. Then

$$\begin{aligned} &\frac{\|u^{n+1}-u^n\|_2^2}{\tau} \\ =&-\nu^2\frac{1}{2}(\|\partial_x u^{n+1}\|_2^2-\|\partial_x u^n\|_2^2+\|\partial_x(u^{n+1}-u^n)\|_2^2)-(\Pi_N(f(u^n)),u^{n+1}-u^n) \\ =&-\nu^2\frac{1}{2}(\|\partial_x u^{n+1}\|_2^2-\|\partial_x u^n\|_2^2+\|\partial_x(u^{n+1}-u^n)\|_2^2)-(f(u^n),u^{n+1}-u^n), \end{aligned}$$

where we have denoted $f(z)=F'(z)=z^3-z$.

Now note that

$$F(u^{n+1})=F(u^n)+f(u^n)\cdot(u^{n+1}-u^n)+\frac{1}{2}f'(\xi)(u^{n+1}-u^n)^2,$$

where ξ lies between u^n and u^{n+1}. The desired inequality then easily follows since $f'(z)=3z^2-1$. □

Proposition 3.1. *Assume* $u_0\in H^1(\mathbb{T})$ *and* $\|u_0\|_{L^\infty(\mathbb{T})}\leq 1$. *Let* $0<\tau_*\leq\tau\leq\frac{1}{2}$. *Let* $0<\alpha_0\leq 1$. *If* $N\geq N_0=N_0(\tau_*,\nu,\|u_0\|_{H^1},\alpha_0)$, *then*

$$\sup_{n\geq 0}\|u^n\|_\infty\leq 1+\alpha_0.$$

Remark 3.1. The condition $u_0\in H^1(\mathbb{T})$ may be weakened to $u_0\in H^s(\mathbb{T})$ $s>\frac{1}{2}$.

Remark 3.2. The dependence of N_0 on ν (neglecting $\|u_0\|_{H^1}$ and α_0) is rather mild especially in the regime $\nu\ll 1$. From the proof below, one can see that it suffices to take N_0 with $N_0^{-2}\log N_0\ll\nu^2$ which is roughly $N_0\gg\nu^{-1}\sqrt{|\log\nu|}$. One should note that from a scaling heuristic the typical mesh size should resolve $\Delta x\approx\nu$ which corresponds to the Fourier cut-off $1/\nu$ in the frequency space. Thus up to a logarithm N_0 is optimal.

Remark 3.3. The constraint $\alpha_0>0$ already comes from the fact that $u^0=\Pi_N u_0$ in general does not preserve the strict upper bound as the Dirichlet kernel changes its sign. On the other hand, even if we assume $\|u^0\|_\infty\leq 1$ (note $u^0=\Pi_N u_0$!), this maximum is still not preserved (for small N) as one can see from the counter-example in Proposition 3.2.

Proof. Step 1. Initial data. Write $\Pi_{>N}=I-\Pi_N$. Then

$$\begin{aligned}\|\Pi_N u_0\|_\infty&\leq\|u_0\|_\infty+\|\Pi_{>N}u_0\|_\infty\\&\leq 1+N^{-\frac{1}{2}}\|u_0\|_{H^1(\mathbb{T})}\leq 1+\alpha_0,\end{aligned}$$

if $N\geq N_0(\alpha_0,\|u_0\|_{H^1})$. Here one should note that we must have $\alpha_0>0$.

Step 2. Induction. Rewrite

$$(1-\tau\nu^2\partial_{xx})u^{n+1}=\Pi_N\big((1+\tau)u^n-\tau(u^n)^3\big).$$

The inductive assumption is

$$\|u^n\|_\infty\leq 1+\alpha_0.$$

In the following we shall show $u^{n+1}\leq 1+\alpha_0$. By repeating the argument for $-u^{n+1}$ we also get the lower bound.

Denote $u^n=1+\eta^n$. Then clearly by the inductive assumption

$$-(2+\alpha_0)\leq\eta^n\leq\alpha_0.$$

For $\eta^{n+1}=u^{n+1}-1$ we have the equation

$$\begin{aligned}(1-\tau\nu^2\partial_{xx})\eta^{n+1}=&\Pi_N\big((1+\tau)(1+\eta^n)-\tau(1+\eta^n)^3-1\big)\\=&\Pi_N\big((1-2\tau)\eta^n-3\tau(\eta^n)^2-\tau(\eta^n)^3\big)\\=&(1-2\tau)\eta^n-\tau(3+\eta^n)(\eta^n)^2+\tau\Pi_{>N}\big(3(\eta^n)^2+(\eta^n)^3\big).\end{aligned}$$

Here we used the fact that $\Pi_N\eta^n=\eta^n$. Note that $3+\eta^n\geq 1-\alpha_0\geq 0$ (here we need $\alpha_0\leq 1$). By using the maximum principle, we then get

$$\begin{aligned}\max\eta^{n+1}\leq&(1-2\tau)\alpha_0+\tau\|(1-\tau\nu^2\partial_{xx})^{-1}\Pi_{>N}\big(3(\eta^n)^2+(\eta^n)^3\big)\|_\infty\\\leq&(1-2\tau)\alpha_0+\tau\cdot\frac{\text{const}}{1+4\pi^2\tau\nu^2N^2}\log(N+2)\cdot\text{const}\\\leq&(1-2\tau)\alpha_0+\text{const}\cdot\frac{1}{\nu^2N^2}\log(N+2)\leq\alpha_0,\end{aligned}$$

if $N\geq N_0$ (here note that in the last inequality we need the lower bound on τ so that N_0 remains independent of τ). Note that in the second inequality above we have used the L^1 bound for the operator

$$(1-\tau\nu^2\partial_{xx})^{-1}\Pi_{>N},$$

which was established in the previous section. □

Proposition 3.2 (Lack of strict maximum principle)**.** *There exist* $\nu>0$, $\tau>0$, $N\geq 1$, u^0 *with* $\|u^0\|_\infty\leq 1$ *such that*

$$\|u^1\|_\infty>1.$$

Remark 3.4. More elaborate (covering more regimes of the parameters) construction of counter-examples is possible but we shall not dwell on this issue here.

Proof. Take $\tau=\frac{1}{2}$, $N=2$, and

$$u^0=1+\eta^0=1-2\delta\cdot\cos^2(2\pi x),$$

where $\delta>0$ will be taken sufficiently small. Write $u^1=1+\eta^1$. Then

$$(1-\tau\nu^2\partial_{xx})\eta^1=-\tau\Pi_N\big(3(\eta^0)^2+(\eta^0)^3\big).$$

Thus

$$\begin{aligned}\eta^1=&-\tau\Pi_N(3(\eta^0)^2)-\tau^2\frac{\nu^2\partial_{xx}}{1-\tau\nu^2\partial_{xx}}\Pi_N(3(\eta^0)^2)+\frac{-\tau}{1-\tau\nu^2\partial_{xx}}\Pi_N((\eta^0)^3)\\=&-3\tau\Pi_N((\eta^0)^2)+\mathcal{O}(\nu^2)+\mathcal{O}(\delta^3).\end{aligned}$$

Now note that $\eta^0=-\delta(1+\cos 4\pi x)$, and

$$(\eta^0)^2=\delta^2\Big(\frac{3}{2}+2\cos 4\pi x+\frac{1}{2}\cos 8\pi x\Big),$$
$$\Pi_N((\eta^0)^2)=\delta^2\Big(\frac{3}{2}+2\cos 4\pi x\Big).$$

Clearly then

$$\max\big(-3\tau\Pi_N((\eta^0)^2)\big)\geq\frac{3}{4}\delta^2.$$

By taking δ small and ν small, we then obtain that

$$\max\eta^1>\frac{3}{8}\delta^2.$$

Thus, we complete the proof. □

Proposition 3.3. *Assume $\|u^0\|_\infty\leq 1+\alpha_0$ for some $0\leq\alpha_0\leq 1$. Let $0<\tau_*\leq\tau\leq\frac{1}{2}$. If $N\geq N_0=N_0(\tau_*,\nu)$ (note that N_0 is independent of α_0), then for all $n\geq 1$,*

$$\|u^n\|_\infty\leq 1+\theta^n\alpha_0+\frac{\beta}{\nu^2N^2}\log(N+2)\cdot\frac{1-\theta^n}{1-\theta},$$

where $\theta=1-2\tau$, and $\beta>0$ is an absolute constant.

Remark 3.5. It follows that

$$\limsup_{n\to\infty}\|u^n\|_\infty\leq 1+\frac{\beta}{2\tau\nu^2N^2}\log(N+2).$$

In yet other words the true maximum 1 is asymptotically preserved up to the spectral truncation error! One should also note that the decay is exponential in n.

Proof. We adopt the same notation as in Proposition 3.1 and the proof is a minor variation. First by using the proof of Proposition 3.1 with $\alpha_0=1$, one has the weak bound

$$\sup_{n\geq 0}\|u^n\|_\infty\leq 2. \tag{3.1}$$

We should emphasize here that N_0 can be taken to be independent of u^0 since we assumed $\|u^0\|_\infty\leq 2$.

Define

$$\alpha_n=\max\eta_n.$$

Then by repeating the derivation in Proposition 3.1, we obtain

$$\begin{aligned}\alpha_{n+1}\leq&(1-2\tau)\alpha_n+\tau\cdot\frac{\text{const}}{1+4\pi^2\tau\nu^2N^2}\log(N+2)\cdot\text{const}\\ \leq&(1-2\tau)\alpha_n+\frac{\beta}{\nu^2N^2}\log(N+2),\end{aligned}$$

where $\beta>0$ is an absolute constant. Note that in deriving the above bound we have used the fact that $\|\eta^n\|_\infty\lesssim 1$. This is where (3.1) is needed.

Iterating in n then gives the upper bound. The proof for the lower bound for $\tilde{\eta}_n=u^n+1$ is similar and thus we obtained the desired upper bound for $\|u^n\|_\infty$. □

Proposition 3.4 (Energy stability and L^∞-stability for $\tau_*\leq\tau\leq\frac{1}{2}$)**.** *Assume $u_0\in H^1(\mathbb{T})$ and*

$$\|u^0\|_{L^\infty(\mathbb{T})}\leq 2.$$

Let N_0 be the same as in Proposition 3.3*, and assume $N\geq N_0$ satisfies*

$$\frac{\beta}{\nu^2N^2}\log(N+2)\cdot\frac{1}{2\tau_*}\leq\frac{1}{10},$$

where the constant $\beta>0$ is the same as in Proposition 3.3*. Let*

$$n_0=-\frac{\log(100)}{\log(1-2\tau_*)}.$$

Then for any $n\geq n_0$, we have

$$\begin{aligned}\|u^n\|_{L^\infty(\mathbb{T})}\leq&1.11,\\ E(u^{n+1})\leq&E(u^n),\end{aligned}$$

where $F(x)=\frac{1}{4}(x^2-1)^2$, and

$$E(u)=\frac{1}{2}\nu^2\|\partial_x u\|^2_{L^2(\mathbb{T})}+\int_{\mathbb{T}}F(u)dx.$$

Proof. Easy to check that by using Proposition 3.3, we have for $n\geq n_0$,

$$\|u^n\|_{L^\infty(\mathbb{T})}\leq 1.11.$$

Now by a standard discrete energy estimate, we have

$$\begin{aligned}&E(u^{n+1})-E(u^n)+\frac{1}{2}\nu^2\|\partial_x(u^{n+1}-u^n)\|_2^2+\left(\frac{1}{\tau}+\frac{1}{2}\right)\|u^{n+1}-u^n\|_2^2\\ \leq&\frac{3}{2}\max\{\|u^n\|_\infty^2,\|u^{n+1}\|_\infty^2\}\cdot\|u^{n+1}-u^n\|_2^2.\end{aligned}$$

Note that $\frac{1}{\tau}+\frac{1}{2}\geq\frac{5}{2}$, and for $n\geq n_0$,

$$\|u^n\|_\infty^2\leq 1.11^2=1.2321<5/3.$$

It clearly follows that

$$E(u^{n+1})\leq E(u^n).$$

Thus, we complete the proof. □

Lemma 3.2. *Let $\tau>0$ and consider the cubic polynomial $p(x)=(1+\tau)x-\tau x^3$. Then the following hold:*

- *If $0<\tau\leq\frac{1}{2}$, then $\max_{|x|\leq 1}|p(x)|=1$. Actually for any $1\leq\alpha\leq\sqrt{1+\frac{2}{\tau}}$, we have*

$$\max_{|x|\leq\alpha}|p(x)|\leq\alpha.$$

  Furthermore for any $1<\alpha<\sqrt{1+\frac{2}{\tau}}$, we have the strict inequality

$$\max_{|x|\leq\alpha}|p(x)|<\alpha.$$

- *If $\frac{1}{2}\leq\tau\leq 2$, then for any*

$$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}\leq\alpha\leq\sqrt{\frac{2+\tau}{\tau}},$$

  we have

$$\max_{|x|\leq\alpha}|p(x)|\leq\alpha.$$

  Furthermore if

$$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}<\alpha<\sqrt{\frac{2+\tau}{\tau}},$$

  then we have the strict inequality

$$\max_{|x|\leq\alpha}|p(x)|<\alpha.$$

- *For $\tau>2$, define $x_0=\sqrt{\frac{1+\tau}{3\tau}}\in(0,1)$ and $x_{n+1}=p(x_n)$. Then $|x_n|\to\infty$ as $n\to\infty$.*

Remark 3.6. For $\tau=\frac{1}{2}$, $\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}=1$, and for $\tau=2$, $\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}=\sqrt{2}$.

Proof. Straightforward. For the first two results, note that a critical point of p is $x_c=\sqrt{\frac{1+\tau}{3\tau}}$ such that

$$p(x_c)=\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}.$$

Solving $p(x)=-x$ gives $x_1=\sqrt{\frac{2+\tau}{\tau}}$. It is easy to verify that if $\tau<2$, then $p(x_c)<-p(x_1)$ and when $\tau=2$ we have $p(x_c)=-p(x_1)$.

For the last result, one notes that

$$x_1=p(x_0)=\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}>\sqrt{\frac{2+\tau}{\tau}}.$$

It is then easy to verify that x_n oscillates to infinity. □

Loosely speaking, with the help of Lemma 3.2, one can reduce the problem of estimating $\|u^{n+1}\|_\infty$ in terms of $\|u^n\|_\infty$ (when $\tau\geq\tau_*$ is not close to zero) to the recurrent relation

$$\alpha_{n+1}=\max_{|x|\leq\alpha_n}|p(x)|+\eta,$$

where $\eta>0$ denotes the spectral error. The following Lemma plays a key role in the proof of maximum principle later. From a practical point of view, we do not state the results in its most general form. For example, below α_0 corresponds to the upper bound of $\|u^0\|_\infty$ in the original numerical scheme, and it is convenient to assume α_0 to be around 1 in practice.

Lemma 3.3 (Prototype iterative system for the maximum principle)**.** *Let $\tau>0$ and $p(x)=(1+\tau)x-\tau x^3$. Consider the recurrent relation*

$$\alpha_{n+1}=\max_{|x|\leq\alpha_n}|p(x)|+\eta,\quad n\geq 0,$$

where $\eta>0$.

1. *Case $0<\tau\leq\frac{1}{2}$. Let $\alpha_0=2$. There exists an absolute constant $\eta_0>0$ sufficiently small, such that for all $0\leq\eta\leq\eta_0$, we have $1\leq\alpha_n\leq 2$ for all n. Furthermore for all $n\geq 1$,*

$$1+\eta\leq\alpha_n\leq 1+\theta^n+\frac{1-\theta^n}{1-\theta}\eta, \tag{3.2}$$

where $\theta=1-2\tau$.

2. Case $\frac{1}{2}\leq\tau\leq 2-\epsilon_0$ *where* $0<\epsilon_0\leq 1$. *Let*

$$\alpha_0=\frac{1}{2}\Big(\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\sqrt{\frac{2+\tau}{\tau}}\Big).$$

Then there exists a constant $\eta_0>0$ *depending only on* ϵ_0, *such that if* $0\leq\eta\leq\eta_0$, *then for all* $n\geq 1$, *we have*

$$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\eta\leq\alpha_n\leq\alpha_0.$$

Remark 3.7. For $\tau=2$, even if $\alpha_0=1$, we have $\alpha_n\to\infty$ as $n\to\infty$. Also note that in Case (2) we do not discuss the decay estimate at all since in practice we are only interested in the regime where the maximum is around 1.

Proof. (1) Observe that for $\eta>0$ sufficiently small the values of α_n are trapped in $[1,2]$ by a simple induction argument. By another induction we obtain $\alpha_n\geq 1+\eta$ for all $n\geq 1$.

For the decay estimate, we discuss three cases.

Case 1: $1\leq\alpha_n\leq\sqrt{\frac{1+\tau}{3\tau}}$. Then

$$\max_{|x|\leq\alpha_n}|p(x)|=p(\alpha_n),$$

and

$$\begin{aligned}\alpha_{n+1}&=1+p(\alpha_n)-1+\eta\\&=1+p'(\xi_n)(\alpha_n-1)+\eta\quad(\xi_n\in(1,\alpha_n),\quad p'(\xi_n)=1+\tau-3\tau(\xi_n)^2)\\&\leq 1+(1-2\tau)(\alpha_n-1)+\eta.\end{aligned}$$

Case 2: $\sqrt{\frac{1+\tau}{3\tau}}<\alpha_n\leq\sqrt{\frac{1+\tau}{\tau}}$. Then

$$\max_{|x|\leq\alpha_n}|p(x)|=p\Big(\sqrt{\frac{1+\tau}{3\tau}}\Big),$$

and

$$\begin{aligned}\alpha_{n+1}&=1+p\Big(\sqrt{\frac{1+\tau}{3\tau}}\Big)-1+\eta\\&\leq 1+(1-2\tau)\Big(\sqrt{\frac{1+\tau}{3\tau}}-1\Big)+\eta\\&\leq 1+(1-2\tau)(\alpha_n-1)+\eta.\end{aligned}$$

Case 3: $\sqrt{\frac{1+\tau}{\tau}}\leq\alpha_n\leq 2$. Observe that

$$\max_{|x|\leq\alpha_n}|p(x)|=\max\Big\{p\Big(\sqrt{\frac{1+\tau}{3\tau}}\Big),|p(\alpha_n)|\Big\}.$$

Let $\tilde{p}(x)=|p(x)|$. Note that in the interval $[\sqrt{\frac{1+\tau}{\tau}},\sqrt{\frac{2+\tau}{\tau}}]$, the function $\tilde{p}(x)$ monotonically increases from 0 to $\sqrt{\frac{2+\tau}{\tau}}$. It is easy to check that $x_*=2\sqrt{\frac{1+\tau}{3\tau}}$ is the unique point in this interval such that

$$\tilde{p}(x_*)=p\Big(\sqrt{\frac{1+\tau}{3\tau}}\Big).$$

Note that $x_*\geq 2$ since $0<\tau\leq\frac{1}{2}$. Thus

$$|p(\alpha_n)|\leq p\Big(\sqrt{\frac{1+\tau}{3\tau}}\Big)$$

and we obtain

$$\begin{aligned}\alpha_{n+1}&=1+p\Big(\sqrt{\frac{1+\tau}{3\tau}}\Big)-1+\eta\\&\leq 1+(1-2\tau)(\alpha_n-1)+\eta.\end{aligned}$$

Concluding from all three cases, an iteration in n then gives (3.2).

(2) First it is easy to check that for all $n\geq 0$ the values of α_n are trapped in $[1,\alpha_0]$. Note that

$$\max_{|x|\leq 1}|p(x)|=p\Big(\sqrt{\frac{1+\tau}{3\tau}}\Big)=\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}.$$

Clearly then the lower bound for α_n, $n\geq 1$ holds. □

A direct corollary of Lemma 3.2 is the following.

Corollary 3.1 (No stability for $\tau>2$)**.** *Let $\tau>2$. Define $u_0(x)\equiv\sqrt{\frac{1+\tau}{3\tau}}\in(0,1)$. Then for any $N\geq 1$, we have $\|u^n\|_\infty\to\infty$ as $n\to\infty$.*

Observe that for $\tau=2$ we have

$$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}=\sqrt{\frac{2+\tau}{\tau}}=\sqrt{2}.$$

Interestingly we shall show that for $\tau=2$, the maximum $\alpha=\sqrt{2}$ cannot be preserved even for moderately large N.

Proposition 3.5 (No maximum principle for $\tau=2$ and N not large)**.** *Let $\tau=2$. Let $0<\nu\ll 1$. For any*

$$\frac{1}{2\sqrt{3}\pi\nu}\leq N\lesssim\frac{1}{\sqrt{\nu}}e^{\frac{1}{12\nu}},$$

we can find u_0 with $\|u_0\|_\infty\leq\sqrt{2}$, but

$$\|u^1\|_\infty>\sqrt{2}.$$

Remark 3.8. A further interesting issue is to consider u_0 spectrally localized with $\|u^0\|_\infty\leq\sqrt{2}$ and $\|u_0\|_\infty\leq\sqrt{2}$. But we do not dwell on this here.

Remark 3.9. Similar results hold for any $\tau_*\leq\tau\leq 2$, $\alpha=\sqrt{1+\frac{2}{\tau}}$. Recall that $p(x)=(1+\tau)x-\tau x^3$, $p(\alpha)=-\alpha$, and $p'(\alpha)<0$. For the argument below, the main point is that the kernel K_N has a nontrivial negative part. One can find $u_0=\alpha+\eta$ with $\|u_0\|_\infty\leq\alpha$, but

$$\begin{aligned}u^1(0)=&p(\alpha)+\big(K_N*(p'(\alpha)\eta)\big)(0)+\mathcal{O}(\eta^2)\\=&-\alpha+p'(\alpha)\big(K_N*(\eta)\big)(0)+\mathcal{O}(\eta^2)<-\alpha,\end{aligned}$$

i.e., $\|u^1\|_\infty>\alpha$. Thus for moderately large N, the upper bound $\alpha=\sqrt{1+\frac{2}{\tau}}$ cannot be preserved.

Proof. Recall that

$$u^1=(1-\tau\nu^2\partial_{xx})^{-1}\Pi_N\big((1+\tau)u^0-\tau(u^0)^3\big).$$

Let $u_0=\sqrt{2}+\delta\eta$, where the function η will be chosen momentarily and $\delta>0$ will be taken sufficiently small. Since $u^0=\Pi_N u_0$, we have

$$\begin{aligned}u^1=&-\sqrt{2}+(1-\tau\nu^2\partial_{xx})^{-1}\Pi_N\big(-9\delta\cdot\eta\big)+\mathcal{O}(\delta^2)\\=&-\sqrt{2}-9\delta(1-\tau\nu^2\partial_{xx})^{-1}\Pi_N(\eta)+\mathcal{O}(\delta^2).\end{aligned}$$

From Proposition 2.8, we observe that the kernel K_N corresponding to the operator $(1-\tau\nu^2\partial_{xx})^{-1}\Pi_N$ can be written as $K_N=K_N^+-K_N^-$ where $K_N^+=\max\{K_N,0\}$, $K_N^-=\max\{-K_N,0\}$. Both K_N^+ and K_N^- have nontrivial L^1 mass. Now note that K_N is an even function, and by the definition of convolution, we have

$$\begin{aligned}&-\Big((1-\tau\nu^2\partial_{xx})^{-1}\Pi_N(\eta)\Big)(0)=-\int_0^1 K_N(-y)\eta(y)dy\\=&-\int_0^1 K_N(y)\eta(y)dy=\int_0^1 K_N^-(y)\eta(y)dy-\int_0^1 K_N^+(y)\eta(y)dy.\end{aligned}$$

One can then choose $\eta(x)=-K_N^-(x)$ and let δ be sufficiently small. Obviously $\|u_0\|_\infty\leq\sqrt{2}$ and

$$-u^1(0)>\sqrt{2}.$$

Thus, we complete the proof. □

Proposition 3.6 (No loss in spectral cut-off when N is large)**.** *Let* $0<\tau_*\leq\tau\leq\frac{1}{2}$. *Assume* $\|u^0\|_\infty\leq 1$. *If* $N\geq N_1=N_1(\tau_*,\nu)$, *then*

$$\sup_{n\geq 0}\|u^n\|_\infty\leq 1.$$

More generally, for any $1\leq\alpha\leq\sqrt{1+\frac{2}{\tau}}$, *if* $\|u^0\|_\infty\leq\alpha$ *and* $N\geq N_2=N_2(\tau_*,\nu)$, *then*

$$\sup_{n\geq 0}\|u^n\|_\infty\leq\alpha.$$

Remark 3.10. Neglecting the dependence on τ_*, we have $N_i(\tau_*,\nu)=\mathcal{O}(e^{\frac{\text{const}}{\nu}})$ $(i=1,2)$, which is of $\mathcal{O}(1)$ for large ν.

Proof. Recall that

$$u^{n+1}=(1-\tau\nu^2\partial_{xx})^{-1}\Pi_N\big((1+\tau)u^n-\tau(u^n)^3\big).$$

The result can then be easily proved using Lemma 3.2, Theorem 2.2 and induction on n. Note that the condition $\tau\geq\tau_*$ is needed for controlling the operator $(1-\tau\nu^2\partial_{xx})^{-1}\Pi_N$ by taking N large to be independent of τ. □

Similarly we have the following proposition. The proof is omitted since it is similar.

Proposition 3.7. *Let* $\frac{1}{2}<\tau\leq 2$. *For any*

$$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}\leq\alpha\leq\sqrt{\frac{2+\tau}{\tau}},$$

if $\|u^0\|_\infty\leq\alpha$ *and*

$$N\geq N_3=N_3(\nu)=\mathcal{O}(e^{\frac{\text{const}}{\nu}}),$$

then

$$\sup_{n\geq 0}\|u^n\|_\infty\leq\alpha.$$

Remark 3.11. Note that the bound $N_3(\nu)=\mathcal{O}(e^{\frac{\text{const}}{\nu}})$ is not good when $\nu\to 0$. This will be improved below by a sligthly different method for $\frac{1}{2}<\tau<2$. Note that for $\tau=2$ such an improvement is impossible by Proposition 3.5.

Proposition 3.8. *Let $\frac{1}{2}<\tau<2-\epsilon_0$ for some $0<\epsilon_0\leq 1$. Denote*

$$M_0=\frac{1}{2}\Big(\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\sqrt{\frac{2+\tau}{\tau}}\Big),$$

if $\|u^0\|_\infty\leq M_0$ and

$$N\geq N_4=N_4(\nu)=C(\epsilon_0)\cdot\nu^{-1}\sqrt{|\log\nu|},$$

when $0<\nu\ll 1$ ($C(\epsilon_0)$ is a constant depending only on ϵ_0), then

$$\sup_{n\geq 0}\|u^n\|_\infty\leq M_0.$$

Remark 3.12. The bound

$$M_0=\frac{1}{2}\Big(\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\sqrt{\frac{2+\tau}{\tau}}\Big)$$

can be replaced by any number

$$\widetilde{M_0}\in\Big(\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3},\sqrt{\frac{2+\tau}{\tau}}\Big).$$

In this general case N_4 will also depend on $\widetilde{M_0}$, or more precisely, its distance to the end-points $\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}$ and $\sqrt{\frac{2+\tau}{\tau}}$.

Proof. Write

$$\begin{aligned}u^{n+1}&=(1-\tau\nu^2\partial_{xx})^{-1}\Pi_N\big((1+\tau)u^n-\tau(u^n)^3\big)\\&=(1-\tau\nu^2\partial_{xx})^{-1}\big((1+\tau)u^n-\tau(u^n)^3\big)+(1-\tau\nu^2\partial_{xx})^{-1}\Pi_{>N}\big((1+\tau)u^n-\tau(u^n)^3\big).\end{aligned}$$

Recall $p(x)=(1+\tau)x-\tau x^3$. Then

$$\|u^{n+1}\|_\infty\leq\|p(u^n)\|_\infty+\frac{\text{const}}{1+\nu^2N^2}\log(N+2)\cdot\|p(u^n)\|_\infty.$$

The result then follows from Lemma 3.3 and induction. □

3.1 The case $0<\tau<\tau_*$

We now consider the analogue of Proposition 3.1 for the case $0<\tau<\tau_*$.

Proposition 3.9 (Energy stability and L^∞-stability for small time step)**.** *Assume $u_0\in H^1(\mathbb{T})$ and $\|u_0\|_{L^\infty(\mathbb{T})}\le 1$. For some $\tau_*=\tau_*(\nu,\|u_0\|_{H^1})$ the following holds for any $0<\tau<\tau_*$: Let $0<\alpha_0\le 1$. If $N\ge N_0=N_0(\nu,\|u_0\|_{H^1},\alpha_0)$, then*

$$\begin{aligned}&\sup_{n\ge 0}\|u^n\|_\infty\le 1+\alpha_0,\\ &E(u^{n+1})\le E(u^n),\qquad \forall n\ge 0,\end{aligned}$$

where $F(x)=\frac{1}{4}(x^2-1)^2$, and

$$E(u)=\frac{1}{2}\nu^2\|\partial_x u\|^2_{L^2(\mathbb{T})}+\int_{\mathbb{T}}F(u)dx.$$

Proof. It is necessary to modify the proof of Proposition 3.1 and we shall outline the main changes.

The main inductive assumption is:

$$\begin{aligned}&\|u^n\|_\infty\le 1+\alpha_0,\\ &E(u^n)\le E(u^{n-1}).\end{aligned}$$

We tacitly assume that $u^{-1}=u^0$ and start the induction for $n\ge 0$. Note that $\|u^0\|_{H^1}\le\|u_0\|_{H^1}$ and $E(u^0)\lesssim E(u_0)$ since $u^0=\Pi_N u_0$.

We now justify the inductive assumption for u^{n+1}. This is divided into the following steps.

Step 1: Energy stability. By a standard discrete energy estimate, we have

$$\begin{aligned}&E(u^{n+1})-E(u^n)+\frac{1}{2}\nu^2\|\partial_x(u^{n+1}-u^n)\|_2^2+\Big(\frac{1}{\tau}+\frac{1}{2}\Big)\|u^{n+1}-u^n\|_2^2\\ \le&\frac{3}{2}\max\{\|u^n\|_\infty^2,\|u^{n+1}\|_\infty^2\}\cdot\|u^{n+1}-u^n\|_2^2.\end{aligned}$$

Recall $p(x)=(1+\tau)x-\tau x^3$ and

$$u^{n+1}=(1-\tau\nu^2\partial_{xx})^{-1}\Pi_N(p(u^n)).$$

Then

$$\begin{aligned}&\|u^{n+1}\|_\infty\lesssim\|u^{n+1}\|_{H^1}\\ \le&\|p(u^n)\|_{H^1}\lesssim_{\nu,\|u_0\|_{H^1}}1.\end{aligned}$$

By inductive assumption we have $\|u^n\|_\infty \le 2$. Then by taking $\tau_* > 0$ sufficiently small (depending only on ν and $\|u_0\|_{H^1}$), we clearly have $E(u^{n+1}) \le E(u^n)$.

Step 2: L^∞-estimate. This step is similar to that in Proposition 3.1. Denote $u^n = 1+\eta^n$. Then from the inductive assumption

$$-(2+\alpha_0) \le \eta^n \le \alpha_0.$$

For $\eta^{n+1} = u^{n+1} - 1$, we then have

$$\begin{aligned}\max \eta^{n+1} &\le (1-2\tau)\alpha_0 + \tau \|(1-\tau\nu^2\partial_{xx})^{-1}\Pi_{>N}\big(3(\eta^n)^2+(\eta^n)^3\big)\|_\infty \\ &\le (1-2\tau)\alpha_0 + \tau \|\Pi_{>N}\big(3(\eta^n)^2+(\eta^n)^3\big)\|_\infty.\end{aligned}$$

By the inequality $\|\Pi_{>N} g\|_\infty \lesssim N^{-\frac{1}{2}}\|g\|_{H^1}$, we have

$$\|\Pi_{>N}((\eta^n)^2)\|_\infty \lesssim N^{-\frac{1}{2}}\|(\eta^n)^2\|_{H^1} \lesssim_{\nu,\|u_0\|_{H^1}} N^{-\frac{1}{2}} \le \frac{1}{2}\alpha_0 \quad \text{for} \quad N \ge N_0.$$

Similar estimates hold for the term corresponding to $(\eta^n)^3$. We then clearly obtain

$$\|u^{n+1}\|_\infty \le 1+\alpha_0.$$

Thus, we complete the proof. □

Now we discuss the decay estimate for the case $0 < \tau < \tau_*$. Note that in the following N_0 is independent of α_0.

Corollary 3.2. *Assume $u_0 \in H^1(\mathbb{T})$ and $\|u^0\|_{L^\infty(\mathbb{T})} \le 1+\alpha_0$ for some $0 < \alpha_0 \le 1$. For some $\tau_* = \tau_*(\nu, \|u_0\|_{H^1}) > 0$ the following holds for any $0 < \tau < \tau_*$: If $N \ge N_0 = N_0(\nu, \|u_0\|_{H^1})$, then for any $n \ge 1$,*

$$\|u^n\|_\infty \le 1 + \theta^n\alpha_0 + \frac{1-\theta^n}{1-\theta}\cdot\tau\cdot N^{-\frac{1}{2}}\cdot C_{\nu,\|u_0\|_{H^1}},$$

where $\theta = 1-2\tau$, and the constant $C_{\nu,\|u_0\|_{H^1}} > 0$ depends only on ν and $\|u_0\|_{H^1}$. Consequently

$$\limsup_{n\to\infty} \|u^n\|_\infty \le 1 + \frac{1}{2}\cdot N^{-\frac{1}{2}}\cdot C_{\nu,\|u_0\|_{H^1}}.$$

Proof. First by taking $\alpha_0 = 1$ in Proposition 3.9 we can achieve for $N \ge N_0 = N_0(\nu, \|u_0\|_{H^1})$ that

$$\begin{aligned}&\sup_{n\ge 0}\|u^n\|_\infty \le 2, \\ &\sup_{n\ge 0} E(u^n) \le E(u^0).\end{aligned}$$

Now denote $\alpha_n=\max\eta^n=\max(u^n-1)$. Then by repeating the proof in Proposition 3.9, we obtain

$$\alpha_{n+1}\leq(1-2\tau)\alpha_n+\tau\cdot N^{-\frac{1}{2}}\cdot C_{\nu,\|u_0\|_{H^1}},$$

where the constant $C_{\nu,\|u_0\|_{H^1}}>0$ depends only on ν and $\|u_0\|_{H^1}$. Similar estimates also hold for $\tilde{\alpha}_n=\max(-1-u^n)$. Thus for $\theta=1-2\tau$

$$\|u^n\|_\infty\leq 1+\theta^n\alpha_0+\frac{1-\theta^n}{1-\theta}\cdot\tau\cdot N^{-\frac{1}{2}}\cdot C_{\nu,\|u_0\|_{H^1}}.$$

Thus, we complete the proof. □

3.2 New energy stability results for $\frac{1}{2}<\tau<\tau_1\approx 0.86$

Consider the equation

$$\frac{1}{2}+\frac{1}{x}=\frac{3}{2}\cdot\Big(\frac{2}{3}\cdot\frac{(1+x)^{\frac{3}{2}}}{\sqrt{3x}}\Big)^2.$$

Easy to check that

$$x=\tau_1=\frac{1}{2}(-2+(9-3\sqrt{6})^{\frac{1}{3}}+(9+3\sqrt{6})^{\frac{1}{3}})\approx 0.860018$$

is the unique real-valued solution.

It follows that if $\frac{1}{2}<\tau\leq\tau_1-\epsilon_0$ where $0<\epsilon_0\leq 0.1$, then

$$\frac{1}{2}+\frac{1}{\tau}\geq\frac{3}{2}\cdot\Big(\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\eta(\epsilon_0)\Big)^2, \tag{3.3}$$

where $\eta(\epsilon_0)>0$ depends only on ϵ_0. For example, it is not difficult to check that for all $\frac{1}{2}\leq\tau\leq 0.86$,

$$\frac{1}{2}+\frac{1}{\tau}\geq\frac{3}{2}\cdot\Big(\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\eta_0\Big)^2,\quad \eta_0=10^{-5}.$$

Proposition 3.10 (Energy stability for $\frac{1}{2}<\tau<\tau_1$)**.** *Let $\frac{1}{2}<\tau\leq\tau_1-\epsilon_0$ for some $0<\epsilon_0\leq 0.1$. Suppose*

$$\|u^0\|_\infty\leq\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}.$$

Denote

$$M_3=\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\eta(\epsilon_0),$$

where $\eta(\epsilon_0)$ *is the same as in* (3.3). *Then for* $N\geq N_0(\nu,\epsilon_0)$, *we have*

$$\begin{aligned}&\sup_{n\geq 0}\|u^n\|_\infty\leq M_3,\\&E(u^{n+1})\leq E(u^n),\quad \forall n\geq 0.\end{aligned}$$

Proof. The L^∞-stability estimate is established in the same way as in Proposition 3.8 (see also the remark after). Now recall the discrete energy estimate

$$\begin{aligned}&E(u^{n+1})-E(u^n)+\frac{1}{2}\nu^2\|\partial_x(u^{n+1}-u^n)\|_2^2+\Big(\frac{1}{\tau}+\frac{1}{2}\Big)\|u^{n+1}-u^n\|_2^2\\\leq&\frac{3}{2}\max\{\|u^n\|_\infty^2,\|u^{n+1}\|_\infty^2\}\cdot\|u^{n+1}-u^n\|_2^2.\end{aligned}$$

Thus it suffices to check

$$\frac{1}{2}+\frac{1}{\tau}\geq\frac{3}{2}\cdot M_3^2=\frac{3}{2}\cdot\Big(\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\eta(\epsilon_0)\Big)^2,$$

which is exactly (3.3). □

4 General theory for dimension $d\geq 1$

Consider the scheme for Allen-Cahn on the torus $\mathbb{T}=[0,1)^d$ in the physical dimensions $d\leq 3$:

$$\begin{cases}\dfrac{u^{n+1}-u^n}{\tau}=\nu^2\Delta u^{n+1}-\Pi_N((u^n)^3-u^n),\quad n\geq 0,\\u^0=\Pi_N u_0,\end{cases}\tag{4.1}$$

where $\nu>0$.

Theorem 4.1 (Energy stability for $0<\tau\leq 0.86$)**.** *Consider* (4.1) *with* $u_0\in H^s(\mathbb{T}^d)\cap H^1(\mathbb{T}^d)$, $s>\frac{d}{2}$ *and* $d\leq 3$. *Let* $0<\tau\leq 0.86$ *and assume*

$$\|u^0\|_\infty\leq\sqrt{\frac{5}{3}}\quad\Big(\textit{note that}\ \sqrt{\frac{5}{3}}\approx 1.29099\Big).$$

Then we have energy stability for any $N \geq N_0 = N_0(s,d,\nu,u_0)$*:*

$$E(u^{n+1}) \leq E(u^n), \quad \forall n \geq 0.$$

Furthermore,

$$\sup_{n\geq 0} \|u^n\|_{H^s} \leq U_1 < \infty,$$

where $U_1 > 0$ *is a constant depending only on* (s, d, ν, u_0).

Remark 4.1. For $0 < \nu \ll 1$, the dependence of N_0 on ν is only power like which is a very mild constraint in practice. In dimension $d=1$ we have

$$N_0 = \mathcal{O}\Big(\nu^{-1}\Big(\log\Big(\frac{1}{\nu}+1\Big)\Big)^{\frac{1}{2}}\Big)$$

for $\tau = \mathcal{O}(1)$ as shown in the previous section. For $0 < \tau \ll 1$, by using concurrently the energy conservation, a rough estimate gives $\|u^n\|_\infty = \mathcal{O}(\nu^{-\frac{1}{4}})$ which yields $\tau_* = \nu^{\frac{1}{2}}$. This in turn renders the estimate

$$N_0 = \mathcal{O}\Big(\nu^{-1.25}\Big(\log\Big(\frac{1}{\nu}+1\Big)\Big)^{\frac{1}{2}}\Big)$$

for $d=1$. In dimension $d \geq 2$ this dependence gets a bit worse since we have to use discrete smoothing estimates.

Proof of Theorem 4.1. We proceed in several steps.

1. By using the theory developed in [3], it is not difficult to check that for some $0 < \tau_* = \tau_*(s,\nu,u_0,d) \leq \frac{1}{2}$, we have energy stability $E(u^{n+1}) \leq E(u^n)$ for all $0 < \tau \leq \tau_*$ and $n \geq 0$. Furthermore the uniform in time H^s estimate follows from discrete smoothing estimates.

2. We now consider the regime $\tau_* \leq \tau \leq \frac{1}{2}$. Since we assumed $\|u^0\|_\infty \leq \sqrt{\frac{5}{3}}$ and $\tau \geq \tau_*$, we can adapt the proof in Proposition 3.1. Observe that by Theorem 2.4, the kernel $(1-\tau\nu^2\Delta)^{-1}\Pi_{>N}$ has a bound $N^{-2}\nu^{-2}\log^d(N+2)$. We then clearly have (for N sufficiently large)

$$\sup_{n\geq 0} \|u^n\|_\infty \leq \sqrt{\frac{5}{3}}.$$

It is obvious that

$$\frac{1}{\tau} + \frac{1}{2} \geq 2.5 \geq \frac{3}{2} \sup_{n\geq 0} \|u^n\|_\infty^2.$$

Thus we have $E(u^{n+1})\leq E(u^n)$ for all $n\geq 0$.

3. Finally we consider the regime $\frac{1}{2}\leq\tau\leq 0.86$. In this case we essentially repeat the proof of Proposition 3.10. Note that

$$\frac{1}{2}(-2+(9-3\sqrt{6})^{\frac{1}{3}}+(9+3\sqrt{6})^{\frac{1}{3}})\approx 0.860018>0.86.$$

The desired result clearly follows.

4. The uniform H^s estimate for $\tau_*\leq\tau\leq 0.86$ follows easily from smoothing estimates. We omit further details. □

Theorem 4.2 (Almost sharp maximum principle for $0<\tau\leq\frac{1}{2}$)**.** *Consider* (4.1) *with* $u_0\in H^s(\mathbb{T}^d)\cap H^1(\mathbb{T}^d)$, $s>\frac{d}{2}$ *and* $d\leq 3$. *Let* $0<\tau\leq\frac{1}{2}$. *Assume*

$$\|u^0\|_\infty=\|\Pi_N u_0\|_\infty\leq 1+\alpha_0$$

for some $0\leq\alpha_0\leq\sqrt{\frac{5}{3}}-1$ (*note that* $\sqrt{\frac{5}{3}}-1\approx 0.29099$). *Then for* $N\geq N_1=N_1(s,d,\nu,u_0)$, *we have*

$$\|u^n\|_\infty\leq 1+(1-2\tau)^n\alpha_0+N^{-(s-\frac{d}{2})}C_{s,d,\nu,u_0},\quad\forall n\geq 1,$$

where $C_{s,d,\nu,u_0}>0$ *depends only on* (s, d, ν, u_0).

Proof of Theorem 4.2. First by using Theorem 4.1, we clearly have (in this part we determine N_0)

$$\sup_{n\geq 0}\|u^n\|_{H^s(\mathbb{T}^d)}\lesssim 1.$$

We then adapt the L^∞-part of the proof of Proposition 3.9 and also the proof of Corollary 3.2. The key is to observe that

$$\|\Pi_{>N}g\|_{L^\infty(\mathbb{T}^d)}\lesssim N^{-(s-\frac{d}{2})}\|g\|_{H^s(\mathbb{T}^d)}.$$

Thus

$$\begin{aligned}\max\eta^{n+1}&\leq(1-2\tau)\max\eta^n+\tau\|\Pi_{>N}(3(\eta^n)^2+(\eta^n)^3)\|_\infty\\&\leq(1-2\tau)\max\eta^n+\tau\cdot N^{-(s-\frac{d}{2})}\cdot\text{const}.\end{aligned}$$

The desired result clearly holds. Note that we allow the full range $\alpha_0\in[0,\sqrt{\frac{5}{3}}-1]$ (i.e., we do not insist $\alpha_0>0$) since we record explicitly the spectral error term. □

Theorem 4.3 (L^∞-stability for $\frac{1}{2}<\tau\leq 2$)**.** *Consider* (4.1) *with* $d\leq 3$ *and* $\frac{1}{2}<\tau\leq 2$. *Then the following hold:*

1. *Let* $\frac{1}{2}<\tau<2-\epsilon_0$ *for some* $0<\epsilon_0\leq 1$. *Denote*

$$M_0=\frac{1}{2}\Big(\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}+\sqrt{\frac{2+\tau}{\tau}}\Big).$$

If $\|u^0\|_\infty\leq M_0$ *and*

$$N\geq N_2=N_2(\nu)=C(\epsilon_0)\cdot\nu^{-1}(|\log\nu|)^{\frac{d}{2}},$$

when $0<\nu\ll 1$ ($C(\epsilon_0)$ *is a constant depending only on* ϵ_0)*, then*

$$\sup_{n\geq 0}\|u^n\|_\infty\leq M_0.$$

2. *Let* $d=1,2$. *For any*

$$\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}\cdot\frac{2}{3}\leq\alpha\leq\sqrt{\frac{2+\tau}{\tau}},$$

if $\|u^0\|_\infty\leq\alpha$ *and*

$$N\geq N_3=N_3(\nu)=\mathcal{O}(e^{\frac{\text{const}}{\nu}}),$$

then

$$\sup_{n\geq 0}\|u^n\|_\infty\leq\alpha.$$

Remark 4.2. For $\tau=2$ and N not large, there are counterexamples as shown in Proposition 3.5.

Proof of Theorem 4.3. For the first statement, we repeat the proof of Proposition 3.8. Note that for general dimension $d\leq 3$, we use Theorem 2.4 to control the operator $(\mathrm{I}-\nu^2\tau\Delta)^{-1}(\mathrm{Id}-\Pi_N)$. The second statement follows from a similar argument as in Proposition 3.7. Here we also use the sharp maximum principle established in Theorem 2.4 for dimension $d=1,2$. □

We now consider the pure spectral truncation for Allen-Cahn on the torus $\mathbb{T}=[0,1)^d$ in dimensions $d\geq 1$:

$$\begin{cases}\partial_t u=\nu^2\Delta u-\Pi_N(u^3-u), & t>0,\\ u\big|_{t=0}=v_0=\Pi_N u_0,\end{cases}\tag{4.2}$$

where $\nu>0$.

Theorem 4.4 (Maximum principle for the continuous in time system with spectral truncation)**.** *Consider* (4.2) *with* $\nu>0$, $d\geq 1$. *Assume* $\|u_0\|_\infty\leq 1$ *and* $u_0\in H^s(\mathbb{T}^d)$, $s>\frac{d}{2}$. *Then for* $N\geq N_0=N_0(s,d,\nu,u_0)$, *we have*

$$\sup_{0\leq t<\infty}\|u(t,\cdot)\|_\infty\leq 1+N^{-\gamma_0}C_1,$$

where $\gamma_0=\min\{s-\frac{d}{2},1\}$, *and* $C_1>0$ *depends only on* (s, d, ν, u_0).

Remark 4.3. The assumption $u_0\in H^s$ is needed so that we can have a uniform choice of the spectral cut-off N_0. One can also use Theorem 4.2 and take $\tau\to 0$ to derive the result for $d\leq 3$.

Proof of Theorem 4.4. First observe that for some $T_0=T_0(s,d,\nu,\|u_0\|_{H^s})>0$, we can construct a unique local solution which satisfies

$$\max_{0\leq t\leq T_0}\|u(t)\|_{H^s}\leq 2\|u_0\|_{H^s},$$
$$\max_{0\leq t\leq T_0}\|u(t)\|_\infty\leq B_0\|u_0\|_{H^s},$$

where $B_0>0$ depends only on (s, d). By using Energy conservation and taking advantage of the Fourier projection, it is clear that the local solution can be extended for all time (the bounds on various norms will depend on N without further estimates). Note that $u\in C_t^0L_x^\infty$, and in particular $\|u(t)\|_\infty$ is a continuous function of t. We first show that if $N\geq N_0=N_0(s,d,\nu,u_0)$ (the choice of N_0 will become clear momentarily)

$$\sup_{0\leq t<\infty}\|u(t)\|_\infty\leq B_0\|u_0\|_{H^s}+2=:\beta_0. \tag{4.3}$$

Indeed assume that $T_*>T_0$ is the first moment that $\|u(t)\|_\infty$ achieves the upper bound. If T_* does not exist, we are done. Otherwise $T_0<T_*<\infty$. By a simple bootstrapping estimate, we have

$$\max_{T_0\leq t\leq T_*}\|\nabla u(t)\|_\infty\leq B_1,$$

where $B_1>0$ depends only on (s, d, ν, u_0). It follows that

$$\max_{T_0\leq t\leq T_*}\|\Pi_{>N}(u^3-u)\|_\infty\leq N^{-1}B_2,$$

where $B_2>0$ depends only on (s, d, ν, u_0). Now without loss of generality we can assume $u(T_*,x_*)=\beta_0$ for some $x_*\in\mathbb{T}^d$. Clearly we have $(\partial_t u)(T_*,x_*)=0$. On the other hand, we have

$$\begin{aligned}&\nu^2(\Delta u)(T_*,x_*)+\Pi_N(u-u^3)\big|_{(T_*,x_*)}\\ \leq&\beta_0-\beta_0^3+N^{-1}B_2\leq -6+N^{-1}B_2<0,\end{aligned}$$

if $N \geq B_2$.

Thus (4.3) holds. Furthermore by a bootstrapping estimate, we have

$$\sup_{T_0 \leq t < \infty} \|\Pi_{>N}(u^3-u)\|_\infty \leq N^{-1} B_3,$$
$$\sup_{0 \leq t \leq T_0} \|\Pi_{>N}(u^3-u)\|_\infty \leq N^{-(s-\frac{d}{2})} B_4,$$

where $B_3>0$, $B_4>0$ depend only on $(s,\, d,\, \nu,\, u_0)$. Also observe that

$$\|v_0\|_\infty \leq 1+\mathcal{O}(N^{-(s-\frac{d}{2})}).$$

Our desired result then follows by running again the maximum principle argument (similar to the proof of (4.3)) and choosing N sufficiently large. □

We now drop the assumption $u_0 \in H^s$ and derive another form of the maximum principle.

Theorem 4.5 (Maximum principle for the continuous in time system with spectral truncation)**.** *Consider* (4.2) *with* $\nu>0$, $d\geq 1$. *Assume* $\|u_0\|_\infty \leq 1$. *Then for* $N \geq N_1 = N_1(d,\nu,u_0)$, *we have*

$$\sup_{0 \leq t < \infty} \|u(t,\cdot)\|_\infty \leq \max\{\|v_0\|_\infty, 1\} + N^{-\frac{1}{2}} C_2,$$

where $C_2>0$ *depends only on* $(d,\, \nu,\, u_0)$.

Proof of Theorem 4.5. The proof is similar to that in Theorem 4.4 and we shall only sketch the needed modifications. Since Π_N is the product of one-dimensional Fourier multipliers, it is not difficult to verify that

$$\sup_{N \geq 1} \|\Pi_N g\|_q \leq C_{q,d} \|g\|_q, \quad \forall 1<q<\infty,$$

where $C_{q,d}$ depends only on $(q,\, d)$. Fix $4d<q<\infty$ sufficiently large (the needed largeness will become clear momentarily.) It is not difficult to construct a local solution in $C_t^0 L_x^q$. Moreover the solution can be extended globally in time. By using the local wellposedness, we can find $t_0 = t_0(\|u_0\|_q, d, \nu) > 0$ sufficiently small such that

$$\sup_{0 \leq t \leq t_0} \|u(t)\|_q \leq 2\|v_0\|_q \leq C_{q,d}\|u_0\|_q \leq C_{q,d},$$
$$\sup_{0 \leq t \leq t_0} t^{-0.99} \|u(t) - e^{t\nu^2\Delta} v_0\|_\infty \leq B_1,$$
$$\sup_{0 \leq t \leq t_0} t^{0.51} \|\nabla u(t)\|_q \leq B_2,$$

where $B_1>0$, $B_2>0$ are constants depending on (d,ν,u_0,q). Clearly for $t_1<t_0$ (t_1 will be chosen later) and q sufficiently large, we have

$$\sup_{t_1\le t\le t_0}\|\Pi_{>N}(u(t)-u(t)^3)\|_\infty\le N^{-0.99}t_1^{-0.51}B_3, \tag{4.4}$$

where $B_3>0$ depends on (d,ν,u_0). Note that after this step is finished we can fix the value of q. By using the smoothing estimates and similar maximum principle estimates as used in Theorem 4.4 we also have

$$\sup_{t\ge t_0}\|\Pi_{>N}(u(t)-u(t)^3)\|_\infty\le N^{-1}B_4, \tag{4.5}$$

where $B_4>0$ depends on (d,ν,u_0).

Now note that for $0\le t\le t_1$, we have

$$\|u(t)\|_\infty\le\|v_0\|_\infty+B_1t_1^{0.9}.$$

Choosing $t_1=N^{-\frac{49}{51}}$ and running a simple maximum principle argument using (4.4), (4.5) then yields the result. □

5 1D Allen-Cahn: fully discrete case using DFT

In this section we consider the analysis of the 1D Allen-Cahn on the periodic torus $\mathbb{T}=[0,1)$ using discrete Fourier transform. We discretize the domain $[0,1)$ using $x_j=\frac{j}{N}$, $j=0,1,\cdots,N-1$, where N is taken to be an even number. In typical FFT simulations, N is usually taken to be a dyadic number. We use $u^n=(u_0^n,\cdots,u_{N-1}^n)^T$ to denote the approximation of $u(t,x_j)=u(t,\frac{j}{N})$, $t=n\tau$, $0\le j\le N-1$. We shall adopt the following convention for discrete Fourier transform:

$$\tilde{U}_k=\frac{1}{N}\sum_{j=0}^{N-1}U_je^{-2\pi ik\cdot\frac{j}{N}},$$

$$U_j=\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}}\tilde{U}_ke^{2\pi ik\cdot\frac{j}{N}},$$

where $\tilde{U}=(\tilde{U}_0,\cdots,\tilde{U}_{N-1})^T$ is the (approximate) Fourier coefficient vector of the input data $U=(U_0,\cdots,U_{N-1})^T$. Note that $\tilde{U}_{k\pm N}=\tilde{U}_k$ for any $k\in\mathbb{Z}$. The discrete Laplacian operator Δ_h corresponds to the Fourier multiplier $-(2\pi k)^2$ (when k is restricted to $-\frac{N}{2}<k\le\frac{N}{2}$).

For $U=(U_0,\cdots,U_{N-1})^T\in\mathbb{R}^N$, $V=(V_0,\cdots,V_{N-1})^T\in\mathbb{R}^N$, we define the inner product

$$\langle U,V\rangle=\frac{1}{N}\sum_{j=0}^{N-1}U_jV_j.$$

It is easy to check that

$$\langle U,V\rangle=\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}\tilde{U}_k\tilde{V}_{-k},$$

which corresponds to the Plancherel formula.

For any real-valued $g\in H^s(\mathbb{T})$ with $s>\frac{1}{2}$, we have

$$\tilde{g}_k=\frac{1}{N}\sum_{j=0}^{N-1}g\Big(\frac{j}{N}\Big)e^{-2\pi ik\cdot\frac{j}{N}}=\sum_{l\in\mathbb{Z}}\hat{g}(k+lN),\quad -\frac{N}{2}<k\leq\frac{N}{2}.$$

Note that the convergence of the series is not a problem thanks to Cauchy-Schwartz. We define

$$\begin{aligned}(Q_Ng)(x)=&\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}\tilde{g}_ke^{2\pi ik\cdot x}\Big)\\=&\sum_{|k|<\frac{N}{2}}\tilde{g}_ke^{2\pi ik\cdot x}+\frac{1}{2}\tilde{g}_{\frac{N}{2}}\cdot(e^{2\pi i\cdot\frac{N}{2}x}+e^{-2\pi i\cdot\frac{N}{2}x}).\end{aligned}$$

Here in the above we used the fact that $\tilde{g}_k^*=\tilde{g}_{-k}$ (here $*$ denotes complex conjugate) since g is real-valued. In particular

$$\tilde{g}_{-\frac{N}{2}}=\tilde{g}_{\frac{N}{2}}=\frac{1}{N}\sum_{j=0}^{N-1}g\Big(\frac{j}{N}\Big)(-1)^j$$

is real-valued. Note that

$$\|Q_Ng\|_2^2=\sum_{|k|<\frac{N}{2}}|\tilde{g}_k|^2+\frac{1}{2}|\tilde{g}_{\frac{N}{2}}|^2,\tag{5.1a}$$

$$\frac{1}{4\pi^2}\|\partial_xQ_Ng\|_2^2=\sum_{|k|<\frac{N}{2}}|k|^2|\tilde{g}_k|^2+\frac{1}{2}\cdot\frac{N^2}{4}\cdot|\tilde{g}_{\frac{N}{2}}|^2.\tag{5.1b}$$

On the Fourier side,

$$\widehat{Q_N g}(k)=\begin{cases}\displaystyle\sum_{l\in\mathbb{Z}}\hat{g}(k+lN), & -\frac{N}{2}<k<\frac{N}{2},\\ \displaystyle\frac{1}{2}\sum_{l\in\mathbb{Z}}\hat{g}(k+lN), & k=\pm\frac{N}{2},\\ 0, & \text{otherwise.}\end{cases}$$

In yet other words, for smooth $g:\mathbb{T}\to\mathbb{R}$ we have

$$g(x)=\sum_{l\in\mathbb{Z}}\sum_{-\frac{N}{2}<k\le\frac{N}{2}}\hat{g}(k+lN)e^{2\pi i(k+lN)x},$$

$$(Q_N g)(x)=\sum_{|k|<\frac{N}{2}}\Big(\sum_{l\in\mathbb{Z}}\hat{g}(k+lN)\Big)e^{2\pi ik\cdot x}+\Big(\sum_{l\in\mathbb{Z}}\hat{g}\Big(\frac{N}{2}+lN\Big)\Big)\cos\Big(2\pi\cdot\frac{N}{2}x\Big).$$

In particular if $\hat{g}$ is supported in $\{-\frac{N}{2}\le k\le\frac{N}{2}\}$ with

$$\hat{g}\Big(\frac{N}{2}\Big)=\hat{g}\Big(-\frac{N}{2}\Big),$$

then clearly $Q_N g=g$ and one can perfectly reconstruct g from its node values $g(\frac{j}{N})$, $0\le j\le N-1$. One should note, However, that the operator Q_N in general *does not* commute with the usual differentiation operators or frequency projection operators.

We shall be using the following estimates without explicit mentioning:

$$\|(Q_N-\mathrm{Id})g\|_\infty=\|Q_N g-g\|_\infty\lesssim\sum_{|k|\ge\frac{N}{2}}|\widehat{g}(k)|\lesssim N^{-\frac{1}{2}}\|\partial_x g\|_{L^2(\mathbb{T})}.$$

Remark 5.1. Note that the operator Q_N is also defined for bounded $g:\mathbb{T}\to\mathbb{R}$ by using only the expression

$$\begin{aligned}(Q_N g)(x)&=\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\le\frac{N}{2}}\Big(\frac{1}{N}\sum_{j=0}^{N-1}g\Big(\frac{j}{N}\Big)e^{-2\pi ik\frac{j}{N}}\Big)e^{2\pi ik\cdot x}\Big)\\&=\sum_{j=0}^{N-1}g\Big(\frac{j}{N}\Big)G_N\Big(x-\frac{j}{N}\Big),\end{aligned}\tag{5.2}$$

where

$$G_N(y)=\frac{\sin N\pi y}{N\tan\pi y}.$$

In general given $U=(U_0,\cdots,U_{N-1})^T\in\mathbb{R}^N$, we can define

$$\begin{aligned}(Q_NU)(x)=&\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}\Big(\frac{1}{N}\sum_{j=0}^{N-1}U_je^{-2\pi ik\frac{j}{N}}\Big)e^{2\pi ik\cdot x}\Big)\\=&\sum_{j=0}^{N-1}U_jG_N\Big(x-\frac{j}{N}\Big).\end{aligned}\tag{5.3}$$

Clearly if $\|U\|_\infty\leq 1$, then

$$\|Q_NU\|_{L^\infty}\leq\sup_{x\in\mathbb{T}}\sum_{j=0}^{N-1}\Big|G_N\Big(\frac{j}{N}-x\Big)\Big|\lesssim\log N.$$

On the other hand, by taking $x=\frac{\epsilon_0}{N}$ with $0<\epsilon_0<1$, $U_j=1$ for $\frac{N}{100}\leq j\leq\frac{N}{10}$ with j being even, $U_j=0$ otherwise, one can easily check that

$$|(Q_NU)(x)|\gtrsim\log N.$$

Thus the mere condition $\|U\|_\infty\leq 1$ cannot guarantee a good bound on Q_NU. In a similar vein, one can also consider the function

$$\phi_N(x)=\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}\frac{1}{1+\nu^2(2\pi k)^2\tau}e^{2\pi ik\cdot x}\Big).$$

For $\tau>0$ sufficiently small (τ can depend on N), we have

$$\|\phi_N\|_{L^1(\mathbb{T})}\gtrsim\log N.$$

Note that as $\tau\to 0$, $\phi_N(\frac{j}{N})\to 0$ for all $1\leq|j|\leq N-1$, whereas $\phi_N(0)\to 1$.

Now consider the fully discrete system:

$$\frac{U^{n+1}-U^n}{\tau}=\nu^2\Delta_hU^{n+1}+U^n-(U^n)^{\cdot 3},\tag{5.4}$$

where $(U^n)^{\cdot 3}=((U_0^n)^3,\cdots,(U_{N-1}^n)^3)^T$. We recast it as

$$U^{n+1}=(\mathrm{I}-\nu^2\tau\Delta_h)^{-1}\Big((1+\tau)U^n-\tau(U^n)^{\cdot 3}\Big).\tag{5.5}$$

One can view (5.4) as the time discretization of the continuous-in-time ODE system:

$$\begin{cases} \dfrac{d}{dt}U=\nu^2\Delta_h U+U-U^{\cdot 3}, \\ U\big|_{t=0}=U^0\in\mathbb{R}^N. \end{cases} \tag{5.6}$$

For small $\tau>0$ one can study the nearness of the solutions to (5.5) and (5.6). However we shall not explore it here. On the other hand, we can consider the following system

$$\begin{cases} \partial_t u=\nu^2\Delta u+Q_N(u-u^3), \\ u\big|_{t=0}=Q_N(u_{\text{init}}), \end{cases} \tag{5.7}$$

where u_{init} is a bounded real-valued function on $\mathbb{T}$. If we require

$$U_j^0=u_{\text{init}}\Big(\frac{j}{N}\Big),\quad \forall 0\le j\le N-1,$$

then it is not difficult to check that the system (5.6) and (5.7) are equivalent, i.e.,

$$U(t)_j=u\Big(t,\frac{j}{N}\Big),\quad \forall 0\le j\le N-1.$$

Thanks to the spectral localization, the function $u(t,x)$ is uniquely determined by its node values $u(t,\frac{j}{N})$ and thus the above equivalence is not a problem.

Lemma 5.1. *Let*

$$N\ge 4+\frac{1}{\pi^2\nu\sqrt{\tau}}e^{\frac{1}{2\nu\sqrt{\tau}}}.$$

Suppose $f\in\mathbb{R}^N$, *then*

$$\|(\mathrm{I}-\nu^2\tau\Delta_h)^{-1}f\|_\infty\le\|f\|_\infty.$$

Proof. It suffices to examine the kernel K_h corresponding to $(I-\nu^2\Delta_h)^{-1}$. Denote $\beta=\nu\sqrt{\tau}$. Note that for $x=\frac{l}{N}$, $l\in\mathbb{Z}$ with $|l|\le N-1$,

$$K_h(x)=K_h\Big(\frac{l}{N}\Big)=\sum_{|k|\le\frac{N}{2}}\frac{1}{1+\beta^2(2\pi k)^2}e^{2\pi ik\frac{l}{N}}-\frac{1}{1+\beta^2(N\pi)^2}(-1)^l$$

is real-valued. On the other hand, for such x we clearly have

$$\begin{aligned} K_h(x)&\ge\sum_{k\in\mathbb{Z}}\frac{1}{1+\beta^2(2\pi k)^2}e^{2\pi ikx}-\sum_{|k|\ge\frac{N}{2}}\frac{1}{1+\beta^2(2\pi k)^2} \\ &>\frac{1}{\beta}e^{-\frac{1}{2\beta}}-\frac{1}{4\pi^2\beta^2}\cdot 2\cdot\frac{1}{\frac{N}{2}-1}. \end{aligned}$$

Thus $K_h(x)$ is positive if $N\ge 4+\frac{1}{\pi^2\beta}e^{\frac{1}{2\beta}}$ and $x=\frac{l}{N}$ with $l\in\mathbb{Z}$, $|l|\le N-1$. The desired result then easily follows from the discrete convolution inequality. □

Theorem 5.1 (Strict maximum principle)**.** *Consider* (5.5) *with initial data* $U^0\in\mathbb{R}^N$. *Assume* $0<\tau\leq\frac{1}{2}$. *Suppose* $\|U^0\|_\infty\leq 1$. *If*

$$N\geq 4+\frac{1}{\pi^2\nu\sqrt{\tau}}e^{\frac{1}{2\nu\sqrt{\tau}}},$$

then

$$\|U^n\|_\infty\leq 1,\quad \forall n\geq 1.$$

Proof. This follows from an easy induction using Lemma 5.1 and Lemma 3.2. □

Remark 5.2. Consider the kernel function

$$\phi_N(x)=\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}\frac{1}{1+\nu^2(2\pi k)^2\tau}e^{2\pi ik\cdot x}\Big).$$

As was already pointed out in Remark 5.1, in general ϕ_N can have $\mathcal{O}(\log N)$ L^1 mass when $\tau\to 0$. On the other hand, if we restrict to the grid points $x_j=\frac{j}{N}$ and consider the discrete L^1 mass

$$m_N=\frac{1}{N}\sum_{j=0}^{N-1}\Big|\phi_N\Big(\frac{j}{N}\Big)\Big|,$$

one can have unit discrete L^1 mass for some special values of N but this does not hold in general. For example consider $N=2$, we have (below we denote $s=4\pi^2\nu^2\tau$)

$$\phi_2(0)=1+\frac{1}{1+s},\quad \phi_2\Big(\frac{1}{2}\Big)=1-\frac{1}{1+s}>0,\quad \forall s>0.$$

Thus $m_2=1$. On the other hand, for $N=4$, we have

$$\phi_4(0)=1+\frac{2}{1+s}+\frac{1}{1+4s},\qquad \phi_4\Big(\frac{1}{4}\Big)=1-\frac{1}{1+4s},$$
$$\phi_4\Big(\frac{1}{2}\Big)=1-\frac{2}{1+s}+\frac{1}{1+4s},\qquad \phi_4\Big(\frac{3}{4}\Big)=1-\frac{1}{1+4s}.$$

For $s>0$ sufficiently small, we have $\phi_4(\frac{1}{2})<0$ and it follows that $m_4>1$. In general, consider $N=4K$ with K being an integer, then

$$\phi_N\Big(\frac{1}{2}\Big)=\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}\frac{1}{1+k^2s}(-1)^k.$$

Note that

$$\frac{d}{ds}\phi_N\Big(\frac{1}{2}\Big)\Big|_{s=0}=\sum_{-\frac{N}{2}+1\le k\le\frac{N}{2}}k^2(-1)^{k+1}=2\sum_{\substack{1\le k\le\frac{N}{2}-1\\ k\text{ is odd}}}k^2-2\sum_{\substack{1\le k\le\frac{N}{2}-1\\ k\text{ is even}}}k^2-\Big(\frac{N}{2}\Big)^2$$
$$=2\sum_{l=1}^{K}\Big((2l-1)^2-(2l)^2\Big)+4K^2=-2K=-\frac{N}{2}.$$

Thus for small $s>0$ we have $\phi_N(\frac{1}{2})<0$ and this renders $m_N>1$. More quantitatively we have for $s\le\alpha_1N^{-2}$ ($\alpha_1>0$ is a sufficiently small absolute constant),

$$\phi_N\Big(\frac{1}{2}\Big)\le-\alpha_2N\cdot s,$$

where $\alpha_2>0$ is a small absolute constant.

Our next set of results will be concerned with generalized maximum principles. We first consider the ODE system (5.6).

Theorem 5.2 (Almost sharp maximum principle)**.** *Consider* (5.6). *Suppose* $u_{\text{init}}:\mathbb{T}\to\mathbb{R}$ *satisfies* $u_{\text{init}}\in H^{\frac{3}{2}}(\mathbb{T})$ *and* $\|u_{\text{init}}\|_\infty\le1$. *Take the initial data* U^0 *such that* $U_j^0=u_{\text{init}}(\frac{j}{N})$ *for* $0\le j\le N-1$. *Then for* $N\ge N_1=N_1(u_{\text{init}},\nu)>0$, *we have*

$$\|U(t)\|_\infty\le\|u(t)\|_\infty\le1+C_1\cdot N^{-\frac{1}{2}},$$

where $C_1>0$ *depends only on* (u_{init}, ν). *In the above* u *is the solution to* (5.7) *with* Q_Nu_{init} *as initial data.*

Remark 5.3. Since we are concerned with the generalized maximum principe, the dependence of N_1 on ν will only be power-type which is much better than the dependence in Theorem 5.1. These can be easily inferred from the computations in the proof below. However we shall not state the explicit dependence here for the ease of notation.

Proof. Step 1. Energy estimate and global wellposedness. Concerning the system (5.6), it is not difficult to check that $\frac{d}{dt}E(U)\le0$, where

$$E(U)=\frac{1}{2}\nu^2\|\nabla_hU\|_2^2+\frac{1}{4}\cdot\frac{1}{N}\sum_{j=0}^{N-1}(U_j^2-1)^2$$
$$=\frac{1}{2}\nu^2\langle-\Delta_hU,U\rangle+\frac{1}{4}\langle U^{\cdot2}-1,U^{\cdot2}-1\rangle.\tag{5.8}$$

It follows easily that the system (5.6) admits a global solution.

Step 2. H^1 Estimate of u. We first consider the initial data. Observe that

$$\begin{aligned}&\frac{1}{4\pi^2}\|\partial_x Q_N u_{\text{init}}\|_2^2\\&=\sum_{|k|<\frac{N}{2}} k^2\Big|\sum_{l\in\mathbb{Z}}\widehat{u_{\text{init}}}(k+lN)\Big|^2+2\cdot\frac{N^2}{4}\Big|\frac{1}{2}\sum_{l\in\mathbb{Z}}\widehat{u_{\text{init}}}\Big(\frac{N}{2}+lN\Big)\Big|^2\\&\lesssim\|\partial_x u_{\text{init}}\|_2^2+\|u_{\text{init}}\|_{H^{\frac{3}{2}}(\mathbb{T})}^2\lesssim\|u_{\text{init}}\|_{H^{\frac{3}{2}}(\mathbb{T})}^2.\end{aligned}$$

Similarly $\|Q_N u_{\text{init}}\|_2\lesssim\|u_{\text{init}}\|_{H^{\frac{3}{2}}(\mathbb{T})}$. Thus

$$\|Q_N u_{\text{init}}\|_2+\|Q_N u_{\text{init}}\|_\infty\lesssim\|Q_N u_{\text{init}}\|_{H^1(\mathbb{T})}\lesssim\|u_{\text{init}}\|_{H^{\frac{3}{2}}(\mathbb{T})}.$$

Note that $\|U^0\|_\infty\le\|Q_N u_{\text{init}}\|_\infty$, and by (5.1)

$$\|\nabla_h U^0\|_2^2=\sum_{-\frac{N}{2}<k\le\frac{N}{2}} k^2|\widetilde{U_k^0}|^2\sim\|\partial_x Q_N u_{\text{init}}\|_2^2.$$

Thus

$$E(U^0)\le A_1=A_1(\|u_{\text{init}}\|_{H^{\frac{3}{2}}},\nu).$$

Now by using the inequality $U_j^2\le\frac{1}{2}(U_j^2-1)^2+\frac{3}{2}$, it is then clear that

$$\|u(t)\|_{H^1}^2\lesssim\sum_{-\frac{N}{2}<k\le\frac{N}{2}}(1+k^2)|\widetilde{U(t)_k}|^2\lesssim 1+E(U(t))\le 1+E(U^0)\le 1+A_1,\qquad\forall t\ge 0.$$

Step 3. L^∞-estimate of u. For the initial data, observe that

$$\|Q_N u_{\text{init}}-u_{\text{init}}\|_\infty\lesssim\sum_{|k|\ge\frac{N}{2}}|\widehat{u_{\text{init}}}(k)|\lesssim N^{-\frac{1}{2}}\|u_{\text{init}}\|_{H^1(\mathbb{T})}.$$

Thus

$$\|Q_N u_{\text{init}}\|_\infty\le 1+\mathcal{O}(N^{-\frac{1}{2}}),$$

where the implied constant depends on $(u_{\text{init}},\,\nu)$.

Now we consider the nonlinear term. By using Step 2, we have $\|u(t)^3\|_{H^1}\lesssim 1$ uniformly in $t\ge 0$. Clearly we have

$$\|Q_N(u^3-u)-(u^3-u)\|_\infty=\mathcal{O}(N^{-\frac{1}{2}}).$$

Then the equation for u can be rewritten as

$$\begin{cases}\partial_t u=\nu^2\Delta u+u-u^3+\mathcal{O}(N^{-\frac{1}{2}}),\\ \|u(0)\|_\infty\le 1+\mathcal{O}(N^{-\frac{1}{2}}).\end{cases}$$

A simple maximum principle argument then yields the desired result. □

Theorem 5.3 (Almost sharp maximum principle, version 2)**.** *Consider* (5.6). *Suppose* $u_{\text{init}}:\mathbb{T}\to\mathbb{R}$ *has Fourier support in* $\{|k|\le\frac{N}{2}\}$ *with*

$$\widehat{u_{\text{init}}}\Big(\frac{N}{2}\Big)=\widehat{u_{\text{init}}}\Big(-\frac{N}{2}\Big)\quad\text{and}\quad\|u_{\text{init}}\|_\infty\le 1.$$

Note that $Q_N(u_{\text{init}})=u_{\text{init}}$. *Take the initial data* U^0 *such that* $U^0_j=u_{\text{init}}(\frac{j}{N})$ *for* $0\le j\le N-1$. *Then for* $N\ge N_2=N_2(u_{\text{init}},\nu)>0$, *we have*

$$\|U(t)\|_\infty\le\|u(t)\|_\infty\le 1+C_2\cdot N^{-\frac{1}{4}},$$

where $C_2>0$ *depends only on* $(u_{\text{init}},\ \nu)$. *In the above* u *is the solution to* (5.7) *with* $Q_N u_{\text{init}}=u_{\text{init}}$ *as initial data.*

Remark 5.4. The dependence of N_2 on ν is only power-like.

Remark 5.5. By Remark 5.1, one cannot deduce a good bound on $\|Q_N u_{\text{init}}\|_\infty$ for general u_{init} assuming only $\|u_{\text{init}}\|_\infty\le 1$.

Proof. For simplicity of notation write $u_0=u_{\text{init}}$. Note that $u(t)$, $t\ge 0$ all have frequency supported in $[-\frac{N}{2},\frac{N}{2}]$. In particular $Q_N u(t)=u(t)$ for all $t\ge 0$.

Step 1. Local in time estimate near $t=0$. By a contraction estimate, for some sufficiently small $t_0=t_0(u_0,\nu)>0$ we have

$$\begin{aligned}&\|u(t)\|_\infty\le 1+\beta_1 t^{\frac{1}{2}},\qquad\forall 0\le t\le t_0,\\ &\sup_{0\le t\le t_0} t^{\frac{1}{2}}\|u(t)\|_{H^1}\le\beta_2,\end{aligned}$$

where the constants $\beta_1>0$, $\beta_2>0$ only depend on $(u_0,\ \nu)$. Now pick $t_1=N^{-\frac{1}{2}}\le t_0$ and we clearly have

$$\begin{aligned}&\sup_{0\le t\le t_1}\|u(t)\|_\infty\le 1+\mathcal{O}(N^{-\frac{1}{4}}),\\ &\|u(t_1)\|_{H^1}\le\mathcal{O}(N^{\frac{1}{4}}).\end{aligned}$$

For $t>t_1$, we can regard $u(t_1)$ as initial data and observe that by using energy conservation (similar to the argument in the preceding theorem, but note that we no longer need the $H^{\frac{3}{2}}$ bound), we have

$$\sup_{t\geq t_1}\|u(t)\|_{H^1}\leq\mathcal{O}(N^{\frac{1}{4}}).$$

Furthermore, for $t\geq t_1$, assuming that $\|u(t)\|_\infty\leq 2$, it is easy to check that

$$\|Q_N(u(t)^3)-u(t)^3\|_\infty\leq\mathcal{O}(N^{-\frac{1}{4}}).$$

One can then run a maximum principle argument to finish the proof (i.e., consider some $t_*>t_1$ as the first time that $\|u(t)\|_\infty$ exceeds the bound $1+C\cdot N^{-\frac{1}{4}}$ and show a contradiction). □

Theorem 5.4 (Almost sharp maximum principle for (5.6), general rough initial data)**.** *Consider* (5.6). *Suppose $U^0\in\mathbb{R}^N$ satisfies $\|U^0\|_\infty\leq 1$. If $N\geq N_3(\nu)>0$, it holds that*

$$\sup_{0\leq t<\infty}\|U(t)\|_\infty\leq 1+\epsilon_1,$$

where $0<\epsilon_1<10^{-2}$ is an absolute constant. More precisely the following hold. (Below we shall write $X=O(Y)$ if $|X|\leq CY$ where the constant C only depends on ν.)

1. *For $t\geq T_0=T_0(\nu)>0$,*

$$\|U(t)\|_\infty\leq\|u(t)\|_\infty\leq 1+\mathcal{O}(N^{-\frac{1}{8}}(\log N)^2),$$
$$\|\partial_x u(t)\|_2\leq\mathcal{O}(1),\quad E(U(t))\leq\mathcal{O}(1),$$

   where u solves (5.9)*, and $E(U)$ was defined in* (5.8)*.*

2. *For $C_1=C_1(\nu)>0$ and $C_1\cdot N^{-\frac{3}{4}}\leq t\leq T_0$, it holds that*

$$\|U(t)\|_\infty\leq\|u(t)\|_\infty\leq 1+\mathcal{O}(N^{-\frac{1}{8}}(\log N)^2).$$

3. *For $C_2=C_2(\nu)>0$ and $C_2\cdot N^{-2}\log N\leq t\leq C_1\cdot N^{-\frac{3}{4}}$,*

$$\|U(t)\|_\infty\leq\|u(t)\|_\infty\leq 1+\mathcal{O}(N^{-\frac{3}{16}}).$$

4. *For $0<t\leq C_2N^{-2}\log N$, we have*

$$\|u(t)-e^{\nu^2t\partial_{xx}}v_0\|_\infty\leq\mathcal{O}(N^{-\frac{1}{2}}(\log N)^{\frac{1}{4}}),$$
$$\sup_{0\leq l\leq N-1}\left|(e^{\nu^2t\partial_{xx}}Q_NU^0)\left(t,\frac{l}{N}\right)\right|\leq\left(1+\frac{1}{2}\epsilon_1\right)\|U^0\|_\infty,$$
$$\|U(t)\|_\infty\leq\left(1+\frac{1}{2}\epsilon_1\right)\|U^0\|_\infty+\mathcal{O}(N^{-\frac{1}{2}}(\log N)^{\frac{1}{4}}).$$

5. *If* $\|U^0\|_\infty \le \frac{1}{1+\frac{1}{2}\epsilon_1}$ *(note that* $\frac{1}{1+\frac{1}{2}\epsilon_1} > 0.995$*), then for all* $t \ge 0$*, we have*

$$\|U(t)\|_\infty \le 1 + \mathcal{O}(N^{-\frac{1}{8}}(\log N)^2).$$

6. *Suppose* $f:\mathbb{T}\to\mathbb{R}$ *is continuous and* $\|f\|_\infty \le 1$*. If* $(U^0)_l = f(\frac{l}{N})$ *for all* $0 \le l \le N-1$*, then*

$$\sup_{t\ge 0}\|U(t)\|_\infty \le 1 + \mathcal{O}(\omega_f(N^{-\frac{2}{3}})) + \mathcal{O}(N^{-c}),$$

where $c > 0$ *is an absolute constant, and* ω_f *is defined in* (B.5)*.*

7. *Suppose* $f:\mathbb{T}\to\mathbb{R}$ *is* C^α*-continuous* (*see* (B.6)) *for some* $0 < \alpha < 1$ *and* $\|f\|_\infty \le 1$*. If* $(U^0)_l = f(\frac{l}{N})$ *for all* $0 \le l \le N-1$*, then*

$$\sup_{t\ge 0}\|U(t)\|_\infty \le 1 + \mathcal{O}(N^{-c_1}),$$

where $c_1 > 0$ *is a constant depending only on* α*.*

Moreover we have the following result which shows the sharpness of our estimates above. There exists a function $f:\ \mathbb{T}\to\mathbb{R}$*, continuous at all of* $\mathbb{T}\setminus\{x_*\}$ *for some* $x_* \in \mathbb{T}$ (*i.e., continuous at all* $x \ne x_*$) *and has the bound* $\|f\|_\infty \le 1$ *such that the following hold: for a sequence of even numbers* $N_m \to \infty$*,* $t_m = \nu^{-2}N_m^{-2}$ *and* $x_m = j_m/N_m$ *with* $0 \le j_m \le N_m - 1$*, if* $\tilde{U}_m(t) \in \mathbb{R}^{N_m}$ *solves* (5.6) *with* $\tilde{U}_m(0) = v_m \in \mathbb{R}^{N_m}$ *satisfying* $(v_m)_j = f(\frac{j}{N_m})$ *for all* $0 \le j \le N_m - 1$*. Then*

$$|\tilde{U}_m(t_m, x_m)| \ge 1 + \eta_*, \quad \forall m \ge 1,$$

where $\eta_* > 0.001$ *is an absolute constant.*

Remark 5.6. One should compare our last result with the Fourier spectral Galerkin truncation case (see for example Theorem 4.5) where we dealt with initial data bounded by one and proved a sharp maximum principle. There thanks to spectral localization the initial data is smooth. Here in the collocation case, our counterexample shows that there is non-approximation if we work with mere L^∞ initial data which is bounded by one and everywhere continuous except at one point. This is certainly connected with the non-smooth cut-off in the Fourier space.

Remark 5.7. The dependence of N_3 on ν is only power-like.

Proof. To simplify the notation we shall tacitly assume $\nu=1$ below. It is not difficult to keep track of the explicit dependence of ν but we shall not do it here in order to simplify the presentation. Let $u=u(t,x)$ solve the following system (below $\nu=1$):

$$\begin{cases}\partial_t u=\nu^2\partial_{xx}u+Q_N(u-u^3),\\ u\big|_{t=0}=Q_NU^0=:v_0,\end{cases}\tag{5.9}$$

where Q_NU^0 was defined in (5.3). It is not difficult to check that $u(t,\frac{j}{N})=U(t)_j$ for all $0\le j\le N-1$ and $t\ge 0$. Furthermore

$$\begin{aligned}&\|v_0\|_{L^2_x(\mathbb{T})}=\|U^0\|_{l^2}=\Big(\frac{1}{N}\sum_j(U^0_j)^2\Big)^{\frac12}\le 1,\\ &\|v_0\|_{L^\infty_x(\mathbb{T})}\lesssim\log N.\end{aligned}$$

In the argument below we shall take $N\ge N_0$ where N_0 is a sufficiently large absolute constant. The needed largeness of N_0 can be easily worked out from the argument.

Step 1. Local in time estimate. Introduce the norm

$$\|u\|_{X_T}:=\frac{\|u\|_{L^\infty_tL^\infty_x([0,T]\times\mathbb{T})}}{\log N}+\|t^{0.5}\partial_xu\|_{L^\infty_tL^2_x([0,T]\times\mathbb{T})}+\|u\|_{L^\infty_tL^2_x([0,T]\times\mathbb{T})}.$$

By using the mild formulation (note that $Q_Nu=u$)

$$u(t)=e^{t\partial_{xx}}v_0+\int_0^t e^{(t-s)\partial_{xx}}(u-u^3)ds-\int_0^t e^{(t-s)\partial_{xx}}(Q_N-\mathrm{Id})(u^3)ds,$$

we have

$$\begin{aligned}\|u(t)\|_\infty&\lesssim\log N+t\max_{0\le s\le t}\|u(s)\|_\infty+\int_0^t(t-s)^{-\frac12}\|u^3\|_1ds+\int_0^tN^{-\frac12}\|\partial_x(u^3)\|_2ds\\ &\lesssim\log N+t\log N+t^{\frac12}\log N+N^{-\frac12}(\log N)^2t^{0.5}\lesssim\log N.\end{aligned}$$

Also by using the inequality $\|u\|_3\lesssim\|u\|_2+\|u\|_2^{\frac56}\|\partial_xu\|_2^{\frac16}$, we have

$$\begin{aligned}\|u(t)\|_2&\lesssim 1+t+\int_0^t(t-s)^{-\frac14}\|u^3\|_1ds+\int_0^t\|(Q_N-\mathrm{Id})(u^3)\|_{L^2(\mathbb{T})}ds\\ &\lesssim 1+t+\int_0^t(t-s)^{-\frac14}\|u^3\|_1ds+\int_0^t\|(Q_N-\mathrm{Id})(u^3)\|_{L^\infty(\mathbb{T})}ds\\ &\lesssim 1+t+\int_0^t(t-s)^{-\frac14}\|u^3\|_1ds+\int_0^tN^{-\frac12}\|\partial_x(u^3)\|_2ds\\ &\lesssim 1+t+t^{\frac34}+\int_0^t(t-s)^{-\frac14}s^{-\frac14}ds+N^{-\frac12}(\log N)^2t^{0.5}\lesssim 1,\end{aligned}$$

and

$$\begin{aligned}t^{0.5}\|\partial_x u(t)\|_2 \lesssim & 1+t^{0.5}\int_0^t (t-s)^{-\frac{3}{4}}\|u^3\|_1 ds+t^{0.5}\int_0^t (t-s)^{-\frac{1}{2}}N^{-\frac{1}{2}}\|\partial_x(u^3)\|_2 ds\\ \lesssim & 1+t^{0.5}\int_0^t (t-s)^{-\frac{3}{4}}s^{-\frac{1}{4}}ds+t^{0.5}\int_0^t (t-s)^{-\frac{1}{2}}N^{-\frac{1}{2}}(\log N)^2 s^{-\frac{1}{2}}ds\\ \lesssim & 1+t^{0.5}\lesssim 1.\end{aligned}$$

By carefully tracking the constants and similar estimates as done in the above, it is not difficult to check that for $T=T_0>0$ (T_0 is a sufficiently small absolute constant) we have contraction in the ball $\{\|u\|_{X_T}\leq C_1\}$ where C_1 is an absolute constant.

Step 2. Refined estimate. For any $0<t\leq T_0$, we have

$$\begin{aligned}\|u(t)-e^{t\partial_{xx}}v_0\|_\infty \leq & \int_0^t \|e^{(t-s)\partial_{xx}}(u-u^3)\|_\infty ds+\int_0^t \|e^{(t-s)\partial_{xx}}(Q_N-\mathrm{Id})(u^3)\|_\infty ds\\ \lesssim & \int_0^t (t-s)^{-\frac{1}{4}}\|u(s)\|_2 ds+\int_0^t (t-s)^{-\frac{1}{2}}\|u^3\|_1 ds+\int_0^t N^{-\frac{1}{2}}\|\partial_x(u^3)\|_2 ds\\ \lesssim & t^{\frac{3}{4}}+t^{\frac{1}{4}}+N^{-\frac{1}{2}}(\log N)^2 t^{\frac{1}{2}}.\end{aligned}$$

Now take $T_1=100N^{-2}\log N\leq T_0$, we have

$$\|u(T_1)-e^{T_1\partial_{xx}}v_0\|_\infty \lesssim N^{-\frac{1}{2}}(\log N)^{\frac{1}{4}}.$$

On the other hand, since

$$(e^{t\partial_{xx}}v_0)(t,x)=\frac{1}{N}\sum_{j=0}^{N-1}(U^0)_j \mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}} e^{-(2\pi k)^2 t+2\pi i k(x-\frac{j}{N})}\Big),$$

it is clear that

$$\begin{aligned}\|e^{T_1\partial_{xx}}v_0\|_\infty \leq & \Big\|\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}} e^{-(2\pi k)^2 T_1+2\pi i k\cdot x}\Big)\Big\|_\infty\\ \leq & 1+\int_{|k|>\frac{N}{2}-1} e^{-(2\pi k)^2 T_1}dk\\ \leq & 1+N^{-1}.\end{aligned}$$

Thus

$$\|u(T_1)\|_\infty \leq 1+\mathcal{O}(N^{-\frac{1}{2}}(\log N)^{\frac{1}{4}}).$$

By a similar estimate, we have for any $T_1 \le t \le T_2 = N^{-\frac{3}{4}} \le T_0$,

$$\|u(t)\|_\infty \le 1 + \mathcal{O}(t^{\frac{1}{4}}) \le 1 + \mathcal{O}(N^{-\frac{3}{16}}).$$

Step 3. The regime $t \ge T_2$. Note that at $t = T_2$, we have

$$\|u(T_2)\|_\infty \le 1 + \mathcal{O}(N^{-\frac{3}{16}}),$$
$$\|\partial_x u(T_2)\|_2 \le \mathcal{O}(T_2^{-\frac{1}{2}}) = \mathcal{O}(N^{\frac{3}{8}}).$$

It follows easily that for the system (5.6),

$$E(U(t)) \le E(U(T_2)) \le \mathcal{O}(N^{\frac{3}{8}}), \quad \forall t \ge T_2,$$

where $E(U)$ was defined in (5.8). This in turn implies that

$$\|\partial_x u(t)\|_2 \le \mathcal{O}(N^{\frac{3}{8}}), \quad \forall t \ge T_2.$$

One should note that for $t \ge T_0$ we have $\|\partial_x u(t)\|_2 \lesssim 1$. Thus

$$\|(Q_N - \mathrm{Id})(u(t)^3)\|_\infty \lesssim N^{-\frac{1}{2}} \cdot N^{\frac{3}{8}} (\log N)^2 \le \mathcal{O}(N^{-\frac{1}{8}} (\log N)^2), \quad \forall t \ge T_2.$$

By using a maximum principle argument, we then obtain

$$\sup_{t \ge T_2} \|u(t)\|_\infty \le 1 + \mathcal{O}(N^{-\frac{1}{8}} (\log N)^2).$$

Step 4. The regime $t \le T_1$. In this regime we observe that

$$\|u(t) - e^{t\partial_{xx}} v_0\|_\infty \lesssim N^{-\frac{1}{2}} (\log N)^{\frac{1}{4}}, \quad \forall 0 \le t \le T_1.$$

By Remark 5.1, in general we have $\|v_0\|_\infty \lesssim \log N$ which is optimal. Thus we need to restrict to the grid points $x_j = \frac{j}{N}$, $0 \le j \le N-1$ and work with the function $U(t)$. Now denote $U^{\mathrm{lin}}(t) \in \mathbb{R}^N$ such that

$$U^{\mathrm{lin}}(t)_j = (e^{t\partial_{xx}} v_0)\Big(t, \frac{j}{N}\Big), \quad \forall 0 \le j \le N-1.$$

Clearly

$$\|U(t) - U^{\mathrm{lin}}(t)\|_\infty \lesssim N^{-\frac{1}{2}} (\log N)^{\frac{1}{4}}, \quad \forall 0 \le t \le T_1.$$

The remaining results then follow from Theorem B.1, Theorem B.2 and Theorem B.3 proved in the appendix. □

Theorem 5.5 (Generalized L^∞-stability for (5.4), case $\frac{1}{2}\leq\tau\leq 1.99$)**.** *Consider* (5.4). *Suppose* $U^0\in\mathbb{R}^N$ *satisfies* $\|U^0\|_\infty\leq 1$. *Assume* $\frac{1}{2}\leq\tau\leq 1.99$. *If* $N\geq N_4(\nu)>0$ *(the dependence of* N_4 *on* ν *is only power-like), then the following hold*

- *For all* $n\geq 0$, *we have*

$$\|U^n\|_\infty\leq M_a=\frac{1}{2}\Big(\frac{2}{3}\cdot\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}+\sqrt{\frac{2+\tau}{\tau}}\Big). \tag{5.10}$$

- *If* $\frac{1}{2}\leq\tau\leq 0.86$, *then for all* $n\geq 0$, *we have*

$$\|U^n\|_\infty\leq M_b=\frac{2}{3}\cdot\frac{(1+\tau)^{\frac{3}{2}}}{\sqrt{3\tau}}+\eta_0,\quad \eta_0=10^{-5}, \tag{5.11a}$$

$$E(U^{n+1})\leq E(U^n), \tag{5.11b}$$

where $E(U)$ *was defined in* (5.8). *Moreover, for* $n\geq 1$,

$$E(U^n)\leq E(U^1)\leq\mathcal{E}_1, \tag{5.12}$$

where $\mathcal{E}_1>0$ *depends only on* ν.

Remark 5.8. The cut-off $\tau=1.99$ is for convenience only. One can replace it by any number less than 2 but N_4 will have to be adjusted correspondingly.

Proof. Denote

$$K_N(x)=\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}\frac{1}{1+\tau(2\pi k\nu)^2}e^{2\pi ik\cdot x}\Big), \tag{5.13a}$$

$$K_\infty(x)=\sum_{k\in\mathbb{Z}}\frac{1}{1+\tau(2\pi k\nu)^2}e^{2\pi ik\cdot x}. \tag{5.13b}$$

For $\frac{1}{2}\leq\tau\leq 2$, we have

$$\|K_N-K_\infty\|_{L^\infty(\mathbb{T})}\leq\mathcal{O}(N^{-1})$$

and

$$\frac{1}{N}\sum_{j=0}^{N-1}\Big|(K_N-K_\infty)\Big(\frac{j}{N}\Big)\Big|\leq\mathcal{O}(N^{-1}).$$

Denote $p(x)=(1+\tau)x-\tau x^3$. By (5.5), we have

$$\|U^{n+1}\|_\infty \le \|p(U^n)\|_\infty + \frac{\text{const}}{N}\cdot\|p(U^n)\|_\infty.$$

Then (5.10) follows from Lemma 3.3 and induction.

The proof of (5.11a)–(5.11b) is similar to that in the proof of Proposition 3.10. The main point is to use the inequality

$$\frac{1}{2}+\frac{1}{\tau}\ge\frac{3}{2}M_b^2,$$

which holds for all $\frac{1}{2}\le\tau\le 0.86$.

Finally (5.12) follows from (5.5) and the simple estimate

$$\|\Delta_h U^{n+1}\|_2 \lesssim \|p(U^n)\|_2 \lesssim 1.$$

Thus, we complete the proof. □

Theorem 5.6 (Almost sharp maximum principle for (5.4), case $N^{-0.3}\le\tau\le\frac{1}{2}$)**.** *Consider* (5.4). *Suppose* $U^0\in\mathbb{R}^N$ *satisfies* $\|U^0\|_\infty\le 1$. *Assume* $N^{-0.3}\le\tau\le\frac{1}{2}$. *If* $N\ge N_5(\nu)>0$ *(the dependence of* N_5 *on* ν *is only power-like), then the following hold.*

For all $n\ge 0$, *we have*

$$\|U^n\|_\infty \le 1+\mathcal{O}(N^{-\frac{1}{3}}), \tag{5.14a}$$
$$E(U^{n+1})\le E(U^n), \tag{5.14b}$$

where $E(U)$ *was defined in* (5.8). *Moreover, for* $n\ge\frac{1}{\tau}$,

$$E(U^n)\le\mathcal{E}_2, \tag{5.15}$$

where $\mathcal{E}_2>0$ *depends only on* ν.

Proof. We define K_N and K_∞ as in (5.13). Then for $N^{-0.3}\le\tau\le\frac{1}{2}$, it is clear that

$$\frac{1}{N}\sum_{j=0}^{N-1}\left|(K_N-K_\infty)\Big(\frac{j}{N}\Big)\right| \le \|K_N-K_\infty\|_{L^\infty(\mathbb{T})}\le\mathcal{O}(N^{-1}\tau^{-1}).$$

Set $\delta>0$ which we will take to be $\mathcal{O}(N^{-\frac{1}{3}})$. Inductively assume $\|U^n\|_\infty\le 1+\delta$. By (5.5) and the preceding estimate, we have

$$\|U^{n+1}\|_\infty \le \max_{|x|\le 1+\delta}|p(x)|+\mathcal{O}(N^{-1}\tau^{-1}),$$

where $p(x)=(1+\tau)x-\tau x^3$. Observe that $p'(1)=1-2\tau$ and for $0<\tau\leq\frac{1}{2}$,

$$\max_{|x|\leq 1+\delta}|p(x)|\leq 1+(1-2\tau)\delta+\mathcal{O}(\delta^2).$$

Since we assume $\delta=\mathcal{O}(N^{-\frac{1}{3}})$, it follows easily that

$$\mathcal{O}(\delta^2)\ll\tau\delta,$$
$$\mathcal{O}(N^{-1}\tau^{-1})\ll\tau\delta,$$

and

$$\|U^{n+1}\|_\infty\leq 1+(1-2\tau)\delta+\mathcal{O}(\delta^2)+\mathcal{O}(N^{-1}\tau^{-1})\leq 1+\delta.$$

Thus (5.14a) holds.

Now (5.14b) follows from the fact that for $\tau\leq\frac{1}{2}$,

$$\frac{1}{\tau}+\frac{1}{2}\geq\frac{5}{2}\geq\frac{3}{2}\max\{\|U^n\|_\infty^2,\|U^{n+1}\|_\infty^2\},\quad\forall n\geq 0.$$

One should note again here that the restriction for τ for energy stability is quite mild, i.e., τ must be smaller than 1 (provided $\|U^n\|_\infty$ is close to 1). The restriction for $\tau\leq\frac{1}{2}$ is was due to the fact that we need to have $\|U^n\|_\infty\leq 1+$small error.

Finally to show (5.15) we shall use discrete smoothing estimates. To this end we assume with no loss that

$$\nu=\frac{1}{2\pi}\quad\text{ and }\quad A=(\mathrm{I}-\nu^2\tau\Delta_h)^{-1}.$$

Note that on the Fourier side

$$\widehat{A}(k)=(1+k^2\tau)^{-1}$$

for $-\frac{N}{2}<k\leq\frac{N}{2}$. It is clear that if $\tau\sim 1$ we have $E(U^1)\leq\mathcal{O}(1)$. Thus we only need to consider the regime $0<\tau\ll 1$. We shall show for all $1\leq n\leq 1/\tau$,

$$\sqrt{n\tau}\|\partial_h U^n\|_2\leq C_0,\tag{5.16}$$

where $C_0>0$ is an absolute constant (for general $\nu>0$, C_0 will depend on ν; here we assumed $\nu=1/2\pi$ for simplicity). Here we use the notation ∂_h to denote the Fourier multiplier $2\pi ik$ in the discrete Fourier transform formula. In particular we have for $U\in\mathbb{R}^N$,

$$\|\partial_h U\|_2^2=\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}(2\pi k)^2|\tilde{U}_k|^2.$$

It is not difficult to check that for $n\geq 1$,

$$U^n = A^n U^0 + \tau \sum_{j=0}^{n-1} A^{n-j}\Big(f(U^j)\Big),$$

where $f(z)=z-z^3$ and $f(U)=U-U^{\cdot 3}$.

Observe that for any $j\geq 1$ and $\tau>0$, we have

$$\sup_{k\in\mathbb{Z}} \sqrt{j\tau}2\pi|k|(1+\tau k^2)^{-j} \leq 2\pi\cdot \sup_{x\geq 0} x\Big(1+\frac{x^2}{j}\Big)^{-j} \leq C_1 < \infty,$$

where $C_1>0$ is an absolute constant.

Now first for $n=1$, we have

$$\begin{aligned}\sqrt{\tau}\|\partial_h U^1\|_2 &\leq \sqrt{\tau}\|\partial_x A U^0\|_2 + \tau^{\frac{3}{2}}\|\partial_x A\big(f(U^0)\big)\|_2\\ &\lesssim \|U^0\|_2 + \tau\|f(U^0)\|_2 \leq C_0,\end{aligned}$$

where clearly we can take C_0 sufficiently large to fulfill the last bound.

Similarly for $n\geq 2$ with $n\tau\leq 1$,

$$\begin{aligned}\sqrt{n\tau}\|\partial_h U^n\|_2 \leq& \sqrt{n\tau}\|\partial_x A^n U^0\|_2 + \tau\cdot\sqrt{n\tau}\|\partial_x A^n\Big(f(U^0)\Big)\|_2\\ &+\tau\sqrt{n\tau}\sum_{j=1}^{n-1}\|\partial_x A^{n-j}\big(f(U^j)\big)\|_2\\ \leq& C_1\|U^0\|_2 + C_1\|f(U^0)\|_2 + \sup_{1\leq j\leq n-1}\|f(U^j)\|_2\cdot C_1\tau\cdot\sqrt{n\tau}\sum_{j=1}^{n-1}\frac{1}{\sqrt{(n-j)\tau}}\\ \leq& C_0.\end{aligned}$$

Thus (5.16) is proved. □

Our next result is concerned with the regime $0<\tau\leq N^{-0.3}$. Let $u^n=u^n(x):\mathbb{T}\to\mathbb{R}$ solve the system

$$\begin{cases}(\mathrm{Id}-\nu\tau\Delta)u^{n+1} = u^n + \tau Q_N(f(u^n)),\\ u^0 = Q_N U^0,\end{cases} \tag{5.17}$$

where $f(z)=z-z^3$. It is not difficult to check that u^n and (5.4) are equivalent in the sense that $u^n(\frac{j}{N})=(U^n)_j$ for all $n\geq 0$ and $0\leq j\leq N-1$.

Theorem 5.7 (Almost sharp maximum principle for (5.4), case $0<\tau\le N^{-0.3}$)**.** *Consider* (5.4)*. Suppose* $U^0\in\mathbb{R}^N$ *satisfies* $\|U^0\|_\infty\le 1$*. Assume* $0<\tau\le N^{-0.3}$*. If* $N\ge N_6(\nu)>0$ *(the dependence of* N_6 *on* ν *is only power-like), then it holds that*

$$\sup_{n\ge 0}\|U^n\|_\infty\le 1+\epsilon_1,$$

where $0<\epsilon_1<10^{-2}$ *is an absolute constant. More precisely the following hold. (Below we shall write* $X=\mathcal{O}(Y)$ *if* $|X|\le CY$ *and the constant* C *depends only on* ν*.)*

1. $E(U^{n+1})\le E(U^n)$ *for all* $n\ge 0$*.*
2. *For some* $T_0=T_0(\nu)>0$ *sufficiently small and for all* $n\ge T_0/\tau$*, we have*

$$\|\partial_x u^n\|_{L^2_x(\mathbb{T})}\le\mathcal{O}(1),$$
$$\|U^n\|_\infty\le\|u^n\|_\infty\le 1+\mathcal{O}(N^{-\frac{3}{40}}),$$

   where u^n *solves* (5.17)*.*
3. *For all* $n\ge 2N^{-0.3}/\tau$*, we have*

$$\|U^n\|_\infty\le\|u^n\|_\infty\le 1+\mathcal{O}(N^{-\frac{3}{40}}).$$

4. *For* $1\le n\le 2N^{-0.3}/\tau$*, we have*

$$\|u^n-(\mathrm{Id}-\nu\tau\partial_{xx})^{-n}u^0\|_\infty\le\mathcal{O}(N^{-\frac{3}{40}}),$$
$$\sup_{0\le l\le N-1}\left|\left((\mathrm{Id}-\nu\tau\partial_{xx})^{-n}u^0\right)\left(\frac{l}{N}\right)\right|\le\left(1+\frac{1}{2}\epsilon_1\right)\|U^0\|_\infty,$$
$$\|U^n\|_\infty\le\left(1+\frac{1}{2}\epsilon_1\right)\|U^0\|_\infty+\mathcal{O}(N^{-\frac{3}{40}}).$$

5. *If* $\|U^0\|_\infty\le\frac{1}{1+\frac{1}{2}\epsilon_1}$ *(note that* $1/(1+\frac{1}{2}\epsilon_1)>0.995$*), then*

$$\sup_{n\ge 0}\|U^n\|_\infty\le 1+\mathcal{O}(N^{-\frac{3}{40}}).$$

6. *Suppose* $f:\mathbb{T}\to\mathbb{R}$ *is continuous and* $\|f\|_\infty\le 1$*. If* $(U^0)_l=f(\frac{l}{N})$ *for all* $0\le l\le N-1$*, then*

$$\sup_{n\ge 0}\|U^n\|_\infty\le 1+\mathcal{O}(\omega_f(N^{-\frac{2}{3}}))+\mathcal{O}(N^{-c}),$$

   where $c>0$ *is an absolute constant, and* ω_f *is defined in* (B.5)*.*

7. *Suppose* $f:\mathbb{T}\to\mathbb{R}$ *is* C^α*-continuous for some* $0<\alpha<1$ *and* $\|f\|_\infty\le 1$. *If* $(U^0)_l=f(\frac{l}{N})$ *for all* $0\le l\le N-1$, *then*

$$\sup_{n\ge 0}\|U^n\|_\infty\le 1+\mathcal{O}(N^{-c_1}),$$

where $c_1>0$ *is a constant depending only on* α.

Moreover we have the following result which shows the sharpness of our estimates above. There exists a function $f:\mathbb{T}\to\mathbb{R}$, *continuous at all of* $\mathbb{T}\backslash\{x_*\}$ *for some* $x_*\in\mathbb{T}$ (*i.e., continuous at all* $x\neq x_*$) *and has the bound* $\|f\|_\infty\le 1$ *such that the following hold: for a sequence of even numbers* $N_m\to\infty$, $\tau_m=\frac{1}{4}\nu^{-2}N_m^{-2}$ *and* $x_m=j_m/N_m$ *with* $0\le j_m\le N_m-1$, *if* $\tilde{U}_m\in\mathbb{R}^{N_m}$ *solves*

$$(\mathrm{Id}-\tau\nu^2\Delta_h)\tilde{U}_m=V_m+\tau_m f(V_m),$$

where $(V_m)_l=f(\frac{l}{N_m})$ *for all* $0\le l\le N_m-1$. *Then*

$$|\tilde{U}_m(x_m)|\ge 1+\eta_*,\quad \forall m\ge 1,$$

where $\eta_*>0.001$ *is an absolute constant.*

Proof. We shall use the system (5.17). Denote $B=(\mathrm{Id}-\nu\tau\Delta)^{-1}$. Clearly we have (below $f_1(z)=-z^3$)

$$u^n=B^{n-1}u^1+\tau\sum_{j=1}^{n-1}B^{n-j}\Big(f(u^j)\Big)+\tau\sum_{j=1}^{n-1}B^{n-j}(Q_N-\mathrm{Id})\Big(f_1(u^j)\Big),\quad n\ge 1, \tag{5.18}$$

with

$$\|u^0\|_{L^2(\mathbb{T})}\le 1,\quad \|u^0\|_{L^\infty(\mathbb{T})}\lesssim\log N.$$

In the argument below we shall assume $N\ge N_6=N_6(\nu)$ where N_6 is a sufficiently large constant depending only on ν. The needed largeness of N_6 as well as the precise (power) dependence on ν can be worked out from the context with more detailed computations. However for simplicity of notation we shall not present it here.

Step 1. Local estimate. We show that for $T_0=T_0(\nu)>0$ sufficiently small and all $n\ge 1$ with $n\tau\le T_0$, it holds that

$$\frac{\|u^n\|_\infty}{\log N}+\sqrt{n\tau}\|\partial_x u^n\|_2+\|u^n\|_2\le C_0, \tag{5.19}$$

where $C_0>0$ is a constant depending only on ν.

For $n=1$, we shall start with

$$U^1=(\mathrm{Id}-\tau\nu^2\Delta_h)^{-1}U^0+\tau(\mathrm{Id}-\tau\nu^2\Delta_h)^{-1}(f(U^0)).$$

Clearly

$$\begin{aligned}&\sqrt{\tau}\|\partial_x u^1\|_2\leq\sqrt{\tau}\|\partial_h U^1\|_2\lesssim\|U^0\|_2+\tau\|f(U^0)\|_2\leq\frac{1}{3}C_0,\\&\|u^1\|_2\leq\|U^1\|_2\lesssim\|U^0\|_2+\tau\|f(U^0)\|_2\leq\frac{1}{3}C_0.\end{aligned}$$

On the other hand, noting that

$$\Big(f(u^0)\Big)\Big(\frac{j}{N}\Big)=\Big(f(U^0)\Big)_j$$

for all $0\leq j\leq N-1$, we have

$$\begin{aligned}\|u^1\|_\infty\leq&\|u^0\|_\infty+\tau\|Q_N(f(u^0))\|_\infty\\=&\|Q_NU^0\|_\infty+\tau\|Q_N(f(U^0))\|_\infty\\\lesssim&\log N.\end{aligned}$$

Thus

$$\frac{\|u^1\|_\infty}{\log N}+\sqrt{\tau}\|\partial_x u^1\|_2+\|u^1\|_2\leq C_0.$$

Next for $n\geq2$ with $n\tau\leq T_0$ and for all $1\leq j\leq n-1$, we inductively assume that

$$\frac{\|u^j\|_\infty}{\log N}+\sqrt{j\tau}\|\partial_x u^j\|_2+\|u^j\|_2\leq C_0.$$

Then by (5.18) and Lemma C.1, we have (note that $j\tau\leq T_0\ll1$)

$$\begin{aligned}\|u^n\|_\infty\leq&\|u^1\|_\infty+\tau b_1\sum_{j=1}^{n-1}((n-j)\tau)^{-\frac{1}{2}}\|f(u^j)\|_1+\tau\sum_{j=1}^{n-1}\|(Q_N-\mathrm{Id})(f_1(u^j))\|_\infty\\\leq&b_2\log N+b_3\cdot\sqrt{n\tau}\cdot C_0(C_0+1)\log N+\tau\sum_{j=1}^{n-1}N^{-\frac{1}{2}}b_4\cdot\|f_1'(u^j)\partial_x u^j\|_2\\\leq&b_2\log N+b_3\cdot\sqrt{n\tau}\cdot C_0(C_0+1)\log N+\tau\sum_{j=1}^{n-1}N^{-\frac{1}{2}}b_4\cdot(\log N)^2(j\tau)^{-\frac{1}{2}}\cdot C_0^3\\\leq&\frac{1}{3}C_0\log N,\end{aligned}$$

where in the above we use b_i to denote various constants depending only on ν, and we take T_0 sufficiently small and N sufficiently large. Similarly, by using the inequality

$$\|u\|_3 \lesssim \|u\|_2 + \|u\|_2^{\frac{5}{6}} \|\partial_x u\|_2^{\frac{1}{6}},$$

we have

$$\begin{aligned}
\|u^n\|_2 \leq & \|u^1\|_2 + \tau b_5 \sum_{j=1}^{n-1} ((n-j)\tau)^{-\frac{1}{4}} \|f(u^j)\|_1 + \tau \sum_{j=1}^{n-1} \|(Q_N - \mathrm{Id})(f_1(u^j))\|_2 \\
\leq & b_6 + b_7 \tau \sum_{j=1}^{n-1} ((n-j)\tau)^{-\frac{1}{4}} \cdot (C_0 + C_0^3 + C_0^{\frac{5}{2}} \cdot (j\tau)^{-\frac{1}{4}}) + \tau \sum_{j=1}^{n-1} \|(Q_N - \mathrm{Id})(f_1(u^j))\|_\infty \\
\leq & b_6 + b_8 \cdot ((n\tau)^{\frac{3}{4}} + (n\tau)^{\frac{1}{2}}) + \tau \sum_{j=1}^{n-1} N^{-\frac{1}{2}} b_4 \cdot (\log N)^2 (j\tau)^{-\frac{1}{2}} \cdot C_0^3 \\
\leq & \frac{1}{3} C_0,
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{n\tau} \|\partial_x u^n\|_2 \leq & \sqrt{n\tau} \|\partial_x B^{n-1} u^1\|_2 + \sqrt{n\tau}\tau \sum_{j=1}^{n-1} \|\partial_x B^{n-j} (f(u^j))\|_2 \\
& + \sqrt{n\tau}\tau \sum_{j=1}^{n-1} \|\partial_x B^{n-j} (Q_N - \mathrm{Id})(f_1(u^j))\|_2 \\
\leq & b_9 + \sqrt{n\tau}\tau \sum_{j=1}^{n-1} ((n-j)\tau)^{-\frac{3}{4}} \|f(u^j)\|_1 \\
& + \sqrt{n\tau}\tau \sum_{j=1}^{n-1} ((n-j)\tau)^{-\frac{1}{2}} N^{-\frac{1}{2}} b_4 \|f_1'(u^j)\partial_x u^j\|_2 \\
\leq & \frac{1}{3} C_0.
\end{aligned}$$

Hence (5.19) is proved.

Step 2. Refined estimate. First we note that

$$\|Q_N(f(u^0))\|_\infty = \|Q_N(f(U^0))\|_\infty \lesssim \log N.$$

Since $\tau \leq N^{-0.3}$, we obtain for any $n \geq 1$,

$$\tau \|B^n Q_N(f(u^0))\|_\infty \lesssim N^{-0.3} \log N. \tag{5.20}$$

Now for any $1\leq n\leq T_0/\tau$, by using (5.20) and the fact that

$$\|u^j\|_3^3\lesssim\|u^j\|_2^3+\|u^j\|_2^{\frac{5}{2}}\|\partial_x u^j\|_2^{\frac{1}{2}}\lesssim 1+(j\tau)^{-\frac{1}{4}},\quad\forall 1\leq j\leq T_0/\tau,$$

we have

$$\begin{aligned}
&\|u^n-B^n u^0\|_\infty\\
\leq&\tau\|B^nQ_N(f(u^0))\|_\infty+\tau\sum_{j=1}^{n-1}\|B^{n-j}f(u^j)\|_\infty+\tau\sum_{j=1}^{n-1}\|(Q_N-\mathrm{Id})(f_1(u^j))\|_\infty\\
\lesssim&N^{-0.3}\log N+\tau\sum_{j=1}^{n-1}((n-j)\tau)^{-\frac{1}{4}}\|u^j\|_2+\tau\sum_{j=1}^{n-1}((n-j)\tau)^{-\frac{1}{2}}\|(u^j)^3\|_1+N^{-\frac{1}{2}}(\log N)^2(n\tau)^{\frac{1}{2}}\\
\lesssim&N^{-0.3}\log N+(n\tau)^{\frac{1}{4}}+(n\tau)^{\frac{3}{4}}\lesssim N^{-0.3}\log N+(n\tau)^{\frac{1}{4}}.
\end{aligned}$$

Step 3. The regime $n\tau\geq 2N^{-0.3}$. Set $n_0=[2N^{-0.3}/\tau]\geq 2$ where $[x]$ denotes the usual integer part of the real number $x>0$. Clearly $n_0\tau\sim N^{-0.3}$ and

$$\|u^{n_0}-B^{n_0}u^0\|_\infty\lesssim N^{-\frac{3}{40}}.$$

On the other hand, since

$$(B^{n_0}u^0)(x)=\frac{1}{N}\sum_{j=0}^{N-1}(U^0)_j\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}(1+(2\pi k\nu)^2\tau)^{-n_0}e^{2\pi ik(x-\frac{j}{N})}\Big),$$

we have

$$\begin{aligned}
\|B^{n_0}u^0\|_\infty\leq&1+\sum_{|k|>\frac{N}{2}-1}(1+(2\pi k\nu)^2\tau)^{-n_0}\\
\leq&1+\sum_{|k|>\frac{N}{2}-1}\frac{1}{1+(2\pi\nu)^2n_0\tau k^2}\\
\leq&1+\mathcal{O}((n_0\tau N)^{-1})=1+\mathcal{O}(N^{-0.7}).
\end{aligned}$$

Thus

$$\|u^{n_0}\|_\infty\leq 1+\mathcal{O}(N^{-\frac{3}{40}}).$$

Note that

$$\|\partial_x u^{n_0}\|_2\lesssim\frac{1}{\sqrt{n_0\tau}}\leq\mathcal{O}(N^{0.15}).$$

Thus for $n_0 \le n \le T_0/\tau$,

$$E(U^n) \le E(U^{n_0}) \le \mathcal{O}(N^{0.3}).$$

This implies for all $n_0 \le n \le T_0/\tau$,

$$\begin{aligned}
&\|\partial_x u^n\|_2 \le \mathcal{O}(N^{0.15}),\\
&\|(Q_N - \mathrm{Id})(f_1(u^n))\|_\infty \le \mathcal{O}(N^{-\frac{1}{2}} \cdot N^{0.15}) = \mathcal{O}(N^{-0.35}).
\end{aligned}$$

Now consider the system

$$(\mathrm{Id} - \nu^2 \tau \Delta) u^{n+1} = u^n + \tau f(u^n) + \tau (Q_N - \mathrm{Id})(f_1(u^n)).$$

If we assume $\|u^n\|_\infty \le 1+\delta$, then (note again that for $p(x) = x + \tau(x - x^3)$, $p''(x) < 0$ for x near ± 1 so that $\mathcal{O}(\delta^2)$ does not appear below)

$$\|u^{n+1}\|_\infty \le 1 + (1-2\tau)\delta + \tau \cdot \mathcal{O}(N^{-0.35}) \le 1 + \delta$$

provided we take $\delta \ge \mathcal{O}(N^{-0.25})$.

Thus it is clear that for all $n_0 \le n \le T_0/\tau$,

$$\|u^n\|_\infty \le 1 + \mathcal{O}(N^{-\frac{3}{40}}).$$

For $n \ge T_0/\tau$, note that $\|\partial_x u^n\|_2 \lesssim 1$. By another maximum principle argument, we obtain

$$\sup_{n \ge n_0} \|u^n\|_\infty \le 1 + \mathcal{O}(N^{-\frac{3}{40}}).$$

Note that we did not optimize the exponent (in N) here for simplicity of presentation. These can certainly be improved by more refined estimates but we do not dwell on this issue here.

Step 4. The regime $n\tau \le 2N^{-0.3}$. In this regime we observe that

$$\|u^n - B^n u^0\|_\infty \le \mathcal{O}(N^{-\frac{3}{40}}).$$

The main contribution of the L^∞-estimate of u^n then comes from the linear flow piece. In this case we have to restrict to the grid points $x_j = j/N$ for $0 \le j \le N-1$ and work with $U^n_{\mathrm{lin}} \in \mathbb{R}^N$ such that

$$(U^n_{\mathrm{lin}})_j = (B^n u^0)\Big(\frac{j}{N}\Big), \quad \forall 0 \le j \le N-1.$$

The remaining results then follow from Theorem C.4 proved in the appendix and a simple perturbation argument. □

6 Strang splitting with Fourier collocation for Allen-Cahn

In this section we consider the Strang splitting with Fourier collocation for Allen-Cahn on $\mathbb{T}$. The PDE is again

$$\partial_t u=\nu^2\Delta u+f(u),\quad f(u)=u-u^3.$$

We shall adopt the same notation as in Section 5. In particular we slightly abuse the notation and denote for $U\in\mathbb{R}^N$

$$f(U)=U-U^{\cdot 3}.$$

We consider the time splitting as follows. Let $\tau>0$ be the time step. Consider the ODE

$$\begin{cases}\dfrac{d}{dt}U=f(U),\\ U\big|_{t=0}=a\in\mathbb{R}^N.\end{cases}$$

We define the solution operator $\mathcal{N}_\tau:\mathbb{R}^N\to\mathbb{R}^N$ as the map $a\to U(\tau)$. Thanks to the explicit form of $f(U)$, we have ($a=(a_0,\cdots,a_{N-1})^T$ and with no loss we shall assume that $N\geq 2$)

$$(\mathcal{N}_\tau a)_j=U(\tau)_j=\frac{a_j}{\sqrt{e^{-2\tau}+(1-e^{-2\tau})(a_j)^2}},\quad j=0,1,\cdots,N-1.$$

Define $\mathcal{S}_\tau=e^{\frac{\tau}{2}\nu^2\Delta_h}$. Then U^{n+1}, $U^n\in\mathbb{R}^N$ are related via the relation:

$$U^{n+1}=\mathcal{S}_\tau\mathcal{N}_\tau\mathcal{S}_\tau U^n,\quad n\geq 0. \tag{6.1}$$

For $U\in\mathbb{R}^N$, we recall

$$\|U\|_2^2=\frac{1}{N}\sum_{j=0}^{N-1}U_j^2=\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}|\tilde{U}_k|^2.$$

Denote

$$\|\partial_h U\|_2^2=\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}(2\pi k)^2|\tilde{U}_k|^2.$$

Lemma 6.1. *Consider* (6.1). *We have*

$$\sup_{n\geq 0}\|U^n\|_2\leq\max\{1,\|U^0\|_2\}. \tag{6.2}$$

Furthermore assuming $\|U^0\|_2\leq 1$, *there are* $\tau_0=\tau_0(\nu)>0$, $t_0=t_0(\nu)>0$ *sufficiently small such that the following hold: If* $\tau\geq\tau_0$, *then*

$$\sup_{n\geq 1}\|\partial_h U^n\|_2\leq C_1, \tag{6.3}$$

where $C_1>0$ *depends only on* ν. *If* $0<\tau<\tau_0$, *then*

$$\sup_{1\leq n\leq t_0/\tau}\sqrt{n\tau}\|\partial_h U^n\|_2+\sup_{n\geq t_0/\tau}\|\partial_h U^n\|_2\leq C_2, \tag{6.4}$$

where $C_2>0$ *depends only on* (ν, t_0).

Suppose $\|U^0\|_\infty\leq 1$, *then for all* $\tau>0$, *we have*

$$\sup_{n\geq 1}\|U^n\|_\infty\leq C_3, \tag{6.5}$$

where $C_3>0$ *depends only on* ν. *Also for any* $t_1\geq\tau$ *and* $n\tau\leq t_1$, *we have*

$$\sup_{n\leq t_1/\tau}\|U^n-\mathcal{S}_{2n\tau}U^0\|_\infty\leq C_4 t_1, \tag{6.6}$$

where $C_4>0$ *depends only on* ν.

Proof. For the basic L^2 bound we can just use the ODE. Clearly

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|U\|_2^2)&=\|U\|_2^2-\|U\|_4^4\\&\leq\|U\|_2^2-\|U\|_2^4.\end{aligned}$$

Thus $\|U(t)\|_2\leq\max\{1,\|U(0)\|_2\}$. This immediately yields (6.2).

For (6.3), we note that since $\tau\gtrsim 1$, it follows easily that

$$\|\partial_h U^{n+1}\|_2\lesssim\|\mathcal{N}_\tau\mathcal{S}_\tau U^n\|_2\lesssim 1.$$

For (6.4), we first rewrite

$$\begin{aligned}U^{n+1}&=\mathcal{S}_{2\tau}U^n+\mathcal{S}_\tau(\mathcal{N}_\tau-\mathrm{Id})\mathcal{S}_\tau U^n\\&=:\mathcal{S}_{2\tau}U^n+\tau\mathcal{S}_\tau f_n,\end{aligned} \tag{6.7}$$

where

$$f_n=\frac{\mathcal{N}_\tau-\mathrm{Id}}{\tau}\mathcal{S}_\tau U^n.$$

Now by using the inequality

$$\Big|\frac{x}{\sqrt{e^{-2\tau}+(1-e^{-2\tau})x^2}}-x\Big|\lesssim\tau(|x|+|x|^3),\quad\forall 0<\tau\lesssim 1,\quad x\in\mathbb{R}.$$

We have

$$|f_n|\lesssim|\mathcal{S}_\tau U^n|+|\mathcal{S}_\tau U^n|^3.$$

By (6.7), we have

$$\begin{aligned}U^n=&\mathcal{S}_{2n\tau}U^0+\tau\sum_{j=0}^{n-1}\mathcal{S}_{(2(n-j)-1)\tau}f_j\\=&\mathcal{S}_{2n\tau}U^0+\tau\mathcal{S}_{(2n-1)\tau}f_0+\tau\sum_{j=1}^{n-1}\mathcal{S}_{(2(n-j)-1)\tau}f_j,\quad\forall n\geq 1.\end{aligned}\tag{6.8}$$

Note that in the above, the empty summation when $n=1$ is defined to be zero. For $1\leq j\leq n-1$, note that

$$\begin{aligned}\|f_j\|_1\lesssim&\|\mathcal{S}_\tau U^j\|_1+\|\mathcal{S}_\tau U^j\|_3^3\\\lesssim&\|\mathcal{S}_\tau U^j\|_2+\|\mathcal{S}_\tau U^j\|_\infty\|U^j\|_2^2\\\lesssim&1+\|\mathcal{S}_\tau U^j\|_\infty.\end{aligned}$$

To bound $\|\mathcal{S}_\tau U^j\|_\infty$, we observe that for $U\in\mathbb{R}^N$,

$$\begin{aligned}\|U\|_\infty\leq&\|Q_N U\|_{L_x^\infty(\mathbb{T})}\lesssim\|Q_N U\|_{L_x^2(\mathbb{T})}+\|Q_N U\|_{L_x^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_x Q_N U\|_{L_x^2(\mathbb{T})}^{\frac{1}{2}}\\\lesssim&\|U\|_2+\|U\|_2^{\frac{1}{2}}\|\partial_h U\|_2^{\frac{1}{2}}.\end{aligned}$$

Thus for $1\leq j\leq n-1$,

$$\|f_j\|_1\lesssim 1+\|\partial_h U^j\|_2^{\frac{1}{2}}.$$

For $j=0$, we have

$$\begin{aligned}\|f_0\|_1\lesssim&1+\|\mathcal{S}_\tau U^0\|_\infty\lesssim 1+\|\partial_h S_\tau U^0\|_2^{\frac{1}{2}}\\\lesssim&1+\tau^{-\frac{1}{2}}.\end{aligned}$$

Denote the kernel corresponding to $\mathcal{S}_{l\tau}$ as $K_{l\tau}$ $(l\geq 1)$, then clearly

$$\begin{aligned}&\|K_{l\tau}\|_2\lesssim(l\tau)^{-\frac{1}{4}},\quad \|\partial_h K_{l\tau}\|_2\lesssim(l\tau)^{-\frac{3}{4}}, && \forall l\tau\lesssim 1,\\ &\|\partial_h\mathcal{S}_{l\tau}U\|_2\lesssim(l\tau)^{-\frac{3}{4}}\|U\|_1, && \forall U\in\mathbb{R}^N,\quad l\tau\lesssim 1.\end{aligned}$$

We then obtain the recurrent relation,

$$F_n=\sqrt{n\tau}\|\partial_h U^n\|_2\lesssim 1+\tau\sqrt{n\tau}\sum_{j=1}^{n-1}((n-j)\tau)^{-\frac{3}{4}}\big(1+(j\tau)^{-\frac{1}{4}}F_j^{\frac{1}{2}}\big),$$

where $F_j=(j\tau)^{\frac{1}{2}}\|\partial_h U^j\|_2$. Let n_0 be the first integer such that $n_0\tau\geq t_0$. Our desired result for U^{n_0} then easily follows from a simple induction procedure. Note that we need $n_0\tau\leq t_0+\tau\ll 1$. For general U^n we need to estimate it using U^{n_0} where $(n-n_0)\tau=\mathcal{O}(t_0)$.

Next for (6.5), we only need to consider the regime $0<\tau\ll 1$ and $n\tau\leq t_1$ for some $t_1\ll 1$. By Theorem B.1 and using (6.8), we have

$$\|U^n\|_\infty\lesssim\|U^0\|_\infty+\tau\sum_{j=0}^{n-1}(\|U^j\|_\infty+\|U^j\|_\infty^3).$$

A simple induction argument then yields the desired estimate. Finally the estimate (6.6) follows easily from (6.5). □

Theorem 6.1 (Almost sharp maximum principle for (6.1), case $N^{-0.4}\leq\tau<\infty$)**.** *Consider* (6.1). *Suppose* $U^0\in\mathbb{R}^N$ *satisfies* $\|U^0\|_\infty\leq 1$. *Assume* $N^{-0.4}\leq\tau<\infty$. *If* $N\geq N_1(\nu)>0$ *(the dependence of* N_1 *on* ν *is only power-like), then for all* $n\geq 0$, *we have*

$$\|U^n\|_\infty\leq 1+\mathcal{O}(N^{-0.5}).$$

Proof. Since $\tau\geq N^{-0.4}$, by Theorem B.1, we have

$$\|\mathcal{S}_\tau U\|_\infty\leq(1+\mathcal{O}(N^{-1}))\|U\|_\infty,\quad \forall U\in\mathbb{R}^N.$$

On the other hand, if $\|U\|_\infty\leq 1+\epsilon_1$ where $\epsilon_1\ll 1$, then it is not difficult to check that for $\tau>0$,

$$\|\mathcal{N}_\tau U\|_\infty\leq 1+e^{-2\tau}\cdot\Big|\|U\|_\infty-1\Big|+e^{-4\tau}\cdot C_1\cdot\Big|\|U\|_\infty-1\Big|^2,$$

where $C_1>0$ is an absolute constant.

Our main induction hypothesis is $\|U^n\|_\infty \le 1+\delta$, where $\delta = \mathcal{O}(N^{-0.5})$. Then it suffices to verify the inequality:

$$(1+\mathcal{O}(N^{-1}))\Big(1+e^{-2\tau}(\mathcal{O}(N^{-1})+\delta)+e^{-4\tau}C_1\cdot(\mathcal{O}(N^{-2})+\delta^2)\Big)\le 1+\delta.$$

This amounts to showing

$$\mathcal{O}(N^{-1})+\mathcal{O}(\delta^2)\le(1-e^{-2\tau})\delta.$$

Since $\delta=\mathcal{O}(N^{-0.5})$ and $\tau\ge N^{-0.4}$, this last inequality is obvious for N sufficiently large. □

Lemma 6.2. *Assume $N\ge 2$. Suppose u is a smooth solution to the equation:*

$$\begin{cases}\partial_t u = Q_N(f(u)),\\ u(0)=a,\end{cases}$$

where $f(z)=z-z^3$, and the initial data a satisfies $\|a\|_{H^1(\mathbb{T})}\le A_0$ for some constant $A_0\ge 1$. We also assume that $\widehat{a}$ is supported in $\{k:-\frac{N}{2}<k\le\frac{N}{2}\}$ (in particular this implies $Q_N a=a$). Then for some $t_0>0$ depending only on A_0 (one can take $t_0\sim A_0^{-2}$), it holds that

$$\sup_{0\le t\le t_0}\|u(t)\|_{H^1}\le \alpha A_0,$$

where $\alpha>0$ is an absolute constant.

Proof. Denote $J=\{k\in\mathbb{Z}:-\frac{N}{2}<k\le\frac{N}{2}\}$. Observe that for any $k\in J$, we have

$$\begin{aligned}|\hat{u}(t,k)|\le{}&|\hat{a}(k)|+\int_0^t|\hat{u}(s,k)|ds\\ &+\sum_{|l|\le 2}\int_0^t\sum_{k_1,k_2\in J}|\hat{u}(s,k+lN-k_1-k_2)||\hat{u}(s,k_1)||\hat{u}(s,k_2)|ds.\end{aligned}$$

Clearly for $|l|\le 2$ and $k,k_1,k_2\in J$, we have

$$|k_1|+|k_2|+|k+lN-k_1-k_2|\ge|k+lN|\ge\frac{1}{4}|k|.$$

Define

$$A(t)=1+\sum_{k\in J}|\hat{u}(t,k)|^2\cdot(1+k^2).$$

It is then not difficult to show that

$$A(t)\leq A(0)+\int_0^t C\cdot A(s)^{\frac{3}{2}}ds,$$

where $C>0$ is an absolute constant. The desired result then follows easily by taking t sufficiently small. □

Theorem 6.2 (Almost sharp maximum principle for (6.1), case $0<\tau<N^{-0.4}$)**.** *Consider* (6.1). *Suppose* $U^0\in\mathbb{R}^N$ *satisfies* $\|U^0\|_\infty\leq 1$. *Assume* $0<\tau<N^{-0.4}$. *If* $N\geq N_2(\nu)>0$ *(the dependence of* N_2 *on* ν *is only power-like), then*

$$\sup_{n\geq 0}\|U^n\|_\infty\leq 1+\epsilon_1,$$

where $0<\epsilon_1<10^{-2}$ *is an absolute constant. More precisely the following hold.*

1. *For some* $T_0=\mathcal{O}(N^{-0.3})$ *sufficiently small and for all* $n\geq T_0/\tau$, *we have*

$$\|U^n\|_\infty\leq\|u^n\|_\infty\leq 1+\mathcal{O}(N^{-0.29}),$$

where u^n *solves* (6.9).

2. *For* $1\leq n\leq T_0/\tau$, *we have*

$$\begin{aligned}&\|U^n-\mathcal{S}_{2n\tau}U^0\|_\infty\leq\mathcal{O}(N^{-0.3}),\\&\|\mathcal{S}_{2n\tau}U^0\|_\infty\leq\Big(1+\frac{1}{2}\epsilon_1\Big)\|U^0\|_\infty,\\&\|U^n\|_\infty\leq\Big(1+\frac{1}{2}\epsilon_1\Big)\|U^0\|_\infty+\mathcal{O}(N^{-0.3}).\end{aligned}$$

3. *If* $\|U^0\|_\infty\leq\frac{1}{1+\frac{1}{2}\epsilon_1}$ *(note that* $1/(1+\frac{1}{2}\epsilon_1)>0.995$*), then*

$$\sup_{n\geq 0}\|U^n\|_\infty\leq 1+\mathcal{O}(N^{-0.3}).$$

4. *Suppose* $f:\mathbb{T}\to\mathbb{R}$ *is continuous and* $\|f\|_\infty\leq 1$. *If* $(U^0)_l=f(\frac{l}{N})$ *for all* $0\leq l\leq N-1$, *then*

$$\sup_{n\geq 0}\|U^n\|_\infty\leq 1+\mathcal{O}(\omega_f(N^{-\frac{2}{3}}))+\mathcal{O}(N^{-c}),$$

where $c>0$ *is an absolute constant, and* ω_f *is defined in* (B.5).

5. *Suppose* $f:\mathbb{T}\to\mathbb{R}$ *is* C^α*-continuous for some* $0<\alpha<1$ *and* $\|f\|_\infty\le 1$. *If* $(U^0)_l=f(\frac{l}{N})$ *for all* $0\le l\le N-1$, *then*

$$\sup_{n\ge 0}\|U^n\|_\infty\le 1+\mathcal{O}(N^{-c_1}),$$

where $c_1>0$ *is a constant depending only on* α.

Moreover we have the following result which shows the sharpness of our estimates above. There exists a function $f:\ \mathbb{T}\to\mathbb{R}$, *continuous at all of* $\mathbb{T}\backslash\{x_*\}$ *for some* $x_*\in\mathbb{T}$ (*i.e., continuous at all* $x\neq x_*$) *and has the bound* $\|f\|_\infty\le 1$ *such that the following hold: for a sequence of even numbers* $N_m\to\infty$, $\tau_m=\frac{1}{4}\nu^{-2}N_m^{-2}$, $n_m\ge 1$, *and* $x_m=j_m/N_m$ *with* $0\le j_m\le N_m-1$, *if* $\tilde{U}_m\in\mathbb{R}^{N_m}$ *satisfies*

$$\tilde{U}_m=\underbrace{e^{\frac{1}{2}\tau_m\Delta_h}\mathcal{N}_{\tau_m}e^{\frac{1}{2}\tau_m\Delta_h}\cdots e^{\frac{1}{2}\tau_m\Delta_h}\mathcal{N}_{\tau_m}e^{\frac{1}{2}\tau_m\Delta_h}}_{iterate\ n_m\ times}a_m,$$

where $a_m\in\mathbb{R}^{N_m}$ *satisfies* $(a_m)_l=f(\frac{l}{N_m})$ *for all* $0\le l\le N_m-1$. *Then*

$$|\tilde{U}_m(x_m)|\ge 1+\eta_*,\quad\forall m\ge 1,$$

where $\eta_*>0.001$ *is an absolute constant.*

Proof. We consider u^{n+1}, $u^n:\mathbb{T}\to\mathbb{R}$ defined via the relation:

$$u^{n+1}=e^{\frac{\tau}{2}\partial_{xx}}V_\tau e^{\frac{\tau}{2}\partial_{xx}}u^n,\tag{6.9}$$

where V_τ is the time-τ solution operator to the problem:

$$\partial_t u=Q_N(f(u)).$$

It is not difficult to check that if $u^0=Q_NU^0$, then $u^n=Q_NU^n$ for all $n\ge 1$.

Choose $n_0\ge 1$ such that $n_0\tau\sim N^{-0.3}$. By Lemma 6.1, we have

$$\|U^{n_0}-\mathcal{S}_{2n_0\tau}U^0\|_\infty\le\mathcal{O}(N^{-0.3}).$$

Thus

$$\|u^{n_0}-Q_N(\mathcal{S}_{2n_0\tau}U^0)\|_\infty\le\mathcal{O}(N^{-0.3}\log N).$$

Now

$$\Big(Q_N(\mathcal{S}_{2n_0\tau}U^0)\Big)(x)=\frac{1}{N}\sum_{j=0}^{N-1}U_j^0\,\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\le\frac{N}{2}}e^{-(2\pi k\nu)^2 2n_0\tau}e^{2\pi ik\cdot(x-\frac{j}{N})}\Big).$$

Since $\|U^0\|_\infty \le 1$ and $n_0\tau \sim N^{-0.3}$, it easily follows that

$$\|Q_{\mathcal{S}_{2n_0\tau}}U^0\|_\infty \le 1+\sum_{|k|\ge\frac{N}{2}} e^{-(2\pi k\nu)^2 2n_0\tau} \le 1+\mathcal{O}(N^{-10}).$$

Thus

$$\|u^{n_0}\|_\infty \le 1+\mathcal{O}(N^{-0.3}\log N).$$

Then for $n \ge n_0$, we consider the problem

$$\begin{cases} \partial_t u = Q_N(f(u)) = f(u)+(Q_N-\mathrm{Id})(f(u)), \\ u(0) = e^{\frac{\tau}{2}\partial_{xx}}u^n = e^{\frac{\tau}{2}\partial_{xx}}Q_N U^n. \end{cases}$$

By Lemma 6.2 (note that $\tau < N^{-0.4}$ can be made sufficiently small by taking N large), we have

$$\sup_{0\le t\le\tau}\|u(t)\|_{H^1} = \mathcal{O}(1).$$

By using Lemma 6.1, we have

$$\|(Q_N-\mathrm{Id})(f(u))\|_\infty \lesssim N^{-\frac{1}{2}}.$$

If we assume that $\|u(0)\|_\infty \le 1+\delta_0$ for some $\delta_0 = \mathcal{O}(N^{-0.29})$, then a simple maximum principle argument yields that

$$\sup_{0\le t\le\tau}\|u(t)\|_\infty \le 1+\delta_0.$$

Consequently

$$\sup_{n\ge n_0}\|u^n\|_\infty \le 1+\mathcal{O}(N^{-0.29}).$$

Finally we consider the regime $1 \le n \le n_0$. Clearly we have

$$\begin{aligned} &\sup_{1\le n\le n_0}\|U^n-\mathcal{S}_{2n\tau}U^0\|_\infty \le \mathcal{O}(N^{-0.3}), \\ &\sup_{1\le n\le n_0}\|u^n-Q_N(\mathcal{S}_{2n\tau}U^0)\|_\infty \le \mathcal{O}(N^{-0.3}\log N). \end{aligned}$$

We are then in a situation similar to that in Theorem 5.4. The argument is then similar and thus omitted. □

Remark 6.1. We should point out that, by using the argument in the preceding proof, it is also possible to obtain the result

$$\|U^n\|_\infty \le \|u^n\|_\infty \le 1+\mathcal{O}(N^{-c})$$

for the case $N^{-0.4} \le \tau \le \tau_1$, where $\tau_1 \ll 1$ is a constant determined in Lemma 6.2 ($\tau_1 = t_0$ in the notation therein). We briefly sketch the argument as follows. Again consider u^{n+1}, $u^n : \mathbb{T} \to \mathbb{R}$ defined via the relation:

$$u^{n+1} = e^{\frac{\tau}{2}\partial_{xx}} V_\tau e^{\frac{\tau}{2}\partial_{xx}} u^n,$$

where V_τ is the time-τ solution operator to the problem:

$$\partial_t u = Q_N(f(u)).$$

Denote $v^1 = e^{\frac{\tau}{2}\partial_{xx}} u^0$. Clearly since $\tau \ge N^{-0.4}$,

$$\|v^1\|_\infty \le 1 + \sum_{|k| \ge \frac{N}{2}} e^{-(2\pi k\nu)^2 \tau} \le 1+\mathcal{O}(N^{-10}).$$

Then we consider the problem

$$\begin{cases} \partial_t u = Q_N(f(u)) = f(u) + (Q_N - \mathrm{Id})(f(u)), \\ u(0) = e^{\frac{\tau}{2}\partial_{xx}} Q_N U^n. \end{cases}$$

By using Lemma 6.1 and Lemma 6.2, we have

$$\begin{aligned} \|(Q_N - \mathrm{Id})(f(u))\|_\infty &\lesssim N^{-\frac{1}{2}} \|\partial_x u\|_2 (1+\|u\|_\infty^2) \\ &\lesssim N^{-\frac{1}{2}}(1+(\log N)^2) + N^{-\frac{1}{2}}(1+(\log N)^2)\cdot \tau^{-\frac{1}{2}}. \end{aligned}$$

In the above last inequality, the first term is due to the case $\tau \sim 1$, and the second term accounts for the case $\tau \ll 1$. Since we assumed $\tau \ge N^{-0.4}$, we then obtain

$$\|(Q_N - \mathrm{Id})(f(u))\|_\infty \le \mathcal{O}(N^{-0.29}).$$

A maximum principle argument then yields that $\|u^n\|_\infty \le 1+\mathcal{O}(N^{-c})$.

Remark 6.2. Denote $\theta = e^{-2\tau}$ and for $u : \mathbb{T} \to \mathbb{R}$,

$$(T_\theta u)(x) = \frac{u(x)}{\sqrt{\theta + (1-\theta)u(x)^2}}, \quad x \in \mathbb{T}.$$

Clearly we have $Q_N\mathcal{N}_\tau U = Q_N T_\theta Q_N U$ for any $U\in\mathbb{R}^N$. It is not difficult to check that for any $f\in H^1(\mathbb{T})$, we have

$$\|Q_N f\|_{H^1(\mathbb{T})}\lesssim\|f\|_{H^1(\mathbb{T})}.$$

Note that the function $g(z)=z/\sqrt{\theta+(1-\theta)z^2}$ satisfies

$$\sup_{\theta\gtrsim 1}\|g'\|_{L^\infty(\mathbb{R})}\lesssim 1.$$

It follows that for $0<\tau\lesssim 1$ (i.e., $\theta\gtrsim 1$),

$$\|Q_N\mathcal{N}_\tau U\|_{H^1(\mathbb{T})}\lesssim\|T_\theta Q_N U\|_{H^1(\mathbb{T})}\lesssim\|Q_N U\|_{H^1(\mathbb{T})}\lesssim\|\partial_h U\|_2+\|U\|_2.$$

In particular this implies that we can improve the uniform-in-time H^1 estimate in Lemma 6.2 to any finite $t_0\sim 1$. A further interesting issue is to understand the regime $t_0\gg 1$. However in this regime one should not expect uniform-in-time H^1 bounds. A counterexample is as follows. Take $\phi(x)=\cos 2\pi Ax$ where $A\gg 1$ is an integer. Take $0<\theta\ll A^{-10}$ and $u=\sqrt{\theta}\phi$. Then clearly $\|u\|_{H^1(\mathbb{T})}\ll 1$ while

$$\|\partial_x T_\theta u\|_{L^2_x(\mathbb{T})}\gtrsim A\gg 1.$$

By taking N sufficiently large, we obtain

$$\|\partial_x Q_N T_\theta Q_N u\|_{L^2_x(\mathbb{T})}\gtrsim A\gg 1.$$

Thus in the regime $t_0\gg 1$ one does not have uniform-in-time H^1-norm bounds.

7 Maximum principle for spectral Burgers

Consider

$$\begin{cases}\partial_t u+\Pi_N(uu_x)=\nu^2 u_{xx}, & (t,x)\in(0,\infty)\times\mathbb{T},\\ u\big|_{t=0}=u^0=\Pi_N f,\end{cases}\tag{7.1}$$

where $f\in L^\infty(\mathbb{T})$.

Theorem 7.1. *Let $\nu>0$ and u be the solution to* (7.1). *We have for all $N\geq 2$,*

$$\sup_{t\geq 0}\|u(t)\|_\infty\leq\|u^0\|_\infty+\alpha\cdot N^{-c},$$

where $c>0$ is an absolute constant, and $\alpha>0$ depends only on $(\|f\|_\infty,\ \nu)$.

Remark 7.1. The dependence of α on $1/\nu$ is at most power-like for $0<\nu\lesssim 1$.

Remark 7.2. If one work with $u^0=\tilde{P}_N f$, where $\tilde{P}_N$ is a nice Fourier projector such as the Fejer's kernel which satisfies $\|\tilde{P}_N f\|_\infty\leq\|f\|_\infty$ and $\text{supp}(\tilde{P}_N f)\subset\{k:-\frac{N}{2}<k\leq\frac{N}{2}\}$, then we obtain in this case

$$\sup_{t\geq 0}\|u(t)\|_\infty\leq\|f\|_\infty+\mathcal{O}(N^{-c}).$$

Such pre-processing of initial data is quite easy to implement in practice.

Proof. Denote $\bar{u}$ as the average of u on $\mathbb{T}$ which is clearly preserved in time. Let $v=u-\bar{u}$. Then

$$\frac{1}{2}\frac{d}{dt}\Big(\|v\|_2^2\Big)=-\nu^2\|\partial_x v\|_2^2\leq-4\pi^2\nu^2\|v\|_2^2.$$

Thus

$$\|v(t)\|_2\leq e^{-4\pi^2\nu^2 t}\|v(0)\|_2,\quad\forall t\geq 0.$$

By using this and standard smoothing estimates, we then obtain

$$\begin{aligned}
&\|v(t)\|_\infty\leq C_1 t^{-\epsilon_0}e^{-c_1 t\nu^2}, && \forall t>0,\\
&\|\partial_x v(t)\|_2\leq C_1 t^{-\frac{1}{2}}e^{-c_1 t\nu^2}, && \forall t>0,\\
&\||\partial_x|^{\frac{1}{8}}\partial_x v\|_\infty\leq C_1 t^{-\epsilon_0-\frac{7}{16}}e^{-c_1 t\nu^2}, && \forall t>0,
\end{aligned}$$

where the constant $C_1>0$ depend only on $(\|f\|_\infty,\,\nu)$, $c_1>0$ is an absolute constant, and $0<\epsilon_0\ll 1$ is a small absolute constant. In the above the pre-factor $t^{-\epsilon_0}$ is due to the fact that we work with $\|v(0)\|_p\lesssim\|f\|_p$ for some p sufficiently large (instead of using $\|v(0)\|_\infty$). It follows that

$$\|(\text{Id}-\Pi_N)(uu_x)\|_\infty=\|(\text{Id}-\Pi_N)((\bar{u}+v)v_x)\|_\infty\leq C_2 N^{-c}t^{-\epsilon_1}e^{-c_1 t\nu^2},\quad\forall t>0,$$

where $c>0$ is an absolute constant, $C_2>0$ depends only on $(\|f\|_\infty,\,\nu)$, and $0<\epsilon_1<1$ is another absolute constant. Thus we may rewrite the original equation as

$$\partial_t u+uu_x=\nu^2 u_{xx}+F,$$

where F obeys

$$\|F\|_\infty\leq C_2 N^{-c}t^{-\epsilon_1}e^{-c_1 t\nu^2},\quad\forall t>0.$$

Since by spectral truncation u^0 is clearly smooth, the function u is also smooth. Consider

$$g(t,x)=u(t,x)-\Big(\|u^0\|_\infty+a(t)\Big),$$

where

$$a(t)=C_2N^{-c}\int_0^t s^{-\epsilon_1}e^{-c_1s\nu^2}ds.$$

Clearly for any $t_0>0$ and $t\geq t_0$, we have g is smooth and

$$\partial_t g+ug_x\leq\nu^2g_{xx},\quad t\geq t_0.$$

A simple maximum principle argument then yields

$$\max_{x\in\mathbb{T}}g(t,x)\leq\max_{x\in\mathbb{T}}g(t_0,x),\quad\forall t\geq t_0.$$

Sending $t_0\to 0+$ then yields the desired estimate for the upper bound. The proof for the lower bound is similar. We omit the details. □

Consider

$$\begin{cases}\dfrac{u^{n+1}-u^n}{\tau}+\Pi_N(u^n\partial_xu^n)=\nu^2\partial_{xx}u^{n+1},\quad(t,x)\in(0,\infty)\times\mathbb{T},\\ u\big|_{t=0}=u^0=\Pi_Nf,\end{cases}\tag{7.2}$$

where $f\in L^\infty(\mathbb{T})$.

Theorem 7.2. *Let $\nu>0$ and u^n be the solution to (7.2). We have for all $0<\tau<\tau_0=\tau_0(\nu,\|f\|_\infty)$ and $N\geq 2$,*

$$\sup_{n\geq 0}\|u^n\|_\infty\leq\|u^0\|_\infty+\alpha_1\cdot N^{-\frac{1}{2}}+\alpha_2\tau^{0.49},$$

where $\alpha_1,\alpha_2>0$ depend only on ($\|f\|_\infty$, ν).

Remark 7.3. For convenience we do not spell out the explicit dependence of τ_0 on ν.

Proof. To ease the notation we shall set $\nu=1$. Rewrite (7.2) as

$$u^{n+1}=Au^n-\frac{1}{2}\tau A\Pi_N\partial_x((u^n)^2),\tag{7.3}$$

where $A=(\mathrm{Id}-\tau\partial_{xx})^{-1}$.

Denote $v^n=u^n-\bar{u}$, where (below we denote $\int=\int_{\mathbb{T}}dx$)

$$\bar{u}=\int u^0=\int f.$$

Then clearly

$$\frac{v^{n+1}-v^n}{\tau}+\Pi_N(u^n\partial_x v^n)=\partial_{xx}v^{n+1}. \tag{7.4}$$

We shall proceed in several steps.

Step 1. Local smoothing estimates. Let $t_0>0$ be chosen sufficiently small (depending only on $\|f\|_\infty$) so that the nonlinear part is dominated by the linear part when $n\tau\le t_0$. Choose $1\le n_0\in\mathbb{Z}$ such that $n_0\tau\sim t_0$. By using (7.3) and discrete smoothing estimates, we have

$$\sup_{1\le n\le n_0}(\sqrt{n\tau}\|\partial_x u^n\|_2+\|u^n\|_2)\le C_1, \tag{7.5a}$$

$$\|\partial_x u^{n_0}\|_\infty+\|u^{n_0}\|_\infty\le C_1, \tag{7.5b}$$

where $C_1>0$ depends only on $\|f\|_\infty$.

Step 2. L^2 estimate. Multiplying both sides of (7.4) by v^{n+1} and integrating, we obtain

$$\frac{1}{2\tau}(\|v^{n+1}\|_2^2-\|v^n\|_2^2+\|v^{n+1}-v^n\|_2^2)+\|\partial_x v^{n+1}\|_2^2+N_1\le 0,$$

where

$$\begin{aligned}N_1&=\int\Pi_N(u^n\partial_x v^n)v^{n+1}=\int u^n\partial_x v^n(v^{n+1}-v^n)\\&=\int u^n(\partial_x v^{n+1})(v^{n+1}-v^n)+\int u^n\partial_x(v^n-v^{n+1})\cdot(v^{n+1}-v^n)dx.\end{aligned}$$

Clearly it holds that

$$\begin{aligned}|N_1|&\le\|u^n\|_\infty\cdot\|\partial_x v^{n+1}\|_2\|v^{n+1}-v^n\|_2+\frac{1}{2}\|\partial_x u^n\|_\infty\|v^{n+1}-v^n\|_2^2\\&\le\frac{1}{2}\|\partial_x v^{n+1}\|_2^2+\Big(\frac{1}{2}\|u^n\|_\infty^2+\frac{1}{2}\|\partial_x u^n\|_\infty\Big)\cdot\|v^{n+1}-v^n\|_2^2.\end{aligned}$$

Thus if

$$\frac{1}{\tau}\ge\|u^n\|_\infty^2+\|\partial_x u^n\|_\infty, \tag{7.6}$$

then

$$(1+4\pi^2\tau)\|v^{n+1}\|_2^2 \le \|v^{n+1}\|_2^2+\tau\|\partial_x v^{n+1}\|_2^2 \le \|v^n\|_2^2.$$

By a similar estimate and under the same condition (7.6), we also obtain

$$\|u^{n+1}\|_2 \le \|u^n\|_2.$$

Step 3. Induction. For $n \ge n_0$, we inductively assume

$$\|\partial_x u^n\|_\infty \le A_1\tau^{-\frac{3}{4}}, \quad \|u^n\|_\infty \le A_1\tau^{-\frac{1}{4}}, \quad \|u^n\|_2 \le C_1,$$

where A_1 is a suitably large constant. Throughout the argument we shall assume $\tau>0$ is sufficiently small, in particular it has to satisfy

$$\tau^{-1} \ge A_1^2\tau^{-\frac{1}{2}}+A_1\tau^{-\frac{3}{4}}$$

and some additional mild constraints in the argument below. Now clearly by Step 2 we have $\|u^{n+1}\|_2 \le C_1$. By using (7.3), we have (below β_i denotes absolute constants)

$$\begin{aligned}
\|\partial_x u^{n+1}\|_\infty &\le \tau^{-\frac{3}{4}}\cdot\beta_1\cdot C_1+\beta_2\|\Pi_N((u^n)^2)\|_\infty \\
&\le \tau^{-\frac{3}{4}}\beta_1 C_1+\beta_3\Big(1+\|u^n\|_\infty^2\log(10+\|\partial_x u^n\|_\infty\|u^n\|_\infty)\Big) \\
&\le \tau^{-\frac{3}{4}}\beta_1 C_1+\beta_3\Big(1+\tau^{-\frac{1}{2}}A_1^2\log(10+A_1^2\tau^{-1})\Big) \\
&\le A_1\tau^{-\frac{3}{4}},
\end{aligned}$$

where in the last step we choose $A_1 \ge 2\beta_1 C_1$ and take τ sufficiently small. By a similar estimate, it is not difficult to check that

$$\|u^{n+1}\|_\infty \le A_1\tau^{-\frac{1}{4}}.$$

This then completes the induction proof and we obtain

$$\begin{aligned}
&\|u^{n+1}\|_2 \le \|u^n\|_2 \le \|u^{n_0}\|_2, && \forall n \ge n_0, \\
&(1+4\pi^2\tau)\|v^{n+1}\|_2^2 \le \|v^n\|_2^2, && \forall n \ge n_0.
\end{aligned}$$

It follows that

$$\|v^n\|_2 \le C_2 e^{-\beta n\tau}, \quad \forall n \ge 1,$$

where $\beta>0$ is an absolute constant, and $C_2>0$ depends only on $\|f\|_\infty$.

Step 4. Time-global estimates. By using the estimates derived in Step 1 to Step 3 and further bootstrapping estimates, we then obtain

$$\begin{aligned}
&\|v^n\|_\infty \le C_3\cdot(n\tau)^{-\epsilon_1}e^{-\gamma_1 n\tau}, && \forall n\ge 1,\\
&\|\partial_x u^n\|_\infty \le C_3\cdot(n\tau)^{-\frac{1}{2}-\epsilon_2}e^{-\gamma_2 n\tau}, && \forall n\ge 1,\\
&\|\partial_{xx} u^n\|_\infty \le C_3\cdot(n\tau)^{-1-\epsilon_3}e^{-\gamma_3 n\tau}, && \forall n\ge 1,\\
&\|\partial_{xxx} u^n\|_\infty \le C_3\cdot(n\tau)^{-\frac{3}{2}-\epsilon_4}e^{-\gamma_4 n\tau}, && \forall n\ge 1,
\end{aligned}$$

where $0<\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\ll 1$ are absolute constants, $\gamma_i>0$ are absolute constants, and $C_3>0$ depends only on $\|f\|_\infty$. By using (7.2), we have

$$\tau\|u^n\partial_x(u^{n+1}-u^n)\|_\infty \le C_5\cdot\tau^2\cdot(n\tau)^{-\frac{3}{2}-\epsilon_5},\quad \forall n\ge 1,$$

where $0<\epsilon_5\ll 1$ is an absolute constant, and $C_5>0$ depends only on $\|f\|_\infty$. Also

$$\tau\|(\mathrm{Id}-\Pi_N)(u^n\partial_x u^n)\|_\infty \le C_6\cdot\tau\cdot N^{-\frac{1}{2}}\cdot(n\tau)^{-\frac{3}{4}-\epsilon_6}e^{-\gamma_5 n\tau},\quad \forall n\ge 1,$$

where $0<\epsilon_6\ll 1$ is an absolute constant, $\gamma_5>0$ is an absolute constant, and $C_6>0$ depends only on $\|f\|_\infty$. Now for $n\ge 1$ we rewrite the equation for u^{n+1} as

$$(\mathrm{Id}-\tau\partial_{xx})u^{n+1}=u^n+\tau u^n\partial_x u^{n+1}+\tau u^n\partial_x(u^n-u^{n+1})+\tau(\mathrm{Id}-\Pi_N)(u^n\partial_x u^n).$$

Clearly by a maximum principle argument, we have

$$\begin{aligned}
\|u^{n+1}\|_\infty &\le \|u^n\|_\infty+\tau\|u^n\partial_x(u^{n+1}-u^n)\|_\infty+\tau\|(\mathrm{Id}-\Pi_N)(u^n\partial_x u^n)\|_\infty\\
&\le \|u^n\|_\infty+C_5\cdot\tau^2\cdot(n\tau)^{-\frac{3}{2}-\epsilon_5}+C_6\cdot\tau\cdot N^{-\frac{1}{2}}\cdot(n\tau)^{-\frac{3}{4}-\epsilon_6}e^{-\gamma_5 n\tau},\quad \forall n\ge 1.
\end{aligned}$$

Iterating in n, we obtain

$$\|u^n\|_\infty \le \|u^1\|_\infty+C_7\tau^{\frac{1}{2}-\epsilon_5}+C_8N^{-\frac{1}{2}},\quad \forall n\ge 1,$$

where $C_7,C_8>0$ are constants depending only on $\|f\|_\infty$. Finally we observe that

$$\begin{aligned}
\|u^1\|_\infty &\le \|u^0\|_\infty+\frac{1}{2}\|\tau\partial_x A\Pi_N((u^0)^2)\|_\infty\\
&\le \|u^0\|_\infty+C\tau^{\frac{1}{2}-\epsilon_6}\|f\|_\infty^2,
\end{aligned}$$

where $0<\epsilon_6\ll 1$ is an absolute constant, and $C>0$ is an absolute constant. Our desired result then follows easily. □

We now consider the Fourier collocation case. Let $U(t)\in\mathbb{R}^N$, $N\geq 2$ solve

$$\begin{cases}\dfrac{d}{dt}U+\dfrac{1}{2}\partial_h(U^{\cdot 2})=\Delta_h U, \quad t>0,\\ U\big|_{t=0}=U^0\in\mathbb{R}^N,\end{cases} \tag{7.7}$$

where for convenience we have set the viscosity coefficient $\nu=1$, and the operators ∂_h, Δ_h correspond to the Fourier multipliers $2\pi ik$, $-4\pi^2k^2$ for $-\frac{N}{2}<k\leq\frac{N}{2}$ respectively in the DFT formula.

Set $u=Q_NU$. We obtain the following reformulation of (7.7):

$$\begin{cases}\partial_t u+\dfrac{1}{2}\partial_x Q_N(u^2)=\partial_{xx}u, \quad (t,x)\in(0,\infty)\times\mathbb{T},\\ u\big|_{t=0}=u^0=Q_NU^0.\end{cases} \tag{7.8}$$

Remark 7.4. At this point we should point out a subtle technical difficulty associated with the analysis of (7.7) and the equivalent system (7.8). Namely in general we have

$$\langle\partial_h(U^{\cdot 2}),U\rangle\neq 0,$$

or in terms of u:

$$\int_{\mathbb{T}}Q_N(u^2)\partial_x u dx\neq 0.$$

An example can be constructed as follows. Let $N=6k_0\geq 6$ where k_0 is an integer. Set $m=N/3$ and $u(x)=\sin 2\pi mx$. It is not difficult to check that

$$Q_N(u^2)=\frac{1}{2}-\frac{1}{2}\cos 4\pi mx.$$

Clearly then

$$\int_{\mathbb{T}}Q_N(u^2)\partial_x u dx\neq 0.$$

Theorem 7.3. *Consider* (7.7) *with* $N\geq 2$. *Suppose* f *is a real-valued function with* $f=Q_Nf$. *Let* $(U^0)_j=f(\frac{j}{N})$ *for all* $0\leq j\leq N-1$. *Then for all* $N\geq N_0=N_0(\|f\|_2)$, *we have*

$$\sup_{t\geq 0}\|U(t)\|_\infty\leq\sup_{t\geq 0}\|u(t)\|_\infty\leq\|f\|_\infty+\gamma_1\cdot N^{-\frac{1}{2}}\cdot(1+\|\partial_x f\|_2^2),$$

where $\gamma_1>0$ *depends only on* $\|f\|_2$, *and* u *solves* (7.8).

Proof. Step 1. L^2 estimate. Set $v=u-\bar{u}$ and note that $\bar{u}$ is clearly preserved in time. Then

$$\partial_t v+\frac{1}{2}\partial_x Q_N(v^2+2\bar{u}v)=\partial_{xx}v.$$

Multiplying both sides by v and integrating, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_2^2=&\frac{1}{2}\langle(Q_N-\mathrm{Id})(v^2+2v\bar{u}),\partial_x v)-\|\partial_x v\|_2^2\\ \leq&\frac{1}{2}\|(Q_N-\mathrm{Id})(v^2+2v\bar{u})\|_2\|\partial_x v\|_2-\|\partial_x v\|_2^2.\end{aligned}$$

Now note that for g with Fourier support localized in $\{k:|k|\leq 10N\}$ we have

$$\|(Q_N-\mathrm{Id})g\|_2\lesssim N^{-1}\|\partial_x g\|_2\lesssim N^{-\frac{1}{2}}\|\partial_x g\|_1.$$

Thus

$$\|(Q_N-\mathrm{Id})(v^2+2v\bar{u})\|_2\|\lesssim N^{-1}|\bar{u}|\|\partial_x v\|_2+N^{-\frac{1}{2}}\|v\|_2\|\partial_x v\|_2.$$

We then obtain

$$\frac{d}{dt}\|v\|_2^2\leq(\alpha_0 N^{-1}|\bar{u}|+\alpha_0 N^{-\frac{1}{2}}\|v\|_2-2)\|\partial_x v\|_2^2,$$

where $\alpha_0>0$ is an absolute constant. Clearly for $N\geq N_0=N_0(\|f\|_2)$ sufficiently large, we have

$$\|v(t)\|_2\leq\|v(0)\|_2 e^{-2\pi^2 t},\quad\forall t\geq 0.$$

Step 2. Higher order estimates. Observe that for $t\gtrsim 1$, by smoothing estimates we have

$$\|\partial_x u(t)\|_{H^3(\mathbb{T})}\leq C\cdot e^{-\beta_1 t},$$

where the constant $C>0$ depends only on $\|f\|_2$, and $\beta_1>0$ is an absolute constant. Now we only need to focus on the local in time estimate. We rewrite the equation for u as

$$\partial_t u+u\partial_x u=\frac{1}{2}\partial_x(\mathrm{Id}-Q_N)(u^2)+\partial_{xx}u. \tag{7.9}$$

Clearly for $T>0$, we have

$$\begin{aligned}\|\partial_x u\|_{L_t^\infty L_x^2([0,T]\times\mathbb{T})}\lesssim&\|\partial_x u_0\|_2+T^{\frac{1}{4}}\|\partial_x u\|_{L_t^\infty L_x^2([0,T]\times\mathbb{T})}\cdot\|u\|_{L_t^\infty L_x^2}\\ &+T^{\frac{1}{4}}\||\partial_x|^{\frac{1}{2}}(Q_N-\mathrm{Id})(u^2)\|_{L_t^\infty L_x^2([0,T]\times\mathbb{T})}.\end{aligned} \tag{7.10}$$

Since u is spectrally localized to $\{|k|\leq N\}$, we have

$$\begin{aligned}\||\partial_x|^{\frac{1}{2}}(Q_N-\mathrm{Id})(u^2)\|_{L^2_x}\lesssim& N^{-\frac{1}{2}}\|\partial_x(u^2)\|_{L^2_x}\\ \lesssim& N^{-\frac{1}{2}}\|\partial_x u\|_{L^2_x}\|u\|_{L^\infty_x}\lesssim\|\partial_x u\|_{L^2_x}\|u\|_{L^2_x}.\end{aligned}$$

Plugging this estimate into (7.10), we obtain for $T_0=T_0(\|f\|_2)$ sufficiently small that

$$\|\partial_x u\|_{L^\infty_t L^2_x([0,T_0]\times\mathbb{T})}\lesssim\|\partial_x f\|_2.$$

Now on the same time interval $[0,T_0]$, we have

$$\begin{aligned}\|t^{\frac{1}{4}}\partial_x u\|_{L^\infty_t L^\infty_x([0,T_0]\times\mathbb{T})}\lesssim&\|\partial_x f\|_2+T_0^{\frac{1}{4}}\cdot\|\partial_x u\|_{L^\infty_t L^2_x([0,T_0]\times\mathbb{T})}\cdot\|u\|_{L^\infty_t L^\infty_x([0,T_0]\times\mathbb{T})}\\ \leq& C\cdot\|\partial_x f\|_2\cdot(1+\|\partial_x f\|_2^{\frac{1}{2}}),\end{aligned}$$

where $C>0$ depends only on $\|f\|_2$. Next take $\epsilon_0<1$ to be an absolute constant which is sufficiently close to 1. We have

$$\begin{aligned}&\|t^{\frac{\epsilon_0}{2}}\partial_x|\partial_x|^{\epsilon_0}u\|_{L^\infty_t L^2_x([0,T_0]\times\mathbb{T})}\\ \lesssim&\|\partial_x f\|_2+T_0^{\frac{1}{4}}\cdot\|\partial_x u\|_{L^\infty_t L^\infty_x([0,T_0]\times\mathbb{T})}\cdot\|u\|_{L^\infty_t L^2_x([0,T_0]\times\mathbb{T})}\\ \leq& C\cdot\|\partial_x f\|_2\cdot(1+\|\partial_x f\|_2^{\frac{1}{2}}),\end{aligned}$$

where $C>0$ depends only on $\|f\|_2$. Now observe that (for the second inequality below we used the fact that u is spectrally localized to $\{|k|\leq N\}$)

$$\begin{aligned}&\||\partial_x|^{\epsilon_0}\partial_x(u^2)\|_{L^2_x}\lesssim\|u\|_{L^2_x}^2+\||\partial_x|^{\epsilon_0}\partial_x u\|_{L^2_x}\cdot\|u\|_{L^\infty_x},\\ &\||\partial_x|^{\epsilon_0}\partial_x(Q_N-\mathrm{Id})(u^2)\|_{L^2_x}\lesssim N^{-(1-\epsilon_0)}\|\partial_x^2(u^2)\|_{L^2_x}\\ &\lesssim N^{-(1-\epsilon_0)}(\|\partial_{xx}u\|_{L^2_x}\|u\|_{L^\infty_x}+\|\partial_x u\|_{L^2_x}\|\partial_x u\|_{L^\infty_x})\\ &\lesssim\||\partial_x|^{\epsilon_0}\partial_x u\|_{L^2_x}\cdot(\|u\|_{L^2_x}+\|u\|_{L^2_x}^{\frac{1}{2}}\|\partial_x u\|_{L^2_x}^{\frac{1}{2}})+\|\partial_x u\|_{L^2_x}^2.\end{aligned}$$

By using the preceding estimates, we then compute

$$\begin{aligned}&\|t^{\frac{1}{4}}\partial_x^2 u\|_{L^\infty_t L^2_x([0,T_0]\times\mathbb{T})}\\ \lesssim&\|\partial_x f\|_2+C+C\cdot\|\partial_x f\|_2(1+\|\partial_x f\|_2)\\ \leq& C+C\|\partial_x f\|_2^2,\end{aligned}$$

where $C>0$ depends only on $\|f\|_2$.

Step 3. Conclusion. By using the estimate from Step 2, we have for any $t>0$,

$$\begin{aligned}\|\partial_x(\mathrm{Id}-Q_N)(u^2)\|_\infty\lesssim& N^{-\frac{1}{2}}\|\partial_x^2(u^2)\|_2\\ \leq& C_2(1+\|\partial_x f\|_2^2)\cdot N^{-\frac{1}{2}}t^{-\frac{1}{4}}e^{-\beta t},\end{aligned}$$

where $\beta>0$ is an absolute constant, and $C_2>0$ depends only on $\|f\|_2$. Note that

$$\|u^0\|_\infty \le \|f\|_\infty + \|(\mathrm{Id}-Q_N)f\|_\infty \le \|f\|_\infty + N^{-\frac{1}{2}}\cdot\gamma\cdot\|\partial_x f\|_2,$$

where $\gamma>0$ is an absolute constant. The desired result then follows from a maximum principle argument. □

Theorem 7.4. *Consider* (7.7) *with* $N\ge 2$. *Then for all* $N\ge N_1=N_1(\|U^0\|_\infty)$, *we have*

$$\sup_{t\ge 0}\|U(t)\|_\infty \le (1+\epsilon_1)\|U^0\|_\infty,$$

where $0<\epsilon_1<10^{-2}$ *is an absolute constant. More precisely the following hold. (Below we write* $X=O(Y)$ *if* $|X|\le CY$ *and the constant* C *depends only on* $\|U^0\|_\infty$.*)*

1. *For any* $t>0$, *we have*

$$\frac{d}{dt}(\|U(t)\|_2^2)=\frac{d}{dt}(\|u(t)\|_2^2)\le 0,$$

where u *solves* (7.8). *In particular for any* $0\le t_1<t_2<\infty$, *we have*

$$\|U(t_2)\|_2\le\|U(t_1)\|_2.$$

2. *For* $t\ge T_1=N^{-\frac{2}{5}}$, *we have*

$$\sup_{t\ge T_1}\|U(t)\|_\infty\le\sup_{t\ge T_1}\|u(t)\|_\infty\le\|U^0\|_\infty+\mathcal{O}(N^{-0.1}).$$

3. *For* $0<t\le T_1$, *we have*

$$\begin{aligned}
&\|u(t)-e^{t\partial_{xx}}Q_NU^0\|_\infty\le\mathcal{O}(N^{-0.1}),\\
&\sup_{0\le l\le N-1}\left|(e^{t\partial_{xx}}Q_NU^0)\Big(t,\frac{l}{N}\Big)\right|\le\Big(1+\frac{1}{2}\epsilon_1\Big)\|U^0\|_\infty,\\
&\|U(t)\|_\infty\le\Big(1+\frac{1}{2}\epsilon_1\Big)\|U^0\|_\infty+\mathcal{O}(N^{-0.1}).
\end{aligned}$$

4. *Suppose* $f:\mathbb{T}\to\mathbb{R}$ *is continuous. If* $(U^0)_l=f(\frac{l}{N})$ *for all* $0\le l\le N-1$, *then*

$$\sup_{t\ge 0}\|U(t)\|_\infty\le\|f\|_\infty+\mathcal{O}(\omega_f(N^{-\frac{2}{3}}))+\mathcal{O}(N^{-c}),$$

where $c>0$ *is an absolute constant, and* ω_f *is defined in* (B.5).

5. *Suppose* $f:\mathbb{T}\to\mathbb{R}$ *is* C^α*-continuous* (*see* (B.6)) *for some* $0<\alpha<1$. *If* $(U^0)_l=f(\frac{l}{N})$ *for all* $0\le l\le N-1$, *then*

$$\sup_{t\ge 0}\|U(t)\|_\infty\le\|f\|_\infty+\mathcal{O}(N^{-c_1}),$$

where $c_1>0$ *is a constant depending only on* α.

Moreover we have the following result which shows the sharpness of our estimates above. There exists a function $f:\ \mathbb{T}\to\mathbb{R}$, *continuous at all of* $\mathbb{T}\backslash\{x_*\}$ *for some* $x_*\in\mathbb{T}$ *(i.e., continuous at all* $x\neq x_*$*) and has the bound* $\|f\|_\infty\le 1$ *such that the following hold: for a sequence of even numbers* $N_m\to\infty$, $t_m=N_m^{-2}$ *and* $x_m=j_m/N_m$ *with* $0\le j_m\le N_m-1$, *if* $\tilde{U}_m(t)\in\mathbb{R}^{N_m}$ *solves* (7.7) *with* $\tilde{U}_m(0)=v_m\in R^{N_m}$ *satisfying* $(v_m)_j=f(\frac{j}{N_m})$ *for all* $0\le j\le N_m-1$. *Then*

$$|\tilde{U}_m(t_m,x_m)|\ge 1+\eta_*,\quad\forall m\ge 1,$$

where $\eta_*>0.001$ *is an absolute constant.*

Proof. We proceed in several steps.

Step 1. We note that statement (1) follows from the same proof as in Theorem 7.3.

Step 2. Local in time estimate. To establish the remaining results, we first perform a local in time estimate. First we show that for some $T_0=T_0(\|U^0\|_2)$ sufficiently small, we have contraction using the norm

$$\|u\|_{L_t^\infty L_x^2([0,T_0]\times\mathbb{T})}+\|t^{\frac{1}{2}}\partial_x u\|_{L_t^\infty L_x^2([0,T_0]\times\mathbb{T})}.$$

Indeed by using the estimate

$$\begin{aligned}\||\partial_x|^{\frac{1}{2}}(Q_N-\mathrm{Id})(u^2)\|_{L_x^2}&\lesssim N^{-\frac{1}{2}}\|\partial_x u\|_{L_x^2}\|u\|_{L_x^\infty}\\&\lesssim\|\partial_x u\|_{L_x^2}\cdot\|u\|_{L_x^2},\end{aligned}$$

we have

$$\|t^{\frac{1}{2}}\partial_x u\|_{L_t^\infty L_x^2([0,T_0]\times\mathbb{T})}\lesssim\|u(0)\|_2+T_0^{\frac{1}{4}}\|t^{\frac{1}{2}}\partial_x u\|_{L_t^\infty L_x^2([0,T_0]\times\mathbb{T})}\cdot\|u\|_{L_t^\infty L_x^2}.$$

Thus for $T_0=T_0(\|U^0\|_2)$ sufficiently small, we have

$$\|u\|_{L_t^\infty L_x^2([0,T_0]\times\mathbb{T})}+\|t^{\frac{1}{2}}\partial_x u\|_{L_t^\infty L_x^2([0,T_0]\times\mathbb{T})}\lesssim\|U^0\|_2.$$

Now for any $0<T\leq T_0$, we have

$$\begin{aligned}&\|u(T)-e^{T\partial_{xx}}Q_NU^0\|_{L_x^\infty}\\ \lesssim&\|t^{\frac{1}{2}}\partial_x u\|_{L_t^\infty L_x^2([0,T_0]\times\mathbb{T})}\cdot\|u\|_{L_t^\infty L_x^2([0,T_0]\times\mathbb{T})}\\ \lesssim&\|U^0\|_2^2.\end{aligned}$$

For $0<t\leq T_0$, we write

$$u(t)=e^{t\partial_{xx}}Q_NU^0+h_1=:\Theta(t)+h_1,$$

where $\|h_1\|_{L_t^\infty L_x^\infty}\lesssim\|U^0\|_2^2$. Note that both Θ and h_1 are spectrally localized to $\{|k|\leq N\}$. It follows that for $0<t\leq T_0$,

$$\begin{aligned}\|Q_N(u(t)^2)\|_{L_x^2(\mathbb{T})}\lesssim&\|Q_N(h_1^2)\|_{L_x^2(\mathbb{T})}+\|Q_N(h_1\Theta(t))\|_{L_x^2(\mathbb{T})}+\|Q_N(\Theta(t)^2)\|_{L_x^2(\mathbb{T})}\\ \lesssim&\|U^0\|_2^4+\|U^0\|_2^2\|\Theta(t)\|_{L_x^2(\mathbb{T})}+\|e^{t\Delta_h}U^0\|_2\|e^{t\Delta_h}U^0\|_\infty\\ \lesssim&\|U^0\|_2^4+\|U^0\|_2^3+\|U^0\|_2\|U^0\|_\infty.\end{aligned}$$

By using this improved estimate, we then obtain for any $0<T\leq T_0$,

$$\|u(T)-e^{T\partial_{xx}}Q_NU^0\|_{L_x^\infty}\lesssim T^{\frac{1}{4}}\Big(\|U^0\|_2^4+\|U^0\|_2^3+\|U^0\|_2\|U^0\|_\infty\Big).$$

Step 3. The regime $t\geq T_1=N^{-\frac{2}{5}}$. Clearly

$$\begin{aligned}\|u(T_1)\|_\infty\leq&\|e^{T_1\partial_{xx}}Q_NU^0\|_\infty+\mathcal{O}(N^{-0.1})\\ \leq&\|U^0\|_\infty+\mathcal{O}(N^{-0.1}),\end{aligned}$$

where we have used the fact that

$$\sum_{|k|\geq\frac{N}{2}}e^{-(2\pi k)^2T_1}\leq\mathcal{O}(N^{-10}).$$

By Step 2, we also have

$$\|\partial_x u(T_1)\|_2=\mathcal{O}(N^{\frac{1}{5}}).$$

By Theorem 7.3, it follows that

$$\sup_{t\geq T_1}\|U(t)\|_\infty\leq\sup_{t\geq T_1}\|u(t)\|_\infty\leq\|U^0\|_\infty+\mathcal{O}(N^{-0.1}).$$

Step 4. The regime $t<T_1$. Note here that our nonlinear solution is dominated by the linear part. The situation is then similar to that in Theorem 5.4 with minor changes. We omit the details. □

Remark 7.5. We should mention that by using the machinery developed above, it is also possible to give a complete rigorous analysis of the fully discrete scheme

$$\frac{U^{n+1}-U^n}{\tau}+\frac{1}{2}\partial_h((U^n)^{\cdot 2})=\Delta_h U^{n+1},\quad n\geq 0.$$

Yet another variant is

$$\frac{U^{n+1}-U^n}{\tau}+U^n\partial_h U^n=\Delta_h U^{n+1},\quad n\geq 0,$$

where $U^n\partial_h U^n$ takes point-wise product of U^n and $\partial_h U^n$. We shall address these and similar other schemes elsewhere.

8 Two dimensional Navier-Stokes

Consider on the torus $\mathbb{T}^2=[0,1)^2$ the two dimensional Navier-Stokes system expressed in the vorticity form:

$$\begin{cases}\partial_t\omega+\Pi_N(u\cdot\nabla\omega)=\Delta\omega,\\ \omega\big|_{t=0}=\omega^0=\Pi_N f,\end{cases}\tag{8.1}$$

where $f\in L^\infty(\mathbb{T}^2)$ and has mean zero. Here and below we assume ω has mean zero which is clearly preserved in time. The velocity u is connected to the vorticity ω through the Biot-Savart law: $u=\nabla^\perp\Delta^{-1}\omega$.

Lemma 8.1. *Let the dimension $d\geq 1$. Suppose $g:\mathbb{T}^d\to\mathbb{R}$ is in $C^\gamma(\mathbb{T}^d)$ for some $0<\gamma<1$. Then for all $N\geq 2$, it holds that*

$$\|(\mathrm{Id}-\Pi_N)g\|_\infty\leq C_1\cdot N^{-\gamma}(\log N)^d\|g\|_{C^\gamma},$$

where $C_1>0$ depends only on $(d,\ \gamma)$.

Proof. By using smooth Fourier cut-offs, we can rewrite

$$(\mathrm{Id}-\Pi_N)g=(\mathrm{Id}-\Pi_N)P_N g+P_{\gtrsim N}g,$$

where P_N, $P_{\gtrsim N}$ are smooth Fourier projectors localized to the regimes $\{|k|\sim N\}$ and $\{|k|\gtrsim N\}$ respectively. The result then follows from Theorem 2.4 together with the fact that

$$\|(\mathrm{Id}-\Delta)P_N g\|_\infty\lesssim N^{2-\gamma}\|g\|_{C^\gamma}.$$

Thus, we complete the proof. □

Theorem 8.1. *Let ω be the solution to* (8.1)*. We have for all $N\geq 2$,*

$$\sup_{t\geq 0}\|\omega(t)\|_\infty \leq \|\omega^0\|_\infty + \alpha\cdot N^{-\frac{1}{2}},$$

where $\alpha>0$ depends only on $\|f\|_\infty$.

Proof. First we compute the L^2-norm. Clearly

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}\omega^2 dx = -\|\nabla\omega\|_2^2 \leq -(2\pi)^2\|\omega\|_2^2.$$

Thus

$$\|\omega(t)\|_2 \leq \|f\|_2\cdot e^{-4\pi^2 t}, \quad t\geq 0.$$

Using this and further smoothing estimates, we obtain

$$\|u(t)\cdot\nabla\omega(t)\|_{C^{0.52}} \leq C_1 t^{-\epsilon_1}e^{-c_1 t}, \qquad \forall t>0,$$

where $C_1>0$ depends only on $\|f\|_\infty$, and $0<\epsilon_1<1$, $c_1>0$ are absolute constants. By Lemma 8.1, it follows that

$$\|(\mathrm{Id}-\Pi_N)(u(t)\cdot\nabla\omega(t))\|_\infty \leq C_2 N^{-\frac{1}{2}} t^{-\epsilon_1}e^{-c_1 t}, \quad \forall t>0,$$

where $C_2>0$ depends only on $\|f\|_\infty$. Our desired result then follows from a maximum principle argument similar to that in Burger's case. □

Consider

$$\begin{cases} \frac{\omega^{n+1}-\omega^n}{\tau}+\Pi_N(u^n\cdot\nabla\omega^n)=\Delta\omega^{n+1}, \quad (t,x)\in(0,\infty)\times\mathbb{T}^2, \\ \omega\big|_{t=0}=\omega^0=\Pi_N f, \end{cases} \tag{8.2}$$

where $f\in L^\infty(\mathbb{T}^2)$ and has mean zero.

Theorem 8.2. *Let ω^n be the solution to* (8.2)*. We have for all $0<\tau<\tau_0=\tau_0(\|f\|_\infty)$ and $N\geq 2$,*

$$\sup_{n\geq 0}\|\omega^n\|_\infty \leq \|\omega^0\|_\infty + \alpha_1\cdot N^{-\frac{1}{2}} + \alpha_2\tau^{0.4},$$

where $\alpha_1,\alpha_2>0$ depend only on $\|f\|_\infty$.

Proof. Rewrite (8.2) as

$$\omega^{n+1}=A\omega^n-\tau A\Pi_N(u^n\cdot\nabla\omega^n), \tag{8.3}$$

where $A=(\mathrm{Id}-\tau\Delta)^{-1}$.

We shall proceed in several steps.

Step 1. L^2 estimate. Multiplying both sides of (8.2) by ω^{n+1} and integrating, we obtain

$$\frac{1}{2\tau}(\|\omega^{n+1}\|_2^2-\|\omega^n\|_2^2+\|\omega^{n+1}-\omega^n\|_2^2)+\|\nabla\omega^{n+1}\|_2^2+N_1\le 0,$$

where

$$\begin{aligned}N_1&=\int\Pi_N(u^n\cdot\nabla\omega^n)\omega^{n+1}=\int u^n\cdot\nabla\omega^n(\omega^{n+1}-\omega^n)\\&=\int(u^n\cdot\nabla\omega^{n+1})(\omega^{n+1}-\omega^n)+\int u^n\cdot\nabla(\omega^n-\omega^{n+1})(\omega^{n+1}-\omega^n)dx.\end{aligned}$$

The second term above vanishes thanks to incompressibility. Then

$$\begin{aligned}|N_1|&\le\|u^n\|_\infty\cdot\|\nabla\omega^{n+1}\|_2\|\omega^{n+1}-\omega^n\|_2\\&\le\frac{1}{2}\|\nabla\omega^{n+1}\|_2^2+\frac{1}{2}\|u^n\|_\infty^2\cdot\|\omega^{n+1}-\omega^n\|_2^2.\end{aligned}$$

Thus if

$$\frac{1}{\tau}\ge\|u^n\|_\infty^2,$$

then

$$(1+8\pi^2\tau)\|\omega^{n+1}\|_2^2\le\|\omega^{n+1}\|_2^2+\tau\|\nabla\omega^{n+1}\|_2^2\le\|\omega^n\|_2^2.$$

Step 2. Induction.

For $n=0$, we have

$$\|u^0\|_\infty\le C_1\|\omega^0\|_4\le C_2\|f\|_4,$$

where $C_1>0$, $C_2>0$ are absolute constants. Clearly if

$$\frac{1}{\tau}\ge C_2\|f\|_4,$$

then

$$(1+8\pi^2\tau)\|\omega^1\|_2^2\leq\|\omega^0\|_2^2.$$

Now for $n\geq 0$, we assume

$$(1+8\pi^2\tau)\|\omega^{j+1}\|_2^2\leq\|\omega^j\|_2^2,\quad\forall 1\leq j\leq n.$$

Then for $n+1$ by using (8.3), we have

$$\begin{aligned}\|\omega^{n+1}\|_4\lesssim&\||\nabla|^{\frac{1}{2}}A\omega^n\|_2+\tau\|\Delta A(u^n\omega^n)\|_{\frac{4}{3}}\\ \lesssim&\tau^{-\frac{1}{4}}\|f\|_2+\|u^n\|_4\|\omega^n\|_2\\ \leq&C_3\tau^{-\frac{1}{4}}\|f\|_2+C_4\|f\|_2^2,\end{aligned}$$

where $C_3>0$, $C_4>0$ are absolute constants. Thus if

$$\frac{1}{\tau}\geq C_3\tau^{-\frac{1}{4}}\|f\|_2+C_4\|f\|_2^2,$$

then clearly we have

$$(1+8\pi^2\tau)\|\omega^{n+2}\|_2^2\leq\|\omega^{n+1}\|_2^2.$$

This then completes the induction proof and we obtain for sufficiently small τ,

$$(1+8\pi^2\tau)\|\omega^{n+1}\|_2^2\leq\|\omega^n\|_2^2,\quad\forall n\geq 0.$$

It follows that

$$\|\omega^n\|_2\leq\gamma_1 e^{-\beta n\tau},\quad\forall n\geq 1,$$

where $\beta>0$ is an absolute constant, and $\gamma_1>0$ depends only on $\|f\|_\infty$.

Step 3. Smoothing estimates. Let $t_0>0$ be chosen sufficiently small (depending only on $\|f\|_\infty$) so that the nonlinear part is dominated by the linear part when $n\tau\leq t_0$. Choose $1\leq n_0\in\mathbb{Z}$ such that $n_0\tau\sim t_0$. By using (8.3) and discrete smoothing estimates, we have

$$\begin{aligned}&\sup_{1\leq n\leq n_0}((n\tau)^{1.6}\||\nabla|^{3.2}\omega^n\|_{20}+(n\tau)^{0.9}\||\nabla|^{1.8}\omega^n\|_{20}+\|\omega^n\|_{20})\leq D_1,\\ &\sup_{n\geq n_0}\|\nabla^4\omega^n\|_\infty\leq D_1e^{-\beta_1 n\tau},\end{aligned}$$

where $D_1>0$ depends only on $\|f\|_\infty$, and $\beta_1>0$ is an absolute constant. It follows that

$$\||\nabla|^{1.8}\omega^n\|_{20}\leq D_2\cdot(n\tau)^{-0.9}e^{-\beta_2 n\tau},\quad \forall n\geq 1,$$

where $D_2>0$ depends only on $\|f\|_\infty$, and $\beta_2>0$ is an absolute constant. By Lemma 8.1, we obtain

$$\tau\|(\mathrm{Id}-\Pi_N)(u^n\cdot\nabla\omega^n)\|_\infty\leq D_3\cdot\tau\cdot N^{-\frac{1}{2}}\cdot(n\tau)^{-0.9}e^{-\beta_2 n\tau},\quad \forall n\geq 1,$$

where $D_3>0$ depends only on $\|f\|_\infty$. By using (8.2), we obtain for $n\geq 1$,

$$\begin{aligned}&\tau\|u^n\cdot\nabla(\omega^{n+1}-\omega^n)\|_\infty\\ \lesssim&\tau^2\|u^n\|_\infty\|\nabla(\Pi_N(u^n\cdot\nabla\omega^n))\|_\infty+\tau^2\|u^n\|_\infty\|\nabla^3\omega^{n+1}\|_\infty\\ \leq&D_4\tau^2\cdot(n\tau)^{-1.6},\end{aligned}$$

where $D_4>0$ depends only on $\|f\|_\infty$.

Step 4. Time-global estimates. Now for $n\geq 1$ we rewrite the equation for ω^{n+1} as

$$(\mathrm{Id}-\tau\Delta)\omega^{n+1}=u^n+\tau u^n\cdot\nabla\omega^{n+1}+\tau u^n\cdot\nabla(\omega^n-\omega^{n+1})+\tau(\mathrm{Id}-\Pi_N)(u^n\cdot\nabla\omega^n).$$

Clearly by a maximum principle argument and the estimates derived in earlier steps, we have

$$\begin{aligned}\|\omega^{n+1}\|_\infty\leq&\|\omega^n\|_\infty+\tau\|u^n\cdot\nabla(\omega^{n+1}-\omega^n)\|_\infty+\tau\|(\mathrm{Id}-\Pi_N)(u^n\cdot\nabla\omega^n)\|_\infty\\ \leq&\|\omega^n\|_\infty+D_4\cdot\tau^2\cdot(n\tau)^{-1.6}+D_3\cdot\tau\cdot N^{-\frac{1}{2}}\cdot(n\tau)^{-0.9}e^{-\beta_2 n\tau},\quad \forall n\geq 1.\end{aligned}$$

Iterating in n, we obtain

$$\|\omega^n\|_\infty\leq\|\omega^1\|_\infty+D_5\tau^{0.4}+D_6N^{-\frac{1}{2}},\quad \forall n\geq 1,$$

where $D_5,D_6>0$ are constants depending only on $\|f\|_\infty$. Finally we observe that

$$\begin{aligned}\|\omega^1\|_\infty\leq&\|\omega^0\|_\infty+\|\tau A\Pi_N\nabla\cdot(u^0\omega^0)\|_\infty\\ \leq&\|\omega^0\|_\infty+C\tau^{0.4}\|f\|_\infty^2,\end{aligned}$$

where $C>0$ is an absolute constant. Our desired result then follows easily. □

Remark 8.1. We should mention that similar to the Burgers case, we can also develop the corresponding analysis for the Fourier collocation method applied to the Navier-Stokes system and many other similar fluid models. All these and related issues will be investigated elsewhere.

Appendix A: some auxiliary estimates

Lemma A.1. *Consider for $z>0$,*

$$F(z)=e^{-z}\int_0^\infty e^{-zt}\Big(t+\frac{t^2}{2}\Big)^{-\frac{1}{2}}dt.$$

Then for $0<z\leq\frac{1}{4}$, we have

$$F(z)=c_1-c_2\log z+f_1(z),$$

where $c_1,c_2>0$ are constants, and

$$|f_1(z)|\lesssim z|\log z|,\quad |f_1'(z)|\lesssim|\log z|.$$

Proof of Lemma A.1. We shall use the notation "NICE" to denote those terms which can be included in c_1 and f_1. It is not difficult to check that

$$\begin{aligned}F(z)=&e^{-z}\int_0^\infty e^{-t}\Big(tz+\frac{t^2}{2}\Big)^{-\frac{1}{2}}dt\\=&e^{-z}\int_0^1 e^{-t}\Big(tz+\frac{t^2}{2}\Big)^{-\frac{1}{2}}dt+\text{NICE}\\=&e^{-z}\int_0^1\Big(tz+\frac{t^2}{2}\Big)^{-\frac{1}{2}}dt+\text{NICE}\\=&\int_0^1\Big(tz+\frac{t^2}{2}\Big)^{-\frac{1}{2}}dt+\text{NICE}.\end{aligned}$$

Now by a simple change of variable, we have

$$\begin{aligned}&\int_0^1(2tz+t^2)^{-\frac{1}{2}}dt=\int_0^{\frac{1}{z}}(2t+t^2)^{-\frac{1}{2}}dt\\=&\int_0^3(2t+t^2)^{-\frac{1}{2}}dt+\int_3^{\frac{1}{z}}t^{-1}\cdot\Big(1+\frac{2}{t}\Big)^{-\frac{1}{2}}dt\\=&\int_0^3(2t+t^2)^{-\frac{1}{2}}dt+\int_3^{\frac{1}{z}}t^{-1}dt+\int_3^\infty t^{-1}\cdot\Big(\Big(1+\frac{2}{t}\Big)^{-\frac{1}{2}}-1\Big)dt\\&-\int_{\frac{1}{z}}^\infty t^{-1}\cdot\Big(\Big(1+\frac{2}{t}\Big)^{-\frac{1}{2}}-1\Big)dt.\end{aligned}$$

Note that

$$\int_0^3(2t+t^2)^{-\frac{1}{2}}dt=2\sinh^{-1}\Big(\sqrt{\frac{3}{2}}\Big)\approx2.0634>\log3\approx1.0986.$$

The desired result then clearly follows. □

Proof of Statement (3) *of Theorem* 2.4, *2D case.* For simplicity we take $\beta=1$. We first consider the case $d=2$. It suffices for us to work with the expression

$$I(x)=\sum_{n\in\mathbb{Z}^2}\int_{|y|_\infty<\frac{1}{2}}F(x-y+n)\frac{\sin M\pi y_1}{\sin\pi y_1}\frac{\sin M\pi y_2}{\sin\pi y_2}dy_1dy_2,$$

where $M=2N+1$, and for $z\neq 0$,

$$F(z)=e^{-|z|}\int_0^\infty e^{-|z|t}\Big(t+\frac{t^2}{2}\Big)^{-\frac{1}{2}}dt.$$

We shall show that for $|x|_\infty\leq\frac{1}{2}$, $I(x)$ has a strictly positive lower bound if M is sufficiently large, namely

$$I(x)\gtrsim\log\Big(1+\frac{1}{|x|}\Big),\quad\forall x\neq 0.$$

Case 1: $|n|_\infty\geq 2$. It is not difficult to check that for $|x|_\infty\leq\frac{1}{2}$, $|y|_\infty<\frac{1}{2}$, the function

$$F_1(y)=\Big(\sum_{|n|_\infty\geq 2}F(x-y+n)\Big)\cdot\frac{y_1}{\sin\pi y_1}\cdot\frac{y_2}{\sin\pi y_2}$$

is C^∞ smooth and has bounded derivatives of all orders. We then have

$$\begin{aligned}
&\int_{|y|_\infty<\frac{1}{2}}F_1(y)\frac{\sin M\pi y_1}{y_1}\frac{\sin M\pi y_2}{y_2}dy\\
=&\int_{|y|_\infty<\frac{1}{2}M}F_1\Big(\frac{y}{M}\Big)\frac{\sin\pi y_1}{y_1}\frac{\sin\pi y_2}{y_2}dy\\
=&\mathcal{O}(M^{-1}(\log M)^2)+\frac{1}{\pi^2}\int_{|y|_\infty<\frac{1}{2}M}F_1\Big(\frac{y}{M}\Big)\frac{1-\cos\pi y_1}{y_1^2}\cdot\frac{1-\cos\pi y_2}{y_2^2}dy\\
=&\mathcal{O}(M^{-1}(\log M)^2)+\frac{1}{\pi^2}F_1(0)\int_{|y|_\infty<\frac{1}{2}M}\frac{1-\cos\pi y_1}{y_1^2}\cdot\frac{1-\cos\pi y_2}{y_2^2}dy\\
=&\mathcal{O}(M^{-1}(\log M)^2)+\frac{1}{\pi^2}F_1(0)\Big(\int_{\mathbb{R}}\frac{1-\cos\pi y_1}{y_1^2}dy_1\Big)^2.
\end{aligned}\tag{A.1}$$

Thus the contribution of this piece is acceptable for us.

Case 2: $|n|_\infty\leq 1$. There are five terms $n=(\pm1,\pm1)$ and $n=(0,0)$. We consider only the case $n=(0,0)$ as the others can be similarly treated. Let $\chi\in C_c^\infty(\mathbb{R}^2)$ be

such that $0\leq\chi(z)\leq 1$ for all z, $\chi(z)=1$ for $|z|\leq\frac{1}{50}$ and $\chi(z)=0$ for $|z|\geq\frac{1}{49}$. By Lemma A.1, we decompose F as

$$\begin{aligned}F(z)=&F(z)\chi(z)+F(z)(1-\chi(z))\\=&-c_2\chi(z)\log|z|+f_1(|z|)\chi(z)+c_1\chi(z)+F(z)(1-\chi(z)).\end{aligned}$$

By a computation similar to that in (A.1), it is not difficult to check that the contribution due to the term $c_1\chi(z)+F(z)(1-\chi(z))$ is acceptable for us. We now consider the contribution due to the term $f_1(|z|)\chi(z)$. For this we need to work with the expression

$$I_1=\int_{|y|_\infty<\frac{1}{2}}f_1(|y-x|)\underbrace{\chi(y-x)\cdot\frac{y_1}{\sin\pi y_1}\cdot\frac{y_2}{\sin\pi y_2}}_{=:\chi_1(y,x)}\cdot\frac{\sin M\pi y_1\sin M\pi y_2}{y_1y_2}dy_1dy_2.$$

We then write

$$I_1=\int_{|y|_\infty<\frac{1}{2}M}f_1\Big(\Big|\frac{y}{M}-x\Big|\Big)\chi_1\Big(\frac{y}{M},x\Big)\cdot\frac{\sin\pi y_1\sin\pi y_2}{y_1y_2}dy.$$

By repeating a similar integration by parts argument in (A.1) and using Lemma A.1, we then only need to control the main error term (the boundary terms are easily controlled)

$$\mathrm{er}_1:=\int_{|y|_\infty<\frac{1}{2}M}\frac{1}{M}\cdot\Big|\log\Big|\frac{y-Mx}{M}\Big|\Big|\cdot\frac{1}{(1+|y_1|)(1+|y_2|)}dy.$$

By the simple 1D inequality (below $|z_1|\lesssim M$, and one can use the fact that $|\log|y_1-z_1||\lesssim\log(2+|y_1|)+\log|z_1|$ for $|y_1-z_1|>1$):

$$\int_{|y_1|<\frac{1}{2}M}|\log|y_1-z_1||\cdot\frac{1}{1+|y_1|}dy_1\lesssim(\log M)^2,$$

we have

$$\mathrm{er}_1\lesssim M^{-1}(\log M)^4.$$

Thus the contribution of I_1 is acceptable for us. Finally we consider the contribution due to the term $-\chi(z)\log|z|$. For this we need to work with the expression

$$I_2=-\int_{|y|_\infty<\frac{1}{2}}\log|y-x|\cdot\underbrace{\chi(y-x)\cdot\frac{y_1}{\sin\pi y_1}\cdot\frac{y_2}{\sin\pi y_2}}_{=:\chi_1(y,x)}\cdot\frac{\sin M\pi y_1\sin M\pi y_2}{y_1y_2}dy_1dy_2.$$

Consider $0<|x|\ll 1$. We have

$$\begin{aligned}
I_2=&-\int_{|y|_\infty<\frac{1}{2}M}\log|\frac{y}{M}-x|\cdot\chi_1\Big(\frac{y}{M},x\Big)\cdot\frac{\sin\pi y_1\sin\pi y_2}{y_1y_2}dy\\
=&-\int_{|y|_\infty<\frac{1}{2}M}\log|y-Mx|\cdot\chi_1\Big(\frac{y}{M},x\Big)\cdot\frac{\sin\pi y_1\sin\pi y_2}{y_1y_2}dy\\
&+\log M\int_{|y|_\infty<\frac{1}{2}M}\chi_1(\frac{y}{M},x)\cdot\frac{\sin\pi y_1\sin\pi y_2}{y_1y_2}dy\\
=&-\int_{|y|_\infty<\frac{1}{2}M}\log|y-Mx|\cdot\chi_1\Big(\frac{y}{M},x\Big)\chi(y-Mx)\cdot\frac{\sin\pi y_1\sin\pi y_2}{y_1y_2}dy \qquad \text{(A.2)}\\
&-\int_{|y|_\infty<\frac{1}{2}M}(\log|y-Mx|)(1-\chi(y-Mx))\cdot\chi_1\Big(\frac{y}{M},x\Big)\cdot\frac{\sin\pi y_1\sin\pi y_2}{y_1y_2}dy \qquad \text{(A.3)}\\
&+(\log M)\chi_1(0,x)\int_{|y|_\infty<\frac{1}{2}M}\frac{\sin\pi y_1\sin\pi y_2}{y_1y_2}dy \qquad \text{(A.4)}\\
&+\log M\int_{|y|_\infty<\frac{1}{2}M}\Big(\chi_1\Big(\frac{y}{M},x\Big)-\chi_1(0,x)\Big)\cdot\frac{\sin\pi y_1\sin\pi y_2}{y_1y_2}dy. \qquad \text{(A.5)}
\end{aligned}$$

For (A.2), since $\chi(y-Mx)$ is compactly supported, it follows easily that

$$|\text{(A.2)}|\lesssim 1.$$

For (A.3), by successive integration by parts, we have

$$\begin{aligned}
\text{(A.3)}=&-(\log|Mx|)(1-\chi(Mx))\cdot\chi_1(0,x)\cdot\int_{|y|_\infty<\frac{1}{2}M}\frac{(1-\cos\pi y_1)(1-\cos\pi y_2)}{\pi^2y_1^2y_2^2}dy\\
&+\mathcal{O}(1).
\end{aligned}$$

We can rewrite (A.4) as

$$\begin{aligned}
\text{(A.4)}=&(\log M)\chi_1(0,x)\int_{|y|_\infty<\frac{1}{2}M}\frac{(1-\cos\pi y_1)(1-\cos\pi y_2)}{\pi^2y_1^2y_2^2}dy\\
&+\mathcal{O}(M^{-1}(\log M)^5).
\end{aligned}$$

For (A.5), one can use integration by parts to obtain that

$$|\text{(A.5)}|\lesssim M^{-1}(\log M)^5.$$

Collecting the estimates, it is then not difficult to check that for $0<|x|\ll 1$ and M sufficiently large,

$$I_2\geq -c_1\log|x|,$$

where $c_1>0$ is an absolute constant.

Next consider the case $|x|\sim 1$. In this case we note that $M|x|=\mathcal{O}(M)$. In estimating (A.2), we note that when $|y-Mx|\lesssim 1$, we have $|y|=\mathcal{O}(M)$ which implies that

$$|(\text{A.2})|=\mathcal{O}(M^{-1}).$$

On the other hand, it is not difficult to check that

$$\begin{aligned}&(\text{A.3})\\ =&-(\log|Mx|)(1-\chi(Mx))\cdot\chi_1(0,x)\cdot\int_{|y|_\infty<\frac{1}{2}M}\frac{(1-\cos\pi y_1)(1-\cos\pi y_2)}{\pi^2y_1^2y_2^2}dy\\ &+\mathcal{O}(M^{-1}(\log M)^5).\end{aligned}$$

Here when estimating the error terms, we used the fact that for $M|x|=\mathcal{O}(M)$,

$$\int_{1\le|y_1|,|y_2|<M}\frac{1}{1+|y-Mx|}\frac{1}{(1+|y_1|)(1+|y_2|)}dy\lesssim M^{-1}(\log M)^2.$$

Collecting the estimates, we obtain that for the case $|x|\sim 1$,

$$I_2\ge -c_2(\log|x|)\chi(x)+\mathcal{O}(M^{-1}(\log M)^5),$$

where c_2 is an absolute constant and we assume M is sufficiently large. Thus for all $x\neq 0$ with $|x|_\infty<\frac{1}{2}$, we have

$$I_2\ge -c_3(\log|x|)\chi(x)+\mathcal{O}(M^{-1}(\log M)^5).$$

One need not worry about the case $\chi(x)=0$ (i.e., when $|x|\ge\frac{1}{50}$), since by (A.1), the main order is a constant.

Collecting all the estimates from Case 1 and Case 2, we then obtain the desired conclusion for 2D. □

Appendix B: estimate of $e^{\nu^2t\partial_{xx}}Q_NU^0$

Recall that

$$(e^{\nu^2t\partial_{xx}}Q_NU^0)(t,x)=\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\le\frac{N}{2}}\Big(\frac{1}{N}\sum_{j=0}^{N-1}U_j^0e^{-2\pi i\frac{kj}{N}}e^{-(2\pi k)^2\nu^2t}\Big)e^{2\pi ik\cdot x}\Big).$$

We need to bound

$$\sup_{0\le l\le N-1}\Big|(e^{\nu^2t\partial_{xx}}Q_NU^0)\Big(t,\frac{l}{N}\Big)\Big|$$

under the assumption that $\|U^0\|_\infty\le 1$. Clearly we have

$$\sup_{0\le l\le N-1}\Big|(e^{\nu^2t\partial_{xx}}Q_NU^0)\Big(t,\frac{l}{N}\Big)\Big|\le\frac{1}{N}\sum_{j=0}^{N-1}\Big|\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\le\frac{N}{2}}e^{-4\pi^2\nu^2tk^2+2\pi ik\cdot\frac{j}{N}}\Big)\Big|=:A_{N,t}.$$

Below we shall write $X=O(Y)$ if $|X|\le CY$ and $C>0$ is a constant depending only on ν.

Theorem B.1. *It holds that*

$$A_{N,t}\le\begin{cases}1+\mathcal{O}(N^{-1}), & \text{if } N^2\nu^2t\ge 100\log N,\\ 1+\epsilon_a+\mathcal{O}(N^{-\frac{1}{3}}\sqrt{\log N}), & \text{if } N^2\nu^2t<100\log N.\end{cases}$$

Here $0<\epsilon_a<0.01$ *is an absolute constant. Furthermore, we have for* $t=t_0=\frac{1}{4}N^{-2}\nu^{-2}$,

$$A_{N,t_0}>1.001+\mathcal{O}(N^{-\frac{1}{3}}\sqrt{\log N}).$$

The rest of this section is devoted to the proof of Theorem B.1. To ease the notation we shall assume $\nu=1$.

First observe that if $N^2t\ge 100\log N$, then

$$\begin{aligned}A_{N,t}&\le 1+\sum_{|k|\ge\frac{N}{2}}e^{-4\pi^2tk^2}\\&\le 1+\int_{|k|\ge\frac{N}{2}-1}e^{-4\pi^2tk^2}dk\le 1+\mathcal{O}(N^{-1}).\end{aligned}$$

Thus we only need to consider the regime $N^2t<100\log N$. Denote $k_0=2N\sqrt{t}$ and note that $k_0\le 20\sqrt{\log N}$. Denote

$$Q_j=\sum_{-\frac{N}{2}<k\le\frac{N}{2}}e^{-4\pi^2tk^2+2\pi ik\cdot\frac{j}{N}}.$$

It is easy to check that

$$\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\le\frac{N}{2}}e^{2\pi ik\cdot x}\Big)=\frac{\sin N\pi x}{\tan\pi x}.$$

Then

$$\begin{aligned}Q_j=&\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}e^{-\frac{(\frac{j}{N}-y)^2}{4t}}\frac{\sin N\pi y}{\tan\pi y}dy\\=&\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-y^2}\frac{\sin 2N\sqrt{t}\pi y}{\tan\pi(\frac{j}{N}+2\sqrt{t}y)}dy(-1)^j\\=&\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-y^2}\frac{\sin\pi k_0 y}{\tan\pi(\frac{j+k_0y}{N})}dy(-1)^j \qquad \text{(B.1)}\\=&\frac{1}{\sqrt{\pi}k_0}\int_{\mathbb{R}}e^{-\frac{y^2}{k_0^2}}\frac{\sin\pi y}{\tan\pi\frac{j+y}{N}}dy(-1)^j. \qquad \text{(B.2)}\end{aligned}$$

We shall use either (B.1) or (B.2). Note that

$$|Q_j|=|Q_{N-j}|,\quad \forall 1\le j\le N-1.$$

Case 1: The regime $N^{\frac{1}{3}}<j<N-N^{\frac{1}{3}}$. This is the tail piece. Observe that

$$\int_{|y|\ge 100\sqrt{\log N}}e^{-y^2}\Big|\frac{\sin\pi k_0y}{\tan\pi\frac{j+k_0y}{N}}\Big|dy\le N^{-5}.$$

Then we only need to estimate

$$\begin{aligned}&\Big|\int_{0<y\le 100\sqrt{\log N}}e^{-y^2}\sin\pi k_0y\Big(\cot\pi\frac{j+k_0y}{N}-\cot\pi\frac{j-k_0y}{N}\Big)dy\Big|\\\lesssim&\frac{N^2}{j_1^2}\cdot\frac{k_0}{N}=Nj_1^{-2}k_0,\quad j_1=\min\{j,N-j\}.\end{aligned}$$

Then

$$\frac{1}{N}\sum_{N^{\frac{1}{3}}<j<N-N^{\frac{1}{3}}}|Q_j|\lesssim\sum_{N^{\frac{1}{3}}<j<N-N^{\frac{1}{3}}}j_1^{-2}k_0\lesssim N^{-\frac{1}{3}}\sqrt{\log N}.$$

Thus the regime $N^{\frac{1}{3}}<j<N-N^{\frac{1}{3}}$ is OK for us.

Case 2: The regime $0\le j\le N^{\frac{1}{3}}$. This is the main piece. Note that since $Q_j=Q_{N-j}$, the regime $N-N^{\frac{1}{3}}\le j\le N-1$ does not need any further analysis. Now for $0\le j\le N^{\frac{1}{3}}$, by using (B.2), we have

$$|Q_j|=\frac{1}{\sqrt{\pi}k_0}\Big|\int_{\mathbb{R}}e^{-\frac{(y+j)^2}{k_0^2}}\frac{\sin\pi y}{\tan\pi\frac{y}{N}}dy\Big|.$$

Observe that

$$k_0^{-1}\Big|\int_{|y|\geq N^{\frac{1}{2}}} e^{-\frac{(y+j)^2}{k_0^2}}\cdot\frac{\sin\pi y}{\tan\pi\frac{y}{N}}dy\Big|\lesssim e^{-cN}.$$

Note that for $|x|\ll 1$, we have

$$\Big|\cot x-\frac{1}{x}\Big|\lesssim x.$$

We obtain

$$\frac{1}{N}\sum_{j\leq N^{\frac{1}{3}}}k_0^{-1}\Big|\int_{|y|\leq N^{\frac{1}{2}}} e^{-\frac{(y+j)^2}{k_0^2}}\cdot\frac{|y|}{N}dy\Big|\lesssim\frac{1}{N}\cdot N^{\frac{1}{3}}\cdot N^{-\frac{1}{2}}\lesssim N^{-\frac{7}{6}}.$$

We then obtain the main piece

$$\begin{aligned}&\frac{1}{N}\sum_{j\leq N^{\frac{1}{3}}}k_0^{-1}\Big|\int_{|y|\leq N^{\frac{1}{2}}} e^{-\frac{(y+j)^2}{k_0^2}}\cdot\frac{\sin\pi y}{y}Ndy\Big|\\=&\sum_{j\leq N^{\frac{1}{3}}}k_0^{-1}\Big|\int_{\mathbb{R}} e^{-\frac{(y+j)^2}{k_0^2}}\cdot\frac{\sin\pi y}{y}dy-\int_{|y|>N^{\frac{1}{2}}} e^{-\frac{(y+j)^2}{k_0^2}}\cdot\frac{\sin\pi y}{y}dy\Big|\\=&\Big(\sum_{j\leq N^{\frac{1}{3}}}k_0^{-1}\Big|\int_{\mathbb{R}} e^{-\frac{(y+j)^2}{k_0^2}}\cdot\frac{\sin\pi y}{y}dy\Big|\Big)+\mathcal{O}(e^{-cN}).\end{aligned}$$

Thus

$$\frac{1}{N}\sum_{j\leq N^{\frac{1}{3}}}|Q_j|=\pi^{-\frac{3}{2}}k_0^{-1}\sum_{j\leq N^{\frac{1}{3}}}\Big|\int_{\mathbb{R}} e^{-\frac{(y+j)^2}{k_0^2}}\cdot\frac{\sin\pi y}{y}dy\Big|+\mathcal{O}(N^{-\frac{7}{6}}).$$

Fix j and consider

$$F_j(1)=\pi^{-\frac{3}{2}}k_0^{-1}\int_{\mathbb{R}} e^{-\frac{(y+j)^2}{k_0^2}}\frac{\sin\pi y}{y}dy=\int_0^1 F_j'(s)ds,$$

where

$$\begin{aligned}F_j'(s)=&\pi^{-\frac{1}{2}}k_0^{-1}\mathrm{Re}\Big(\int_{\mathbb{R}} e^{-\frac{(y+j)^2}{k_0^2}}e^{i\pi sy}dy\Big)\\=&e^{-\frac{1}{4}\pi^2k_0^2s^2}\cos\pi sj.\end{aligned}$$

It follows that

$$\pi^{-\frac{3}{2}}k_0^{-1}\int_{\mathbb{R}}e^{-\frac{(y+j)^2}{k_0^2}}\cdot\frac{\sin\pi y}{y}dy=\int_0^1 e^{-\frac{1}{4}\pi^2k_0^2s^2}\cos\pi jsds.$$

Collecting the estimates, we then obtain for $k_0\leq 20\sqrt{\log N}$,

$$\begin{aligned}&\frac{1}{N}\sum_{j=0}^{N-1}|Q_j|\\ =&\mathcal{O}(N^{-\frac{1}{3}}\sqrt{\log N})+\sum_{|j|\leq N^{\frac{1}{3}}}\Big|\underbrace{\int_0^1 e^{-\frac{1}{4}\pi^2k_0^2s^2}\cos\pi jsds}_{=:\beta_j}\Big|.\end{aligned}$$

Here β_j depends on k_0 but we suppress this explicit dependence for simplicity of notation. By using successive integration by parts (three times), it is not difficult to check that uniformly for all $k_0\leq 20\sqrt{\log N}$,

$$|\beta_j|\lesssim j^{-2},\quad \forall |j|\geq N^{\frac{1}{3}}.$$

Thus we can rewrite

$$\frac{1}{N}\sum_{j=0}^{N-1}|Q_j|=\mathcal{O}(N^{-\frac{1}{3}}\sqrt{\log N})+\sum_{j\in\mathbb{Z}}|\beta_j|.$$

We now focus on analyzing the quantity

$$A(k_0)=\sum_{j\in\mathbb{Z}}|\beta_j|.$$

First we consider $k_0=1$. For $k_0=1$, a rigorous numerical computation gives

$$\begin{aligned}&\beta_0\approx 0.54934,\\ &\beta_1=\beta_{-1}\approx 0.219568,\\ &\beta_2=\beta_{-2}\approx 0.0031275,\\ &\beta_3=\beta_{-3}\approx 0.00413676.\end{aligned}$$

It follows that

$$A(1)\geq\sum_{|j|\leq 3}\beta_j\geq 1.002.$$

Next we shall give more precise bounds on $A(k_0)$ depending on the size of k_0.

Remark B.1. Observe that

$$\frac{1}{2}\int_{-1}^{1} e^{-\frac{1}{4}\pi^2 k_0^2 s^2}\cos j\pi s ds = \int_{|s|\leq\frac{1}{2}} e^{-\pi^2 k_0^2 s^2}\cos 2\pi j s ds.$$

We may then rewrite

$$A(k_0) = \sum_{j\in\mathbb{Z}} \Big| \int_{|s|\leq\frac{1}{2}} e^{-\pi^2 k_0^2 s^2}\cos 2\pi j s ds \Big|.$$

Interestingly, this is exactly the l^1 norm of the Fourier coefficients of the function $e^{-\pi^2 k_0^2 x^2}$ on the periodic torus $\mathbb{T}=[-\frac{1}{2},\frac{1}{2})$.

Remark B.2. Interestingly, if we consider

$$S = \sum_{j\in\mathbb{Z}} \Big| \int_{|s|\leq\frac{1}{2}} e^{-\beta|s|}\cos 2\pi j s ds \Big| =: \sum_{j\in\mathbb{Z}} |P_j|,$$

we can obtain the closed form of P_j. Indeed we have

$$\begin{aligned} P_j &= \int_{|s|\leq\frac{1}{2}} e^{-\beta|s|}\cos 2\pi j s ds = 2\mathrm{Re}\Big(\int_0^{\frac{1}{2}} e^{-s(\beta-2\pi i j)} ds\Big) \\ &= 2\mathrm{Re}\left(\frac{e^{-s(\beta-2\pi i j)}}{-(\beta-2\pi i j)}\Big|_{s=0}^{\frac{1}{2}}\right) \\ &= 2\beta\frac{e^{-\frac{1}{2}\beta}(-1)^j - 1}{-(\beta^2+(2\pi j)^2)} \\ &= 2\beta\frac{1-e^{-\frac{1}{2}\beta}(-1)^j}{\beta^2+(2\pi j)^2} \geq 0. \end{aligned}$$

Thus $S=1$.

B.1 Estimate of $A(k_0)$ when k_0 is large

We first give a bound on $A(k_0)$ when $k_0\geq 1$. Rewrite

$$A(k_0) = \sum_{j\in\mathbb{Z}} \Big| \int_{|s|\leq\frac{1}{2}} e^{-as^2}\cos 2\pi j s ds \Big|,$$

where $a=\pi k_0^2\geq\pi$. Denote

$$f_0(s) = \sum_{n\in\mathbb{Z}} e^{-a(s+n)^2}.$$

Clearly

$$\int_{|s|\leq\frac{1}{2}} f_0(s)\cos 2\pi jsds = \int_{\mathbb{R}} e^{-as^2}\cos 2\pi jsds \geq 0,$$

and thus

$$\sum_{j\in\mathbb{Z}}\left|\int_{|s|\leq\frac{1}{2}} f_0(s)\cos 2\pi jsds\right| = \sum_{j\in\mathbb{Z}}\int_{|s|\leq\frac{1}{2}} f_0(s)\cos 2\pi jsds = f_0(0) = \sum_{n\in\mathbb{Z}} e^{-an^2}.$$

Now for $|n|\geq 1$ and $|j|\geq 1$, we have

$$\begin{aligned}
&-\int_{|s|\leq\frac{1}{2}} e^{-a(s+n)^2}\cos 2\pi jsds = \frac{1}{2\pi j}\int_{|s|\leq\frac{1}{2}} \frac{d}{ds}\left(e^{-a(s+n)^2}\right)\sin 2\pi jsds \\
=&(2\pi j)^{-2}\frac{d}{ds}\left(e^{-a(s+n)^2}\right)(-\cos 2\pi js)\Big|_{s=-\frac{1}{2}}^{\frac{1}{2}} + (2\pi j)^{-2}\int_{|s|\leq\frac{1}{2}} \frac{d^2}{ds^2}\left(e^{-a(s+n)^2}\right)\cos 2\pi jsds.
\end{aligned}$$

Now consider the function e^{-x^2}. Clearly

$$(e^{-x^2})'' = e^{-x^2}(4x^2-2) \geq 0, \qquad \forall |x| \geq \frac{1}{\sqrt{2}}.$$

It follows that $\frac{d^2}{ds^2}e^{-a(s+n)^2} \geq 0$ since $(n+s)^2 \geq \frac{1}{4}$ and $a \geq \pi$. Thus

$$\begin{aligned}
&(2\pi j)^{-2}\left|\int_{|s|\leq\frac{1}{2}} \frac{d^2}{ds^2}\left(e^{-a(s+n)^2}\right)\cos 2\pi jsds\right| \\
\leq&(2\pi j)^{-2}2a\left|e^{-a(n+\frac{1}{2})^2}\left(n+\frac{1}{2}\right) - e^{-a(n-\frac{1}{2})^2}\left(n-\frac{1}{2}\right)\right|, \\
&(2\pi j)^{-2}\left|\frac{d}{ds}\left(e^{-a(s+n)^2}\right)(-\cos 2\pi js)\Big|_{s=-\frac{1}{2}}^{\frac{1}{2}}\right| \\
\leq&(2\pi j)^{-2}2a\left|e^{-a(n+\frac{1}{2})^2}\left(n+\frac{1}{2}\right) - e^{-a(n-\frac{1}{2})^2}\left(n-\frac{1}{2}\right)\right|.
\end{aligned}$$

It follows that

$$\begin{aligned}
A(k_0) \leq& \sum_{n\in\mathbb{Z}} e^{-an^2} + \sum_{|j|\geq 1,|n|\geq 1} 2(2\pi j)^{-2}2a\left|e^{-a(n+\frac{1}{2})^2}\left(n+\frac{1}{2}\right) - e^{-a(n-\frac{1}{2})^2}\left(n-\frac{1}{2}\right)\right| \\
&+ \int_{|s|\leq\frac{1}{2}} \sum_{|n|\geq 1} e^{-a(n+s)^2}ds
\end{aligned}$$

$$\leq 1+2\sum_{n\geq 1}e^{-an^2}+\int_{|s|\geq\frac{1}{2}}e^{-as^2}ds+\frac{a}{3}\sum_{|n|\geq 1}\left|e^{-a(n+\frac{1}{2})^2}\left(n+\frac{1}{2}\right)-e^{-a(n-\frac{1}{2})^2}\left(n-\frac{1}{2}\right)\right|$$
$$\leq 1+2\sum_{n\geq 1}e^{-an^2}+\frac{2}{\sqrt{a}}\int_{s\geq\frac{1}{2}\sqrt{a}}e^{-s^2}ds+\frac{a}{3}e^{-\frac{a}{4}},\quad a=\pi k_0^2\geq\pi.$$

This bound is particularly effective when a is large.

Remark B.3. We should mention that it is possible to extract more information on the Fourier coefficients β_j in the regime $k_0\gtrsim 1$. Consider for $b=\frac{1}{4}\pi k_0^2\geq\frac{1}{2}$,

$$A(k_0)=\sum_{j\in\mathbb{Z}}|\beta_j|,\quad \beta_j=\int_0^1 e^{-bs^2}\cos\pi jsds.$$

We rewrite

$$\begin{aligned}\beta_j&=\int_0^\infty e^{-bs^2}\cos\pi jsds-\int_1^\infty e^{-bs^2}\cos\pi jsds\\&=\frac{1}{2}\sqrt{\pi}b^{-\frac{1}{2}}e^{-\frac{\pi^2j^2}{4b}}-\int_1^\infty e^{-bs^2}\cos\pi jsds.\end{aligned}$$

Clearly for $j\neq 0$,

$$\begin{aligned}&-\int_1^\infty e^{-bs^2}\cos\pi jsds\\&=(j\pi)^{-2}\Big(\partial_s(e^{-bs^2})\big|_{s=1}\cdot(-1)^j+\int_1^\infty\partial_{ss}(e^{-bs^2})\cos j\pi sds\Big).\end{aligned}$$

Observe that $\partial_{ss}e^{-bs^2}=e^{-bs^2}2b(2bs^2-1)\geq 0$ for $s\geq 1$, $b\geq\frac{1}{2}$. Thus $\beta_j\geq 0$ if j is odd.

B.2 Estimate of $A(k_0)$ when k_0 is small

In this subsection we give the precise form of $A(k_0)$ when k_0 is small. First we rewrite

$$A(k_0)=\sum_{j\in\mathbb{Z}}|\beta_j|,\quad \beta_j=\int_0^1 e^{-bs^2}\cos\pi jsds,$$

where $b=\frac{1}{4}\pi^2k_0^2\leq\frac{1}{2}$. Clearly for $j\geq 1$,

$$\begin{aligned}-\beta_j=&-\int_0^1 e^{-bs^2}\cos\pi jsds\\ =&(j\pi)^{-1}\int_0^1\partial_s(e^{-bs^2})\sin j\pi sds\\ =&(j\pi)^{-2}\partial_s(e^{-bs^2})(-\cos j\pi s)\Big|_{s=0}^1+(j\pi)^{-2}\int_0^1\partial_{ss}(e^{-bs^2})\cos j\pi sds.\end{aligned}$$

Now denote

$$g(s)=(j\pi)^{-2}(-1)\partial_s(e^{-bs^2})$$

and note that $g'(s)\geq 0$ whenever $0<b\leq\frac{1}{2}$ and $0\leq s\leq 1$. Then

$$\beta_j=g(1)(-1)^{j+1}+\int_0^1 g'(s)\cos j\pi sds.$$

It follows that $\beta_j\leq 0$ for j even, and $\beta_j\geq 0$ for j odd. Note that $\beta_0>0$.

We then write

$$\begin{aligned}A(k_0)=&\beta_0+\sum_{j\text{ odd}}\beta_j-\sum_{0\neq j\text{ even}}\beta_j\\ =&2\beta_0+\sum_{j\in\mathbb{Z}}\beta_j-2\Big(\beta_0+\sum_{0\neq j\text{ even}}\beta_j\Big)\\ =&2\int_0^1 e^{-bs^2}ds+1-(1+e^{-b})\\ =&2\int_0^1 e^{-bs^2}ds-e^{-b}.\end{aligned}$$

Thus for $0<b=\frac{1}{4}\pi^2k_0^2\leq\frac{1}{2}$,

$$A(k_0)=2\int_0^1 e^{-bs^2}ds-e^{-b}.$$

Remark B.4. We explain how to rigorously compute $\sum_{l\in\mathbb{Z}}\beta_l$ and $\sum_{l\in\mathbb{Z}}\beta_{2l}$ as follows. Firstly we rewrite

$$\beta_l=\int_{|s|\leq\frac{1}{2}}f(s)\cos 2\pi lsds,$$

where $f(s)=e^{-as^2}$. Recall the Dirichlet kernel

$$D_N(x)=\sum_{|j|\leq N} e^{2\pi ijx}=\frac{\sin(2N+1)\pi x}{\sin\pi x}.$$

Clearly

$$\begin{aligned}
&\int_{|x|\leq\frac{1}{2}}(f(x)-f(0))D_N(x)dx\\
=&\frac{f(x)-f(0)}{\sin\pi x}\frac{1}{2N+1}(-\cos(2N+1)\pi x)\Big|_{x=-\frac{1}{2}}^{\frac{1}{2}}\\
&+\frac{1}{2N+1}\int_{|x|\leq\frac{1}{2}}\frac{d}{dx}\Big(\frac{f(x)-f(0)}{\sin\pi x}\Big)\cos(2N+1)\pi x dx\\
=&\mathcal{O}\Big(\frac{1}{N}\Big).
\end{aligned}$$

Thus,

$$\int_{|x|\leq\frac{1}{2}} f(x)D_N(x)dx\to f(0)\quad\text{as}\quad N\to\infty.$$

Alternatively if one does not need the quantitative convergence rate, one can directly obtain

$$\lim_{N\to\infty}\int_{|x|\leq\frac{1}{2}}\frac{f(x)-f(0)}{\sin\pi x}\sin(2N+1)\pi x dx=0$$

since $(f(x)-f(0))/\sin\pi x\in L^1((-\frac{1}{2},\frac{1}{2}))$.

Next we consider

$$\begin{aligned}
\int_{|x|\leq\frac{1}{2}} f(x)D_N(2x)dx=&\frac{1}{2}\int_{|x|\leq 1} f\Big(\frac{x}{2}\Big)D_N(x)dx\\
=&\frac{1}{2}\int_{|x|\leq\frac{1}{2}} f\Big(\frac{x}{2}\Big)D_N(x)dx+\int_{\frac{1}{2}\leq x\leq 1} f\Big(\frac{x}{2}\Big)D_N(x)dx\\
=&\frac{1}{2}\int_{|x|\leq\frac{1}{2}} f\Big(\frac{x}{2}\Big)D_N(x)dx+\int_0^{\frac{1}{2}} f\Big(\frac{1-x}{2}\Big)D_N(x)dx.
\end{aligned}$$

Now consider

$$f\Big(\frac{1-x}{2}\Big)=f\Big(\frac{1}{2}\Big)+\tilde{f}(x).$$

Clearly

$$\begin{aligned}\int_0^{\frac{1}{2}}\tilde{f}(x)D_N(x)dx&=\int_0^{\frac{1}{2}}\frac{\tilde{f}(x)}{\sin\pi x}\sin(2N+1)\pi x dx\\&=\frac{\tilde{f}(x)}{\sin\pi x}\frac{1}{2N+1}\frac{1}{\pi}(-\cos(2N+1)\pi x)\Big|_0^{\frac{1}{2}}+\mathcal{O}\Big(\frac{1}{N}\Big)=\mathcal{O}\Big(\frac{1}{N}\Big).\end{aligned}$$

Thus we obtain

$$\int_{|x|\leq\frac{1}{2}}f(x)D_N(2x)dx=\frac{1}{2}\Big(f(0)+f\Big(\frac{1}{2}\Big)\Big)+\mathcal{O}\Big(\frac{1}{N}\Big).$$

Remark B.5. More generally, suppose $a<b$ and let $(n_j)_{j=0}^L$ be all the possible integer points in the interval $[a,b]$ listed in the ascending order. If a is an integer then $n_0=a$ and similarly if b is an integer then $n_L=b$. Suppose f is a continuous function on $[a,b]$ such that

$$\frac{f(x)-\tilde{f}(x)}{\sin\pi x}\in L^1((a,b)),$$

where

$$\tilde{f}(x)=\sum_{j=0}^{L}f(n_j)\phi(x-n_j).$$

Here $\phi\in C_c^\infty(\mathbb{R})$ is such that $\phi(x)\equiv1$ for $|x|\leq\frac{1}{4}$ and $\phi(x)\equiv0$ for $|x|\geq\frac{1}{3}$. If there are no integers in $[a,b]$ then we define $\tilde{f}(x)\equiv0$. Then we have

$$\begin{aligned}&\lim_{N\to\infty}\int_a^b f(x)\frac{\sin(2N+1)\pi x}{\sin\pi x}dx\\&=\begin{cases}0, & \text{if } [a,b] \text{ contains no integers,}\\ \displaystyle\sum_{j=0}^{L}f(n_j), & \text{if } [a,b] \text{ contain some integers, but } a \text{ and } b \text{ are not integers,}\\ \displaystyle\frac{1}{2}f(a)+\sum_{j=1}^{L}f(n_j), & \text{if } a\in\mathbb{Z},\ b\notin\mathbb{Z},\\ \displaystyle\frac{1}{2}f(b)+\sum_{j=0}^{L-1}f(n_j), & \text{if } b\in\mathbb{Z}, \text{ but } a\notin Z,\\ \displaystyle\frac{1}{2}(f(a)+f(b))+\sum_{j=1}^{L-1}f(n_j), & \text{if } a\in\mathbb{Z} \text{ and } b\in\mathbb{Z}.\end{cases}\end{aligned}$$

Note that the above derivation is a natural extension of the usual identity for Dirac comb:

$$\sum_{j\in\mathbb{Z}}\cos 2\pi js=\lim_{N\to\infty}\sum_{|j|\leq N}\cos 2\pi js=\sum_{n\in\mathbb{Z}}\delta(s+n)\quad\text{in }\ \mathcal{S}'(\mathbb{R}).$$

For a function $f:\mathbb{T}\to\mathbb{R}$, we denote

$$(e^{\nu^2t\partial_{xx}}Q_Nf)(t,x)=\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}\Big(\frac{1}{N}\sum_{j=0}^{N-1}f\Big(\frac{j}{N}\Big)e^{-2\pi i\frac{kj}{N}}e^{-(2\pi k)^2\nu^2t}\Big)e^{2\pi ik\cdot x}\Big).$$

Theorem B.2. *Let $\nu>0$ and identify $\mathbb{T}=[0,1)$. There exists a function $f:\mathbb{T}\to\mathbb{R}$ with $\|f\|_\infty\leq 1$ and satisfying f is continuous at $[0,\frac{1}{4})\bigcup(\frac{1}{4},1)$ (i.e., f is continuous at all $\frac{1}{4}\neq x\in\mathbb{T}$) such that the following hold: for a sequence of even numbers $N_m\to\infty$, $t_m=\frac{1}{4}\nu^{-2}N_m^{-2}$ and $x_m=j_m/N_m$ with $0\leq j_m\leq N_m-1$, we have*

$$\Big|(e^{\nu^2t\partial_{xx}}Q_{N_m}f)(t_m,x_m)\Big|\geq 1+\epsilon_*,\tag{B.3}$$

where $\epsilon_*>0$ is an absolute constant.

Let $u_m(t)\in\mathbb{R}^{N_m}$ solve

$$\begin{cases}\dfrac{d}{dt}u_m=\nu^2\Delta_hu_m+u_m-(u_m)^{\cdot 3},\\ u_m\big|_{t=0}=v_m,\end{cases}$$

where $(v_m)_j=f(\frac{j}{N_m})$ for all $0\leq j\leq N_m-1$. Then

$$|u_m(t_m,x_m)|\geq 1+\frac{1}{2}\epsilon_*,\quad\forall m\geq 1.\tag{B.4}$$

Remark B.6. The point $x_*=\frac{1}{4}$ is for convenience only and we may replace it by any other point $x_*\in\mathbb{T}$.

Proof. We begin by noting that (B.4) follows from (B.3) and a perturbation argument (as in Theorem 5.4). We now show (B.3). With no loss we assume $\nu=1$. Suppose f satisfies $\|f\|_\infty\leq 1$. Then for $N\geq 10$, $\tau_N=\frac{1}{4}N^{-2}$ and $0\leq l\leq N-1$, by using the proof of Theorem B.1, we have

$$\begin{aligned}(e^{t\partial_{xx}}Q_Nf)(\tau_N,\frac{l}{N})&=\sum_{j=0}^{N-1}f\Big(\frac{l-j}{N}\Big)\cdot\frac{1}{N}\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}e^{-4\pi^2k^2\tau_N}e^{2\pi ik\frac{j}{N}}\Big)\\&=\mathcal{O}(N^{-\frac{1}{3}}\sqrt{\log N})+f\Big(\frac{l}{N}\Big)c_0+\sum_{1\leq|j|\leq N^{\frac{1}{3}}}f\Big(\frac{l+j}{N}\Big)c_j,\end{aligned}$$

where

$$c_j=\int_0^1 e^{-\frac{1}{16}\pi^2 s^2}\cos(\pi j s)ds(-1)^j.$$

By the computation in the proof of Theorem B.1, we have

$$\sum_{|j|\leq 3}|c_j|\geq 1.002.$$

Thus if f satisfies

$$f\Big(\frac{l+j}{N}\Big)=\mathrm{sgn}(c_j)\quad \text{for}\quad |j|\leq 3,\quad \text{and}\quad f\Big(\frac{l+j}{N}\Big)=0\quad \text{for}\quad 4\leq|j|\leq N^{\frac{1}{3}},$$

then

$$(e^{t\partial_{xx}}Q_N f)\Big(\tau_N,\frac{l}{N}\Big)\geq 1.002+\mathcal{O}(N^{-\frac{1}{3}}\sqrt{\log N}).$$

Now for $m=1,2,3,\cdots$, define

$$y_m=\frac{1}{4}+\frac{1}{m+10}.$$

We may then choose large N_m and $0\leq l_m\leq N_m-1$ such that

$$\Big|\frac{l_m}{N_m}-y_m\Big|\ll\frac{1}{m^2}.$$

By a suitable interpolation, it is then obvious to define the function f such that

$$f\Big(\frac{l_m+j}{N_m}\Big)=\mathrm{sgn}(c_j)$$

for $|j|\leq 3$ and for all m satisfying the desired properties. □

To state the next result we need some notation. Suppose $f:\mathbb{T}\to\mathbb{R}$ is continuous. Then we denote the modulus of continuity $\omega_f:[0,1]\to[0,\infty)$ as

$$\omega_f(\delta)=\max_{|x-y|\leq\eta}|f(x)-f(y)|, \tag{B.5}$$

where $|x-y|$ denotes the distance of $x,\ y\in\mathbb{T}$. For $0<\alpha<1$, we say $f:\mathbb{T}\to\mathbb{R}$ is C^α-continuous if

$$\sup_{0<|x-y|\leq 1}\frac{|f(x)-f(y)|}{|x-y|^\alpha}<\infty. \tag{B.6}$$

Theorem B.3. *Let $\nu>0$. Suppose $f:\mathbb{T}\to\mathbb{R}$ is continuous and $\|f\|_\infty\leq 1$. Then*

$$\sup_{t\geq 0}\sup_{0\leq l\leq N-1}\left|(e^{\nu^2 t\partial_{xx}}Q_N f)\Big(t,\frac{l}{N}\Big)\right|\leq 1+\mathcal{O}(\omega_f(N^{-\frac{2}{3}}))+\mathcal{O}(N^{-c}),$$

where $c>0$ is an absolute constant. If $f:\mathbb{T}\to\mathbb{R}$ is C^α-continuous for some $0<\alpha<1$ and $\|f\|_\infty\leq 1$. Then

$$\sup_{t\geq 0}\sup_{0\leq l\leq N-1}\left|(e^{\nu^2 t\partial_{xx}}Q_N f)\Big(t,\frac{l}{N}\Big)\right|\leq 1+\mathcal{O}(N^{-c_1}),$$

where $c_1>0$ is a constant depending only on α.

Proof. With no loss we assume $\nu=1$. Suppose f satisfies $\|f\|_\infty\leq 1$. By Theorem B.1 we only need to consider the regime $N^2t<100\log N$. For $0\leq l\leq N-1$ and $N^2t<100\log N$, by using the proof of Theorem B.1, we have

$$\begin{aligned}(e^{t\partial_{xx}}Q_N f)\Big(t,\frac{l}{N}\Big)&=\sum_{j=0}^{N-1}f\Big(\frac{l-j}{N}\Big)\cdot\frac{1}{N}\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}e^{-4\pi^2k^2t}e^{2\pi ik\frac{j}{N}}\Big)\\&=\mathcal{O}(N^{-\frac{1}{3}}\sqrt{\log N})+f\Big(\frac{l}{N}\Big)c_0+\sum_{1\leq|j|\leq N^{\frac{1}{3}}}f\Big(\frac{l+j}{N}\Big)c_j,\end{aligned}$$

where

$$c_j=\int_0^1 e^{-\frac{1}{4}\pi^2N^2ts^2}\cos(\pi js)ds(-1)^j.$$

Now observe that for $|j|\leq N^{\frac{1}{3}}$,

$$\begin{aligned}&\Big|f\Big(\frac{l+j}{N}\Big)-f\Big(\frac{l}{N}\Big)\Big|\leq\omega_f(N^{-\frac{2}{3}}),\quad\text{if } f \text{ is continuous},\\&\Big|f\Big(\frac{l+j}{N}\Big)-f\Big(\frac{l}{N}\Big)\Big|\leq\mathcal{O}(N^{-\frac{2}{3}\alpha}),\quad\text{if } f \text{ is } C^\alpha \text{ continuous}.\end{aligned}$$

Thus in the main order, the function f can be taken as a constant value (the contributions of f for j in the regime $N^{\frac{1}{3}}<j<N-N^{\frac{1}{3}}$ is unimportant). Since

$$\sum_{j=0}^{N-1}\frac{1}{N}\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}e^{-4\pi^2k^2t}e^{2\pi ik\frac{j}{N}}\Big)=1,$$

the desired result then follows easily. □

Appendix C: estimate of $(\mathrm{Id}-\nu^2\tau\partial_{xx})^{-n}Q_NU^0$ and related estimates

Lemma C.1. *Let $\tau>0$ and $s\geq 1$. Then for any $1\leq p<q\leq\infty$, we have*

$$\|(\mathrm{Id}-\tau\nu^2\partial_{xx})^{-s}f\|_{L^q_x(\mathbb{T})}\leq C_{p,q,\nu}\cdot\Big(1+(s\tau)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}\Big)\|f\|_{L^p_x(\mathbb{T})},\tag{C.1}$$

where $C_{p,q,\nu}>0$ depends only on (p, q, ν). Also

$$\|\partial_x(\mathrm{Id}-\tau\nu^2\partial_{xx})^{-s}f\|_{L^2_x(\mathbb{T})}\leq C_2\cdot\Big(1+(s\tau)^{-\frac{3}{4}}\Big)\|f\|_{L^1_x(\mathbb{T})},\tag{C.2}$$

where $C_2>0$ depends only on ν.

Proof. Denote by K the kernel corresponding to $(\mathrm{Id}-\tau\nu^2\partial_{xx})^{-s}$. Clearly $\|K\|_{L^1_x(\mathbb{T})}=1$ and

$$\|K\|_{L^\infty_x(\mathbb{T})}\leq 1+\sum_{0\neq k\in\mathbb{Z}}\Big(1+\frac{\alpha}{s}k^2\Big)^{-s},$$

where $\alpha=s\tau(2\pi\nu)^2$. Now note that for $s\geq 1$ and $x\geq 0$, we have

$$\Big(1+\frac{x}{s}\Big)^{-s}\leq(1+x)^{-1}.\tag{C.3}$$

It follows that

$$\begin{aligned}&\|K\|_{L^\infty_x(\mathbb{T})}\lesssim 1+\alpha^{-\frac{1}{2}},\\&\|K\|_{L^r_x(\mathbb{T})}\lesssim 1+\alpha^{-\frac{1}{2}(1-\frac{1}{r})},\quad\forall 1\leq r\leq\infty.\end{aligned}$$

Then (C.1) follows from Young's inequality. Now for (C.2), we first note that (by using (C.3))

$$\|\partial_x(\mathrm{Id}-\tau\nu^2\partial_{xx})^{-s}f\|_{L^2_x(\mathbb{T})}\lesssim(\tau s)^{-\frac{1}{2}}\|(\mathrm{Id}-s\tau\nu^2\partial_{xx})^{-\frac{1}{2}}f\|_{L^2_x(\mathbb{T})}.$$

Denote $T=(\mathrm{Id}-s\tau\nu^2\partial_{xx})^{-\frac{1}{2}}$. Since

$$\|Tf\|_2^2=\langle f,T^2f\rangle.$$

The desired estimate follows easily. □

We recall for $n\geq 1$,

$$\begin{aligned}&((\mathrm{Id}-\nu^2\tau\partial_{xx})^{-n}Q_NU^0)(x)\\=&\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}\Big(\frac{1}{N}\sum_{j=0}^{N-1}U_j^0e^{-2\pi i\frac{kj}{N}}(1+\nu^2\tau(2\pi k)^2)^{-n}\Big)e^{2\pi ik\cdot x}\Big).\end{aligned}$$

We need to examine

$$\sup_{0\leq l\leq N-1}\Big|((\mathrm{Id}-\nu^2\tau\partial_{xx})^{-n}Q_NU^0)\Big(\frac{l}{N}\Big)\Big|$$

under the assumption that $\|U^0\|_\infty\leq 1$. Clearly we have

$$\begin{aligned}&\sup_{0\leq l\leq N-1}\Big|((\mathrm{Id}-\nu^2\tau\partial_{xx})^{-n}Q_NU^0)\Big(\frac{l}{N}\Big)\Big|\\\leq&\frac{1}{N}\sum_{j=0}^{N-1}\Big|\mathrm{Re}\Big(\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}(1+(2\pi\nu k)^2\tau)^{-n}e^{2\pi ik\cdot\frac{j}{N}}\Big)\Big|=:B_{N,n}.\end{aligned}$$

Below we shall write $X=O(Y)$ if $|X|\leq CY$ and $C>0$ is a constant depending only on ν.

Theorem C.4. *Let $\nu>0$, $\tau>0$ and $n\geq 1$. It holds that*

$$B_{N,n}\leq\begin{cases}1+\mathcal{O}(N^{-1}), & \text{if } \tau\geq\frac{1}{2},\\ 1+\mathcal{O}(N^{-0.7}), & \text{if } 0<\tau<\frac{1}{2} \text{ and } n\tau\nu^2\geq N^{-0.3},\\ 1+\epsilon_a+\mathcal{O}(N^{-\frac{1}{3}}\sqrt{\log N}), & \text{if } 0<\tau<\frac{1}{2} \text{ and } n\tau\nu^2\leq N^{-0.3}.\end{cases}$$

Here $0<\epsilon_a<0.01$ is an absolute constant.

If $\tau=\frac{1}{4N^2\nu^2}$, then for $n=1$ we have

$$B_{N,1}>1.001+\mathcal{O}(N^{-c}),$$

where $c>0$ is an absolute constant.

Suppose $f:\mathbb{T}\to\mathbb{R}$ is continuous and $\|f\|_\infty\leq 1$. Then

$$\sup_{0\leq l\leq N-1}\Big|((\mathrm{Id}-\nu^2\tau\partial_{xx})^{-n}Q_Nf)\Big(\frac{l}{N}\Big)\Big|\leq 1+\mathcal{O}(\omega_f(N^{-\frac{2}{3}}))+\mathcal{O}(N^{-c}),\tag{C.4}$$

where $c>0$ is an absolute constant, and ω_f is defined in (B.5).

Suppose $f:\mathbb{T}\to\mathbb{R}$ is C^α-continuous for some $0<\alpha<1$ and $\|f\|_\infty\leq 1$. If $(U^0)_l=f(\frac{l}{N})$ for all $0\leq l\leq N-1$, then

$$\sup_{0\leq l\leq N-1}\left|((\mathrm{Id}-\nu^2\tau\partial_{xx})^{-n}Q_Nf)\Big(\frac{l}{N}\Big)\right|\leq 1+\mathcal{O}(N^{-c_1}), \tag{C.5}$$

where $c_1>0$ is a constant depending only on α.

There exists a function $f:\ \mathbb{T}\to\mathbb{R}$, continuous at all of $\mathbb{T}\setminus\{x_*\}$ for some $x_*\in\mathbb{T}$ (i.e., continuous at all $x\neq x_*$) and has the bound $\|f\|_\infty\leq 1$ such that the following hold: for a sequence of even numbers $N_m\to\infty$, $\tau_m=\frac{1}{4}\nu^{-2}N_m^{-2}$ and $x_m=j_m/N_m$ with $0\leq j_m\leq N_m-1$, we have

$$\left|((\mathrm{Id}-\nu^2\tau\partial_{xx})^{-1}Q_Nf)(x_m)\right|\geq 1+\eta_*,\quad \forall m\geq 1, \tag{C.6}$$

where $\eta_*>0.001$ is an absolute constant.

We now give the proof of Theorem C.4. To ease the notation we shall assume $\nu=1$. We discuss several cases.

Case 1: $\tau\geq\frac{1}{2}$. Observe that for all $n\geq 1$,

$$\begin{aligned}\sum_{|k|\geq\frac{N}{2}}\frac{1}{(1+(2\pi k)^2\tau)^n}&\lesssim\int_{|k|\geq\frac{N}{2}-1}\frac{1}{k^2}dk\\&\leq\mathcal{O}(N^{-1}).\end{aligned}$$

Thus in this case

$$|B_{N,n}|\leq 1+\mathcal{O}(N^{-1}).$$

Case 2: $0<\tau<\frac{1}{2}$ and $n\tau\geq N^{-0.3}$. We have

$$\begin{aligned}\sum_{|k|\geq\frac{N}{2}}\frac{1}{(1+(2\pi k)^2\tau)^n}&\lesssim\int_{|k|\geq\frac{N}{2}-1}\frac{1}{1+n\tau k^2}dk\\&\leq\mathcal{O}(N^{-1}(n\tau)^{-1})\leq\mathcal{O}(N^{-0.7}).\end{aligned}$$

Thus in this case

$$|B_{N,n}|\leq 1+\mathcal{O}(N^{-0.7}).$$

Case 3: $0<\tau<\frac{1}{2}$ and $n\tau<N^{-0.3}$. In this case we shall use Theorem B.1. Note that for $a=1+\beta k^2$ ($\beta>0$ is a constant) and $n\geq 1$, we have

$$a^{-n}=\frac{1}{(n-1)!}\int_0^\infty e^{-sa}s^{n-1}ds. \tag{C.7}$$

The desired estimate then follows from this and Theorem B.1.

By a similar reasoning, we note that (C.4)–(C.5) follow from (C.7) and Theorem B.3.

Next we shall derive more detailed estimates for $B_{N,1}$. We begin with an identity: for $\tau>0$,

$$\int_{\mathbb{R}}\frac{1}{1+(2\pi k)^2\tau}e^{2\pi ikx}dk=\frac{1}{2\sqrt{\tau}}e^{-\frac{|x|}{\sqrt{\tau}}}.$$

We denote $B_{N,1}$ as B_N:

$$B_N=\frac{1}{N}\sum_{j=0}^{N-1}|Q_j|,$$

where

$$\begin{aligned}Q_j&=\sum_{-\frac{N}{2}<k\leq\frac{N}{2}}\frac{1}{1+(2\pi k)^2\tau}e^{2\pi ik\frac{j}{N}}\\&=\frac{1}{2\sqrt{\tau}}\int_{\mathbb{R}}e^{-\frac{|y-\frac{j}{N}|}{\sqrt{\tau}}}\frac{\sin N\pi y}{\tan\pi y}dy\\&=(-1)^j\frac{1}{2\sqrt{\tau}}\int_{\mathbb{R}}e^{-\frac{|y|}{\sqrt{\tau}}}\frac{\sin N\pi y}{\tan\pi(\frac{j}{N}+y)}dy\\&=(-1)^j\frac{1}{2}\int_{\mathbb{R}}e^{-|y|}\frac{\sin k_1y}{\tan\pi(\frac{j+k_1y}{N})}dy.\end{aligned}$$

Here we denote $k_1=N\sqrt{\tau}$.

Case 1: $k_1\geq N^{\frac{1+\epsilon_0}{2}}$ where $0<\epsilon_0\ll 1$ is a small constant. Then $N\tau>N^{\epsilon_0}$ and

$$B_N\leq 1+\sum_{|k|\geq\frac{N}{2}}\frac{1}{1+(2\pi k)^2\tau}\leq 1+\mathcal{O}(N^{-\epsilon_0}).$$

Case 2: $k_1<N^{\frac{1+\epsilon_0}{2}}$. Note that we may restrict the integral in dy to the regime $|y|\leq 100\log N$ since the contribution due to the other regime is negligible.

First consider $N^{\frac{1}{2}+2\epsilon_0}\leq j\leq N-N^{\frac{1}{2}+2\epsilon_0}$. Clearly for $j_1=\min\{j,|N-j|\}$, we have

$$\int_{0<y<100\log N}e^{-y}\Big|\cot\Big(\frac{j+k_1y}{N}\Big)-\cot\Big(\frac{j-k_1y}{N}\Big)\Big|dy\lesssim\frac{N^2}{j_1^2}\cdot\frac{k_1}{N}.$$

Then

$$\frac{1}{N}\sum_{N^{\frac{1}{2}+2\epsilon_0}\leq j\leq N-N^{\frac{1}{2}+2\epsilon_0}}|Q_j|\lesssim k_1\cdot N^{-\frac{1}{2}-2\epsilon_0}\leq\mathcal{O}(N^{-\epsilon_0}).$$

Next consider $0\leq j<N^{\frac{1}{2}+2\epsilon_0}$. We rewrite

$$Q_j=\frac{1}{2k_1}\int_{\mathbb{R}}e^{-\frac{|y+j|}{k_1}}\frac{\sin\pi y}{\tan\pi\frac{y}{N}}dy.$$

In the above integral, observe that the regime $|y|\geq N^{\frac{2}{3}}$ is negligible. On the other hand for $|y|<N^{\frac{2}{3}}$, we observe that

$$|\cot(\frac{y}{N})-\frac{N}{y}|\lesssim\frac{|y|}{N}.$$

Clearly the contribution due to the error term $\mathcal{O}(\frac{|y|}{N})\leq\mathcal{O}(\frac{|y|}{k_1})\cdot\frac{k_1}{N}$ is bounded by

$$\frac{1}{N}\sum_{j\leq N^{\frac{1}{2}+2\epsilon_0}}\frac{k_1}{N}\lesssim\mathcal{O}(N^{-c}).$$

We then obtain

$$B_N=\mathcal{O}(N^{-c})+\sum_{|j|\leq N^{\frac{1}{2}+2\epsilon_0}}\frac{1}{2k_1}\frac{1}{\pi}\Big|\int_{\mathbb{R}}e^{-\frac{|y+j|}{k_1}}\frac{\sin\pi y}{y}dy\Big|$$

Now we have

$$F_j(1)=\frac{1}{2k_1\pi}\int_{\mathbb{R}}e^{-\frac{|y+j|}{k_1}}\frac{\sin\pi y}{y}dy=\int_0^1F_j'(s)ds,$$

where

$$\begin{aligned}F_j'(s)=&\frac{1}{2k_1}\int_{\mathbb{R}}e^{-\frac{|y+j|}{k_1}}\cos(\pi sy)dy\\=&(\cos\pi js)\frac{1}{2k_1}\int_{\mathbb{R}}e^{-\frac{|y|}{k_1}}e^{i\pi sy}dy\\=&(\cos\pi js)\cdot\frac{1}{1+(k_1\pi s)^2}.\end{aligned}$$

Thus we obtain

$$B_N = \mathcal{O}(N^{-c}) + \sum_{|j|\le N^{\frac{1}{2}+2\epsilon_0}} \Big| \int_0^1 \frac{1}{1+(k_1\pi s)^2} \cos \pi j s ds \Big|$$
$$= \mathcal{O}(N^{-c}) + \sum_{j\in\mathbb{Z}} \Big| \int_0^1 \frac{1}{1+(k_1\pi s)^2} \cos \pi j s ds \Big|.$$

Now for $k_1 = \frac{1}{2}$, we denote

$$\beta_j = \int_0^1 \frac{1}{1+(k_1\pi s)^2} \cos \pi j s ds.$$

By rigorous numerical computation, we have

$$\begin{aligned}
&\beta_0 \approx 0.639093,\\
&\beta_1 = \beta_{-1} \approx 0.165881,\\
&\beta_2 = \beta_{-2} \approx 0.00883796,\\
&\beta_3 = \beta_{-3} \approx 0.00691018,\\
&\beta_4 = \beta_{-4} \approx -0.00220351.
\end{aligned}$$

In particular

$$\sum_{|j|\le 3} |\beta_j| > 1.0023.$$

Finally we show how to derive (C.6). With no loss we assume $\nu = 1$. Suppose f satisfies $\|f\|_\infty \le 1$. Then for $N \ge 10$, $\tau = \frac{1}{4}N^{-2}$ and $0 \le l \le N-1$, we have

$$((\mathrm{Id} - \tau\partial_{xx})^{-1} Q_N f)\Big(\frac{l}{N}\Big) = \sum_{j=0}^{N-1} f\Big(\frac{l-j}{N}\Big) \cdot \frac{1}{N} \mathrm{Re}\Big(\sum_{-\frac{N}{2} < k \le \frac{N}{2}} (1+(2\pi)^2 \tau k^2)^{-1} e^{2\pi i k \frac{j}{N}} \Big)$$
$$= \mathcal{O}(N^{-c}) + f\Big(\frac{l}{N}\Big)\beta_0 + \sum_{1\le|j|\le N^{\frac{1}{2}+2\epsilon_0}} f\Big(\frac{l+j}{N}\Big)\beta_j,$$

where

$$\beta_j = \int_0^1 \frac{1}{1+(\frac{1}{2}\pi s)^2} \cos \pi j s ds.$$

It is then clear to choose f similar to the proof in Theorem B.2. We omit further details.

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