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Tatsien Li · Yi Zhou

Nonlinear Wave Equations

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Preface

Nonlinear wave equations belong to a typical category of nonlinear evolutionary equations that is of great theoretical significance and practical value. Research on the global existence and the blow-up phenomenon of classical solutions to the Cauchy problem with small initial data for nonlinear wave equations relates to the asymptotic stability of the null solution or the stabilization of the corresponding control system of such equations, and is a highly meaningful and challenging subject of study. The research in this field was initiated by Prof. F. John in the late 1970s and early 1980s when he gave some examples to reveal the blow-up phenomenon of solutions to nonlinear wave equations. Later on, Profs. S. Klainerman, D. Christodoulou, and L. Hörmander together with Prof. F. John, as well as some other Profs. like M. Kovalyov, H. Lindblad, G. Ponce, J. Shatah, T.C. Sideris, gave various results about the global existence and the lower bound estimates of life-span of classical solutions in different space dimensions and with different powers of nonlinear terms on the right-hand side, which formed a frontier research direction of great significance and attraction. Although the results obtained by these mathematicians were profound, they did not cover all the possible important situations at the time, and there remained a lack about the sharpness of the established lower bound estimates of life-span for classical solutions. The whole research is still, so to speak, in the initial stages of development. On the other hand, there is a great deal of diversity in the methods adopted by these mathematicians, each having its own characteristics and some being rather complex. So there does not seem to be a unified and easy approach to tackling such type of problems.

During my stay in France, I visited Heriot-Watt University of Great Britain in 1980 and encountered Prof. F. John who at the time happened to visit that university, too. So I got the chance to consult him face to face, which turned out to be a most instructive experience. At the beginning of 1981 when I paid a visit to Courant Institute of Mathematical Science in the USA, I met Prof. S. Klainerman and had careful discussions with him. He presented me with the preprint of his long article close to 60 pages. All these aroused my concern and interest in nonlinear wave equations and prompted our studies in this field. Some of my doctor students in the earlier years, including Yunmei Chen, Xin Yu, and Yi Zhou, chose this as the subject of their doctoral dissertations and made valuable contributions. Thanks to their participation and efforts, especially Yi Zhou's long persistent hard work, we have managed to carry on with this research subject in Fudan University up to now and obtained fruitful results. Our limited accomplishment in this area can be generally summarized into two aspects. One is to have established the complete lower bound estimates of life-span (including the result of global existence) for classical solutions to the Cauchy problem of nonlinear wave equations with small initial data in all possible space dimensions and with all possible powers of nonlinear terms on the right-hand side, and the estimates are the best ones that are unlikely to be improved, that is to say, we in principle draw the conclusion for research in this regard. The other is to have proposed the unified and straightforward approach to handling such problems, that is, the global iteration method, which applies the simple contraction mapping principle and requires roughly the same amount of work done to prove the local existence of classical solutions.

In the book Nonlinear Evolution Equations (in Chinese) coauthored by Yunmei Chen and me and published by Science Press (Beijing) in 1989, we proved the global existence of classical solutions to the Cauchy problem with small initial data for nonlinear wave equations by the global iteration method. Later, in another book Global Classical Solutions for Nonlinear Evolution Equations coauthored also by Yunmei Chen and me and published by Longman Scientific & Technical Press in 1992, the method was further employed to make some lower bound estimates of life-span for classical solutions. However, restricted by the progress of scientific research at that time, we did not touch upon or get the best results about the important situations of space dimensions $n = 2$ and $n = 4$, etc. Besides, we failed to deal with theories related to the null condition and the sharpness of some lower bound estimates of life-span. Since the two books both mentioned nonlinear evolution equations including also nonlinear heat equations and nonlinear wave equations are only a part of them, the length of the discussion was inevitably limited, which to a certain extent caused the deficiency mentioned above. Around 1995, we basically finished work on the global existence and the lower bound estimates of life-span of classical solutions to the Cauchy problem with small initial data for nonlinear wave equations, so Yi Zhou and I began to think about a monograph on nonlinear wave equations. In fact, the Shanghai Scientific and Technical Publishers had invited us to write on this long before, but with too many errands to go, we wrote on and off or sometimes even put off for quite a long period. Another important reason that hindered us from finishing the book soon was that some of the lower bound estimates of life-span we got then had not been proved to be the best ones immune to further improvement, so if we wrapped up the book in haste and delivered it for publication, it might never be consummate, which is a fact we would not be content with. In the recent years, the sharpness of all the lower bound estimates of the life-span has been obtained, and thus we felt the urgency of finishing the book as quickly as possible. Meanwhile, throughout the years we have found that some of the previous proofs can be simplified or improved so as to be presented in a relatively new form, which can be seen as an additional achievement. Although we were determined to pull our forces together and start afresh, it took yet another two to three years for us to complete the final version of the book in 2014, as many proofs needed to be rewritten. Seeing the book finally come out after such a long track of time, the gratification of the authors can well be imagined.

The whole book has got fifteen chapters in all. The first seven serve as a prelude for later discussions, but still have their own meanings and values. Among the later eight chapters, five discuss the global existence and the lower bound estimates of life-span for classical solutions in all possible situations by adopting the global iteration method, including the proof of global existence of classical solutions under the hypothesis of the null condition; two focus on demonstrating the sharpness of the obtained lower bound estimates of life-span; and the last chapter entails relevant applications and extensions. Most of the references listed in the bibliography are cited in the body part of the book, while a few of them, though not formally quoted, are more or less related to the content, from which we hope readers can get some necessary information. Dr. Ke Wang has been responsible for the typewriting and typesetting of the book in Chinese, while the English version is to be translated by Prof. Yachun Li and published by the Springer-Verlag. The authors would like to express their deep-felt gratitude to them all for their hearty devotion, earnest support, and strong help.

Owing to the limitation of the authors' knowledge, there must be mistakes and careless omissions in the book, so the authors hereby sincerely invite readers to make frank comments and criticism in any respect.

Shanghai, China | Tatsien Li
February 2015

Contents

Chapter 1
Introduction and Overview

1.1 Objectives

Nonlinear wave equations are a kind of important infinite-dimensional dynamical systems. The so-called infinite-dimensional dynamical system is a system described by nonlinear evolutionary partial differential equations (nonlinear evolutionary equations for short). While, the nonlinear evolution equation is the common name of nonlinear partial differential equations whose solutions depend not only on the spatial variables but also on a special argument t (time). For example, the nonlinear heat equations appearing in the phenomenon of heat flux and reaction-diffusion (including reaction-diffusion equations), the nonlinear wave equations appearing in the vibration and electromagnetics, the nonlinear Schrödinger equation in quantum mechanics, the Navier-Stokes equation describing the incompressible fluids, the Yang-Mills equation in the gauge field theory, the hyperbolic conservation laws, the KdV equation, and so on. All of these are equations with wide applications and essential significance in related disciplines.

To make clear the problems which is going to be studied in this book, we first investigate the finite-dimensional dynamical systems, i.e., the nonlinear ordinary differential equations (systems). In this situation, the solution is a function of the time variable t only, and does not depend on the spatial variables.

Let us first consider the simplest case: the following Cauchy problem of nonlinear ordinary differential equation:

$$\frac{du}{dt} = u^{1+\alpha}, \tag{1.1.1}$$

$$t = 0 : u = \varepsilon, \tag{1.1.2}$$

where α is a positive constant, and $\varepsilon > 0$ is a small parameter. The solution of this problem can be expressed explicitly as

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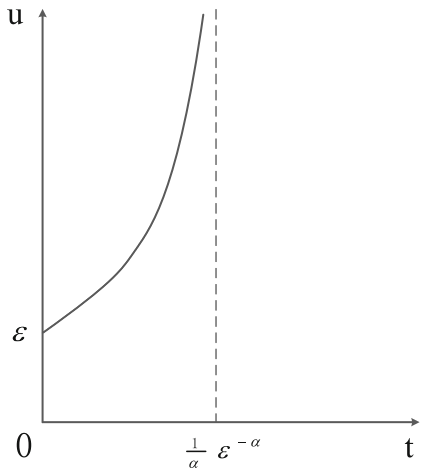


Fig. 1.1 Blow-up of solution

$$u(t) = \frac{\varepsilon}{(1 - \alpha t \varepsilon^{\alpha})^{1/\alpha}}. \tag{1.1.3}$$

Therefore, as $t \nearrow \frac{1}{\alpha}\varepsilon^{-\alpha}$, $u(t) \to +\infty$, as shown in Fig. 1.1.

Thus, for this problem, the solution does not exist for all the time $t \geq 0$, i.e., there does not exist a **global classical solution** (The so-called classical solution means a solution in the normal sense; while, the global solution means that the solution exists for all the time $t \geq 0$). This implies the formation of singularity for solution after a certain period of time (the solution itself or its derivative $\to \infty$; here both the solution and its derivative $\to \infty$), called the **blow-up of solution**. Compared with the linear case, for the Cauchy problem of nonlinear ordinary equations, the solution may blow up generically. In current situation, knowing that the solution will blow up, let us show how long the solution will exist. Obviously, if we denote by $\widetilde{T}(\varepsilon)$ the **life-span of solution**, i.e., the maximum time for the solution to exist, then we have

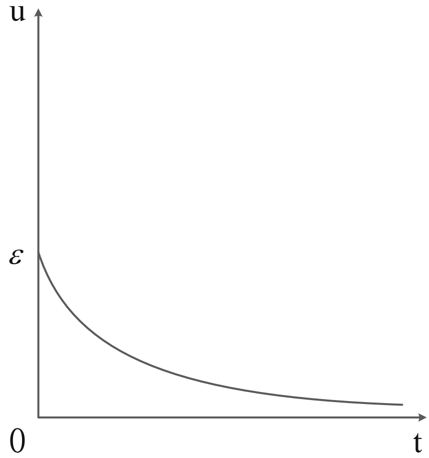
$$\widetilde{T}(\varepsilon) = \frac{1}{\alpha}\varepsilon^{-\alpha} \approx \varepsilon^{-\alpha}. \tag{1.1.4}$$

This shows that, for small initial data, the higher the order of the nonlinear term on the right-hand side, namely, the larger the α, the larger the life-span of solution. This is because that, for a small solution, the nonlinear term on the right-hand side has smaller influence when it has higher order.

However, if the equation involves a dissipative term, things will change greatly. Regarding u as the velocity and assuming that there exists a damping force proportional to the velocity, the above Eq. (1.1.1), for example, will be replaced by the following equation (where the proportional constant is taken to be 1):

$$\frac{du}{dt} = -u + u^{1+\alpha}, \tag{1.1.5}$$

Fig. 1.2 Exponential decay of solution



and the initial value is still (1.1.2). Then it is easy to know that the solution of Cauchy problem (1.1.5) and (1.1.2) is

$$u(t) = \frac{\varepsilon}{[e^{\alpha t}(1-\varepsilon^\alpha)+\varepsilon^\alpha]^{1/\alpha}}, \tag{1.1.6}$$

as shown in Fig. 1.2. This Cauchy problem admits a unique solution for all the time $t \geq 0$ as long as $\varepsilon > 0$ is sufficiently small ($\varepsilon < 1$), namely, the life-span of solution is

$$\widetilde{T}(\varepsilon) = +\infty, \tag{1.1.7}$$

and this solution is exponentially decaying as $t \to +\infty$.

Why there is so big difference between these two situations? A fundamental reason is that the corresponding linearized equations are quite different. The linearized equation of (1.1.5) is $\frac{du}{dt} = -u$, all of its solutions decay exponentially as $t \to +\infty$; While, the linearized equation of (1.1.1) is $\frac{du}{dt} = 0$, all of its nonzero solutions do not decay. It is just this essential distinction that yields different results on whether the global solutions exist or not.

As a matter of fact, there is a rather general conclusion in the situation of ordinary differential equations. We consider the following system of ordinary differential equations:

$$\frac{dU}{dt} = f(U), \tag{1.1.8}$$

where $U = (u_1, \ldots, u_n)^T$ is the unknown vector function, and $f(U) = (f_1(U), \cdots, f_n(U))^T$ is a given suitably smooth function of U, and assume that

$$f(0) = 0, \tag{1.1.9}$$

i.e., $U \equiv 0$ is an equilibrium (the zero solution) of the system. The linearized system of (1.1.8) can be written as

$$\frac{dU}{dt} = AU, \tag{1.1.10}$$

where

$$A = \nabla f(0) \tag{1.1.11}$$

is the Jacobian matrix of the nonlinear term $f(U)$ on the right-hand side of (1.1.8) at $U = 0$.

Assume that each eigenvalue of A has negative real part, equivalently, each solution of the linearized system (1.1.10) decays exponentially as $t \to +\infty$, then the Cauchy problem of the original nonlinear system (1.1.8) with small initial data

$$t = 0 : U = U_0 \ \ (U_0 \text{ small}) \tag{1.1.12}$$

admits a global solution $U = U(t)$ for $t \geq 0$, and $U(t)$ decays exponentially as $t \to +\infty$.

Here, "each eigenvalue of A has negative real part" is equivalent to saying that there is a certain dissipative mechanism in system (1.1.8); While, "the Cauchy problem with small initial data admits a global solution for $t \geq 0$ and the solution decays exponentially as $t \to +\infty$" implies that: if the null solution is perturbed a little bit at the initial time, this small perturbation will finally disappear very quickly as $t \to +\infty$ in a way of exponential decay, that is to say, the zero solution has the asymptotic stability. See You (1982) in the theory of ordinary differential equations.

Hence, for the Cauchy problem of nonlinear evolution equations with small initial data, to study the global existence and uniqueness of (classical) solutions on $t \geq 0$ and the (exponential) decay of solutions as $t \to +\infty$ is, equivalently, to study the asymptotic stability of zero solution from the point of view of differential equations, to study whether the zero solution is an attractor from the point of view of dynamical systems, and to study the stabilization of the system from the point of view of control theory. Therefore, this is a research topic of important theoretical and practical significance.

This theory, in the case of ordinary differential equations (systems), namely, in the case of finite dimensional dynamical systems, can be guaranteed, as stated above, only when the equations (systems) have the above mentioned dissipative mechanism. A natural question is how to generalize this theory to the case of infinite dimensional dynamical systems, namely, to the case of nonlinear partial differential equations, and figure out when the zero solution is asymptotically stable, i.e., when the Cauchy problem with small initial data admits a unique global classical solution on $t \geq 0$ with a certain decay as $t \to +\infty$.

Comparing the dynamical systems of the infinite and finite dimensional cases, they have something in common and there are also essential differences.

There is only one main common point, that is, roughly speaking, in the case of small initial data, the higher the order $1 + \alpha$ of the nonlinear term, namely, the larger the α, the larger the life-span of classical solution.

But in the infinite dimensional case, the solution depends not only on the time variable t but also on the space variable $x = (x_1, \ldots, x_n)$ $(n = 1, 2$ or 3 in various specific applications), and this brings forth great complexity and very rich topics to the discussion in the infinite dimensional case.

First of all, as mentioned above, there are many different types of evolutionary partial differential equations. Each type corresponds to different physical phenomena, with its own essential features and methods, which needs to be dealt with individually.

In this book we focus on the following Cauchy problem of nonlinear wave equations with small initial data:

$$\Box u = F(u, Du, D_x Du), \tag{1.1.13}$$

$$t = 0 : u = \varepsilon\phi(x), u_t = \varepsilon\psi(x), \tag{1.1.14}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \triangle \quad \left(\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right) \tag{1.1.15}$$

is the wave operator,

$$D_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \quad D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \tag{1.1.16}$$

ϕ and ψ are sufficiently smooth functions with compact support, assume without loss of generality that $\phi, \psi \in C_0^\infty(I\!R^n)$, and $\varepsilon > 0$ is a small parameter.

Denote

$$\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \cdots, n; (\lambda_{ij}), i, j = 0, 1, \cdots, n, i + j \geq 1). \tag{1.1.17}$$

Assume that the nonlinear term $F(\hat{\lambda})$ on the right-hand side is sufficiently smooth in a neighborhood of $\hat{\lambda} = 0$ and satisfies

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}), \tag{1.1.18}$$

in which $\alpha \geq 1$ is an integer.

In addition to the importance of the wave phenomenon itself and its various applications, the motivation that we focus on the nonlinear wave equation lies also in that this is the first nonlinear evolution equation which involves in the study on the asymptotic stability of the zero solution. The first work dates back to that of Segal 1968 in 1968. However, arising from the study of the late mathematician John 1979–1981 on the blow-up phenomena of solutions to the nonlinear wave equations at the end of 1970's, it was really developed by the works of Klainerman 1980–1982 in the beginning of 1980's and a series of subsequent works by John and Klainerman (1984) Klainerman (1983) Klainerman (1985), and Klainerman and Ponce (1983). Meanwhile, since the hyperbolic case is much more complicated than many other cases, which is more challenging in mathematics, there are still many problems worth further research and thinking. It is also worth pointing out that the general framework of solving method that we are going to introduce in the following is applicable not only to nonlinear wave equations,but also to some other nonlinear evolution equations, such as nonlinear heat equations, nonlinear Schrödinger equations, and so on (see Li and Chen 1992).

Second of all, different from the case of ordinary differential equations (systems), the linearized equation of the above nonlinear wave equation, namely, the usual wave equation

$$\Box u = 0, \tag{1.1.19}$$

does not contain a dissipative term (its energy is conserved, while for the case with dissipative terms, see Sect. 1.4), however its solutions may still have some decay. For instance, it can be proved that (see Li and Chen 1992): any solution $u = u^0(t, x)$ of equation (1.1.19) satisfies

$$\|u^0(t, \cdot)\|_{L^\infty(I\!R^n)} \le C(1+t)^{-\frac{n-1}{2}}, \quad \forall t \ge 0, \tag{1.1.20}$$

where C is a positive constant depending on the solution but independent of t. Therefore, if the space dimension $n \ge 2$, then any solution of (1.1.19) decays as $t \to +\infty$. There is one more thing that is different from the case of ordinary differential equations (systems), that is, even if the solution decays, it may not decay exponentially, it may decay polynomially like (1.1.20); and the higher the space dimension n, the larger the decay rate. To illustrate this fact, F. John has quoted the following motto from Shakespeare's Henry VI:

Glory is like a circle in the water,
Which never ceaseth to enlarge itself,
Till by broad spreading it disperse to naught.

As a matter of fact, when the space dimension is higher, the wave has more space to evacuate and then decays faster.

Based on the above observation, roughly speaking, when the space dimension n and α are larger, the solution may have larger decay rate and larger life-span, then the global solution on $t \ge 0$ for Cauchy problem may exist for small initial data and may have some decay as $t \to +\infty$, i.e., the zero solution is asymptotically stable; Otherwise, when n and α are smaller, the classical solutions only exists locally in general, that is, the classical solutions may blow up in a finite time and then the zero solution is not stable.

Since whether the classical solutions exist or not is undetermined, we can generally study the life-span $\widetilde{T}(\varepsilon)$ of classical solutions to the Cauchy problem (1.1.13)–(1.1.14) for any given space dimension $n \ge 1$ and integer $\alpha \ge 1$. If $\widetilde{T}(\varepsilon) = +\infty$, then we have the global existence and uniqueness of classical solutions on $t \ge 0$. Meanwhile, we can further study the asymptotic stability of the solution as $t \to +\infty$, especially its decay. Otherwise, if $\widetilde{T}(\varepsilon)$ is finite, then we can only have the local existence of solutions in the finite interval $[0, \widetilde{T}(\varepsilon))$, and the solution will blow up as $t \to \widetilde{T}(\varepsilon)$, on this occasion, we hope to obtain a sharp estimate on the lower bound of $\widetilde{T}(\varepsilon)$. In other words, one of our objectives of study is to establish a sharp estimate on the lower bound of the life-span $\widetilde{T}(\varepsilon)$ for any given $n \geq 1$ and $\alpha \geq 1$. The so-called sharpness means that the estimate cannot be improved for general F, namely, we can always find some special F and initial data such that the life-span of the solution has an upper bound estimate of the same type.

Our second objective is to provide a concise and unified framework to deal with this kind of research — the **global iteration method**.

In the sequel we will see that, for various important results in this research field, which were obtained by using different methods by many famous mathematicians for different particular cases since 1980's, we can handle them in a unified and simple method, as long as we realize the above two objectives; moreover, we can essentially improve some results or fill gaps of research for some important situations, so that the whole problem can be solved thoroughly.

1.2 Past and Present

We first consider the special case that the nonlinear term F does not depend on u explicitly:

$$F = F(Du, D_x Du). \tag{1.2.1}$$

The first general results were given by Klainerman (1980) and Klainerman (1982). With the aid of the L^∞ decay estimate (1.1.20) of solutions to the wave equation and the energy estimates, using Nash–Moser–Hörmander iteration he proved that, for sufficiently small $\varepsilon > 0$, the Cauchy problem (1.1.13)–(1.1.14) admits a global classical solution $u = u(t, x)$ on $t \geq 0$ and the solution has some decay as $t \to +\infty$ if n and α satisfy the following relations:

$\alpha =$	1	2	$3, 4, \cdots$
$n \geq$	6	3	2

The same result was proved in tandem by Shatah (1982) and Klainerman & Ponce (1983) in simpler methods, while the later used the local solution extension method and the former used the simple contraction mapping principle. They both used the $L^q (q > 2)$ decay estimate of solutions to the wave equation.

The case $\alpha = 1$, corresponding to the quadratic nonlinearity, being the first nonlinear power in the Taylor's expansion of the nonlinear term, appears most naturally. On this occasion, the restriction $n \geq 6$ on the space dimension in the above result is not sharp. In 1985, Klainerman used some Lorentz invariant operators like momentum, angular momentum, and time–space expansion to establish some related decay estimates and improve this restriction to $n \geq 4$ by using the local solution extension method. Then the above table which ensures the global existence of classical solutions is turned into

$\alpha =$	1	2	$3, 4, \cdots$
$n \geq$	4	3	2

Now we consider the case that $\alpha = 1$ and $n = 3$. John (1981) already proved that the classical solution to the equation

$$\Box u = u_t^2 \quad (n = 3) \tag{1.2.2}$$

with any given compactly supported nontrivial initial data must blow up in a finite time, so in general we cannot expect the global existence of classical solutions and have to estimate the life-span of solutions. Based on a series of research given by John, Sideris, and Klainerman, finally Klainerman (1983), John & Klainerman (1984) obtained the following lower bound estimate to the life-span in general cases:

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-1}\}, \tag{1.2.3}$$

where a is a positive constant independent of ε. At this moment, the classical solution does not exist globally in general, but the life-span $\widetilde{T}(\varepsilon)$ grows exponentially as $\varepsilon \to 0$, namely, when $\varepsilon > 0$ is very small, the life-span is considerably large from the perspective of practical applications. Hence, such solutions are called **almost global solutions**.

Furthermore, we consider the cases that $\alpha = 1, n = 2$ and $\alpha = 2, n = 2$. For the case $n = 2$, Kovalyov (1987) proved that

$$\widetilde{T}(\varepsilon) \geq \begin{cases} b(\varepsilon \ln \varepsilon)^{-2}, & \alpha = 1, \\ \exp\{a\varepsilon^{-2}\}, & \alpha = 2, \end{cases} \tag{1.2.4}$$

where a, b are both positive constants independent of ε. But the above result is not sharp when $\alpha = 1$ and can be improved by

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-2}, \quad \alpha = 1, \tag{1.2.5}$$

which has been mentioned in the lecture notes of Hörmander in 1985, Li Tatsien and Yu Xin also proved this independently in 1989.

When $n = 1$, things are easier because of D'Alembert formula, and it can be proved that

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-\alpha}, \quad \forall \text{ integer } \alpha \geq 1, \tag{1.2.6}$$

where b is a positive constant independent of ε (see Li Tatsien and Yu Xin (1989).

In summary, we have the following table

	$\widetilde{T}(\varepsilon) \geq$				
$n \quad \alpha =$	1	2	$\cdots$	α	$\cdots$
1	$b\varepsilon^{-1}$	$b\varepsilon^{-2}$	$\cdots$	$b\varepsilon^{-\alpha}$	$\cdots$
2	$b\varepsilon^{-2}$	$\exp\{a\varepsilon^{-2}\}$			
3	$\exp\{a\varepsilon^{-1}\}$		$+\infty$		
$4, 5, \cdots$					

We point out that, the results in the above table are all sharp with respect to general nonlinear term F without any additional restrictions.

Now we consider the general case that the nonlinear term F on the right-hand side depends on u:

$$F = F(u, Du, D_x Du). \tag{1.2.7}$$

Since for wave equations, the energy estimates can give only the L^2 estimates to partial derivatives of the solution but not the L^2 estimate of the solution itself, things become much more complicated. To obtain the sharp lower bound estimate of the life-span, some delicate estimates need to be established to the solution itself of the wave equation.

For the most important case $\alpha = 1$, with the aid of the conformal mapping from $I\!R^{n+1}$ to $R \times S^n$, Christodoulou (1986) first proved the global existence of classical solutions with small initial data under the condition that $n \geq 5$ is odd. The essential restriction that the space dimension is odd was removed by Li Tatsien and Chen Yunmei (1987 1988b) by using the following concise and unified method (the **global iteration method**), they proved the global existence of classical solutions with small initial data under the following conditions satisfied by n and α:

$\alpha =$	1	$2, 3, \cdots$
$n \geq$	5	3

Now we turn to the case that $\alpha = 1$ and $n = 4$. After carefully analyzing the results and method by Li Tatsien and Chen Yunmei in the above paper, Hörmander, the Fields Medal winner, proved in Hörmander (1991) that

$$\widetilde{T}(\varepsilon) \geq \begin{cases} \exp\{a\varepsilon^{-1}\}, \\ +\infty, \qquad \text{when } F''_{uu}(0,0,0) = 0; \end{cases} \tag{1.2.8}$$

while, for the case that $\alpha = 1$ and $n = 3$, Lindblad (1990b) proved that

$$\widetilde{T}(\varepsilon) \geq \begin{cases} b\varepsilon^{-2}, \\ \exp\{a\varepsilon^{-1}\}, \ \text{when } F''_{uu}(0,0,0) = 0, \end{cases} \tag{1.2.9}$$

where a, b are both positive constants independent of ε. We can see from their results that: even if F depends on u, as long as it does not contain u^2, the life-span has the same lower bound estimate as in the special case that F does not depend on u when $n \geq 3$. This shows that the "worst" term is u^2.

The above full results for the cases that $n \geq 3$ and $\alpha \geq 1$ can also be obtained by the unified global iteration method (see Li Tatsien, Yu Xin and Zhou Yi 1991a, 1992b), which can be represented in the following table:

<table>
<tr><td></td><td colspan="2">$\widetilde{T}(\varepsilon) \geq$</td></tr>
<tr><td>$n \quad \alpha =$</td><td>1</td><td>2, 3, $\cdots$</td></tr>
<tr><td rowspan="2">3</td><td>$b\varepsilon^{-2}$</td><td rowspan="5">$+\infty$</td></tr>
<tr><td>$\exp\{a\varepsilon^{-1}\}$, when $F''_{uu}(0,0,0) = 0$</td></tr>
<tr><td rowspan="2">4</td><td>$\exp\{a\varepsilon^{-1}\}$</td></tr>
<tr><td>$+\infty$, when $F''_{uu}(0,0,0) = 0$</td></tr>
<tr><td>5,6,$\cdots$</td><td></td></tr>
</table>

Now we consider the rest cases. The case that $n = 1$ and $\alpha \geq 1$ is relatively simple; but the case that $n = 2$ and $\alpha \geq 1$ is more complicated. When $n = 1$ and $\alpha \geq 1$, we have

$$\widetilde{T}(\varepsilon) \geq \begin{cases} b\varepsilon^{-\frac{\alpha}{2}}, & \text{in general;} \\ b\varepsilon^{-\frac{\alpha(1+\alpha)}{2+\alpha}}, & \text{when } \int_{-\infty}^{+\infty} \psi(x)dx = 0; \\ b\varepsilon^{-\alpha}, & \text{when } \partial_u^\beta F(0,0,0) = 0, \ \forall\ 1+\alpha \leq \beta \leq 2\alpha, \end{cases} \tag{1.2.10}$$

where b is a positive constant independent of ε (see Li Tatsien, Yu Xin, and Zhou Yi 1991b, 1992a). When $n = 2$ and $\alpha \geq 1$, through rather delicate discussions, we have the following table:

<table>
<tr><td>$n = 2$</td><td colspan="3">$\widetilde{T}(\varepsilon) \geq$</td></tr>
<tr><td>$\alpha =$</td><td>1</td><td>2</td><td>3, 4, $\cdots$</td></tr>
<tr><td rowspan="3"></td><td>$be(\varepsilon)$</td><td rowspan="2">$b\varepsilon^{-6}$</td><td rowspan="3">$+\infty$</td></tr>
<tr><td>$b\varepsilon^{-1}$, when $\int \psi(x)dx = 0$</td></tr>
<tr><td>$b\varepsilon^{-2}$, when
$\partial_u^2 F(0,0,0) = 0$</td><td>$\exp\{a\varepsilon^{-2}\}$, when
$\partial_u^\beta F(0,0,0) = 0$
$(\beta = 3, 4)$</td></tr>
</table>

where a and b are both positive constants independent of ε, and $e(\varepsilon)$ is defined by

$$\varepsilon^2 e^2(\varepsilon) \ln(1 + e(\varepsilon)) = 1 \tag{1.2.11}$$

(see Li Tatsien and Zhou Yi 1993, 1994b).

We point out that all these results are sharp except for the case that $n = 4$ and $\alpha = 1$, for which the estimate

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-1}\} \tag{1.2.12}$$

obtained by Hörmander, can be improved by (see Li Tatsien and Zhou Yi (1995b–1995c), see also Lindblad and Sogge (1996) who simplified the proof to a certain extent)

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-2}\}, \tag{1.2.13}$$

which was already proved to be sharp.

The consideration in the above cases is made for very general nonlinear term F on the right-hand side. However, even the general nonlinear term on the right-hand side cannot ensure the global existence of classical solutions, it is still possible to obtain global classical solutions for nonlinear terms on the right-hand side, satisfying some special requirements, in particular, when the nonlinear term on the right-hand side has a certain compatibility with the wave operator.

To ensure the existence of global classical solutions, there is a kind of additional requirements on the nonlinear terms called the **null condition**, which is applicable to quite a lot important practical applications. Roughly speaking, the so-called null condition means that every small plane wave solution to the linearized equation (namely, the homogeneous linear wave equation) is still a solution to the corresponding nonlinear equation (namely, the nonlinear wave equation under consideration). For instance,

$$\Box u = u_t^2 - |\nabla u|^2 \quad (\text{here } \nabla \text{ is } D_x) \tag{1.2.14}$$

is exactly a nonlinear wave equation satisfying the null condition. See Christodoulou (1986), Christodoulou and Klainerman (1993), Klainerman (1986), Sogge (1995) for reference.

1.3 Methods

From the above historical survey we can see how a full result was obtained based on step-by-step efforts of many mathematicians in a considerably long period of time. Most of previous researches were conducted individually for a variety of nonlinear evolution equations and for different cases (different space dimensions, different values of α, special forms of F or general form of F, global existence or life-span estimates of classical solutions, $\cdots$) of the same equation, where various methods were adopted. To construct a theory edifice on an existing base, this base has to be cleaned up first. It turns out that this kind of problems can be unitedly treated by simply using the contraction mapping principle. This enables us to put forward a normalized treatment method, called the **global iteration method**, based on the contraction mapping principle to deal with the global existence and the life-span estimates of classical solutions. It shows up as a solving process composed of several pieces as follows:

estimates of solutions to the linearized problem

$\Downarrow$

selection of the solution space

$\Downarrow$

the contraction mapping principle

$\Downarrow$

the lower bound estimate of life-span (including the global existence).

This framework has some obvious features and advantages.

1. Universality

a. It is applicable to various types of nonlinear evolution equations. In addition to nonlinear wave equations, it is also applicable to nonlinear heat equations, nonlinear Schrödinger equations, and quite a few some other nonlinear evolution equations and coupled systems.

b. It imposes no additional restrictions to the specific form of the nonlinear term other than the power $1 + \alpha$ in a neighborhood of the origin.

c. It combines the treatment on both the global existence and the lower bound estimate of life-span of solutions in a unified way.

2. Conciseness

a. To use the global iteration method to the nonlinear evolution equation under consideration, it needs only to know clearly about the properties and related estimates of the solution to the corresponding linearized equation (mainly, the decay estimates as well as the energy estimates), and then construct a suitable function space where we can use the contraction mapping principle. The result for the nonlinear case can be obtained directly by dealing with everything on the basis of linearized problem.

b. All the results are represented by a simple relation between the space dimension $n(\geq 1)$ and $\alpha(\geq 1)$, moreover, in many situations the results can be obtained for all the corresponding cases of n and α in a unified way.

c. The workload when using this method equates roughly to that of proving the local existence of classical solutions.

3. Accuracy

Since the whole solving framework is based on the understanding on the solutions to the corresponding linearized problem, in the case of the global classical solutions, the decay rate of the obtained solution as $t \to +\infty$ remains exactly the same as that for the linearized problem.

To be specific, for the Cauchy problem (1.1.13)–(1.1.14) of nonlinear wave equations, in the special case that the right-hand side F does not depend on u explicitly (see (1.2.2)), it can be proved in a unified and concise way that (See Li and Yu (1989))

$$\widetilde{T}(\varepsilon) \geq \begin{cases} +\infty, & \text{if } K_0 > 1, \\ \exp\{a\varepsilon^{-\alpha}\}, & \text{if } K_0 = 1, \\ b\varepsilon^{-\frac{\alpha}{1-K_0}}, & \text{if } K_0 < 1, \end{cases} \tag{1.3.1}$$

where

$$K_0 = \frac{n-1}{2}\alpha, \tag{1.3.2}$$

and a and b are both positive constants independent of ε. This immediately gives us the first table in Sect. 1.2.

In the general case that F depends explicitly on u (see (1.2.7)), as stated before, the key point is to establish appropriate and subtle estimates on the solution itself to the Cauchy problem of the linear wave equation

$$\Box u = F(t, x), \tag{1.3.3}$$

$$t = 0 : u = f(x), u_t = g(x), \tag{1.3.4}$$

which then yield the accurate result for the lower bound estimate on the life-span. There is a known L^2 estimate for the solution when $n \geq 3$, called Von Wahl Inequality (See Wahl (1970)). Applying this inequality in the global iteration method leads to the global existence result of Li Tatsien and Chen Yunmei (1988b); but it is not sufficient for obtaining the accurate lower bound estimate of life-span. After analyzing the method in Li and Chen (1988b), Hörmander (1991) improved this inequality and then obtained the lower bound estimate (1.2.12) of life-span when $\alpha = 1$ and $n = 4$. Li Tatsien and Yu Xin (1991) observed that the solution of the wave equation has larger decay rate as $t \to +\infty$ inside the light cone, so they divided the whole space into two parts and introduced a new type of Banach space, established a new inequality — the generalized Von Wahl Inequality, and finally the global iteration method is applied to obtain uniformly and briefly the result shown in the second table in Sect. 1.2 in the case that $n \geq 3$ and $\alpha \geq 1$, that is,

$$\widetilde{T}(\varepsilon) \geq \begin{cases} +\infty, & \text{if } K > 1, \\ \exp\{a\varepsilon^{-\alpha}\}, & \text{if } K = 1, \\ b\varepsilon^{-\frac{\alpha}{1-K}}, & \text{if } K < 1, \end{cases} \tag{1.3.5}$$

where

$$K = \frac{(n-1)\alpha - 1}{2}, \tag{1.3.6}$$

and a and b are both positive constants independent of ε.

To obtain the result when $n = 2$, the corresponding results (see Li and Zhou 1993, 1994b) can only be obtained by the global iteration method after extending the above mentioned Von Wahl Inequality and generalized Von Wahl Inequality from the case of L^2 to $L^p (p \geq 1)$. In the case that $n = 4$ and $\alpha = 1$, more dlicate estimates (see Li and Zhou 1995b–1995c) are necessary to improve the estimate (1.2.12) of Hörmander.

We can figure out that, to obtain good results, the global iteration method, as a universal method, should be used in accordance with appropriate and subtle estimates of solutions to the linearized equations of the nonlinear evolution equations under consideration. In addition, it also has independent value in establishing the delicate estimates of solutions to the linear problem, which can be further applied to other occasion.

1.4 Supplements

The application of the global iteration method to the nonlinear wave equations has been stated as above and will be demonstrated in details in the main body of this book. Here, to help the reader has a more macroscopic view, we cite some results obtained by applying the global iteration method to some other nonlinear evolution equations, although they will be kept away from this book.

For the following Cauchy problem of nonlinear heat equations with small initial data

$$u_t - \triangle u = F(u, D_x u, D_x^2 u), \tag{1.4.1}$$

$$t = 0 : u = \varepsilon\phi(x), \tag{1.4.2}$$

the lower bound estimate on the life-span $\widetilde{T}(\varepsilon)$ of classical solutions can be expressed in the following table (see Zheng Songmu and Chen Yunmei 1986, Li Tatsien and Chen Yunmei 1988a, 1992):

	$\widetilde{T}(\varepsilon) \geq$		
$n \quad \alpha =$	1	2	$3, 4, \cdots$
1	$b\varepsilon^{-2}$	$\exp\{a\varepsilon^{-2}\}$	
2	$\exp\{a\varepsilon^{-1}\}$		
$3, 4, \cdots$		$+\infty$	

As a matter of fact, for each solution $u = u^0(t, x)$ to the heat equation

$$u_t - \triangle u = 0, \tag{1.4.3}$$

by using the heat kernel expression, it is easy to prove that

$$|u^0(t, x)| \leq C(1 + t)^{-\frac{n}{2}}, \quad \forall t \geq 0, \tag{1.4.4}$$

where C is a positive constant independent of t. Compared with (1.1.20), it can be found that the decay rate of solutions is bigger than that for the wave equation, then for nonlinear heat equations, the asymptotic stability of zero solution can be obtained in more situations. By using the global iteration method, it can be obtained at once that

$$\widetilde{T}(\varepsilon) \geq \begin{cases} +\infty, & \text{if } \overline{K} > 1, \\ \exp\{a\varepsilon^{-\alpha}\}, & \text{if } \overline{K} = 1, \\ b\varepsilon^{-\frac{\alpha}{1-\overline{K}}}, & \text{if } \overline{K} < 1, \end{cases} \tag{1.4.5}$$

where

$$\overline{K} = \frac{\alpha n}{2}. \tag{1.4.6}$$

This gives all the contents contained in the above table.

Now let us look at the following Cauchy problem of nonlinear dissipative wave equations with small initial data:

$$\Box u + u_t = F(u, Du, D_x Du), \tag{1.4.7}$$

$$t = 0 : u = \varepsilon\phi(x), u_t = \varepsilon\psi(x). \tag{1.4.8}$$

Regarding u as the displacement, the presence of the dissipation term u_t here is caused by the assumption that the damping force proportional to the velocity exists in the process of vibration. Thanks to this dissipation term, the decay rate of solutions becomes larger than that for the normal wave equations. In fact, for the linearized equation of (1.4.7)

$$\Box u + u_t = 0, \tag{1.4.9}$$

any given solution $u = u^0(t, x)$ has the same decay rate (1.4.4) as that for the heat equation, and then the life-span of solutions to equation (1.4.7) has the same lower bound estimate (1.4.5) as that for the heat equation (see Li Yachun 1996), and the results given are all sharp (see Li Tatsien and Zhou Yi 1995a).

There are also corresponding results for nonlinear Schrödinger equations, we will not go into details here.

1.5 Arrangement of Contents

This book includes 15 chapters.

This chapter (Chap. 1) is the introduction and overview. In addition to remaining the independent value of their own, Chaps. 2–7 are mainly in preparation for introducing basic results and contents of this book in later chapters. Thereinto, we introduce the solving formulas for linear wave equation in Chap. 2, Chap. 3 is focused on introducing some Sobolev type inequalities with decay factors, Chap. 4 is aimed at establishing various estimates to solutions of the linear wave equation, some estimates on product functions and composite functions are given in Chap. 5, Chap. 6 is devoted to establishing the general theory of the Cauchy problem for the second-order linear hyperbolic equations, and in Chap. 7 the Cauchy problem of nonlinear wave equations is reduced to the Cauchy problem of second-order quasi-linear hyperbolic systems in general, acting as a necessary preparation for later discussion.

In Chaps. 8–11, we fully have our discussion focused on the global existence and lower bound estimates of life-span for classical solutions to the Cauchy problem of nonlinear wave equations in the cases that the dimension $n = 1$, $n \geq 3$, $n = 2$ and $n = 4$, respectively, and all the results announced in Chap. 1 are proved. In Chap. 12, we focus on the null condition which is showed to have positive influence on the global existence of classical solutions to the Cauchy problem of nonlinear wave equations, and then we improve some results in the previous chapters accordingly.

In Chaps. 13 and 14, the upper bound estimates on the life-span of classical solutions are established for some typical examples, which show that all the above mentioned results about the lower bound estimates on the life-span of classical solutions to the Cauchy problem of nonlinear wave equations are sharp, and so the whole theory is close to the level of perfection.

In the end, in Chap. 15, examples are given to show some important applications and generalizations of the aforementioned results.

Here we point out that C or C_i $(i = 1, 2, \cdots)$ appearing in quite a few estimates stand for some generic positive constants and are not always mentioned each time they appear.

Chapter 2
Linear Wave Equations

2.1 Expression of Solutions

In this section we consider the following Cauchy problem of linear wave equations:

$$\Box u = F(t, x), \quad (t, x) \in I\!R \times I\!R^n, \tag{2.1.1}$$
$$t = 0 : u = f(x), u_t = g(x), \quad \in I\!R^n, \tag{2.1.2}$$

where $x = (x_1, \ldots, x_n)$,

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2} \tag{2.1.3}$$

is the $n-$dimensional wave operator, and F, f and g are given functions with suitable regularities.

According to the superposition principle and the Duhamel's principle based on this, to solve the Cauchy problem (2.1.1)–(2.1.2), it suffices to solve the Cauchy problem of the following homogeneous wave equation:

$$\Box u = 0, \quad (t, x) \in I\!R \times I\!R^n, \tag{2.1.4}$$
$$t = 0 : u = 0, u_t = g(x), \quad x \in I\!R^n. \tag{2.1.5}$$

Denote the solution of this problem by

$$u = S(t)g. \tag{2.1.6}$$

Here

$$S(t) : g \to u(t, \cdot), \tag{2.1.7}$$

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being a linear operator whose specific properties reflect the nature of wave equations, is the key object of study of this chapter.

If the solution of the Cauchy problem (2.1.4)–(2.1.5) is known, then it is easy to know that the solution to the Cauchy problem

$$\Box u = 0, \quad (t, x) \in I\!R \times I\!R^n, \tag{2.1.8}$$

$$t = 0 : u = f(x), u_t = 0, \quad x \in I\!R^n \tag{2.1.9}$$

can be expressed by

$$u = \frac{\partial}{\partial t}(S(t) f); \tag{2.1.10}$$

while, the solution to the Cauchy problem of the inhomogeneous wave equation:

$$\Box u = F(t, x), \quad (t, x) \in I\!R \times I\!R^n, \tag{2.1.11}$$

$$t = 0 : u = 0, u_t = 0, \quad x \in I\!R^n \tag{2.1.12}$$

can be expressed, according to the Duhamel's principle, by

$$u = \int_0^t S(t - \tau) F(\tau, \cdot) d\tau. \tag{2.1.13}$$

Therefore, in general the solution to the Cauchy problem (2.1.1)–(2.1.2) of wave equations can be represented uniformly by

$$u = \frac{\partial}{\partial t}(S(t) f) + S(t)g + \int_0^t S(t - \tau) F(\tau, \cdot) d\tau. \tag{2.1.14}$$

On the other hand, the solution to the Cauchy problem (2.1.4)–(2.1.5) can also be obtained by solving the Cauchy problem of the forms (2.1.8)–(2.1.9) or (2.1.11)–(2.1.12). In fact, if the solution v to the Cauchy problem

$$\Box v = 0, \quad (t, x) \in I\!R \times I\!R^n, \tag{2.1.15}$$

$$t = 0 : v = g, v_t = 0, \quad x \in I\!R^n \tag{2.1.16}$$

is already known, then

$$u = \int_0^t v(\tau, \cdot) d\tau \tag{2.1.17}$$

is exactly the solution to the Cauchy problem (2.1.4)–(2.1.5). Moreover, it is easy to show from (2.1.14) that the solution to the Cauchy problem

$$\Box u = g(x)\delta(t), \quad (t, x) \in I\!R \times I\!R^n, \tag{2.1.18}$$

$$t = -1 : u = 0, u_t = 0, \quad x \in I\!R^n \tag{2.1.19}$$

is exactly the solution to the Cauchy problem (2.1.4)–(2.1.5), where δ is the Dirac function.

2.1.1 Expression of Solutions When $n \leq 3$

When $n = 1$, as $t \geq 0$, the solution to the Cauchy problem (2.1.4)–(2.1.5) of the one-dimensional wave equation is given by the well-known d'Alembert formula:

$$u(t, x) = \frac{1}{2}\int_{x-t}^{x+t} g(y)dy. \tag{2.1.20}$$

When $n = 2$, as $t \geq 0$, the solution to the Cauchy problem (2.1.4)–(2.1.5) of the two-dimensional wave equation is given by the two-dimensional Poisson formula:

$$u(t, x) = \frac{1}{2\pi}\int_{|y-x|\leq t} \frac{g(y)}{\sqrt{t^2 - |y - x|^2}}dy, \tag{2.1.21}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, and

$$|y - x| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

When $n = 3$, as $t \geq 0$, the solution to the Cauchy problem (2.1.4)–(2.1.5) of the three-dimensional wave equation is given by the three-dimensional Poisson formula:

$$u(t, x) = \frac{1}{4\pi t}\int_{|y-x|=t} g(y)dS_y, \tag{2.1.22}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$,

$$|y - x| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2},$$

and dS_y stands for the area element of the sphere $|y - x| = t$.

The derivation of formulas (2.1.20)–(2.1.22) can be found, say, in Gu Chaohao, Li Tatsien et al. 1987.

We can find out from (2.1.20)–(2.1.22) that, when the space dimension $n \leq 3$, the expressions of the solution $u = u(t, x)$ to the Cauchy problem (2.1.4)–(2.1.5) involve only $g(x)$ itself but not its derivatives. Besides, when

$$g(x) \geq 0, \quad \forall x \in I\!R^n, \tag{2.1.23}$$

we always have

$$u(t,x) \geq 0, \quad \forall (t,x) \in I\!R_+ \times I\!R^n, \tag{2.1.24}$$

where $n = 1$, 2 and 3. This property is called the **positivity of the fundamental solution** (See Remark 2.2).

When $n \geq 4$, the fundamental solution does not have the positivity any longer. This can be shown by the expression of solutions, which will be derived later soon.

2.1.2 Method of Spherical Means

Here and throughout this section, we always assume that $n > 1$.

For any given function $\psi(x) = \psi(x_1, \ldots, x_n)$, denote by

$$h(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} \psi(y) dS_y \tag{2.1.25}$$

the integral mean of ψ on the sphere centered at $x = (x_1, \ldots, x_n)$ with radius r, where ω_n stands for the area of the unit sphere S^{n-1} in $I\!R^n$, dS_y is the area element of the sphere $|y - x| = r$, and $\omega_n r^{n-1}$ is the area of this sphere. The above formula can be easily rewritten as

$$h(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} \psi(x + r\xi) d\omega_\xi, \tag{2.1.26}$$

where $d\omega_\xi$ is the area element of the unit sphere S^{n-1}, and $\xi = (\xi_1, \ldots, \xi_n)$.

From the above formula, it turns out that the function $h(x,r)$ is well-defined not only for $r \geq 0$ but also for $r < 0$, and is an even function of r.

If $\psi \in C^2$, then it is obvious that $h \in C^2$, and

$$h(x,0) = \psi(x), \tag{2.1.27}$$

and since h is an even function of r, we have

$$\frac{\partial h}{\partial r}(x,0) = 0. \tag{2.1.28}$$

In addition, from (2.1.26) we have

$$\frac{\partial h(x,r)}{\partial r} = \frac{1}{\omega_n} \int_{|\xi|=1} \sum_{i=1}^{n} \psi_{x_i}(x + r\xi) \xi_i d\omega_\xi$$

$$= \frac{1}{\omega_n r^{n-1}} \int_{|\tilde{\xi}|=r} \sum_{i=1}^{n} \psi_{x_i}(x+\tilde{\xi})\xi_i dS,$$

where $\tilde{\xi} = r\xi$, and dS stands for the area element of the sphere $|\tilde{\xi}| = r$. Then, from the Green's formula we get

$$\frac{\partial h(x,r)}{\partial r} = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|\le r} \triangle\psi(y)dy, \tag{2.1.29}$$

where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{2.1.30}$$

is the n-D Laplacian operator.

Differentiating (2.1.29) once with respect to r, and using (2.1.29) again, we obtain

$$\begin{aligned} \frac{\partial^2 h(x,r)}{\partial r^2} &= -\frac{n-1}{\omega_n r^n} \int_{|y-x|\le r} \triangle\psi(y)dy \\ &\quad + \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} \triangle\psi(y)dS_y \\ &= -\frac{n-1}{r}\frac{\partial h(x,r)}{\partial r} \\ &\quad + \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} \triangle\psi(y)dS_y. \end{aligned} \tag{2.1.31}$$

On the other hand, from (2.1.26) we have

$$\begin{aligned} \triangle_x h(x,r) &= \frac{1}{\omega_n} \int_{|\xi|=1} \triangle_x \psi(x+r\xi)d\omega_\xi \\ &= \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} \triangle\psi(y)dS_y, \end{aligned} \tag{2.1.32}$$

where $\triangle_x$ stands for the Laplacian operator with respect to x (see (2.1.30)).

Combining (2.1.31)–(2.1.32) and noting (2.1.27)–(2.1.28), we obtain the following

Lemma 2.1 *Assume that $\psi(x) \in C^2$, then its spherical mean function $h(x,r) \in C^2$, and satisfies the following Darboux equation*

$$\frac{\partial^2 h(x,r)}{\partial r^2} + \frac{n-1}{r}\frac{\partial h(x,r)}{\partial r} = \triangle_x h(x,r) \tag{2.1.33}$$

and the initial condition

$$r = 0 : h = \psi(x), \frac{\partial h}{\partial r} = 0. \tag{2.1.34}$$

In particular, taking

$$\psi(x_1, \ldots, x_n) = \phi(x_1) \tag{2.1.35}$$

as a function depending only on x_1 *but not on* $x_2, \ldots, x_n$*, we can prove that its spherical mean function has the expression*

$$h(x, r) = \frac{\omega_{n-1}}{\omega_n} \int_{-1}^{1} \phi(x_1 + r\mu)(1 - \mu^2)^{\frac{n-3}{2}} d\mu, \tag{2.1.36}$$

where, ω_{n-1} *is taken to be* 2 *when* $n = 2$*, i.e., it is defined artificially that* $\omega_1 = 2$*, and the same below. This coincides with the value of* ω_1 *deduced by using (2.4.7) in this chapter when* $n = 2$*.*

In fact, from (2.1.26) we easily get

$$\begin{aligned} h(x, r) &= \frac{1}{\omega_n r^{n-1}} \int_{|y|=r} \psi(x + y) dS \\ &= \frac{1}{\omega_n r^{n-1}} \frac{\partial}{\partial r} \int_{|y| \le r} \psi(x + y) dy. \end{aligned} \tag{2.1.37}$$

Noticing (2.1.35), we have

$$\int_{|y| \le r} \psi(x + y) dy = \int_{\lambda^2 + |\tilde{y}|^2 \le r^2} \phi(x_1 + \lambda) d\lambda d\tilde{y},$$

where $\tilde{y} = (y_2, \ldots, y_n)$. Adopting polar coordinates to the variable $\tilde{y}$ and denoting $\rho = |\tilde{y}|$, the above formula can be rewritten as

$$\begin{aligned} &\int_{|y| \le r} \psi(x + y) dy \\ &= \omega_{n-1} \int_{\lambda^2 + \rho^2 \le r^2} \phi(x_1 + \lambda) \rho^{n-2} d\lambda d\rho \\ &= \omega_{n-1} \int_{-r}^{r} d\lambda \int_0^{\sqrt{r^2 - \lambda^2}} \phi(x_1 + \lambda) \rho^{n-2} d\rho, \end{aligned}$$

then it is easy to show that

$$\frac{\partial}{\partial r}\int_{|y|\le r}\psi(x+y)dy$$

$$=\omega_{n-1}r\int_{-r}^{r}\phi(x_1+\lambda)(r^2-\lambda^2)^{\frac{n-3}{2}}d\lambda$$

$$=\omega_{n-1}r^{n-1}\int_{-1}^{1}\phi(x_1+r\mu)(1-\mu^2)^{\frac{n-3}{2}}d\mu.$$

Thus, (2.1.36) follows from (2.1.37).

The spherical mean function $h(x,r)$ given by (2.1.36) depends only on x_1 and r, then the corresponding Darboux equation (2.1.33) is reduced to

$$\frac{\partial^2 h}{\partial r^2}+\frac{n-1}{r}\frac{\partial h}{\partial r}=\frac{\partial^2 h}{\partial x_1^2}, \tag{2.1.38}$$

moreover,

$$\frac{\partial^2 h}{\partial x_1^2}=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}\phi''(x_1+r\mu)(1-\mu^2)^{\frac{n-3}{2}}d\mu. \tag{2.1.39}$$

Taking $x_1=0$ in (2.1.38)–(2.1.39), we obtain

Lemma 2.2 *Suppose that*

$$h(r)=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}\phi(r\mu)(1-\mu^2)^{\frac{n-3}{2}}d\mu, \tag{2.1.40}$$

then we have

$$h''(r)+\frac{n-1}{r}h'(r)=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}\phi''(r\mu)(1-\mu^2)^{\frac{n-3}{2}}d\mu. \tag{2.1.41}$$

Now we apply the above results to solving the Cauchy problem of wave equations.

Suppose that $v=v(t,x)$ is the solution to the Cauchy problem (2.1.15)–(2.1.16). It is clear that v is an even function of t. Let

$$w(x,r)=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}v(r\mu,x)(1-\mu^2)^{\frac{n-3}{2}}d\mu. \tag{2.1.42}$$

Regarding x as a parameter, from Lemma 2.2 and using equation (2.1.15), it yields

$$\begin{aligned}&\frac{\partial^2 w(x,r)}{\partial r^2}+\frac{n-1}{r}\frac{\partial w(x,r)}{\partial r}\\&=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}v_{tt}(r\mu,x)(1-\mu^2)^{\frac{n-3}{2}}d\mu\\&=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}\triangle_x v(r\mu,x)(1-\mu^2)^{\frac{n-3}{2}}d\mu\\&=\triangle_x w(x,r),\end{aligned}$$

i.e., $w=w(x,r)$ satisfies the Darboux equation (2.1.33). Meanwhile, from (2.1.16), and taking particularly $\phi\equiv 1$ (thus its spherical mean is $h\equiv 1$) in (2.1.36), we have

$$\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}(1-\mu^2)^{\frac{n-3}{2}}d\mu=1, \tag{2.1.43}$$

then it is clear that

$$r=0:w=g(x),\ \frac{\partial w}{\partial r}=0. \tag{2.1.44}$$

Hence, it follows from Lemma 2.1 that

$$w(x,r)=\frac{1}{\omega_n}\int_{|\xi|=1}g(x+r\xi)d\omega_\xi. \tag{2.1.45}$$

Combining (2.1.42) and (2.1.45) and noting that v is an even function of t, we obtain

$$\frac{2\omega_{n-1}}{\omega_n}\int_{0}^{1}v(r\mu,x)(1-\mu^2)^{\frac{n-3}{2}}d\mu=\frac{1}{\omega_n}\int_{|\xi|=1}g(x+r\xi)d\omega_\xi. \tag{2.1.46}$$

Equation (2.1.46) is an integral equation satisfied by the solution $v=v(t,x)$ to the Cauchy (2.1.15)–(2.1.16). Therefore, the Cauchy problem (2.1.15)–(2.1.16) can be solved through inversion of (2.1.46).

Applying in (2.1.46) the variable substitution

$$r=\sqrt{s},\ r\mu=\sqrt{\sigma}, \tag{2.1.47}$$

and denoting

$$Q(r,x)=\frac{1}{\omega_n}\int_{|\xi|=1}g(x+r\xi)d\omega_\xi, \tag{2.1.48}$$

we obtain

$$\frac{\omega_{n-1}}{\omega_n}\int_0^s \frac{v(\sqrt{\sigma},x)}{\sqrt{\sigma}}(s-\sigma)^{\frac{n-3}{2}}d\sigma = s^{\frac{n-2}{2}}Q(\sqrt{s},x). \tag{2.1.49}$$

Ignoring for the time being the dependence with respect to x, and denoting

$$w(s) = s^{\frac{n-2}{2}}Q(\sqrt{s},x),\ \ \chi(\sigma) = \frac{v(\sqrt{\sigma},x)}{\sqrt{\sigma}}, \tag{2.1.50}$$

Equation (2.1.49) can be rewritten as

$$\frac{\omega_{n-1}}{\omega_n}\int_0^s \chi(\sigma)(s-\sigma)^{\frac{n-3}{2}}d\sigma = w(s). \tag{2.1.51}$$

Next we will solve the integral equation (2.1.51) so as to derive the expression of solutions to the Cauchy problem of wave equations as $n > 1$.

2.1.3 Expression of Solutions When $n(> 1)$ Is Odd

When $n(> 1)$ is odd, $\frac{n-3}{2}$ is a nonnegative integer, by taking derivatives of order $\frac{n-1}{2}$ on both sides of (2.1.51), we can solve that

$$\chi(s) = \frac{\omega_n}{\omega_{n-1}\cdot(\frac{n-3}{2})!}\left(\frac{d}{ds}\right)^{\frac{n-1}{2}} w(s), \tag{2.1.52}$$

thus, noting (2.1.50), we have

$$\frac{v(\sqrt{s},x)}{\sqrt{s}} = \frac{\omega_n}{\omega_{n-1}\cdot(\frac{n-3}{2})!}\left(\frac{d}{ds}\right)^{\frac{n-1}{2}}(s^{\frac{n-2}{2}}Q(\sqrt{s},x)). \tag{2.1.53}$$

Taking $s = t^2$ in the above formula, we get that the solution to the Cauchy problem (2.1.15)–(2.1.16) is

$$v(t,x) = \frac{\omega_n}{\omega_{n-1}\cdot(\frac{n-3}{2})!}t\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}}(t^{n-2}Q(t,x)),$$
$$(t,x)\in I\!R_+\times I\!R^n. \tag{2.1.54}$$

Using Theorem 2.5 in the appendix (Sect. 2.4) of this chapter, namely,

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}, \tag{2.1.55}$$

the above formula can also be written as

$$v(t,x) = \frac{\sqrt{\pi}}{\Gamma(\frac{n}{2})} t \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} (t^{n-2}Q(t,x)),$$
$$(t,x) \in I\!R \times I\!R^n. \tag{2.1.56}$$

Finally, using (2.1.17), we obtain the following

Theorem 2.1 *When $n(>1)$ is odd, the solution to the Cauchy problem (2.1.4)–(2.1.5) is*

$$u(t,x) = \frac{\sqrt{\pi}}{2\Gamma(\frac{n}{2})} \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} (t^{n-2}Q(t,x)), \tag{2.1.57}$$

where

$$Q(t,x) = \frac{1}{\omega_n}\int_{|\xi|=1} g(x+t\xi)d\omega_\xi. \tag{2.1.58}$$

Taking particularly $n=3$ in Theorem 2.1, and noting that $\omega_3 = 4\pi$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the three-dimensional Poisson formula (2.1.22) follows immediately.

2.1.4 Expression of Solutions When n(≥ 2) Is Even

When $n(\geq 2)$ is even, to obtain the solution $u = u(t,x)$ to the Cauchy problem (2.1.4)–(2.1.5), we can add an argument x_{n+1} artificially, and regard u as the solution to the following Cauchy problem

$$\Box_{n+1} u = 0, \tag{2.1.59}$$
$$t=0: u=0, u_t = g(x), \tag{2.1.60}$$

where $x = (x_1, \ldots, x_n)$, and

$$\Box_{n+1} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_{n+1}^2} \tag{2.1.61}$$

is the $(n+1)$-dimensional wave operator.

Applying Theorem 2.1 to the Cauchy (2.1.59)–(2.1.60), we get

$$u(t,x) = \frac{\sqrt{\pi}}{2\Gamma(\frac{n+1}{2})} \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} (t^{n-1}\overline{Q}(t,x)), \tag{2.1.62}$$

where

$$\overline{Q}(t,x) = \frac{1}{\omega_{n+1}} \int_{|\xi'|=1} g(x_1 + t\xi_1, \ldots, x_n + t\xi_n) d\omega_{\xi'}, \tag{2.1.63}$$

and $\xi' = (\xi, \xi_{n+1}) = (\xi_1, \ldots, \xi_n, \xi_{n+1})$.

Denote $y' = (y, y_{n+1})$. It is clear that

$$\begin{aligned}
\overline{Q}(t,x) &= \frac{1}{\omega_{n+1} t^n} \int_{|y'|=t} g(x+y) dS_{y'} \\
&= \frac{1}{\omega_{n+1} t^n} \frac{\partial}{\partial t} \int_{|y'|\le t} g(x+y) dy' \\
&= \frac{1}{\omega_{n+1} t^n} \frac{\partial}{\partial t} \int_{|y|\le t} \int_{-\sqrt{t^2-|y|^2}}^{\sqrt{t^2-|y|^2}} g(x+y) dy_{n+1} dy \\
&= \frac{2}{\omega_{n+1} t^n} \frac{\partial}{\partial t} \int_{|y|\le t} \sqrt{t^2-|y|^2}\, g(x+y) dy \\
&= \frac{2}{\omega_{n+1} t^{n-1}} \int_{|y|\le t} \frac{g(x+y)}{\sqrt{t^2-|y|^2}} dy \\
&= \frac{2}{\omega_{n+1} t^{n-1}} \int_{|y-x|\le t} \frac{g(y)}{\sqrt{t^2-|y-x|^2}} dy.
\end{aligned} \tag{2.1.64}$$

Thus, using (2.4.7) in the appendix (Sect. 2.4) of this chapter, namely,

$$\frac{\omega_{n+1}}{\omega_n} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \sqrt{\pi}, \tag{2.1.65}$$

we obtain

Theorem 2.2 *When $n (\geq 2)$ is even, the solution to the Cauchy problem (2.1.4)–(2.1.5) is*

$$u(t,x) = \frac{1}{\omega_n \Gamma(\frac{n}{2})} \left(\frac{1}{2t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} R(t,x), \tag{2.1.66}$$

where

$$R(t,x) = \int_{|y-x|\le t} \frac{g(y)}{\sqrt{t^2-|y-x|^2}} dy. \tag{2.1.67}$$

Taking particularly $n = 2$ in Theorem 2.2, and noting that $\omega_2 = 2\pi$, the two-dimensional Poisson formula (2.1.21) follows immediately.

Some of the results in Sects. 2.1.2–2.1.4 can be found in Courant and Hilbert (1989).

2.2 Expression of Fundamental Solutions

The solution $E = E(t, x)$ of the following Cauchy problem of wave equation

$$\Box E = 0, \quad (t, x) \in I\!R_+ \times I\!R^n, \tag{2.2.1}$$

$$t = 0 : E = 0,\ E_t = \delta(x), \quad x \in I\!R^n \tag{2.2.2}$$

in the sense of distributions, is called the **fundamental solution** of the wave operator. In (2.2.2), $\delta(x)$ is the Dirac function.

Obviously, when we find the fundamental solution $E = E(t, x)$, the solution to the Cauchy problem (2.1.4–2.1.5) can be expressed by

$$S(t)g = E(t, \cdot) * g, \quad \forall t \geq 0, \tag{2.2.3}$$

where $*$ stands for the convolution of distributions.

Conversely, if there exists a distribution E such that (2.2.3) holds for any given function g, then E must be the fundamental solution of the wave operator.

Now we derive the expression of the fundamental solution of the wave operator.

For any given $a > 0$, define the function

$$\chi_+^a(y) = \frac{(\max(y, 0))^a}{\Gamma(a+1)} = \begin{cases} \frac{y^a}{\Gamma(a+1)}, & y \geq 0, \\ 0, & y < 0. \end{cases} \tag{2.2.4}$$

$\chi_+^a(y)$ is a continuous function of y, whose support is $\{y \geq 0\}$. It is easy to show that, as $a > 0$ we have

$$\frac{d}{dy}\chi_+^{a+1}(y) = \chi_+^a(y). \tag{2.2.5}$$

Since one can keep differentiating a continuous function in the sense of distributions, $\chi_+^a(y)$ can be defined inductively for $a \leq 0$ in the category of distributions by using the above formula. Hence, for any given real number a, the function $\chi_+^a(y)$ with support $\subseteq \{y \geq 0\}$ can be defined. It is easy to know that $\chi_+^a(y)$ is a homogeneous function of degree a with respect to y, and

$$\text{sing supp}\chi_+^a \subseteq \{y = 0\}, \tag{2.2.6}$$

where sing supp stands for the singular support of distributions.

In particular, we have

$$\chi_+^0(y) = \frac{d}{dy}\chi_+^1(y) = H(y), \tag{2.2.7}$$

where

$$H(y) = \begin{cases} 1, & y > 0, \\ 0, & y < 0 \end{cases} \tag{2.2.8}$$

is the Heaviside function. Then

$$\chi_+^{-1}(y) = \frac{d}{dy}\chi_+^0(y) = \delta(y). \tag{2.2.9}$$

In addition, noticing that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, it is easy to show that

$$\chi_+^{-\frac{1}{2}}(y) = \frac{d}{dy}\chi_+^{\frac{1}{2}}(y) = \begin{cases} \frac{1}{\sqrt{\pi y}}, & y > 0, \\ 0, & y < 0. \end{cases} \tag{2.2.10}$$

Theorem 2.3 *The fundamental solution of the $n(\geq 1)$-dimensional wave operator is*

$$E(t, x) = \frac{1}{2\pi^{\frac{n-1}{2}}}\chi_+^{-\frac{n-1}{2}}(t^2 - |x|^2). \tag{2.2.11}$$

Proof It suffices to verify (2.2.3).

When $n = 1$, from (2.2.11) and noting (2.2.7) we have

$$\begin{aligned} E(t, \cdot) * g &= \frac{1}{2}\int H(t^2 - |x-y|^2)g(y)dy \\ &= \frac{1}{2}\int H(t - |x-y|)g(y)dy \\ &= \frac{1}{2}\int_{|y-x|\leq t} g(y)dy \\ &= \frac{1}{2}\int_{x-t}^{x+t} g(y)dy. \end{aligned}$$

From D'Alembert formula (2.1.20), it yields (2.2.3) as $n = 1$.

When $n(\geq 2)$ is even, noting that due to (2.2.10) we have

$$\chi_+^{-\frac{1}{2}}(t^2 - |\cdot|^2) * g = \frac{1}{\sqrt{\pi}}\int_{|y-x|\leq t} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}}dy,$$

then from Theorem 2.2 and noting (2.1.55) we have

$$S(t)g = \frac{\sqrt{\pi}}{\omega_n\Gamma(\frac{n}{2})}\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}(\chi_+^{-\frac{1}{2}}(t^2 - |\cdot|^2) * g)$$

$$
\begin{aligned}
&= \frac{\sqrt{\pi}}{\omega_n \Gamma(\frac{n}{2})} \chi_+^{-\frac{n-1}{2}}(t^2 - |\cdot|^2) * g \\
&= E(t, \cdot) * g,
\end{aligned}
$$

i.e., (2.2.3) is satisfied when $n(\geq 2)$ is even.

When $n(\geq 3)$ is odd, noting that due to (2.2.9) we have

$$
\begin{aligned}
&\chi_+^{-1}(t^2 - |\cdot|^2) * g \\
&= \int \delta(t^2 - |x-y|^2) g(y) dy \\
&= \int \delta((t + |x-y|)(t - |x-y|)) g(y) dy \\
&= \int \delta(2t(t - |x-y|)) g(y) dy \\
&= \frac{1}{2t} \int \delta(t - |x-y|) g(y) dy \\
&= \frac{1}{2t} \int_{|y-x|=t} g(y) dS_y \\
&= \frac{t^{n-2}}{2} \int_{|\xi|=1} g(x + t\xi) d\omega_\xi,
\end{aligned}
$$

then from Theorem 2.1 and noting (2.1.55) we have

$$
\begin{aligned}
S(t)g &= \frac{\sqrt{\pi}}{\omega_n \Gamma(\frac{n}{2})} \left(\frac{1}{2t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} (\chi_+^{-1}(t^2 - |\cdot|^2) * g) \\
&= \frac{\sqrt{\pi}}{\omega_n \Gamma(\frac{n}{2})} \chi_+^{-\frac{n-1}{2}}(t^2 - |\cdot|^2) * g \\
&= E(t, \cdot) * g,
\end{aligned}
$$

i.e., (2.2.3) is satisfied when $n(\geq 3)$ is odd.

The proof of Theorem 2.3 is finished. □

Remark 2.1 Noting (2.2.9) and (2.2.5), from Theorem 2.3 we easily know that: when $n(> 1)$ is odd, the support of the fundamental solution $E(t, x)$ is the characteristic cone $|x| = t$.

Remark 2.2 From Theorem 2.3, it is easy to show that the fundamental solution of the wave operator as $n = 1$ is

$$
E(t, x) = \begin{cases} \frac{1}{2}, & |x| \leq t, \\ 0, & |x| > t; \end{cases} \tag{2.2.12}
$$

as $n = 2$ it is

$$E(t,x) = \begin{cases} \frac{1}{2\pi\sqrt{t^2-|x|^2}}, & |x| \le t, \\ 0, & |x| > t, \end{cases} \tag{2.2.13}$$

where $x = (x_1, x_2)$; while, as $n = 3$ it is

$$E(t,x) = \frac{\delta(|x|-t)}{4\pi|x|}, \tag{2.2.14}$$

where $x = (x_1, x_2, x_3)$. These coincide with the results shown by (2.1.20)–(2.1.22), and indicate directly the positivity of fundamental solutions as $n = 1$, 2 and 3 shown in Sect. 2.1.1.

2.3 Fourier Transform

The solution of the Cauchy problem to linear wave equations can also be obtained by the Fourier transform.

Taking the Fourier transform in the Cauchy problem (2.1.4)–(2.1.5) with respect to the argument x, we have

$$\hat{u}_{tt}(t,\xi) + |\xi|^2\hat{u}(t,\xi) = 0, \tag{2.3.1}$$

$$t = 0 : \hat{u} = 0, \hat{u}_t = \hat{g}(\xi), \tag{2.3.2}$$

where $\hat{u}$ and $\hat{g}$ stand for the Fourier transforms of u and g, respectively. Regarding ξ as a parameter and solving the above Cauchy problem of ordinary differential equation, we immediately get

$$\hat{u}(t,\xi) = \frac{\sin(|\xi|t)}{|\xi|}\hat{g}(\xi). \tag{2.3.3}$$

Using (2.1.14), we obtain the following

Theorem 2.4 *Suppose that $u = u(t,x)$ is the solution of the Cauchy problem (2.1.1)–(2.1.2), then the Fourier transform of u with respect to x is*

$$\begin{aligned}\hat{u}(t,\xi) = \cos(|\xi|t)\hat{f}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\hat{g}(\xi) \\ + \int_0^t \frac{\sin(|\xi|(t-\tau))}{|\xi|}\hat{F}(\tau,\xi)d\tau.\end{aligned} \tag{2.3.4}$$

Hereafter, we will utilize Theorem 2.4 to establish some estimates on solutions to the Cauchy problem of wave equations.

2.4 Appendix—The Area of Unit Sphere

It is known that Γ function is defined by (see Chen and Yu 2010):

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \forall z > 0. \tag{2.4.1}$$

We have

$$\Gamma(z+1) = z\Gamma(z), \quad \forall z > 0, \tag{2.4.2}$$

and when z is a positive integer,

$$\Gamma(z+1) = z!. \tag{2.4.3}$$

Moreover

$$\Gamma(1) = 1 \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{2.4.4}$$

B function is defined by (see Chen and Yu 2010)

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \quad \forall p, q > 0, \tag{2.4.5}$$

and we have

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{2.4.6}$$

Taking $x = \mu^2$ in the following operations, and noting (2.4.6) and (2.4.4), when $n > 1$ we have

$$\begin{aligned}
\int_{-1}^{1} (1-\mu^2)^{\frac{n-3}{2}} d\mu &= 2\int_0^1 (1-\mu^2)^{\frac{n-3}{2}} d\mu \\
&= \int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{n-3}{2}} dx \\
&= B\left(\frac{1}{2}, \frac{n-1}{2}\right) \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \\
&= \frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})},
\end{aligned}$$

then from (2.1.43) we obtain: when $n > 1$ we have

$$\frac{\omega_n}{\omega_{n-1}} = \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}\sqrt{\pi}. \tag{2.4.7}$$

This shows that $\{\Gamma(\frac{n}{2})\omega_n\}$ forms a geometric sequence with common ratio $\sqrt{\pi}$. Hence, noticing that $\omega_2 = 2\pi$, we have

$$\Gamma\left(\frac{n}{2}\right)\omega_n = \pi^{\frac{n-2}{2}}(\Gamma(1)\omega_2) = 2\pi^{\frac{n}{2}},$$

then we obtain the following

Theorem 2.5 *The area of the unit sphere S^{n-1} in $n(> 1)$-dimensional space $\mathbb{R}^n$ is*

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \tag{2.4.8}$$

Chapter 3
Sobolev Type Inequalities with Decay Factor

3.1 Preliminaries

In this chapter we are going to establish Sobolev type inequalities with decay factor. The key point is to consider the Lorentz invariance of the wave operator and then introduce a group of first-order partial differential operators instead of the normal differential operators in the differential operations (see Klainerman 1985).

To illustrate this, denote

$$x_0 = t,\ x = (x_1, \ldots, x_n), \tag{3.1.1}$$

and we make the following convention about the range of related letters as superscripts or subscripts, if there are no special instructions:

$$a, b, c, \ldots = 0, 1, \ldots, n; \tag{3.1.2}$$
$$i, j, k, \ldots = 1, \ldots, n. \tag{3.1.3}$$

Introduce the Lorentz metric

$$\eta = (\eta^{ab})_{a,b=0,1,\ldots,n} = \mathrm{diag}\{-1, 1, \ldots, 1\}, \tag{3.1.4}$$

and denote

$$\partial_0 = -\frac{\partial}{\partial t},\ \partial_i = \frac{\partial}{\partial x_i} \quad (i = 1, \ldots, n), \tag{3.1.5}$$

then the n-dimensional wave operator can be written as

$$\Box = -\eta^{ab}\partial_a\partial_b = \partial_0^2 - \partial_1^2 - \cdots - \partial_n^2. \tag{3.1.6}$$

Here and hereafter, we make the convention that repeated indices stand for summation, then

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$$\eta^{ab}\partial_a\partial_b = \sum_{a,b=0}^{n} \eta^{ab}\partial_a\partial_b.$$

Introduce the following first-order partial differential operators:

$$\Omega_{ab} = x_a\partial_b - x_b\partial_a = -\Omega_{ba} \quad (a, b = 0, 1, \dots, n), \tag{3.1.7}$$
$$L_0 = \eta^{ab}x_a\partial_b = t\partial_t + x_1\partial_1 + \cdots + x_n\partial_n, \tag{3.1.8}$$
$$\partial = (\partial_0, \partial_1, \dots, \partial_n) = (-\partial_t, \partial_1, \dots, \partial_n), \tag{3.1.9}$$
$$\partial_x \overset{\text{def.}}{=} D_x = (\partial_1, \dots, \partial_n), \tag{3.1.10}$$

They will play important roles in later discussion. From (3.1.7) we particularly have

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i = -\Omega_{ji} \quad (i, j = 1, \dots, n), \tag{3.1.11}$$
$$\Omega_{0i} = t\partial_i + x_i\partial_t \overset{\text{def.}}{=} L_i \quad (i = 1, \dots, n). \tag{3.1.12}$$

Denote

$$\Omega_x = (\Omega_{ij})_{1\le i<j\le n}, \tag{3.1.13}$$
$$\Omega = (\Omega_{ab})_{0\le a<b\le n}, \tag{3.1.14}$$
$$L = (L_0, L_1, \dots, L_n), \tag{3.1.15}$$
$$\widehat{\Omega}_x = (\Omega_x, \partial_x), \tag{3.1.16}$$
$$\overline{\Omega} = (\Omega, L_0) = (\Omega_x, L), \tag{3.1.17}$$
$$\Gamma = (\Omega, L_0, \partial) = (\Omega_x, L, \partial) = (\overline{\Omega}, \partial). \tag{3.1.18}$$

For future needs, we give some simple and important properties for the sets made up of these first-order partial differential operators.

3.1.1 Commutant Relations

Lemma 3.1.1 *The following commutant relations hold:*

$$[\partial_a, \partial_b] = 0, \tag{3.1.19}$$
$$[\Omega_{ab}, \Omega_{cd}] = \eta^{bc}\Omega_{ad} + \eta^{ad}\Omega_{bc} - \eta^{bd}\Omega_{ac} - \eta^{ac}\Omega_{bd}, \tag{3.1.20}$$
$$[L_0, \Omega_{ab}] = 0, \tag{3.1.21}$$
$$[\Omega_{ab}, \partial_c] = \eta^{bc}\partial_a - \eta^{ac}\partial_b, \tag{3.1.22}$$
$$[L_0, \partial_a] = -\partial_a, \tag{3.1.23}$$

where $[\cdot, \cdot]$ *stands for the Poisson bracket, i.e.,*

$$[A, B] = AB - BA. \tag{3.1.24}$$

Proof Equation (3.1.19) is obvious. To prove other commutant relations, we first point out that we always have

$$\partial_a x_b = \eta^{ab} = \eta^{ba} \quad (a, b = 0, 1, \ldots, n). \tag{3.1.25}$$

Then

$$\begin{aligned}
&[\Omega_{ab}, \Omega_{cd}] \\
&= \Omega_{ab}\Omega_{cd} - \Omega_{cd}\Omega_{ab} \\
&= (x_a\partial_b - x_b\partial_a)(x_c\partial_d - x_d\partial_c) \\
&\quad -(x_c\partial_d - x_d\partial_c)(x_a\partial_b - x_b\partial_a) \\
&= x_a(\partial_b x_c)\partial_d - x_c(\partial_d x_a)\partial_b - x_b(\partial_a x_c)\partial_d + x_c(\partial_d x_b)\partial_a \\
&\quad -x_a(\partial_b x_d)\partial_c + x_d(\partial_c x_a)\partial_b + x_b(\partial_a x_d)\partial_c - x_d(\partial_c x_b)\partial_a \\
&= \eta^{bc}x_a\partial_d - \eta^{da}x_c\partial_b - \eta^{ac}x_b\partial_d + \eta^{db}x_c\partial_a \\
&\quad -\eta^{bd}x_a\partial_c + \eta^{ca}x_d\partial_b + \eta^{ad}x_b\partial_c - \eta^{cb}x_d\partial_a \\
&= \eta^{bc}(x_a\partial_d - x_d\partial_a) + \eta^{ad}(x_b\partial_c - x_c\partial_b) \\
&\quad -\eta^{bd}(x_a\partial_c - x_c\partial_a) - \eta^{ac}(x_b\partial_d - x_d\partial_b) \\
&= \eta^{bc}\Omega_{ad} + \eta^{ad}\Omega_{bc} - \eta^{bd}\Omega_{ac} - \eta^{ac}\Omega_{bd},
\end{aligned}$$

this is just (3.1.20).

$$\begin{aligned}
&[L_0, \Omega_{ab}] \\
&= L_0\Omega_{ab} - \Omega_{ab}L_0 \\
&= \eta^{cd}x_c\partial_d(x_a\partial_b - x_b\partial_a) - (x_a\partial_b - x_b\partial_a)\eta^{cd}x_c\partial_d \\
&= \eta^{cd}x_c(\partial_d x_a)\partial_b - x_a\eta^{cd}(\partial_b x_c)\partial_d \\
&\quad -\eta^{cd}x_c(\partial_d x_b)\partial_a + x_b\eta^{cd}(\partial_a x_c)\partial_d \\
&= \eta^{cd}\eta^{da}x_c\partial_b - \eta^{cd}\eta^{bc}x_a\partial_d - \eta^{cd}\eta^{db}x_c\partial_a + \eta^{cd}\eta^{ac}x_b\partial_d \\
&= x_a\partial_b - x_a\partial_b - x_b\partial_a + x_b\partial_a = 0,
\end{aligned}$$

this is just (3.1.21).

$$\begin{aligned}
[\Omega_{ab}, \partial_c] &= \Omega_{ab}\partial_c - \partial_c\Omega_{ab} \\
&= (x_a\partial_b - x_b\partial_a)\partial_c - \partial_c(x_a\partial_b - x_b\partial_a) \\
&= -(\partial_c x_a)\partial_b + (\partial_c x_b)\partial_a = \eta^{bc}\partial_a - \eta^{ac}\partial_b,
\end{aligned}$$

this is exactly (3.1.22).

Finally,

$$\begin{aligned}[L_0, \partial_a] &= L_0\partial_a - \partial_a L_0 \\ &= (\eta^{cd} x_c \partial_d)\partial_a - \partial_a(\eta^{cd} x_c \partial_d) \\ &= -\eta^{cd}(\partial_a x_c)\partial_d = -\eta^{cd}\eta^{ac}\partial_d = -\partial_a,\end{aligned}$$

this is exactly (3.1.23).

Lemma 3.1.1 is proved. □

By mathematical induction, from (3.1.22)–(3.1.23) we easily obtain the following

Corollary 3.1.1 *For any given multi-index $k = (k_1, \ldots, k_\sigma)$, we have*

$$\begin{aligned}[\partial_a, \Gamma^k] &= \sum_{|i|\le|k|-1} A_{ki}\Gamma^i D \\ &= \sum_{|i|\le|k|-1} \widetilde{A}_{ki} D\Gamma^i \quad (a = 0, 1, \ldots, n),\end{aligned} \tag{3.1.26}$$

where $|k| = k_1 + \cdots + k_\sigma$, σ stands for the number of partial differential operators in the set $\Gamma = (\Gamma_1, \ldots, \Gamma_\sigma)$,

$$\begin{aligned}\Gamma^k &= \Gamma_1^{k_1} \cdots \Gamma_\sigma^{k_\sigma}, \\ D &= \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right),\end{aligned} \tag{3.1.27}$$

$i = (i_1, \ldots, i_\sigma)$ is a multi-index, $|i| = i_1 + \cdots + i_\sigma$, and A_{ki} and $\widetilde{A}_{ki}$ are all constants.

3.1.2 $L^{p,q}(\mathbb{R}^n)$ space

We first introduce the $L^{p,q}(\mathbb{R}^n)$ space (first introduced by Li and Yu (1989, 1991)).

Definition 3.1.1 If

$$g(r, \xi) \triangleq f(r\xi) r^{\frac{n-1}{p}} \in L^p(0, +\infty; L^q(S^{n-1})), \tag{3.1.28}$$

where $r = |x|$, $\xi = (\xi_1, \ldots, \xi_n) \in S^{n-1}$ (S^{n-1} is the unit sphere in $\mathbb{R}^n$: $|\xi| = 1$), $1 \le p, q \le +\infty$, then we say that $f = f(x) \in L^{p,q}(\mathbb{R}^n)$, endowed with the norm

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} \overset{\text{def.}}{=} \|f(r\xi) r^{\frac{n-1}{p}}\|_{L^p(0,+\infty; L^q(S^{n-1}))}. \tag{3.1.29}$$

From (3.1.28)–(3.1.29), for $1 \le p, q < +\infty$ we have

$$\|f\|_{L^{p,q}(I\!R^n)} = \left(\int_0^{\infty} \|f(r\xi)\|^p_{L^q(S^{n-1})} r^{n-1}\mathrm{d}r\right)^{\frac{1}{p}}$$
$$= \left(\int_0^{\infty} \left(\int_{|\xi|=1} |f(r\xi)|^q \mathrm{d}\omega_\xi\right)^{\frac{p}{q}} r^{n-1}\mathrm{d}r\right)^{\frac{1}{p}},$$

where $\mathrm{d}\omega_\xi$ is the area element of S^{n-1}; for $1 \le p < +\infty$ and $q = +\infty$ we have

$$\|f\|_{L^{p,\infty}(I\!R^n)} = \left(\int_0^{\infty} \|f(r\xi)\|^p_{L^\infty(S^{n-1})} r^{n-1}\mathrm{d}r\right)^{\frac{1}{p}}$$
$$= \left(\int_0^{\infty} \left(\text{ess sup}_{|\xi|=1}|f(r\xi)|\right)^p r^{n-1}\mathrm{d}r\right)^{\frac{1}{p}};$$

for $p = +\infty$ and $1 \le q < +\infty$ we have

$$\|f\|_{L^{\infty,q}(I\!R^n)} = \text{ess sup}_{0\le r<\infty}\|f(r\xi)\|_{L^p(S^{n-1})}$$
$$= \text{ess sup}_{0\le r<\infty}\left(\int_{|\xi|=1} |f(r\xi)|^q \mathrm{d}\omega_\xi\right)^{\frac{1}{q}};$$

for $p = q = +\infty$ we have

$$\|f\|_{L^{\infty,\infty}(I\!R^n)} = \operatorname*{ess\,sup}_{\substack{0\le r<\infty\\|\xi|=1}} |f(r\xi)| = \operatorname*{ess\,sup}_{x\in I\!R^n} |f(x)|.$$

Lemma 3.1.2 *Endowed with the norm (3.1.29), $L^{p,q}(I\!R^n)$ is a Banach space. Moreover, when $p = q$, $L^{p,q}(I\!R^n)$ is reduced to the normal $L^p(I\!R^n)$ space:*

$$L^{p,p}(I\!R^n) = L^p(I\!R^n). \tag{3.1.30}$$

As stated above, $L^{p,q}(I\!R^n)$ is a space which is L^p in the radial direction and is L^q on the sphere, and will play an important role in later discussion.

3.1.3 Generalized Sobolev Norms

Lemma 3.1.1 tells us that, the elements of each of the sets Ω_x, Ω, $\widehat{\Omega}_x$, $\bar{\Omega}$ and Γ of first-order partial differential operators can span to a Lie algebra, that is to say, the commutant operator of any two given operators in a corresponding set can be represented by a linear combination of operators in it with constant coefficients. Therefore, the normal differential operators can be replaced by these partial differential operators so as to constitute the corresponding generalized Sobolev norms.

Denote by $A = (A_i)_{1 \le i \le \sigma}$ any given set in the sets Ω_x, Ω, $\widehat{\Omega}_x$, $\overline{\Omega}$ and Γ of partial differential operators, for any given function $u = u(t, x)$, for which the norms on the right-hand side of the following formula are well defined, we can use

$$\|u(t,\cdot)\|_{A,N,p,q} = \sum_{|k| \le N} \|A^k u(t,\cdot)\|_{L^{p,q}(\mathbb{R}^n)}, \quad \forall t \ge 0 \tag{3.1.31}$$

and

$$\|u(t,\cdot)\|_{A,N,p} = \|u(t,\cdot)\|_{A,N,p,p}, \quad \forall t \ge 0 \tag{3.1.32}$$

to define the corresponding generalized Sobolev norms, where N is any given nonnegative integer, $k = (k_1, \ldots, k_\sigma)$ is a multi-index, $|k| = k_1 + \cdots + k_\sigma$, and $A^k = A_1^{k_1} \cdots A_\sigma^{k_\sigma}$.

In particular, since the sets Ω_x and $\widehat{\Omega}_x$ contain only the partial derivatives with respect to x, for any given set A in them, we can also define the corresponding generalized Sobolev norms to any given function $u = u(x)$ depending only on x by

$$\|u(\cdot)\|_{A,N,p,q} = \sum_{|k| \le N} \|A^k u(\cdot)\|_{L^{p,q}(\mathbb{R}^n)} \tag{3.1.33}$$

and

$$\|u(\cdot)\|_{A,N,p} = \|u(\cdot)\|_{A,N,p,p}. \tag{3.1.34}$$

Thanks to the Lie algebraic properties of the operator sets Ω_x, Ω, $\widehat{\Omega}_x$, $\bar{\Omega}$ and Γ, the above defined generalized Sobolev norms corresponding to different orderings of operators in the set A are all equivalent to each other, and moreover, the equivalence of the norms showed by (3.1.31)–(3.1.32) is uniform with respect to t. Therefore, different orderings of operators do not influence the definition of norms substantially.

Specifically, denote by $\bar{A} = (\bar{A}_i)_{1 \le i \le \sigma}$ the same operator set $A = (A_i)_{1 \le i \le \sigma}$ with only different ordering of the operators within it, then we have

$$\begin{aligned} C_1 \|u(t,\cdot)\|_{A,N,p,q} &\le \|u(t,\cdot)\|_{\bar{A},N,p,q} \\ &\le C_2 \|u(t,\cdot)\|_{A,N,p,q}, \quad \forall t \ge 0, \end{aligned} \tag{3.1.35}$$

and so forth, where C_1 and C_2 are positive constants independent of both the choice of $u = u(t, x)$ and t.

In particular, choosing Γ as the operator set A, we obtain the corresponding generalized Sobolev norms $\|u(t,\cdot)\|_{\Gamma,N,p,q}$ and $\|u(t,\cdot)\|_{\Gamma,N,p}$. From Corollary 3.1.1 we have

Lemma 3.1.3 *For any given integer $N \ge 0$, we have*

$$\begin{aligned} c\|Du(t,\cdot)\|_{\Gamma,N,p,q} &\le \sum_{|k| \le N} \|D\Gamma^k u(t,\cdot)\|_{L^{p,q}(\mathbb{R}^n)} \\ &\le C\|Du(t,\cdot)\|_{\Gamma,N,p,q}, \quad \forall t \ge 0, \end{aligned} \tag{3.1.36}$$

where $1 \leq p, q \leq +\infty$, *c and C are positive constants independent of both the choice of* $u = u(t, x)$ *and* t. □

3.1.4 Commutativity with the Wave Operator

Now we prove that all the operators except L_0 in the set Γ of partial differential operators are commutative with the wave operator $\Box$, and the commutant operator of L_0 and $\Box$ is just an amplification of $\Box$. In other words, we want to prove

Lemma 3.1.4 *We have the following commutant relations:*

$$[\partial_a, \Box] = 0, \tag{3.1.37}$$

$$[\Omega_{ab}, \Box] = 0, \tag{3.1.38}$$

$$[L_0, \Box] = -2\Box. \tag{3.1.39}$$

Proof Formula (3.1.37) is obvious. Noting (3.1.6)–(3.1.8) and (3.1.25), we have

$$\begin{aligned}
[\Omega_{ab}, \Box] &= \Omega_{ab}\Box - \Box\Omega_{ab} \\
&= \eta^{cd}\partial_c\partial_d(x_a\partial_b - x_b\partial_a) - (x_a\partial_b - x_b\partial_a)\eta^{cd}\partial_c\partial_d \\
&= \eta^{cd}\partial_c[(\partial_d x_a)\partial_b] + \eta^{cd}(\partial_c x_a)\partial_d\partial_b \\
&\quad -\eta^{cd}\partial_c[(\partial_d x_b)\partial_a] - \eta^{cd}(\partial_c x_b)\partial_d\partial_a \\
&= \eta^{cd}\eta^{da}\partial_c\partial_b + \eta^{cd}\eta^{ca}\partial_d\partial_b - \eta^{cd}\eta^{db}\partial_c\partial_a \\
&\quad -\eta^{cd}\eta^{cb}\partial_d\partial_a \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
[L_0, \Box] &= L_0\Box - \Box L_0 \\
&= \eta^{cd}\partial_c\partial_d(\eta^{ab}x_a\partial_b) - \eta^{ab}x_a\partial_b(\eta^{cd}\partial_c\partial_d) \\
&= \eta^{cd}\eta^{ab}\partial_c[(\partial_d x_a)\partial_b] + \eta^{cd}\eta^{ab}(\partial_c x_a)\partial_d\partial_b \\
&= \eta^{cd}\eta^{ab}\eta^{da}\partial_c\partial_b + \eta^{cd}\eta^{ab}\eta^{ca}\partial_d\partial_b \\
&= 2\eta^{ab}\partial_a\partial_b = -2\Box.
\end{aligned}$$

These are exactly (3.1.38) and (3.1.39), respectively.

Lemma 3.1.4 is proved. □

According to Lemma 3.1.4 and the mathematical induction, it is easy to obtain

Lemma 3.1.5 *For any given multi-index* $k = (k_1, \ldots, k_\sigma)$, *we have*

$$[\Box, \Gamma^k] = \sum_{|i| \le |k|-1} B_{ki} \Gamma^i \Box, \tag{3.1.40}$$

where $i = (i_1, \ldots, i_\sigma)$ is a multi-index, B_{ki} are constants.

3.1.5 Representing Derivatives Under Ordinary Coordinates by Derivatives Under Polar Coordinates

On any given sphere centered at the origin, the set Ω_x given by (3.1.13) is a complete group of tangential differential operators. In fact, the outward normal direction at any given point $x = (x_1, \ldots, x_n)$ on this sphere is $(x_1, \ldots, x_n)$, then the differential operators

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i \quad (1 \le i < j \le n) \tag{3.1.41}$$

at this point give the directional derivatives along the directions $(0, \ldots, \underset{(i)}{-x_j}, 0, \ldots, 0, \underset{(j)}{x_i}, 0, \ldots, 0)$ on the tangential space of this sphere, and these tangential directions can obviously span to the whole tangential space of the sphere at this point. Therefore, the set Ω_x and the radial derivative

$$\partial_r = \frac{1}{r} \sum_{i=1}^{n} x_i \partial_i \quad (r = |x|) \tag{3.1.42}$$

can be regarded as the derivatives under polar coordinates.

In order to represent the derivatives under ordinary coordinates by the derivatives under polar coordinates, we multiply both sides of (3.1.41) by x_i and summing up with respect i, noting (3.1.42) we have

$$\sum_{i=1}^{n} x_i \Omega_{ij} = r^2 \partial_j - r x_j \partial_r,$$

then we obtain

$$\partial_i = \frac{1}{r^2} \left(\sum_{j=1}^{n} x_j \Omega_{ji} + r x_i \partial_r \right) \quad (i = 1, \ldots, n). \tag{3.1.43}$$

Lemma 3.1.6 *We have the following commutant relations:*

$$[\partial_i, \partial_r] = \frac{1}{r} \partial_i - \frac{x_i}{r^2} \partial_r, \tag{3.1.44}$$

$$[\partial_i, r\partial_r] = \partial_i, \tag{3.1.45}$$

$$[\Omega_{ij}, \partial_r] = 0, \tag{3.1.46}$$

$$[\Omega_{ij}, r\partial_r] = 0. \tag{3.1.47}$$

Proof Noting that $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$ $(i = 1, \ldots, n)$, from (3.1.42) we have

$$\begin{aligned}
[\partial_i, \partial_r] &= \partial_i \left(\sum_{j=1}^{n} \frac{x_j}{r} \partial_j \right) - \sum_{j=1}^{n} \frac{x_j}{r} \partial_j \partial_i \\
&= \sum_{j=1}^{n} \frac{\delta_{ij}}{r} \partial_j - \sum_{j=1}^{n} \frac{x_j x_i}{r^3} \partial_j \\
&= \frac{1}{r} \partial_i - \frac{x_i}{r^2} \partial_r,
\end{aligned}$$

this is exactly (3.1.44). Formula (3.1.45) can be obtained similarly.

From (3.1.11), using the obtained (3.1.44), we have

$$\begin{aligned}
[\Omega_{ij}, \partial_r] &= (x_i \partial_j - x_j \partial_i)\partial_r - \partial_r (x_i \partial_j - x_j \partial_i) \\
&= x_i [\partial_j, \partial_r] - x_j [\partial_i, \partial_r] - \frac{x_i}{r} \partial_j + \frac{x_j}{r} \partial_i \\
&= x_i \left(\frac{1}{r} \partial_j - \frac{x_j}{r^2} \partial_r \right) - x_j \left(\frac{1}{r} \partial_i - \frac{x_i}{r^2} \partial_r \right) \\
&\quad - \frac{x_i}{r} \partial_j + \frac{x_j}{r} \partial_i \\
&= 0,
\end{aligned}$$

this is just (3.1.46). Similarly, (3.1.47) can be obtained by using (3.1.45). □

Now we prove the following

Lemma 3.1.7 *For any given multi-index* α $(|\alpha| > 0)$*, we have*

$$|D_x^{\alpha} u(x)| \le C r^{-|\alpha|} \sum_{0 < k + |\beta| \le |\alpha|} r^k |\partial_r^k \Omega_x^{\beta} u(x)|, \tag{3.1.48}$$

where $r = |x| \neq 0$*,* β *is a multi-index, and* C *is a positive constant independent of both* u *and* x*.*

Remark 3.1.1 Noticing the Recursive relations such as

$$(r\partial_r)^2 = r^2 \partial_r^2 + r\partial_r,$$

Formula (3.1.48) can be equivalently rewritten as

$$|D_x^\alpha u(x)| \le Cr^{-|\alpha|} \sum_{0<k+|\beta|\le|\alpha|} |(r\partial_r)^k \Omega_x^\beta u(x)|. \tag{3.1.49}$$

Therefore, due to (3.1.46)–(3.1.47), the different orderings of the operators ∂_r and Ω_x on the right-hand sides of (3.1.48) and (3.1.49) do not affect the outcome.

Proof of Lemma 3.1.7 From (3.1.43) we have

$$|\partial_i u(x)| \le \frac{C}{r}(|\Omega_x u(x)| + |(r\partial_r)u(x)|), \tag{3.1.50}$$

noticing the commutant relations (3.1.22) and (3.1.45), it is easy to prove (3.1.49) by then, mathematical induction, and thus obtain the conclusions in Lemma 3.1.7. The proof is finished. □

The above discussion is carried out in $\mathbb{R}^n$. The operators in Minkowski space $\mathbb{R}^{1+n}$ corresponding to Ω_x and ∂_r can be taken as $\Omega = (\Omega_{ab})$ and L_0. To represent the derivatives in ordinary sense, multiply both sides of

$$\Omega_{ab} = x_a\partial_b - x_b\partial_a \quad (0 \le a < b \le n) \tag{3.1.51}$$

by $\eta^{ca}x_c$ and sum up with respect to a, noting (3.1.8) we have

$$\begin{aligned} \eta^{ca}x_c\Omega_{ab} &= \eta^{ca}x_c x_a\partial_b - \eta^{ca}x_c x_b\partial_a \\ &= -(t^2 - r^2)\partial_b - x_b L_0, \end{aligned}$$

thus

$$\partial_a = -\frac{\eta^{cb}x_c\Omega_{ba} + x_a L_0}{t^2 - |x|^2} \quad (a = 0, 1, \ldots, n). \tag{3.1.52}$$

Lemma 3.1.8 *For any given multi-index* $\alpha(|\alpha| > 0)$*, we have*

$$|D^\alpha u(t,x)| \le C(1 + |t - |x||)^{-|\alpha|} \sum_{0<|\beta|\le|\alpha|} |\Gamma^\beta u(t,x)|, \tag{3.1.53}$$

where D is defined by (3.1.27), β is a multi-index, and C is a positive constant independent of both u and (t,x).

Proof Since the set Γ contains D, (3.1.53) is obvious when $|t - |x|| \le 1$; while, when $|t - |x|| > 1$, we need only to prove

$$|D^\alpha u(t,x)| \le C|t - |x||^{-|\alpha|} \sum_{|\beta|\le|\alpha|} |\overline{\Omega}^\beta u(t,x)|, \tag{3.1.54}$$

where $\overline{\Omega} = (\Omega, L_0)$ (see (3.1.17)).

When $|t - |x|| > 1$, it follows obviously from (3.1.52) that

$$|\partial_a u(t,x)| \le \frac{C}{|t-|x||} |\overline{\Omega} u(t,x)|.$$

Noticing the commutant relations (3.1.22)–(3.1.23), it is easy to prove the required (3.1.54) by mathematical induction. □

Remark 3.1.2 Assume in Lemma 3.1.8 that $|\alpha| = 1$, then from (3.1.53) we have

$$|Du(t,x)| \le C(1 + |t - |x||)^{-1} \sum_{|\beta|=1} |\Gamma^\beta u(t,x)|. \tag{3.1.55}$$

3.2 Some Variations of Classical Sobolev Embedding Theorems

In this section, we always assume that $n > 1$.

3.2.1 Sobolev Embedding Theorems on a Unit Sphere

We know that the set $\Omega_x = (\Omega_{ij})_{1 \le i < j \le n}$ is a complete group of differential operators on the unit sphere S^{n-1}. Thanks to the Lie algebraic properties of Ω_x, we can utilize Ω_x to construct the corresponding Sobolev spaces $W^{s,p}_{\Omega_x}(S^{n-1})$ or $H^s_{\Omega_x}(S^{n-1})$ on S^{n-1} with norms defined by

$$\begin{aligned}\|u\|_{W^{s,p}_{\Omega_x}(S^{n-1})} &= \sum_{|\alpha| \le s} \|\Omega_x^\alpha u\|_{L^p(S^{n-1})} \\ &= \sum_{|\alpha| \le s} \left(\int_{S^{n-1}} |\Omega_x^\alpha u(\xi)|^p \mathrm{d}\omega_\xi \right)^{\frac{1}{p}} \end{aligned} \tag{3.2.1}$$

or

$$\begin{aligned}\|u\|_{H^s_{\Omega_x}(S^{n-1})} &= \sum_{|\alpha| \le s} \|\Omega_x^\alpha u\|_{L^2(S^{n-1})} \\ &= \sum_{|\alpha| \le s} \left(\int_{S^{n-1}} |\Omega_x^\alpha u(\xi)|^2 \mathrm{d}\omega_\xi \right)^{\frac{1}{2}}, \end{aligned} \tag{3.2.2}$$

where $\xi = (\xi_1, \ldots, \xi_n) \in S^{n-1}$, $d\omega_\xi$ is the area element of S^{n-1}, α is a multi-index, and $1 \le p \le +\infty$ (when $p = +\infty$, it is necessary to make corresponding changes to the norm expression on the rightmost-hand side of (3.2.1)).

Applying classical Sobolev embedding theorems to the $(n-1)$-dimensional compact manifold S^{n-1}, we obtain

Theorem 3.2.1 *Suppose that the function $u = u(x) = u(r\xi)$ is such that the quantities on the right-hand sides of the following inequalities are well-defined, where $r = |x|$, and $\xi \in S^{n-1}$,*

1° *If $s > \frac{n-1}{p}$, then we have*

$$|u(x)| = |u(r\xi)| \le C\|u(r\xi)\|_{W^{s,p}_{\Omega_x}(S^{n-1})}; \tag{3.2.3}$$

2° *If $s = \frac{n-1}{p}$, then for any given q satisfying $p \le q < +\infty$, we have*

$$\|u(r\xi)\|_{L^q(S^{n-1})} \le C\|u(r\xi)\|_{W^{s,p}_{\Omega_x}(S^{n-1})}; \tag{3.2.4}$$

3° *If $0 < s < \frac{n-1}{p}$, then (3.2.4) still holds for any given q satisfying $\frac{1}{q} = \frac{1}{p} - \frac{s}{n-1}$.*

In (3.2.3) and (3.2.4), r is regarding as a parameter and C is a positive constant independent of both u and r.

3.2.2 Sobolev Embedding Theorems on a Ball

Denote by B_λ a ball in $\mathbb{R}^n$ centered at x_0 with radius $\lambda\,(> 0)$:

$$B_\lambda = \{x \mid |x - x_0| < \lambda\}. \tag{3.2.5}$$

Applying the Sobolev embedding theorem to functions defined on B_λ, we have

Theorem 3.2.2 *For any given $p \ge 1$, suppose that function $u = u(x)$ is such that the quantities on the right-hand sides of the following inequalities are well-defined,*

1° *If $s > \frac{n}{p}$, then we have*

$$\|u\|_{L^\infty(B_\lambda)} \le C\lambda^{-\frac{n}{p}} \sum_{|\alpha| \le s} \lambda^{|\alpha|} \|D_x^\alpha u\|_{L^p(B_\lambda)}; \tag{3.2.6}$$

2° *If $s = \frac{n}{p}$, then for any given q satisfying $p \le q < +\infty$, we have*

$$\|u\|_{L^q(B_\lambda)} \le C\lambda^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{|\alpha| \le s} \lambda^{|\alpha|} \|D_x^\alpha u\|_{L^p(B_\lambda)}; \tag{3.2.7}$$

3° *If* $0 < s < \frac{n}{p}$, *then (3.2.7) still holds for any given q satisfying* $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$.
In (3.2.6) and (3.2.7), C is a positive constant independent of both u and λ.

Proof Without loss of generality, we need only to prove Theorem 3.2.2 for the case that $x_0 = 0$. When $\lambda = 1$, this is just the usual Sobolev embedding theorem. For general $\lambda > 0$, setting $x = \lambda y$ and denoting

$$v(y) = u(\lambda y) = u(x), \tag{3.2.8}$$

it can be proved by scaling transformation. As a matter of fact, it is clear that

$$\|v\|_{L^\infty(B_1)} = \|u\|_{L^\infty(B_\lambda)}, \tag{3.2.9}$$

$$\|D_y^\alpha v\|_{L^p(B_1)} = \lambda^{|\alpha| - \frac{n}{p}} \|D_x^\alpha u\|_{L^p(B_\lambda)}, \tag{3.2.10}$$

then from estimates (3.2.6) and (3.2.7) satisfied by v in the case $\lambda = 1$, we obtained (3.2.6) and (3.2.7) for general $\lambda > 0$ immediately. □

Remark 3.2.1 The conclusions in Theorem 3.2.2 still hold if B_λ is changed into $I\!R^n \setminus \overline{B}_\lambda$.

3.2.3 Sobolev Embedding Theorems on an Annulus

Let

$$E_{a,\lambda} = \{y \mid ||y| - a| < \lambda a\}, \tag{3.2.11}$$

where $a > 0$, and $0 < \lambda \le \lambda_0 < 1$. $E_{a,\lambda}$ is the annular region enclosed by two spheres centered at the origin with radiuses $(1 - \lambda)a$ and $(1 + \lambda)a$, respectively, that is,

$$E_{a,\lambda} = \{y \mid (1 - \lambda)a < |y| < (1 + \lambda)a\}.$$

Theorem 3.2.3 *If* $s > \frac{n}{p}$, *then for any given* $x_0 \in I\!R^n$ *with* $|x_0| \ne 0$, *we have*

$$|u(x_0)| \le C\lambda^{-\frac{1}{p}} |x_0|^{-\frac{n}{p}} \sum_{k+|\alpha| \le s} \lambda^k |x_0|^k \|\partial_r^k \Omega_x^\alpha u(x)\|_{L^p(E_{|x_0|,\lambda})}, \tag{3.2.12}$$

where C is a positive constant independent of the function u and x_0 *as well as the choice of* λ.

Proof First, we point out that: to prove (3.2.12), it suffices to prove the following estimate at $|x_0| = 1$:

$$|u(x_0)| \le C\lambda^{-\frac{1}{p}} \sum_{k+|\alpha| \le s} \lambda^k \|\partial_r^k \Omega_x^\alpha u\|_{L^p(E_{1,\lambda})}. \tag{3.2.13}$$

In fact, in the general case that $|x_0| \neq 0$, set $\bar{x} = \frac{x}{|x_0|}$, and denote

$$v(\bar{x}) = u(|x_0|\bar{x}) = u(x), \tag{3.2.14}$$

noticing that $\Omega_{\bar{x}} = \Omega_x$ and $\partial_{\bar{r}} = |x_0|\partial_r$, it is easy to show that

$$\begin{aligned}
&\|\partial_{\bar{r}}^k \Omega_{\bar{x}}^\alpha v(\bar{x})\|_{L^p(E_{1,\lambda})} \\
&= \|\partial_{\bar{r}}^k \Omega_{\bar{x}}^\alpha u(|x_0|\bar{x})\|_{L^p(E_{1,\lambda})} \\
&= \left(\int_{||\bar{x}|-1|\le\lambda} |\partial_{\bar{r}}^k \Omega_{\bar{x}}^\alpha u(|x_0|\bar{x})|^p \mathrm{d}\bar{x}\right)^{\frac{1}{p}} \\
&= |x_0|^{k-\frac{n}{p}} \left(\int_{||x|-|x_0||\le\lambda|x_0|} |\partial_r^k \Omega_x^\alpha u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&= |x_0|^{k-\frac{n}{p}} \|\partial_r^k \Omega_x^\alpha u(x)\|_{L^p(E_{|x_0|,\lambda})}.
\end{aligned}$$

Denote $\bar{x}_0 = \frac{x_0}{|x_0|}$, then $|\bar{x}_0| = 1$. Noting the above formula and using estimate (3.2.13) satisfied by $v(\bar{x}_0) = u(x_0)$, (3.2.12) follows immediately.

Second, we point out that: to prove (3.2.13), it suffices to prove the following estimate at $\lambda = \lambda_0 (< 1)$:

$$|u(x_0)| \le C \sum_{k+|\alpha|\le s} \|\partial_r^k \Omega_x^\alpha u\|_{L^p(E_{1,\lambda_0})}. \tag{3.2.15}$$

In fact, in the general case that $0 < \lambda \le \lambda_0$, set

$$v(\bar{x}) = u(x), \tag{3.2.16}$$

where $x = r\xi$, $\bar{x} = \bar{r}\xi$, $\xi \in S^{n-1}$, and

$$\bar{r} = \frac{\lambda_0}{\lambda}(r - 1 + \lambda) + 1 - \lambda_0. \tag{3.2.17}$$

Noticing that $\Omega_{\bar{x}} = \Omega_x$ and $\partial_{\bar{r}} = \frac{\lambda}{\lambda_0}\partial_r$, it is easy to show that

$$\begin{aligned}
&\|\partial_{\bar{r}}^k \Omega_{\bar{x}}^\alpha v(\bar{x})\|_{L^p(E_{1,\lambda_0})} \\
&= \|\partial_{\bar{r}}^k \Omega_{\bar{x}}^\alpha u(x)\|_{L^p(E_{1,\lambda_0})} \\
&= \left(\int_{||\bar{x}|-1|\le\lambda_0} |\partial_{\bar{r}}^k \Omega_{\bar{x}}^\alpha u(x)|^p \mathrm{d}\bar{x}\right)^{\frac{1}{p}} \\
&= \left(\frac{\lambda}{\lambda_0}\right)^k \left(\int_{1-\lambda_0}^{1+\lambda_0} \int_{S^{n-1}} |\partial_r^k \Omega_x^\alpha u(x)|^p \bar{r}^{n-1} \mathrm{d}\bar{r} \mathrm{d}\omega_\xi\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\lambda}{\lambda_0}\right)^{k-\frac{1}{p}} \left(\int_{1-\lambda}^{1+\lambda} \int_{S^{n-1}} |\partial_r^k \Omega_x^\alpha u(x)|^p \left(\frac{\bar{r}}{r}\right)^{n-1} r^{n-1} \mathrm{d}r \mathrm{d}\omega_\xi\right)^{\frac{1}{p}} \\
&\leq \left(\frac{\lambda}{\lambda_0}\right)^{k-\frac{1}{p}} \left(\frac{1+\lambda_0}{1-\lambda}\right)^{\frac{n-1}{p}} \|\partial_r^k \Omega_x^\alpha u(x)\|_{L^p(E_{1,\lambda})} \\
&\leq \left(\frac{\lambda}{\lambda_0}\right)^{k-\frac{1}{p}} \left(\frac{1+\lambda_0}{1-\lambda_0}\right)^{\frac{n-1}{p}} \|\partial_r^k \Omega_x^\alpha u(x)\|_{L^p(E_{1,\lambda})} \\
&\leq C\lambda^{k-\frac{1}{p}} \|\partial_r^k \Omega_x^\alpha u(x)\|_{L^p(E_{1,\lambda})},
\end{aligned}
$$

Then, using estimate (3.2.15) satisfied by $v(\bar{x}_0) = u(x_0)$, (3.2.13) follows immediately.

Thus, to complete the proof of Theorem 3.2.3, it suffices to prove (3.2.15). When $x \in E_{1,\lambda_0}$, we have $1 - \lambda_0 < r = |x| < 1 + \lambda_0$, then, using Lemma 3.1.7 and the Sobolev embedding theorem, we immediately obtain our conclusion. The proof is finished. □

Remark 3.2.2 For the contents in this subsection, please see Klainerman (1985) for reference.

3.2.4 Sobolev Embedding Theorems for Decomposed Dimensions

Denote

$$x = (x', x''), \tag{3.2.18}$$

where

$$x' = (x_1, \ldots, x_m), \ \ x'' = (x_{m+1}, \ldots, x_n), \tag{3.2.19}$$

and $1 \leq m \leq n - 1$.

Theorem 3.2.4 *We have*

$$\|f\|_{L^\infty(\mathbb{R}^m; L^2(\mathbb{R}^{n-m}))} \leq C\|f\|_{H^{s_0}(\mathbb{R}^n)}, \tag{3.2.20}$$

where $s_0 > \frac{m}{2}$*; meanwhile,*

$$\|f\|_{L^p(\mathbb{R}^m; L^2(\mathbb{R}^{n-m}))} \leq C\|f\|_{H^{s_0}(\mathbb{R}^n)}, \tag{3.2.21}$$

where $2 < p < +\infty$*, and* $\frac{1}{p} = \frac{1}{2} - \frac{s_0}{m}$ *(thus* $0 < s_0 < \frac{m}{2}$*). In (3.2.20) and (3.2.21), C is a positive constant independent of f, and* $H^{s_0}(\mathbb{R}^n)$ *is a fractional Sobolev space with norm*

$$\|f\|_{H^{s_0}(\mathbb{R}^n)} = \|(1+|\xi|^2)^{\frac{s_0}{2}} \hat{f}(\xi)\|_{L^2(\mathbb{R}^n)}, \tag{3.2.22}$$

where $\xi = (\xi_1, \ldots, \xi_n)$*, and* $\hat{f}$ *stands for the Fourier transform of* f*.*

Proof We first prove (3.2.21). From the Sobolev embedding theorem and the Parseval inequality we have

$$\begin{aligned}
&\|f\|_{L^p(\mathbb{R}^m;L^2(\mathbb{R}^{n-m}))} \\
&= \left(\int_{\mathbb{R}^m} \|f(x',\cdot)\|^p_{L^2(\mathbb{R}^{n-m})} \mathrm{d}x'\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^{n-m}} f^2(x',x'')\mathrm{d}x''\right)^{\frac{p}{2}} \mathrm{d}x'\right)^{\frac{1}{p}} \\
&= \left\|\int_{\mathbb{R}^{n-m}} f^2(\cdot,x'')\mathrm{d}x''\right\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\mathbb{R}^m)} \\
&\le \left(\int_{\mathbb{R}^{n-m}} \|f^2(\cdot,x'')\|_{L^{\frac{p}{2}}(\mathbb{R}^m)} \mathrm{d}x''\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^{n-m}} \left(\int_{\mathbb{R}^m} |f|^p(x',x'')\mathrm{d}x'\right)^{\frac{2}{p}} \mathrm{d}x''\right)^{\frac{1}{2}} \\
&= \|f\|_{L^2(\mathbb{R}^{n-m};L^p(\mathbb{R}^m))} \\
&\le C\|f\|_{L^2(\mathbb{R}^{n-m};H^{s_0}(\mathbb{R}^m))} \\
&= C\left(\int_{\mathbb{R}^{n-m}} \|f(\cdot,x'')\|^2_{H^{s_0}(\mathbb{R}^m)} \mathrm{d}x''\right)^{\frac{1}{2}} \\
&= C\|(1+|\xi'|^2)^{\frac{s_0}{2}} \hat{f}(\xi',\xi'')\|_{L^2(\mathbb{R}^n)} \\
&\le C\|f\|_{H^{s_0}(\mathbb{R}^n)},
\end{aligned}$$

which is just (3.2.21).

Similarly, we have

$$\begin{aligned}
&\|f\|_{L^\infty(\mathbb{R}^m;L^2(\mathbb{R}^{n-m}))} \\
&= \operatorname*{ess\,sup}_{x'\in\mathbb{R}^m} \|f(x',\cdot)\|_{L^2(\mathbb{R}^{n-m})} \\
&= \operatorname*{ess\,sup}_{x'\in\mathbb{R}^m} \left(\int_{\mathbb{R}^{n-m}} f^2(x',x'')\mathrm{d}x''\right)^{\frac{1}{2}} \\
&\le \left(\int_{\mathbb{R}^{n-m}} \|f(\cdot,x'')\|^2_{L^\infty(\mathbb{R}^m)} \mathrm{d}x''\right)^{\frac{1}{2}} \\
&= \|f\|_{L^2(\mathbb{R}^{n-m};L^\infty(\mathbb{R}^m))}
\end{aligned}$$

$$\leq C\|f\|_{L^2(\mathbb{R}^{n-m};H^{s_0}(\mathbb{R}^m))}$$
$$\leq C\|f\|_{H^{s_0}(\mathbb{R}^n)},$$

which is exactly (3.2.20). □

Remark 3.2.3 From the proof of Theorem 3.2.4 we can see that when $2 < p \leq +\infty$ we always have

$$\|f\|_{L^p(\mathbb{R}^m;L^2(\mathbb{R}^{n-m}))} \leq C\|f\|_{L^2(\mathbb{R}^{n-m};L^p(\mathbb{R}^m))}, \tag{3.2.23}$$

where C is a positive constant independent of f.

In particular, taking $m = 1$ in Theorem 3.2.4, we have

Corollary 3.2.1 *We have the following estimate:*

$$\|f\|_{L^\infty(\mathbb{R};L^2(\mathbb{R}^{n-1}))} \leq C\|f\|_{H^{s_0}(\mathbb{R}^n)}, \tag{3.2.24}$$

where $s_0 > \frac{1}{2}$*; meanwhile,*

$$\|f\|_{L^p(\mathbb{R};L^2(\mathbb{R}^{n-1}))} \leq C\|f\|_{H^{s_0}(\mathbb{R}^n)}, \tag{3.2.25}$$

where $2 < p < +\infty$*, and* $\frac{1}{p} = \frac{1}{2} - s_0$ *(thus* $0 < s_0 < \frac{1}{2}$*).*

3.3 Sobolev Embedding Theorems Based on Binary Partition of Unity

In this section we will review a kind of Sobolev embedding theorems introduced by Li and Zhou (1995b, c), in the derivation of which the binary partition of unity plays an important role.

3.3.1 Binary Partition of Unity

Suppose that $\Phi_0 = \Phi_0(x) \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\Phi_0(x) = \Phi_0(|x|), \tag{3.3.1}$$
$$\text{supp}\Phi_0 \subseteq \{x \mid |x| \leq 2\} \tag{3.3.2}$$

and

$$\Phi_0(x) \equiv 1, \quad |x| \leq 1. \tag{3.3.3}$$

Set

$$\Phi_j(x) = \Phi_1(2^{-(j-1)}x), \quad j = 1, 2, \ldots, \tag{3.3.4}$$

where

$$\Phi_1(x) = \Phi_0(2^{-1}x) - \Phi_0(x). \tag{3.3.5}$$

It is clear that, for $j = 1, 2, \ldots$, $\Phi_j(x) \in C_0^\infty(\mathbb{R}^n)$ and

$$\Phi_j(x) = \Phi_j(|x|), \tag{3.3.6}$$

$$\text{supp}\Phi_j \subseteq \{x \mid 2^{j-1} \le |x| \le 2^{j+1}\}. \tag{3.3.7}$$

We have the following partition of unity:

$$\sum_{j=0}^{\infty} \Phi_j(x) \equiv 1, \quad \forall x \in \mathbb{R}^n.$$

In fact, due to (3.3.4)–(3.3.5) and noting (3.3.3), for any fixed $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\sum_{j=0}^{N} \Phi_j(x) &= \Phi_0(x) + (\Phi_0(2^{-1}x) - \Phi_0(x)) \\ &\quad + (\Phi_0(2^{-2}x) - \Phi_0(2^{-1}x)) + \cdots \\ &\quad + (\Phi_0(2^{-N}x) - \Phi_0(2^{-N+1}x)) \\ &= \Phi_0(2^{-N}x) \to 1\end{aligned}$$

as $N \to \infty$. Thus, we obtain the following

Lemma 3.3.1 (Binary partition of unity) *There exist* $\Phi_j(x) \in C_0^\infty(\mathbb{R}^n)$ $(j = 0, 1, \ldots)$ *satisfying*

(i) $\Phi_j(x) = \Phi_j(|x|),\ j = 0, 1, \ldots;$
(ii) $\text{supp}\Phi_0 \subseteq \{x \mid |x| \le 2\}$, *and* $\Phi_0(x) \equiv 1, |x| \le 1;$
(iii) $\text{supp}\Phi_j \subseteq \{x \mid 2^{j-1} \le |x| \le 2^{j+1}\},\ j = 1, 2, \ldots,$

such that

$$\sum_{j=0}^{\infty} \Phi_j(x) \equiv 1, \quad \forall x \in \mathbb{R}^n. \tag{3.3.8}$$

Noting (3.3.2), from Lemma 3.3.1 and the previous arguments we obtain

Corollary 3.3.1 *There exist* $\Phi_j(x) \in C_0^\infty(\mathbb{R}^n)$ $(j = 1, 2, \ldots)$*, satisfying*

(i) $\Phi_j(x) = \Phi_1(2^{-(j-1)}x),\ j = 1, 2, \ldots;$
(ii) $\Phi_1(x) = \Phi_1(|x|)$, *and* $\text{supp}\Phi_1 \subseteq \{x \mid 1 \le |x| \le 4\},$

such that

$$\sum_{j=1}^{\infty} \Phi_j(x) \equiv 1, \quad \forall x \in I\!R^n, |x| \geq 2. \tag{3.3.9}$$

3.3.2 Sobolev Embedding Theorems Based on Binary partition of Unity

We will prove the following

Theorem 3.3.1 *Let* $\Psi(x)$ *be the characteristic function of the set* $\{x \mid |x| \geq a\}$ $(a > 0)$.

1° *If* $\frac{1}{2} < s_0 < \frac{n}{2}$, *then we have*

$$\|\Psi f\|_{L^{\infty,2}(I\!R^n)} \leq C a^{s_0 - \frac{n}{2}} \|f\|_{\dot{H}^{s_0}(I\!R^n)}; \tag{3.3.10}$$

2° *For any given* $p > 2$, *we have*

$$\|\Psi f\|_{L^{p,2}(I\!R^n)} \leq C a^{-(n-1)s_0} \|f\|_{\dot{H}^{s_0}(I\!R^n)}, \tag{3.3.11}$$

where $s_0 = \frac{1}{2} - \frac{1}{p}$.

In (3.3.10)–(3.3.11), C is a positive constant independent of both f and a, and $\dot{H}^{s_0}(I\!R^n)$ *is a homogeneous Sobolev space with the norm*

$$\|f\|_{\dot{H}^{s_0}(I\!R^n)} = \||\xi|^{s_0} \hat{f}(\xi)\|_{L^2(I\!R^n)}, \tag{3.3.12}$$

where $\hat{f}(\xi)$ *stands for the Fourier transform of* $f(x)$.

Proof 1° Thanks to the scaling transformation, it suffices to prove the corresponding inequalities (3.3.10) and (3.3.11) when $a = 4$.

In fact, in the general case that $a > 0$, set $x = by$, where $b = \frac{a}{4}$, and denote

$$\tilde{f}(y) = f(by) = f(x). \tag{3.3.13}$$

Noting that

$$\widehat{f(by)} = b^{-n} \hat{f}\left(\frac{\eta}{b}\right), \tag{3.3.14}$$

from (3.3.12) it is easy to show that

$$\begin{aligned} \|\tilde{f}(y)\|_{\dot{H}^{s_0}(I\!R^n)} &= \|f(by)\|_{\dot{H}^{s_0}(I\!R^n)} \\ &= b^{-n} \left\| |\eta|^{s_0} \hat{f}\left(\frac{\eta}{b}\right) \right\|_{L^2(I\!R^n)} \end{aligned}$$

$$= b^{s_0-\frac{n}{2}} \| |\xi|^{s_0} \hat{f}(\xi) \|_{L^2(\mathbb{R}^n)}$$
$$= b^{s_0-\frac{n}{2}} \| f \|_{\dot{H}^{s_0}(\mathbb{R}^n)}. \tag{3.3.15}$$

It is also clear from (3.1.29) that, if we denote by $\widetilde{\Psi}$ the characteristic function of the set $\{x \mid |x| \geq 4\}$, then we have

$$\| \widetilde{\Psi} \tilde{f} \|_{L^{\infty,2}(\mathbb{R}^n)} = \| \Psi f \|_{L^{\infty,2}(\mathbb{R}^n)} \tag{3.3.16}$$

and

$$\| \widetilde{\Psi} \tilde{f} \|_{L^{p,2}(\mathbb{R}^n)} = b^{-\frac{n}{p}} \| \Psi f \|_{L^{p,2}(\mathbb{R}^n)}. \tag{3.3.17}$$

Consequently, from the inequality satisfied by $\tilde{f}$ at $a = 4$ we can obtain the inequalities (3.3.10) and (3.3.11) in the general case that $a > 0$.

2° Now we prove (3.3.10) at $a = 4$, i.e., to prove that if $\Psi(x)$ is the characteristic function of the set $\{x \mid |x| \geq 4\}$, then for $\frac{1}{2} < s_0 < \frac{n}{2}$ we have

$$\| \Psi f \|_{L^{\infty,2}(\mathbb{R}^n)} \leq C \| f \|_{\dot{H}^{s_0}(\mathbb{R}^n)}. \tag{3.3.18}$$

According to Corollary 3.3.1, on the set $\{x \mid |x| \geq 4\}$ we have

$$\Psi f(x) \equiv f(x) \equiv \sum_{j=1}^{\infty} \Phi_j(x) f(x) \overset{\text{def.}}{=} \sum_{j=1}^{\infty} f_j(x). \tag{3.3.19}$$

For $f_1(x) = \Phi_1(x) f(x)$, its support is included in $\{x \mid 1 \leq |x| \leq 4\}$. According to (3.2.24) in Corollary 3.2.1 and noting that the norms $\| f_1 \|_{L^\infty(\mathbb{R}; L^2(\mathbb{R}^{n-1}))}$ and $\| f_1 \|_{L^{\infty,2}(\mathbb{R}^n)}$ are equivalent to each other through a homeomorphic transformation between independent variables, for $s_0 > \frac{1}{2}$ we have

$$\| f_1 \|_{L^{\infty,2}(\mathbb{R}^n)} \leq C \| f_1 \|_{H^{s_0}(\mathbb{R}^n)}. \tag{3.3.20}$$

Due to Poincaré inequality and Parseval identity, noting that f_1 is compactly supported, we have

$$\begin{aligned}
&\| f_1 \|_{H^{s_0}(\mathbb{R}^n)} \\
&\leq C \| f_1 \|_{\dot{H}^{s_0}(\mathbb{R}^n)} \\
&= C \| |\xi|^{s_0} \hat{f}_1(\xi) \|_{L^2(\mathbb{R}^n)} \\
&= C \left\| |\xi|^{s_0} \int_{\mathbb{R}^n} \widehat{\Phi}_1(\xi - \eta) \hat{f}(\eta) \mathrm{d}\eta \right\|_{L^2(\mathbb{R}^n)} \\
&\leq C \left(\left\| \int_{\mathbb{R}^n} |\xi - \eta|^{s_0} |\widehat{\Phi}_1(\xi - \eta) \hat{f}(\eta)| \mathrm{d}\eta \right\|_{L^2(\mathbb{R}^n)} + \left\| \int_{\mathbb{R}^n} |\widehat{\Phi}_1(\xi - \eta)| |\eta|^{s_0} |\hat{f}(\eta)| \mathrm{d}\eta \right\|_{L^2(\mathbb{R}^n)} \right) \\
&= C(\| \Phi_1^* f_* \|_{L^2(\mathbb{R}^n)} + \| \Phi_{1*} f^* \|_{L^2(\mathbb{R}^n)}) \\
&\leq C(\| f_* \|_{L^\gamma(\mathbb{R}^n)} \| \Phi_1^* \|_{L^{\frac{n}{s_0}}(\mathbb{R}^n)} + \| \Phi_{1*} \|_{L^\infty(\mathbb{R}^n)} \| f^* \|_{L^2(\mathbb{R}^n)}),
\end{aligned} \tag{3.3.21}$$

where

$$\begin{cases} \widehat{\Phi}_1^*(\xi) = |\xi|^{s_0}|\widehat{\Phi}_1(\xi)|, \\ \hat{f}_*(\xi) = |\hat{f}(\xi)|, \\ \widehat{\Phi}_{1*}(\xi) = |\widehat{\Phi}_1(\xi)|, \\ \hat{f}^*(\xi) = |\xi|^{s_0}|\hat{f}(\xi)| \end{cases} \tag{3.3.22}$$

and

$$\frac{1}{\gamma} + \frac{s_0}{n} = \frac{1}{2} \tag{3.3.23}$$

(here it is necessary to assume furthermore that $s_0 < \frac{n}{2}$). From the Sobolev embedding theorem we have

$$\|f_*\|_{L^\gamma(I\!R^n)} \leq C\|f_*\|_{\dot{H}^{s_0}(I\!R^n)} = C\|f\|_{\dot{H}^{s_0}(I\!R^n)}. \tag{3.3.24}$$

From Parseval identity we have

$$\|f^*\|_{L^2(I\!R^n)} = \|f\|_{\dot{H}^{s_0}(I\!R^n)}. \tag{3.3.25}$$

Due to $s_0 < \frac{n}{2}$, using the Hausdorff–Young inequality and noticing that Φ_1 is compactly supported, we have

$$\begin{aligned} \|\Phi_1^*\|_{L^{\frac{n}{s_0}}(I\!R^n)} &\leq C\|\widehat{\Phi}_1^*\|_{L^{\frac{n}{n-s_0}}(I\!R^n)} \\ &= C\||\xi|^{s_0}\widehat{\Phi}_1(\xi)\|_{L^{\frac{n}{n-s_0}}(I\!R^n)} < +\infty. \end{aligned} \tag{3.3.26}$$

In addition, it is obvious that

$$\|\Phi_{1*}\|_{L^\infty(I\!R^n)} \leq \|\widehat{\Phi}_1\|_{L^1(I\!R^n)} < +\infty. \tag{3.3.27}$$

Plugging (3.3.24)–(3.3.27) into (3.3.21), it follows from (3.3.20) that

$$\|f_1\|_{L^{\infty,2}(I\!R^n)} \leq C\|f\|_{\dot{H}^{s_0}(I\!R^n)}. \tag{3.3.28}$$

Using again the scaling transformation, from (3.3.28) we get

$$\|f_j\|_{L^{\infty,2}(I\!R^n)} \leq 2^{(j-1)(s_0-\frac{n}{2})}C\|f\|_{\dot{H}^{s_0}(I\!R^n)} \quad (j = 1, 2, \ldots). \tag{3.3.29}$$

In fact, for any given $j = 1, 2, \ldots$, set $y = 2^{-(j-1)}x$, and denote

$$\begin{aligned} \tilde{f}_j(y) &= f_j(2^{(j-1)}y) = f_j(x), \\ \tilde{f}(y) &= f(2^{(j-1)}y) = f(x), \end{aligned} \tag{3.3.30}$$

similarly to (3.3.15) and (3.3.16), we have

$$\|\tilde{f}_j\|_{L^{\infty,2}(I\!R^n)} = \|f_j\|_{L^{\infty,2}(I\!R^n)} \tag{3.3.31}$$

and

$$\|\tilde{f}\|_{\dot{H}^{s_0}(I\!R^n)} = 2^{(j-1)(s_0-\frac{n}{2})}\|f\|_{\dot{H}^{s_0}(I\!R^n)}. \tag{3.3.32}$$

Thus, due to (3.3.19) and noting $s_0 < \frac{n}{2}$, we obtain

$$\begin{aligned}\|\Psi f\|_{L^{\infty,2}(I\!R^n)} &\le \sum_{j=1}^{\infty}\|f_j\|_{L^{\infty,2}(I\!R^n)}\\ &\le C\sum_{j=1}^{\infty}2^{(j-1)(s_0-\frac{n}{2})}\|f\|_{\dot{H}^{s_0}(I\!R^n)}\\ &\le C\|f\|_{\dot{H}^{s_0}(I\!R^n)}.\end{aligned}$$

This is just the desired (3.3.18).

3° Now we prove (3.3.11) at $a = 4$, i.e., to prove that if $\Psi(x)$ is the characteristic function of the set $\{x \mid |x| \ge 4\}$, then for any given $p > 2$, when $s_0 = \frac{1}{2} - \frac{1}{p}$ we have

$$\|\Psi f\|_{L^{p,2}(I\!R^n)} \le C\|f\|_{\dot{H}^{s_0}(I\!R^n)}. \tag{3.3.33}$$

This proof is similar to that of (3.3.18), here we only explain some different points between them. Now from (3.2.25) in Corollary 3.2.1 we similarly have

$$\|f_1\|_{L^{p,2}(I\!R^n)} \le C\|f_1\|_{H^{s_0}(I\!R^n)}, \tag{3.3.34}$$

where $s_0 = \frac{1}{2} - \frac{1}{p}$. Therefore, similarly to (3.3.28), we have

$$\|f_1\|_{L^{p,2}(I\!R^n)} \le C\|f\|_{\dot{H}^{s_0}(I\!R^n)}. \tag{3.3.35}$$

Moreover, noticing that in addition to (3.3.31)–(3.3.32), similarly to (3.3.17) we also have

$$\|\tilde{f}_j\|_{L^{p,2}(I\!R^n)} = 2^{-(j-1)\frac{n}{p}}\|f_j\|_{L^{p,2}(I\!R^n)}, \tag{3.3.36}$$

then we can obtain from (3.3.35) by the scaling transformation that

$$\|f_j\|_{L^{p,2}(I\!R^n)} \le 2^{-(j-1)(n-1)s_0}C\|f\|_{\dot{H}^{s_0}(I\!R^n)} \quad (j = 1, 2, \ldots), \tag{3.3.37}$$

consequently we can easily obtain the desired (3.3.33). The proof is finished. □

In order to prove the following theorem, we first review the corresponding Hölder inequality in $L^{p,q}(I\!R^n)$ space. Just like the $L^{p,q}(I\!R^n)$ space is an extension of the

$L^p(\mathbb{R}^n)$ space, this Hölder inequality is also an extension of the usual Hölder inequality and is easy to be proved by using the usual Hölder inequality.

Lemma 3.3.2 (Hölder inequality) *Suppose that* $f_1(x) \in L^{p_1,q_1}(\mathbb{R}^n)$, $f_2(x) \in L^{p_2,q_2}(\mathbb{R}^n)$, *where* $1 \le p_1, p_2, q_1, q_2 \le +\infty$, *and* $\frac{1}{p_1} + \frac{1}{p_2} \le 1$, $\frac{1}{q_1} + \frac{1}{q_2} \le 1$, *then* $f_1 f_2(x) \in L^{p,q}(\mathbb{R}^n)$ *with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \tag{3.3.38}$$

moreover,

$$\|f_1 f_2\|_{L^{p,q}(\mathbb{R}^n)} \le \|f_1\|_{L^{p_1,q_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2,q_2}(\mathbb{R}^n)}. \tag{3.3.39}$$

Theorem 3.3.2 *Let* $\Psi(x)$ *be the characteristic function of the set* $\{x \mid |x| > a\}$ $(a > 0)$, *and* $\overline{\Psi} = 1 - \Psi$. *Then we have*

$$\|f\|_{\dot{H}^{-s_0}(\mathbb{R}^n)} \le C(\|\overline{\Psi} f\|_{L^q(\mathbb{R}^n)} + a^{s_0 - \frac{n}{2}} \|\Psi f\|_{L^{1,2}(\mathbb{R}^n)}), \tag{3.3.40}$$

where

$$\frac{1}{2} < s_0 < \frac{n}{2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{2} + \frac{s_0}{n}, \tag{3.3.41}$$

and C *is a positive constant independent of both* f *and* a.

Proof From the definition we know that

$$\|f\|_{\dot{H}^{-s_0}(\mathbb{R}^n)} = \sup_{\substack{v \in \dot{H}^{s_0}(\mathbb{R}^n) \\ v \ne 0}} \frac{\int f v}{\|v\|_{\dot{H}^{s_0}(\mathbb{R}^n)}}. \tag{3.3.42}$$

Using the Hölder inequality, we have

$$\begin{aligned} \left| \int f v \right| &\le \left| \int (\overline{\Psi} f) v \right| + \left| \int (\Psi f) v \right| \\ &\le \|\overline{\Psi} f\|_{L^q(\mathbb{R}^n)} \|v\|_{L^\gamma(\mathbb{R}^n)} + \|\Psi f\|_{L^{1,2}(\mathbb{R}^n)} \|\Psi v\|_{L^{\infty,2}(\mathbb{R}^n)}, \end{aligned} \tag{3.3.43}$$

where q is determined by (3.3.41) and $\frac{1}{\gamma} = \frac{1}{2} - \frac{s_0}{n}$.

By the Sobolev embedding theorem we have

$$\|v\|_{L^\gamma(\mathbb{R}^n)} \le C \|v\|_{\dot{H}^{s_0}(\mathbb{R}^n)}. \tag{3.3.44}$$

From (3.3.10) in Theorem 3.1 we also have

$$\|\Psi v\|_{L^{\infty,2}(\mathbb{R}^n)} \le C a^{s_0 - \frac{n}{2}} \|v\|_{\dot{H}^{s_0}(\mathbb{R}^n)}. \tag{3.3.45}$$

Substituting (3.3.44)–(3.3.45) into (3.3.43), the wanted (3.3.40) follows from (3.3.42). □

3.4 Sobolev Type Inequalities with Decay Factor

3.4.1 Sobolev Type Inequalities with Decay Factor Inside the Characteristic Cone

Lemma 3.4.1 (Interpolation inequality) *Suppose that $\Omega \subseteq I\!R^n$ is a bounded or unbounded domain, and $f \in L^p(\Omega) \cap L^q(\Omega)$ for $1 \le p \le q \le +\infty$, then for any given r satisfying $p \le r \le q$, $f \in L^r(\Omega)$ and satisfies the following interpolation inequality:*

$$\|f\|_{L^r(\Omega)} \le \|f\|^{\theta}_{L^p(\Omega)} \|f\|^{1-\theta}_{L^q(\Omega)}, \tag{3.4.1}$$

where $0 \le \theta \le 1$ satisfying

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}. \tag{3.4.2}$$

Proof Noting (3.4.2), by Hölder inequality we have

$$\begin{aligned}\|f\|_{L^r(\Omega)} &= \||f|^{\theta}|f|^{1-\theta}\|_{L^r(\Omega)}\\ &\le \||f|^{\theta}\|_{L^{\frac{p}{\theta}}(\Omega)} \||f|^{1-\theta}\|_{L^{\frac{q}{\theta}}(\Omega)}\\ &= \|f\|^{\theta}_{L^p(\Omega)} \|f\|^{1-\theta}_{L^q(\Omega)}.\end{aligned}$$

This is exactly (3.4.1). □

Suppose that $\chi(t,x)$ is the characteristic function of a certain set in $I\!R_+ \times I\!R^n$, similarly to (3.1.31)–(3.1.32), we may define

$$\|u(t,\cdot)\|_{A,N,p,q,\chi} = \sum_{|k|\le N} \|\chi(t,\cdot)A^k u(t,\cdot)\|_{L^{p,q}(I\!R^n)}, \quad \forall t \ge 0 \tag{3.4.3}$$

and

$$\|u(t,\cdot)\|_{A,N,p,\chi} = \|u(t,\cdot)\|_{A,N,p,p,\chi}, \quad \forall t \ge 0. \tag{3.4.4}$$

This is actually the generalized Sobolev norm restricted on this set. In particular, we may define $\|u(t,\cdot)\|_{\Gamma,N,p,q,\chi}$ and $\|u(t,\cdot)\|_{\Gamma,N,p,\chi}$ etc.

Now, as shown in the following theorem, we give the Sobolev type inequalities with decay factor on a cone inside the characteristic cone of the wave equation.

Theorem 3.4.1 *Let $\chi_1(t,x)$ be the characteristic function of the set $\{(t,x) \mid |x| \le \frac{1+t}{2}\}$. For any given $p \ge 1$,*

1° *if $s > \frac{n}{p}$, then we have*

$$|\chi_1 u(t,x)| \le C(1+t+|x|)^{-\frac{n}{p}} \|u(t,\cdot)\|_{\Gamma,s,p,\chi_1}; \tag{3.4.5}$$

2° *if* $s = \frac{n}{p}$, *then for any given* q *satisfying* $p \le q < +\infty$, *we have*

$$\|\chi_1 u(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C(1+t)^{-n(\frac{1}{p}-\frac{1}{q})}\|u(t,\cdot)\|_{\Gamma,s,p,\chi_1}; \tag{3.4.6}$$

3° *if* $0 < s < \frac{n}{p}$, *then (3.4.6) still holds for any given* q *satisfying* $\frac{1}{q} \ge \frac{1}{p} - \frac{s}{n}$ *and* $q > p$.

In (3.4.5) and (3.4.6), C *is a positive constant independent of both* u *and* $t \ge 0$.

Proof Taking $\lambda = \frac{1+t}{2}$ in Theorem 3.2.2 and using Lemmas 3.1.8 and 3.4.1, we can obtain the required conclusions.

In fact, when $s > \frac{n}{p}$, from (3.2.6) and noting Lemma 3.1.8, we get

$$\begin{aligned}
|\chi_1 u(t,x)| &\le \|u(t,\cdot)\|_{L^\infty\left(B_{\frac{1+t}{2}}\right)} \\
&\le C(1+t)^{-\frac{n}{p}} \sum_{|\alpha|\le s} (1+t)^{|\alpha|}\|D_x^\alpha u(t,\cdot)\|_{L^p\left(B_{\frac{1+t}{2}}\right)} \\
&= C(1+t)^{-\frac{n}{p}} \sum_{|\alpha|\le s} (1+t)^{|\alpha|}\|\chi_1 D_x^\alpha u(t,\cdot)\|_{L^p(\mathbb{R}^n)} \\
&\le C(1+t)^{-\frac{n}{p}} \sum_{|\alpha|\le s} (1+t)^{|\alpha|} \cdot \sum_{|\beta|\le|\alpha|} \|(1+|t-|\cdot||)^{-|\alpha|}\chi_1 \Gamma^\beta u(t,\cdot)\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Noticing that $|x| \le \frac{1+t}{2}$, it is easy to show that

$$C_1(1+t) \le 1+|t-|x|| \le C_2(1+t), \tag{3.4.7}$$

where C_1 and C_2 are both positive constants. From the above formula we obtain

$$|\chi_1 u(t,x)| \le C(1+t)^{-\frac{n}{p}}\|u(t,\cdot)\|_{\Gamma,s,p,\chi_1}. \tag{3.4.8}$$

Noting again that $|x| \le \frac{1+t}{2}$, we have

$$1+t \le 1+t+|x| \le \frac{3}{2}(1+t),$$

which implies the conclusion in case 1°.

In case 2°, from (3.2.7), (3.4.6) can be proved similarly to (3.4.8).

In case 3°, setting

$$\frac{1}{\bar{q}} = \frac{1}{p} - \frac{s}{n},$$

we have

$$\bar{q} \ge q > p.$$

Consequently, from Lemma 3.4.1 we have

$$\|\chi_1 u(t,\cdot)\|_{L^q(I\!R^n)} \le \|\chi_1 u(t,\cdot)\|^{\theta}_{L^p(I\!R^n)} \|\chi_1 u(t,\cdot)\|^{1-\theta}_{L^{\bar{q}}(I\!R^n)}, \tag{3.4.9}$$

where $0 \le \theta \le 1$ satisfying

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{\bar{q}}. \tag{3.4.10}$$

Similarly to (3.4.8), from (3.2.7) we can prove

$$\|\chi_1 u(t,\cdot)\|_{L^{\bar{q}}(I\!R^n)} \le C(1+t)^{-n(\frac{1}{p}-\frac{1}{\bar{q}})} \|u(t,\cdot)\|_{\Gamma,s,p,\chi_1}. \tag{3.4.11}$$

Plugging this in (3.4.9) and noticing (3.4.10), we obtain

$$\begin{aligned}\|\chi_1 u(t,\cdot)\|_{L^q(I\!R^n)} &\le C(1+t)^{-n(\frac{1}{p}-\frac{1}{\bar{q}})(1-\theta)} \|u(t,\cdot)\|^{1-\theta}_{\Gamma,s,p,\chi_1} \|\chi_1 u(t,\cdot)\|^{\theta}_{L^p(I\!R^n)} \\ &\le C(1+t)^{-n(\frac{1}{p}-\frac{1}{q})} \|u(t,\cdot)\|_{\Gamma,s,p,\chi_1}.\end{aligned}$$

This is exactly (3.4.6). The proof is finished. □

From Theorem 3.4.1 we immediately arrive at

Corollary 3.4.1 *For any given $p > 1$ and any given integer $N \ge 0$,*

1° *if $s > \frac{n}{p}$, then*

$$\|u(t,\cdot)\|_{\Gamma,N,\infty,\chi_1} \le C(1+t)^{-\frac{n}{p}} \|u(t,\cdot)\|_{\Gamma,N+s,p,\chi_1}, \quad \forall t \ge 0; \tag{3.4.12}$$

2° *If $s = \frac{n}{p}$, then for any q satisfying $p \le q < +\infty$, we have*

$$\|u(t,\cdot)\|_{\Gamma,N,q,\chi_1} \le C(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|u(t,\cdot)\|_{\Gamma,N+s,p,\chi_1}, \quad \forall t \ge 0; \tag{3.4.13}$$

3° *If $0 < s < \frac{n}{p}$, (3.4.13) still holds for any given q satisfying $\frac{1}{q} \ge \frac{1}{p} - \frac{s}{n}$ and $q > p$.*

Remark 3.4.1 The estimates in Theorem 3.4.1 and Corollary 3.4.1 hold for any given function $u(t,x)$ (as long as the norms appearing are well-defined). The reason why there appear on the right-hand sides of these estimates the decay rates of t is that the Γ operator contains positive power of t and the corresponding norms imply a certain growth with respect to t.

According to Remark 3.2.1, and noting that $1 + |t - |x|| = 1 + |x| - t \ge \frac{5+t}{2}$ when $|x| \ge \frac{3(1+t)}{2}$, we have

Remark 3.4.2 In Theorem 3.4.1, if $\chi_1(t, x)$ is taken to be the characteristic function of the set $\{(t, x) \mid |x| \geq \frac{3(1+t)}{2}\}$, then we still have (3.4.8) for case 1° and (3.4.6) for cases 2° and 3°, respectively. Therefore, Corollary 3.4.1 still holds.

Remark 3.4.3 Please refer to Klainerman (1985) for 1° of Theorem 3.4.1.

3.4.2 Sobolev Type Inequalities with Decay Factor on the Entire Space

Now we prove

Theorem 3.4.2 *For any given $p \geq 1$, when $s > \frac{n}{p}$, we have the following Sobolev type inequality with decay factor:*

$$|u(t,x)| \leq C(1+t+|x|)^{-\frac{n-1}{p}}(1+|t-|x||)^{-\frac{1}{p}} \cdot \|u(t,\cdot)\|_{\Gamma,s,p}, \quad \forall t \geq 0, \forall x \in \mathbb{R}^n, \tag{3.4.14}$$

where C is a positive constant independent of u.

For this purpose, we first prove the following lemma for the radial differential operator ∂_r.

Lemma 3.4.2 *For any given integer $k \geq 1$, we have*

$$\partial_r^k = \sum_{i_1,\ldots,i_k=1}^{n} \frac{x_{i_1}\cdots x_{i_k}}{r^k}\partial_{i_1}\cdots\partial_{i_k}. \tag{3.4.15}$$

Proof Due to (3.1.42), (3.4.15) holds for $k = 1$. By mathematical induction, it suffices to prove that: if (3.4.15) holds for the integer $k \geq 1$, then it also holds for the integer $k + 1$.

According to (3.4.15) and using the definition (3.1.42) of ∂_r, we have

$$\begin{aligned}
\partial_r^{k+1} &= \sum_{i_1,\ldots,i_{k+1}} \frac{x_{i_{k+1}}}{r}\partial_{i_{k+1}}\left(\frac{x_{i_1}\cdots x_{i_k}}{r^k}\partial_{i_1}\cdots\partial_{i_k}\right)\\
&= \sum_{i_1,\ldots,i_{k+1}} \frac{x_{i_1}\cdots x_{i_{k+1}}}{r^{k+1}}\partial_{i_1}\cdots\partial_{i_{k+1}}\\
&\quad + \sum_{i_1,\ldots,i_{k+1}} \frac{x_{i_{k+1}}}{r}\left[\frac{\delta_{i_1 i_{k+1}}x_{i_2}\cdots x_{i_k}}{r^k} + \cdots\right.\\
&\qquad\qquad \left. + \frac{x_{i_1}\cdots x_{i_{k-1}}\delta_{i_k i_{k+1}}}{r^k} - k\frac{x_{i_1}\cdots x_{i_k}}{r^{k+1}}\cdot\frac{x_{i_{k+1}}}{r}\right]\partial_{i_1}\cdots\partial_{i_k}\\
&= \sum_{i_1,\ldots,i_{k+1}} \frac{x_{i_1}\cdots x_{i_{k+1}}}{r^{k+1}}\partial_{i_1}\cdots\partial_{i_{k+1}}.
\end{aligned}$$

This is exactly (3.4.15) for the integer $k+1$.

Corollary 3.4.2 *For any given integer $k \geq 1$, we have*

$$|\partial_r^k u| \leq C \sum_{|\beta|=k} |D_x^\beta u|, \tag{3.4.16}$$

where C is a positive constant independent of u.

Proof of Theorem 3.4.2 We divide the proof into two cases.

1° The case that

$$|t-|x|| \geq \frac{1}{2}(t+|x|) > 0. \tag{3.4.17}$$

Applying (3.2.6) in Theorem 3.2.2 to the ball B_λ centered at x with radius $\lambda = \frac{1}{4}(t+|x|)$ and using Lemma 3.1.8, we obtain

$$\begin{aligned}
|u(t,x)| &\leq C(t+|x|)^{-\frac{n}{p}} \sum_{|\alpha|\leq s} (t+|x|)^{|\alpha|} \|D_y^\alpha u(t,y)\|_{L^p(B_\lambda)} \\
&\leq C(t+|x|)^{-\frac{n}{p}} \sum_{|\alpha|\leq s} (t+|x|)^{|\alpha|} \left\| (1+|t-|y||)^{-|\alpha|} \sum_{|\beta|\leq\alpha} |\Gamma^\beta u(t,y)| \right\|_{L^p(B_\lambda)} \\
&\leq C(t+|x|)^{-\frac{n}{p}} \sum_{|\alpha|\leq s} (t+|x|)^{|\alpha|} \left\| |t-|y||^{-|\alpha|} \sum_{|\beta|\leq\alpha} |\Gamma^\beta u(t,y)| \right\|_{L^p(B_\lambda)}.
\end{aligned} \tag{3.4.18}$$

Since on B_λ we have

$$|y-x| \leq \frac{1}{4}(t+|x|),$$

it is easy to show from (3.4.17) that

$$|t-|y|| \geq \frac{1}{4}(t+|x|),$$

then from (3.4.18) we get

$$|u(t,x)| \leq C(t+|x|)^{-\frac{n}{p}} \|u(t,\cdot)\|_{\Gamma,s,p}. \tag{3.4.19}$$

Consequently, we have

$$|u(t,x)| \leq C(t+|x|)^{-\frac{n-1}{p}} |t-|x||^{-\frac{1}{p}} \|u(t,\cdot)\|_{\Gamma,s,p}. \tag{3.4.20}$$

2° The case that

$$0 < |t - |x|| < \frac{1}{2}(t + |x|). \tag{3.4.21}$$

Now it is obvious that $|x| \neq 0$. Take

$$\lambda = \frac{|t - |x||}{2|x|}. \tag{3.4.22}$$

Then under condition (3.4.21) we have

$$\frac{1}{3}|x| < t < 3|x|, \tag{3.4.23}$$

thus

$$|t - |x|| < 2|x|,$$

therefore, $0 < \lambda < 1$.

Applying Theorem 3.2.3 to the annular domain $E_{|x|,\lambda} = \{y \mid ||y| - |x|| < \lambda|x|\}$ and noticing Corollary 3.4.1, we obtain

$$\begin{aligned} |u(t,x)| &\le C\left(\frac{|t-|x||}{|x|}\right)^{-\frac{1}{p}} |x|^{-\frac{n}{p}} \sum_{k+|\alpha|\le s} |t-|x||^k \|\partial_r^k \Omega_y^\alpha u(t,y)\|_{L^p(E_{|x|,\lambda})} \\ &\le C|t-|x||^{-\frac{1}{p}} |x|^{-\frac{n-1}{p}} \sum_{|\alpha|+|\beta|\le s} |t-|x||^{|\beta|} \|D_y^\beta \Omega_y^\alpha u(t,y)\|_{L^p(E_{|x|,\lambda})}, \end{aligned}$$

which, together with Lemma 3.1.8, implies that

$$\begin{aligned} |u(t,x)| &\le C|t-|x||^{-\frac{1}{p}} |x|^{-\frac{n-1}{p}} \sum_{|\alpha|+|\beta|\le s} |t-|x||^{|\beta|} \\ &\quad \cdot \left\| (1+|t-|y||)^{-|\beta|} \sum_{|\gamma|\le|\beta|} |\Gamma^\gamma \Omega_y^\alpha u(t,y)| \right\|_{L^p(E_{|x|,\lambda})} \\ &\le C|t-|x||^{-\frac{1}{p}} |x|^{-\frac{n-1}{p}} \sum_{|\alpha|+|\beta|\le s} |t-|x||^{|\beta|} \\ &\quad \cdot \left\| |t-|y||^{-|\beta|} \sum_{|\gamma|\le|\beta|} |\Gamma^\gamma \Omega_y^\alpha u(t,y)| \right\|_{L^p(E_{|x|,\lambda})}. \end{aligned} \tag{3.4.24}$$

Since on $E_{|x|,\lambda}$ we have

$$||y| - |x|| < \frac{1}{2}|t - |x||,$$

then we get

$$|t - |y|| \geq |t - |x|| - ||y| - |x|| > \frac{1}{2}|t - |x||. \tag{3.4.25}$$

Thus, (3.4.24) leads to

$$|u(t, x)| \leq C|t - |x||^{-\frac{1}{p}}|x|^{-\frac{n-1}{p}}\|u(t, \cdot)\|_{\Gamma,s,p}. \tag{3.4.26}$$

Then, noting (3.4.23), we have

$$|u(t, x)| \leq C(t + |x|)^{-\frac{n-1}{p}}|t - |x||^{-\frac{1}{p}}\|u(t, \cdot)\|_{\Gamma,s,p}. \tag{3.4.27}$$

Finally, from the usual Sobolev inequality we have

$$|u(t, x)| \leq C\|u(t, \cdot)\|_{\Gamma,s,p}, \quad \forall t \geq 0, \forall x \in I\!R^n. \tag{3.4.28}$$

Combining (3.4.20), (3.4.27) and (3.4.28), the desired (3.4.14) follows. The proof of Theorem 3.4.2 is finished.

Corollary 3.4.3 *For any given* $p \geq 1$*, when* $s > \frac{n}{p}$ *we have*

$$\|u(t, \cdot)\|_{L^\infty(I\!R^n)} \leq C(1 + t)^{-\frac{n-1}{p}}\|u(t, \cdot)\|_{\Gamma,s,p}, \quad \forall t \geq 0, \tag{3.4.29}$$

where C *is a positive constant independent of* u.

Corollary 3.4.4 *For any given integer* $N \geq 0$*, we have*

$$\|u(t, \cdot)\|_{\Gamma,N,\infty} \leq C(1 + t)^{-\frac{n-1}{p}}\|u(t, \cdot)\|_{\Gamma,N+s,p}, \quad \forall t \geq 0, \tag{3.4.30}$$

where $p \geq 1$*,* $s > \frac{n}{p}$*, and* C *is a positive constant independent of* u.

Remark 3.4.4 Please refer to Klainerman (1985) for the conclusion and proof outline of Theorem 3.4.2.

Remark 3.4.5 Comparing (3.4.12) in Corollary 3.4.1 (or the corresponding conclusion in Remark 3.4.2) with Corollary 3.4.4, we can see that: on a cone inside (or outside) the characteristic cone of the wave equation, the corresponding Sobolev type inequalities have faster decay rate.

Chapter 4
Estimates on Solutions to the Linear Wave Equations

4.1 Estimates on Solutions to the One-Dimensional Linear Wave Equations

Consider the following Cauchy problem of one-dimensional linear wave equations:

$$u_{tt} - u_{xx} = F(t, x), \quad (t, x) \in I\!R_+ \times I\!R, \tag{4.1.1}$$

$$t = 0 : u = f(x), \ u_t = g(x), \quad x \in I\!R. \tag{4.1.2}$$

We will establish some related estimates on its solution $u = u(t, x)$ as follows.

Theorem 4.1.1 *For the solution $u = u(t, x)$ of the one-dimensional Cauchy problem (4.1.1)–(4.1.2), the following estimates hold:*

$1°$ *We have*

$$\|u(t,\cdot)\|_{L^\infty(I\!R)} \le \|f\|_{L^\infty(I\!R)} + t\|g\|_{L^\infty(I\!R)} + \int_0^t (t-\tau)\|F(\tau,\cdot)\|_{L^\infty(I\!R)} d\tau, \quad \forall\, t \ge 0; \tag{4.1.3}$$

$2°$ *For any given p $(1 \le p \le +\infty)$, we have*

$$\|u(t,\cdot)\|_{L^p(I\!R)} \le \|f\|_{L^p(R)} + t^{\frac{1}{p}}\|g\|_{L^1(I\!R)} + \int_0^t (t-\tau)^{\frac{1}{p}}\|F(\tau,\cdot)\|_{L^1(I\!R)} d\tau, \quad \forall\, t \ge 0 \tag{4.1.4}$$

and

$$\|Du(t,\cdot)\|_{L^p(I\!R)} \le \|f'\|_{L^p(I\!R)} + \|g\|_{L^p(I\!R)} + \int_0^t \|F(\tau,\cdot)\|_{L^p(I\!R)} d\tau, \quad \forall\, t \ge 0, \tag{4.1.5}$$

where

$$D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right); \tag{4.1.6}$$

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3° *If*

$$\int_{-\infty}^{+\infty} g(x)dx = 0, \tag{4.1.7}$$

then when $F \equiv 0$*, for any given* $p (1 \le p \le +\infty)$ *we have*

$$\|u(t,\cdot)\|_{L^p(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})} + \|G\|_{L^p(\mathbb{R})}, \quad \forall\, t \ge 0, \tag{4.1.8}$$

where

$$G(x) = \int_{-\infty}^{x} g(y)dy \tag{4.1.9}$$

is the primitive function of g*.*

Proof Thanks to (2.1.14) in Chap. 2, it suffices to prove Theorem 4.1.1 for the case that $F \equiv 0$, and in this case, from (2.1.20) in Chap. 2, the solution to the Cauchy problem (4.1.1)–(4.1.2) can be expressed by

$$u(t,x) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy. \tag{4.1.10}$$

Noting that

$$\frac{1}{2}\left\| \int_{x-t}^{x+t} g(y)dy \right\|_{L^\infty(\mathbb{R})} \le \frac{1}{2}\left\| \int_{x-t}^{x+t} dy \right\|_{L^\infty(\mathbb{R})} \cdot \|g\|_{L^\infty(\mathbb{R})} = t\|g\|_{L^\infty(\mathbb{R})}, \tag{4.1.11}$$

Estimate (4.1.3) for the case $F \equiv 0$ follows immediately from (4.1.10).

To prove (4.1.4) for the case $F \equiv 0$, we rewrite (4.1.10) as

$$u(t,x) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int_R H(t - |x-y|)g(y)dy, \tag{4.1.12}$$

where H is the Heaviside function. Noticing that

$$\|H(t - |\cdot - y|)\|_{L^p(\mathbb{R})} = \left(\int_{y-t}^{y+t} dx \right)^{\frac{1}{p}} = (2t)^{\frac{1}{p}},$$

from (4.1.11) we have

$$\begin{aligned}
\|u(t,\cdot)\|_{L^p(\mathbb{R})} &\le \|f\|_{L^p(\mathbb{R})} + \frac{1}{2}\int_R \|H(t - |\cdot - y|)\|_{L^p(\mathbb{R})} |g(y)|dy \\
&= \|f\|_{L^p(\mathbb{R})} + \frac{1}{2}(2t)^{\frac{1}{p}} \|g\|_{L^1(\mathbb{R})} \\
&\le \|f\|_{L^p(\mathbb{R})} + t^{\frac{1}{p}} \|g\|_{L^1(\mathbb{R})}.
\end{aligned}$$

This is exactly (4.1.4) when $F \equiv 0$.

From (4.1.10) we have

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}(f'(x+t) - f'(x-t)) + \frac{1}{2}(g(x+t) + g(x-t)), \tag{4.1.13}$$

$$\frac{\partial u}{\partial x}(t,x) = \frac{1}{2}(f'(x+t) + f'(x-t)) + \frac{1}{2}(g(x+t) - g(x-t)), \tag{4.1.14}$$

then it is easy to get (4.1.5) when $F \equiv 0$.

Finally, under assumption (4.1.7), noting (4.1.9), (4.1.10) can be rewritten as

$$u(t,x) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}(G(x+t) - G(x-t)), \tag{4.1.15}$$

which implies (4.1.8) immediately. The proof is finished. □

4.2 Generalized Huygens Principle

In the following sections, we always assume that the space dimension $n \geq 2$ unless otherwise noted.

Consider the following Cauchy problem of the linear homogeneous wave equation:

$$\Box u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.2.1}$$

$$t = 0 : u = f(x), \ u_t = g(x), \quad x \in \mathbb{R}^n, \tag{4.2.2}$$

and assume that the initial data are compactly supported:

$$\text{supp}\{f, g\} \subseteq \{x \mid |x| \leq \rho\} \ \ (\rho > 0 \ \text{ is a constant}). \tag{4.2.3}$$

When $n(\geq 3)$ is odd, from the expression of solutions showed in Theorem 2.1.1 in Chap. 2 we can see that, when

$$t - |x| \geq \rho, \tag{4.2.4}$$

we always have

$$u(t,x) \equiv 0. \tag{4.2.5}$$

This is just the well-known **Huygens principle**.

When $n(\geq 2)$ is even, the above conclusion is no longer valid (see Theorem 2.1.2 in Chap. 2), but we can establish a corresponding estimate on the solution, called the **generalized Huygens principle** as follows.

Theorem 4.2.1 *Let $u = u(t,x)$ be the solution to the Cauchy problem (4.2.1)–(4.2.2), and (4.2.3) hold. Then when $n(\geq 2)$ is even, and*

$$t - |x| \geq 2\rho, \tag{4.2.6}$$

we have

$$|u(t,x)| \leq C(t+|x|)^{-\frac{n-1}{2}}(t-|x|)^{-\frac{n-1}{2}}\left((t-|x|)^{-1}\|f\|_{L^1(I\!R^n)} + \|g\|_{L^1(I\!R^n)}\right), \tag{4.2.7}$$

where C is a positive constant independent of both (f, g) and ρ.

Proof From Theorem 2.1.2 in Chap. 2, now the solution of the Cauchy problem (4.2.1)–(4.2.2) can be expressed by

$$u(t,x) = \frac{1}{\omega_n \Gamma(\frac{n}{2})}\left(2t\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n}{2}} \int_{|y-x|\leq t} \frac{f(y)}{\sqrt{t^2-|y-x|^2}}dy \right.$$
$$\left. + \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \int_{|y-x|\leq t} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy\right), \tag{4.2.8}$$

where ω_n is the area of the unit sphere in $I\!R^n$. Noting the compact support assumption (4.2.3) on the initial data, the range of integration in the above formula is actually $\{y \mid |y-x| \leq t\} \bigcap \{y \mid |y| \leq \rho\}$.

Due to assumption (4.2.6), on the sphere $|y-x| = t$ we always have

$$|y| \geq |y-x| - |x| = t - |x| \geq 2\rho,$$

then the integrands in (4.2.8) are always zero. Thus, the differential operator $\frac{1}{2t}\frac{\partial}{\partial t}$ can be moved into the integral sign successively so that we get

$$u(t,x) = C_1 t \int_{\substack{|y-x|\leq t \\ |y|\leq\rho}} (t^2-|y-x|^2)^{-\frac{n+1}{2}} f(y)dy + C_2 \int_{\substack{|y-x|\leq t \\ |y|\leq\rho}} (t^2-|y-x|^2)^{-\frac{n-1}{2}} g(y)dy, \tag{4.2.9}$$

where C_1 and C_2 are some constants.

Under condition (4.2.6) and the assumption $|y| \leq \rho$, we have

$$\begin{aligned} t^2 - |y-x|^2 &= t^2 - |x|^2 + 2x\cdot y - |y|^2 \\ &\geq t^2 - |x|^2 - 2|x||y| - |y|^2 \\ &\geq t^2 - |x|^2 - 2\rho|x| - \rho^2 \\ &= (t^2-|x|^2)\left(1 - \frac{(2|x|+\rho)\rho}{(t-|x|)(t+|x|)}\right) \\ &\geq (t^2-|x|^2)\left(1 - \frac{(2|x|+\rho)\rho}{2\rho(2|x|+2\rho)}\right) \\ &\geq \frac{1}{2}(t^2-|x|^2). \end{aligned} \tag{4.2.10}$$

Therefore, (4.2.7) follows immediately from (4.2.9). The proof is finished. □

Remark 4.2.1 According to Theorem 4.2.1, on the domain enclosed by the forward characteristic cone given by (4.2.6) with the vertex $(t, x) = (2\rho, 0)$, we have

$$|u(t,x)| \leq C_\rho (1+t)^{-\frac{n-1}{2}} (\|f\|_{L^1(R^n)} + \|g\|_{L^1(R^n)}), \tag{4.2.11}$$

where C_ρ is a positive constant independent of (f, g) but depending on ρ. Therefore, as $t \to +\infty$ the solution $u = u(t, x)$ has at least the decay rate $(1+t)^{-\frac{n-1}{2}}$ on the domain enclosed by this forward characteristic cone.

Remark 4.2.2 If assumption (4.2.6) is changed into

$$t - |x| \geq a\rho, \tag{4.2.12}$$

where $a > 1$ is a constant, the conclusion of Theorem 4.2.1 is still valid. Since $a(> 1)$ can be infinitely close to 1, in the interior of the forward characteristic cone with vertex $(t, x) = (\rho, 0)$, namely, when

$$t - |x| > \rho, \tag{4.2.13}$$

the solution $u = u(t, x)$ has at least the decay rate $(1+t)^{-\frac{n-1}{2}}$ as $t \to +\infty$. This gives the reason why the conclusion given by Theorem 4.2.1 is called the **generalized Huygens principle** by comparing with the result (4.2.5) for the case that n is odd.

Remark 4.2.3 The result in Theorem 4.2.1 goes back to Hörmander (1988), and the proof can also be found in Li and Zhou (1995c).

From Theorem 4.2.1, it is easy to obtain the following

Corollary 4.2.1 *Under the assumptions of Theorem 4.2.1, for any given l satisfying $0 \leq l \leq \frac{n-1}{2}$, we have*

$$|u(t,x)| \leq C_\rho (1+t+|x|)^{-\frac{n-1}{2}} (1+|t-|x||)^{-l} \left(\|f\|_{L^1(\mathbb{R}^n)} + \|g\|_{L^1(\mathbb{R}^n)}\right), \tag{4.2.14}$$

where C_ρ is a positive constant depending only on ρ.

4.3 Estimates on Solutions to the Two-Dimensional Linear Wave Equations

Consider the following Cauchy problem of the two-dimensional linear homogeneous wave equation:

$$\Box u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \tag{4.3.1}$$

$$t = 0 : u = f(x), \ u_t = g(x), \quad x \in \mathbb{R}^2, \tag{4.3.2}$$

where $x = (x_1, x_2)$.

Theorem 4.3.1 *Let $u = u(t, x)$ be the solution to the two-dimensional Cauchy problem (4.3.1)–(4.3.2).*

1° We have

$$\|u(t,\cdot)\|_{L^2(I\!R^2)} \le \|f\|_{L^2(I\!R^2)} + C\sqrt{\ln(2+t)}\|(1+|\cdot|^2)g\|_{L^2(I\!R^2)}; \tag{4.3.3}$$

2° If

$$\int_{I\!R^2} g(x)dx = 0, \tag{4.3.4}$$

then we have

$$\|u(t,\cdot)\|_{L^2(I\!R^2)} \le \|f\|_{L^2(I\!R^2)} + C\|(1+|\cdot|^2)g\|_{L^2(I\!R^2)}, \tag{4.3.5}$$

where C is a positive constant.

Proof From Theorem 2.3.1 in Chap. 2, the Fourier transform of $u = u(t, x)$ with respect to x is

$$\widehat{u}(t,\xi) = \cos(|\xi|t)\widehat{f}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\widehat{g}(\xi). \tag{4.3.6}$$

Thus, from the Parseval identity we have

$$\begin{aligned} \|u(t,\cdot)\|_{L^2_x} &= \|\widehat{u}(t,\cdot)\|_{L^2_\xi} \\ &\le \|\widehat{f}\|_{L^2} + \left\|\frac{\sin(|\xi|t)}{|\xi|}\widehat{g}(\xi)\right\|_{L^2} \\ &= \|f\|_{L^2} + \left\|\frac{\sin(|\xi|t)}{|\xi|}\widehat{g}(\xi)\right\|_{L^2}. \end{aligned} \tag{4.3.7}$$

Adopting polar coordinates to variable ξ: $\xi = r\omega$, where $r = |\xi|$ and $\omega = (\cos\theta, \sin\theta)$, we have

$$I(t) \overset{\text{def.}}{=} \left\|\frac{\sin(|\xi|t)}{|\xi|}\widehat{g}(\xi)\right\|^2_{L^2} = \iint \frac{\sin^2(rt)}{r}\widehat{g}^2(r\omega)drd\theta. \tag{4.3.8}$$

Integrating by parts, we get

$$\begin{aligned} I'(t) &= \iint \sin(2rt)\widehat{g}^2(r\omega)drd\theta \\ &= \frac{1}{t}\iint \cos(2rt)\widehat{g}(r\omega)\partial_r\widehat{g}(r\omega)drd\theta, \end{aligned}$$

then we obtain

$$|I'(t)| \le \frac{1}{t}\left(\iint \widehat{g}^2(r\omega)drd\theta\right)^{\frac{1}{2}}\left(\iint (\partial_r\widehat{g}(r\omega))^2 drd\theta\right)^{\frac{1}{2}}.$$

By integration by parts once in the above two integrals and using Parseval identity, we obtain

$$|I'(t)| \le \frac{C}{t}\|(1+|\cdot|^2)g\|_{L^2}^2,\ \forall\, t > 0, \tag{4.3.9}$$

where C is a positive constant.

Noting that near $t = 0$, say, $0 \le t \le 1$, we have

$$\sin^2(rt) \le (rt)^2 \le r^2,$$

from (4.3.8) and using Parseval identity we have

$$I(t) \le \|g\|_{L^2}^2,\ \forall\, 0 \le t \le 1. \tag{4.3.10}$$

Combining (4.3.9)–(4.3.10), it is easy to obtain that

$$I(t) \le C\ln(2+t)\|(1+|\cdot|^2)g\|_{L^2}^2,\ \forall\, t \ge 0, \tag{4.3.11}$$

then (4.3.3) follows immediately from (4.3.7).

On the other hand, if (4.3.4) holds, then from the definition of Fourier transform, this condition is equivalent to

$$\widehat{g}(0) = 0, \tag{4.3.12}$$

hence the integration by parts yields

$$\begin{aligned}\frac{\widehat{g}(\xi)}{|\xi|} &= \frac{1}{|\xi|}\int_0^1 \partial_s\widehat{g}(s\xi)ds = \int_0^1 \partial_r\widehat{g}(s\xi)ds \\ &= \partial_r\widehat{g}(\xi) - |\xi|\int_0^1 s\partial_r^2\widehat{g}(s\xi)ds,\end{aligned} \tag{4.3.13}$$

where $\xi = r\omega$.

It is easy to show from (4.3.7) that

$$\begin{aligned}\|u(t,\cdot)\|_{L^2} &\le \|f\|_{L^2} + \left\|\frac{\widehat{g}(\xi)}{|\xi|}\right\|_{L^2} \\ &\le \|f\|_{L^2} + \|\widehat{g}\|_{L^2} + \left\|\frac{\widehat{g}(\xi)}{|\xi|}\right\|_{L^2(B_1)},\end{aligned} \tag{4.3.14}$$

where $B_1 = \{\xi |\, |\xi| \leq 1\}$. And from (4.3.13) we have

$$\left\| \frac{\widehat{g}(\xi)}{|\xi|} \right\|_{L^2(B_1)} \leq \|\partial_r \widehat{g}\|_{L^2(B_1)} + \int_0^1 s\|\partial_r^2 \widehat{g}(s\xi)\|_{L^2(B_1)} ds$$
$$= \|\partial_r \widehat{g}\|_{L^2(B_1)} + \int_0^1 s^2\|\partial_r^2 \widehat{g}(\xi)\|_{L^2(B_s)} ds$$
$$\leq \|\partial_r \widehat{g}\|_{L^2} + \|\partial_r^2 \widehat{g}\|_{L^2}.$$

Then, noting (4.3.14) and using Parseval identity, (4.3.5) follows immediately. □

4.4 An L^2 Estimate on Solutions to the $n(\geq 4)$-Dimensional Linear Wave Equations

In this section, based on an estimate in Hidano et al. (2009) we will establish a new L^2 estimate on solutions to the Cauchy problem of the $n(\geq 4)$-dimensional linear wave equations. This estimate will play a crucial role in Chap. 11 when establishing the sharp lower bound estimate on the life-span of solutions to the Cauchy problem of four-dimensional linear wave equations with small initial data.

First of all, we prove the following lemma. The result in this lemma is known as Morawetz estimate.

Lemma 4.4.1 *Suppose that $n \geq 3$ and $u = u(t, x)$ is the solution to the Cauchy problem*

$$\Box u(t, x) = 0, \tag{4.4.1}$$
$$t = 0 : u = 0,\ u_t = g(x). \tag{4.4.2}$$

of the n-dimensional linear wave equation, then we have the following space-time estimate:

$$\||x|^{-s}u\|_{L^2(I\!R \times I\!R^n)} \leq C\|g\|_{\dot{H}^{s-\frac{3}{2}}(I\!R^n)}, \tag{4.4.3}$$

where s satisfies

$$1 < s < \frac{n}{2}, \tag{4.4.4}$$

$\dot{H}^{s-\frac{3}{2}}(I\!R^n)$ is defined by (3.3.12) in Chap. 3, and C is a positive constant.

Proof We first prove that: if s satisfies (4.4.4), then for any given $v \in \dot{H}^s(I\!R^n)$ we have

$$\sup_{r>0} r^{\frac{n}{2}-s}\|v(r\omega)\|_{L^2(S^{n-1})} \leq C\|v\|_{\dot{H}^s(I\!R^n)}, \tag{4.4.5}$$

where $x = r\omega$, $r = |x|$ and $\omega \in S^{n-1}$.

In fact, from 1° of Theorem 3.3.1 (wherein taking $a = 1$) in Chap. 3, for any given $h \in \dot{H}^s(I\!R^n)$, we easily get

$$\|h\|_{L^2(S^{n-1})} \leq C\|h\|_{\dot{H}^s(I\!R^n)}. \tag{4.4.6}$$

For any given $v \in \dot{H}^s(I\!R^n)$, taking $h(x) = v(\lambda x) \stackrel{\text{def.}}{=} h_\lambda(x)$ in the above formula, where λ is any given positive number, we obtain

$$\|h_\lambda\|_{L^2(S^{n-1})} \leq C\|h_\lambda\|_{\dot{H}^s(I\!R^n)}. \tag{4.4.7}$$

whereas

$$\|h_\lambda\|_{L^2(S^{n-1})} = \|v(\lambda\omega)\|_{L^2(S^{n-1})} \tag{4.4.8}$$

and

$$\|h_\lambda\|_{\dot{H}^s(I\!R^n)} = \left\||\xi|^s \hat{h}_\lambda\right\|_{L^2(I\!R^n)} = \left\||\xi|^s \widehat{v(\lambda x)}\right\|_{L^2(I\!R^n)},$$

where $\widehat{\ }$ over a function stands for the Fourier transform of this function. From the definition of Fourier transform we have

$$\widehat{v(\lambda x)} = \lambda^{-n}\hat{v}\Big(\frac{\xi}{\lambda}\Big),$$

then it is easy to show that

$$\begin{aligned} \|h_\lambda\|_{\dot{H}^s(I\!R^n)} &= \left\||\xi|^s \widehat{v(\lambda x)}\right\|_{L^2(I\!R^n)} = \lambda^{-n}\left\||\xi|^s \hat{v}\Big(\frac{\xi}{\lambda}\Big)\right\|_{L^2(I\!R^n)} \\ &= \lambda^{s-\frac{n}{2}}\left\||\xi|^s \hat{v}(\xi)\right\|_{L^2(I\!R^n)} = \lambda^{s-\frac{n}{2}}\|v\|_{\dot{H}^s(I\!R^n)}. \end{aligned} \tag{4.4.9}$$

Plugging (4.4.8)–(4.4.9) in (4.4.7), we immediately get that for any given $\lambda > 0$, we have

$$\|v(\lambda\omega)\|_{L^2(S^{n-1})} \leq C\lambda^{s-\frac{n}{2}}\|v\|_{\dot{H}^s(I\!R^n)}. \tag{4.4.10}$$

Taking $\lambda = r = |x|$ particularly in the above formula, (4.4.5) follows immediately.

Applying (4.4.10) to the Fourier transform $\hat{v}$ of v, we obtain

$$\Big(\int_{S^{n-1}} |\hat{v}(\lambda\omega)|^2 d\omega\Big)^{\frac{1}{2}} \leq C\lambda^{s-\frac{n}{2}}\||x|^s v\|_{L^2(I\!R^n)}. \tag{4.4.11}$$

From this, the duality can be used to get

$$\left\||x|^{-s}\int_{S^{n-1}} e^{i\lambda x\cdot\omega}h(\omega)d\omega\right\|_{L^2(I\!R^n)} \leq C\lambda^{s-\frac{n}{2}}\|h\|_{L^2(S^{n-1})}. \tag{4.4.12}$$

In fact,

$$\text{The left-hand side of (4.4.12)} = \sup_{v\neq 0} \frac{\int_{I\!R^n} v(x)|x|^{-s}\int_{S^{n-1}} e^{i\lambda x\cdot\omega}h(\omega)d\omega dx}{\|v\|_{L^2(I\!R^n)}}. \tag{4.4.13}$$

Set

$$\bar{v}(x) = |x|^{-s}v(x), \tag{4.4.14}$$

we have

$$\begin{aligned}\int_{I\!R^n} v(x)|x|^{-s}\int_{S^{n-1}} e^{i\lambda x\cdot\omega}h(\omega)d\omega dx &= \int_{S^{n-1}}\Big(\int_{I\!R^n} e^{i\lambda x\cdot\omega}\bar{v}(x)dx\Big)h(\omega)d\omega\\ &= \int_{S^{n-1}} \hat{\bar{v}}(\lambda\omega)h(\omega)d\omega,\end{aligned}$$

then

$$\Big|\int_{I\!R^n} v(x)|x|^{-s}\int_{S^{n-1}} e^{i\lambda x\cdot\omega}h(\omega)d\omega dx\Big| \le \|\hat{\bar{v}}(\lambda\omega)\|_{L^2(S^{n-1})}\|h\|_{L^2(S^{n-1})}, \tag{4.4.15}$$

while, using (4.4.11) and noting (4.4.14), we have

$$\|\hat{\bar{v}}(\lambda\omega)\|_{L^2(S^{n-1})} \le C\lambda^{s-\frac{n}{2}}\||x|^s\bar{v}\|_{L^2(I\!R^n)} = C\lambda^{s-\frac{n}{2}}\|v\|_{L^2(I\!R^n)}. \tag{4.4.16}$$

Thus, (4.4.12) follows from (4.4.13).

Now we are ready to consider the solution $u = u(t, x)$ to the Cauchy problem (4.4.1)–(4.4.2). From (2.3.3) in Chap. 2 we have

$$u = \text{Im} v. \tag{4.4.17}$$

Since

$$\hat{v}(t,\xi) = \frac{e^{it|\xi|}}{|\xi|}\hat{g}(\xi), \tag{4.4.18}$$

taking Fourier transform with respect to t for the above formula, we get the space-time Fourier transform of v:

$$v^{\sharp}(\tau,\xi) = \begin{cases} \dfrac{\delta(\tau-|\xi|)}{|\xi|}\hat{g}(\xi), & \tau > 0,\\ 0, & \tau < 0,\end{cases} \tag{4.4.19}$$

thus, the Fourier transform of v with respect to t is: for $\tau > 0$,

$$\tilde{v}(\tau, x) = \int_{I\!R^n} e^{ix\cdot\xi}\frac{\delta(\tau-|\xi|)}{|\xi|}\hat{g}(\xi)\mathrm{d}\xi = \tau^{n-2}\int_{S^{n-1}} e^{ix\cdot\omega\tau}\hat{g}(\tau\omega)d\omega; \tag{4.4.20}$$

and for $\tau < 0$, $\tilde{v}(\tau, x) \equiv 0$. Then, using (4.4.12) we obtain that: for $\tau > 0$, we have

$$\||x|^{-s}\tilde{v}(\tau, x)\|_{L^2(\mathbb{R}^n)} \leq C\tau^{\frac{n}{2}-2+s}\|\hat{g}(\tau\omega)\|_{L^2(S^{n-1})}. \tag{4.4.21}$$

Noting that when $\tau < 0$, $\tilde{v}(\tau, x) \equiv 0$, taking the L^2 norm with respect to τ for the above formula and using Paserval identity, we obtain

$$\begin{aligned}\||x|^{-s}v(t, x)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} &\leq C\Big(\int_0^\infty \tau^{2\left(\frac{n}{2}-2+s\right)}\int_{S^{n-1}}\hat{g}^2(\tau\omega)d\omega d\tau\Big)^{\frac{1}{2}} \\ &= C\Big(\int_{\mathbb{R}^n}|\xi|^{2s-3}\hat{g}^2(\xi)d\xi\Big)^{\frac{1}{2}} = C\|g\|_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^n)},\end{aligned} \tag{4.4.22}$$

thus, noting (4.4.17), we immediately obtain (4.4.3). The proof is finished. □

The following lemma gives the dual estimate of the above Morawetz estimate.

Lemma 4.4.2 *Suppose that $n \geq 3$ and $u = u(t, x)$ is the solution to the Cauchy problem of the n-dimensional linear wave equations:*

$$\Box u(t, x) = F(t, x), \tag{4.4.23}$$
$$t = 0 : u = u_t = 0, \tag{4.4.24}$$

then for any given $T > 0$, we have

$$\sup_{0\leq t\leq T}\|u(t,\cdot)\|_{\dot{H}^{\frac{3}{2}-s}(\mathbb{R}^n)} \leq C\||x|^s F(t, x)\|_{L^2(0,T;L^2(\mathbb{R}^n))}, \tag{4.4.25}$$

where s satisfies (4.4.4), and C is a positive constant.

Proof Thanks to Duhamel principle (2.1.13) in Chap. 2 and Lemma 4.4.1, it is easy to obtain that

$$\||x|^{-s}u\|_{L^2(0,T;L^2(\mathbb{R}^n))} \leq C\int_0^T \|F(\tau,\cdot)\|_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^n)}d\tau. \tag{4.4.26}$$

In addition, by duality we have

$$\sup_{0\leq t\leq T}\|u(t,\cdot)\|_{\dot{H}^{\frac{3}{2}-s}(\mathbb{R}^n)} = \sup_{G\neq 0}\frac{\int_0^T\int_{\mathbb{R}^n}u(t,x)G(t,x)dxdt}{\int_0^T\|G(t,\cdot)\|_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^n)}dt}. \tag{4.4.27}$$

Let $v = v(t, x)$ satisfy

$$\Box v(t, x) = G(t, x), \tag{4.4.28}$$
$$t = T : v = v_t = 0. \tag{4.4.29}$$

From integration by parts we get

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} u(t,x)G(t,x)dxdt &= \int_0^T\int_{\mathbb{R}^n} u(t,x)\Box v(t,x)dxdt\\ &= \int_0^T\int_{\mathbb{R}^n} \Box u(t,x)v(t,x)dxdt\\ &= \int_0^T\int_{\mathbb{R}^n} F(t,x)v(t,x)dxdt, \qquad (4.4.30)\end{aligned}$$

then

$$\left|\int_0^T\int_{\mathbb{R}^n} u(t,x)G(t,x)dxdt\right| \le \||x|^s F\|_{L^2(0,T;L^2(\mathbb{R}^n))}\||x|^{-s}v\|_{L^2(0,T;L^2(\mathbb{R}^n))}. \qquad (4.4.31)$$

Applying (4.4.26) to v, we have

$$\||x|^{-s}v\|_{L^2(0,T;\mathbb{R}^n)} \le C\int_0^T \|G(t,\cdot)\|_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^n)}dt, \qquad (4.4.32)$$

then the dual estimate (4.4.25) follows immediately from (4.4.27). The proof is finished. □

Theorem 4.4.1 *Suppose that $n \ge 4$ and $u = u(t,x)$ is the solutions to the Cauchy problem (4.4.23)–(4.4.24) of the n-dimensional linear wave equations, where, for any given $t \in [0,T]$, the term $F(t,x)$ on the right-hand side is supported in $\{x \mid |x| \le t+\rho\}$ with respect to x, where ρ is a positive constant:*

$$\mathrm{supp}F \subseteq \{(t,x) \mid 0 \le t \le T, |x| \le t+\rho\}. \qquad (4.4.33)$$

Then we have the following L^2 estimate of the solution u:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le C_\rho\Big\{\|(1+t)^s\chi_1 F\|_{L^2(0,T;L^q(\mathbb{R}^n))} + \|(1+t)^{-\frac{n-3}{2}}\chi_2 F\|_{L^2(0,T;L^{1,2}(\mathbb{R}^n))}\Big\},$$
$$0 \le t \le T, \qquad (4.4.34)$$

where

$$\frac{1}{2} < s < 1, \qquad (4.4.35)$$

q $(1 < q < 2)$ is determined by

$$\frac{1}{q} = \frac{1}{2} + \frac{\frac{3}{2}-s}{n}, \qquad (4.4.36)$$

$\chi_1(t,x)$ is the characteristic function of the set $\{(t,x) \mid |x| \leq \frac{1+t}{2}\}$, $\chi_2 = 1 - \chi_1$, and C_ρ is a positive constant possibly depending on ρ.

Proof Denote

$$|D| = \sqrt{-\Delta_x}, \tag{4.4.37}$$

where Δ_x stands for the Laplace operator in $I\!R^n$. Acting $|D|^{s-\frac{3}{2}}$ on both sides of the wave equation (4.4.23) and noticing the commutativity between $|D|$ and the wave operator $\Box$, from Lemma 4.4.2 we easily get

$$\begin{aligned}
&\|u(t,\cdot)\|_{L^2(I\!R^n)}\\
&\leq C\Big\| |x|^s\big(|D|^{s-\frac{3}{2}}F\big)\Big\|_{L^2(0,T;L^2(I\!R^n))}\\
&\leq C\Big\{\Big\| |x|^s\big(|D|^{s-\frac{3}{2}}(\chi_1 F)\big)\Big\|_{L^2(0,T;L^2(I\!R^n))}\\
&+\Big\| |x|^s\big(|D|^{s-\frac{3}{2}}(\chi_2 F)\big)\Big\|_{L^2(0,T;L^2(I\!R^n))}\Big\}, \qquad 0\leq t\leq T.
\end{aligned} \tag{4.4.38}$$

First, we estimate the first term on the right-hand side of the above formula. Noting (4.4.37) and the definition of χ_1, it is easy to show that

$$\Big(|D|^{s-\frac{3}{2}}(\chi_1 F)\Big)(t,x) = \overset{\vee}{|\xi|^{s-\frac{3}{2}}} * (\chi_1 F),$$

where $^\vee$ over a function stands for the inverse Fourier transform of this function, and $*$ stands for the convolution. Noting that in

$$\overset{\vee}{|\xi|^{s-\frac{3}{2}}} = C\int_{I\!R^n} e^{ix\cdot\xi}|\xi|^{s-\frac{3}{2}}d\xi,$$

by setting $\xi = \frac{\eta}{|x|}$ we have

$$\overset{\vee}{|\xi|^{s-\frac{3}{2}}} = C\Big(\int_{I\!R^n} e^{i\frac{x}{|x|}\cdot\eta}|\eta|^{s-\frac{3}{2}}d\eta\Big)|x|^{-(n+s-\frac{3}{2})},$$

and $\int_{I\!R^n} e^{i\frac{x}{|x|}\cdot\eta}|\eta|^{s-\frac{3}{2}}d\eta$ is a constant independent of x, then we finally obtain

$$\Big(|D|^{s-\frac{3}{2}}(\chi_1 F)\Big)(t,x) = C\int_{I\!R^n}\frac{\chi_1 F(t,y)}{|x-y|^{n+s-\frac{3}{2}}}dy = C\int_{|y|\leq\frac{1+t}{2}}\frac{\chi_1 F(t,y)}{|x-y|^{n+s-\frac{3}{2}}}dy. \tag{4.4.39}$$

Then, when $|x| \geq 1+t$, since we have $|x-y| \geq |x| - |y| \geq \frac{|x|}{2}$ from $|y| \leq \frac{1+t}{2}$, we have

$$\begin{aligned}\left|\left(|D|^{s-\frac{3}{2}}(\chi_1 F)\right)(t,x)\right| &\leq C|x|^{-(n+s-\frac{3}{2})}\|\chi_1 F\|_{L^1(I\!R^n)} \\ &\leq C|x|^{-(n+s-\frac{3}{2})}\|\chi_1 F\|_{L^q(I\!R^n)}\|\chi_1\|_{L^{q'}(I\!R^n)} \\ &\leq C(1+t)^{\frac{n}{q'}}|x|^{-(n+s-\frac{3}{2})}\|\chi_1 F\|_{L^q(I\!R^n)}, \end{aligned} \tag{4.4.40}$$

where $q'(>2)$ is determined by $\frac{1}{q} + \frac{1}{q'} = 1$, then from (4.4.36) we have

$$\frac{1}{q'} = \frac{1}{2} - \frac{\frac{3}{2} - s}{n}. \tag{4.4.41}$$

Noting $n \geq 4$ and (4.4.41), from (4.4.40) we easily obtain

$$\begin{aligned}\left\||x|^s\left(|D|^{s-\frac{3}{2}}(\chi_1 F)\right)\right\|_{L^2(|x|\geq 1+t)} &\leq C(1+t)^{\frac{n}{q'}}\|\chi_1 F\|_{L^q(I\!R^n)}\||x|^{-(n-\frac{3}{2})}\|_{L^2(|x|\geq 1+t)} \\ &\leq C(1+t)^{\frac{n}{q'}-\frac{n-3}{2}}\|\chi_1 F\|_{L^q(I\!R^n)} \\ &= C(1+t)^s\|\chi_1 F\|_{L^q(I\!R^n)}. \end{aligned} \tag{4.4.42}$$

On the other hand, when $|x| \leq 1+t$, from (3.3.12) in Chap. 3 and noting (4.4.36), it is clear that

$$\begin{aligned}\left\||x|^s\left(|D|^{s-\frac{3}{2}}(\chi_1 F)\right)\right\|_{L^2(|x|\leq 1+t)} &\leq (1+t)^s\left\||D|^{s-\frac{3}{2}}(\chi_1 F)\right\|_{L^2(I\!R^n)} \\ &= (1+t)^s\|\chi_1 F\|_{\dot{H}^{s-\frac{3}{2}}(I\!R^n)} \\ &\leq C(1+t)^s\|\chi_1 F\|_{L^q(I\!R^n)}, \end{aligned} \tag{4.4.43}$$

Here, we obtained the last inequality by using both the continuity of the embedding

$$H^{\frac{3}{2}-s}(I\!R^n) \subset L^{q'}(I\!R^n)$$

in the Sobolev embedding theorem and the duality, and q' is defined by (4.4.41).

Combining (4.4.42)–(4.4.43), we get

$$\left\||x|^s\left(|D|^{s-\frac{3}{2}}(\chi_1 F)\right)\right\|_{L^2(I\!R^n)} \leq C(1+t)^s\|\chi_1 F\|_{L^q(I\!R^n)}, \tag{4.4.44}$$

then

$$\left\||x|^s\left(|D|^{s-\frac{3}{2}}(\chi_1 F)\right)\right\|_{L^2(0,T;L^2(I\!R^n))} \leq C\left\|(1+t)^s\chi_1 F\right\|_{L^2(0,T;L^q(I\!R^n))}. \tag{4.4.45}$$

Now we estimate the second term on the right-hand side of (4.4.38).

Noticing the definition of χ_2 and assumption (4.4.33) on F (where we assume without loss of generality that $\rho \geq 1$), similarly to (4.4.39), we have

$$\big(|D|^{s-\frac{3}{2}}(\chi_2 F)\big)(t,x) = C\int_{I\!R^n} \frac{\chi_2 F(t,y)}{|x-y|^{n+s-\frac{3}{2}}}dy = C\int_{t+\rho\geq|y|\geq\frac{1+t}{2}} \frac{\chi_2 F(t,y)}{|x-y|^{n+s-\frac{3}{2}}}dy. \tag{4.4.46}$$

Then, when $|x| \geq 2(t+\rho)$, similarly to (4.4.40), we have

$$\big|\big(|D|^{s-\frac{3}{2}}(\chi_2 F)\big)(t,x)\big| \leq C|x|^{-(n+s-\frac{3}{2})}\|\chi_2 F\|_{L^1(I\!R^n)}, \tag{4.4.47}$$

consequently, noting that $n \geq 4$, it is easy to show that

$$\begin{aligned}\big\||x|^s\big(|D|^{s-\frac{3}{2}}(\chi_2 F)\big)\big\|_{L^2(|x|\geq 2(t+\rho))} &\leq C\|\chi_2 F\|_{L^1(I\!R^n)}\||x|^{-(n-\frac{3}{2})}\|_{L^2(|x|\geq 2(t+\rho))}\\ &\leq C(1+t)^{-\frac{n-3}{2}}\|\chi_2 F\|_{L^1(I\!R^n)}\\ &\leq C(1+t)^{-\frac{n-3}{2}}\|\chi_2 F\|_{L^{1,2}(I\!R^n)}.\end{aligned} \tag{4.4.48}$$

On the other hand, when $|x| \leq 2(t+\rho)$, according to (3.3.12) in Chap. 3 and Theorem 3.3.2 in Chap. 3 (in which we take $f = \chi_2 F, \psi = \chi_2, a = \frac{1+t}{2}$ and $s_0 = \frac{3}{2} - s$) and noting (4.4.35), we have

$$\begin{aligned}\big\||x|^s\big(|D|^{s-\frac{3}{2}}(\chi_2 F)\big)\big\|_{L^2(|x|\leq 2(t+\rho))} &\leq C_\rho(1+t)^s\big\||D|^{s-\frac{3}{2}}(\chi_2 F)\big\|_{L^2(I\!R^n)}\\ &= C_\rho(1+t)^s\|\chi_2 F\|_{\dot{H}^{s-\frac{3}{2}}(I\!R^n)}\\ &\leq C_\rho(1+t)^{-\frac{n-3}{2}}\|\chi_2 F\|_{L^{1,2}(I\!R^n)}.\end{aligned} \tag{4.4.49}$$

Combining (4.4.48)–(4.4.49), we obtain

$$\big\||x|^s\big(|D|^{s-\frac{3}{2}}(\chi_2 F)\big)\big\|_{L^2(I\!R^n)} \leq C_\rho(1+t)^{-\frac{n-3}{2}}\|\chi_2 F\|_{L^{1,2}(I\!R^n)}, \tag{4.4.50}$$

thus

$$\big\||x|^s\big(|D|^{s-\frac{3}{2}}(\chi_2 F)\big)\big\|_{L^2(0,T;L^2(I\!R^n))} \leq C_q\|(1+t)^{-\frac{n-3}{2}}\chi_2 F\|_{L^2(0,T;L^{1,2}(I\!R^n))}. \tag{4.4.51}$$

Substituting (4.4.45) and (4.4.51) into (4.4.38), we obtain the desired (4.4.34). The proof of Theorem 4.4.1 is finished. □

4.5 $L^{p,q}$ Estimates on Solutions to the Linear Wave Equations

Consider the following Cauchy problem of the linear wave equations:

$$\Box u(t,x) = F(t,x), \quad (t,x) \in I\!R_+ \times I\!R^n, \tag{4.5.1}$$

$$t = 0 : u = f(x), \; u_t = g(x), \quad x \in I\!R^n. \tag{4.5.2}$$

In this section we will establish some new estimates on the solution by using the $L^{p,q}$ space introduced in Sect. 3.1.2 of Chap. 3.

First we prove

Lemma 4.5.1 *Suppose that $n \geq 1$ and $u = u(t, x)$ is the solution to the Cauchy problem (4.5.1)–(4.5.2). Then for any given real number s, we have*

$$\|u(t, \cdot)\|_{\dot{H}^s(I\!R^n)} \leq \|f\|_{\dot{H}^s(I\!R^n)} + \|g\|_{\dot{H}^{s-1}(I\!R^n)} + \int_0^t \|F(\tau, \cdot)\|_{\dot{H}^{s-1}(I\!R^n)} d\tau, \quad \forall\, t \geq 0; \tag{4.5.3}$$

moreover, for any given real number $\sigma (0 \leq \sigma \leq 1)$, we have

$$\|u(t, \cdot)\|_{L^2(I\!R^n)} \leq \|f\|_{L^2(I\!R^n)} + t^{\sigma}\|g\|_{\dot{H}^{\sigma-1}(I\!R^n)} + \int_0^t (t-\tau)^{\sigma}\|F(\tau, \cdot)\|_{\dot{H}^{\sigma-1}(I\!R^n)} d\tau, \quad \forall\, t \geq 0, \tag{4.5.4}$$

where $\dot{H}^s(I\!R^n)$ is a homogeneous Sobolev space with the norm (see (3.3.12) in Chap. 3)

$$\|f\|_{\dot{H}^s(I\!R^n)} = \||\xi|^s \widehat{f}(\xi)\|_{L^2(I\!R^n)}, \tag{4.5.5}$$

where $\widehat{f}(\xi)$ is the Fourier transform of $f(x)$.

Proof From Theorem 2.3.1 in Chap. 2, the Fourier transform of $u = u(t, x)$ with respect to x is given by

$$\widehat{u}(t, \xi) = \cos(|\xi|t)\widehat{f}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\widehat{g}(\xi) + \int_0^t \frac{\sin(|\xi|(t-\tau))}{|\xi|}\widehat{F}(\tau, \xi) d\tau. \tag{4.5.6}$$

Then it is easy to show that

$$|\widehat{u}(t, \xi)| \leq |\widehat{f}(\xi)| + |\xi|^{-1}|\widehat{g}(\xi)| + \int_0^t |\xi|^{-1}|\widehat{F}(\tau, \xi)| d\tau.$$

Multiplying both sides of the above formula by $|\xi|^s$, and noting (4.5.5), we immediately get (4.5.3).

Moreover, noticing that for any given σ satisfying $0 \leq \sigma \leq 1$, we have

$$|\sin(|\xi|t)| \leq |\sin(|\xi|t)|^{\sigma} \leq (|\xi|t)^{\sigma},$$

we obtain from (4.5.6) that

$$\widehat{u}(t, \xi) \leq |\widehat{f}(\xi)| + t^{\sigma}|\xi|^{\sigma-1}|\widehat{g}(\xi)| + \int_0^t (t-\tau)^{\sigma}|\xi|^{\sigma-1}|\widehat{F}(\tau, \xi)| d\tau,$$

from which (4.5.4) follows immediately. The proof is finished. □

Remark 4.5.1 We note that when $n \geq 3$, according to the Sobolev embedding theorem, the embedding

$$H^1(\mathbb{R}^n) \subset L^{q'}(\mathbb{R}^n)$$

is continuous, where $\frac{1}{q'} = \frac{1}{2} - \frac{1}{n}$. Then by duality the embedding

$$L^q(\mathbb{R}^n) \subset \dot{H}^{-1}(\mathbb{R}^n)$$

is continuous, where q satisfies

$$\frac{1}{q} = \frac{1}{2} + \frac{1}{n}. \tag{4.5.7}$$

From this, taking particularly $s = 0$ in (4.5.3) or taking particularly $\sigma = 0$ in (4.5.4), we obtain

$$\|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} + C\Big(\|g\|_{L^q(\mathbb{R}^n)} + \int_0^t \|F(\tau, \cdot)\|_{L^q(\mathbb{R}^n)} d\tau\Big), \quad \forall\, t \geq 0, \tag{4.5.8}$$

where q is defined by (4.5.7), and C is a positive constant. This is an inequality established for the L^2 norm of the solution $u(t, x)$ when $n \geq 3$, which is known as the Von Wahl inequality (see Wahl (1970)).

Lemma 4.5.2 *Under the assumption of Lemma 4.5.1, we have the following energy estimate:*

$$\|Du(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq \|D_x f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \int_0^t \|F(\tau, \cdot)\|_{L^2(\mathbb{R}^n)} d\tau, \ \forall\, t \geq 0, \tag{4.5.9}$$

where $D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ *and* $D_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$.

Proof From (4.5.6) we have

$$|\xi|\widehat{u}(t, \xi) = \cos(|\xi|t)|\xi|\widehat{f}(\xi) + \sin(|\xi|t)\widehat{g}(\xi) + \int_0^t \sin(|\xi|(t - \tau))\widehat{F}(\tau, \xi) d\tau$$

and

$$\frac{\partial \widehat{u}(t, \xi)}{\partial t} = -\sin(|\xi|t)|\xi|\widehat{f}(\xi) + \cos(|\xi|t)\widehat{g}(\xi) + \int_0^t \cos(|\xi|(t - \tau))\widehat{F}(\tau, \xi) d\tau,$$

from which the energy estimate (4.5.9) follows immediately. □

Next we will give more detailed L^2 estimates to the solution of the Cauchy problem (4.5.1)–(4.5.2).

Theorem 4.5.1 *Let $u = u(t, x)$ be the solution to the Cauchy problem (4.5.1)–(4.5.2). Then*

1° *When* $n \geq 3$*, we have*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} + C\Big\{\|g\|_{L^q(\mathbb{R}^n)} + \int_0^t (\|F(\tau,\cdot)\|_{q,\chi_1} + (1+\tau)^{-\frac{n-2}{2}}\|F(\tau,\cdot)\|_{1,2,\chi_2})d\tau\Big\}, \quad \forall\, t \geq 0, \tag{4.5.10}$$

where q *is defined by (4.5.7).*

2° *When* $n = 2$*, we have*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} + C\Big\{t^\sigma\|g\|_{L^q(\mathbb{R}^n)} + \int_0^t (t-\tau)^\sigma\big(\|F(\tau,\cdot)\|_{q,\chi_1} + (1+\tau)^{-\sigma}\|F(\tau,\cdot)\|_{1,2,\chi_2}\big)d\tau\Big\}, \quad \forall\, t \geq 0, \tag{4.5.11}$$

where $0 < \sigma < \frac{1}{2}$*, and* q *satisfies*

$$\frac{1}{q} = 1 - \frac{\sigma}{2}. \tag{4.5.12}$$

In (4.5.10)–(4.5.11), χ_1 *is the characteristic function of the set* $\{(t,x) \mid |x| \leq \frac{1+t}{2}\}$*,* $\chi_2 = 1 - \chi_1$*,* $\|F(\tau,\cdot)\|_{q,\chi_1} = \|\chi_1 F(\tau,\cdot)\|_{L^q(\mathbb{R}^n)}$*,* $\|F(\tau,\cdot)\|_{1,2,\chi_2} = \|\chi_2 F(\tau,\cdot)\|_{L^{1,2}(\mathbb{R}^n)}$*, and* C *is a positive constant independent of* f, g, F *and* t*.*

Proof First we prove (4.5.10) when $n \geq 3$.

Taking $s = 0$ in (4.5.3), similarly to (4.5.8), we get

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} + C\|g\|_{L^q(\mathbb{R}^n)} + \int_0^t \|F(\tau,\cdot)\|_{\dot{H}^{-1}(\mathbb{R}^n)}d\tau, \quad \forall\, t \geq 0. \tag{4.5.13}$$

Using Theorem 3.2 in Chap. 3 (in which we take $s_0 = 1$ and $a = \frac{1+\tau}{2}$), we have

$$\|F(\tau,\cdot)\|_{\dot{H}^{-1}(\mathbb{R}^n)} \leq C(\|F(\tau,\cdot)\|_{q,\chi_1} + (1+\tau)^{-\frac{n-2}{2}}\|F(\tau,\cdot)\|_{1,2,\chi_2}), \tag{4.5.14}$$

where q is defined by (4.5.7). Plugging (4.5.14) in (4.5.13), we get the desired (4.5.10).

Now we prove (4.5.11) when $n = 2$.

We notice that when $n = 2$, from the Sobolev embedding theorem, for any σ satisfying $0 < \sigma \leq 1$,

$$H^{1-\sigma}(\mathbb{R}^2) \subset L^{q'}(\mathbb{R}^2)$$

is a continuous embedding, where $\frac{1}{q'} = \frac{1}{2} - \frac{1-\sigma}{2} = \frac{\sigma}{2}$. Then, by duality

$$L^q(I\!R^2) \subset \dot{H}^{\sigma-1}(I\!R^2)$$

is a continuous embedding, where q is defined by (4.5.12). Thus, from (4.5.4) we get

$$\|u(t,\cdot)\|_{L^2(I\!R^2)} \le \|f\|_{L^2(I\!R^2)} + Ct^\sigma \|g\|_{L^q(I\!R^2)} + \int_0^t (t-\tau)^\sigma \|F(\tau,\cdot)\|_{\dot{H}^{\sigma-1}(I\!R^2)} d\tau, \quad \forall\, t \ge 0. \tag{4.5.15}$$

When $0 < \sigma < \frac{1}{2}$, using Theorem 3.3.2 in Chap. 3 (in which we take $s_0 = 1 - \sigma$ and $a = \frac{1+\tau}{2}$), we have

$$\|F(\tau,\cdot)\|_{\dot{H}^{\sigma-1}(I\!R^2)} \le C(\|F(\tau,\cdot)\|_{q,\chi_1} + (1+\tau)^{-\sigma}\|F(\tau,\cdot)\|_{1,2,\chi_2}), \tag{4.5.16}$$

where q is defined by (4.5.12). Substituting (4.5.16) into (4.5.15), we get (4.5.11). The proof is finished. □

Noting Lemma 3.1.5 and Corollary 3.1.1 in Chap. 3, from Theorem 4.5.1 we immediately obtain

Corollary 4.5.1 *Under the assumption of Theorem 4.5.1, for any given integer $N \ge 0$,*

$1°$ *when $n \ge 3$, we have*

$$\|u(t,\cdot)\|_{\Gamma,N,2} \le \|u(0,\cdot)\|_{\Gamma,N,2} + C\Big\{\|u_t(0,\cdot)\|_{\Gamma,N,q}$$
$$+ \int_0^t \Big(\|F(\tau,\cdot)\|_{\Gamma,N,q,\chi_1} + (1+\tau)^{-\frac{n-2}{2}}\|F(\tau,\cdot)\|_{\Gamma,N,1,2,\chi_2}\Big)\, d\tau\Big\}, \quad \forall t \ge 0, \tag{4.5.17}$$

where q is defined by (4.5.7).

$2°$ *when $n = 2$, we have*

$$\|u(t,\cdot)\|_{\Gamma,N,2} \le \|u(0,\cdot)\|_{\Gamma,N,2} + C\Big\{t^\sigma\|u_t(0,\cdot)\|_{\Gamma,N,q}$$
$$+ \int_0^t (t-\tau)^\sigma \big(\|F(\tau,\cdot)\|_{\Gamma,N,q,\chi_1} + (1+\tau)^{-\sigma}\|F(\tau,\cdot)\|_{\Gamma,N,1,2,\chi_2}\big)\, d\tau\Big\}, \quad \forall\, t \ge 0, \tag{4.5.18}$$

where $0 < \sigma < \frac{1}{2}$, and q is defined by (4.5.12).

In (4.5.17)–(4.5.18), $\|u(0,\cdot)\|_{\Gamma,N,2}$ represents the value of $\|u(t,\cdot)\|_{\Gamma,N,2}$ at $t = 0$, $\|u_t(0,\cdot)\|_{\Gamma,N,q}$ represents the value of $\|u_t(t,\cdot)\|_{\Gamma,N,q}$ at $t = 0$.

Similarly, from (4.5.8) we get

Corollary 4.5.2 *When $n \ge 3$, suppose that $u = u(t,x)$ is the solution to the Cauchy problem (4.5.1)–(4.5.2), then for any given integer $N \ge 0$, we have*

$$\|u(t,\cdot)\|_{\Gamma,N,2} \le C\Big(\|u(0,\cdot)\|_{\Gamma,N,2} + \|u_t(0,\cdot)\|_{\Gamma,N,q} + \int_0^t \|F(\tau,\cdot)\|_{\Gamma,N,q} d\tau\Big), \quad \forall t \ge 0, \tag{4.5.19}$$

where q satisfies (4.5.7).

Theorem 4.5.2 *Let $n \ge 2$. Under the assumptions of Theorem 4.5.1, for any given $p > 2$ we have*

$$\|u(t,\cdot)\|_{p,2,\chi_2} \le C(1+t)^{-(n-1)(\frac{1}{2}-\frac{1}{p})} \cdot \Big\{\|f\|_{\dot{H}^s(I\!\!R^n)} + \|g\|_{L^\gamma(I\!\!R^n)} + \int_0^t (\|F(\tau,\cdot)\|_{\gamma,\chi_1} + (1+\tau)^{-(\frac{n-2}{2}+s)}\|F(\tau,\cdot)\|_{1,2,\chi_2}) d\tau\Big\}, \quad \forall\, t \ge 0, \tag{4.5.20}$$

where

$$s = \frac{1}{2} - \frac{1}{p}, \ \frac{1}{\gamma} = \frac{1}{2} + \frac{1-s}{n}, \tag{4.5.21}$$

and C is a positive constant independent of f, g, F and t.

Proof From (3.3.11) in Theorem 3.3.1 of Chap. 3, in which we take $a = \frac{1+t}{2}$, we have

$$\|u(t,\cdot)\|_{p,2,\chi_2} \le C(1+t)^{-(n-1)(\frac{1}{2}-\frac{1}{p})}\|u(t,\cdot)\|_{\dot{H}^s(I\!\!R^n)}, \tag{4.5.22}$$

where s is given by the first formula of (4.5.21), so $0 < s < \frac{1}{2}$. Thus, from (4.5.3) we obtain

$$\|u(t,\cdot)\|_{p,2,\chi_2} \le C(1+t)^{-(n-1)(\frac{1}{2}-\frac{1}{p})}\Big\{\|f\|_{\dot{H}^s(I\!\!R^n)} + \|g\|_{\dot{H}^{s-1}(I\!\!R^n)} + \int_0^t \|F(\tau,\cdot)\|_{\dot{H}^{s-1}(I\!\!R^n)} d\tau\Big\}. \tag{4.5.23}$$

From the Sobolev embedding theorem we know that

$$H^{1-s}(I\!\!R^n) \subset L^{\gamma'}(I\!\!R^n)$$

is a continuous embedding, where $\frac{1}{\gamma'} = \frac{1}{2} - \frac{1-s}{n}$, then by duality

$$L^\gamma(I\!\!R^n) \subset \dot{H}^{s-1}(I\!\!R^n)$$

is a continuous embedding, where γ is defined by the second formula of (4.5.21). Then

$$\|g\|_{\dot{H}^{s-1}(I\!\!R^n)} \le C\|g\|_{L^\gamma(I\!\!R^n)}. \tag{4.5.24}$$

Furthermore, by Theorem 3.3.2 in Chap. 3, in which we take $s_0 = 1 - s$ and $a = \frac{1+\tau}{2}$, we have

$$\|F(\tau,\cdot)\|_{\dot{H}^{s-1}(I\!R^n)} \le C(\|F(\tau,\cdot)\|_{\gamma,\chi_1} + (1+\tau)^{-(\frac{n-2}{2}+s)}\|F(\tau,\cdot)\|_{1,2,\chi_2}), \tag{4.5.25}$$

where γ is defined by the second formula of (4.5.21). Plugging (4.5.24)–(4.5.25) in (4.5.23), we get the desired (4.5.20). □

Using Lemma 3.1.5 and Corollary 3.1.1 in Chap. 3, from Theorem 4.5.2 we immediately have

Corollary 4.5.3 *Under the assumptions of Theorem 4.5.2, for any given integer $N \ge 0$, we have*

$$\|u(t,\cdot)\|_{\Gamma,N,p,2,\chi_2} \le C(1+t)^{-(n-1)(\frac{1}{2}-\frac{1}{p})}\Big\{\sum_{|k|\le N}\|\Gamma^k u(0,\cdot)\|_{\dot{H}^s(I\!R^n)} + \|u_t(0,\cdot)\|_{\Gamma,N,\gamma}$$
$$+\int_0^t (\|F(\tau,\cdot)\|_{\Gamma,N,\gamma,\chi_1} + (1+\tau)^{-(\frac{n-2}{2}+s)}\|F(\tau,\cdot)\|_{\Gamma,N,1,2,\chi_2})d\tau\Big\}, \quad \forall t \ge 0, \tag{4.5.26}$$

where $p > 2$, s and γ are defined by (4.5.21), $\|\Gamma^k u(0,\cdot)\|_{\dot{H}^s(I\!R^n)}$ stands for the value of $\|\Gamma^k u(t,\cdot)\|_{\dot{H}^s(I\!R^n)}$ at $t = 0$, and C is a positive constant.

Remark 4.5.2 Estimate (4.5.10) in Theorem 4.5.1 was established by Li and Yu (1991).

4.6 L^1–L^∞ Estimates on Solutions to the Linear Wave Equations

This section aims to estimate the L^∞ norm of solutions to the Cauchy problem of the linear wave equations by using the L^1 norm of the initial data and their partial derivatives of several orders. Such estimates are called the L^1–L^∞ estimates.

4.6.1 L^1–L^∞ Estimates on Solutions to the Homogeneous Linear Wave Equation

Theorem 4.6.1 *Let $n \ge 2$. If $u = u(t,x)$ is the solution to the Cauchy problem*

$$\begin{cases} \Box u(t,x) = 0, \quad (t,x) \in I\!R^+ \times I\!R^n, & (4.6.1)\\ t = 0:\ u = f(x),\ u_t = g(x), \quad x \in I\!R^n, & (4.6.2)\end{cases}$$

then we have

$$|u(t,x)| \le C(1+t)^{-\frac{n-1}{2}}(\|f\|_{W^{n,1}(I\!R^n)} + \|g\|_{W^{n-1,1}(I\!R^n)}), \quad \forall (t,x) \in I\!R^+ \times I\!R^n \tag{4.6.3}$$

or

$$\|u(t,\cdot)\|_{L^\infty(I\!R^n)} \le C(1+t)^{-\frac{n-1}{2}}(\|f\|_{W^{n,1}(I\!R^n)} + \|g\|_{W^{n-1,1}(I\!R^n)}), \quad \forall t \ge 0, \quad (4.6.3')$$

where C is a positive constant independent of (f,g) and t.

Proof From the expressions (2.1.14), (2.2.3) and (2.2.11) of solutions given in Chap. 2, noticing also (2.2.5) in Chap. 2, it is easy to show that

$$\begin{aligned} u(t,x) &= C_n\left\{\frac{d}{dt}\int_{I\!R^n}\chi_+^{-\frac{n-1}{2}}(t^2-|x-y|^2)f(y)dy + \int_{I\!R^n}\chi_+^{-\frac{n-1}{2}}(t^2-|x-y|^2)g(y)dy\right\} \\ &= C_n\left\{2t\int_{I\!R^n}\chi_+^{-\frac{n+1}{2}}(t^2-|x-y|^2)f(y)dy + \int_{I\!R^n}\chi_+^{-\frac{n-1}{2}}(t^2-|x-y|^2)g(y)dy\right\} \\ &= C_n\left\{2t\int_{I\!R^n}\chi_+^{-\frac{n+1}{2}}(t^2-|y|^2)f(x-y)dy + \int_{I\!R^n}\chi_+^{-\frac{n-1}{2}}(t^2-|y|^2)g(x-y)dy\right\}, \end{aligned} \quad (4.6.4)$$

where $C_n = \dfrac{1}{2\pi^{\frac{n-1}{2}}}$, $\chi_+^a(y)$ is defined by (2.2.4) and (2.2.5) in Chap. 2, and the integral here stands for the convolution in the sense of distributions.

First, we estimate

$$I = \int_{I\!R^n}\chi_+^{-\frac{n-1}{2}}(t^2-|y|^2)g(x-y)dy. \quad (4.6.5)$$

Set $y = r\xi$, where $r = |y|$ and $\xi \in S^{n-1}$, we have

$$I = \int_{I\!R^n}\chi_+^{-\frac{n-1}{2}}(t^2-r^2)g(x-r\xi)r^{n-1}drd\omega_\xi, \quad (4.6.6)$$

where $d\omega_\xi$ stands for the area element of S^{n-1}.

From (2.2.5) in Chap. 2, for any given integer $m \ge 0$, it is easy to show that

$$\left(\frac{1}{2r}\partial_r\right)^m\chi_+^{-\frac{n-1}{2}+m}(t^2-r^2) = (-1)^m\chi_+^{-\frac{n-1}{2}}(t^2-r^2), \quad (4.6.7)$$

then

$$I = (-1)^m\int_{I\!R^n}\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^m\chi_+^{-\frac{n-1}{2}+m}(t^2-r^2)\cdot g(x-r\xi)r^{n-1}drd\omega_\xi. \quad (4.6.8)$$

Noticing (4.2.5) and that the support of $\chi_+^a(y) \subseteq \{y \ge 0\}$, it is easy to verify that: if we particularly take

$$m = \begin{cases} \frac{n-3}{2}, & \text{if } n(\ge 3) \text{ is odd;} \\ \frac{n-2}{2}, & \text{if } n(\ge 2) \text{ is even,} \end{cases} \quad (4.6.9)$$

then, for any integer a satisfying $1 \le a \le m$, we always have

$$\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^{m-a} \chi_+^{-\frac{n-1}{2}+m}(t^2-r^2)\cdot\frac{1}{2r}\left(\partial_r\frac{1}{2r}\right)^{a-1}(g(x-r\xi)r^{n-1})\Big|_{r=0}^{r=+\infty}=0. \tag{4.6.10}$$

Hence, integrating m times by parts with respect to r on the right-hand side of (4.6.8), we obtain

$$I=\int_{\mathbb{R}^n}\chi_+^{-\frac{n-1}{2}+m}(t^2-r^2)\left(\partial_r\frac{1}{2r}\right)^m(g(x-r\xi)r^{n-1})drd\omega_\xi, \tag{4.6.11}$$

where m is determined by (4.6.9).

From (4.6.9) we have

$$-\frac{n-1}{2}+m=\begin{cases}-1, & \text{if } n(\ge 3) \text{ is odd;}\\ -\frac{1}{2}, & \text{if } n(\ge 2) \text{ is even,}\end{cases} \tag{4.6.12}$$

then, from (2.2.9) and (2.2.10) in Chap. 2, $\chi_+^{-\frac{n-1}{2}+m}$ is a positive measure. Hence, from (4.6.11) it is easy to show that

$$\begin{aligned}|I| &\le C\sum_{l\le m}\int_{\mathbb{R}^n}\chi_+^{-\frac{n-1}{2}+m}(t^2-r^2)|\partial_r^l g(x-r\xi)|r^{n-1-2m+l}drd\omega_\xi\\ &\le C\sum_{|k|\le m}\int_{\mathbb{R}^n}\chi_+^{-\frac{n-1}{2}+m}(t^2-|y|^2)\big|D_x^k g(x-y)\big||y|^{-2m+|k|}dy,\end{aligned} \tag{4.6.13}$$

here and hereafter, C and C_k etc. represent some positive constants, and $D_x = (\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n})$.

Now for any given $k(|k|\le m)$, we consider the integral

$$I_k=\int_{\mathbb{R}}\chi_+^{-\frac{n-1}{2}+m}(t^2-|y|^2)\big|D_x^k g(x-y)\big||y|^{-2m+|k|}dy. \tag{4.6.14}$$

Let

$$y=(y',y''), \tag{4.6.15}$$

where

$$y'=(y_1,\cdots,y_{|k|+1}),\ \ y''=(y_{|k|+2},\cdots,y_n). \tag{4.6.16}$$

Noting (4.6.9), when $|k|\le m$, we always have $|k|+1<n$, so y' and y'' are both nonempty. Since $|y''|\le|y|$, from (4.6.14) we get

$$
\begin{aligned}
I_k &\le \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}+m}(t^2-|y'|^2-|y''|^2)|y''|^{-2m+|k|}\left|D_x^k g(x-y)\right|dy \\
&\le \int\left(\int \chi_+^{-\frac{n-1}{2}+m}(t^2-|y'|^2-|y''|^2)|y''|^{-2m+|k|}dy''\right)\cdot\sup_{y''}\left|D_x^k g(x-y)\right|dy'.
\end{aligned}
\tag{4.6.17}
$$

□

Lemma 4.6.1 *For any given k ($|k|\le m$), m being given by (4.6.9), and for any given real number R, we have*

$$
\int \chi_+^{-\frac{n-1}{2}+m}(R-|y''|^2)|y''|^{-2m+|k|}dy'' \le C_{|k|}, \tag{4.6.18}
$$

where $C_{|k|}$ is a positive constant independent of R but possibly depending on $|k|$.

Proof When $R\le 0$, from the property of the support of $\chi_+^a(y)$ we know that the integral on the left-hand side of (4.6.18) is always zero, the conclusion in the lemma is obvious.

When $R>0$, setting $y''=\sqrt{R}z''$ and noticing that $\chi_+^a(y)$ is a homogeneous function of degree a with respect to y, it is clear that

$$
\int \chi_+^{-\frac{n-1}{2}+m}(R^2-|y''|^2)|y''|^{-2m+|k|}dy'' = \int \chi_+^{-\frac{n-1}{2}+m}(1-|z''|^2)|z''|^{-2m+|k|}dz'', \tag{4.6.19}
$$

whose value is independent of R. Therefore, to prove Lemma 4.6.1, it suffices to prove that the integral on the right-hand side of the above formula is finite.

When $n(\ge 3)$ is odd, from (2.6.12) and (2.2.9) in Chap. 2, the integral on the right-hand side of (4.6.19) is reduced to

$$
\begin{aligned}
\int \chi_+^{-1}(1-|z''|^2)|z''|^{-2m+|k|}dz'' &= \int \delta(1-|z''|^2)|z''|^{-2m+|k|}dz'' \\
&= \int \delta(2(1-|z''|))|z''|^{-2m+|k|}dz'' \\
&= \frac{1}{2}\int \delta(1-|z''|)|z''|^{-2m+|k|}dz'' < +\infty;
\end{aligned}
$$

while, when $n(\ge 2)$ is even, from (2.6.12) and (2.2.10) in Chap. 2, the integral on the right-hand side of (4.6.19) is reduced to

$$
\begin{aligned}
\int \chi_+^{-\frac{1}{2}}(1-|z''|^2)|z''|^{-2m+|k|}dz'' &= \int_{|z''|\le 1}\frac{|z''|^{-2m+|k|}}{\sqrt{\pi(1-|z''|^2)}}dz'' = C\int_0^1 \frac{dr}{\sqrt{1-r^2}} \\
&= \frac{\pi}{2}C < +\infty,
\end{aligned}
$$

where C is a positive constant depending on $|k|$.

This proves Lemma 4.6.1. □

By Lemma 4.6.1, from (4.6.19)) we can obtain

$$I_k \leq C_k \int \sup_{y''} \left| D_x^k g(x-y) \right| dy'. \tag{4.6.20}$$

Noting the following identity:

$$h(y', y'') = (-1)^{n-|k|-1} \int_{y_{|k|+2}}^{\infty} \cdots \int_{y_n}^{\infty} \partial_{|k|+2} \cdots \partial_n h(y', z'') dz'',$$

and taking $h(y', y'') = D_x^k g(x-y)$ in it, from (4.6.20) we immediately have

$$I_k \leq C_k \|g\|_{W^{n-1,1}(\mathbb{R}^n)}, \tag{4.6.21}$$

then from (4.6.13) we get

$$|I| \leq C \|g\|_{W^{n-1,1}(\mathbb{R}^n)}. \tag{4.6.22}$$

Now we prove that: when $t \geq 2$ we have

$$|I| \leq C t^{-\frac{n-1}{2}} \|g\|_{W^{n-1,1}(\mathbb{R}^n)}, \tag{4.6.23}$$

where C is a positive constant independent of t.

We divide the proof into three steps.

(i) For any given $t \geq 2$, if it is satisfied on the support of g that

$$t - r \geq \frac{1}{2}, \tag{4.6.24}$$

where $r = |y|$, then

$$t^2 - r^2 \geq \frac{t}{2} > 0, \tag{4.6.25}$$

thus from (4.6.5) and noting that $\chi_+^a(y)$ is a homogeneous function of degree a with respect to y, we have

$$I = \int_{\mathbb{R}^n} \chi_+^{-\frac{n-1}{2}} (t^2 - r^2) g(x-y) dy = \chi_+^{-\frac{n-1}{2}}(1) \int_{\mathbb{R}^n} (t^2 - r^2)^{-\frac{n-1}{2}} g(x-y) dy.$$

Therefore, noting (4.6.25), we have

$$|I| \leq C t^{-\frac{n-1}{2}} \|g\|_{L^1(\mathbb{R}^n)}. \tag{4.6.26}$$

(ii) For any given $t \geq 2$, if it is satisfied on the support of g that

$$t - r \leq 1, \tag{4.6.27}$$

where $r = |y|$, then

$$r \geq t - 1 \geq \frac{t}{2} \geq 1, \tag{4.6.28}$$

thus from (4.6.13) (in which m is replaced by $m + 1$, and it is obvious that the inequality still holds), we obtain

$$\begin{aligned} |I| &\leq C \sum_{|k| \leq m+1} \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}+m+1}(t^2 - r^2) \big| D_x^k g(x-y) \big| r^{-2m-2+|k|} dy \\ &\leq C \sum_{|k| \leq m+1} \left(\frac{t}{2}\right)^{-2m-2+|k|} \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}+m+1}(t^2 - r^2) \big| D_x^k g(x-y) \big| dy \\ &\leq C t^{-m-1} \sum_{|k| \leq m+1} \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}+m+1}(t^2 - r^2) \big| D_x^k g(x-y) \big| dy. \end{aligned} \tag{4.6.29}$$

When $n(\geq 3)$ is odd, from (4.6.12) and using (2.2.7) in Chap. 2, we have

$$\chi_+^{-\frac{n-1}{2}+m+1}(t^2 - r^2) = H(t^2 - r^2),$$

here H is the Heaviside function, thus from (4.6.29) and noting (4.6.9), we immediately get

$$|I| \leq C t^{-\frac{n-1}{2}} \|g\|_{W^{\frac{n-1}{2},1}(I\!R^n)}. \tag{4.6.30}$$

When $n(\geq 2)$ is even, from (4.6.12), using (2.2.4) in Chap. 2, and noticing (4.6.27), it is easy to show that

$$\chi_+^{-\frac{n-1}{2}+m+1}(t^2 - r^2) = C_0\sqrt{t^2 - r^2} \leq C_0\sqrt{t + r} \leq C_0\sqrt{2t},$$

so C_0 is a positive constant, thus from (4.6.29) and noting (4.6.9), we immediately get

$$|I| \leq C t^{-\frac{n-1}{2}} \|g\|_{W^{\frac{n}{2},1}(I\!R^n)}. \tag{4.6.31}$$

(iii) Combining estimates (4.6.30)–(4.6.31) and using the partition of unity, it is easy to yield (4.6.23). Then, noticing (4.6.22), we obtain

$$|I| \leq C(1 + t)^{-\frac{n-1}{2}} \|g\|_{W^{n-1,1}(I\!R^n)}. \tag{4.6.32}$$

Similarly, we can prove

$$\left| \int_{I\!R^n} \chi_+^{-\frac{n+1}{2}} (t^2 - |y|^2) f(x-y) dy \right| \leq C(1+t)^{-\frac{n+1}{2}} \|f\|_{W^{n,1}(I\!R^n)}. \tag{4.6.33}$$

Hence, the desired (4.6.3) and (6.3)' follow from (4.6.4).
The proof of Theorem 4.6.1 is finished. □

Corollary 4.6.1 *Under the assumptions of Theorem 4.6.1, assume furthermore that* (f, g) *has the following compact support:*

$$supp\{f, g\} \subseteq \{x \mid |x| \leq \rho\}, \tag{4.6.34}$$

where ρ *is a positive number, then we have*

$$|u(t,x)| \leq C_\rho (1+t+|x|)^{-\frac{n-1}{2}} \left(1 + \left|t - |x|\right|\right)^{-l} (\|f\|_{W^{n,1}(I\!R^n)} + \|g\|_{W^{n-1,1}(I\!R^n)}), \tag{4.6.35}$$

where C_ρ *is a positive constant depending only on* ρ*. Moreover, when* $n(\geq 3)$ *is odd,* $l \geq 0$*; while, when* $n(\geq 2)$ *is even,* $0 \leq l \leq \frac{n-1}{2}$.

Proof Due to the finite speed of propagation of the wave, on the support of the solution $u = u(t,x)$, $t - |x| \geq -\rho$.

When

$$-\rho \leq t - |x| \leq 2\rho,$$

it is easy to show that for any given $l \geq 0$, (4.6.35) can be deduced from Theorem 4.6.1. When

$$t - |x| \geq 2\rho,$$

if $n(\geq 3)$ is odd, then from the Huygens principle we know that $u(t,x) \equiv 0$(see (4.2.4)–(4.2.5)); while, if $n(\geq 2)$ is even, and $0 \leq l \leq \frac{n-1}{2}$, then (4.6.35) can also be derived from Corollary 4.2.1. □

4.6.2 L^1–L^∞ Estimates on Solutions to the Inhomogeneous Linear Wave Equations

Lemma 4.6.2 (J.-L.Lions extension) *Suppose that* $\Omega \subset I\!R^n$ *is a domain with a* C^m *boundary,* $m \geq 0$ *is an integer, and* $1 \leq p \leq +\infty$*. Then there exists an extension operator* P *from* $W^{m,p}(\Omega)$ *to* $W^{m,p}(I\!R^n)$*, such that for any given* $u \in W^{m,p}(\Omega)$*,*

$$Pu \in W^{m,p}(I\!R^n), \tag{4.6.36}$$

and

$$\|Pu\|_{W^{m,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{m,p}(\Omega)}, \tag{4.6.37}$$

where C ia a positive constant independent of u.

Proof See Lions (1980) for reference. □

Lemma 4.6.3 *Suppose that B_t is the ball in $\mathbb{R}^n$ centered at the origin with radius t. Then for any given integer $m \geq 0$, there exists an extension operator P_t^m such that for any given function $f \in W^{m,1}(B_t)$, $P_t^m f$ is a function defined on the entire space $\mathbb{R}^n$ and satisfies*

$$\|D^\alpha(P_t^m f)\|_{L^1(\mathbb{R}^n)} \leq C \sum_{|\beta|\leq|\alpha|} t^{|\beta|-|\alpha|}\|D^\beta f\|_{L^1(B_t)}, \quad \forall|\alpha| \leq m, \tag{4.6.38}$$

where C is a positive constant independent of f.

Proof When $t = 1$, take P_1^m as the Lions extension operator mentioned in the above lemma, (4.6.38) is then an immediate consequence of (4.6.37)(in which we take $p = 1$). In general, set $\bar{x} = \frac{x}{t}$, $\bar{f}(\bar{x}) = f(\frac{x}{t})$, and take $P_t^m f(x) = P_1^m \bar{f}(\bar{x}) = P_1^m f(\frac{x}{t})$, then from

$$\|D_{\bar{x}}^\alpha(P_1^m \bar{f})\|_{L^1(\mathbb{R}^n)} \leq C \sum_{|\beta|\leq|\alpha|} \|D_{\bar{x}}^\beta \bar{f}\|_{L^1(B_1)}, \quad \forall|\alpha| \leq m$$

satisfied at $t = 1$, (4.6.38) is obtained immediately by scaling. □

Corollary 4.6.2 *Under the assumptions of Lemma 4.6.3, for any given $t \geq 1$, we have*

$$\|P_t^m f\|_{W^{m,1}(\mathbb{R}^n)} \leq C_m\|f\|_{W^{m,1}(B_t)}, \tag{4.6.39}$$

where C_m ia a positive constant independent of both f and $t \geq 1$.

Lemma 4.6.4 *Suppose that $n \geq 2$ and $u = u(t, x)$ is the solution to the Cauchy problem*

$$\begin{cases} \Box u(t, x) = F(x)\delta(t - |x|), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, & (4.6.40) \\ t = 0 : \ u = 0, \ u_t = 0. & (4.6.41) \end{cases}$$

If

$$supp\ F \subseteq \{x | 1 \leq |x| \leq 2\}, \tag{4.6.42}$$

then we have

$$|u(t,x)| \le C(1+t+|x|)^{-\frac{n-1}{2}}\left(1+\left|t-|x|\right|\right)^{-l}\|F\|_{W^{n-1,1}(I\!R^n)}, \tag{4.6.43}$$

where C *is a positive constant. Moreover, when* $n(\ge 3)$ *is odd,* $l \ge 0$*; while, when* $n(\ge 2)$ *is even,* $0 \le l \le \frac{n-1}{2}$.

Proof From the special form of the right-hand side of the wave equation (4.6.40) and noting (4.6.42), it is easy to show that $t \ge 1$ and $t - |x| \ge 0$ on the support of the solution $u = u(t,x)$.

(i) First we prove that: when $t - |x| \ge 6$, if $n(\ge 3)$ is odd, then we have

$$u(t,x) \equiv 0; \tag{4.6.44}$$

while, if $n(\ge 2)$ is even, then we have

$$|u(t,x)| \le C(t^2 - |x|^2)^{-\frac{n-1}{2}}\|F\|_{L^1(I\!R^n)}. \tag{4.6.45}$$

Thus, it is clear that (4.6.43) holds in this case.

By Duhamel principle, the solution to the Cauchy problem (4.6.40)–(4.6.41) can be written as

$$u = u(t,x) = \int_0^t v(t,x;\tau)d\tau, \tag{4.6.46}$$

where $v = v(t,x;\tau)$ is the solution to the following Cauchy problem:

$$\begin{cases} \Box v(t,x;\tau) = 0, & (4.6.47)\\ t = \tau:\ v = 0,\ v_t = F(x)\delta(\tau - |x|). & (4.6.48)\end{cases}$$

Due to (4.6.42), $v = v(t,x;\tau)$ is identically equal to zero except when $1 \le \tau \le 2$, then, when $t \ge 2$, (4.6.46) can be written as

$$u = u(t,x) = \int_1^2 v(t,x;\tau)d\tau. \tag{4.6.49}$$

When $t - |x| \ge 6$, if $1 \le \tau \le 2$, then $(t-\tau) - |x| \ge 4$. Thus, due to (4.2.4)–(4.2.5), if $n(\ge 3)$ is odd, then $v(t,x;\tau) \equiv 0$, and then (4.6.49) implies (4.6.44). Moreover, according to Theorem 4.2.1, if $n(\ge 2)$ is even, it is easy to show that

$$\begin{aligned}|v(t,x;\tau)| &\le C((t-\tau)^2 - |x|^2)^{-\frac{n-1}{2}}\|F(\cdot)\delta(\tau - |\cdot|)\|_{L^1(I\!R^n)}\\ &\le C(t^2 - |x|^2)^{-\frac{n-1}{2}}\|F(\cdot)\delta(\tau - |\cdot|)\|_{L^1(I\!R^n)},\end{aligned}$$

then (4.6.45) can be obtained easily from (4.6.49).

(ii) When $0 \le t - |x| \le 6$, since $t \ge 1$, it is easy to show that (4.6.43) that we want to prove is equivalent to

$$|u(t,x)| \le Ct^{-\frac{n-1}{2}} \|F\|_{W^{n-1,1}(I\!R^n)}. \tag{4.6.50}$$

According to Theorem 2.2.1 in Chap. 2, using Duhamel principle, and noticing that χ_+^a is a homogeneous function of degree a, we have

$$\begin{aligned}
u(t,x) &= \int_0^t \int_{I\!R^n} E(t-\tau, x-y)F(y)\delta(\tau - |y|)dyd\tau \\
&= C_n \int_0^t \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}}((t-\tau)^2 - |x-y|^2)F(y)\delta(\tau-|y|)dyd\tau \\
&= C_n \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}}((t-|y|)^2 - |x-y|^2)F(y)dy \\
&= C_n \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}}(t^2 - |x|^2 + 2\langle x, y\rangle - 2t|y|)F(y)dy \\
&= C_n (2t)^{-\frac{n-1}{2}} \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}}\left(b + a\frac{\langle x, y\rangle}{|x|} - |y|\right)F(y)dy \\
&\overset{\text{def.}}{=} C_n (2t)^{-\frac{n-1}{2}} I,
\end{aligned} \tag{4.6.51}$$

where E is the fundamental solution, $C_n = \dfrac{1}{2\pi^{\frac{n-1}{2}}}$,

$$I = \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}}\left(b + a\frac{\langle x, y\rangle}{|x|} - |y|\right)F(y)dy, \tag{4.6.52}$$

$\langle x, y\rangle$ is the inner product between x and y, and

$$a = \frac{|x|}{t},\ b = \frac{1}{2}(t - |x|)\left(1 + \frac{|x|}{t}\right) \tag{4.6.53}$$

are both independent of y.

When $0 \le t - |x| \le 6$, it is obvious that

$$0 \le a \le 1, \quad 0 \le b \le 6. \tag{4.6.54}$$

Thus, to prove (4.6.50), it suffices to prove that: for I given by (4.6.52) we have

$$|I| \le C\|F\|_{W^{n-1,1}(I\!R^n)}, \tag{4.6.55}$$

where C is a positive constant independent of both a and b.

Thanks to the rotational symmetry, we can assume without loss of generality that

$$x = (|x|, \overbrace{0, \cdots, 0}^{n-1}).$$

Then

$$\frac{\langle x, y\rangle}{|x|} = y_1,$$

hence (4.6.52) can be written as

$$I = \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}}(b + ay_1 - |y|)F(y)dy. \tag{4.6.56}$$

Set

$$y = (y_1, y')$$

and

$$y' = q\widetilde{\omega},$$

where $q = |y'|$ and $\widetilde{\omega} \in S^{n-2}$. From (2.2.5) in Chap. 2, it is easy to get

$$\left(-\frac{|y|}{q}\partial_q\right)^m \chi_+^{-\frac{n-1}{2}+m}(b + ay_1 - |y|) = \chi_+^{-\frac{n-1}{2}}(b + ay_1 - |y|), \tag{4.6.57}$$

where m is given by (4.6.9). Similarly to the derivation of (4.6.11), by integration by parts, (4.6.56) can be written as

$$I = \int dy_1 \int_{I\!R^{n-1}} \chi_+^{-\frac{n-1}{2}+m}(b + ay_1 - |y|) \cdot \left(\partial_q \frac{|y|}{q}\right)^m (F(y_1, y')q^{n-2})dqd\widetilde{\omega}. \tag{4.6.58}$$

We note that $\chi_+^{-\frac{n-1}{2}+m}$ is a positive measure due to (4.6.12), and $q \le 2$ holds on the support of F due to (4.6.42), then, noticing that χ_+^a is a homogeneous function of degree a, we obtain

$$\begin{aligned}
|I| &\le C \sum_{0\le l\le m} \int dy_1 \int_{I\!R^{n-1}} \chi^{-\frac{n-1}{2}+m}(b + ay_1 - |y|)|\partial_q^l F|q^{n-2-2m+l}dqd\widetilde{\omega} \\
&\le C \sum_{0\le l\le m} \int dy_1 \int_{I\!R^{n-1}} \chi^{-\frac{n-1}{2}+m}(b + ay_1 - |y|)|D^l F|q^{n-2-2m+l}dqd\widetilde{\omega} \\
&= C \sum_{0\le l\le m} \int_{I\!R^n} (b + ay_1 + |y|)^{\frac{n-1}{2}-m}\chi_+^{-\frac{n-1}{2}+m}((b + ay_1)^2 - |y|^2)|D^l F||y'|^{-2m+l}dy \\
&\le C \sum_{0\le l\le m} \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}+m}((b + ay_1)^2 - |y|^2)|D^l F||y'|^{-2m+l}dy,
\end{aligned} \tag{4.6.59}$$

where (4.6.12) and (4.6.42) have been used to obtain the last inequality.

Now for any given l $(0 \le l \le m)$, we consider the integral

$$I_l = \int_{I\!R^n} \chi_+^{-\frac{n-1}{2}+m}((b+ay_1)^2 - |y|^2)\left|D^l F\right||y'|^{-2m+l}dy. \tag{4.6.60}$$

Set

$$\widetilde{y}' = (y_1, \ldots, y_{l+1}),\ \ \widetilde{y}'' = (y_{l+2}, \ldots, y_n), \tag{4.6.61}$$

and denote

$$(b+ay_1)^2 - |y|^2 = R(\widetilde{y}') - |\widetilde{y}''|^2, \tag{4.6.62}$$

where

$$R(\widetilde{y}') = (b+ay_1)^2 - |\widetilde{y}'|^2. \tag{4.6.63}$$

Noting that $|\widetilde{y}''| \le |y'|$, from (4.6.60) we have

$$I_l \le \int \left(\int \chi_+^{-\frac{n-1}{2}+m}(R - |\widetilde{y}''|^2)|\widetilde{y}''|^{-2m+l}d\widetilde{y}''\right) \sup_{\widetilde{y}''} |D^l F| d\widetilde{y}', \tag{4.6.64}$$

then similarly to the proof of (4.6.21), we can obtain

$$I_l \le C_l \|F\|_{W^{n-1,1}(I\!R^n)}. \tag{4.6.65}$$

Thus, (4.6.55) follows from (4.6.59)–(4.6.60), and then (4.6.50), namely, (4.6.43) holds.

Combining (i) and (ii), Lemma 4.6.4 is proved. □

Remark 4.6.1 If assumption (4.6.42) is changed to

$$\text{supp } F \subseteq \left\{x \mid r_1 \le |x| \le r_2\right\}, \tag{4.6.66}$$

where r_1 and r_2 are positive constants satisfying $r_1 < r_2$, then the conclusions of Lemma 4.6.4 still hold.

Lemma 4.6.5 *Under the assumptions of Lemma 4.6.4 we have*

$$|u(t,x)| \le C\big(1+t+|x|\big)^{-\frac{n-1}{2}}\big(1+\big|t-|x|\big|\big)^{-l}\|F\|_{W^{n-1,1}(B_t)}, \quad \forall t \ge 0, \tag{4.6.67}$$

where C is a positive constant independent of both F and t, B_t is the ball in $I\!R^n$ centered at the origin with radius t. Moreover, when $n(\ge 3)$ is odd, $l \ge 0$; while, when $n(\ge 2)$ is even, $0 \le l \le \frac{n-1}{2}$.

Proof Noting that $t \geq 1$ on the support of the solution $u = u(t,x)$, we need only to prove (4.6.67) when $t \geq 1$.

Assume that $F^{(t)}$ is the restriction of F on B_t, and let $G = P_t^{n-1} F^{(t)}$ be the Lions extension operator given in Corollary 4.6.2. Suppose that $v = v(\tau, x)$ is the solution of the Cauchy problem

$$\begin{cases} \Box v(\tau,x) = G(x)\delta(\tau - |x|), \quad (\tau,x) \in I\!\!R^+ \times I\!\!R^n, & (4.6.68) \\ \tau = 0: \ v = 0, \ v_\tau = 0. & (4.6.69) \end{cases}$$

From the construction of the Lions extension (see Lions (1980)) it is easy to see that, if F satisfies (4.6.42), then its extension G must satisfy a condition of form (4.6.66). Hence, from Lemma 4.6.4 and Remark 4.6.1 we obtain

$$|v(\tau,x)| \leq C\big(1+\tau+|x|\big)^{-\frac{n-2}{2}}\big(1+\big|\tau-|x|\big|\big)^{-l}\|G\|_{W^{n-1,1}(I\!\!R^n)}, \tag{4.6.70}$$

in particular, we have

$$|v(t,x)| \leq C\big(1+t+|x|\big)^{-\frac{n-2}{2}}\big(1+\big|t-|x|\big|\big)^{-l}\|G\|_{W^{n-1,1}(I\!\!R^n)}. \tag{4.6.71}$$

From Corollary 4.6.2 we have

$$\|G\|_{W^{n-1,1}(I\!\!R^n)} \leq C\|F\|_{W^{n-1,1}(I\!\!R^n)}. \tag{4.6.72}$$

In addition, for any given $t \geq 1$, by definition, $G(x) \equiv F(x)$ on $|x| \leq t$, then for any given τ satisfying $0 \leq \tau \leq t$, Eq. (4.6.68) can be written as

$$\Box v(\tau,x) = F(x)\delta(\tau - |x|), \tag{4.6.73}$$

then it is clear that

$$v(t,x) = u(t,x), \quad \forall x \in I\!\!R^n. \tag{4.6.74}$$

Plugging (4.6.72) and (4.6.74) in (4.6.71), we arrive at the desired (4.6.67). The proof is finished. □

Lemma 4.6.6 *Suppose that $n \geq 2$ and $u = u(t,x)$ is the solution of the Cauchy problem*

$$\begin{cases} \Box u(t,x) = F(t,x), \quad (t,x) \in I\!\!R^+ \times I\!\!R^n, & (4.6.75) \\ t = 0: \ u = 0, \ u_t = 0, & (4.6.76) \end{cases}$$

where the function $F(t, x)$ *on the right-hand side satisfies*

$$supp\ F \subseteq \Big\{(t, x)\Big|1 \le |x| \le 2,\ \big|t - |x|\big| \le \frac{1}{2}\Big\}, \tag{4.6.77}$$

then we have

$$|u(t, x)| \le C\big(1 + t + |x|\big)^{-\frac{n-1}{2}}\big(1 + \big|t - |x|\big|\big)^{-l}\int_0^t \|F(\tau, \cdot)\|_{\overline{\Omega}, n-1,1} d\tau, \tag{4.6.78}$$

where C *is a positive constant,* $\overline{\Omega}$ *is defined by (3.1.17) in Chap. 3. Moreover, when* $n(\ge 3)$ *is odd,* $l \ge 0$*; while, when* $n(\ge 2)$ *is even,* $0 \le l \le \frac{n-1}{2}$.

Proof For any given $q \in I\!R$, let

$$F_q(t, x) = F(|x| - q, x)\delta(t - |x|). \tag{4.6.79}$$

By assumption (4.6.77), when $|q| \ge \frac{1}{2}$, $F_q \equiv 0$.

Let

$$\tau_q(t, x) = \delta(t + q)\delta(x). \tag{4.6.80}$$

We have

$$\begin{aligned}\int F_q * \tau_q dq &= \int \tau_q * F_q dq \\ &= \iiint \delta(t - \tau + q)\delta(x - y)F(|y| - q, y)\delta(\tau - |y|)d\tau dy dq \\ &= \iint \delta(t - \tau + q)F(|x| - q, x)\delta(\tau - |x|)d\tau dq \\ &= \int \delta(t - |x| + q)F(|x| - q, x)dq = F(t, x).\end{aligned} \tag{4.6.81}$$

Therefore, for the fundamental solution $E = E(t, x)$ of the wave equation (see Sect. 2.2 in Chap. 2), we have

$$\begin{aligned}E * F =& E * \int F_q * \tau_q dq \\ =& \int E * F_q * \tau_q dq \\ =& \iiint \delta(t - \tau + q)\delta(x - y)(E * F_q)(\tau, y)dy d\tau dq \\ =& \int (E * F_q)(t + q, x)dq,\end{aligned} \tag{4.6.82}$$

thus, the solution to the Cauchy problem (4.6.75)–(4.6.76) is

$$\begin{aligned} u(t,x) &= \int_0^t (E * F)(\tau, x) d\tau \\ &= \int_0^t \int (E * F_q)(\tau + q, x) dq d\tau \\ &= \int \int_0^t (E * F_q)(\tau + q, x) d\tau dq \\ &= \int \int_q^{t+q} (E * F_q)(\tau, x) d\tau dq \\ &= \int u_q(t + q, x) d\tau dq, \end{aligned} \tag{4.6.83}$$

where we denote

$$u_q(t,x) = \int_q^t (E * F_q)(\tau, x) d\tau dq. \tag{4.6.84}$$

Thanks to (4.6.79), $u_q = u_q(t,x)$ is the solution of the Cauchy problem

$$\begin{cases} \Box u_q = F_q(t,x) = F(|x| - q, x)\delta(t - |x|), & (4.6.85) \\ t = q : \ u_q = 0, \ (u_q)_t = 0. & (4.6.86) \end{cases}$$

According to Lemma 4.6.4 and noting that $|q| \le \frac{1}{2}$, it is easy to show that

$$\begin{aligned} |u_q(t+q,x)| \le\ & C(1 + t + q + |x|)^{-\frac{n-1}{2}} (1 + \big|t + q - |x|\big|)^{-l} \\ & \cdot \sum_{|\alpha| \le n-1} \int_{|x| \le t+q} |D_x^l F(|x| - q, x)| dx \\ \le\ & C(1 + t + |x|)^{-\frac{n-1}{2}} (1 + \big|t - |x|\big|)^{-l} \\ & \cdot \sum_{|\alpha| \le n-1} \int_{|x| \le t+q} |D_x^l F(|x| - q, x)| dx, \end{aligned} \tag{4.6.87}$$

then from (4.6.83) we have

$$\begin{aligned} |u(t,x)| \le\ & C(1 + t + |x|)^{-\frac{n-1}{2}} (1 + \big|t - |x|\big|)^{-l} \cdot \\ & \cdot \sum_{|\alpha| \le n-1} \int \int_{|x| \le t+q} |D_x^l F(|x| - q, x)| dx dq. \end{aligned} \tag{4.6.88}$$

Furthermore, according to Lemma 3.1.7 in Chap. 3 and noting (4.6.77), we get

$$|u(t,x)| \le C(1+t+|x|)^{-\frac{n-1}{2}}(1+\big|t-|x|\big|)^{-l} \cdot \sum_{k+|\beta|\le n-1}\int\int_{|x|\le t+q} |\partial_r^k \Omega_x^\beta F(|x|-q,x)|dxdq. \tag{4.6.89}$$

But noting that $\partial_r = \frac{1}{r}\sum_{i=1}^{n} x_i\partial_i$ $(r=|x|)$, it is easy to know that

$$\begin{aligned}\partial_r F(|x|-q,x) =& F_t(|x|-q,x)+F_r(|x|-q,x)\\ =& \frac{(tF_t+rF_r)+(rF_t+tF_r)}{t+r}\\ =& \frac{L_0F+L_rF}{t+r},\end{aligned} \tag{4.6.90}$$

where we denote

$$L_r = \sum_{i=1}^{n}\frac{x_i}{r}L_i, \tag{4.6.91}$$

and L_i $(i=1,\dots,n)$ are given by (3.1.12) in Chap. 3. Thus, from (4.6.89) we obtain

$$\begin{aligned}|u(t,x)| &\le C(1+t+|x|)^{-\frac{n-1}{2}}(1+\big|t-|x|\big|)^{-l}\sum_{|\alpha|\le n-1}\int_0^t\int|\overline{\Omega}^\alpha F(\tau,x)|dxd\tau\\ &= C(1+t+|x|)^{-\frac{n-1}{2}}(1+\big|t-|x|\big|)^{-l}\int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n-1,1}d\tau,\end{aligned} \tag{4.6.92}$$

where $\overline{\Omega}$ is given by (3.1.17) in Chap. 3, and $\|\cdot\|_{\overline{\Omega},n-1,1}$ is defined by (3.1.32) in Chap. 3. This is exactly (4.6.78) that we want. The proof of Lemma 4.6.6 is finished. □

4.6.3 L^1–L^∞ Estimates on Solutions to the Linear Wave Equations

Based on the previous two subsections, now we prove the following two important theorems related to the L^1–L^∞ estimates on solutions to the linear wave equations.

Theorem 4.6.2 *Suppose that $n \ge 2$ and $u = u(t,x)$ is the solution to the following Cauchy problem:*

$$\begin{cases}\Box u(t,x) = F(t,x), \quad (t,x)\in I\!R^+\times I\!R^n, & (4.6.93)\\ t=0:\ u=0,\ u_t=0, & (4.6.94)\end{cases}$$

then we have

$$|u(t,x)| \le C\big(1+t+|x|\big)^{-\frac{n-1}{2}}\big(1+\big|t-|x|\big|\big)^{-l} \cdot \int_0^t \sum_{|\alpha|\le n-1} \Big\|(1+|\cdot|+\tau)^{-\frac{n-1}{2}+l}\Gamma^\alpha F(\tau,\cdot)\Big\|_{L^1(\mathbb{R}^n)} d\tau, \tag{4.6.95}$$

where C is a positive constant, Γ is defined by (3.1.18) in Chap. 3. Moreover, when $n(\ge 3)$ is odd, $l \ge 0$; while, when $n(\ge 2)$ is even, $0 \le l \le \frac{n-1}{2}$.

Theorem 4.6.3 *Suppose that $n \ge 2$ and $u = u(t,x)$ is the solution to the following Cauchy problem:*

$$\begin{cases} \Box u(t,x) = \sum_{a=0}^{n} C_a \partial_a F(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, & (4.6.96) \\ t=0: \ u=0, \ u_t=0, & (4.6.97) \end{cases}$$

where $C_a (a = 0, 1, \cdots, n)$ are constants, then we have

$$|u(t,x)| \le C\big(1+t+|x|\big)^{-\frac{n-1}{2}}\Big(\int_0^t (1+\tau)^{\frac{n-1}{2}} \|F(\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)} d\tau + (1+\tau)^{-\frac{n+1}{2}} \|F(\tau,\cdot)\|_{\Gamma,n+1,1} d\tau\Big), \tag{4.6.98}$$

where C is a positive constant, and Γ is defined by (3.1.18) in Chap. 3.

In order to prove Theorem 4.6.2, we first introduce the following two lemmas.

Lemma 4.6.7 *Suppose that $n \ge 2$, $u = u(t,x)$ is the solution to Cauchy problem (4.6.93)–(4.6.94), and*

$$supp\ F \subseteq \big\{(t,x)\big|t^2+|x|^2 \le 4\big\}, \tag{4.6.99}$$

then we have

$$|u(t,x)| \le C\big(1+t+|x|\big)^{-\frac{n-1}{2}}\big(1+\big|t-|x|\big|\big)^{-l} \int_0^t \|F(\tau,\cdot)\|_{W^{n-1,1}(\mathbb{R}^n)} d\tau, \tag{4.6.100}$$

where C is a positive constant. Moreover, when $n(\ge 3)$ is odd, $l \ge 0$; while, when $n(\ge 2)$ is even, $0 \le l \le \frac{n-1}{2}$.

Proof From Duhamel principle we have

$$u(t,x) = \int_0^t v(t,x;\tau) d\tau, \tag{4.6.101}$$

where $v = v(t,x;\tau)$ is the solution of the Cauchy problem

$$\begin{cases} \Box v(t,x)=0, & (4.6.102)\\ t=\tau:\ v=0,\ v_t=F(\tau,x). & (4.6.103)\end{cases}$$

From assumption (4.6.99) we have supp $F\subseteq\{(t,x)|0\le t\le 2, |x|\le 2\}$, then $v=v(t,x;\tau)$ is identically equal to zero except when $0\le\tau\le 2$, and the corresponding supp $F(\tau,x)\subseteq\{x||x|\le 2\}$. Thus, from Corollary 4.6.1 we have

$$|v(t,x;\tau)|\le C\big(1+t-\tau+|x|\big)^{-\frac{n-1}{2}}\big(1+\big|t-\tau-|x|\big|\big)^{-l}\|F(\tau,\cdot)\|_{W^{n-1,1}(\mathbb{R}^n)}. \tag{4.6.104}$$

Noticing that when $0\le\tau\le 2$ we have

$$1+t-\tau+|x|\le 1+t+|x|$$

and

$$1+t-\tau+|x|\ge 1+\frac{t-\tau}{3}+|x|\ge 1+\frac{t-2}{3}+|x|\ge\frac{1}{3}(1+t+|x|),$$

we also have

$$1+\big|t-\tau-|x|\big|\le 1+\big|t-|x|\big|+\tau\le 3+\big|t-|x|\big|\le 3\big(1+\big|t-|x|\big|\big)$$

and

$$1+\big|t-\tau-|x|\big|\ge 1+\frac{\big|t-\tau-|x|\big|}{3}\ge 1+\frac{\big|t-|x|\big|-\tau}{3}\ge\frac{1}{3}\big(1+\big|t-|x|\big|\big),$$

(4.6.101) and (4.6.104) immediately yield (4.6.100) that we want. □

Lemma 4.6.8 *Suppose that $n\ge 2$, $u=u(t,x)$ is the solution to Cauchy problem (4.6.93)-(4.6.94), and*

$$supp\ F\subseteq\big\{(t,x)\big|1\le t^2+|x|^2\le 4\big\}, \tag{4.6.105}$$

then we have

$$|u(t,x)|\le C\big(1+t+|x|\big)^{-\frac{n-1}{2}}\big(1+\big|t-|x|\big|\big)^{-l}\int_0^t\|F(\tau,\cdot)\|_{\overline{\Omega},n-1,1}d\tau, \tag{4.6.106}$$

where C is a positive constant, and $\overline{\Omega}$ is given by (3.1.17) in Chap. 3. Moreover, when $n(\ge 3)$ is odd, $l\ge 0$; while, when $n(\ge 2)$ is even, $0\le l\le\frac{n-1}{2}$.

Proof (i) Assume that $\big|t-|x|\big|\ge\frac{1}{4}$ holds on the support of F. From (3.1.54) in Chap. 3, for any given multi-index α, we have

$$|D^\alpha F(t,x)| \le C \sum_{0\le|\beta|\le|\alpha|} \left|\overline{\Omega}^\beta F(t,x)\right|, \tag{4.6.107}$$

then Lemma 4.6.7 immediately leads to (4.6.106).

(ii) Assume that $\left|t-|x|\right| \le \frac{1}{2}$ holds on the support of F. Then (4.6.106) follows immediately from Lemma 4.6.6.

(iii) Based on (i) and (ii), in the general case, our conclusions can be obtained by partition of unity. □

Now we prove Theorem 4.6.2.

Proof Under assumption (4.6.105), from (4.6.106) in Lemma 4.6.8, it is clear that

$$|u(t,x)| \le C(t+|x|)^{-\frac{n-1}{2}}\left|t-|x|\right|^{-l}\int_0^t \sum_{|\alpha|\le n-1} \|(\tau+|\cdot|)^{-\frac{n-1}{2}+l}\overline{\Omega}^\alpha F(\tau,\cdot)\|_{L^1(I\!R^n)}d\tau. \tag{4.6.108}$$

For any given $\lambda>0$, let $u=u(t,x)$ be the solution of Cauchy problem (4.6.93)–(4.6.94) corresponding to the function $F(t,x)$ on the right-hand side. Turning $F(t,x)$ into a function of (t,x), depending on the parameter λ: $F_\lambda(t,x)=F(\lambda t,\lambda x)$, by scaling, the solution of the corresponding Cauchy problem should be $u_\lambda(t,x)=\lambda^{-2}u(\lambda t,\lambda x)$. Assume that $F_\lambda(t,x)$, as a function of (t,x), still satisfies (4.6.105), then from (4.6.108) we have

$$\begin{aligned}|u_\lambda(t,x)| &= \lambda^{-2}|u(\lambda t,\lambda x)|\\ &\le C(t+|x|)^{-\frac{n-1}{2}}\left|t-|x|\right|^{-l}\int_0^t \sum_{|\alpha|\le n-1} \|(\tau+|\cdot|)^{-\frac{n-1}{2}+l}\overline{\Omega}^\alpha F_\lambda(\tau,\cdot)\|_{L^1(I\!R^n)}d\tau.\end{aligned}$$

In the above formula, set $\bar{t}=\lambda t$, $\bar{x}=\lambda x$, and note that $\overline{\Omega}\ (t,x)$ is a scaling invariant differential operator, still denoting $(\bar{t},\bar{x})$ as (t,x) in the resulting formula, we can easily see that (4.6.108) is of scaling invariant form, i.e., for any given $\lambda>0$, (4.6.108) is still valid when

$$\text{supp } F \subseteq \{(t,x)\big|\lambda^2 \le t^2+|x|^2 \le 4\lambda^2\}, \tag{4.6.109}$$

then from the binary partition of unity (see Sect. 3.1 in Chap. 3) we know that (4.6.108) still holds when

$$\text{supp } F \subseteq \{(t,x)\big|t^2+|x|^2 \ge 1\}. \tag{4.6.110}$$

Taking particularly $l=0$ in (4.6.108), under assumption (4.6.110) we have

$$|u(t,x)| \le C(t+|x|)^{-\frac{n-1}{2}}\int_0^t \sum_{|\alpha|\le n-1} \|(\tau+|\cdot|)^{-\frac{n-1}{2}}\overline{\Omega}^\alpha F(\tau,\cdot)\|_{L^1(I\!R^n)}d\tau. \tag{4.6.111}$$

Since we have $\tau + |y| \geq 1$ (where y is the integral variable) in the integral on the right-hand side of (4.6.111), due to (4.6.110), for l given in Lemma 4.6.8 it is obvious that

$$|u(t,x)| \leq C(t+|x|)^{-\frac{n-1}{2}} \int_0^t \sum_{|\alpha|\leq n-1} \|(\tau + |\cdot|)^{-\frac{n-1}{2}+l} \overline{\Omega}^\alpha F(\tau,\cdot)\|_{L^1(I\!R^n)} d\tau. \tag{4.6.112}$$

Thus, from (4.6.108) and (4.6.112) we obtain

$$|u(t,x)| \leq C(t+|x|)^{-\frac{n-1}{2}} \left(1 + \big|t - |x|\big|\right)^{-l} \cdot \int_0^t \sum_{|\alpha|\leq n-1} \|(\tau + |\cdot|)^{-\frac{n-1}{2}+l} \overline{\Omega}^\alpha F(\tau,\cdot)\|_{L^1(I\!R^n)} d\tau. \tag{4.6.113}$$

Noting that we have $t + |x| \geq 1$ on the support of the solution $u = u(t,x)$ due to (4.6.110), under assumption (4.6.110), from the above formula we get

$$|u(t,x)| \leq C(1+t+|x|)^{-\frac{n-1}{2}} \left(1 + \big|t - |x|\big|\right)^{-l} \cdot \int_0^t \sum_{|\alpha|\leq n-1} \|(\tau + |\cdot|)^{-\frac{n-1}{2}+l} \overline{\Omega}^\alpha F(\tau,\cdot)\|_{L^1(I\!R^n)} d\tau. \tag{4.6.114}$$

On the other hand, by Lemma 4.6.7, (4.6.100) holds under assumption (4.6.99). Using the partition of unity, the desired (4.6.95) follows from the combination of (4.6.114) and (4.6.100). The proof of Theorem 4.6.2 is finished. □

In order to prove Theorem 4.6.3, we first prove the following lemma.

Lemma 4.6.9 *Suppose that $n \geq 2$ and $u = u(t,x)$ is the solution to the Cauchy problem (4.6.96)–(4.6.97), in which $F(t,x)$ satisfies (4.6.105), then we have*

$$|u(t,x)| \leq C(1+t+|x|)^{-\frac{n-1}{2}} \int_0^t (\|F(\tau,\cdot)\|_{L^\infty(I\!R^n)} + \|F(\tau,\cdot)\|_{\overline{\Omega},n+1,1}) d\tau, \tag{4.6.115}$$

where C is a positive constant, and $\overline{\Omega}$ is given by (3.1.17) in Chap. 3.

Proof By Lemma 4.6.7 we have

$$|u(t,x)| \leq C(1+t+|x|)^{-\frac{n-1}{2}} \int_0^t \|F(\tau,\cdot)\|_{W^{n-1,1}(I\!R^n)} d\tau. \tag{4.6.116}$$

Therefore, if $\big|\tau - |y|\big| \geq \frac{1}{4}$ (where (τ, y) are the integral variables on the right-hand side of the above formula) on the support of F, then from (3.1.54) in Chap. 3 it is easy to show that

$$|u(t,x)| \le C(1+t+|x|)^{-\frac{n-1}{2}} \int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n-1,1} d\tau, \tag{4.6.117}$$

then (4.6.115) follows.

Thus, by partition of unity, it suffices to discuss the case that $\big|\tau - |y|\big| \le \frac{1}{2}$ on the support of F. Since we always have $\tau^2 + |y|^2 \ge 1$ due to (4.6.105), it is easy to show that $\tau \ge \frac{1}{4}$ and $|y| \ge \frac{1}{4}$ on the support of F. In fact, from $\big|\tau - |y|\big| \le \frac{1}{2}$, we have $\tau \le |y| + \frac{1}{2}$, and then

$$1 \le \tau^2 + |y|^2 \le \Big(|y| + \frac{1}{2}\Big)^2 + |y|^2 = 2|y|^2 + |y| + \frac{1}{4} \le \Big(2|y| + \frac{1}{2}\Big)^2,$$

i.e., $|y| \ge \frac{1}{4}$. Similarly, we can prove $\tau \ge \frac{1}{4}$.

Suppose that $v = v(t,x)$ is the solution of the Cauchy problem

$$\begin{cases} \Box v = F(t,x), \quad (t,x) \in I\!R^+ \times I\!R^n, & (4.6.118)\\ t = 0: \ v = 0, \ v_t = 0. & (4.6.119)\end{cases}$$

Noting that $\tau \ge \frac{1}{4}$ on the support of F, it easy easy to show that the solution $u = u(t,x)$ of the Cauchy problem (4.6.96)–(4.6.97) can be written as

$$u = u(t,x) = \sum_{a=0}^{n} C_a \partial_a v. \tag{4.6.120}$$

We first consider the case that $\big|t - |x|\big| \ge \frac{1}{8}$. Now from (3.1.54) in Chap. 3 we have

$$|u(t,x)| \le C|Dv(t,x)| \le C|\overline{\Omega} v(t,x)|. \tag{4.6.121}$$

Using (3.1.38) and (3.1.39) in Lemma 3.1.4 of Chap. 3 and noting that $\tau \ge \frac{1}{4}$ on the support of F, it is clear that $\overline{\Omega} v(t,x)$ satisfies a Cauchy problem similar to (4.6.118)–(4.6.119), while there are additional terms containing $\overline{\Omega} F(t,x)$ on the right-hand side of equation (4.6.118). Thus, using Lemma 4.6.8, it follows from the above formula that

$$|u(t,x)| \le C(1+t+|x|)^{-\frac{n-1}{2}} \int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n,1} d\tau. \tag{4.6.122}$$

Now we consider the case that $\big|t - |x|\big| \le \frac{1}{8}$.

Construct a function $\psi \in C_0^\infty(I\!R^+ \times I\!R^n)$ such that

$$\psi(t,x) = \psi(t,|x|), \tag{4.6.123}$$

$$\psi(t,x) \equiv 0, \text{ if } t+|x| \le \frac{1}{6}; \text{ or if } t+|x| \ge \frac{1}{4} \text{ and } \big|t-|x|\big| \ge \frac{1}{6}, \tag{4.6.124}$$

$$\psi(t,x) \equiv 1, \text{ if } t+|x| \geq \frac{1}{4} \text{ and } \big|t-|x|\big| \leq \frac{1}{8}, \tag{4.6.125}$$

and

$$0 \leq \psi \leq 1. \tag{4.6.126}$$

Let

$$v_1(t,x) = \psi(t,x)v(t,x). \tag{4.6.127}$$

We have

$$\begin{cases} \Box v_1 = \psi F + 2Q(\psi, v) + v\Box\psi, & (4.6.128)\\ t=0: \ v_1 = 0, \ v_{1t} = 0, & (4.6.129)\end{cases}$$

where

$$Q(\psi, v) = \partial_t\psi\partial_t v - \sum_{i=1}^n \partial_i\psi\partial_i v. \tag{4.6.130}$$

Noticing the definition of Ω_{ij} (see (3.1.11)) in Chap. 3) and using (3.1.43) in Chap. 3, we can verify directly the following identity:

$$\Box v_1 = r^{-\frac{n-1}{2}}(\partial_t^2 - \partial_r^2)(r^{\frac{n-1}{2}}v_1) + \frac{(n-1)(n-3)}{4}r^{-2}v_1 - \sum_{i,j,k=1}^n \frac{x_j x_k}{r^4}\Omega_{ji}\Omega_{ki}v_1, \tag{4.6.131}$$

then from (4.6.128)–(4.6.129) we get

$$(\partial_t^2 - \partial_r^2)(r^{\frac{n-1}{2}}v_1) = -\frac{(n-1)(n-3)}{4}r^{\frac{n-1}{2}-2}v_1 + r^{\frac{n-1}{2}}\sum_{i,j,k=1}^n \frac{x_j x_k}{r^4}\Omega_{ji}\Omega_{ki}v_1$$
$$+ r^{\frac{n-1}{2}}(\psi F + 2Q(\psi, v) + v\Box\psi) \overset{\text{def.}}{=} \widetilde{G}(t,x), \tag{4.6.132}$$

$$t=0: \ r^{\frac{n-1}{2}}v_1 = 0, \ \partial_t(r^{\frac{n-1}{2}}v_1) = 0. \tag{4.6.133}$$

Noting that $t \geq \frac{1}{4}$ on the support of $v(t,x)$, from (4.6.124) we know that on the support of $v_1(t,x)$ we have $\big|t-|x|\big| \leq \frac{1}{6}$, and then $\frac{t}{3} \leq |x| \leq t + \frac{1}{6}$. Thus, it is clear that

$$\left| -\frac{(n-1)(n-3)}{4}r^{\frac{n-1}{2}-2}v_1 + r^{\frac{n-1}{2}}\sum_{i,j,k=1}^n \frac{x_j x_k}{r^4}\Omega_{ji}\Omega_{ki}v_1 \right| \leq C(1+t)^{\frac{n-1}{2}-2}\sum_{|\alpha|\leq 2}|\overline{\Omega}^\alpha v_1|. \tag{4.6.134}$$

By (3.1.52) in Chap. 3 we easily have

$$|\partial_a v| \le \frac{C}{\big|t-|x|\big|} \sum_{|\alpha|=1} |\overline{\Omega}^\alpha v| \quad (a=0,1,\ldots,n), \tag{4.6.135}$$

where C is a positive constant. Observing that $t \ge \frac{1}{4}$ on the support of $v = v(t,x)$, due to (4.6.125), we now have $\partial_a \psi \equiv 0$ unless $\big|t-|x|\big| \ge \frac{1}{8}$. Then, from (4.6.135) we easily get

$$|2Q(\psi, v) + v\Box\psi| \le C \sum_{|\alpha|=1} |\overline{\Omega}^\alpha v|. \tag{4.6.136}$$

On the other hand, from (3.1.8) and (3.1.12) in Chap. 3 it is easy to have

$$tQ(\psi, v) = L_0\psi \cdot \partial_t v - \sum_{i=1}^n \partial_{x_i}\psi \cdot L_i v \tag{4.6.137}$$

and

$$t\Box\psi = L_0\psi_t - \sum_{i=1}^n L_i\psi_{x_i}, \tag{4.6.138}$$

and then similarly, we can get

$$|2Q(\psi, v) + v\Box\psi| \le Ct^{-1} \sum_{|\alpha|\le 1} |\overline{\Omega}^\alpha v|. \tag{4.6.139}$$

Combining (4.6.136) and (4.6.138), we obtain

$$|2Q(\psi, v) + v\Box\psi| \le C(1+t)^{-1} \sum_{|\alpha|\le 1} |\overline{\Omega}^\alpha v|. \tag{4.6.140}$$

Using (4.1.5) (in which we take $p = +\infty$) and (4.1.3) in Theorem 4.1.1, from (4.6.132)–(4.6.133) we obtain, respectively,

$$|r^{\frac{n-1}{2}}\partial_t v_1|,\ |\partial_r(r^{\frac{n-1}{2}} v_1)| \le \int_0^t \|\widetilde{G}(\tau,\cdot)\|_{L^\infty(I\!R^n)} d\tau \tag{4.6.141}$$

and

$$|r^{\frac{n-1}{2}} v_1| \le t\int_0^t \|\widetilde{G}(\tau,\cdot)\|_{L^\infty(I\!R^n)} d\tau \le Cr\int_0^t \|\widetilde{G}(\tau,\cdot)\|_{L^\infty(I\!R^n)} d\tau, \tag{4.6.142}$$

where we used in the last formula the above stated fact: $r = |x| \geq \frac{t}{3}$ on the support of $v_1(t, x)$.

Observing that

$$r^{\frac{n-1}{2}} \partial_r v_1 = \partial_r (r^{\frac{n-1}{2}} v_1) - \frac{n-1}{2} r^{\frac{n-1}{2}-1} v_1,$$

from (4.6.141)–(4.6.141) we obtain

$$|\partial_t v_1(t, r)|, \ |\partial_r v_1(t, x)| \leq C r^{-\frac{n-1}{2}} \int_0^t \|\widetilde{G}(\tau, \cdot)\|_{L^\infty(I\!R^n)} d\tau, \tag{4.6.143}$$

then, using (4.6.132), (4.6.134) and (4.6.140), noting (4.6.105) and that $t + |x| \leq \frac{1}{4}$ on the support of $\widetilde{G}(t, x)$ so that all the related estimates are made accordingly, it is easy to obtain that

$$\begin{aligned} |\partial_t v_1|, \ |\partial_r v_1| \leq C r^{-\frac{n-1}{2}} \Bigg(& \int_0^t \|F(\tau, \cdot)\|_{L^\infty(I\!R^n)} d\tau \\ & + \int_0^t \left((1+\tau)^{-1} + (1+\tau)^{\frac{n-1}{2}-2} \right) \sum_{|\alpha| \leq 2} \|\overline{\Omega}^\alpha v(\tau, \cdot)\|_{L^\infty(I\!R^n)} d\tau \Bigg). \end{aligned} \tag{4.6.144}$$

According to the Lorentz invariance of the wave operator (see (3.1.38)–(3.1.39) in Lemma 3.1.4 of Chap. 3), and noting that $t \geq \frac{1}{4}$ on the support of $v = v(t, x)$, it is easy to know that every $\overline{\Omega}^\alpha v$ $(|\alpha| \leq 2)$ satisfies a Cauchy problem similar to the Cauchy problem (4.6.118)–(4.6.119) satisfied by v, but now the right-hand side of equation (4.6.118) is a linear combination of $\overline{\Omega}^\alpha F$ $(|\alpha| \leq 2)$. Then, according to Theorem 4.6.2 and noticing that $|x| \geq \frac{t}{3}$ and $t + |x| \leq \frac{1}{4}$ on the support of $\widetilde{G}(t, x)$ so that all the related estimates are made accordingly, from (3.1.52) in Chap. 3 we obtain

$$|\Gamma f| \leq C |\overline{\Omega} F|.$$

Therefore, when $0 \leq \tau \leq t$ we have

$$\begin{aligned} \sum_{|\alpha| \leq 2} \|\overline{\Omega}^\alpha v(\tau, \cdot)\|_{L^\infty(I\!R^n)} & \leq C(1+\tau)^{-\frac{n-1}{2}} \int_0^\tau \|F(s, \cdot)\|_{\overline{\Omega}, n+1, 1} ds \\ & \leq C(1+\tau)^{-\frac{n-1}{2}} \int_0^t \|F(s, \cdot)\|_{\overline{\Omega}, n+1, 1} ds, \end{aligned} \tag{4.6.145}$$

then

$$\begin{aligned}&\int_0^t \Big((1+\tau)^{-1}+(1+\tau)^{\frac{n-1}{2}-2}\Big)\sum_{|\alpha|\le 2}\|\overline{\Omega}^\alpha v(\tau,\cdot)\|_{L^\infty(I\!R^n)}d\tau\\ &\le C\int_0^t \Big((1+\tau)^{-\frac{n-1}{2}-1}+(1+\tau)^{-2}\Big)d\tau\cdot\int_0^t \|F(s,\cdot)\|_{\overline{\Omega},n+1,1}ds\\ &\le C\int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n+1,1}ds. \qquad (4.6.146)\end{aligned}$$

Hence, from (4.6.144) we have

$$|\partial_t v_1|,\ |\partial_r v_1| \le Cr^{-\frac{n-1}{2}}\Big(\int_0^t \|F(\tau,\cdot)\|_{L^\infty(I\!R^n)}d\tau+\int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n+1,1}d\tau\Big). \qquad (4.6.147)$$

As stated above, on the support of $v_1(t,x)$ we have $t\ge\frac14$ and $\frac t3\le|x|\le t+\frac16$, then from the above formula we immediately get

$$|\partial_t v_1|,\ |\partial_r v_1| \le C(1+t+|x|)^{-\frac{n-1}{2}}\Big(\int_0^t \|F(\tau,\cdot)\|_{L^\infty(I\!R^n)}d\tau+\int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n+1,1}d\tau\Big). \qquad (4.6.148)$$

By (4.6.125) and noting that $t\ge\frac14$ on the support of $v(t,x)$, it is easy to show that $v_1(t,x)\equiv v(t,x)$ when $\big|t-|x|\big|\le\frac18$. Therefore, in the situation that $\big|t-|x|\big|\le\frac18$, from (4.6.148) we have

$$\begin{aligned}|\partial_t v(t,x)|,\ |\partial_r v(t,x)| \le C(1+t+|x|)^{-\frac{n-1}{2}}\Big(&\int_0^t \|F(\tau,\cdot)\|_{L^\infty(I\!R^n)}d\tau\\ &+\int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n+1,1}d\tau\Big). \qquad (4.6.149)\end{aligned}$$

In addition, from (4.6.145) and noticing that $\big|t-|x|\big|\le\frac18$, similarly we have

$$\begin{aligned}\sum_{|\alpha|\le 2}|\overline{\Omega}^\alpha v(t,\cdot)| &\le C(1+t)^{-\frac{n-1}{2}}\int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n+1,1}d\tau\\ &\le C(1+t+|x|)^{-\frac{n-1}{2}}\int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n+1,1}d\tau. \qquad (4.6.150)\end{aligned}$$

Thus, using Lemma 3.1.7 in Chap. 3, and noting that when $t\ge\frac14$ and $\big|t-|x|\big|\le\frac18$ we have $r=|x|\ge\frac18$, from (4.6.149)–(4.6.150) we get

$$|\partial_a v(t,x)| \le C(1+t+|x|)^{-\frac{n-1}{2}}\left(\int_0^t \|F(\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)}d\tau+\int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n+1,1}d\tau\right),$$
$$a=0,1,\ldots,n. \quad (4.6.151)$$

Then, in the situation that $\left|t-|x|\right| \le \frac{1}{8}$, from (4.6.120) we obtain

$$|u(t,x)| \le C(1+t+|x|)^{-\frac{n-1}{2}}\left(\int_0^t \|F(\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)}d\tau+\int_0^t \|F(\tau,\cdot)\|_{\overline{\Omega},n+1,1}d\tau\right). \quad (4.6.152)$$

Combining (4.6.122) and (4.6.152), we finish the proof of Lemma 4.6.9. □

Now we prove Theorem 4.6.3.

Proof of Theorem 4.6.3 For any given $\lambda > 0$, let $u = u(t,x)$ be the solution to the Cauchy problem (4.6.96)–(4.6.97). If we change the function $F(t,x)$ on right-hand side into function $F_\lambda(t,x) = F(\lambda t, \lambda x)$ depending on a parameter λ, the solution to the corresponding Cauchy problem should be $u_\lambda(t,x) = \lambda^{-1}u(\lambda t, \lambda x)$.

Similarly to (4.6.108), we rewrite (4.6.115) obtained under assumption (4.6.105) into the following scaling invariant form:

$$|u(t,x)| \le C(t+|x|)^{-\frac{n-1}{2}}\left(\int_0^t \|(\tau+|\cdot|)^{\frac{n-1}{2}}F(\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)}d\tau \right.$$
$$\left. +\int_0^t \sum_{|\alpha|\le n+1}\|(\tau+|\cdot|)^{-\frac{n+1}{2}}\overline{\Omega}^\alpha F(\tau,\cdot)\|_{L^1(\mathbb{R}^n)}d\tau\right), \quad (4.6.153)$$

and then we find that this formula still holds under assumption (4.6.110). Noticing that $t \ge \frac{1}{4}$ on the support of $u(t,x)$ due to assumption (4.6.105), under assumption (4.6.110), from (4.6.153) we obtain

$$|u(t,x)| \le C(1+t+|x|)^{-\frac{n-1}{2}}\left(\int_0^t (1+\tau)^{\frac{n-1}{2}}\|F(\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)}d\tau \right.$$
$$\left. +\int_0^t (1+\tau)^{-\frac{n+1}{2}}\|F(\tau,\cdot)\|_{\overline{\Omega},n+1,1}d\tau\right). \quad (4.6.154)$$

But if (4.6.99) is satisfied, then $t \le 2$ on the support of $F(t,x)$, and from Theorem 4.6.2 we easily know that

$$|u(t,x)| \le C(1+t+|x|)^{-\frac{n-1}{2}}\int_0^t (1+\tau)^{-\frac{n-1}{2}}\|F(\tau,\cdot)\|_{\Gamma,n,1}d\tau$$
$$\le C(1+t+|x|)^{-\frac{n-1}{2}}\int_0^t (1+\tau)^{-\frac{n+1}{2}}\|F(\tau,\cdot)\|_{\Gamma,n,1}d\tau. \quad (4.6.155)$$

Based on (4.6.154)–(4.6.155), (4.6.98) can be obtained by partition of unity. Theorem 4.6.3 is proved. □

Observing Lemma 3.1.5 in Chap. 3, from Theorems 4.6.1 and 4.6.2 (in which we take $l = 0$) it is easy to obtain

Corollary 4.6.3 *Let $n \geq 2$. If $u = u(t, x)$ is the solution to Cauchy problem (4.6.1)–(4.6.2), then for any given integer $N \geq 0$, we have*

$$(1+t)^{\frac{n-1}{2}} \|u(t,\cdot)\|_{\Gamma,N,\infty} \leq C\left\{\|u(0,\cdot)\|_{\Gamma,N+n,1} + \int_0^t (1+\tau)^{-\frac{n-1}{2}} \|F(\tau,\cdot)\|_{\Gamma,N+n-1,1} d\tau\right\}, \quad \forall t \geq 0, \tag{4.6.156}$$

where C is a positive constant, and $\|u(0,\cdot)\|_{\Gamma,N+n,1}$ is the value of $\|u(t,\cdot)\|_{\Gamma,N+n,1}$ at $t = 0$.

Similarly, from Theorem 4.6.1 and Theorem 4.6.3 we obtain

Corollary 4.6.4 *Let $n \geq 2$. If $u = u(t, x)$ is the solution to Cauchy problem (4.6.96) and (4.6.2), then for any given integer $N \geq 0$, we have*

$$\begin{aligned}(1+t)^{\frac{n-1}{2}} &\|u(t,\cdot)\|_{\Gamma,N,\infty} \\ &\leq C\left\{\|u(0,\cdot)\|_{\Gamma,N+n,1} + \int_0^t \left((1+\tau)^{\frac{n-1}{2}} \|F(\tau,\cdot)\|_{\Gamma,N,\infty} + (1+\tau)^{-\frac{n+1}{2}} \|F(\tau,\cdot)\|_{\Gamma,N+n+1,1}\right) d\tau\right\}, \\ &\qquad \forall t \geq 0,\end{aligned} \tag{4.6.157}$$

where C is a positive constant, and $\|u(0,\cdot)\|_{\Gamma,N+n,1}$ is the value of $\|u(t,\cdot)\|_{\Gamma,N+n,1}$ at $t = 0$.

Remark 4.6.2 The heart of this section is Theorems 4.6.1, 4.6.2 and 4.6.3. Where Theorem 4.6.1 belongs to Klainerman (see Klainerman (1980)), while, Theorem 4.6.2 is an improvement and extension of Hörmander (see Hörmander (1988)) to a similar theorem of Klainerman (see Klainerman (1986)), and Theorem 4.6.3 essentially belongs to Lindblad(he considered the case $n = 3$, see Lindblad (1990b)). The proofs of these three theorems in this chapter follow in principle the arguments of Hörmander.

Chapter 5
Some Estimates on Product Functions and Composite Functions

For the needs of the following chapters, in this chapter we will give some estimates about product functions and composite functions (see Li and Chen 1989, 1992).

5.1 Some Estimates on Product Functions

For the space $L^{p,q}(R^n)$ introduced in Sect. 3.1.2 of Chap. 3, Similarly to Lemma 2.3.2 in Chap. 2, we have the following

Lemma 5.1.1 (Hölder inequality) *If* $f_i \in L^{p_i,q_i}(I\!R^n)$, $1 \le p_i, q_i \le +\infty (i = 1, \dots, M)$ *and*

$$\frac{1}{p} = \sum_{i=1}^{M} \frac{1}{p_i}, \quad \frac{1}{q} = \sum_{i=1}^{M} \frac{1}{q_i}, \quad 1 \le p, q \le +\infty, \tag{5.1.1}$$

then

$$\prod_{i=1}^{M} f_i \in L^{p,q}(I\!R^n), \tag{5.1.2}$$

and

$$\Big\| \prod_{i=1}^{M} f_i \Big\|_{L^{p,q}(I\!R^n)} \le \prod_{i=1}^{M} \| f_i \|_{L^{p_i,q_i}(I\!R^n)}. \tag{5.1.3}$$

Utilizing the set Γ of partial differential operators defined by (3.1.18) in Chap. 3, for any given integer $N \ge 0$, we define

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$$\|f(t,\cdot)\|_{\Gamma,N,p,q,\chi} = \sum_{|k|\le N} \|\chi(t,\cdot)\Gamma^k f(t,\cdot)\|_{L^{p,q}(I\!R^n)}, \tag{5.1.4}$$

where $1 \le p, q \le +\infty$, $\chi(t, x)$ is the characteristic function of any given set in $R_+ \times I\!R^n$, $k = (k_1, \ldots, k_\sigma)$ is a multi-index, $|k| = k_1+\cdots+k_\sigma$, σ is the number of the partial differential operators included in Γ: $\Gamma = (\Gamma_1, \ldots, \Gamma_\sigma)$, and $\Gamma^k = \Gamma_1^{k_1} \cdots \Gamma_\sigma^{k_\sigma}$.

In the sequel, we simply denote

$$\|f(t,\cdot)\|_{\Gamma,N,p,q,\chi} = \begin{cases} \|f(t,\cdot)\|_{\Gamma,N,p,\chi}, & \text{if } p = q; \\ \|f(t,\cdot)\|_{\Gamma,N,p,q}, & \text{if } \chi \equiv 1; \\ \|f(t,\cdot)\|_{p,q,\chi}, & \text{if } N = 0; \\ \|f(t,\cdot)\|_{L^{p,q}(R^n)}, & \text{if } N = 0, \text{ and } \chi \equiv 1 \end{cases} \tag{5.1.5}$$

and so on.

Lemma 5.1.2 *Suppose that* $1 \le p, q, p_i, q_i \le +\infty (i = 1, \ldots, 4)$ *satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}, \tag{5.1.6}$$

and the norms appearing on the right-hand sides of the following formulas are all well-defined. Then for any given integer $N > 0$ *we have*

$$\begin{aligned} &\|fg(t,\cdot)\|_{\Gamma,N,p,q,\chi} \\ &\le C(\|f(t,\cdot)\|_{\Gamma,[\frac{N-1}{2}],p_1,q_1,\chi}\|\Gamma g(t,\cdot)\|_{\Gamma,N-1,p_2,q_2,\chi} \\ &\quad +\|f(t,\cdot)\|_{\Gamma,N,p_3,q_3,\chi}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_4,q_4,\chi}), \quad \forall t \ge 0; \end{aligned} \tag{5.1.7}$$

and for any given multi-index $k(|k| = N > 0)$ *we have*

$$\begin{aligned} &\|(\Gamma^k(fg) - f\Gamma^k g)(t,\cdot)\|_{p,q,\chi} \\ &\le C(\|f(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1,\chi}\|g(t,\cdot)\|_{\Gamma,N-1,p_2,q_2,\chi} \\ &\quad +\|\Gamma f(t,\cdot)\|_{\Gamma,N-1,p_3,q_3,\chi}\|g(t,\cdot)\|_{\Gamma,[\frac{N-1}{2}],p_4,q_4,\chi}), \quad \forall t \ge 0, \end{aligned} \tag{5.1.8}$$

where C *is a positive constant, and* [ ] *stands for the integer part of a real number.*

Proof First we prove (5.1.7).

From the definition of the operator set Γ, it is easy to know that we still have the chain rule:

$$\Gamma(fg) = (\Gamma f)g + f(\Gamma g), \tag{5.1.9}$$

therefore, for any given multi-index $k(|k| \le N)$ we have

$$\begin{aligned}\Gamma^k(fg) = & \sum_{|i|+|j|\le N} C_{ij}\Gamma^i f \cdot \Gamma^j g \\ = & \sum_{\substack{|i|+|j|\le N \\ |i|<|j|}} C_{ij}\Gamma^i f \cdot \Gamma^j g + \sum_{\substack{|i|+|j|\le N \\ |i|\ge|j|}} C_{ij}\Gamma^i f \cdot \Gamma^j g \\ \stackrel{\text{def}}{=} & \ \mathrm{I} + \mathrm{II},\end{aligned} \tag{5.1.10}$$

where C_{ij} are constants.

In I, we should have $|i| \le [\frac{N-1}{2}]$. In fact, when N is even, due to $|i| < \frac{N}{2} = [\frac{N}{2}]$, we have $|i| \le [\frac{N-1}{2}]$; while, when N is odd, due to $|i| < \frac{N}{2} = [\frac{N-1}{2}] + \frac{1}{2}$, we still have $|i| \le [\frac{N-1}{2}]$. Thus, from Hölder inequality (5.1.3), and noting that $\chi^2 = \chi$ and $|j| > 0$, it is easy to show that

$$\|\mathrm{I}\|_{p,q,\chi} \le C\|f(t,\cdot)\|_{\Gamma,[\frac{N-1}{2}],p_1,q_1,\chi}\|\Gamma g(t,\cdot)\|_{\Gamma,N-1,p_2,q_2,\chi}. \tag{5.1.11}$$

While, in II, due to $|j| \le [\frac{N}{2}]$, from Hölder inequality (5.1.3) we then have

$$\|\mathrm{II}\|_{p,q,\chi} \le C\|f(t,\cdot)\|_{\Gamma,N,p_3,q_3,\chi}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_4,q_4,\chi}, \tag{5.1.12}$$

where C is a positive constant. (5.1.7) follows by combining (5.1.11) and (5.1.12).

Moreover, for $|k| = N(> 0)$, since

$$\Gamma^k(fg) - f\Gamma^k g = \sum_{|i|+|j|=N-1} C_{ij}\Gamma^i(\Gamma f)\Gamma^j g, \tag{5.1.13}$$

we can get (5.1.8) similarly. The proof is finished. □

Remark 5.1.1 Noting (5.1.5), from Lemma 5.1.2 we immediately obtain that: for any given integer $N > 0$, if the norms appearing on the right-hand sides of the following formulas are all well-defined, then we have

$$\begin{aligned}\|fg(t,\cdot)\|_{\Gamma,N,r,\chi} \le C(&\|f(t,\cdot)\|_{\Gamma,[\frac{N-1}{2}],p_1,\chi}\|\Gamma g(t,\cdot)\|_{\Gamma,N-1,q_1,\chi} \\ &+\|f(t,\cdot)\|_{\Gamma,N,p_2,\chi}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}],q_2,\chi}), \quad \forall t \ge 0\end{aligned} \tag{5.1.14}$$

and

$$\begin{aligned}\|fg(t,\cdot)\|_{\Gamma,N,r} \le C(&\|f(t,\cdot)\|_{\Gamma,[\frac{N-1}{2}],p_1}\|\Gamma g(t,\cdot)\|_{\Gamma,N-1,q_1} \\ &+\|f(t,\cdot)\|_{\Gamma,N,p_2}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}],q_2}), \quad \forall t \ge 0;\end{aligned} \tag{5.1.15}$$

moreover, for any given multi-index $k(|k| = N > 0)$, we have

$$\|(\Gamma^k(fg) - f\Gamma^k g)(t,\cdot)\|_{r,\chi} \le C(\|f(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,\chi}\|g(t,\cdot)\|_{\Gamma,N-1,q_1,\chi} + \|\Gamma f(t,\cdot)\|_{\Gamma,N-1,p_2,\chi}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}],q_2,\chi}), \quad \forall t \ge 0 \tag{5.1.16}$$

and

$$\|(\Gamma^k(fg) - f\Gamma^k g)(t,\cdot)\|_{L^r(I\!R^n)} \le C(\|f(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1}\|g(t,\cdot)\|_{\Gamma,N-1,q_1} + \|\Gamma f(t,\cdot)\|_{\Gamma,N-1,p_2}\|g(t,\cdot)\|_{\Gamma,[\frac{N-1}{2}],q_2}), \quad \forall t \ge 0, \tag{5.1.17}$$

where C is a positive constant, $1 \le p_1, q_1, p_2, q_2, r \le +\infty$, and

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}. \tag{5.1.18}$$

Remark 5.1.2 From (5.1.7) we can obtain that: for any given integer $N \ge 0$ we have

$$\|fg(t,\cdot)\|_{\Gamma,N,p,q,\chi} \le C(\|f(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1,\chi}\|g(t,\cdot)\|_{\Gamma,N,p_2,q_2,\chi} + \|f(t,\cdot)\|_{\Gamma,N,p_3,q_3,\chi}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_4,q_4,\chi}), \quad \forall t \ge 0, \tag{5.1.19}$$

where C is a positive constant.

Moreover, similarly to (5.1.8), we can prove that: for any given multi-index $k(|k| = N > 0)$ we have

$$\begin{aligned} &\|(\Gamma^k(fg) - f\Gamma^k g)(t,\cdot)\|_{p,q,\chi} \\ &\le C(\|\Gamma f(t,\cdot)\|_{\Gamma,[\frac{N-1}{2}],p_1,q_1,\chi}\|g(t,\cdot)\|_{\Gamma,N-1,p_2,q_2,\chi} \\ &\quad + \|\Gamma f(t,\cdot)\|_{\Gamma,N-1,p_3,q_3,\chi}\|g(t,\cdot)\|_{\Gamma,[\frac{N-1}{2}],p_4,q_4,\chi}), \quad \forall t \ge 0, \end{aligned} \tag{5.1.20}$$

where C is a positive constant.

Equations (5.1.19) and (5.1.20) take the forms with more symmetry. Corresponding to Remark 5.1.1, we can obtain some similar estimates.

Remark 5.1.3 Similarly to (5.1.19), using Hölder inequality (5.1.3) it can be proved that: for any given integer $N \ge 0$ we have

$$\Big\|\prod_{i=0}^{\beta} f_i(t,\cdot)\Big\|_{\Gamma,N,p,q,\chi} \le C\sum_{i=0}^{\beta}\|f_i(t,\cdot)\|_{\Gamma,N,p_{ii},q_{ii},\chi}\prod_{j\ne i}\|f_j(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_{ij},q_{ij},\chi}, \quad \forall t \ge 0, \tag{5.1.21}$$

where $1 \le p_0, q_0, p_{ij}, q_{ij} \le +\infty (i, j = 0, \cdots, \beta)$,

$$\frac{1}{p} = \sum_{j=0}^{\beta} \frac{1}{p_{ij}}, \quad \frac{1}{q} = \sum_{j=0}^{\beta} \frac{1}{q_{ij}} \quad (i = 0, \cdots, \beta), \tag{5.1.22}$$

and C is a positive constant.

Remark 5.1.4 In Lemma 5.1.2, replacing Γ by $D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, the conclusions still hold. Then, for any given integer $N > 0$ we have

$$\begin{aligned} \|fg(t,\cdot)\|_{D,N,p,q,\chi} \le C(&\|f(t,\cdot)\|_{D,[\frac{N-1}{2}],p_1,q_1,\chi}\|Dg(t,\cdot)\|_{D,N-1,p_2,q_2,\chi} \\ &+\|f(t,\cdot)\|_{D,N,p_3,q_3,\chi}\|g(t,\cdot)\|_{D,[\frac{N}{2}],p_4,q_4,\chi}), \quad \forall t \ge 0; \end{aligned} \tag{5.1.23}$$

and for any given multi-index $k(|k| = N > 0)$ we have

$$\begin{aligned} &\|(D^k(fg) - fD^k g)(t,\cdot)\|_{p,q,\chi} \\ \le\ &C(\|f(t,\cdot)\|_{D,[\frac{N}{2}],p_1,q_1,\chi}\|g(t,\cdot)\|_{D,N-1,p_2,q_2,\chi} \\ &+\|Df(t,\cdot)\|_{D,N-1,p_3,q_3,\chi}\|g(t,\cdot)\|_{D,[\frac{N-1}{2}],p_4,q_4,\chi}), \quad \forall t \ge 0, \end{aligned} \tag{5.1.24}$$

where C is a positive constant. We also have similar conclusions for Remarks 5.1.1–5.1.3.

Lemma 5.1.3 *Let $n \ge 2$. Suppose that functions $f = f(t,x)$ and $g = g(t,x)$ have compact support $\{x \mid |x| \le t + \rho\}$ with respect to variable x for any given $t \ge 0$, and the norms appearing on the right-hand sides of the following formulas are well-defined, then for $a = 0, 1, \ldots, n$ we have*

$$\|f\partial_a g(t,\cdot)\|_{L^{p,q}(\mathbb{R}^n)} \le C_\rho \|D_x f(t,\cdot)\|_{L^2(\mathbb{R}^n)} \sum_{|I|=1} \|\Gamma^I g(t,\cdot)\|_{L^{p_1,q_1}(\mathbb{R}^n)}, \quad \forall t \ge 0, \tag{5.1.25}$$

where

$$\partial_0 = -\frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i} \quad (i = 1, \ldots, n), \tag{5.1.26}$$

C_ρ is a positive constant depending on ρ, and $1 \le p, q, p_1, q_1 \le +\infty$ satisfy

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{p_1}, \quad \frac{1}{q} = \frac{1}{2} + \frac{1}{q_1}. \tag{5.1.27}$$

Proof First we prove

$$|\partial_a g(t,x)| \le C(\rho)(2\rho + t - r)^{-1} \sum_{|I|=1} |\Gamma^I g(t,x)|. \tag{5.1.28}$$

Here and hereafter, we denote by $C(\rho)$ a positive constant depending on ρ, and by C a positive constant independent of ρ.

Thanks to the compact support assumption, it suffices to give the proof in the case that $r \le t+\rho$, where $r = |x|$.

From the definition of Γ (see (3.1.18) in Chap. 3), (5.1.28) is obvious when $|t-r| \le \rho$; while, when $t - r \ge \rho$, (5.1.28) still holds due to (3.1.52) in Chap. 3. Thus, from Hölder inequality (5.1.3) we have

$$\|f\partial_a g(t,\cdot)\|_{L^{p,q}(I\!R^n)} \le C(\rho)\left\|\frac{f(t,\cdot)}{2\rho+t-r}\right\|_{L^2(I\!R^n)} \sum_{|I|=1} \|\Gamma^I g(t,\cdot)\|_{L^{p_1,q_1}(I\!R^n)}. \tag{5.1.29}$$

On the other hand, using the integration by parts and the Hölder inequality, we have

$$\begin{aligned}
\int_0^{t+\rho} \frac{f^2(t,\cdot)}{(2\rho+t-r)^2} r^{n-1} dr &= \int_0^{t+\rho} f^2(t,\cdot) r^{n-1} d\left(\frac{1}{2\rho+t-r}\right) \\
&= -\int_0^{t+\rho} \frac{f^2(t,\cdot)}{2\rho+t-r} d(r^{n-1}) - \int_0^{t+\rho} \frac{2ff_r(t,\cdot)}{2\rho+t-r} r^{n-1} dr \\
&\le \int_0^{t+\rho} \frac{2|f||f_r|(t,\cdot)}{2\rho+t-r} r^{n-1} dr \\
&\le C\left(\int_0^{t+\rho} \frac{f^2(t,\cdot)}{(2\rho+t-r)^2} r^{n-1} dr\right)^{1/2} \left(\int_0^{t+\rho} f_r^2(t,\cdot) r^{n-1} dr\right)^{1/2},
\end{aligned}$$

Then (noting (3.1.42) in Chap. 3), we get

$$\left\|\frac{f(t,\cdot)}{2\rho+t-r}\right\|_{L^2(I\!R^n)} \le C\|\partial_r f(t,\cdot)\|_{L^2(I\!R^n)} \le C\|D_x f(t,\cdot)\|_{L^2(I\!R^n)}. \tag{5.1.30}$$

Estimate (5.1.25) follows by combining (5.1.29) and (5.1.30). □

Lemma 5.1.4 *Under the assumptions of Lemma 5.1.3, for any given integer $N \ge 0$ we have*

$$\begin{aligned}
\|f\partial_a g(t,\cdot)\|_{\Gamma,N,p,q} \le C_\rho\{&\|f(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1}\|Dg(t,\cdot)\|_{\Gamma,N,2} \\
&+\|D_x f(t,\cdot)\|_{\Gamma,N,2}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+1,p_1,q_1}\}, \quad \forall t \ge 0;
\end{aligned} \tag{5.1.31}$$

and for any given multi-index $k(|k| = N > 0)$ we have

$$\begin{aligned}
\|(\Gamma^k(f\partial_a g) - f\Gamma^k\partial_a g)(t,\cdot)\|_{L^{p,q}(I\!R^n)} \le C_\rho(&\|f(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1}\|Dg(t,\cdot)\|_{\Gamma,N-1,2} \\
&+\|D_x f(t,\cdot)\|_{\Gamma,N,2}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+1,p_1,q_1}), \quad \forall t \ge 0
\end{aligned} \tag{5.1.32}$$

and

$$\|(\Gamma^k\partial_a(fg) - f\Gamma^k\partial_a g)(t,\cdot)\|_{L^{p,q}(\mathbb{R}^n)} \le C_\rho(\|Df(t,\cdot)\|_{\Gamma,N,2}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+1,p_1,q_1} + \|f(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+1,p_1,q_1}\|g(t,\cdot)\|_{\Gamma,N,2}), \quad \forall t \ge 0, \tag{5.1.33}$$

where C_ρ is a positive constant depending on ρ.

Proof For any given multi-index $k(|k| \le N)$, similarly to (5.1.10) we have

$$\Gamma^k(f\partial_a g) = \sum_{|i|+|j|\le N} C_{ij}\Gamma^i f \cdot \Gamma^j \partial_a g, \tag{5.1.34}$$

where C_{ij} are constants.

We notice that at most one of $|i|$ and $|j|$ may be greater than $[\frac{N}{2}]$:

(i) if $|i| \le [\frac{N}{2}]$, then from Hölder inequality (5.1.3) we have

$$\begin{aligned} &\|\Gamma^i f \cdot \Gamma^j \partial_a g(t,\cdot)\|_{L^{p,q}(\mathbb{R}^n)} \\ \le &\|\Gamma^i f(t,\cdot)\|_{L^{p_1,q_1}(\mathbb{R}^n)}\|\Gamma^j\partial_a g(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ \le &C\|f(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1}\|\partial_a g(t,\cdot)\|_{\Gamma,N,2}; \end{aligned} \tag{5.1.35}$$

(ii) if $|j| \le [\frac{N}{2}]$, then from Lemma 3.1.3 and Corollary 3.1.1 in Chap. 3, it is easy to show that

$$\begin{aligned} &\|\Gamma^i f \cdot \Gamma^j \partial_a g(t,\cdot)\|_{L^{p,q}(\mathbb{R}^n)} \\ \le &C \sum_{|\tilde{j}|\le|j|} \|\Gamma^i f \cdot D\Gamma^{\tilde{j}} g(t,\cdot)\|_{L^{p,q}(\mathbb{R}^n)} \\ \le &C(\rho)\|D_x\Gamma^i f(t,\cdot)\|_{L^2(\mathbb{R}^n)} \sum_{\substack{|I|=1 \\ |\tilde{j}|\le|j|}} \|\Gamma^I\Gamma^{\tilde{j}} g(t,\cdot)\|_{L^{p_1,q_1}(\mathbb{R}^n)} \\ \le &C(\rho)\|D_x f(t,\cdot)\|_{\Gamma,N,2}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+1,p_1,q_1}. \end{aligned} \tag{5.1.36}$$

Combining (5.1.35) and (5.1.36), we obtain the estimate (5.1.31) that we want.

Moreover, for any given multi-index $k(|k| = N > 0)$ we have

$$\Gamma^k(f\partial_a g) - f\Gamma^k\partial_a g = \sum_{\substack{|i|+|j|=N \\ |i|>0}} C_{ij}\Gamma^i f \cdot \Gamma^j\partial_a g, \tag{5.1.37}$$

then we can get (5.1.32) similarly.

Finally, noting Corollary 3.1.1 in Chap. 3 we have

$$\begin{aligned} & \Gamma^k \partial_a (fg) - f\Gamma^k \partial_a g \\ = \ & \Gamma^k(\partial_a f \cdot g) + \Gamma^k(f\partial_a g) - f\Gamma^k\partial_a g \\ = \ & \sum_{|i|+|j|\le N} C_{ij}\Gamma^i\partial_a f \cdot \Gamma^j g + \sum_{\substack{|i|+|j|\le N \\ |i|>0}} D_{ij}\Gamma^i f \cdot \Gamma^j \partial_a g \\ = \ & \sum_{|i|+|j|\le N} C_{ij}\Gamma^i\partial_a f \cdot \Gamma^j g + \sum_{\substack{|i|+|j|\le N \\ |i|>0, |\tilde{j}|\le|j|}} \overline{D}_{ij}\Gamma^i f \cdot D\Gamma^{\tilde{j}} g \\ \overset{\text{def}}{=} \ & \text{I} + \text{II}, \end{aligned} \tag{5.1.38}$$

where C_{ij}, D_{ij} and $\overline{D}_{ij}$ are constants.

Using Hölder inequality (5.1.3) in I, if $|i| \le \frac{N}{2}$, then we have

$$\|\Gamma^i\partial_a f \cdot \Gamma^j g(t,\cdot)\|_{L^{p,q}(I\!R^n)} \le C\|Df(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1}\|g(t,\cdot)\|_{\Gamma,N,2};$$

while, if $|j| \le \frac{N}{2}$, then we have

$$\|\Gamma^i\partial_a f \cdot \Gamma^j g(t,\cdot)\|_{L^{p,q}(I\!R^n)} \le C\|Df(t,\cdot)\|_{\Gamma,N,2}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1}.$$

Combining them we get

$$\begin{aligned} \|\text{I}\|_{L^{p,q}(I\!R^n)} \le \ & C(\|Df(t,\cdot)\|_{\Gamma,N,2}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1} \\ & + \|Df(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1}\|g(t,\cdot)\|_{\Gamma,N,2}), \quad \forall\, t \ge 0. \end{aligned} \tag{5.1.39}$$

In II, when $|i| \le [\frac{N}{2}]$, from Hölder inequality (5.1.3) we easily know that

$$\|\Gamma^i f \cdot \Gamma^j \partial_a g(t,\cdot)\|_{L^{p,q}(I\!R^n)} \le C\|f(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1}\|\partial_a g(t,\cdot)\|_{\Gamma,N-1,2};$$

while, when $|j| \le [\frac{N}{2}]$, by Lemma 5.1.3 and noting Corollary 3.1.1 in Chap. 3, we have

$$\begin{aligned} \|\Gamma^i f \cdot D\Gamma^{\tilde{j}} g(t,\cdot)\|_{L^{p,q}(I\!R^n)} & \le C(\rho)\|D_x\Gamma^i f(t,\cdot)\|_{L^2(I\!R^n)} \sum_{|I|=1} \|\Gamma^I\Gamma^{\tilde{j}} g(t,\cdot)\|_{L^{p_1,q_1}(I\!R^n)} \\ & \le C(\rho)\|Df(t,\cdot)\|_{\Gamma,N,2}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+1,p_1,q_1}. \end{aligned}$$

Combining them we obtain

$$\begin{aligned} \|\text{II}\|_{L^{p,q}(I\!R^n)} \le \ & C(\rho)(\|Df(t,\cdot)\|_{\Gamma,N,2}\|g(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+1,p_1,q_1} \\ & + \|f(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_1,q_1}\|\partial_a g(t,\cdot)\|_{\Gamma,N-1,2}), \quad \forall t \ge 0. \end{aligned} \tag{5.1.40}$$

Equation (5.1.33) can be proved by combining (5.1.39) and (5.1.40). The proof is finished. □

5.2 Some Estimates on Composite Functions

Lemma 5.2.1 *Suppose that $G = G(w)$ is a sufficiently smooth function of $w = (w_1, \ldots, w_M)$, and when*

$$|w| \le \nu_0, \tag{5.2.1}$$

we have

$$G(w) = O(|w|^{1+\beta}), \tag{5.2.2}$$

where ν_0 is a positive constant, and β is a nonnegative integer. For any given integer $N \ge 0$, if the vector function $w = w(t, x)$ satisfies

$$\|w(t, \cdot)\|_{\Gamma, [\frac{N}{2}], \infty} \le \nu_0, \quad \forall\, t \ge 0, \tag{5.2.3}$$

then for any given multi-index k $(|k| \le N)$, we have

$$|\Gamma^k G(w(t, x))| \le C(\nu_0) \sum_{\substack{|l_0| + \cdots + |l_\beta| \le |k| \\ 1 \le i_j \le M (j = 0, \cdots, \beta)}} \prod_{j=0}^{\beta} |\Gamma^{l_j} w_{i_j}(t, x)|, \tag{5.2.4}$$

where $C(\nu_0)$ is a positive constant depending on ν_0.

Proof From (5.2.2) we have

$$G(w) = \sum_{\substack{\sum_{j=1}^{M} i_j = 1+\beta \\ i_j \ge 0 (j = 1, \ldots, M)}} \widetilde{G}_{i_1 \cdots i_M}(w) w_1^{i_1} \ldots w_M^{i_M}, \tag{5.2.5}$$

simply denoted by

$$G(w) = \widetilde{G}(w) w^{1+\beta}. \tag{5.2.6}$$

In the proof below we also adopt similar simplified expressions.

From this we easily know that (5.2.4) is obvious when $|k| = 0$.

For any given multi-index $k (0 < |k| \le N)$, we have

$$\begin{aligned} \Gamma^k G(w(t, x)) &= \Gamma^k (\widetilde{G}(w) w^{1+\beta}) \\ &= \sum_{|i| + |j| = |k|} C_{ij} \Gamma^i (\widetilde{G}(w)) \Gamma^j (w^{1+\beta}), \end{aligned} \tag{5.2.7}$$

in which

$$\Gamma^i \widetilde{G}(w) = \sum_{\substack{\sum_{j=1}^{M} \gamma_j = \rho \\ 1 \le \rho \le |i|}} \frac{\partial^\rho \widetilde{G}(w)}{\partial^{\gamma_1} w_1 \dots \partial^{\gamma_M} w_M} (\Gamma w)^{\alpha_1} \dots (\Gamma^{|i|} w)^{\alpha_{|i|}}, \tag{5.2.8}$$

where

$$|\alpha_1| + \cdots + |\alpha_{|i|}| = \rho \tag{5.2.9}$$

and

$$1 \cdot |\alpha_1| + \cdots + |i| \cdot |\alpha_{|i|}| = |i|. \tag{5.2.10}$$

In (5.2.7), if $|i| \le [\frac{|k|}{2}]$, noting (5.2.3), from (5.2.8) we then have

$$|\Gamma^i \widetilde{G}(w(t,x))| \le C_{\nu_0}, \tag{5.2.11}$$

where C_{ν_0} is a positive constant depending on ν_0, then it is easy to show that (5.2.4) is satisfied for the corresponding part (denoted by I) of the sum in (5.2.7).

In (5.2.7), if $|i| \ge [\frac{|k|}{2}] + 1$, from (5.2.10) we easily know that either $|\alpha_{[\frac{|i|}{2}]+1}|, \dots,$ $|\alpha_{|i|}|$ are all zeros; or only one of them is 1 and others are all zeros. Therefore, except at most one factor $\Gamma^{|h|} w$($|h|$ is a certain number among $[\frac{|i|}{2}] + 1, \dots, |i|$), each term in the sum on the right-hand side of (5.2.8) can be estimated by (5.2.3). On the other hand, now since it is easy to know that $|j| \le [\frac{|k|}{2}]$, each term in $\Gamma^j(w^{1+\beta})$ can also be estimated by (5.2.3). Thus, it is clear that (5.2.4) is also satisfied for this part (denoted by II) of the sum in (5.2.7).

This proves (5.2.4). □

Lemma 5.2.2 *Under the assumptions of Lemma 5.2.1, for any given integer $N \ge 0$, when $\beta = 0$, we have*

$$\|G(w(t,\cdot))\|_{\Gamma,N,p,q,\chi} \le C(\nu_0) \|w(t,\cdot)\|_{\Gamma,N,p,q,\chi}, \quad \forall t \ge 0; \tag{5.2.12}$$

while, when $\beta \ge 1$, we have

$$\|G(w(t,\cdot))\|_{\Gamma,N,p,q,\chi} \le C(\nu_0) \left(\prod_{i=1}^{\beta} \|w(t,\cdot)\|_{\Gamma,[\frac{N}{2}],p_i,q_i,\chi} \right) \|w(t,\cdot)\|_{\Gamma,N,p_0,q_0,\chi}, \quad \forall t \ge 0, \tag{5.2.13}$$

where $1 \le p, q, p_i, q_i \le +\infty (i = 0, 1, \dots, \beta)$,

$$\frac{1}{p} = \sum_{i=0}^{\beta} \frac{1}{p_i}, \quad \frac{1}{q} = \sum_{i=0}^{\beta} \frac{1}{q_i}, \tag{5.2.14}$$

$\chi(t,x)$ is the characteristic function of any given set in $R_+ \times R^n$, and $C(\nu_0)$ is a positive constant depending on ν_0.

Proof Estimate (5.2.12) is an obvious consequence of (5.2.4) when $\beta = 0$. While, when $\beta \geq 1$, using (5.1.21), (5.2.13) follows from (5.2.4). The proof is finished. □

Remark 5.2.1 In Lemmas 5.2.1 and 5.2.2, if we replace Γ by $D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, then the conclusions still hold.

In addition, it is not difficult to prove that: when $\beta > 0$, for any given integer $N > 0$ we have

$$\|DG(w)(t,\cdot)\|_{D,N-1,p} \leq C\|w(t,\cdot)\|^{\beta}_{D,[\frac{N}{2}],\infty}\|Dw(t,\cdot)\|_{D,N-1,p}, \quad \forall t \geq 0, \tag{5.2.15}$$

where $1 \leq p \leq +\infty$, and C is a positive constant.

Lemma 5.2.3 *Under the assumptions of Lemma 5.2.1, for any given integer $N \geq [\frac{n}{2}]+1$, if the vector function $w = w(t,x)$ satisfies (5.2.3), then we have*

$$\|DG(w)(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n-1}{2}(1+\beta)}\|w(t,\cdot)\|^{\beta}_{\Gamma,N,2}\|Dw(t,\cdot)\|_{\Gamma,N,2}, \quad \forall t \geq 0, \tag{5.2.16}$$

where $D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, and C is a positive constant.

Proof Adopting the simplified notations, we have

$$DG(w) = G'(w)Dw. \tag{5.2.17}$$

By (5.2.2) we can write

$$G'(w) = \widetilde{G}(w)w^{\beta}, \tag{5.2.18}$$

and $\widetilde{G}(w)$ is a sufficiently smooth function on $|w| \leq \nu_0$. From (5.2.3) we have

$$\|\widetilde{G}(w)(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C. \tag{5.2.19}$$

Therefore, noting that $N \geq [\frac{n}{2}]+1$, from estimate (3.4.29) (in which we take $p = 2, s = [\frac{n}{2}]+1$) with decay factor in Chap. 3, we have

$$\begin{aligned}
&\|DG(w(t,\cdot))\|_{L^\infty(\mathbb{R}^n)} \\
\leq\, & C\|w(t,\cdot)\|^{\beta}_{L^\infty(\mathbb{R}^n)}\|Dw(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \\
\leq\, & C(1+t)^{-\frac{n-1}{2}(1+\beta)}\|w(t,\cdot)\|^{\beta}_{\Gamma,[\frac{n}{2}]+1,2}\|Dw(t,\cdot)\|_{\Gamma,[\frac{n}{2}]+1,2} \\
\leq\, & C(1+t)^{-\frac{n-1}{2}(1+\beta)}\|w(t,\cdot)\|^{\beta}_{\Gamma,N,2}\|Dw(t,\cdot)\|_{\Gamma,N,2}, \quad \forall t \geq 0.
\end{aligned}$$

This is exactly (5.2.16). □

Lemma 5.2.4 (Interpolation inequality) *Suppose that* $f \in L^{p_1,q_1}(I\!R^n) \bigcap L^{p_2,q_2}(I\!R^n)$, *and* $1 \le p_1, p_2, q_1, q_2 \le +\infty$. *Then* $f \in L^{p,q}(I\!R^n)$, *where*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \tag{5.2.20}$$

and θ *is any given constant satisfying* $0 \le \theta \le 1$; *Moreover, we have the following interpolation inequality:*

$$\|f\|_{L^{p,q}(I\!R^n)} \le \|f\|^{\theta}_{L^{p_1,q_1}(I\!R^n)} \|f\|^{1-\theta}_{L^{p_2,q_2}(I\!R^n)}. \tag{5.2.21}$$

Proof Similarly to the proof of Lemma 3.4.1 in Chap. 3, using Hölder inequality and noting the definition of the norm in $L^{p,q}(I\!R^n)$ space (see (3.1.29) in Chap. 3), we have

$$\begin{aligned} \|f\|_{L^{p,q}(I\!R^n)} &\le \||f|^{\theta}\|_{L^{\frac{p_1}{\theta},\frac{q_1}{\theta}}(I\!R^n)} \||f|^{1-\theta}\|_{L^{\frac{p_2}{1-\theta},\frac{q_2}{1-\theta}}(I\!R^n)} \\ &= \|f\|^{\theta}_{L^{p_1,q_1}(I\!R^n)} \|f\|^{1-\theta}_{L^{p_2,q_2}(I\!R^n)}. \end{aligned}$$

This proves (5.2.21). □

Lemma 5.2.5 *Let* $n \ge 2$. *Suppose that* $G = G(w)$ *is a sufficiently smooth function of* $w = (w_1, \ldots, w_M)$ *with*

$$G(0) = 0. \tag{5.2.22}$$

For any given integer $N \ge n+2$ *and any given real number* $r (1 \le r \le 2)$, *if the vector function* $w = w(t,x) = (w_1, \ldots, w_M)(t,x)$ *satisfies (5.2.3) and the norms appearing on the right-hand sides of the following formulas are well-defined, then we have*

$$\Big\|G(w)\prod_{i=1}^{\beta} u_i(t,\cdot)\Big\|_{\Gamma,N,r,2} \le C(1+t)^{-\frac{n-1}{2}(1-\frac{2}{\beta p})\beta} \|w(t,\cdot)\|_{\Gamma,N,2} \prod_{i=1}^{\beta} \|u_i(t,\cdot)\|_{\Gamma,N,2}, \quad \forall t \ge 0 \tag{5.2.23}$$

and

$$\Big\|G(w)\prod_{i=1}^{\beta} u_i(t,\cdot)\Big\|_{\Gamma,N,r,2,\chi_1} \le C(1+t)^{-\frac{n}{2}(1-\frac{2}{\beta p})\beta} \|w(t,\cdot)\|_{\Gamma,N,2} \prod_{i=1}^{\beta} \|u_i(t,\cdot)\|_{\Gamma,N,2}, \quad \forall t \ge 0, \tag{5.2.24}$$

where $\chi_1(t,x)$ *is the characteristic function of the set*

$$\left\{(t,x) \,\middle|\, |x| \le \frac{1+t}{2}, t \ge 0\right\}, \tag{5.2.25}$$

$\beta \geq 1$ is an integer,

$$\frac{1}{p} = \frac{1}{r} - \frac{1}{2}, \tag{5.2.26}$$

and C is a positive constant.

Proof For any given multi-index $k(|k| \leq N)$, by the chain rule we have

$$\Gamma^k \left(G(w) \prod_{i=1}^{\beta} u_i(t,\cdot) \right) = \sum_{\sum_{i=0}^{\beta} |k_i| = |k|} C_{k_0 k_1 \cdots k_\beta} \Gamma^{k_0} G(w(t,\cdot)) \prod_{i=1}^{\beta} \Gamma^{k_i} u_i(t,\cdot), \tag{5.2.27}$$

where $C_{k_0 k_1 \dots k_\beta}$ are constants, and $k_0, \dots, k_\beta$ are multi-indexes.

Noting that at most one of $|k_0|, \dots, |k_\beta|$ may be greater than $[\frac{N}{2}]$, we have

(i) if $|k_j| > [\frac{N}{2}]$ for a certain $j (1 \leq j \leq \beta)$, using Hölder inequality (5.1.3), the embedding theorem (see (3.2.3) in Chap. 3) on the sphere S^{n-1} and the estimate (3.4.29) (in which we take $p = 2, s = [\frac{n}{2}] + 1$) with decay factor in Chap. 3, and noting that $[\frac{N}{2}] + [\frac{n}{2}] + 1 \leq N$ when $N \geq n + 2$, we then have

$$\begin{aligned}
&\left\| \Gamma^{k_0} G(w) \prod_{i=1}^{\beta} \Gamma^{k_i} u_i(t,\cdot) \right\|_{L^{1,2}(I\!R^n)} \\
&\leq C \|\Gamma^{k_0} G(w(t,\cdot))\|_{L^{2,\infty}(I\!R^n)} \left(\prod_{\substack{i=1 \\ i \neq j}}^{\beta} \|\Gamma^{k_i} u_i(t,\cdot)\|_{L^\infty(I\!R^n)} \right) \|\Gamma^{k_j} u_j(t,\cdot)\|_{L^2(I\!R^n)} \\
&\leq C \|G(w(t,\cdot))\|_{\Gamma, |k_0| + [\frac{n-1}{2}] + 1, 2} \left(\prod_{\substack{i=1 \\ i \neq j}}^{\beta} (1+t)^{-\frac{n-1}{2}} \|\Gamma^{k_i} u_i(t,\cdot)\|_{\Gamma, [\frac{n}{2}]+1, 2} \right) \|u_j(t,\cdot)\|_{\Gamma, |k_j|, 2} \\
&\leq C(1+t)^{-\frac{n-1}{2}(\beta-1)} \|w(t,\cdot)\|_{\Gamma, N, 2} \left(\prod_{\substack{i=1 \\ i \neq j}}^{\beta} \|u_i(t,\cdot)\|_{\Gamma, N, 2} \right) \|u_j(t,\cdot)\|_{\Gamma, |k_j|, 2}
\end{aligned} \tag{5.2.28}$$

and

$$\begin{aligned}
&\left\| \Gamma^{k_0} G(w) \prod_{i=1}^{\beta} \Gamma^{k_i} u_i(t,\cdot) \right\|_{L^2(I\!R^n)} \\
&\leq C \|\Gamma^{k_0} G(w(t,\cdot))\|_{L^\infty(I\!R^n)} \left(\prod_{\substack{i=1 \\ i \neq j}}^{\beta} \|\Gamma^{k_i} u_i(t,\cdot)\|_{L^\infty(I\!R^n)} \right) \|\Gamma^{k_j} u_j(t,\cdot)\|_{L^2(I\!R^n)} \\
&\leq C(1+t)^{-\frac{n-1}{2}\beta} \|w(t,\cdot)\|_{\Gamma, N, 2} \left(\prod_{\substack{i=1 \\ i \neq j}}^{\beta} \|u_i(t,\cdot)\|_{\Gamma, N, 2} \right) \|u_j(t,\cdot)\|_{\Gamma, |k_j|, 2}.
\end{aligned} \tag{5.2.29}$$

Then, using the interpolation inequality (5.2.21) (in which we take $p = r, q = 2, p_1 = 1, q_1 = p_2 = q_2 = 2$, and thus $\theta = \frac{2}{r} - 1$), we have

$$\begin{aligned}
&\left\| \Gamma^{k_0} G(w) \prod_{i=1}^{\beta} \Gamma^{k_i} u_i(t,\cdot) \right\|_{L^{r,2}(I\!R^n)} \\
&\le \| \cdot \|_{L^{1,2}(I\!R^n)}^{\frac{2}{r}-1} \| \cdot \|_{L^2(I\!R^n)}^{2-\frac{2}{r}} \\
&\le C(1+t)^{-\frac{n-1}{2}(1-\frac{2}{\beta p})\beta} \|w(t,\cdot)\|_{\Gamma,N,2} \left(\prod_{\substack{i=1 \\ i \ne j}}^{\beta} \|u_i(t,\cdot)\|_{\Gamma,N,2} \right) \|u_j(t,\cdot)\|_{\Gamma,|k_j|,2}, \quad \forall 1 \le r \le 2,
\end{aligned} \tag{5.2.30}$$

where p is defined by (5.2.26).

(ii) if $|k_j| \le [\frac{N}{2}]$ for all $j (1 \le j \le \beta)$, similarly we have

$$\begin{aligned}
&\left\| \Gamma^{k_0} G(w) \prod_{i=1}^{\beta} \Gamma^{k_i} u_i(t,\cdot) \right\|_{L^{r,2}(I\!R^n)} \\
&\le C(1+t)^{-\frac{n-1}{2}(1-\frac{2}{\beta p})\beta} \|w(t,\cdot)\|_{\Gamma,|k_0|,2} \prod_{i=1}^{\beta} \|u_i(t,\cdot)\|_{\Gamma,N,2}, \quad \forall 1 \le r \le 2.
\end{aligned} \tag{5.2.31}$$

Noting that $|k_i| \le N$ $(i = 0, 1, \ldots, \beta)$, from (5.2.30)–(5.2.31) we immediately get (5.2.23).

Using estimate (3.4.12) with decay factor in Chap. 3, we can similarly prove (5.2.24). □

Lemma 5.2.6 *Under the assumptions of Lemma 5.2.5, for any given multi-index k $(|k| \le N)$ we have*

$$\begin{aligned}
&\left\| \Gamma^k \left(G(w) \prod_{i=1}^{\beta} u_i(t,\cdot) \right) - G(w) \left(\prod_{i=1}^{\beta-1} u_i \right) \Gamma^k u_\beta(t,\cdot) \right\|_{L^{r,2}(I\!R^n)} \\
&\le C(1+t)^{-\frac{n-1}{2}(1-\frac{2}{\beta p})\beta} \|w(t,\cdot)\|_{\Gamma,N,2} \left(\prod_{i=1}^{\beta-1} \|u_i(t,\cdot)\|_{\Gamma,N,2} \right) \|u_\beta(t,\cdot)\|_{\Gamma,N-1,2}, \quad \forall t \ge 0
\end{aligned} \tag{5.2.32}$$

and

$$\left\| \Gamma^k \left(G(w) \prod_{i=1}^{\beta} u_i(t,\cdot) \right) - G(w) \left(\prod_{i=1}^{\beta-1} u_i \right) \Gamma^k u_\beta(t,\cdot) \right\|_{r,2,\chi_1}$$

$$\leq C(1+t)^{-\frac{n}{2}(1-\frac{2}{\beta p})\beta}\|w(t,\cdot)\|_{\Gamma,N,2}\left(\prod_{i=1}^{\beta-1}\|u_i(t,\cdot)\|_{\Gamma,N,2}\right)\|u_\beta(t,\cdot)\|_{\Gamma,N-1,2},\quad \forall\, t\geq 0, \tag{5.2.33}$$

where C is a positive constant.

Proof Since

$$\Gamma^k\left(G(w)\prod_{i=1}^{\beta}u_i(t,\cdot)\right)-G(w)\left(\prod_{i=1}^{\beta-1}u_i\right)\Gamma^k u_\beta(t,\cdot)$$
$$=\sum_{\substack{\sum_{i=0}^{\beta}|k_i|=|k|\\ |k_\beta|<|k|}}C_{k_0k_1\cdots k_\beta}\Gamma^{k_0}G(w(t,\cdot))\prod_{i=1}^{\beta}\Gamma^{k_i}u_i(t,\cdot), \tag{5.2.34}$$

repeating the proof of Lemma 5.2.5, and noticing $|k_\beta|\leq N-1$, we can obtain our conclusions. □

Lemma 5.2.7 *Suppose that function $G=G(w)$ satisfies the conditions stated in Lemma 5.2.1. For any given integer $N\geq n+2$, if the vector functions $\overline{w}(t,x)=(\overline{w}_1(t,x),\ldots,\overline{w}_M(t,x))$ and $\overline{\overline{w}}(t,x)=(\overline{\overline{w}}_1(t,x),\ldots,\overline{\overline{w}}_M(t,x))$ both satisfy (5.2.3) and the norms appearing on the right-hand sides of the following formulas are all well-defined, then for any given real number $r\,(1\leq r\leq 2)$ we have*

$$\begin{aligned}&\|(G(\overline{w})-G(\overline{\overline{w}}))u(t,\cdot)\|_{\Gamma,N,r,2}\\ \leq\ & C(1+t)^{-\frac{n-1}{2}(1-\frac{2}{\alpha p})\alpha}(1+\|\widetilde{w}(t,\cdot)\|_{\Gamma,N,2})\|\widetilde{w}(t,\cdot)\|_{\Gamma,N,2}^{\beta}\\ &\cdot\|w^*(t,\cdot)\|_{\Gamma,N,2}\|u(t,\cdot)\|_{\Gamma,N,2},\quad \forall t\geq 0\end{aligned} \tag{5.2.35}$$

and

$$\begin{aligned}&\|(G(\overline{w})-G(\overline{\overline{w}}))u(t,\cdot)\|_{\Gamma,N,r,2,\chi_1}\\ \leq\ & C(1+t)^{-\frac{n}{2}(1-\frac{2}{\alpha p})\alpha}(1+\|\widetilde{w}(t,\cdot)\|_{\Gamma,N,2})\|\widetilde{w}(t,\cdot)\|_{\Gamma,N,2}^{\beta}\\ &\cdot\|w^*(t,\cdot)\|_{\Gamma,N,2}\|u(t,\cdot)\|_{\Gamma,N,2},\quad \forall\, t\geq 0,\end{aligned} \tag{5.2.36}$$

where $\alpha=1+\beta$, p satisfies (5.2.26), $\chi_1(t,x)$ is the characteristic function of the set (5.2.25),

$$w^*=\overline{w}-\overline{\overline{w}}, \tag{5.2.37}$$

and

$$\|\widetilde{w}(t,\cdot)\|_{\Gamma,N,2}=\|\overline{w}(t,\cdot)\|_{\Gamma,N,2}+\|\overline{\overline{w}}(t,\cdot)\|_{\Gamma,N,2}. \tag{5.2.38}$$

Proof Adopting the simplified notations, we have

$$G(\overline{w}) - G(\overline{\overline{w}}) = \widehat{G}(\overline{w}, \overline{\overline{w}})w^*, \tag{5.2.39}$$

where $\widehat{G}(\overline{w}, \overline{\overline{w}})$ is a sufficiently smooth function, and when

$$|\overline{w}|, |\overline{\overline{w}}| \le \nu_0, \tag{5.2.40}$$

we have

$$\widehat{G}(\overline{w}, \overline{\overline{w}}) = O(|\overline{w}|^\beta + |\overline{\overline{w}}|^\beta). \tag{5.2.41}$$

Therefore, if $\beta \ge 1$, from Lemma 5.2.5 it is clear that

$$\begin{aligned} &\|(G(\overline{w}) - G(\overline{\overline{w}}))u(t,\cdot)\|_{\Gamma,N,r,2} \\ &\le C(1+t)^{-\frac{n-1}{2}(1-\frac{2}{\alpha p})\alpha} \|\widetilde{w}(t,\cdot)\|^\beta_{\Gamma,N,2} \|\widetilde{w}(t,\cdot)\|_{\Gamma,N,2} \|u(t,\cdot)\|_{\Gamma,N,2} \end{aligned} \tag{5.2.42}$$

and

$$\begin{aligned} &\|(G(\overline{w}) - G(\overline{\overline{w}}))u(t,\cdot)\|_{\Gamma,N,r,2,\chi_1} \\ &\le C(1+t)^{-\frac{n}{2}(1-\frac{2}{\alpha p})\alpha} \|\widetilde{w}(t,\cdot)\|^\beta_{\Gamma,N,2} \|w^*(t,\cdot)\|_{\Gamma,N,2} \|u(t,\cdot)\|_{\Gamma,N,2}, \end{aligned} \tag{5.2.43}$$

then (5.2.35) and (5.2.36) hold; while, if $\beta = 0$, rewriting (5.2.39) as

$$G(\overline{w}) - G(\overline{\overline{w}}) = (\widehat{G}(\overline{w}, \overline{\overline{w}}) - \widehat{G}(0,0))w^* + \widehat{G}(0,0)w^*, \tag{5.2.44}$$

and noting that $\left(1 - \frac{2}{\alpha p}\right)\alpha = \alpha - \frac{2}{p}$, it is easy to show that (5.2.35) and (5.2.36) still hold. The proof is finished. □

Remark 5.2.2 The factor $(1 + \|\tilde{w}(t,x)\|_{\Gamma,N,2})$ on the right-hand sides of (5.2.35) and (5.2.36) appears only when $\beta = 0$.

Similarly to Lemma 5.2.7 we can obtain

Lemma 5.2.8 *Suppose that function $G = G(w)$ satisfies the conditions stated in Lemma 5.2.1, and $\beta \ge 1$ is an integer. For any given integer $N \ge n+2$, if the vector functions $\overline{w} = \overline{w}(t,x)$ and $\overline{\overline{w}} = \overline{\overline{w}}(t,x)$ both satisfy the conditions in Lemma 5.2.7 and the norms appearing on the right-hand sides of the following formulas are all well-defined, then for any given real number $r (1 \le r \le 2)$ we have*

$$\begin{aligned} &\|(G(\overline{w}) - G(\overline{\overline{w}}))(t,\cdot)\|_{\Gamma,N,r,2} \\ &\le C(1+t)^{-\frac{n-1}{2}(1-\frac{2}{\beta p})\beta} \|\widetilde{w}(t,\cdot)\|^\beta_{\Gamma,N,2} \|w^*(t,\cdot)\|_{\Gamma,N,2} \end{aligned} \tag{5.2.45}$$

and

$$\begin{aligned} &\|(G(\overline{w}) - G(\overline{\overline{w}}))(t,\cdot)\|_{\Gamma,N,r,2,\chi_1} \\ &\le C(1+t)^{-\frac{n}{2}(1-\frac{2}{\beta p})\beta} \|\widetilde{w}(t,\cdot)\|^\beta_{\Gamma,N,2} \|w^*(t,\cdot)\|_{\Gamma,N,2}, \end{aligned} \tag{5.2.46}$$

where p is defined by (5.2.26), and $\chi_1(t,x)$ is the characteristic function of the set (5.2.25).

Remark 5.2.3 Since $1 \le r \le 2$, we have

$$\|f\|_{L^r(I\!R^n)} \le C\|f\|_{L^{r,2}(I\!R^n)}, \tag{5.2.47}$$

where C is a positive constant. Therefore, replacing the space $L^{r,2}(I\!R^n)$ on the left-hand sides of the estimates in Lemmas 5.2.5–5.2.8 by $L^r(I\!R^n)$, the conclusions still hold.

5.3 Appendix—A Supplement About the Estimates on Product Functions

In this section, for the need of Chap. 6, we will prove the estimates on product functions as follows.

Lemma 5.3.1 *Suppose that*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 1 \le p, q, r \le +\infty. \tag{5.3.1}$$

For any given integer $s \ge 1$, if the norms appearing on the right-hand sides of the following formulas are all well-defined, then we have

$$\|D^s(fg)\|_{L^r(I\!R^n)} \le C(\|f\|_{L^p(I\!R^n)}\|D^s g\|_{L^q(I\!R^n)} + \|D^s f\|_{L^q(I\!R^n)}\|g\|_{L^p(I\!R^n)}) \tag{5.3.2}$$

and

$$\|D^s(fg) - fD^s g\|_{L^r(I\!R^n)} \le C\left(\|Df\|_{L^p(I\!R^n)}\|D^{s-1}g\|_{L^q(I\!R^n)} + \|D^s f\|_{L^q(I\!R^n)}\|g\|_{L^p(I\!R^n)}\right), \tag{5.3.3}$$

where $D = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$, C is a positive constant, and $D^s f$ stands for the set consisted of all the s-order partial derivatives of f.

To prove Lemma 5.3.1, we first give

Lemma 5.3.2 (Nirenberg inequality) *Suppose that $f \in L^p(I\!R^n)$, $D^s f \in L^q(I\!R^n)$, where $s \ge 1$ is an integer, and $1 \le p, q \le +\infty$. Then, for any given integer i satisfying $0 \le i \le s$ we have*

$$\|D^i f\|_{L^r(I\!R^n)} \le C\|f\|^{1-\frac{i}{s}}_{L^p(I\!R^n)}\|D^s f\|^{\frac{i}{s}}_{L^q(I\!R^n)}, \tag{5.3.4}$$

where

$$\frac{1}{r} = \left(1 - \frac{i}{s}\right)\frac{1}{p} + \frac{i}{s}\frac{1}{q}, \tag{5.3.5}$$

and C is a positive constant.

The proof of Lemma 5.3.2 can be found in Nirenberg (1959).
Now we prove Lemma 5.3.1.
For any given integer $s \geq 1$, it is obvious that

$$D^s(fg) = \sum_{\substack{i+j=s \\ i,j\geq 0}} C_{ij} D^i f \cdot D^j g, \tag{5.3.6}$$

where C_{ij} are constants. Using Hölder inequality (5.1.3), we have

$$\|D^s(fg)\|_{L^r(I\!R^n)} \leq C \sum_{i+j=s} \|D^i f\|_{L^{r_1}(I\!R^n)} \|D^j g\|_{L^{r_2}(I\!R^n)}, \tag{5.3.7}$$

where $1 \leq r_1, r_2 \leq +\infty$, and

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}. \tag{5.3.8}$$

Particularly taking r_1 and r_2 such that

$$\frac{1}{r_1} = \left(1 - \frac{i}{s}\right)\frac{1}{p} + \frac{i}{s}\frac{1}{q} \tag{5.3.9}$$

and

$$\frac{1}{r_2} = \left(1 - \frac{j}{s}\right)\frac{1}{p} + \frac{j}{s}\frac{1}{q}, \tag{5.3.10}$$

respectively, from Nirenberg inequality (5.3.4) we have

$$\|D^i f\|_{L^{r_1}(I\!R^n)} \leq C\|f\|_{L^p(I\!R^n)}^{1-\frac{i}{s}} \|D^s f\|_{L^q(I\!R^n)}^{\frac{i}{s}} \tag{5.3.11}$$

and

$$\|D^j g\|_{L^{r_2}(I\!R^n)} \leq C\|g\|_{L^p(I\!R^n)}^{1-\frac{j}{s}} \|D^s g\|_{L^q(I\!R^n)}^{\frac{j}{s}}. \tag{5.3.12}$$

Plugging (5.3.11)–(5.3.12) in (5.3.7), and noting that

$$\frac{i}{s} + \frac{j}{s} = 1, \tag{5.3.13}$$

we obtain

$$\|D^s(fg)\|_{L^r(I\!R^n)} \leq C \sum_{i+j=s} (\|D^s f\|_{L^q(I\!R^n)}\|g\|_{L^p(I\!R^n)})^{\frac{i}{s}} (\|f\|_{L^p(I\!R^n)}\|D^s g\|_{L^q(I\!R^n)})^{\frac{j}{s}}. \tag{5.3.14}$$

Then, using the inequality

$$ab \leq \frac{1}{\bar{p}} a^{\bar{p}} + \frac{1}{\bar{q}} b^{\bar{q}}, \tag{5.3.15}$$

where $a, b \geq 0$, $\frac{1}{\bar{p}} + \frac{1}{\bar{q}} = 1$, and $1 \leq \bar{p}, \bar{q} \leq +\infty$, and taking particularly $\bar{p} = \frac{s}{i}$, $\bar{q} = \frac{s}{j}$ in (5.3.13), we get (5.3.2) from (5.3.14).

As to (5.3.3), as long as we notice that

$$D^s(fg) - fD^s g = \sum_{\substack{i+j=s-1\\ i,j\geq 0}} C_{ij} D^i(Df) D^j g, \tag{5.3.16}$$

it can be proved similarly.

Chapter 6
Cauchy Problem of the Second-Order Linear Hyperbolic Equations

6.1 Introduction

In order to solve the Cauchy problem of nonlinear wave equations later (see Chap. 7), we will consider in this chapter the following Cauchy problem of n-dimensional linear hyperbolic equations:

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_i x_j} - 2\sum_{j=1}^{n} a_{0j}(t,x)u_{tx_j} = F(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{6.1.1}$$

$$t = 0: u = f(x),\, u_t = g(x), \quad x \in \mathbb{R}^n, \tag{6.1.2}$$

and prove the existence, uniqueness and regularity of solutions (see Li and Chen 1989, 1992). Here we assume that for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$,

$$a_{ij}(t,x) = a_{ji}(t,x) \quad (i,j = 1,\ldots,n) \tag{6.1.3}$$

and

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geq m_0|\xi|^2, \quad \forall \xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n, \tag{6.1.4}$$

where $m_0 > 0$ is a constant.

Remark 6.1.1 Under assumptions (6.1.3)–(6.1.4), Eq. (6.1.1) is a second-order linear hyperbolic equation.

To explain this, we only need to notice that for any given $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, denoting $a_{ij} = a_{ij}(t,x)$ and $a_{0j} = a_{0j}(t,x)$, the corresponding characteristic quadratic form

$$\lambda_0^2 - 2\sum_{j=1}^{n} a_{0j}\lambda_0\lambda_j - \sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_j \tag{6.1.5}$$

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can be written to the sum of squares with one positive coefficient and n negative ones. Due to (6.1.3)–(6.1.4), (a_{ij}) $(i, j=1, \dots, n)$ is a symmetric positively definite matrix, and through an orthogonal transformation we can first change $(\lambda_1, \dots, \lambda_n)$ into $(\bar{\lambda}_1, \dots, \bar{\lambda}_n)$, such that the quadratic form (6.1.5) can be reduced to

$$\lambda_0^2 - 2\sum_{j=1}^{n} \bar{a}_{0j}\lambda_0\bar{\lambda}_j - \sum_{j=1}^{n} \bar{a}_{jj}\bar{\lambda}_j^2, \tag{6.1.6}$$

where

$$\bar{a}_{jj} \geq m_0 > 0 \quad (j = 1, \dots, n). \tag{6.1.7}$$

Then we can write the quadratic form (6.1.6) as

$$\left(1 + \sum_{j=1}^{n} \frac{\bar{a}_{0j}^2}{\bar{a}_{jj}}\right)\lambda_0^2 - \sum_{j=1}^{n} \bar{a}_{jj}\left(\bar{\lambda}_j + \frac{\bar{a}_{0j}}{\bar{a}_{jj}}\lambda_0\right)^2, \tag{6.1.8}$$

which is exactly the required form.

6.2 Existence and Uniqueness of Solutions

We will use the Galerkin method to prove the following

Lemma 6.2.1 *For any given positive number $T > 0$, if we assume that*

$$f \in H^{s+1}(I\!\!R^n), \;\; g \in H^s(I\!\!R^n), \tag{6.2.1}$$

$$a_{ij} \in L^\infty((0, T) \times I\!\!R^n), \tag{6.2.2}$$

$$\frac{\partial a_{ij}}{\partial t}, \frac{\partial a_{ij}}{\partial x_k} \in L^\infty(0, T; H^{s-1}(I\!\!R^n)) \;\; (i, j, k = 1, \dots, n), \tag{6.2.3}$$

$$a_{0j} \in L^\infty(0, T; H^s(I\!\!R^n)) \;\; (j = 1, \dots, n) \tag{6.2.4}$$

and

$$F \in L^2(0, T; H^s(I\!\!R^n)), \tag{6.2.5}$$

where $s \geq [\frac{n}{2}] + 2$ is an integer, then Cauchy problem (6.1.1)–(6.1.2) admits a unique solution $u = u(t, x)$ satisfying

$$u \in L^\infty(0, T; H^{s+1}(I\!\!R^n)), \tag{6.2.6}$$

$$u_t \in L^\infty(0, T; H^s(I\!\!R^n)), \tag{6.2.7}$$

$$u_{tt} \in L^2(0, T; H^{s-1}(I\!\!R^n)), \tag{6.2.8}$$

and we have the following estimates:

$$\|u(t,\cdot)\|^2_{H^{s+1}(I\!R^n)} + \|u_t(t,\cdot)\|^2_{H^s(I\!R^n)}$$
$$\le C_0(T)\left(\|f\|^2_{H^{s+1}(I\!R^n)} + \|g\|^2_{H^s(I\!R^n)} + \int_0^t \|F(\tau,\cdot)\|^2_{H^s(I\!R^n)} d\tau\right),$$
$$\forall t \in [0,T], \qquad (6.2.9)$$

where $C_0(T)$ is a positive constant depending on T and the norms of a_{ij} and $a_{0j}(i,j=1,\ldots,n)$ in the spaces shown in (6.2.2)–(6.2.4).

Proof Let $\{w_h\}(h=1,2,\ldots)$ be a basis in the space $H^{s+1}(I\!R^n)$. For any given $m \in N$, find an approximate solution

$$u_m(t) = \sum_{l=1}^m g_{lm}(t) w_l \qquad (6.2.10)$$

to Cauchy problem (6.1.1)–(6.1.2) such that

$$(u''_m(t), w_h)_{H^s(I\!R^n)} - 2\sum_{j=1}^n \left(a_{0j}(t,x)\frac{\partial u'_m(t)}{\partial x_j}, w_h\right)_{H^s(I\!R^n)}$$
$$- \sum_{i,j=1}^n \left\langle a_{ij}(t,x)\frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}, w_h\right\rangle_{H^{s-1}(I\!R^n), H^{s+1}(I\!R^n)}$$
$$= (F(t), w_h)_{H^s(I\!R^n)} \ (1 \le h \le m), \ \forall t \in [0,T] \qquad (6.2.11)$$

and

$$u_m(0) = u_{0m} \overset{\text{def.}}{=} \sum_{l=1}^m \xi_{lm} w_l, \qquad (6.2.12)$$

$$u'_m(0) = u_{1m} \overset{\text{def.}}{=} \sum_{l=1}^m \eta_{lm} w_l, \qquad (6.2.13)$$

and assume that as $m \to \infty$,

$$u_{0m} \to f \text{ strongly in } H^{s+1}(I\!R^n), \qquad (6.2.14)$$
$$u_{1m} \to g \text{ strongly in } H^s(I\!R^n). \qquad (6.2.15)$$

In (6.2.11), $\langle\cdot,\cdot\rangle_{H^{s-1}(I\!R^n), H^{s+1}(I\!R^n)}$ stands for the dual inner product between the spaces $H^{s-1}(I\!R^n)$ and $H^{s+1}(I\!R^n)$, and $(\cdot,\cdot)_{H^s(I\!R^n)}$ the inner product in the space $H^s(I\!R^n)$.

By (6.2.10), (6.2.11)–(6.2.13) can be rewritten as

$$\sum_{l=1}^{m} g''_{lm}(t)(w_l, w_h)_{H^s(I\!R^n)} - \sum_{l=1}^{m} g_{lm}(t) \sum_{i,j=1}^{n} \left\langle a_{ij}(t,x)\frac{\partial^2 w_l}{\partial x_i \partial x_j}, w_h\right\rangle_{H^{s-1}(I\!R^n), H^{s+1}(I\!R^n)}$$
$$-2\sum_{l=1}^{m} g'_{lm}(t) \sum_{j=1}^{n} \left(a_{0j}(t,x)\frac{\partial w_l}{\partial x_j}, w_h\right)_{H^s(I\!R^n)}$$
$$= (F(t), w_h)_{H^s(I\!R^n)} \ \ (1 \le h \le m), \quad \forall t \in [0,T] \tag{6.2.16}$$

and

$$g_{lm}(0) = \xi_{lm}, \quad g'_{lm}(0) = \eta_{lm} \ \ (1 \le l \le m). \tag{6.2.17}$$

By assumptions (6.2.2)–(6.2.5), and noting that the space $H^M(I\!R^n)$ is an algebra when $M \ge [\frac{n}{2}] + 1$, it is clear that the inner products appearing in (6.2.16) all make sense. Then, we get a Cauchy problem of second-order linear ordinary differential equations in terms of unknowns $\{g_{lm}(t)(1 \le l \le m)\}$. From the linear independence of $w_1, \ldots, w_m$ we have

$$\det |(w_l, w_h)_{H^s(I\!R^n)}| \ne 0, \tag{6.2.18}$$

therefore, from the theory of linear ordinary differential equations we know that Cauchy problem (6.2.16)–(6.2.17) admits on the interval $[0, T]$ a unique solution

$$g_{lm}(t) \in H^2(0,T) \ \ (1 \le l \le m), \tag{6.2.19}$$

then we can determine uniquely the approximate solution $u_m(t)$ from (6.2.10), and

$$u_m(t) \in H^2(0,T; H^{s+1}(I\!R^n)). \tag{6.2.20}$$

Now we estimate the approximate solution sequence $\{u_m(t)\}$.
Multiplying (6.2.11) by $g'_{hm}(t)$ and summing over h, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u'_m(t)\|^2_{H^s(I\!R^n)} - \sum_{i,j=1}^{n} \left\langle a_{ij}(t,x)\frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}, u'_m(t)\right\rangle_{H^{s-1}(I\!R^n), H^{s+1}(I\!R^n)}$$
$$-2\sum_{j=1}^{n} \left(a_{0j}(t,x)\frac{\partial u'_m(t)}{\partial x_j}, u'_m(t)\right)_{H^s(I\!R^n)}$$
$$= (F(t), u'_m(t))_{H^s(I\!R^n)}, \quad \forall t \in [0,T]. \tag{6.2.21}$$

By carefully examining the second term on the left-hand side of (6.2.21), we have

$$\left\langle a_{ij}(t,x)\frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}, u'_m(t)\right\rangle_{H^{s-1}(I\!R^n), H^{s+1}(I\!R^n)}$$

$$
\begin{aligned}
&= \sum_{|k|\le s} \left\langle D_x^k\left(a_{ij}(t,x)\frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}\right), D_x^k u_m'(t)\right\rangle_{H^{-1}(I\!R^n),H^1(I\!R^n)} \\
&= \sum_{|k|\le s} \left\langle a_{ij}(t,x) D_x^k \frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}, D_x^k u_m'(t)\right\rangle_{H^{-1}(I\!R^n),H^1(I\!R^n)} \\
&\quad + \sum_{|k|\le s} \left\langle D_x^k\left(a_{ij}(t,x)\frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}\right) - a_{ij}(t,x) D_x^k \frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}, D_x^k u_m'(t)\right\rangle_{H^{-1}(I\!R^n),H^1(I\!R^n)} \\
&= \sum_{|k|\le s} \left\langle a_{ij}(t,x) D_x^k \frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}, D_x^k u_m'(t)\right\rangle_{H^{-1}(I\!R^n),H^1(I\!R^n)} \\
&\quad + \sum_{|k|\le s} \left(D_x^k\left(a_{ij}(t,x)\frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}\right) - a_{ij}(t,x) D_x^k \frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}, D_x^k u_m'(t)\right)_{L^2(I\!R^n)},
\end{aligned}
\tag{6.2.22}
$$

where $\langle \cdot, \cdot \rangle_{H^{-1}(I\!R^n),H^1(I\!R^n)}$ stands for the dual inner product between the spaces $H^{-1}(I\!R^n)$ and $H^1(I\!R^n)$, and $(\cdot, \cdot)_{L^2(I\!R^n)}$ the inner product in the space $L^2(I\!R^n)$.

For the first term on the right-hand side of (6.2.22), it is obvious that

$$
\begin{aligned}
&\sum_{|k|\le s} \left\langle a_{ij}(t,x) D_x^k \frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}, D_x^k u_m'(t)\right\rangle_{H^{-1}(I\!R^n),H^1(I\!R^n)} \\
&= \sum_{|k|\le s} \left\langle \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j} D_x^k u_m(t)\right), D_x^k u_m'(t)\right\rangle_{H^{-1}(I\!R^n),H^1(I\!R^n)} \\
&\quad - \sum_{|k|\le s} \left\langle \frac{\partial a_{ij}(t,x)}{\partial x_j} \frac{\partial}{\partial x_j} D_x^k u_m(t), D_x^k u_m'(t)\right\rangle_{H^{-1}(I\!R^n),H^1(I\!R^n)} \\
&= - \sum_{|k|\le s} \left(a_{ij}(t,x)\frac{\partial}{\partial x_j} D_x^k u_m(t), \frac{\partial}{\partial x_i} D_x^k u_m'(t)\right)_{L^2(I\!R^n)} \\
&\quad - \sum_{|k|\le s} \left(\frac{\partial a_{ij}(t,x)}{\partial x_j} \frac{\partial}{\partial x_j} D_x^k u_m(t), D_x^k u_m'(t)\right)_{L^2(I\!R^n)}.
\end{aligned}
\tag{6.2.23}
$$

Since

$$
\begin{aligned}
&\left(a_{ij}(t,x)\frac{\partial}{\partial x_j} D_x^k u_m(t), \frac{\partial}{\partial x_i} D_x^k u_m'(t)\right)_{L^2(I\!R^n)} \\
&= \frac{d}{dt}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j} D_x^k u_m(t), \frac{\partial}{\partial x_i} D_x^k u_m(t)\right)_{L^2(I\!R^n)} \\
&\quad - \left(a_{ij}(t,x)\frac{\partial}{\partial x_j} D_x^k u_m'(t), \frac{\partial}{\partial x_i} D_x^k u_m(t)\right)_{L^2(I\!R^n)} \\
&\quad - \left(\frac{\partial a_{ij}(t,x)}{\partial t}\frac{\partial}{\partial x_j} D_x^k u_m(t), \frac{\partial}{\partial x_i} D_x^k u_m(t)\right)_{L^2(I\!R^n)},
\end{aligned}
\tag{6.2.24}
$$

and also noticing the symmetry of a_{ij} (see (6.1.3)), it is easy to get

$$\begin{aligned}&\sum_{i,j=1}^{n}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j}D_x^k u_m(t),\ \frac{\partial}{\partial x_i}D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}\\&=\frac{1}{2}\frac{d}{dt}\sum_{i,j=1}^{n}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j}D_x^k u_m(t),\ \frac{\partial}{\partial x_i}D_x^k u_m(t)\right)_{L^2(\mathbb{R}^n)}\\&\quad-\frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial a_{ij}(t,x)}{\partial t}\frac{\partial}{\partial x_j}D_x^k u_m(t),\ \frac{\partial}{\partial x_i}D_x^k u_m(t)\right)_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{6.2.25}$$

From (6.2.22)–(6.2.23) and (6.2.25), we can rewrite the second term on the left-hand side of (6.2.21) as

$$\begin{aligned}&-\sum_{i,j=1}^{n}\left\langle a_{ij}(t,x)\frac{\partial^2 u_m(t)}{\partial x_i\partial x_j},u_m'(t)\right\rangle_{H^{s-1}(\mathbb{R}^n),H^{s+1}(\mathbb{R}^n)}\\&=\frac{1}{2}\frac{d}{dt}\sum_{|k|\le s}\sum_{i,j=1}^{n}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j}D_x^k u_m(t),\ \frac{\partial}{\partial x_i}D_x^k u_m(t)\right)_{L^2(\mathbb{R}^n)}\\&-\frac{1}{2}\sum_{|k|\le s}\sum_{i,j=1}^{n}\left(\frac{\partial a_{ij}(t,x)}{\partial t}\frac{\partial}{\partial x_j}D_x^k u_m(t),\ \frac{\partial}{\partial x_i}D_x^k u_m(t)\right)_{L^2(\mathbb{R}^n)}\\&+\sum_{|k|\le s}\sum_{i,j=1}^{n}\left(\frac{\partial a_{ij}(t,x)}{\partial x_i}\frac{\partial}{\partial x_j}D_x^k u_m(t),D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}\\&-\sum_{|k|\le s}\sum_{i,j=1}^{n}\left(D_x^k\left(a_{ij}(t,x)\frac{\partial^2 u_m(t)}{\partial x_i\partial x_j}\right)-a_{ij}(t,x)D_x^k\frac{\partial^2 u_m(t)}{\partial x_i\partial x_j},D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{6.2.26}$$

In addition, we have

$$\begin{aligned}&\left(a_{0j}(t,x)\frac{\partial u_m'(t)}{\partial x_j},u_m'(t)\right)_{H^s(\mathbb{R}^n)}\\&=\sum_{|k|\le s}\left(D_x^k\left(a_{0j}(t,x)\frac{\partial u_m'(t)}{\partial x_j}\right),D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}\\&=\sum_{|k|\le s}\left(a_{0j}(t,x)D_x^k\frac{\partial u_m'(t)}{\partial x_j},D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}\\&\quad+\sum_{|k|\le s}\left(D_x^k\left(a_{0j}(t,x)\frac{\partial u_m'(t)}{\partial x_j}\right)-a_{0j}(t,x)D_x^k\frac{\partial u_m'(t)}{\partial x_j},D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{6.2.27}$$

While, for the first term on the right-hand side of the above formula, we have

$$\begin{aligned}
&\left(a_{0j}(t,x)D_x^k\frac{\partial u_m'(t)}{\partial x_j}, D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}\\
=&\left(\frac{\partial}{\partial x_j}\left(a_{0j}(t,x)D_x^k u_m'(t)\right), D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}-\left(\frac{\partial a_{0j}(t,x)}{\partial x_j}D_x^k u_m'(t), D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}\\
=&-\left(a_{0j}(t,x)D_x^k u_m'(t), D_x^k\frac{\partial u_m'(t)}{\partial x_j}\right)_{L^2(\mathbb{R}^n)}-\left(\frac{\partial a_{0j}(t,x)}{\partial x_j}D_x^k u_m'(t), D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)},
\end{aligned}$$

hence

$$\left(a_{0j}(t,x)D_x^k\frac{\partial u_m'(t)}{\partial x_j}, D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}=-\frac{1}{2}\left(\frac{\partial a_{0j}(t,x)}{\partial x_j}D_x^k u_m'(t), D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}. \tag{6.2.28}$$

From (6.2.27)–(6.2.28), the third term on the left-hand of (6.2.21) can be rewritten as

$$\begin{aligned}
&-2\sum_{j=1}^n\left(a_{0j}(t,x)\frac{\partial u_m'(t)}{\partial x_j}, u_m'(t)\right)_{H^s(\mathbb{R}^n)}\\
=&\sum_{|k|\le s}\sum_{j=1}^n\left(\frac{\partial a_{0j}(t,x)}{\partial x_j}D_x^k u_m'(t), D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}\\
&-2\sum_{|k|\le s}\sum_{j=1}^n\left(D_x^k\left(a_{0j}(t,x)\frac{\partial u_m'(t)}{\partial x_j}\right)-a_{0j}(t,x)D_x^k\frac{\partial u_m'(t)}{\partial x_j}, D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{6.2.29}$$

Thus, combining (6.2.26) and (6.2.29), we can rewrite (6.2.21) as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Bigg(\|u_m'(t)\|^2_{H^s(\mathbb{R}^n)}\\
&+\sum_{|k|\le s}\sum_{i,j=1}^n\left(a_{ij}(t,x)\frac{\partial}{\partial x_j}D_x^k u_m(t), \frac{\partial}{\partial x_i}D_x^k u_m(t)\right)_{L^2(\mathbb{R}^n)}\Bigg)\\
=&\frac{1}{2}\sum_{|k|\le s}\sum_{i,j=1}^n\left(\frac{\partial a_{ij}(t,x)}{\partial t}\frac{\partial}{\partial x_j}D_x^k u_m(t), \frac{\partial}{\partial x_i}D_x^k u_m(t)\right)_{L^2(\mathbb{R}^n)}\\
&-\sum_{|k|\le s}\sum_{i,j=1}^n\left(\frac{\partial a_{ij}(t,x)}{\partial x_i}\frac{\partial}{\partial x_j}D_x^k u_m(t), D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}\\
&+\sum_{|k|\le s}\sum_{i,j=1}^n\left(D_x^k\left(a_{ij}(t,x)\frac{\partial^2 u_m(t)}{\partial x_i\partial x_j}\right)-a_{ij}(t,x)D_x^k\frac{\partial^2 u_m(t)}{\partial x_i\partial x_j}, D_x^k u_m'(t)\right)_{L^2(\mathbb{R}^n)}
\end{aligned}$$

$$-\sum_{|k|\le s}\sum_{j=1}^{n}\left(\frac{\partial a_{0j}(t,x)}{\partial x_j}D_x^k u_m'(t),\, D_x^k u_m'(t)\right)_{L^2(I\!R^n)}$$
$$+2\sum_{|k|\le s}\sum_{j=1}^{n}\left(D_x^k\left(a_{0j}(t,x)\frac{\partial u_m'(t)}{\partial x_j}\right)-a_{0j}(t,x)D_x^k\frac{\partial u_m'(t)}{\partial x_j},\, D_k^x u_m'(t)\right)_{L^2(I\!R^n)}$$
$$+(F(t),u_m'(t))_{H^s(I\!R^n)},\quad \forall t\in[0,T]. \tag{6.2.30}$$

Integrating the above formula with respect to t, and noting (6.2.12)–(6.2.13), when $0\le t\le T$ we get

$$\|u_m'(t)\|^2_{H^s(I\!R^n)}+\sum_{|k|\le s}\sum_{i,j=1}^{n}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j}D_x^k u_m(t),\,\frac{\partial}{\partial x_i}D_x^k u_m(t)\right)_{L^2(I\!R^n)}$$
$$=\|u_{1m}\|^2_{H^s(I\!R^n)}+\sum_{|k|\le s}\sum_{i,j=1}^{n}\left(a_{ij}(0,x)\frac{\partial}{\partial x_j}D_x^k u_{0m},\,\frac{\partial}{\partial x_i}D_x^k u_{0m}\right)_{L^2(I\!R^n)}$$
$$+\sum_{|k|\le s}\sum_{i,j=1}^{n}\int_0^t\left(\frac{\partial a_{ij}(\tau,x)}{\partial\tau}\frac{\partial}{\partial x_j}D_x^k u_m(\tau),\,\frac{\partial}{\partial x_i}D_x^k u_m(\tau)\right)_{L^2(I\!R^n)}d\tau$$
$$-2\sum_{|k|\le s}\sum_{i,j=1}^{n}\int_0^t\left(\frac{\partial a_{ij}(\tau,x)}{\partial x_i}\frac{\partial}{\partial x_j}D_x^k u_m(\tau),\,D_x^k u_m'(\tau)\right)_{L^2(I\!R^n)}d\tau$$
$$+2\sum_{|k|\le s}\sum_{i,j=1}^{n}\int_0^t\left(D_x^k\left(a_{ij}(\tau,x)\frac{\partial^2 u_m(\tau)}{\partial x_i\partial x_j}\right)-a_{ij}(\tau,x)D_x^k\frac{\partial^2 u_m(\tau)}{\partial x_i\partial x_j},\,D_x^k u_m'(\tau)\right)_{L^2(I\!R^n)}d\tau$$
$$-2\sum_{|k|\le s}\sum_{j=1}^{n}\int_0^t\left(\frac{\partial a_{0j}(\tau,x)}{\partial x_j}D_x^k u_m'(\tau),\,D_x^k u_m'(\tau)\right)_{L^2(I\!R^n)}d\tau$$
$$+4\sum_{|k|\le s}\sum_{j=1}^{n}\int_0^t\left(D_x^k\left(a_{0j}(\tau,x)\frac{\partial u_m'(\tau)}{\partial x_j}\right)-a_{0j}(\tau,x)D_x^k\frac{\partial u_m'(\tau)}{\partial x_j},\,D_x^k u_m'(\tau)\right)_{L^2(I\!R^n)}d\tau$$
$$+2\int_0^t(F(\tau),u_m'(\tau))_{H^s(I\!R^n)}d\tau$$
$$\overset{\text{def.}}{=}\|u_{1m}\|^2_{H^s(I\!R^n)}+\sum_{|k|\le s}\sum_{i,j=1}^{n}\left(a_{ij}(0,x)\frac{\partial}{\partial x_j}D_x^k u_{0m},\,\frac{\partial}{\partial x_i}D_x^k u_{0m}\right)_{L^2(I\!R^n)}$$
$$+\text{I}+\text{II}+\text{III}+\text{IV}+\text{V}+\text{VI}. \tag{6.2.31}$$

Noticing that when $s\ge[\frac{n}{2}]+2$, the Sobelev embedding theorem tells us that

$$H^{s-1}(I\!R^n)\subset L^\infty(I\!R^n) \tag{6.2.32}$$

is a continuous embedding, by assumptions (6.2.3)–(6.2.4) it is obvious that

$$\text{I+II+IV} \le C_1 \int_0^t \Big(\|u_m'(\tau)\|^2_{H^s(I\!R^n)} + \|D_x u_m(\tau)\|^2_{H^s(I\!R^n)}\Big) d\tau, \tag{6.2.33}$$

where constant $C_1 > 0$ depends only on the norms of $\frac{\partial a_{ij}}{\partial t}$, $\frac{\partial a_{ij}}{\partial x_i}$ and $\frac{\partial a_{0j}}{\partial x_j}(i, j = 1, \dots, n)$ in $L^\infty(0, T; H^{s-1}(I\!R^n))$.

From (5.3.4) in Lemma 5.3.1 of Chap. 5 (in which we take $p = +\infty, q = r = 2$) we have

$$\begin{aligned}
&\left\| D_x^k\Big(a_{ij}(\tau, x)\frac{\partial^2 u_m(\tau)}{\partial x_i \partial x_j}\Big) - a_{ij}(\tau, x) D_x^k \frac{\partial^2 u_m(\tau)}{\partial x_i \partial x_j}\right\|_{L^2(I\!R^n)} \\
&\le C\Bigg(\|D_x a_{ij}(\tau, x))\|_{L^\infty(I\!R^n)} \left\| D_x^{|k|-1}\frac{\partial^2 u_m(\tau)}{\partial x_i \partial x_j}\right\|_{L^2(I\!R^n)} \\
&\qquad + \|D_x^{|k|} a_{ij}(\tau, x)\|_{L^2(I\!R^n)} \left\|\frac{\partial^2 u_m(\tau)}{\partial x_i \partial x_j}\right\|_{L^\infty(I\!R^n)}\Bigg),
\end{aligned} \tag{6.2.34}$$

where C is a positive constant, and $D_x^{|k|}$ represents the set of all partial derivatives of $|k|$-th order, and so on. Noting also (6.2.32), when $|k| \le s$ we have

$$\begin{aligned}
&\left\| D_x^k\Big(a_{ij}(\tau, x)\frac{\partial^2 u_m(\tau)}{\partial x_i \partial x_j}\Big) - a_{ij}(\tau, x) D_x^k \frac{\partial^2 u_m(\tau)}{\partial x_i \partial x_j}\right\|_{L^2(I\!R^n)} \\
&\le C\|D_x a_{ij}(\tau, x)\|_{H^{s-1}(I\!R^n)} \|D_x u_m(\tau)\|_{H^s(I\!R^n)},
\end{aligned} \tag{6.2.35}$$

Then, by assumption (6.2.3) we obtain

$$\text{III} \le C_2 \int_0^t \Big(\|u_m'(\tau)\|^2_{H^s(I\!R^n)} + \|D_x u_m(\tau)\|^2_{H^s(I\!R^n)}\Big) d\tau, \tag{6.2.36}$$

where constant $C_2 > 0$ depends only on norms of $\frac{\partial a_{ij}}{\partial x_k}$ $(i, j, k = 1, \dots, n)$ in $L^\infty(0, T;\ H^{s-1}(I\!R^n))$. Similarly we have

$$\text{V} \le C_3 \int_0^t \|u_m'(\tau)\|^2_{H^s(I\!R^n)} d\tau, \tag{6.2.37}$$

where constant $C_3 > 0$ depends only on norms of $\frac{\partial a_{0j}}{\partial x_k}(j, k = 1, \dots, n)$ in $L^\infty(0, T;$ H^{s-1} $(I\!R^n))$.

Moreover, it is obvious that

$$\text{VI} \le \int_0^t \|u_m'(\tau)\|^2_{H^s(I\!R^n)} d\tau + \int_0^t \|F(\tau)\|^2_{H^s(I\!R^n)} d\tau. \tag{6.2.38}$$

By assumption (6.1.4) we have

$$\sum_{|k|\le s}\sum_{i,j=1}^{n}\left(a_{ij}(t,x)\frac{\partial}{\partial x_j}D_x^k u_m(t),\ \frac{\partial}{\partial x_i}D_x^k u_m(t)\right)_{L^2(I\!R^n)} \ge m_0\|D_x u_m(t)\|^2_{H^s(I\!R^n)}. \tag{6.2.39}$$

Therefore, using (6.2.33), (6.2.36)–(6.2.39) and noting (6.2.2), it follows from (6.2.31) that

$$\begin{aligned}&\|u_m'(t)\|^2_{H^s(I\!R^n)} + \|D_x u_m(t)\|^2_{H^s(I\!R^n)}\\ \le\ & C_4\Bigg(\|u_{1m}\|^2_{H^s(I\!R^n)} + \|D_x u_{0m}\|^2_{H^s(I\!R^n)} + \int_0^t \|F(\tau)\|^2_{H^s(I\!R^n)}d\tau\\ &+\int_0^t\left(\|u_m'(\tau)\|^2_{H^s(I\!R^n)} + \|D_x u_m(\tau)\|^2_{H^s(I\!R^n)}\right)d\tau\Bigg),\quad \forall t\in[0,T],\end{aligned} \tag{6.2.40}$$

where constant $C_4 > 0$ depends only on norms of $a_{ij}(i,\ j = 1,\dots,n)$ in $L^\infty((0,T)\times I\!R^n)$ and norms of $\frac{\partial a_{ij}}{\partial t}$, $\frac{\partial a_{ij}}{\partial x_k}$ and $\frac{\partial a_{0j}}{\partial x_k}(i,\ j, k = 1,\dots,n)$ in $L^\infty(0,T;\ H^{s-1}(I\!R^n))$.

Combining (6.2.14)–(6.2.15) and assumptions (6.2.1) and (6.2.5), and using Gronwall inequality,we have

$$\|u_m'(t)\|^2_{H^s(I\!R^n)} + \|D_x u_m(t)\|^2_{H^s(I\!R^n)} \le C(T),\quad \forall t\in[0,T], \tag{6.2.41}$$

where $C(T)$ is a positive constant depending on T but independent of m. Furthermore, from

$$u_m(t) = u_m(0) + \int_0^t u_m'(\tau)d\tau = u_{0m} + \int_0^t u_m'(\tau)d\tau, \tag{6.2.42}$$

we have

$$\|u_m(t)\|_{H^s(I\!R^n)} \le \|u_{0m}\|_{H^s(I\!R^n)} + \int_0^t \|u_m'(\tau)\|_{H^s(I\!R^n)}d\tau, \tag{6.2.43}$$

then it is easy to know that

$$\|u_m(t)\|_{H^s(I\!R^n)} \le C(T),\quad \forall t\in[0,T]. \tag{6.2.44}$$

Hence, we now obtain

$$\{u_m(t)\} \in \text{a bounded set in } L^\infty(0,T;\ H^{s+1}(I\!R^n)) \tag{6.2.45}$$

$$\{u_m'(t)\} \in \text{a bounded set in } L^\infty(0,T;\ H^{s}(I\!R^n)). \tag{6.2.46}$$

Along with (6.2.35) and noting assumption (6.2.3), for any given multi-index k with $|k| \le s$, we have

$$\left\{D_x^k\left(a_{ij}(t,x)\frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}\right) - a_{ij}(t,x)D_x^k\frac{\partial^2 u_m(t)}{\partial x_i \partial x_j}\right\}$$
$$\in \text{ a bounded set in } L^\infty(0,T;L^2(I\!R^n)). \tag{6.2.47}$$

Similarly, for $|k| \leq s$, we have

$$\left\{D_x^k\left(a_{0j}(t,x)\frac{\partial u'_m(t)}{\partial x_j}\right) - a_{0j}(t,x)D_x^k\frac{\partial u'_m(t)}{\partial x_j}\right\}$$
$$\in \text{a bounded set in } L^\infty(0,T;L^2(I\!R^n)). \tag{6.2.48}$$

Therefore, from the weak compactness we know that: there exists a subsequence $\{u_\mu(t)\}$ of $\{u_m(t)\}$, such that when $\mu \to \infty$ we have

$$u_\mu(t) \overset{*}{\rightharpoonup} u(t) \text{ weak } * \text{ in } L^\infty(0,T;H^{s+1}(I\!R^n)), \tag{6.2.49}$$
$$u'_\mu(t) \overset{*}{\rightharpoonup} u'(t) \text{ weak } * \text{ in } L^\infty(0,T;H^{s}(I\!R^n)), \tag{6.2.50}$$

and for $|k| \leq s$, we have

$$D_x^k\left(a_{ij}(t,x)\frac{\partial^2 u_\mu(t)}{\partial x_i \partial x_j}\right) - a_{ij}(t,x)D_x^k\frac{\partial^2 u_\mu(t)}{\partial x_i \partial x_j}$$
$$\overset{*}{\rightharpoonup} \quad D_x^k\left(a_{ij}(t,x)\frac{\partial^2 u(t)}{\partial x_i \partial x_j}\right) - a_{ij}(t,x)D_x^k\frac{\partial^2 u(t)}{\partial x_i \partial x_j}$$
$$\text{weak } * \text{ in } L^\infty(0,T;L^2(I\!R^n)), \tag{6.2.51}$$
$$D_x^k\left(a_{0j}(t,x)\frac{\partial u'_\mu(t)}{\partial x_j}\right) - a_{0j}(t,x)D_x^k\frac{\partial u'_\mu(t)}{\partial x_j}$$
$$\overset{*}{\rightharpoonup} \quad D_x^k\left(a_{0j}(t,x)\frac{\partial u'(t)}{\partial x_j}\right) - a_{0j}(t,x)D_x^k\frac{\partial u'(t)}{\partial x_j}$$
$$\text{weak } * \text{ in } L^\infty(0,T;L^2(I\!R^n)). \tag{6.2.52}$$

Moreover, similarly to (6.2.22) and (6.2.27), we have

$$\left\langle a_{ij}(t,x)\frac{\partial^2 u_\mu(t)}{\partial x_i \partial x_j}, w_h\right\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}$$
$$= \sum_{|k|\leq s}\left\langle a_{ij}(t,x)D_x^k\frac{\partial^2 u_\mu(t)}{\partial x_i \partial x_j}, D_x^k w_h\right\rangle_{H^{-1}(I\!R^n),H^1(I\!R^n)}$$
$$+ \sum_{|k|\leq s}\left(D_x^k\left(a_{ij}(t,x)\frac{\partial^2 u_\mu(t)}{\partial x_i \partial x_j}\right) - a_{ij}(t,x)D_x^k\frac{\partial^2 u_\mu(t)}{\partial x_i \partial x_j}, D_x^k w_h\right)_{L^2(I\!R^n)} \tag{6.2.53}$$

and

$$\begin{aligned}
&\left\langle a_{0j}(t,x)\frac{\partial u'_\mu(t)}{\partial x_j}, w_h\right\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\
=&\left(a_{0j}(t,x)\frac{\partial u'_\mu(t)}{\partial x_j}, w_h\right)_{H^s(I\!R^n)}\\
=&\sum_{|k|\le s}\left(a_{0j}(t,x)D_x^k\frac{\partial u'_\mu(t)}{\partial x_j}, D_x^k w_h\right)_{L^2(I\!R^n)}\\
&+\sum_{|k|\le s}\left(D_x^k\left(a_{0j}(t,x)\frac{\partial u'_\mu(t)}{\partial x_j}\right)-a_{0j}(t,x)D_x^k\frac{\partial u'_\mu(t)}{\partial x_j}, D_x^k w_h\right)_{L^2(I\!R^n)}\\
=&\sum_{|k|\le s}\left\langle a_{0j}(t,x)D_x^k\frac{\partial u'_\mu(t)}{\partial x_j}, D_x^k w_h\right\rangle_{H^{-1}(I\!R^n),H^1(I\!R^n)}\\
&+\sum_{|k|\le s}\left(D_x^k\left(a_{0j}(t,x)\frac{\partial u'_\mu(t)}{\partial x_j}\right)-a_{0j}(t,x)D_x^k\frac{\partial u'_\mu(t)}{\partial x_j}, D_x^k w_h\right)_{L^2(I\!R^n)}. \qquad (6.2.54)
\end{aligned}$$

Taking $\mu\to\infty$ in (6.2.53) and (6.2.54), it follows from (6.2.49)–(6.2.52) that

$$\begin{aligned}
&\left\langle a_{ij}(t,x)\frac{\partial^2 u_\mu(t)}{\partial x_i\partial x_j}, w_h\right\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\
\overset{*}{\rightharpoonup}\ &\left\langle a_{ij}(t,x)\frac{\partial^2 u(t)}{\partial x_i\partial x_j}, w_h\right\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\
&\text{weak } * \text{ in } L^\infty(0,T) \qquad (6.2.55)
\end{aligned}$$

and

$$\begin{aligned}
&\left(a_{0j}(t,x)\frac{\partial u'_\mu(t)}{\partial x_j}, w_h\right)_{H^s(I\!R^n)}=\left\langle a_{0j}(t,x)\frac{\partial u'_\mu(t)}{\partial x_j}, w_h\right\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\
\overset{*}{\rightharpoonup}\ &\left\langle a_{0j}(t,x)\frac{\partial u'(t)}{\partial x_j}, w_h\right\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\
&\text{weak } * \text{ in } L^\infty(0,T). \qquad (6.2.56)
\end{aligned}$$

Thus, taking $m=\mu\to\infty$ in (6.2.11), for any given $h\in I\!N$ we obtain that

$$\begin{aligned}
&\frac{d^2}{dt^2}\langle u_\mu(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}=\langle u''_\mu(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\
\overset{*}{\rightharpoonup}\ &\left\langle \sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2 u(t)}{\partial x_i\partial x_j}+2\sum_{j=1}^n a_{0j}(t,x)\frac{\partial u'(t)}{\partial x_j}+F(t), w_h\right\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\
&\text{weak } * \text{ in } L^\infty(0,T). \qquad (6.2.57)
\end{aligned}$$

On the other hand, noting (6.2.50), for any given $h \in I\!N$, when $\mu \to \infty$ we have

$$\begin{aligned}&\frac{d}{dt}\langle u_\mu(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)} = \langle u'_\mu(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\ \overset{*}{\rightharpoonup}\ &\frac{d}{dt}\langle u(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)} = \langle u'(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\ &\text{weak } * \text{ in } L^\infty(0,T), \end{aligned} \tag{6.2.58}$$

then

$$\begin{aligned}&\frac{d^2}{dt^2}\langle u_\mu(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\ \to\ &\frac{d^2}{dt^2}\langle u(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)} = \langle u''(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\ &\text{in } \mathcal{D}'(0,T). \end{aligned} \tag{6.2.59}$$

Together with (6.2.57) and (6.2.59), we obtain: for any given $h \in I\!N$, it holds in $\mathcal{D}'(0,T)$ that

$$\left\langle u''(t) - \sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2 u(t)}{\partial x_i \partial x_j} - 2\sum_{j=1}^n a_{0j}(t,x)\frac{\partial u'(t)}{\partial x_j} - F(t), w_h\right\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)} = 0, \tag{6.2.60}$$

that is,

$$\begin{aligned}&\left\langle \int_0^T \Big(u''(t) - \sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2 u(t)}{\partial x_i \partial x_j}\right.\\ &\qquad\qquad \left. -2\sum_{j=1}^n a_{0j}(t,x)\frac{\partial u'(t)}{\partial x_j} - F(t)\Big)\phi(t)dt, w_h\right\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}\\ &= 0, \quad \forall \phi \in \mathcal{D}(0,T), \quad \forall h \in I\!N. \end{aligned} \tag{6.2.61}$$

Since $\{w_h\}(h = 1, 2, \ldots)$ is a set of basis in $H^{s+1}(I\!R^n)$, the above formula yields that in $H^{s-1}(I\!R^n)$,

$$\int_0^T \left(u''(t) - \sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2 u(t)}{\partial x_i \partial x_j} - 2\sum_{j=1}^n a_{0j}(t,x)\frac{\partial u'(t)}{\partial x_j} - F(t)\right)\phi(t)dt = 0, \quad \forall \phi \in \mathcal{D}(0,T), \tag{6.2.62}$$

consequently, in $\mathcal{D}'(0,T; H^{s-1}(I\!R^n))$ we have

$$u''(t) - \sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2 u(t)}{\partial x_i \partial x_j} - 2\sum_{j=1}^n a_{0j}(t,x)\frac{\partial u'(t)}{\partial x_j} = F(t), \tag{6.2.63}$$

namely, u is a solution to Eq. (6.1.1). Furthermore, noting (6.2.49)–(6.2.50), (6.2.2) and (6.2.4)–(6.2.5), and using Eq. (6.2.63), we get (6.2.6)–(6.2.8). As a result, (6.2.63) holds in $L^2(0, T; H^{s-1}(I\!R^n))$. In addition, thanks to (6.2.57) and (6.2.63), when $\mu \to \infty$ we also have

$$\begin{aligned} &\frac{d}{dt}\langle u'_\mu(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)} = \frac{d^2}{dt^2}\langle u_\mu(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)} \\ \overset{*}{\rightharpoonup}\ &\frac{d}{dt}\langle u'(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)} = \frac{d^2}{dt^2}\langle u(t), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)} \\ &\text{weak } * \text{ in } L^\infty(0, T). \end{aligned} \tag{6.2.64}$$

Now we prove that u satisfies the initial condition (6.1.2). It follows from (6.2.49)–(6.2.50) that, when $\mu \to \infty$, we have

$$u_\mu(0) = u_{0\mu} \overset{*}{\rightharpoonup} u(0) \text{ weak } * \text{ in } H^s(I\!R^n), \tag{6.2.65}$$

then we immediately get from (6.2.14) that

$$u(0) = f. \tag{6.2.66}$$

This is exactly the first formula in (6.1.2). Similarly, from (6.2.58) and (6.2.64), when $\mu \to \infty$, we have

$$\langle u'_\mu(0), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)} \to \langle u'(0), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}, \quad \forall h \in I\!N, \tag{6.2.67}$$

then it follows from (6.2.15) that

$$\langle u'(0), w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)} = \langle g, w_h\rangle_{H^{s-1}(I\!R^n),H^{s+1}(I\!R^n)}, \quad \forall h \in I\!N. \tag{6.2.68}$$

Since $\{w_h\}$ is a set of basis in $H^{s+1}(I\!R^n)$, we get

$$u'(0) = g. \tag{6.2.69}$$

This is just the second formula in (6.1.2).

In consequence, the u obtained from (6.2.49) gives the solution to Cauchy problem (6.1.1)–(6.1.2), and satisfies (6.2.6)–(6.2.8). This proves the existence of solutions.

Now we prove estimate (6.2.9) for any solution $u = u(t, x)$ to Cauchy problem (6.1.1)–(6.1.2), satisfying (6.2.6)–(6.2.8).

Taking the inner product of u_t with both sides of Eq. (6.1.1) in the space $H^s(I\!R^n)$, integrating with respect to t over the interval $[0, t]$, and using almost the same arguments as in establishing estimate (6.2.40) to the approximate solution $u_m(t)$ previously, we obtain

$$\begin{aligned}&\|D_x u(t,\cdot)\|^2_{H^s(I\!R^n)} + \|u_t(t,\cdot)\|^2_{H^s(I\!R^n)}\\ &\le C_5\Big(\|D_x f\|^2_{H^s(I\!R^n)} + \|g\|^2_{H^s(I\!R^n)} + \int_0^t \|F(\tau,\cdot)\|^2_{H^s(I\!R^n)}d\tau\\ &\qquad + \int_0^t \Big(\|D_x u(\tau,\cdot)\|^2_{H^s(I\!R^n)} + \|u_\tau(\tau,\cdot)\|^2_{H^s(I\!R^n)}\Big)d\tau\Big), \quad \forall t\in[0,T], \end{aligned}\tag{6.2.70}$$

where constant $C_5 > 0$ depends only on the norm of $a_{ij}(i,j=1,\ldots,n)$ in $L^\infty((0,T)\times I\!R^n)$ and norms of $\frac{\partial a_{ij}}{\partial t}$, $\frac{\partial a_{ij}}{\partial x_k}$ and $\frac{\partial a_{0j}}{\partial x_k}(i,j,k=1,\ldots,n)$ in $L^\infty(0,T;H^{s-1}(I\!R^n))$. Moreover, since

$$u(t,\cdot) = f(\cdot) + \int_0^t u_\tau(\tau,\cdot)d\tau,\tag{6.2.71}$$

we get

$$\|u(t,\cdot)\|_{H^s(I\!R^n)} \le \|f\|_{H^s(I\!R^n)} + \int_0^t \|u_\tau(\tau,\cdot)\|_{H^s(I\!R^n)}d\tau, \quad \forall t\in[0,T].\tag{6.2.72}$$

Combining (6.2.70) and (6.2.72) and using Gronwall inequality, we can obtain the required estimate (6.2.9).

The uniqueness of solutions to Cauchy problem (6.1.1)–(6.1.2), satisfying (6.2.6)–(6.2.8), follows immediately from estimate (6.2.9). As a result, the whole approximate sequence $\{u_m(t)\}$ converges.

The proof of Lemma 2.1.1 is finished. □

6.3 Regularity of Solutions

In this section, we will use a mollifying argument to improve Lemma 6.2.1 as follows.

Theorem 6.3.1 *Under the assumptions of Lemma 6.2.1, for the solution $u = u(t,x)$ to Cauchy problem (6.1.1)–(6.1.2), after possible change of values on a zero-measure set of the interval $[0,T]$, we have*

$$u \in C([0,T];H^{s+1}(I\!R^n)),\tag{6.3.1}$$

$$u_t \in C([0,T];H^s(I\!R^n)).\tag{6.3.2}$$

To prove Theorem 6.3.1, we need to use some properties about the mollifying operator.

Denote by J_δ the mollifier with respect to the variable $x\in I\!R^n$:

$$J_\delta f = j_\delta * f,\tag{6.3.3}$$

where $f = f(x)$, $\delta > 0$, and j_δ can be taken as, say,

$$j_\delta(x) = \frac{1}{\delta^n} j\left(\frac{x}{\delta}\right), \tag{6.3.4}$$

where

$$j(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right), & |x| \le 1 \\ 0, & |x| \ge 1 \end{cases} \in \mathcal{D}(I\!R^n) = C_0^\infty(I\!R^n), \tag{6.3.5}$$

and the constant C is chosen such that

$$\int_{I\!R^n} j(x)dx = 1. \tag{6.3.6}$$

Lemma 6.3.1 *Assume that*

$$f \in H^s(I\!R^n), \tag{6.3.7}$$

where $s \ge 0$ is any given integer, then we have

(i) *for any given $\delta > 0$,*

$$J_\delta f \in C^\infty(I\!R^n), \tag{6.3.8}$$

and for any given integer $N \ge 0$,

$$J_\delta f \in H^N(I\!R^n). \tag{6.3.9}$$

(ii) *for any given multi-index $k = (k_1, \ldots, k_n)$ with $|k| \le s$,*

$$J_\delta D_x^k f = D_x^k J_\delta f. \tag{6.3.10}$$

(iii) *for any given $\delta > 0$,*

$$\|J_\delta f\|_{H^s(I\!R^n)} \le C\|f\|_{H^s(I\!R^n)}, \tag{6.3.11}$$

where C is a positive constant independent of δ; and when $\delta \to 0$,

$$J_\delta f \to f \text{ strongly in } H^s(I\!R^n). \tag{6.3.12}$$

(iv) *for any given $\delta > 0$ and for any given integer $N > s$,*

$$\|J_\delta f\|_{H^N(I\!R^n)} \le C_N(\delta)\|f\|_{H^s(I\!R^n)}, \tag{6.3.13}$$

where $C_N(\delta)$ is a positive constant depending on δ and N.

Proof See Hörmander (1963).

Lemma 6.3.2 (Friedrichs Lemma) *Assume that*

$$a \in W^{1,\infty}(\mathbb{R}^n), \quad f \in L^2(\mathbb{R}^n), \tag{6.3.14}$$

then we have

$$\|[J_\delta, L]f\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)}, \tag{6.3.15}$$

where $C > 0$ is a positive constant independent of δ; and when $\delta \to 0$,

$$[J_\delta, L]f \to 0 \text{ strongly in } L^2(\mathbb{R}^n), \tag{6.3.16}$$

in which

$$L = a(x)\frac{\partial}{\partial x_i} \tag{6.3.17}$$

is a partial differential operator, and

$$[J_\delta, L] = J_\delta L - LJ_\delta \tag{6.3.18}$$

is the corresponding commutant operator.

Proof See Hörmander (1963). □

Now we use Lemma 6.3.2 to prove the following

Lemma 6.3.3 *For any given integer $s \ge [\frac{n}{2}] + 2$, assume that*

$$a \in L^\infty(\mathbb{R}^n), \tag{6.3.19}$$

$$D_x a \in H^{s-1}(\mathbb{R}^n), \tag{6.3.20}$$

$$f \in H^s(\mathbb{R}^n), \tag{6.3.21}$$

then

$$\|[J_\delta, L]f\|_{H^s(\mathbb{R}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)}, \tag{6.3.22}$$

and when $\delta \to 0$,

$$[J_\delta, L]f \to 0 \text{ strongly in } H^s(\mathbb{R}^n), \tag{6.3.23}$$

where L is still defined by (6.3.17), and C is a positive constant independent of δ.

Proof Noticing (6.2.32), due to Lemma 6.3.2, it suffices to prove, for any given multi-index k with $0 < |k| \le s$, that

$$\|D_x^k[J_\delta, L]f\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)}, \tag{6.3.24}$$

and when $\delta \to 0$,

$$D_x^k[J_\delta, L]f \to 0 \text{ strongly in } L^2(\mathbb{R}^n). \tag{6.3.25}$$

We have

$$D_x^k[J_\delta, L]f = [J_\delta, L]D_x^k f + [D_x^k, [J_\delta, L]]f. \tag{6.3.26}$$

By Lemma 6.3.2 and assumptions (6.3.19)–(6.3.20), noting (6.2.32), it is obvious that

$$\|[J_\delta, L]D_x^k f\|_{L^2(I\!R^n)} \le C\|f\|_{H^s(I\!R^n)}, \tag{6.3.27}$$

and when $\delta \to 0$,

$$[J_\delta, L]D_x^k f \to 0 \text{ strongly in } L^2(I\!R^n). \tag{6.3.28}$$

Therefore, it remains to check the second term on the right-hand side of (6.3.26).

By the property

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \tag{6.3.29}$$

of the commutant operator, and noting that (6.3.10) implies

$$[D_x^k, J_\delta] = 0, \tag{6.3.30}$$

we obtain

$$[D_x^k, [J_\delta, L]]f = [J_\delta, [D_x^k, L]]f = (J_\delta[D_x^k, L]f - [D_x^k, L]f) - [D_x^k, L](J_\delta f - f). \tag{6.3.31}$$

Since

$$[D_x^k, L]f = D_x^k\left(a(x)\frac{\partial f}{\partial x_i}\right) - a(x)D_x^k\left(\frac{\partial f}{\partial x_i}\right), \tag{6.3.32}$$

similarly to (6.2.34), and noticing (6.2.32), we get

$$\begin{aligned}\|[D_x^k, L]f\|_{L^2(I\!R^n)} &\le C\left(\|D_x a\|_{L^\infty(I\!R^n)}\left\|D_x^{|k|-1}\left(\frac{\partial f}{\partial x_i}\right)\right\|_{L^\infty(I\!R^n)} + \|D_x^{|k|}a\|_{L^2(I\!R^n)}\left\|\frac{\partial f}{\partial x_i}\right\|_{L^\infty(I\!R^n)}\right)\\ &\le C\|D_x a\|_{H^{s-1}(I\!R^n)}\|f\|_{H^s(I\!R^n)},\end{aligned} \tag{6.3.33}$$

where $D^{|k|}$ stands for the set of all partial derivatives of order $|k|$, etc. Similarly we have

$$\|[D_x^k, L](J_\delta f - f)\|_{L^2(I\!R^n)} \le C\|D_x a\|_{H^{s-1}(I\!R^n)}\|J_\delta f - f\|_{H^s(I\!R^n)}. \tag{6.3.34}$$

Then, using (6.3.11)–(6.3.12) in Lemma 6.3.1, and noting (6.3.20), it follows from (6.3.31) that

$$\|[D_x^k, [J_\delta, L]]f\|_{L^2(I\!R^n)} \le C\|D_x a\|_{H^{s-1}(I\!R^n)}\|f\|_{H^s(I\!R^n)} \le C\|f\|_{H^s(I\!R^n)}, \tag{6.3.35}$$

and when $\delta \to 0$,

$$[D_x^k, [J_\delta, L]f] \to 0 \text{ strongly in } L^2(I\!R^n). \tag{6.3.36}$$

This is exactly what we want to prove. We finish the proof of Lemma 6.3.3. □

Now we prove Theorem 6.3.1.

Denote

$$u^\delta(t, \cdot) = J_\delta u(t, \cdot), \tag{6.3.37}$$

where $u = u(t, x)$ is the solution to Cauchy problem (6.1.1)–(6.1.2).

Using (i) and (iv) in Lemma 6.3.1, for any given $\delta > 0$, it follows from (6.2.6)–(6.2.8) that

$$u^\delta \in L^\infty(0, T; H^{s+2}(I\!R^n)), \tag{6.3.38}$$

$$u_t^\delta \in L^\infty(0, T; H^{s+1}(I\!R^n)), \tag{6.3.39}$$

$$u_{tt}^\delta \in L^\infty(0, T; H^{s}(I\!R^n)), \tag{6.3.40}$$

then, after possible change of values on a zero-measure set of the interval $[0, T]$, we have

$$u^\delta \in C([0, T]; H^{s+1}(I\!R^n)), \tag{6.3.41}$$

$$u_t^\delta \in C([0, T]; H^{s}(I\!R^n)). \tag{6.3.42}$$

Acting the mollifier J_δ on both sides of both Eq. (6.1.1) and initial condition (6.1.2), respectively, and noticing (6.3.10), we obtain

$$u_{tt}^\delta - \sum_{i,j=1}^n a_{ij}(t, x) u_{x_i x_j}^\delta - 2\sum_{j=1}^n a_{0j}(t, x) u_{t x_j}^\delta = F^\delta(t, x) + G^\delta, \tag{6.3.43}$$

$$t = 0 : u^\delta = f^\delta, \quad u_t^\delta = g^\delta, \tag{6.3.44}$$

where

$$F^\delta(t, \cdot) = J_\delta F(t, \cdot), \tag{6.3.45}$$

$$f^\delta = J_\delta f, \quad g^\delta = J_\delta g, \tag{6.3.46}$$

and

$$\begin{aligned} G^\delta &= G^\delta(t, x) \\ &= \sum_{i,j=1}^n (J_\delta(a_{ij}(t, x) u_{x_i x_j}) - a_{ij}(t, x) J_\delta u_{x_i x_j}) \\ &\quad + 2\sum_{j=1}^n (J_\delta(a_{0j}(t, x) u_{t x_j}) - a_{0j}(t, x) J_\delta u_{t x_j}) \end{aligned}$$

$$= \sum_{i,j=1}^{n} \left(J_\delta \left(a_{ij}(t,x) \frac{\partial u_{x_i}}{\partial x_j} \right) - a_{ij}(t,x) \frac{\partial}{\partial x_j} (J_\delta u_{x_i}) \right)$$

$$+2 \sum_{j=1}^{n} \left(J_\delta \left(a_{0j}(t,x) \frac{\partial u_t}{\partial x_j} \right) - a_{0j}(t,x) \frac{\partial}{\partial x_j} (J_\delta u_t) \right). \tag{6.3.47}$$

According to (6.3.11)–(6.3.12) in Lemma 6.3.1, and noticing assumptions (6.2.1) and (6.2.5), when $\delta \to 0$ we have

$$f^\delta \to f \text{ strongly in } H^{s+1}(I\!R^n), \tag{6.3.48}$$

$$g^\delta \to g \text{ strongly in } H^s(I\!R^n), \tag{6.3.49}$$

and from Lebesgue dominated convergence theorem we have

$$F^\delta \to F \text{ strongly in } L^2(0,T; H^s(I\!R^n)). \tag{6.3.50}$$

In addition, from Lemma 6.3.3, and noting (6.2.2)–(6.2.4) and (6.2.6)-(6.2.7), we have

$$\|G^\delta(t,\cdot)\|_{H^s(I\!R^n)} \le C\left(\|D_x u(t,\cdot)\|_{H^s(I\!R^n)} + \|u_t(t,\cdot)\|_{H^s(I\!R^n)} \right)$$

$$\le C\left(\|u(t,\cdot)\|_{H^{s+1}(I\!R^n)} + \|u_t(t,\cdot)\|_{H^s(I\!R^n)} \right), \quad \forall t \in [0,T], \tag{6.3.51}$$

and when $\delta \to 0$, for any given $t \in [0,T]$,

$$G^\delta(t,\cdot) \to 0 \text{ strongly in } H^s(I\!R^n). \tag{6.3.52}$$

Hence, from Lebesgue dominated convergence theorem we have, as $\delta \to 0$,

$$G^\delta \to 0 \text{ strongly in } L^2(0,T; H^s(I\!R^n)). \tag{6.3.53}$$

Due to the established estimate (6.2.9), for any given $\delta, \delta' > 0$, it is easy to show by (6.3.43)–(6.3.44) that

$$\|u^\delta(t,\cdot) - u^{\delta'}(t,\cdot)\|^2_{H^{s+1}(I\!R^n)} + \|u^\delta_t(t,\cdot) - u^{\delta'}_t(t,\cdot)\|^2_{H^s(I\!R^n)}$$

$$\le C(T)\Big(\|f^\delta - f^{\delta'}\|^2_{H^{s+1}(I\!R^n)} + \|g^\delta - g^{\delta'}\|^2_{H^s(I\!R^n)}$$

$$+ \int_0^T \|F^\delta(\tau,\cdot) - F^{\delta'}(\tau,\cdot)\|^2_{H^s(I\!R^n)} d\tau$$

$$+ \int_0^T \left(\|G^\delta(\tau,\cdot)\|^2_{H^s(I\!R^n)} + \|G^{\delta'}(\tau,\cdot)\|^2_{H^s(I\!R^n)} \right) d\tau \Big), \quad \forall t \in [0,T]. \tag{6.3.54}$$

Then, using (6.3.48)–(6.3.50) and (6.3.53), and noting (6.3.41)–(6.3.42), we obtain that when $\delta \to 0$,

$$u^{\delta} \text{ strongly in } C([0,T]; H^{s+1}(I\!R^n)), \tag{6.3.55}$$
$$u_t^{\delta} \text{ strongly in } C([0,T]; H^{s}(I\!R^n)). \tag{6.3.56}$$

While, due to (6.2.6)–(6.2.7), similarly to (6.3.50), we obtain that when $\delta \to 0$,

$$u^{\delta} \to u \text{ strongly in } L^2(0,T; H^{s+1}(I\!R^n)), \tag{6.3.57}$$
$$u_t^{\delta} \to u_t \text{ strongly in } L^2(0,T; H^{s}(I\!R^n)). \tag{6.3.58}$$

Hence, when $\delta \to 0$ we have

$$u^{\delta} \to u \text{ strongly in } C([0,T]; H^{s+1}(I\!R^n)), \tag{6.3.59}$$
$$u_t^{\delta} \to u_t \text{ strongly in } C([0,T]; H^{s}(I\!R^n)). \tag{6.3.60}$$

This proves Theorem 6.3.1.

Corollary 6.3.1 *Using Eq. (6.1.1) we obtain that: under the assumptions of Theorem 6.3.1 we also have*

$$u_{tt} \in L^2(0,T; H^{s-1}(I\!R^n)). \tag{6.3.61}$$

Corollary 6.3.2 *If we assume furthermore in Theorem 6.3.1 that*

$$F \in L^{\infty}(0,T; H^{s-1}(I\!R^n)), \tag{6.3.62}$$

then for the solution $u = u(t,x)$ to Cauchy problem (6.1.1)–(6.1.2), we have

$$u_{tt} \in L^{\infty}(0,T; H^{s-1}(I\!R^n)). \tag{6.3.63}$$

Corollary 6.3.3 *If we assume furthermore in Theorem 6.3.1 that*

$$a_{ij} \in C([0,T] \times I\!R^n),\ D_x a_{ij} \in C([0,T]; H^{s-2}(I\!R^n)), \tag{6.3.64}$$
$$a_{0j} \in C([0,T]; H^{s-1}(I\!R^n)) \tag{6.3.65}$$

and

$$F \in C([0,T]; H^{s-1}(I\!R^n)), \tag{6.3.66}$$

then for the solution $u = u(t,x)$ to Cauchy problem (6.1.1)–(6.1.2), we have

$$u_{tt} \in C([0,T]; H^{s-1}(I\!R^n)). \tag{6.3.67}$$

Chapter 7
Reduction of Nonlinear Wave Equations to a Second-Order Quasi-linear Hyperbolic System

7.1 Introduction

As stated before, this book is concerned with the Cauchy problem of nonlinear wave equations with small initial data:

$$\Box u = F(u, Du, D_x Du), \tag{7.1.1}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{7.1.2}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \triangle \ \Big(\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\Big) \tag{7.1.3}$$

is the n-dimensional wave operator,

$$D_x = \Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big),$$

$$D = \Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big) = \Big(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big), \tag{7.1.4}$$

here we denote $x_0 = t$ for convenience, $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$, and $\varepsilon > 0$ is a small parameter.

Let

$$\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \dots, n; (\lambda_{ij}), i, j = 0, 1, \dots, n, i + j \geq 1). \tag{7.1.5}$$

Assume that in a neighborhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \leq \nu_0$ (ν_0 is a suitably small positive number), the nonlinear right-hand side term $F(\hat{\lambda})$ is a sufficiently smooth function satisfying

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$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}), \tag{7.1.6}$$

and $\alpha \geq 1$ is an integer.

This chapter is aimed to show that Cauchy problem (7.1.1)–(7.1.2) of nonlinear wave equations with small initial data can be essentially reduced to studying on the Cauchy problem of quasi-linear hyperbolic equations as

$$\Box u = \sum_{i,j=1}^{n} b_{ij}(u, Du)u_{x_i x_j} + 2\sum_{j=1}^{n} a_{0j}(u, Du)u_{tx_j} + F(u, Du) \tag{7.1.7}$$

with the corresponding initial data (7.1.2). In (7.1.7), if we denote

$$\tilde{\lambda} = (\lambda, (\lambda_i), i = 0, 1, \ldots, n), \tag{7.1.8}$$

then for $|\tilde{\lambda}| \leq \nu_0$, $b_{ij}(\tilde{\lambda})$, $a_{0j}(\tilde{\lambda})$ $(i, j = 1, \ldots, n)$ and $F(\tilde{\lambda})$ are all sufficiently smooth functions satisfying

$$b_{ij}(\tilde{\lambda}) = b_{ji}(\tilde{\lambda}) \ \ (i, j = 1, \ldots, n), \tag{7.1.9}$$

$$b_{ij}(\tilde{\lambda}),\ a_{0j}(\tilde{\lambda}) = O(|\tilde{\lambda}|^{\alpha}) \ \ (i, j = 1, \ldots, n), \tag{7.1.10}$$

$$F(\tilde{\lambda}) = O(|\tilde{\lambda}|^{1+\alpha}), \tag{7.1.11}$$

and $\alpha \geq 1$ is the integer appearing in (7.1.6), and we have

$$\sum_{i,j=1}^{n} a_{ij}(\tilde{\lambda})\xi_i\xi_j \geq m_0|\xi|^2, \ \ \forall \xi \in \mathbb{R}^n, \tag{7.1.12}$$

where m_0 is a positive constant.

$$a_{ij}(\tilde{\lambda}) = \delta_{ij} + b_{ij}(\tilde{\lambda}), \tag{7.1.13}$$

and δ_{ij} is the Kronecker symbol.

For this purpose, we will prove that Cauchy problem (7.1.1)–(7.1.2) can be equivalently reduced to a Cauchy problem of a system of second-order quasi-linear hyperbolic equations of form (7.1.7) with small initial data like (7.1.2) (see Sect. 7.2).

Moreover, on the occasion that the nonlinear right-hand side term F in (7.1.1) satisfies some special requirements, the obtained system of second-order quasi-linear hyperbolic equations of form (7.1.7) also owns some special forms (see Sect. 7.3). For example, when F does not depend on u explicitly:

$$F = F(Du, D_x Du), \tag{7.1.14}$$

the corresponding system (7.1.7) of second-order quasi-linear hyperbolic equations has the following special form:

$$\Box u = \sum_{i,j=1}^{n} b_{ij}(Du)u_{x_i x_j} + 2\sum_{j=1}^{n} a_{0j}(Du)u_{tx_j} + F(Du). \tag{7.1.15}$$

For another example, if F satisfies

$$\partial_u^\beta F(0,0,0) = 0, \quad 1+\alpha \le \beta \le \beta_0, \tag{7.1.16}$$

where $\beta_0 > \alpha$ is an integer, and $\alpha \ge 1$ is the integer appearing in (7.1.6), then the corresponding term $F(u, Du)$ in system (7.1.7) of second-order quasi-linear hyperbolic equations has to satisfy a similar condition

$$\partial_u^\beta F(0,0) = 0, \quad 1+\alpha \le \beta \le \beta_0. \tag{7.1.17}$$

Remark 7.1.1 From (7.1.6) we always have

$$\partial_u^\beta F(0,0,0) = 0, \quad 0 \le \beta \le \alpha, \tag{7.1.18}$$

and (7.1.16) is an additional condition.

According to the results obtained in this chapter, this book will focus on discussing Cauchy problem (7.1.7) and (7.1.2), or its special form (7.1.15) and (7.1.2), of the second-order quasi-linear hyperbolic equation with small initial data.

7.2 Case of a General Nonlinear Right-Hand Side Term F

This section is concerned with Cauchy problem (7.1.1)–(7.1.2) with a general nonlinear right-hand side term

$$F = F(u, Du, D_x Du), \tag{7.2.1}$$

Proposition 7.2.1 *Under assumption (7.1.6), Cauchy problem (7.1.1)–(7.1.2) of the nonlinear wave equation with small initial data can be reduced equivalently to a Cauchy problem of a second-order quasi-linear hyperbolic system of form (7.1.7) satisfying (7.1.9)–(7.1.13) with small initial data like (7.1.2).*

Proof Let $u = u(t,x)$ be a solution to Cauchy problem (7.1.1)–(7.1.2). Setting

$$u_i = \frac{\partial u}{\partial x_i} \quad (i = 1, \dots, n), \tag{7.2.2}$$

and denoting

$$U = (u, u_1, \dots, u_n)^T. \tag{7.2.3}$$

(7.1.1) can be written as

$$\Box u = F(u, DU) \overset{\text{def.}}{=} F(u, Du, Du_1, \ldots, Du_n). \tag{7.2.4}$$

Differentiating the above formula with respect to $x_i (i = 1, \ldots, n)$, respectively, we obtain

$$\Box u_i = \frac{\partial F}{\partial u}(u, DU)\frac{\partial u}{\partial x_i} + \nabla F(u, DU)\frac{\partial DU}{\partial x_i} \quad (i = 1, \ldots, n), \tag{7.2.5}$$

where $\nabla F(u, DU)$ stands for the gradient of F with respect to the variable DU.

It is easy to know from (7.1.6) that (7.2.4)–(7.2.5) can always be written as a second-order quasi-linear hyperbolic system of form (7.1.7) for the vector function U, satisfying the corresponding (7.1.9)–(7.1.13). In addition, the initial conditions corresponding to the vector function U are given by (7.1.2) and

$$t = 0: \ u_i = \varepsilon\frac{\partial \varphi(x)}{\partial x_i}, (u_i)_t = \varepsilon\frac{\partial \psi(x)}{\partial x_i} \ (i = 1, \ldots, n). \tag{7.2.6}$$

Thus, Cauchy problem (7.1.1)–(7.1.2) of the nonlinear wave equation with small initial data is reduced to a Cauchy problem of the second-order quasi-linear hyperbolic system (7.2.4)–(7.2.5) with small initial data (7.1.2) and (7.2.6).

On the contrary, if $U = (u, u_1, \ldots, u_n)^T$ is a solution to the Cauchy problem of the second-order quasi-linear hyperbolic system (7.2.4)–(7.2.5) with small initial data (7.1.2) and (7.2.6), satisfying (7.1.9)–(7.1.13), then u can be proved to be a solution to the original Cauchy problem (7.1.1)–(7.1.2) of the nonlinear wave equation satisfying (7.1.6).

To this end, it suffices to prove (7.2.2). Let

$$\bar{u}_i = \frac{\partial u}{\partial x_i} \quad (i = 1, \ldots, n). \tag{7.2.7}$$

Similarly to the above derivation, differentiating (7.2.4) with respect to $x_i (i = 1, \ldots, n)$, respectively, we have

$$\Box \bar{u}_i = \frac{\partial F}{\partial u}(u, DU)\frac{\partial u}{\partial x_i} + \nabla F(u, DU)\frac{\partial DU}{\partial x_i} (i = 1, \ldots, n), \tag{7.2.8}$$

while, from (7.1.2) we have

$$t = 0: \ \bar{u}_i = \varepsilon\frac{\partial \varphi(x)}{\partial x_i}, (\bar{u}_i)_t = \varepsilon\frac{\partial \psi(x)}{\partial x_i} \ (i = 1, \cdots, n). \tag{7.2.9}$$

Noting the uniqueness of solutions to the Cauchy problem of wave equations, from (7.2.5)–(7.2.6) and (7.2.8)–(7.2.9) we immediately get

$$\bar{u}_i = u_i \quad (i = 1, \ldots, n), \tag{7.2.10}$$

this is just the desired (7.2.2).

The proof of Proposition 2.1 is finished.

Remark 7.2.1 If the initial value $\psi(x)$ of Cauchy problem (7.1.1)–(7.1.2) satisfies

$$\int_{I\!R^n} \psi(x) dx = 0, \tag{7.2.11}$$

then, by the assumption $\psi \in C_0^\infty(I\!R^n)$ and noting (7.2.6), the initial value of the reduced Cauchy problem of form (7.1.7) still satisfies the same assumption as (7.2.11).

7.3 Cases of Special Nonlinear Right-Hand Side Terms F

We first consider the case that the nonlinear right-hand side term F does not depend on u explicitly:

$$F = F(Du, D_x Du). \tag{7.3.1}$$

At this moment, (7.2.4)–(7.2.5), a second-order quasi-linear hyperbolic system obtained in Sect. 7.2, can be simplified to the following form:

$$\Box u = F(DU) \overset{\text{def.}}{=} F(Du, Du_1, \ldots, Du_n) \tag{7.3.2}$$

and

$$\Box u_i = \nabla F(DU) \frac{\partial DU}{\partial x_i} \quad (i = 1, \ldots, n), \tag{7.3.3}$$

in which the right-hand side terms do not depend on u_i $(i = 1, \ldots, n)$ explicitly just like in (7.2.4)–(7.2.5), and they do not depend on u explicitly either. Then, corresponding to Proposition 2.1, we have

Proposition 7.3.1 *For case (7.3.1) that the nonlinear right-hand side term F does not depend on u explicitly, under the corresponding assumption (7.1.6), the Cauchy problem of nonlinear wave equation*

$$\Box u = F(Du, D_x Du) \tag{7.3.4}$$

with small initial data (7.1.2) can be reduced equivalently to a Cauchy problem of a second-order quasi-linear hyperbolic system of form (7.1.15) satisfying (7.1.9)–(7.1.13) with small initial data like (7.1.2).

Now we consider the case that nonlinear right-hand side term $F = F(u, Du, D_x Du)$ satisfies not only (7.1.6) but also the following conditions on its partial derivatives with respect to u:

$$\partial_u^\beta F(0, 0, 0) = 0, \quad 1 + \alpha \leq \beta \leq \beta_0, \tag{7.3.5}$$

where $\beta_0 > \alpha$ is an integer, and $\alpha \geq 1$ in the integer appearing in (7.1.6).

Under the special assumption that (7.3.5) is satisfied, from (7.2.4)–(7.2.5) it is easy to show that: when Cauchy problem (7.1.1)–(7.1.2) of the nonlinear wave equation is reduced to a Cauchy problem of a second-order quasi-linear hyperbolic system of form (7.1.7) with small initial data like (7.1.2), the term $F(u, Du)$ in (7.1.7) has to satisfy a similar condition like (7.3.5):

$$\partial_u^\beta F(0, 0) = 0, \quad 1 + \alpha \leq \beta \leq \beta_0. \tag{7.3.6}$$

Thus, we obtain

Proposition 7.3.2 *Under the assumption that the nonlinear right-hand side term F satisfies not only (7.1.6) but also (7.3.5), Cauchy problem (7.1.1)–(7.1.2) of the nonlinear wave equation with small initial data can be reduced equivalently to a Cauchy problem of a second-order quasi-linear hyperbolic system of form (7.1.7) satisfying (7.1.9)–(7.1.13) with small initial data like (7.1.2), and the term $F(u, Du)$ in (7.1.7) satisfies condition (7.3.6) similar to (7.3.5).*

Remark 7.3.1 $\beta_0 = 2\alpha$ is an important special case. On this occasion, (7.3.5) and (7.3.6) can be written, respectively, as

$$\partial_u^\beta F(0, 0, 0) = 0, \quad 1 + \alpha \leq \beta \leq 2\alpha \tag{7.3.7}$$

and

$$\partial_u^\beta F(0, 0) = 0, \quad 1 + \alpha \leq \beta \leq 2\alpha. \tag{7.3.8}$$

Remark 7.3.2 For the important special case $\alpha = 1$, (7.3.7) and (7.3.8) can be reduced, respectively, to

$$F''_{uu}(0, 0, 0) = 0 \tag{7.3.9}$$

and

$$F''_{uu}(0, 0) = 0. \tag{7.3.10}$$

Chapter 8
Cauchy Problem of One-Dimensional Nonlinear Wave Equations

8.1 Introduction

In this chapter we consider the following Cauchy problem of one-dimensional fully nonlinear wave equations with small initial data:

$$u_{tt} - u_{xx} = F(u, Du, Du_x), \tag{8.1.1}$$

$$t = 0 : u = \varepsilon\phi(x),\ u_t = \varepsilon\psi(x), \tag{8.1.2}$$

where

$$D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right), \tag{8.1.3}$$

$$\phi, \psi \in C_0^\infty(\mathbb{R}), \tag{8.1.4}$$

and $\varepsilon > 0$ is a small parameter.

Let

$$\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1; (\lambda_{ij}), i, j = 0, 1, i + j \geq 1). \tag{8.1.5}$$

Assume that in a neighborhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \leq \nu_0$, the nonlinear term $F(\hat{\lambda})$ is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}), \tag{8.1.6}$$

and $\alpha \geq 1$ is an integer.

This chapter is aimed at studying the life-span $\widetilde{T}(\varepsilon)$ of classical solutions to Cauchy problem (8.1.1)–(8.1.2) for any given integer $\alpha \geq 1$. From the definition of life-span, $\widetilde{T}(\varepsilon)$ is the upper bound of all the values of τ, such that Cauchy problem (8.1.1)–(8.1.2) admits classical solutions on $0 \leq t \leq \tau$, so the maximum interval of existence for classical solutions to Cauchy problem (8.1.1)–(8.1.2) is $[0, \widetilde{T}(\varepsilon))$.

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We will prove that: there exists a suitably small positive number ε_0 such that for any given $\varepsilon \in (0, \varepsilon_0]$, we have the following estimates on the lower bound of life-span (Li et al. 1991b, 1992a):

(i) In general,

$$\widetilde{T}(\varepsilon) \geq a\varepsilon^{-\frac{\alpha}{2}}. \tag{8.1.7}$$

(ii) If

$$\int_{-\infty}^{\infty} \psi(x)dx = 0, \tag{8.1.8}$$

then

$$\widetilde{T}(\varepsilon) \geq a\varepsilon^{-\frac{\alpha(1+\alpha)}{2+\alpha}}. \tag{8.1.9}$$

(iii) If

$$\partial_u^\beta F(0,0,0) = 0, \quad \forall 1+\alpha \leq \beta \leq \beta_0, \tag{8.1.10}$$

then

$$\widetilde{T}(\varepsilon) \geq a\varepsilon^{-\min(\frac{\beta_0}{2},\alpha)}. \tag{8.1.11}$$

When $\beta_0 \geq 2\alpha$, (8.1.11) is reduced to

$$\widetilde{T}(\varepsilon) \geq a\varepsilon^{-\alpha}. \tag{8.1.12}$$

In particular, when the nonlinear right-hand side term F does not depend on u explicitly:

$$F = F(Du, Du_x), \tag{8.1.13}$$

(8.1.12) is also true.

In the above estimates, a stands for a positive constant independent of ε, and $\beta_0 > \alpha$ is an integer.

From the results in Chaps. 13 and 14 we know that the above lower bound estimates of life-span are all sharp.

By Chap. 7, to investigate Cauchy problem (8.1.1)–(8.1.2) of one-dimensional nonlinear wave equation, it suffices, essentially, to consider the following Cauchy problem of one-dimensional quasi-linear hyperbolic equation:

$$u_{tt} - u_{xx} = b(u, Du)u_{xx} + 2a_0(u, Du)u_{tx} + F(u, Du), \tag{8.1.14}$$

$$t = 0 : u = \varepsilon\phi(x),\ u_t = \varepsilon\psi(x), \tag{8.1.15}$$

where (ϕ, ψ) is assumed to satisfy (8.1.4), and $\varepsilon > 0$ is a small parameter. Let

$$\tilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1). \tag{8.1.16}$$

Assume that for $|\tilde{\lambda}| \le \nu_0$, $b(\tilde{\lambda})$, $a_0(\tilde{\lambda})$ and $F(\tilde{\lambda})$ are all sufficiently smooth functions satisfying

$$b(\tilde{\lambda}),\ a_0(\tilde{\lambda}) = O(|\tilde{\lambda}|^{\alpha}), \tag{8.1.17}$$

$$F(\tilde{\lambda}) = O(|\tilde{\lambda}|^{1+\alpha}), \tag{8.1.18}$$

where $\alpha \ge 1$ is an integer. Then, for suitably small ν_0, we have

$$a(\tilde{\lambda}) \overset{\text{def.}}{=} 1 + b(\tilde{\lambda}) \ge m_0, \tag{8.1.19}$$

and m_0 is a positive constant. Moreover, condition (8.1.10) is now reduced to (see Proposition 7.3.2 in Chap. 7)

$$\partial_u^{\beta} F(0,0) = 0, \quad \forall 1 + \alpha \le \beta \le \beta_0. \tag{8.1.20}$$

8.2 Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (8.1.14)–(8.1.15)

8.2.1 Metric Space $X_{S,E,T}$. Main results

In this section, we will prove the lower bound estimates given by (8.1.7) and (8.1.9) for the life-span of classical solutions to Cauchy problem (8.1.14)–(8.1.15) of one-dimensional quasi-linear hyperbolic equation.

From the Sobolev embedding theorem,there exists a suitably small $E_0 > 0$, such that

$$\|f\|_{L^\infty(I\!R)} \le \nu_0, \quad \forall f \in H^1(I\!R), \|f\|_{H^1(I\!R)} \le E_0. \tag{8.2.1}$$

For any given integer $S \ge 4$, and any given positive numbers $E(\le E_0)$ and T, we introduce the following set of functions:

$$X_{S,E,T} = \{v(t,x) | D_{S,T}(v) \le E, \partial_t^l v(0,x) = u_l^{(0)}(x) \ (l = 0, 1, \dots, S)\}, \tag{8.2.2}$$

where

$$D_{S,T}(v) = \sup_{0 \le t \le T} \|v(t,\cdot)\|_{L^\infty(I\!R)} + \sup_{0 \le t \le T} g^{-1}(t) \|v(t,\cdot)\|_{L^{1+\alpha}(I\!R)} + \sup_{0 \le t \le T} \|Dv(t,\cdot)\|_{D,S,2} \tag{8.2.3}$$

with

$$g(t) = \begin{cases} (1+t)^{\frac{1}{1+\alpha}}, & \text{if } \int \psi dx \ne 0; \\ 1, & \text{if } \int \psi dx = 0 \end{cases} \tag{8.2.4}$$

and

$$\|w(t,\cdot)\|_{D,S,2} = \sum_{|k|\le S} \|D^k w(t,\cdot)\|_{L^2(I\!R)}, \quad \forall t \ge 0. \tag{8.2.5}$$

Besides, $u_0^{(0)} = \varepsilon\phi(x)$, $u_1^{(0)} = \varepsilon\psi(x)$, and for every $l = 2, \ldots, S$, $u_l^{(0)}(x)$ is the value of $\partial_t^l u(t,x)$ at $t = 0$, which is uniquely determined by equation (8.1.14) and initial condition (8.1.15). Obviously, $u_l^{(0)}(l = 0, 1, \ldots, S)$ are all sufficiently smooth functions with compact support.

Introduce the following metric on $X_{S,E,T}$:

$$\rho(\overline{v}, \overline{\overline{v}}) = D_{S,T}(\overline{v} - \overline{\overline{v}}), \quad \forall \overline{v}, \overline{\overline{v}} \in X_{S,E,T}. \tag{8.2.6}$$

We want to prove

Lemma 8.2.1 *For suitably small $\varepsilon > 0$, $X_{S,E,T}$ is a nonempty complete metric space.*

Proof Choosing an infinitely differentiable function $a(t)$ on $[0, T]$, such that

$$a(t) = \begin{cases} 0, & \text{if } t \ge \min(T, 1), \\ 1, & \text{near } t = 0, \end{cases} \tag{8.2.7}$$

it is easy to know that for suitably small $\varepsilon > 0$ (the choosing of ε depends on E but not on T), the function

$$v = v(t,x) = a(t) \sum_{l=0}^{S} \frac{t^l}{l!} u_l^{(0)}(x) \tag{8.2.8}$$

belongs to $X_{S,E,T}$, so $X_{S,E,T}$ is not empty.

It is easy to know that $X_{S,E,T}$ constitutes a metric space with respect to the metric (8.2.6). To prove its completeness, let $\{v_i\}$ be a Cauchy sequence in it:

$$\rho(v_i, v_j) \to 0, \quad i, j \to \infty. \tag{8.2.9}$$

Noticing that $L^\infty(0, T; L^\infty(I\!R))$, $L^\infty(0, T; L^{1+\alpha}(I\!R))$ and $L^\infty(0, T; H^{S-l}(I\!R))$ $(l = 0, 1, \ldots, S)$ are all complete Banach spaces, from (8.2.9) we easily know that there exists a function v such that

$$v_i \to v \text{ strongly in } L^\infty(0, T; L^\infty(I\!R)), \tag{8.2.10}$$

$$g(t)v_i \to g(t)v \text{ strongly in } L^\infty(0, T; L^{1+\alpha}(I\!R)), \tag{8.2.11}$$

and for $l = 0, 1, \ldots, S$, we have

$$\partial_t^l D v_i \to \partial_t^l D v \text{ strongly in } L^\infty(0, T; H^{S-l}(I\!R)). \tag{8.2.12}$$

From this it easily yields that

$$\rho(v_i, v) \to 0, \quad i \to \infty \tag{8.2.13}$$

and

$$v \in X_{S,E,T}. \tag{8.2.14}$$

The proof of Lemma 8.2.1 is finished. □

Noting $S > 4$ and the definition of $X_{S,E,T}$, from the Sobolev embedding theorem we know that

$$H^1(I\!R) \subset L^\infty(I\!R) \tag{8.2.15}$$

is a continuous embedding, then, using the interpolation it is easy to prove

Lemma 8.2.2 *For any given $v \in X_{S,E,T}$ with $S \geq 4$, we have*

$$\|v(t,\cdot)\|_{D,[\frac{S}{2}]+2,\infty} \leq CE, \quad \forall t \in [0,T], \tag{8.2.16}$$

and for any given p satisfying $2 \leq p \leq +\infty$, we have

$$\|Dv(t,\cdot)\|_{L^p(I\!R)} \leq CE, \quad \forall t \in [0,T], \tag{8.2.17}$$

where C is positive constant.

The main theorem in this section is as follows.

Theorem 8.2.1 *Under assumptions (8.1.4) and (8.1.17)–(8.1.18), for any given integer $S \geq 4$, there exist positive constants ε_0 and C_0 satisfying $C_0\varepsilon_0 \leq E_0$, such that for any given $\varepsilon \in (0, \varepsilon_0]$ there exists a positive number $T(\varepsilon)$ such that Cauchy problem (8.1.14)–(8.1.15) admits a unique classical solution $u \in X_{S,C_0\varepsilon,T(\varepsilon)}$ on $[0, T(\varepsilon)]$, and $T(\varepsilon)$ can be chosen as*

$$T(\varepsilon) = \begin{cases} a\varepsilon^{-\frac{\alpha}{2}} - 1, & \text{if } \int \psi dx \neq 0; \\ a\varepsilon^{-\frac{\alpha(1+\alpha)}{2+\alpha}} - 1, & \text{if } \int \psi dx = 0, \end{cases} \tag{8.2.18}$$

where a is a positive constant independent of ε.

Moreover, after a possible change of values on a zero-measure set with respect to t on the interval $[0, T(\varepsilon)]$, we have

$$u \in C([0, T(\varepsilon)]; H^{S+1}(I\!R)), \tag{8.2.19}$$

$$u_t \in C([0, T(\varepsilon)]; H^{S}(I\!R)), \tag{8.2.20}$$

$$u_{tt} \in C([0, T(\varepsilon)]; H^{S-1}(I\!R)). \tag{8.2.21}$$

Remark 8.2.1 From the Sobolev embedding theorem we know that $H^1(I\!R) \subset C(I\!R)$ is a continuous embedding. Therefore, the solution $u = u(t, x)$ satisfying (8.2.19)–(8.2.21) is a twice continuously differentiable classical solution to Cauchy problem (8.1.14)–(8.1.15).

Remark 8.2.2 Noticing that $\widetilde{T}(\varepsilon) > T(\varepsilon)$, it follows from (8.2.18) that estimates (8.1.7) and (8.1.9) are true for Cauchy problem (8.1.13)–(8.1.14).

8.2.2 Framework to Prove Theorem 8.2.1—The Global Iteration Method

To prove Theorem 8.2.1, we solve the following Cauchy problem of linear hyperbolic equation for any given $v \in X_{S,E,T}$:

$$u_{tt} - u_{xx} = \hat{F}(v, Dv, Du_x) \overset{\text{def.}}{=} b(v, Dv)u_{xx} + 2a_0(v, Dv)u_{tx} + F(v, Dv), \quad (8.2.22)$$

$$t = 0 : u = \varepsilon\phi(x), \; u_t = \varepsilon\psi(x). \quad (8.2.23)$$

Define a mapping

$$M : v \to u = Mv. \quad (8.2.24)$$

We want to prove that: for suitably small $\varepsilon > 0$, we can find a positive constant C_0 such that when $E = C_0\varepsilon$ and $T = T(\varepsilon)$ is defined by (8.2.18), the mapping M maps $X_{S,E,T}$ into itself and possesses a certain contractive property.

Lemma 8.2.3 *When $E > 0$ is suitably small, for any given $v \in X_{S,E,T}$, after a possible change of values on a zero-measure set of t, we have*

$$u = Mv \in C([0, T]; H^{S+1}(I\!R)), \quad (8.2.25)$$

$$u_t \in C([0, T]; H^S(I\!R)), \quad (8.2.26)$$

$$u_{tt} \in L^\infty(0, T; H^{S-1}(I\!R)). \quad (8.2.27)$$

Proof From the definition of $X_{S,E,T}$, for any given $v \in X_{S,E,T}$ we have

$$Dv \in L^\infty(0, T; H^S(I\!R)). \quad (8.2.28)$$

Hence, since

$$v(t, \cdot) = v(0, \cdot) + \int_0^t v_\tau(\tau, \cdot)d\tau = \varepsilon\phi(x) + \int_0^t v_\tau(\tau, \cdot)d\tau, \quad (8.2.29)$$

noting (8.1.4), it is easy to get

$$v \in L^\infty(0, T; H^S(I\!R)). \tag{8.2.30}$$

So, by (8.1.17)–(8.1.18), using Lemma 5.2.2 and Remark 5.2.1 of Chap. 5 and noticing (8.2.2) and (8.2.15), we have

$$\begin{aligned} & b(v, Dv), a_0(v, Dv), F(v, Dv) \in L^\infty(0, T; H^S(I\!R)), \\ & Db(v, Dv) \in L^\infty(0, T; H^{S-1}(I\!R)). \end{aligned} \tag{8.2.31}$$

In addition, when $E > 0$ is suitably small, we have

$$a(v, Dv) \stackrel{\text{def.}}{=} 1 + b(v, Dv) \geq m_0, \tag{8.2.32}$$

and m_0 is a positive constant.

Therefore, from Theorem 6.3.1 and Corollary 6.3.2 of Chap. 6, the conclusion we want immediately follows. The proof is finished. □

It is easy to prove the following

Lemma 8.2.4 *For $u = u(t, x) = Mv$, the values of $\partial_t^l u(0, x) (l = 0, 1, \ldots, S+1)$ are independent of the choosing of $v \in X_{S,E,T}$, and*

$$\partial_t^l u(0, x) = u_l^{(0)}(x) \quad (l = 0, 1, \ldots, S). \tag{8.2.33}$$

Moreover,

$$\|u(0, \cdot)\|_{D,S+1,p} \leq C_p \varepsilon, \tag{8.2.34}$$

where $1 \leq p \leq +\infty$, C_p is a positive constant, and $\|u(0, \cdot)\|_{D,S+1,p}$ stands for the value of $\|u(t, \cdot)\|_{D,S+1,p}$ at $t = 0$.

The following two lemmas are crucial to the proof of Theorem 8.2.1.

Lemma 8.2.5 *Under the assumptions of Theorem 8.2.1, when $E > 0$ is suitably small, for any given $v \in X_{S,E,T}$, $u = Mv$ satisfies*

$$D_{S,T}(u) \leq C_1\{\varepsilon + (R + \sqrt{R})(E + D_{S,T}(u))\}, \tag{8.2.35}$$

where C_1 is a positive constant independent of E and T, and

$$R = R(E, T) \stackrel{\text{def.}}{=} \begin{cases} E^\alpha (1+T)^2, & \text{if } \int \psi dx \neq 0; \\ E^\alpha (1+T)^{\frac{2+\alpha}{1+\alpha}}, & \text{if } \int \psi dx = 0. \end{cases} \tag{8.2.36}$$

Lemma 8.2.6 *Under the assumptions of Lemma 8.2.5, for any given $\overline{v}, \overline{\overline{v}} \in X_{S,E,T}$, if both $\overline{u} = M\overline{v}$ and $\overline{\overline{u}} = M\overline{\overline{v}}$ also satisfy $\overline{u}, \overline{\overline{u}} \in X_{S,E,T}$, then we have*

$$D_{S-1,T}(\overline{u}-\overline{\overline{u}}) \leq C_2(R+\sqrt{R})(D_{S-1,T}(\overline{u}-\overline{\overline{u}})+D_{S-1,T}(\overline{v}-\overline{\overline{v}})), \quad (8.2.37)$$

where C_2 is a positive constant independent of E and T, and $R = R(E, T)$ is still defined by (8.2.36).

The proofs of Lemmas 8.2.5 and 8.2.6 will be given later. Now we first use these two lemmas to prove Theorem 8.2.1.

Proof of Theorem 8.2.1 Take

$$C_0 = 3\max(C_1, C_2), \quad (8.2.38)$$

where C_1 and C_2 are the positive constants appearing in Lemmas 8.2.5 and 8.2.6, respectively.

From the proof of Lemma 8.2.1 we can show that, if we take $E(\varepsilon) = C_0\varepsilon$ and $T(\varepsilon)$ as shown in (8.2.18), then $X_{S,E(\varepsilon),T(\varepsilon)}$ is not empty as long as $\varepsilon > 0$ is sufficiently small.

First we prove that we can choose a suitable constant a in (8.2.18), such that

$$R(E(\varepsilon), T(\varepsilon)) + \sqrt{R(E(\varepsilon), T(\varepsilon))} \leq \frac{1}{C_0}. \quad (8.2.39)$$

In fact, from (8.2.36) and (8.2.18), when $\int \psi dx \neq 0$ or $\int \psi dx = 0$, we have, respectively,

$$R(E(\varepsilon), T(\varepsilon)) + \sqrt{R(E(\varepsilon), T(\varepsilon))} = C_0^{\alpha} a^2 + C_0^{\frac{\alpha}{2}} a$$

or

$$R(E(\varepsilon), T(\varepsilon)) + \sqrt{R(E(\varepsilon), T(\varepsilon))} = C_0^{\alpha} a^{\frac{2+\alpha}{1+\alpha}} + C_0^{\frac{\alpha}{2}} a^{\frac{2+\alpha}{2(1+\alpha)}}.$$

Then, (8.2.39) is true as long as $a > 0$ is suitably small.

Using (8.2.39), from Lemma 8.2.5 we easily obtain that: there exists a suitably small $\varepsilon_0 > 0$, such that for all $\varepsilon (0 < \varepsilon \leq \varepsilon_0)$, for any given $v \in X_{S,E(\varepsilon),T(\varepsilon)}$, $u = Mv$ satisfies

$$D_{S,T(\varepsilon)}(u) \leq E(\varepsilon). \quad (8.2.40)$$

Noticing Lemma 8.2.4, we have $u = Mv \in X_{S,E(\varepsilon),T(\varepsilon)}$, i.e., M maps $X_{S,E(\varepsilon),T(\varepsilon)}$ into itself. Furthermore, from Lemma 8.2.6 we obtain that: for all $\varepsilon (0 < \varepsilon \leq \varepsilon_0)$, for any given $\overline{v}, \overline{\overline{v}} \in X_{S,E(\varepsilon),T(\varepsilon)}$, setting $\overline{u} = M\overline{v}$, $\overline{\overline{u}} = M\overline{\overline{v}}$, we then have

$$D_{S-1,T(\varepsilon)}(\overline{u}-\overline{\overline{u}}) \leq \frac{1}{2} D_{S-1,T(\varepsilon)}(\overline{v}-\overline{\overline{v}}), \quad (8.2.41)$$

that is, M is a contract mapping with respect to the metric of space $X_{S-1,E(\varepsilon),T(\varepsilon)}$.

Lemma 8.2.7 *$X_{S,E,T}$ is a closed subset of $X_{S-1,E,T}$.*

Proof It suffices to prove that if

$$v_i \in X_{S,E,T}, \tag{8.2.42}$$

and as $i \to \infty$, we have

$$v_i \to v \text{ in } X_{S-1,E,T}, \tag{8.2.43}$$

then

$$v \in X_{S,E,T}. \tag{8.2.44}$$

In fact, from (8.2.43) and the definition of the metric in $X_{S-1,E,T}$ we have

$$v_i \to v \text{ strongly in } L^\infty(0, T; L^\infty(I\!R)), \tag{8.2.45}$$

$$g(t)v_i \to g(t)v \text{ strongly in } L^\infty(0, T; L^{1+\alpha}(I\!R)), \tag{8.2.46}$$

and for $l = 0, 1, \ldots, S-1$, we have

$$\partial_t^l D v_i \to \partial_t^l D v \text{ strongly in } L^\infty(0, T; H^{S-1-l}(I\!R)). \tag{8.2.47}$$

By (8.2.42) and the definition of $X_{S,E,T}$, we have

$$D_{S,T}(v_i) \le E, \quad i = 1, 2, \ldots. \tag{8.2.48}$$

Thus, noting (8.2.47), for $l = 0, 1, \ldots, S$, we get

$$\partial_t^l D v_i \overset{*}{\rightharpoonup} \partial_t^l D v \text{ weak} * \text{ in } L^\infty(0, T; H^{S-l}(I\!R)). \tag{8.2.49}$$

From this, noting that for $i = 1, 2, \ldots$, we have

$$\partial_t^l v_i(0, x) = u_l^{(0)}(x) \quad (l = 0, 1, \ldots, S), \tag{8.2.50}$$

it is easy to show that

$$\partial_t^l v(0, x) = u_l^{(0)}(x) \quad (l = 0, 1, \ldots, S). \tag{8.2.51}$$

Combining (8.2.45)–(8.2.46) and (8.2.49), from (8.2.48) we then have

$$D_{S,T}(v) \le E. \tag{8.2.52}$$

(8.2.44) is proved. □

Now we prove that: for all $\varepsilon (0 < \varepsilon \le \varepsilon_0)$, the mapping M has a unique fixed point $u \in X_{S,E(\varepsilon),T(\varepsilon)}$ on $X_{S,E(\varepsilon),T(\varepsilon)}$:

$$u = Mu, \tag{8.2.53}$$

so $u = u(t, x)$ is a classical solution to Cauchy problem (8.1.14)–(8.1.15) on $[0, T(\varepsilon)]$.

The uniqueness of this fixed point is an obvious consequence of the contract property of M under the metric of space $X_{S-1,E(\varepsilon),T(\varepsilon)}$. To prove the existence, we take any

$$u^{(0)} \in X_{S,E(\varepsilon),T(\varepsilon)} \tag{8.2.54}$$

as the zero-order approximation and use

$$u^{(i+1)} = Mu^{(i)} \quad (i = 0, 1, 2, \ldots) \tag{8.2.55}$$

to construct an iteration sequence. Since M maps $X_{S,E(\varepsilon),T(\varepsilon)}$ into itself, we have

$$u^{(i)} \in X_{S,E(\varepsilon),T(\varepsilon)} \quad (i = 0, 1, 2, \ldots). \tag{8.2.56}$$

From the contraction of M in $X_{S-1,E(\varepsilon),T(\varepsilon)}$, this iteration generates a fixed point in $X_{S-1,E(\varepsilon),T(\varepsilon)}$:

$$u \in X_{S-1,E(\varepsilon),T(\varepsilon)}, \tag{8.2.57}$$

such that (8.2.53) is satisfied, and as $i \to \infty$ we have

$$u^{(i)} \to u \text{ in } X_{S-1,E(\varepsilon),T(\varepsilon)}. \tag{8.2.58}$$

Then, from Lemma 8.2.7 we immediately get

$$u \in X_{S,E(\varepsilon),T(\varepsilon)}, \tag{8.2.59}$$

which is the unique fixed point of M in the space $X_{S,E(\varepsilon),T(\varepsilon)}$. Furthermore, by Lemma 8.2.3, (8.2.19)–(8.2.20) hold, so it is easy to show that

$$b(u, Du), a_0(u, Du), F(u, Du) \in C([0, T]; H^S(\mathbb{R})), \tag{8.2.60}$$

then (8.2.21) follows immediately from Corollary 6.3.3 of Chap. 6. The proof of Theorem 8.2.1 is finished.

8.2.3 Proof of Lemma 8.2.5

We first estimate $\|u(t, \cdot)\|_{L^\infty(\mathbb{R})}$.

Using (8.1.4) in Theorem 4.1.1 of Chap. 4 (in which we take $p = +\infty$), from (8.2.22)–(8.2.23) we get

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \le \varepsilon(\|\phi\|_{L^\infty(\mathbb{R})} + \|\psi\|_{L^1(\mathbb{R})}) + \int_0^t \|\hat{F}(v, Dv, Du_x)(\tau, \cdot)\|_{L^1(\mathbb{R})} d\tau$$

$$\leq C\varepsilon + \int_0^t \|\hat{F}(v, Dv, Du_x)(\tau, \cdot)\|_{L^1(I\!R)} d\tau, \tag{8.2.61}$$

Here and hereafter, C stands for a positive constant independent of ε.

By Hölder inequality (see Lemma 5.1.1 in Chap. 5), noting (8.1.17)–(8.1.18), (8.2.15), (8.2.17) and the definition of $X_{S,E,T}$, and using Lemma 5.2.2 and Remark 5.2.1 in Chap. 5 as well as Lemma 3.4.1 in Chap. 3, it is easy to have

$$\begin{aligned}
&\|(b(v, Dv)u_{xx} + 2a_0(v, Dv)u_{tx})(\tau, \cdot)\|_{L^1(I\!R)} \\
&\leq C\|(v, Dv)(\tau, \cdot)\|^{\alpha}_{L^{1+\alpha}(I\!R)} \|Du_x(\tau, \cdot)\|_{L^{1+\alpha}(I\!R)} \\
&\leq CE^{\alpha} g^{\alpha}(\tau) \|Du_x(\tau, \cdot)\|^{1-\frac{2}{1+\alpha}}_{L^{\infty}(I\!R)} \|Du_x(\tau, \cdot)\|^{\frac{2}{1+\alpha}}_{L^2(I\!R)} \\
&\leq CE^{\alpha} g^{\alpha}(\tau) \|Du_x(\tau, \cdot)\|_{H^1(I\!R)} \\
&\leq CE^{\alpha} g^{\alpha}(\tau) D_{S,T}(u)
\end{aligned} \tag{8.2.62}$$

and

$$\|F(v, Dv)(\tau, \cdot)\|_{L^1(I\!R)} \leq C\|(v, Dv)\|^{1+\alpha}_{L^{1+\alpha}(I\!R)} \leq CE^{1+\alpha} g^{1+\alpha}(\tau). \tag{8.2.63}$$

Then, noticing (8.2.4), from (8.2.61) we obtain

$$\|u(t, \cdot)\|_{L^{\infty}(I\!R)} \leq C\varepsilon + CE^{\alpha}(1+t)^{\kappa}(E + D_{S,T}(u)), \tag{8.2.64}$$

where

$$\kappa = \begin{cases} 2, & \text{if } \int \psi dx \neq 0; \\ 1, & \text{if } \int \psi dx = 0. \end{cases} \tag{8.2.65}$$

Noticing (8.2.36), from (8.2.64) we get

$$\sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^{\infty}(I\!R)} \leq C\{\varepsilon + R(E, T)(E + D_{S,T}(u))\}. \tag{8.2.66}$$

Now we estimate $\|u(t, \cdot)\|_{L^{1+\alpha}(I\!R)}$.

Using (8.1.4) and (8.1.8) in Theorem 4.1.1 of Chap. 4 (in which we take $p = 1+\alpha$) and noting (8.2.4), similarly to (8.2.66), it follows from (8.2.22)–(8.2.23) that

$$\begin{aligned}
\|u(t, \cdot)\|_{L^{1+\alpha}(I\!R)} &\leq C\varepsilon g(t) + C\int_0^t (t-\tau)^{\frac{1}{1+\alpha}} \|\hat{F}(v, Dv, Du_x)(\tau, \cdot)\|_{L^1(I\!R)} d\tau \\
&\leq C\varepsilon g(t) + CE^{\alpha}(1+t)^{\frac{2+\alpha}{1+\alpha}} g^{1+\alpha}(t)(E + D_{S,T}(u)) \\
&\leq Cg(t)\{\varepsilon + R(E, T)(E + D_{S,T}(u))\},
\end{aligned} \tag{8.2.67}$$

so

$$\sup_{0 \leq t \leq T} g^{-1}(t)\|u(t, \cdot)\|_{L^{1+\alpha}(I\!R)} \leq C\{\varepsilon + R(E, T)(E + D_{S,T}(u))\}. \tag{8.2.68}$$

Finally, we estimate $\|Du(t,\cdot)\|_{D,S,2}$.

For any given double-index $k=(k_1,k_2)$ with $0\le|k|\le S$, acting D^k on both sides of (8.2.22), taking the inner product with $D^k u_t$ in L^2, and integrating with respect to t, similarly to (6.2.31) in Chap. 6, we obtain the following energy integral formula:

$$
\begin{aligned}
&\|D^k u_t(t,\cdot)\|^2_{L^2(I\!R)}+\int_{I\!R} a(v,Dv)(t,\cdot)(D^k u_x(t,\cdot))^2dx\\
=&\|D^k u_t(0,\cdot)\|^2_{L^2(I\!R)}+\int_{I\!R} a(v,Dv)(0,\cdot)(D^k u_x(0,\cdot))^2dx\\
&+\int_0^t\int_{I\!R}\frac{\partial b(v,Dv)(\tau,\cdot)}{\partial\tau}(D^k u_x(\tau,\cdot))^2dxd\tau\\
&-2\int_0^t\int_{I\!R}\frac{\partial b(v,Dv)(\tau,\cdot)}{\partial x}D^k u_x(\tau,\cdot)D^k u_\tau(\tau,\cdot)dxd\tau\\
&-2\int_0^t\int_{I\!R}\frac{\partial a_0(v,Dv)(\tau,\cdot)}{\partial x}(D^k u_\tau(\tau,\cdot))^2dxd\tau\\
&+2\int_0^t\int_{I\!R}G_k(\tau,\cdot)D^k u_\tau(\tau,\cdot)dxd\tau\\
&+2\int_0^t\int_{I\!R}g_k(\tau,\cdot)D^k u_\tau(\tau,\cdot)dxd\tau\\
\stackrel{\text{def.}}{=}&\|D^k u_t(0,\cdot)\|^2_{L^2(I\!R)}+\int_{I\!R} a(v,Dv)(0,\cdot)(D^k u_x(0,\cdot))^2dx\\
&+\text{I}+\text{II}+\text{III}+\text{IV}+\text{V},
\end{aligned}
\tag{8.2.69}
$$

where the function $a(\cdot)$ is defined by (8.1.19), and

$$
\begin{aligned}
G_k&=D^k(b(v,Dv)u_{xx})-b(v,Dv)D^k u_{xx}\\
&\quad+2[D^k(a_0(v,Dv)u_{tx})-a_0(v,Dv)D^k u_{tx}],
\end{aligned}
\tag{8.2.70}
$$

$$
g_k=D^kF(v,Dv). \tag{8.2.71}
$$

Noticing (8.1.17) and (8.2.16), it is easy to have

$$
|\text{I}|,|\text{II}|,|\text{III}|\le CE^\alpha(1+t)D^2_{S,T}(u)\le CR(E,T)D^2_{S,T}(u). \tag{8.2.72}
$$

Using (5.1.24) in Chap. 5 (in which we take $p=q=p_2=q_2=p_3=q_3=2,\ p_1=q_1=p_4=q_4=+\infty$, and $\chi\equiv1$) and Lemma 5.2.2 and Remark 5.2.1 in Chap. 5 (in which we take $p=q=p_i=q_i=+\infty$ $(i=0,1,\dots,\beta)$ or $p=q=p_0=q_0=2$, and $p_i=q_i=+\infty(i=1,\dots,\beta)$) and noting (8.2.15) and (8.2.16), we obtain

$$
\|G_k(\tau,\cdot)\|_{L^2(I\!R)}\le CE^\alpha D_{S,T}(u). \tag{8.2.73}
$$

When $|k| > 0$, using (5.2.15) in Chap. 5 and noting (8.2.16), from (8.1.18) we get

$$\|g_k(\tau,\cdot)\|_{L^2(I\!R)} \le CE^{\alpha+1}. \tag{8.2.74}$$

While, when $|k| = 0$, from Hölder inequality we have

$$\begin{aligned}\|g_0(\tau,\cdot)\|_{L^2(I\!R)} &= \|F(v, Dv)(\tau,\cdot)\|_{L^2(I\!R)}\\ &\le C\|(v, Dv)(\tau,\cdot)\|^{1+\alpha}_{L^{2(1+\alpha)}(I\!R)}.\end{aligned} \tag{8.2.75}$$

Furthermore, using the interpolation inequality (see Lemma 3.4.1 in Chap. 3) and noticing the definition of $X_{S,E,T}$, we have

$$\begin{aligned}\|v(\tau,\cdot)\|_{L^{2(1+\alpha)}(I\!R)} &\le \|v(\tau,\cdot)\|^{\frac{1}{2}}_{L^\infty(I\!R)}\|v(\tau,\cdot)\|^{\frac{1}{2}}_{L^{1+\alpha}(I\!R)}\\ &\le CE(g(\tau))^{\frac{1}{2}},\end{aligned} \tag{8.2.76}$$

Then by (8.2.17) we get

$$\|g_0(\tau,\cdot)\|_{L^2(I\!R)} \le CE^{1+\alpha}(g(\tau))^{\frac{1+\alpha}{2}}. \tag{8.2.77}$$

From (8.2.73)–(8.2.74) and (8.2.77) we obtain

$$|IV| \le CE^\alpha(1+t)D^2_{S,T}(u) \le CR(E,T)D^2_{S,T}(u) \tag{8.2.78}$$

and

$$\begin{aligned}|\mathrm{V}| &\le CE^{1+\alpha}(g(t))^{\frac{1+\alpha}{2}}(1+t)D_{S,T}(u)\\ &\le CR(E,T)ED_{S,T}(u).\end{aligned} \tag{8.2.79}$$

Thus, using (8.2.72) and (8.2.78)–(8.2.79), and noticing (8.2.34) and (8.2.32), from (8.2.69) we have

$$\sup_{0\le t\le T}\|Du(t,\cdot)\|_{D,S,2} \le C\{\varepsilon + \sqrt{R(E,T)}(E + D_{S,T}(u))\}. \tag{8.2.80}$$

Combining (8.2.66), (8.2.68) and (8.2.80), we get the desired estimate (8.2.35). The proof of Lemma 8.2.5 is finished.

8.2.4 Proof of Lemma 8.2.6

Let

$$u^* = \overline{u} - \overline{\overline{u}}, \quad v^* = \overline{v} - \overline{\overline{v}}. \tag{8.2.81}$$

From the definition of mapping M we have

$$u^*_{tt} - a(\overline{v}, D\overline{v})u^*_{xx} - 2a_0(\overline{v}, D\overline{v})u^*_{tx} = F^*, \tag{8.2.82}$$

$$t = 0 : u^* = u^*_t = 0, \tag{8.2.83}$$

where the function $a(\cdot)$ is defined by (8.1.19), and

$$F^* = (b(\overline{v}, D\overline{v}) - b(\overline{\overline{v}}, D\overline{\overline{v}}))\overline{\overline{u}}_{xx} + 2(a_0(\overline{v}, D\overline{v}) - a_0(\overline{\overline{v}}, D\overline{\overline{v}}))\overline{\overline{u}}_{tx} + F(\overline{v}, D\overline{v}) - F(\overline{\overline{v}}, D\overline{\overline{v}}). \tag{8.2.84}$$

Similarly to (8.2.66) and (8.2.68), now we have

$$\sup_{0\le t\le T} \|u^*(t,\cdot)\|_{L^\infty(I\!R)} \le CR(E,T)(D_{S-1,T}(u^*) + D_{S-1,T}(v^*)) \tag{8.2.85}$$

and

$$\sup_{1\le t\le T} g^{-1}(t)\|u^*(t,\cdot)\|_{L^{1+\alpha}(I\!R)} \le CR(E,T)(D_{S-1,T}(u^*) + D_{S-1,T}(v^*)). \tag{8.2.86}$$

Now we estimate $\|Du^*(t,\cdot)\|_{D,S-1,2}$.

For any given double-index $k = (k_0, k_1)$ with $0 \le |k| \le S-1$, similarly to (8.2.69), we have

$$\begin{aligned}
&\|D^k u^*_t(t,\cdot)\|^2_{L^2(I\!R)} + \int_{I\!R} a(\overline{v}, D\overline{v})(t,\cdot)(D^k u^*_x(t,\cdot))^2 dx\\
&= \int_0^t \int_{I\!R} \frac{\partial b(\overline{v}, D\overline{v})(\tau,\cdot)}{\partial \tau}(D^k u^*_x(\tau,\cdot))^2 dx d\tau\\
&\quad - 2\int_0^t \int_{I\!R} \frac{\partial b(\overline{v}, D\overline{v})(\tau,\cdot)}{\partial x} D^k u^*_x(\tau,\cdot) D^k u^*_\tau(\tau,\cdot) dx d\tau\\
&\quad - 2\int_0^t \int_{I\!R} \frac{\partial a_0(\overline{v}, D\overline{v})(\tau,\cdot)}{\partial x}(D^k u^*_\tau(\tau,\cdot))^2 dx d\tau\\
&\quad + 2\int_0^t \int_{I\!R} \overline{G}_k(\tau,\cdot) D^k u^*_\tau(\tau,\cdot) dx d\tau\\
&\quad + 2\int_0^t \int_{I\!R} \overline{g}_k(\tau,\cdot) D^k u^*_\tau(\tau,\cdot) dx d\tau\\
&\overset{\text{def.}}{=} \text{I+II+III+IV+V},
\end{aligned} \tag{8.2.87}$$

where

$$\begin{aligned}
\overline{G}_k &= D^k(b(\overline{v}, D\overline{v})u^*_{xx}) - b(\overline{v}, D\overline{v})D^k u^*_{xx}\\
&\quad + 2(D^k(a_0(\overline{v}, D\overline{v})u^*_{tx}) - a_0(\overline{v}, D\overline{v})D^k u^*_{tx}),
\end{aligned} \tag{8.2.88}$$

$$\overline{g}_k = D^k F^*. \tag{8.2.89}$$

Similarly to (8.2.72) and (8.2.78), now we have

$$|\mathrm{I}|+|\mathrm{II}|+|\mathrm{III}|+|\mathrm{IV}| \le CE^{\alpha}(1+t)D^2_{S-1,T}(u^*) \\ \le CR(E,T)D^2_{S-1,T}(u^*). \qquad (8.2.90)$$

Moreover, similarly to (8.2.74) and (8.2.77), respectively, we have, for $|k|>0$,

$$\|\overline{g}_k(\tau,\cdot)\|_{L^2(I\!R)} \le CE^{\alpha}D_{S-1,T}(v^*), \qquad (8.2.91)$$

while

$$\|\overline{g}_0(\tau,\cdot)\|_{L^2(I\!R)} \le CE^{\alpha}(g(\tau))^{\frac{1+\alpha}{2}}D_{S-1,T}(v^*), \qquad (8.2.92)$$

hence

$$|\mathrm{V}| \le CE^{\alpha}(g(t))^{\frac{1+\alpha}{2}}(1+t)D_{S-1,T}(u^*)D_{S-1,T}(v^*) \\ \le CR(E,T)D_{S-1,T}(u^*)D_{S-1,T}(v^*). \qquad (8.2.93)$$

So, by (8.2.90) and (8.2.93), from (8.2.87) we get

$$\sup_{0\le t\le T}\|Du^*(t,\cdot)\|_{D,S-1,T} \le C\sqrt{R(E,T)}(D_{S-1,T}(u^*)+D_{S-1,T}(v^*)). \qquad (8.2.94)$$

Combining (8.2.85)–(8.2.86) and (8.2.94), we get the desired (8.2.37). The proof of Lemma 8.2.6 is finished.

8.3 Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (8.1.14)–(8.1.15) (Continued)

8.3.1 Metric Space $X_{S,E,T}$. Main results

In this section, we investigate the special case satisfying assumption (8.1.20), and prove that Cauchy problem (8.1.14)–(8.1.15) has the lower bound estimate of form (8.1.11) for the life-span of classical solutions. We use arguments similar to the previous section, and here we only present some essentially different points.

In this case, no matter (8.1.8) is true or not, we still introduce the function set $X_{S,E,T}$ by (8.2.2), but instead of (8.2.3), we take

$$D_{S,T}(v)=\sup_{0\le t\le T}\|v(t,\cdot)\|_{L^\infty(I\!R)}+\sup_{0\le t\le T}(1+t)^{-\frac{1}{1+\beta_0}}\|v(t,\cdot)\|_{L^{1+\beta_0}(I\!R)}+\sup_{0\le t\le T}\|Dv(t,\cdot)\|_{D,S,2}, \qquad (8.3.1)$$

where $\beta_0 > \alpha$ is the integer appearing in (8.1.20). We have

Theorem 8.3.1 *Under the assumptions of Theorem 8.2.1, assume furthermore that (8.1.20) holds, we have the same conclusion as Theorem 8.2.1, but instead of (8.2.18), we now take*

$$T(\varepsilon) = a\varepsilon^{-\min(\frac{\beta_0}{2},\alpha)} - 1, \tag{8.3.2}$$

where α is a positive constant independent of ε.

To prove Theorem 8.3.1, it suffices to prove the following two lemmas.

Lemma 8.3.1 *Under the assumptions of Theorem 8.3.1, when $E > 0$ is suitably small, for any given $v \in X_{S,E,T}$, $u = Mv$ satisfies*

$$D_{S,T}(u) \le C_1\{\varepsilon + (R^2 + R + \sqrt{R})(E + D_{S,T}(u))\}, \tag{8.3.3}$$

where C_1 is a positive constant independent of E and T, and

$$R = R(E,T) \overset{\text{def.}}{=} E^{\min(\frac{\beta_0}{2},\alpha)}(1+T). \tag{8.3.4}$$

Lemma 8.3.2 *Under the assumptions of Lemma 8.3.1, for any given $\overline{v}, \overline{\overline{v}} \in X_{S,E,T}$, if both $\overline{u} = M\overline{v}$ and $\overline{\overline{u}} = M\overline{\overline{v}}$ satisfy $\overline{u}, \overline{\overline{u}} \in X_{S,E,T}$, then we have*

$$D_{S-1,T}(\overline{u} - \overline{\overline{u}}) \le C_2(R^2 + R + \sqrt{R})(D_{S-1,T}(\overline{u} - \overline{\overline{u}}) + D_{S-1,T}(\overline{v} - \overline{\overline{v}})), \tag{8.3.5}$$

where C_2 is a positive constant independent of E and T, and $R = R(E,T)$ is still defined by (8.3.4).

8.3.2 Proof of Lemma 8.3.1

Noting (8.1.20), we can rewrite $\hat{F}$ as

$$\begin{aligned}
&\hat{F}(v, Dv, Du_x)\\
=\;& (b(v,0)u_x)_x - b_x(v,0)u_x + (b(v,Dv) - b(v,0))u_{xx}\\
&+ 2(a_0(v,0)u_x)_t - 2a_{0t}(v,0)u_x + 2(a_0(v,Dv) - a_0(v,0))u_{xt}\\
&+ (F(v,Dv) - F(v,0) - F_{Dv}(v,0)Dv) + F(v,0) + F_{Dv}(v,0)Dv\\
=\;& \sum_{i=0}^{1}\partial_i G_i(v,u_x) + \sum_{i=0}^{1} A_i(v)v_{x_i}u_x + \sum_{i,j=0}^{1} B_{ij}(v,Dv)v_{x_i}u_{xx_j} + \sum_{i,j=0}^{1} C_{ij}(v,Dv)v_{x_i}v_{x_j} + F(v,0),
\end{aligned} \tag{8.3.6}$$

where $(x_0, x_1) = (t, x)$, $(\partial_0, \partial_1) = (\partial_t, \partial_x) = D$, and by (8.1.17)–(8.1.18) and (8.1.20), in a neighborhood of $\overline{\lambda} = 0$ we have

$$G_i(\overline{\lambda}) = O(|\overline{\lambda}|^{1+\alpha}) \quad (i = 0, 1; \overline{\lambda} = (\lambda, \lambda_1)), \tag{8.3.7}$$

and $G_i(\overline{\lambda})$ is affine with respect to the variable λ_1,

$$A_i(\lambda) = O(|\lambda|^{\alpha-1}) \quad (i = 0, 1), \tag{8.3.8}$$
$$B_{ij}(\tilde{\lambda}), C_{ij}(\tilde{\lambda}) = O(|\tilde{\lambda}|^{\alpha-1}) \quad (i, j = 0, 1; \tilde{\lambda} = (\lambda, \lambda_0, \lambda_1)) \tag{8.3.9}$$

and

$$F(\lambda, 0) = O(|\lambda|^{1+\beta_0}). \tag{8.3.10}$$

Thus, the solution $u = Mv$ to Cauchy problem (8.2.22)–(8.2.23) can be expressed by

$$u = w^{(0)} + \sum_{i=0}^{1} \partial_i w^{(i)} - u^{(0)} + u^{(1)} + u^{(2)}, \tag{8.3.11}$$

where $w^{(0)}$ is the solution to Cauchy problem

$$w_{tt}^{(0)} - w_{xx}^{(0)} = 0, \tag{8.3.12}$$
$$t = 0 : w^{(0)} = \varepsilon\phi(x), w_t^{(0)} = \varepsilon\psi(x), \tag{8.3.13}$$

and $w^{(i)}(i = 0, 1)$, $u^{(0)}$, $u^{(1)}$ and $u^{(2)}$ satisfy equations

$$w_{tt}^{(i)} - w_{xx}^{(i)} = G_i(v, u_x), \quad (i = 0, 1), \tag{8.3.14}$$
$$u_{tt}^{(0)} - u_{xx}^{(0)} = 0, \tag{8.3.15}$$
$$u_{tt}^{(1)} - u_{xx}^{(1)} = \sum_{i=0}^{1} A_i(v) v_{x_i} u_x + \sum_{i,j=0}^{1} B_{ij}(v, Dv) v_{x_i} u_{x x_j} + \sum_{i,j=0}^{1} C_{ij}(v, Dv) v_{x_i} v_{x_j} \tag{8.3.16}$$

and

$$u_{tt}^{(2)} - u_{xx}^{(2)} = F(v, 0), \tag{8.3.17}$$

respectively; moreover, $u^{(0)}$ satisfies the initial condition

$$t = 0 : u^{(0)} = 0, u_t^{(0)} = G_0(v, u_x)(0, x), \tag{8.3.18}$$

while, $w^{(i)}(i = 0, 1)$, $u^{(1)}$ and $u^{(2)}$ all satisfy the zero initial condition.

We first estimate $\|u(t, \cdot)\|_{L^\infty(I\!R)}$.

By (4.1.5) in Chap. 4, noting (8.3.7) and (8.2.15), it is easy to know that, when $E > 0$ is suitably small, for $i = 0, 1$ we have

$$\|\partial_i w^{(i)}(t, \cdot)\|_{L^\infty(I\!R)} \le \int_0^t \|G_i(v, u_x)(\tau, \cdot)\|_{L^\infty(I\!R)} d\tau$$

$$
\begin{aligned}
&\le CE^{\alpha}(1+t)(E+D_{S,T}(u))\\
&\le CR(E,T)(E+D_{S,T}(u)),
\end{aligned}
\tag{8.3.19}
$$

here and hereafter, C always stands for a positive constant independent of E and T, and $R(E,T)$ is defined by (8.3.4).

By (4.1.4) in Chap. 4, noting (8.3.7) and (8.2.34), we have

$$
\|u^{(0)}(t,\cdot)\|_{L^{\infty}(I\!R)} \le C\varepsilon. \tag{8.3.20}
$$

Using (4.1.4) in Chap. 4 again, and noticing (8.3.8)–(8.3.10) and the definition of $X_{S,E,T}$, it is easy to show

$$
\begin{aligned}
\|u^{(1)}(t,\cdot)\|_{L^{\infty}(I\!R)} \le\ & C\Bigg\{\int_0^t \sum_{i=0}^{1}\|A_i(v)v_{x_i}u_x(\tau,\cdot)\|_{L^1(I\!R)}d\tau\\
&+\int_0^t \sum_{i,j=0}^{1}\|B_{ij}(v,Dv)v_{x_i}u_{xx_j}(\tau,\cdot)\|_{L^1(I\!R)}d\tau\\
&+\int_0^t \sum_{i,j=0}^{1}\|C_{ij}(v,Dv)v_{x_i}v_{x_j}(\tau.\cdot)\|_{L^1(I\!R)}d\tau\Bigg\}\\
\le\ & CE^{\alpha}(1+t)(E+D_{S,T}(u))\\
\le\ & CR(E,T)(E+D_{S,T}(u))
\end{aligned}
\tag{8.3.21}
$$

and

$$
\begin{aligned}
\|u^{(2)}(t,\cdot)\|_{L^{\infty}(I\!R)} &\le C\int_0^t \|F(v,0)(\tau,\cdot)\|_{L^1(I\!R)}d\tau\\
&\le C\int_0^t \|v(\tau,\cdot)\|^{1+\beta_0}_{L^{1+\beta_0}(I\!R)}d\tau\\
&\le CE^{1+\beta_0}(1+t)^2 \le CER^2(E,T).
\end{aligned}
\tag{8.3.22}
$$

Moreover, still from (4.1.4) in Chap. 4, it is obvious that

$$
\|w^{(0)}(t,\cdot)\|_{L^{\infty}(I\!R)} \le C\varepsilon. \tag{8.3.23}
$$

Combining (8.3.19)–(8.3.23), it follows from (8.3.11) that

$$
\sup_{0\le t\le T}\|u(t,\cdot)\|_{L^{\infty}(I\!R)} \le C\{\varepsilon+(R^2(E,T)+R(E,T))(E+D_{S,T}(u))\}. \tag{8.3.24}
$$

Now we estimate $\|u(t,\cdot)\|_{L^{1+\beta_0}(I\!R)}$.

By (4.1.5) in Chap. 4, similarly to (8.3.19), for $i = 0, 1$, we get

$$\begin{aligned} \|\partial_i w^{(i)}(t,\cdot)\|_{L^{1+\beta_0}(\mathbb{R})} &\le \int_0^t \|G_i(\upsilon, u_x)(\tau,\cdot)\|_{L^{1+\beta_0}(\mathbb{R})} d\tau \\ &\le C\int_0^t \|\upsilon(\tau,\cdot)\|^{\alpha}_{L^\infty(\mathbb{R})}\|(\upsilon, u_x)(\tau,\cdot)\|_{L^{1+\beta_0}(\mathbb{R})} d\tau \\ &\le CE^\alpha(1+t)^{1+\frac{1}{1+\beta_0}}(E + D_{S,T}(u)) \\ &\le C(1+t)^{\frac{1}{1+\beta_0}} R(E,T)(E + D_{S,T}(u)). \end{aligned} \tag{8.3.25}$$

Similarly, by (4.1.4) in Chap. 4, we obtain

$$\|w^{(0)}(t,\cdot)\|_{L^{1+\beta_0}(\mathbb{R})}, \|u^{(0)}(t,\cdot)\|_{L^{1+\beta_0}(\mathbb{R})} \le C\varepsilon(1+t)^{\frac{1}{1+\beta_0}}, \tag{8.3.26}$$

$$\|u^{(1)}(t,\cdot)\|_{L^{1+\beta_0}(\mathbb{R})} \le C(1+t)^{\frac{1}{1+\beta_0}} R(E,T)(E + D_{S,T}(u)) \tag{8.3.27}$$

and

$$\|u^{(2)}(t,\cdot)\|_{L^{1+\beta_0}(\mathbb{R})} \le C(1+t)^{\frac{1}{1+\beta_0}} ER^2(E,T). \tag{8.3.28}$$

Hence, it follows from (8.3.11) that

$$\sup_{0\le t\le T}(1+t)^{-\frac{1}{1+\beta_0}}\|u(t,\cdot)\|_{L^{1+\beta_0}(\mathbb{R})} \le C\{\varepsilon + (R^2(E,T) + R(E,T))(E + D_{S,T}(u))\}. \tag{8.3.29}$$

Finally, we estimate $\|Du(t,\cdot)\|_{D,S,2}$.

Now we still have (8.2.69)–(8.2.72), (8.2.78) and (8.2.74). Moreover, since

$$F(\upsilon, D\upsilon) = F(\upsilon, 0) + \tilde{F}(\upsilon, D\upsilon)D\upsilon, \tag{8.3.30}$$

where, $\tilde{F}(\tilde{\lambda})$ is sufficiently smooth in a neighborhood of $\tilde{\lambda} = (\lambda, \lambda_0, \lambda_1) = 0$, and

$$\tilde{F}(\tilde{\lambda}) = O(|\tilde{\lambda}|^\alpha), \tag{8.3.31}$$

using the interpolation inequality (see Lemma 3.4.1 in Chap. 3), and noticing (8.3.10) and the definition of $X_{S,E,T}$, we have

$$\begin{aligned} \|F(\upsilon,0)(\tau,\cdot)\|_{L^2(\mathbb{R})} &\le \|F(\upsilon,0)(\tau,\cdot)\|^{\frac12}_{L^\infty(\mathbb{R})}\|F(\upsilon,0)(\tau,\cdot)\|^{\frac12}_{L^1(\mathbb{R})} \\ &\le C\|\upsilon(\tau,\cdot)\|^{\frac{1+\beta_0}{2}}_{L^\infty(\mathbb{R})}\|\upsilon(\tau,\cdot)\|^{\frac{1+\beta_0}{2}}_{L^{1+\beta_0}(\mathbb{R})} \\ &\le CE^{1+\beta_0}(1+\tau)^{\frac12} \end{aligned} \tag{8.3.32}$$

and

$$\begin{aligned} \|\tilde{F}(\upsilon, D\upsilon)D\upsilon(\tau,\cdot)\|_{L^2(\mathbb{R})} &\le \|\tilde{F}(\upsilon, D\upsilon)(\tau,\cdot)\|_{L^\infty(\mathbb{R})}\|D\upsilon(\tau,\cdot)\|_{L^2(\mathbb{R})} \\ &\le CE^{1+\alpha}, \end{aligned} \tag{8.3.33}$$

then

$$\|g_0(\tau,\cdot)\|_{L^2(I\!R)} \le CE^{1+\beta_0}(1+\tau)^{\frac{1}{2}} + CE^{1+\alpha}. \tag{8.3.34}$$

Noticing (8.2.74), we then have

$$\begin{aligned} |V| &\le CE^{1+\beta_0}(1+t)^{\frac{3}{2}} D_{S,T}(u) + CE^{1+\alpha}(1+t)D_{S,T}(u) \\ &\le C(R^2(E,T)+R(E,T))ED_{S,T}(u). \end{aligned} \tag{8.3.35}$$

Using (8.2.72), (8.2.78) and (8.3.35), it follows from (8.2.69) that

$$\sup_{0\le t\le T} \|Du(t,\cdot)\|_{D,S,2} \le C\{\varepsilon + (R(E,T)+\sqrt{R(E,T)})(E+D_{S,T}(u))\}. \tag{8.3.36}$$

Combining (8.3.24), (8.3.29) and (8.3.36), we get the desired (8.3.3). The proof of Lemma 8.3.1 is finished.

8.3.3 Proof of Lemma 8.3.2

We have

$$\begin{aligned} b(\overline{v}, D\overline{v}) - b(\overline{\overline{v}}, D\overline{\overline{v}}) &= b_1(\tilde{v}, D\tilde{v})v^* + b_2(\tilde{v}, D\tilde{v})Dv^* \\ &= b_1(\tilde{v}, 0)v^* + (b_1(\tilde{v}, D\tilde{v}) - b_1(\tilde{v}, 0))v^* + b_2(\tilde{v}, D\tilde{v})Dv^* \\ &= b_1(\tilde{v}, 0)v^* + b_3(\tilde{v}, D\tilde{v})D\tilde{v}v^* + b_2(\tilde{v}, D\tilde{v})Dv^* \end{aligned} \tag{8.3.37}$$

and

$$\begin{aligned} & F(\overline{v}, D\overline{v}) - F(\overline{\overline{v}}, D\overline{\overline{v}}) \\ = \ & F(\overline{v}, 0) - F(\overline{\overline{v}}, 0) + \sum_{i=0}^{1} (F_i(\overline{v}, 0)\partial_i\overline{v} - F_i(\overline{\overline{v}}, 0)\partial_i\overline{\overline{v}}) \\ & + \sum_{i,j=0}^{1} \Big(F_{ij}(\overline{v}, D\overline{v})\partial_i\overline{v}\partial_j\overline{v} - F_{ij}(\overline{\overline{v}}, D\overline{\overline{v}})\partial_i\overline{\overline{v}}\partial_j\overline{\overline{v}} \Big) \\ = \ & \frac{\partial F}{\partial u}(\tilde{v}, 0)v^* + \sum_{i=0}^{1} \partial_i (G_i(\overline{v}) - G_i(\overline{\overline{v}})) \\ & + \sum_{i,j=0}^{1} \Big(F_{ij}(\overline{v}, D\overline{v})\partial_i\overline{v}\partial_j v^* + F_{ij}(\overline{v}, D\overline{v})\partial_i v^* \partial_j\overline{\overline{v}} + (F_{ij}(\overline{v}, D\overline{v}) - F_{ij}(\overline{\overline{v}}, D\overline{\overline{v}}))\partial_i\overline{\overline{v}}\partial_j\overline{\overline{v}} \Big) \\ = \ & \frac{\partial F}{\partial u}(\tilde{v}, 0)v^* + \sum_{i=0}^{1} \partial_i (\hat{G}_i(\tilde{v})v^*) \end{aligned}$$

$$+\sum_{i,j=0}^{1}\Big(F_{ij}(\overline{v},D\overline{v})\partial_i\overline{v}\partial_j v^* + F_{ij}(\overline{v},D\overline{v})\partial_i v^*\partial_j\overline{\overline{v}} + \hat{F}_{ij}(\tilde{v},D\tilde{v})v^*\partial_i\overline{\overline{v}}\partial_j\overline{\overline{v}}\Big)$$
$$+\sum_{i,j,k=0}^{1}F_{ijk}(\tilde{v},D\tilde{v})\partial_i\overline{\overline{v}}\partial_j\overline{\overline{v}}\partial_k v^*, \tag{8.3.38}$$

where

$$\tilde{v}=(\overline{v},\overline{\overline{v}}), \tag{8.3.39}$$

and $G_i(v)$ is a primitive function of $F_i(v,0)$ $(i=0,1)$.

Thus, similarly to the proof of Lemma 8.3.1, we get

$$\begin{aligned}&\sup_{0\le t\le T}\|u^*(t,\cdot)\|_{L^\infty(I\!R)}+\sup_{0\le t\le T}(1+t)^{-\frac{1}{1+\beta_0}}\|u^*(t,\cdot)\|_{L^{1+\beta_0}(I\!R)}\\&\le C(R^2(E,T)+R(E,T))(D_{S-1,T}(u^*)+D_{S-1,T}(v^*)).\end{aligned} \tag{8.3.40}$$

On the other hand, we still have (8.2.87)–(8.2.91). Moreover, noticing that

$$\begin{aligned}&F(\overline{v},D\overline{v})-F(\overline{\overline{v}},D\overline{\overline{v}})\\&=F(\overline{v},0)-F(\overline{\overline{v}},0)+\sum_{i=0}^{1}\Big(F_i(\overline{v},D\overline{v})\partial_i\overline{v}-F_i(\overline{\overline{v}},D\overline{\overline{v}})\partial_i\overline{\overline{v}}\Big)\\&=\frac{\partial F}{\partial u}(\tilde{v},0)v^*+\sum_{i=0}^{1}\Big(F_i(\overline{v},D\overline{v})\partial_i v^*+(F_i(\overline{v},D\overline{v})-F_i(\overline{\overline{v}},D\overline{\overline{v}}))\partial_i\overline{\overline{v}}\Big),\end{aligned} \tag{8.3.41}$$

similarly to (8.3.34), we have

$$\|\overline{g}_0(\tau,\cdot)\|_{L^2(I\!R)}=\|F^*(\tau,\cdot)\|_{L^2(I\!R)}\le C(E^{\beta_0}(1+\tau)^{\frac{1}{2}}+E^{\alpha})D_{S-1,T}(v^*), \tag{8.3.42}$$

then we have

$$|V|\le C(R^2(E,T)+R(E,T))D_{S-1,T}(u^*)D_{S-1,T}(v^*). \tag{8.3.43}$$

Thus, we obtain

$$\sup_{0\le t\le T}\|Du^*(t,\cdot)\|_{D,S-1,2}\le C(R(E,T)+\sqrt{R(E,T)})(D_{S-1,T}(u^*)+D_{S-1,T}(v^*)). \tag{8.3.44}$$

Combining (8.3.40) and (8.3.44) yields the desired (8.3.5). The proof of Lemma 8.3.2 is finished.

Chapter 9
Cauchy Problem of $n(\geq 3)$-Dimensional Nonlinear Wave Equations

9.1 Introduction

In this chapter we investigate the following Cauchy problem of $n(\geq 3)$ dimensional nonlinear wave equations:

$$\Box u = F(u, Du, D_x Du), \tag{9.1.1}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{9.1.2}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{9.1.3}$$

is the n dimensional wave operator,

$$D_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right),\ D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \tag{9.1.4}$$

φ and ψ are sufficiently smooth and compactly supported functions, without loss of generality, we assume that $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$, and $\varepsilon > 0$ is a small parameter.

Let

$$\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \cdots, n; (\lambda_{ij}), i, j = 0, 1, \cdots, n, i + j \geq 1). \tag{9.1.5}$$

Assume that in a neighborhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \leq \nu_0$, the nonlinear term $F(\hat{\lambda})$ is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}), \tag{9.1.6}$$

and $\alpha \geq 1$ is an integer.

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This chapter is aimed at studying the life-span $\widetilde{T}(\varepsilon)$ of the classical solution $u = u(t, x)$ to Cauchy problem (9.1.1)–(9.1.2) in a unified way for any given integer $\alpha \geq 1$ when the space dimension $n \geq 3$.

We will prove that: there exists a suitably small positive number ε_0 such that for any given $\varepsilon \in (0, \varepsilon_0]$, for different values of $\alpha \geq 1$ and $n \geq 3$, we have:

(i) In general, the life-span $\widetilde{T}(\varepsilon)$ has the lower bound estimate as shown in the following table:

<table>
<tr><td></td><td colspan="2">$\widetilde{T}(\varepsilon) \geq$</td></tr>
<tr><td>n $\alpha =$</td><td>1</td><td>2, 3, $\cdots$</td></tr>
<tr><td>3</td><td>$b\varepsilon^{-2}$</td><td rowspan="3">$+\infty$</td></tr>
<tr><td>4</td><td>$\exp\{a\varepsilon^{-1}\}$</td></tr>
<tr><td>5,6,$\cdots$</td><td></td></tr>
</table>

(ii) If we have

$$\partial_u^2 F(0, 0, 0) = 0, \tag{9.1.7}$$

then the life-span $\widetilde{T}(\varepsilon)$ has the lower bound estimate as shown in the following table:

<table>
<tr><td></td><td colspan="2">$\widetilde{T}(\varepsilon) \geq$</td></tr>
<tr><td>n $\alpha =$</td><td>1</td><td>2, 3, $\cdots$</td></tr>
<tr><td>3</td><td>$\exp\{a\varepsilon^{-1}\}$</td><td rowspan="2">$+\infty$</td></tr>
<tr><td>4,5,$\cdots$</td><td></td></tr>
</table>

In particular, when the nonlinear term on the right-hand side does not depend on u explicitly:

$$F = F(Du, D_x Du), \tag{9.1.8}$$

the above estimate is true.

In the above two tables, both a and b stand for positive constants independent of ε.

The first table above gives, in a unified way (see Li Tatsien and Yu Xin 1989, 1991), results on the global existence ($\widetilde{T}(\varepsilon) = +\infty$) from Li Tatsien and Chen Yunmei (1988b), as well as results on the lower bound of life-span from Hörmander (1991) as $n = 4$ and $\alpha = 1$ and from Lindblad (1990b) as $n = 3$ and $\alpha = 1$, respectively. While, the second table above gives, under condition (9.1.7), in a unified way (see Li Tatsien and Zhou Yi 1992b), results on the lower bound of life-span from Hörmander (1991) as $n = 4$ and $\alpha = 1$ and from Lindblad (1990b) as $n = 3$ and $\alpha = 1$, respectively.

From results of Chaps. 13 and 14, the above lower bound estimates on the life-span are all sharp except that of Hörmander (1991) as $n = 4$ and $\alpha = 1$, shown in the first table as follows:

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-1}\}, \tag{9.1.9}$$

Estimate (9.1.9) will be improved to (see Li Tatsien and Zhou Yi 1995b, 1995c; Lindblad and Sogge 1996)

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-2}\}, \tag{9.1.10}$$

and estimate (9.1.10) is sharp as well.

From Chap. 7 we know that, in order to prove the above results for Cauchy problem (9.1.1)–(9.1.2) of nonlinear wave equation, it suffices, essentially, to consider the following Cauchy problem of second-order quasi-linear hyperbolic equation:

$$\Box u = \sum_{i,j=1}^{n} b_{ij}(u, Du)u_{x_i x_j} + 2\sum_{j=1}^{n} a_{0j}(u, Du)u_{t x_j} + F(u, Du), \tag{9.1.11}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{9.1.12}$$

where

$$\varphi, \psi \in C_0^\infty(\mathbb{R}^n), \tag{9.1.13}$$

and $\varepsilon > 0$ is a small parameter. Let

$$\widetilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \cdots, n). \tag{9.1.14}$$

Assume that when $|\widetilde{\lambda}| \leq \nu_0$, $b_{ij}(\widetilde{\lambda})$, $a_{0j}(\widetilde{\lambda})$ and $F(\widetilde{\lambda})$ are all sufficiently smooth functions satisfying

$$b_{ij}(\widetilde{\lambda}) = b_{ji}(\widetilde{\lambda}) \quad (i, j = 1, \cdots, n), \tag{9.1.15}$$

$$b_{ij}(\widetilde{\lambda}), a_{0j}(\widetilde{\lambda}) = O(|\widetilde{\lambda}|^\alpha) \quad (i, j = 1, \cdots, n), \tag{9.1.16}$$

$$F(\widetilde{\lambda}) = O(|\widetilde{\lambda}|^{1+\alpha}) \tag{9.1.17}$$

and

$$\sum_{i,j=1}^{n} a_{ij}(\widetilde{\lambda})\xi_i\xi_j \geq m_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \tag{9.1.18}$$

where $\alpha \geq 1$ is an integer, m_0 is a positive constant, and

$$a_{ij}(\widetilde{\lambda}) = \delta_{ij} + b_{ij}(\widetilde{\lambda}), \tag{9.1.19}$$

with δ_{ij} the Kronecker symbol. In addition, condition (9.1.7) is now reduced to

$$\partial_u^2 F(0, 0) = 0. \tag{9.1.20}$$

9.2 Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (9.1.11)–(9.1.12)

In this section, for the life-span of classical solutions to Cauchy problem (9.1.11)–(9.1.12) of $n(\geq 3)$ dimensional second-order quasi-linear hyperbolic equation, we will prove the lower bound estimates shown by the first table in the previous section.

9.2.1 Metric Space $X_{S,E,T}$. Main Results

Thanks to the Sobolev embedding theorem, there exists a suitably small $E_0 > 0$, such that

$$\|f\|_{L^\infty(I\!R^n)} \leq \nu_0, \quad \forall f \in H^{[\frac{n}{2}]+1}(I\!R^n), \ \|f\|_{H^{[\frac{n}{2}]+1}(I\!R^n)} \leq E_0. \tag{9.2.1}$$

For any given integer $S \geq 2[\frac{n}{2}] + 4$, and any given positive numbers $E(\leq E_0)$ and $T(0 < T \leq +\infty)$, we introduce the set of functions

$$X_{S,E,T} = \{v(t,x) | D_{S,T}(v) \leq E, \partial_t^l v(0,x) = u_l^{(0)}(x) \quad (l = 0, 1, \cdots, S+1)\}, \tag{9.2.2}$$

where

$$D_{S,T}(v) = \sum_{i=0}^{2} \sup_{0 \leq t \leq T} \|D^i v(t,\cdot)\|_{\Gamma,S,2}. \tag{9.2.3}$$

Here, $\|\cdot\|_{\Gamma,S,2}$ is defined in Sect. 3.1.3 of Chap. 3, and if T is finite, then the supremum is taken over $[0, T]$; while, if $T = +\infty$, then the supremum is taken over $[0, +\infty)$. For simplicity, we uniformly denote by $[0, T]$ the corresponding interval below. Moreover, $u_0^{(0)} = \varepsilon\varphi(x)$, $u_1^{(0)} = \varepsilon\psi(x)$, and when $l = 2, \cdots, S+1$, $u_l^{(0)}(x)$ are values of $\partial_t^l u(t,x)$ at $t = 0$, which are determined uniquely by Eq. (9.1.11) and initial condition (9.1.12). Obviously, $u_l^{(0)}(l = 0, 1, \cdots, S+1)$ are all sufficiently smooth functions with compact support.

Introduce the following metric on $X_{S,E,T}$:

$$\rho(\overline{v}, \overline{\overline{v}}) = D_{S,T}(\overline{v} - \overline{\overline{v}}), \quad \forall \overline{v}, \overline{\overline{v}} \in X_{S,E,T}. \tag{9.2.4}$$

Similarly to Lemma 8.2.1 in Chap. 8, it is easy to prove

Lemma 9.2.1 *When $\varepsilon > 0$ is suitably small, $X_{S,E,T}$ is a non-empty complete metric space.*

Noticing $S \geq 2[\frac{n}{2}] + 4$, from (3.4.30) in Chap. 3 (in which we take $p = 2$, $N = [\frac{S}{2}] + 1$ and $s = [\frac{n}{2}] + 1$) we obtain

Lemma 9.2.2 *When $S \geq 2[\frac{n}{2}] + 4$, for any given $v \in X_{S,E,T}$ we have*

$$\|(v, Dv, D^2v)(t, \cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq CE(1+t)^{-\frac{n-1}{2}}, \quad \forall t \in [0, T], \tag{9.2.5}$$

where C is a positive constant.

The main result of this section is the following

Theorem 9.2.1 *Let $n \geq 3$. Under assumptions (9.1.15)–(9.1.19), for any given $S \geq 2[\frac{n}{2}] + 4$, there exist positive constants ε_0 and C_0 satisfying $C_0\varepsilon_0 \leq E_0$, and for any given $\varepsilon \in (0, \varepsilon_0]$, there exists a positive constant $T(\varepsilon)$ such that Cauchy problem (9.1.11)–(9.1.12) admits on $[0, T(\varepsilon)]$ a unique classical solution $u \in X_{S,C_0\varepsilon,T(\varepsilon)}$, and $T(\varepsilon)$ can be taken as*

$$T(\varepsilon) = \begin{cases} +\infty, & \text{if } K > 1, \\ \exp\{a\varepsilon^{-\alpha}\} - 1, & \text{if } K = 1, \\ b\varepsilon^{-\frac{\alpha}{1-K}} - 1, & \text{if } 0 \leq K < 1, \end{cases} \tag{9.2.6}$$

where

$$K = \frac{(n-1)\alpha - 1}{2}, \tag{9.2.7}$$

and a and b are both positive constants depending only on α and n.

Moreover, after a possible change of values for t on a zero-measure set of $[0, T(\varepsilon)]$, for any given finite T_0 satisfying $0 < T_0 \leq T(\varepsilon)$ we have

$$u \in C\left([0, T_0]; H^{S+1}(I\!\!R^n)\right), \tag{9.2.8}$$

$$u_t \in C\left([0, T_0]; H^{S}(I\!\!R^n)\right), \tag{9.2.9}$$

$$u_{tt} \in C\left([0, T_0]; H^{S-1}(I\!\!R^n)\right). \tag{9.2.10}$$

Remark 9.2.1 From Sobolev embedding theorem we know that

$$H^{[\frac{n}{2}]+1}(I\!\!R^n) \subset C(I\!\!R^n)$$

is a continuous embedding. Noting $S \geq 2[\frac{n}{2}] + 4$, function $u = u(t, x)$ satisfying (9.2.8)–(9.2.10) is a classical solution, with at least second-order continuous derivatives, to Cauchy problem (9.1.11)–(9.1.12).

Remark 9.2.2 It is easy to show from (9.2.7) that: $K > 1$ when $\alpha \geq 2$ or $n \geq 5$; $K = 1$ when $n = 4$ and $\alpha = 1$; while, $0 < K < 1$ when $n = 3$ and $\alpha = 1$. Noting that the life-span $\widetilde{T}(\varepsilon) > T(\varepsilon)$, the first table in Sect. 1 follows immediately from (9.2.6).

9.2.2 Framework to Prove Theorem 9.2.1—The Global Iteration Method

To prove Theorem 9.2.1, for any given $v \in X_{S,E,T}$, by solving the following Cauchy problem of linear hyperbolic equation:

$$\begin{aligned}\Box u &= \hat{F}(v, Dv, D_x Du) \stackrel{\text{def.}}{=} \sum_{i,j=1}^{n} b_{ij}(v, Dv)u_{x_i x_j} \\ &\quad +2\sum_{j=1}^{n} a_{0j}(v, Dv)u_{tx_j} + F(v, Dv), \qquad (9.2.11)\\ t = 0 &: u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \qquad (9.2.12)\end{aligned}$$

we define a mapping

$$M : v \longrightarrow u = Mv. \qquad (9.2.13)$$

We want to prove that: when $\varepsilon > 0$ is suitably small, we can find a positive constant C_0 such that when $E = C_0\varepsilon$ and $T = T(\varepsilon)$ is defined by (9.2.6), M not only maps $X_{S,E,T}$ into itself but also possesses a certain contractive property, and thus M has a unique fixed point in $X_{S,E,T}$, which is exactly the classical solution to Cauchy problem (9.1.11)–(9.1.12) on $0 \leq t \leq T(\varepsilon)$.

Using results in Chap. 6, it is easy to prove the following

Lemma 9.2.3 *When $E > 0$ is suitably small, for any given $v \in X_{S,E,T}$, after possible change of values on a zero-measure set of t, for any finite T_0 satisfying $0 < T_0 \leq T$, we have*

$$u = Mv \in C\left([0, T_0]; H^{S+1}(I\!R^n)\right), \qquad (9.2.14)$$

$$u_t \in C\left([0, T_0]; H^{S}(I\!R^n)\right), \qquad (9.2.15)$$

$$u_{tt} \in L^\infty\left(0, T_0; H^{S-1}(I\!R^n)\right). \qquad (9.2.16)$$

Moreover, for any given $t \in [0, T]$, $u = u(t, x)$ is compactly supported with respect to x.

It is easy to prove the following

Lemma 9.2.4 *For $u = u(t, x) = Mv$, the values of $\partial_t^l u(0, \cdot)(l = 0, 1, \cdots, S+2)$ do not depend on the choice of $v \in X_{S,E,T}$, and*

$$\partial_t^l u(0, x) = u_l^{(0)}(x) \quad (l = 0, 1, \cdots, S+1). \qquad (9.2.17)$$

Moreover,

$$\|u(0, \cdot)\|_{\Gamma,S+2,2} + \|u_t(0, \cdot)\|_{\Gamma,S+1,q} \leq C\varepsilon, \qquad (9.2.18)$$

where q satisfies

$$\frac{1}{q} = \frac{1}{2} + \frac{1}{n}, \tag{9.2.19}$$

and C is a positive constant depending only on S.

The following two lemmas are crucial to the proof of Theorem 9.2.1.

Lemma 9.2.5 *Under the assumptions of Theorem 9.2.1, if $E > 0$ is suitably small, then for any given $v \in X_{S,E,T}$, $u = Mv$ satisfies*

$$D_{S,T}(u) \le C_1\{\varepsilon + (R + \sqrt{R})(E + D_{S,T}(u))\}, \tag{9.2.20}$$

where C_1 is a positive constant,

$$R = R(E, T) \overset{\text{def.}}{=} E^{\alpha} \int_0^T (1+t)^{-K} dt, \tag{9.2.21}$$

and K is given by (9.2.7).

Lemma 9.2.6 *Under the assumptions of Lemma 9.2.5, for any given $\overline{v}, \overline{\overline{v}} \in X_{S,E,T}$, if both $\overline{u} = M\overline{v}$ and $\overline{\overline{u}} = M\overline{\overline{v}}$ satisfy $\overline{u}, \overline{\overline{u}} \in X_{S,E,T}$, then we have*

$$D_{S-1,T}(\overline{u} - \overline{\overline{u}}) \le C_2(R + \sqrt{R})(D_{S-1,T}(\overline{u} - \overline{\overline{u}}) + D_{S-1,T}(\overline{v} - \overline{\overline{v}})), \tag{9.2.22}$$

where C_2 is a positive constant, and $R = R(E, T)$ is still defined by (9.2.21).

The proof of Lemmas 9.2.5 and 9.2.6 will be given later. Now we first use these two lemmas to prove Theorem 9.2.1.

Proof of Theorem 9.2.1 Take

$$C_0 = 3\max(C_1, C_2), \tag{9.2.23}$$

where C_1 and C_2 are positive constants given in Lemmas 9.2.5 and 9.2.6, respectively.

We first prove that, if there exists a positive number ε_0 satisfying $C_0\varepsilon_0 \le E_0$, and for any $\varepsilon \in (0, \varepsilon_0]$, $E = E(\varepsilon) = C_0\varepsilon$ and $T = T(\varepsilon) > 0$ satisfy

$$R(E(\varepsilon), T(\varepsilon)) + \sqrt{R(E(\varepsilon), T(\varepsilon))} \le \frac{1}{C_0}, \tag{9.2.24}$$

then the mapping M admits a unique fixed point in $X_{S,E(\varepsilon),T(\varepsilon)}$.

In fact, noting (9.2.24), it is easy to show from Lemmas 9.2.5 and 9.2.6 that for any given $v \in X_{S,E(\varepsilon),T(\varepsilon)}$, $u = Mv$ satisfies

$$D_{S,T(\varepsilon)}(u) \le E(\varepsilon), \tag{9.2.25}$$

and for any given $\overline{v}, \overline{\overline{v}} \in X_{S,E(\varepsilon),T(\varepsilon)}$, $\overline{u} = M\overline{v}$ and $\overline{\overline{u}} = M\overline{\overline{v}}$ satisfy

$$D_{S-1,T(\varepsilon)}(\overline{u} - \overline{\overline{u}}) \leq \frac{1}{2} D_{S-1,T(\varepsilon)}(\overline{v} - \overline{\overline{v}}). \tag{9.2.26}$$

In other words, M maps $X_{S,E(\varepsilon),T(\varepsilon)}$ into itself, and M is contractive with respect to the metric of $X_{S-1,E(\varepsilon),T(\varepsilon)}$. Noticing that $X_{S,E(\varepsilon),T(\varepsilon)}$ is a closed subset of $X_{S-1,E(\varepsilon),T(\varepsilon)}$ (see Lemma 8.2.7 in Chap. 8), from the standard contraction mapping principle we know that, M admits a fixed point

$$u \in X_{S,E(\varepsilon),T(\varepsilon)}, \tag{9.2.27}$$

then by Lemma 9.2.6 this fixed point is also unique.

This fixed point $u = u(t,x)$ is obviously the classical solution to Cauchy problem (9.1.11)–(9.1.12) on $0 \leq t \leq T(\varepsilon)$.

Now we determine $\varepsilon_0 > 0$ and $T(\varepsilon)(0 < \varepsilon \leq \varepsilon_0)$ such that the required (9.2.24) is satisfied. Below we always assume that $\varepsilon_0 > 0$ is sufficiently small so that (9.2.1) is true when $E_0 = C_0\varepsilon_0$.

(i) In the case $K > 1$, because

$$\int_0^T (1+t)^{-K} dt \leq C, \quad \forall T > 0, \tag{9.2.28}$$

where C is a positive constant independent of T, we can always choose

$$T(\varepsilon) = +\infty,$$

and choose $\varepsilon_0 > 0$ to be so small that (9.2.24) is satisfied for any ε satisfying $0 < \varepsilon \leq \varepsilon_0$.

(ii) In the case $K = 1$, we take

$$T(\varepsilon) = \exp\{a\varepsilon^{-\alpha}\} - 1,$$

where a is a positive number satisfying

$$C_0(aC_0^\alpha + \sqrt{aC_0^\alpha}) \leq 1. \tag{9.2.29}$$

Then, we have

$$\begin{aligned} R(E(\varepsilon), T(\varepsilon)) &= E^\alpha(\varepsilon) \int_0^{T(\varepsilon)} (1+t)^{-1} dt \\ &= C_0^\alpha \varepsilon^\alpha \ln(1 + T(\varepsilon)) \\ &= aC_0^\alpha, \end{aligned} \tag{9.2.30}$$

Therefore, noting (9.2.29), we get the required (9.2.24). In this case, we obtain an almost global solution.

(iii) In the case $0 \le K < 1$, we take

$$T(\varepsilon) = b\varepsilon^{-\frac{\alpha}{1-K}} - 1,$$

where b is a positive number satisfying

$$C_0\left(\frac{1}{1-K}C_0^\alpha b^{1-K} + \sqrt{\frac{1}{1-K}C_0^\alpha b^{1-K}}\right) \le 1. \tag{9.2.31}$$

Then, we have

$$\begin{aligned} R(E(\varepsilon), T(\varepsilon)) &= E^\alpha(\varepsilon)\int_0^{T(\varepsilon)}(1+t)^{-K}dt \\ &= \frac{1}{1-K}C_0^\alpha \varepsilon^\alpha[(1+T(\varepsilon))^{1-K} - 1] \\ &\le \frac{1}{1-K}C_0^\alpha b^{1-K}, \end{aligned} \tag{9.2.32}$$

therefore, from (9.2.31) we get the required (9.2.24).

In addition, from Lemma 9.2.3, (9.2.14)–(9.2.15) are satisfied for any given finite T_0 satisfying $0 < T_0 \le T(\varepsilon)$, thus it is easy to prove that

$$b_{ij}(u, Du), a_{0j}(u, Du), F(u, Du) \in C\left([0, T_0]; H^s(I\!R^n)\right) \quad (i, j = 1, \cdots, n). \tag{9.2.33}$$

Then, Corollary 6.3.3 in Chap. 6 immediately yields

$$u_{tt} \in C\left([0, T_0]; H^{s-1}(I\!R^n)\right).$$

Together with the definition of $X_{S,E,T}$, we obtain (9.2.8)–(9.2.10). This completes the proof of Theorem 9.2.1.

9.2.3 Proof of Lemma 9.2.5

We first estimate $\|u(t, \cdot)\|_{\Gamma,S,2}$.

Applying (4.5.17) in Chap. 4 (in which we take $N = S$) to Cauchy problem (9.2.11)–(9.2.12) and noting (9.2.18), we obtain

$$\|u(t,\cdot)\|_{\Gamma,S,2} \leq C\left\{\varepsilon + \int_0^t \Big(\|\hat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,q,\chi_1} + (1+\tau)^{-\frac{n-2}{2}}\|\hat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}\Big)\, d\tau\right\}, \tag{9.2.34}$$

where q satisfies (9.2.19), χ_1 is the characteristic function of set $\{(t,x) \,|\, |x| \leq \frac{1+t}{2}\}$, $\chi_2 = 1-\chi_1$, and C is a positive constant.

Noticing (9.1.16) and the definition of $X_{S,E,T}$, from (9.2.24) in Lemma 5.2.5 of Chap. 5 (in which we take $r = q$, $p = n$), and noting that $L^{q,2}(I\!R^n) \subset L^q(I\!R^n)$ is a continuous embedding, we have

$$\begin{aligned} &\|(b_{ij}(v, Dv)u_{x_i x_j})(\tau,\cdot)\|_{\Gamma,S,q,\chi_1}, \ \|(a_{0j}(v, Dv)u_{tx_j})(\tau,\cdot)\|_{\Gamma,S,q,\chi_1} \\ &\leq C(1+\tau)^{-\frac{n}{2}(1-\frac{2}{\alpha n})\alpha}\|(v, Dv)(\tau,\cdot)\|^{\alpha}_{\Gamma,S,2}\|D^2u(\tau,\cdot)\|_{\Gamma,S,2} \\ &\leq C(1+\tau)^{-K}E^{\alpha}D_{S,T}(u), \quad \forall \tau \in [0,T]. \end{aligned} \tag{9.2.35}$$

Similarly, noting (9.1.17), we get

$$\|F(v, Dv)(\tau,\cdot)\|_{\Gamma,S,q,\chi_1} \leq C(1+\tau)^{-K}E^{1+\alpha}, \quad \forall \tau \in [0,T]. \tag{9.2.36}$$

From (9.2.23) in Lemma 5.2.5 of Chap. 5 (in which we take $r = 1, p = 2$) we have similar estimates for $(1+\tau)^{-\frac{n-2}{2}}\|(b_{ij}(v, Dv)u_{x_i x_j})(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}$, $(1+\tau)^{-\frac{n-2}{2}}\|(a_{0j}(v, Dv) \quad u_{tx_j})(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}$ and $(1+\tau)^{-\frac{n-2}{2}}\|F(v, Dv)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}$. Plugging these estimates into (9.2.34), we obtain

$$\sup_{0\leq t\leq T}\|u(t,\cdot)\|_{\Gamma,S,2} \leq C\{\varepsilon + R(E,T)(E + D_{S,T}(u))\}, \tag{9.2.37}$$

where $R(E,T)$ is defined by (9.2.21).

Now we estimate $\|D^2u(t,\cdot)\|_{\Gamma,S,2}$.

From Lemma 3.1.5 in Chap. 3, for any given multi-index $k(|k| \leq S)$, from (9.2.11) we have

$$\begin{aligned} \Box\Gamma^k Du &= \Gamma^k D\Box u + \sum_{|l|\leq|k|-1} B_{ki}\Gamma^l D\Box u \\ &= \Gamma^k D\hat{F}(v, Dv, D_x Du) + \sum_{|l|\leq|k|-1} B_{kl}\Gamma^l D\hat{F}(v, Dv, D_x Du) \\ &= \sum_{i,j=1}^n b_{ij}(v, Dv)(\Gamma^k Du)_{x_i x_j} + 2\sum_{j=1}^n a_{0j}(v, Dv)(\Gamma^k Du)_{tx_j} + G_k + g_k, \end{aligned} \tag{9.2.38}$$

where

$$
\begin{aligned}
G_k = \sum_{i,j=1}^{n} \Big\{ & \big(\Gamma^k D(b_{ij}(v, Dv)u_{x_i x_j}) - b_{ij}(v, Dv)\Gamma^k Du_{x_i x_j}\big) \\
& + b_{ij}(v, Dv)\ \big(\Gamma^k Du_{x_i x_j} - (\Gamma^k Du)_{x_i x_j}\big)\Big\} \\
+ 2\sum_{j=1}^{n} \Big\{ & \big(\Gamma^k D(a_{0j}(v, Dv)u_{tx_j}) - a_{0j}(v, Dv)\Gamma^k Du_{tx_j}\big) \\
& + a_{0j}(v, Dv)\ \big(\Gamma^k Du_{tx_j} - (\Gamma^k Du)_{tx_j}\big)\Big\},
\end{aligned} \tag{9.2.39}
$$

$$
\begin{aligned}
g_k = \Gamma^k DF(v, Dv) + \sum_{|l|\le|k|-1} B_{kl}\Gamma^l D \Bigg\{ & \sum_{i,j=1}^{n} b_{ij}(v, Dv)u_{x_i x_j} \\
& + 2\sum_{j=1}^{n} a_{0j}(v, Dv)u_{tx_j} + F(v, Dv)\Bigg\} \\
= \Gamma^k DF(v, Dv) + \sum_{|l|\le|k|} \widetilde{B}_{kl}\Gamma^l \Bigg\{ & \sum_{i,j=1}^{n} b_{ij}(v, Dv)u_{x_i x_j} \\
& + 2\sum_{j=1}^{n} a_{0j}(v, Dv)u_{tx_j} + F(v, Dv)\Bigg\},
\end{aligned} \tag{9.2.40}
$$

and B_{kl} and $\widetilde{B}_{kl}$ are some constants.

Thus, taking the inner product of (9.2.38) with $(\Gamma^k Du)_t$ in $L^2(I\!R^n)$, similarly to (8.2.69) in Chap. 8, we have the following energy integral formula:

$$
\begin{aligned}
& \|(\Gamma^k Du(t,\cdot))_t\|^2_{L^2(I\!R^n)} + \sum_{i,j=1}^{n} \int_{I\!R^n} a_{ij}(v, Dv)(t,\cdot)(\Gamma^k Du(t,\cdot))_{x_i}(\Gamma^k Du(t,\cdot)_{x_j})dx \\
= \ & \|(\Gamma^k Du(0,\cdot))_t\|^2_{L^2(I\!R^n)} + \sum_{i,j=1}^{n} \int_{I\!R^n} a_{ij}(v, Dv)(0,\cdot)(\Gamma^k Du(0,\cdot))_{x_i}(\Gamma^k Du(0,\cdot)_{x_j})dx \\
& + \sum_{i,j=1}^{n} \int_0^t \int_{I\!R^n} \frac{\partial b_{ij}(v, Dv)(\tau,\cdot)}{\partial \tau}(\Gamma^k Du(\tau,\cdot))_{x_i}(\Gamma^k Du(\tau,\cdot))_{x_j} dx d\tau \\
& - 2\sum_{i,j=1}^{n} \int_0^t \int_{I\!R^n} \frac{\partial b_{ij}(v, Dv)(\tau,\cdot)}{\partial x_i}(\Gamma^k Du(\tau,\cdot))_{x_j}(\Gamma^k Du(\tau,\cdot))_{\tau} dx d\tau
\end{aligned}
$$

$$
\begin{aligned}
&-2\sum_{j=1}^{n}\int_0^t\int_{I\!R^n}\frac{\partial a_{0j}(v,Dv)(\tau,\cdot)}{\partial x_j}(\Gamma^k Du(\tau,\cdot))_\tau(\Gamma^k Du(\tau,\cdot))_\tau dxd\tau\\
&+2\int_0^t\int_{I\!R^n}G_k(\tau,\cdot)(\Gamma^k Du(\tau,\cdot))_\tau dxd\tau+2\int_0^t\int_{I\!R^n}g_k(\tau,\cdot)(\Gamma^k Du(\tau,\cdot))_\tau dxd\tau\\
\stackrel{\text{def.}}{=}\;&\|(\Gamma^k Du(0,\cdot))_t\|^2_{L^2(I\!R^n)}+\sum_{i,j=1}^{n}\int_{I\!R^n}a_{ij}(v,Dv)(0,\cdot)(\Gamma^k Du(0,\cdot))_{x_i}(\Gamma^k Du(0,\cdot)_{x_j})dx\\
&+\text{I}+\text{II}+\text{III}+\text{IV}+\text{V},
\end{aligned}
\tag{9.2.41}
$$

where a_{ij} are given by (9.1.19).

Noting $K<\frac{n-1}{2}\alpha$ and (9.1.16), from Lemmas 5.2.2 and 5.2.3 in Chap. 5 and Corollary 3.1.1 in Chap. 3, it is easy to get

$$
\begin{aligned}
|\text{I}|,|\text{II}|,|\text{III}|&\leq C\int_0^t(1+\tau)^{-K}E^\alpha\|D^2u(\tau,\cdot)\|^2_{\Gamma,S,2}d\tau\\
&\leq CR(E,T)D^2_{S,T}(u),\quad\forall t\in[0,T].
\end{aligned}
\tag{9.2.42}
$$

Now we estimate the L^2 norm of $G_k(\tau,\cdot)$. Noticing (9.1.16) and Lemma 9.2.2, from Lemmas 5.2.5 and 5.2.6 in Chap. 5 (in which we take $r=2$, then $p=+\infty$), we obtain

$$
\begin{aligned}
&\|(\Gamma^k D(b_{ij}(v,Dv)u_{x_ix_j})-b_{ij}(v,Dv)\Gamma^k Du_{x_ix_j})(\tau,\cdot)\|_{L^2(I\!R^n)}\\
\leq\;&\|(\Gamma^k(b_{ij}(v,Dv)Du_{x_ix_j})-b_{ij}(v,Dv)\Gamma^k Du_{x_ix_j})(\tau,\cdot)\|_{L^2(I\!R^n)}\\
&+\|(\Gamma^k(Db_{ij}(v,Dv)u_{x_ix_j})(\tau,\cdot)\|_{L^2(I\!R^n)}\\
\leq\;&C(1+\tau)^{-\frac{n-1}{2}\alpha}(\|(v,Dv,D^2v)(\tau,\cdot)\|_{\Gamma,S,2})^\alpha\|D^2u(\tau,\cdot)\|_{\Gamma,S,2}\\
\leq\;&C(1+\tau)^{-\frac{n-1}{2}\alpha}E^\alpha D_{S,T}(u),\quad\forall\tau\in[0,T].
\end{aligned}
\tag{9.2.43}
$$

On the other hand, noticing (9.1.16), using (3.4.30) in Corollary 3.4.4 of Chap. 3 (in which we take $N=0$ and $p=2$) and Corollary 3.1.1 of Chap. 3, we have

$$
\begin{aligned}
&\|b_{ij}(v,Dv)(\Gamma^k Du_{x_ix_j}-(\Gamma^k Du)_{x_ix_j})(\tau,\cdot)\|_{L^2(I\!R^n)}\\
\leq\;&C(\|(v,Dv)(\tau,\cdot)\|_{L^\infty(I\!R^n)})^\alpha\|(\Gamma^k Du_{x_ix_j}-(\Gamma^k Du)_{x_ix_j})(\tau,\cdot)\|_{L^2(I\!R^n)}\\
\leq\;&C(1+\tau)^{-\frac{n-1}{2}\alpha}E^\alpha\|D^2u(\tau,\cdot)\|_{\Gamma,S,2}\\
\leq\;&C(1+\tau)^{-K}E^\alpha D_{S,T}(u),\quad\forall\tau\in[0,T].
\end{aligned}
\tag{9.2.44}
$$

For the terms with respect to a_{0j} in G_k, similar estimates hold. Therefore, we have

$$
\|G_k(\tau,\cdot)\|_{L^2(I\!R^n)}\leq C(1+\tau)^{-K}E^\alpha D_{S,T}(u),\quad\forall\tau\in[0,T],
\tag{9.2.45}
$$

then we get

$$
|\text{IV}|\leq CR(E,T)D^2_{S,T}(u).
\tag{9.2.46}
$$

Similarly, by Lemma 5.2.5 in Chap. 5 (in which we take $r = 2$, then $p = +\infty$), we have

$$\begin{aligned}\|g_k(\tau,\cdot)\|_{L^2(I\!R^n)} \le\, & C\Big(\|DF(v,Dv)(\tau,\cdot)\|_{\Gamma,S,2} + \|F(v,Dv)(\tau,\cdot)\|_{\Gamma,S,2} \\ & + \sum_{i,j=1}^n \|b_{ij}(v,Dv)u_{x_ix_j}(\tau,\cdot)\|_{\Gamma,S,2} + 2\sum_{j=1}^n \|a_{0j}(v,Dv)u_{tx_j}(\tau,\cdot)\|_{\Gamma,S,2}\Big) \\ \le\, & C(1+\tau)^{-K}E^{\alpha}(E + D_{S,T}(u)), \quad \forall \tau \in [0,T],\end{aligned} \tag{9.2.47}$$

then

$$|\mathrm{V}| \le CR(E,T)(ED_{S,T}(u) + D^2_{S,T}(u)). \tag{9.2.48}$$

By (9.2.42), (9.2.46) and (9.2.48), and noticing (9.1.18) and Lemma 9.2.4, it follows from (9.2.41) that

$$\sup_{0\le t\le T} \sum_{|k|\le S} \|D\Gamma^k Du(t,\cdot)\|^2_{L^2(I\!R^n)} \le C\{\varepsilon^2 + R(E,T)(ED_{S,T}(u) + D^2_{S,T}(u))\}. \tag{9.2.49}$$

Hence, by Corollary 3.1.1 in Chap. 3, we immediately get

$$\sup_{0\le t\le T} \|D^2u(t,\cdot)\|^2_{\Gamma,S,2} \le C\{\varepsilon^2 + R(E,T)(ED_{S,T}(u) + D^2_{S,T}(u))\}, \tag{9.2.50}$$

then

$$\sup_{0\le t\le T} \|D^2u(t,\cdot)\|_{\Gamma,S,2} \le C\{\varepsilon + \sqrt{R(E,T)}(E + D_{S,T}(u))\}. \tag{9.2.51}$$

Using similar arguments we can obtain

$$\sup_{0\le t\le T} \|Du(t,\cdot)\|_{\Gamma,S,2} \le C\{\varepsilon + \sqrt{R(E,T)}(E + D_{S,T}(u))\}. \tag{9.2.52}$$

In fact, similarly to (9.2.38), we have

$$\begin{aligned}\Box\Gamma^k u &= \Gamma^k\Box u + \sum_{|l|\le|k|-1} B_{kl}\Gamma^l\Box u \\ &= \Gamma^k\hat F(v,Dv,D_xDu) + \sum_{|l|\le|k|-1} B_{kl}\Gamma^l\hat F(v,Dv,D_xDu) \\ &= \sum_{i,j=1}^n b_{ij}(v,Dv)(\Gamma^k u)_{x_ix_j} + 2\sum_{j=1}^n a_{0j}(v,Dv)(\Gamma^k u)_{tx_j} + \overline{G}_k + \overline{g}_k,\end{aligned} \tag{9.2.53}$$

where

$$\overline{G}_k = \sum_{i,j=1}^{n} \left\{ \left(\Gamma^k (b_{ij}(v, Dv) u_{x_i x_j}) - b_{ij}(v, Dv) \Gamma^k u_{x_i x_j} \right) + b_{ij}(v, Dv) \left(\Gamma^k u_{x_i x_j} - (\Gamma^k u)_{x_i x_j} \right) \right\}$$
$$+ 2 \sum_{j=1}^{n} \left\{ \left(\Gamma^k (a_{0j}(v, Dv) u_{t x_j}) - a_{0j}(v, Dv) \Gamma^k u_{t x_j} \right) + a_{0j}(v, Dv) \left(\Gamma^k u_{t x_j} - (\Gamma^k u)_{t x_j} \right) \right\}, \tag{9.2.54}$$

$$\overline{g}_k = \Gamma^k F(v, Dv) + \sum_{|l| \leq |k|-1} B_{kl} \Gamma^l \left\{ \sum_{i,j=1}^{n} b_{ij}(v, Dv) u_{x_i x_j} + 2 \sum_{j=1}^{n} a_{0j}(v, Dv) u_{t x_j} + F(v, Dv) \right\}, \tag{9.2.55}$$

and B_{kl} are some constants. Thus, similarly to (9.2.41), we have

$$\|(\Gamma^k u(t,\cdot))_t\|^2_{L^2(I\!R^n)} + \sum_{i,j=1}^{n} \int_{I\!R^n} a_{ij}(v, Dv)(t,\cdot)(\Gamma^k u(t,\cdot))_{x_i} (\Gamma^k u(t,\cdot))_{x_j} dx$$
$$= \|(\Gamma^k u(0,\cdot))_t\|^2_{L^2(I\!R^n)} + \sum_{i,j=1}^{n} \int_{I\!R^n} a_{ij}(v, Dv)(0,\cdot)(\Gamma^k u(0,\cdot))_{x_i} (\Gamma^k u(0,\cdot))_{x_j} dx$$
$$+ \sum_{i,j=1}^{n} \int_0^t \int_{I\!R^n} \frac{\partial b_{ij}(v, Dv)(\tau,\cdot)}{\partial \tau} (\Gamma^k u(\tau,\cdot))_{x_i} (\Gamma^k u(\tau,\cdot))_{x_j} dx d\tau$$
$$- 2 \sum_{i,j=1}^{n} \int_0^t \int_{I\!R^n} \frac{\partial b_{ij}(v, Dv)(\tau,\cdot)}{\partial x_i} (\Gamma^k u(\tau,\cdot))_{x_j} (\Gamma^k u(\tau,\cdot))_{\tau} dx d\tau$$
$$- 2 \sum_{j=1}^{n} \int_0^t \int_{I\!R^n} \frac{\partial a_{0j}(v, Dv)(\tau,\cdot)}{\partial x_j} (\Gamma^k u(\tau,\cdot))_{\tau} (\Gamma^k u(\tau,\cdot))_{\tau} dx d\tau$$
$$+ 2 \int_0^t \int_{I\!R^n} \overline{G}_k(\tau,\cdot)(\Gamma^k u(\tau,\cdot))_{\tau} dx d\tau + 2 \int_0^t \int_{I\!R^n} \overline{g}_k(\tau,\cdot)(\Gamma^k u(\tau,\cdot))_{\tau} dx d\tau. \tag{9.2.56}$$

From this, (9.2.52) can be obtained by completely similar arguments.

Combining (9.2.37) and (9.2.51)–(9.2.52), we obtain the desired (9.2.20).

The proof of Lemma 9.2.5 is finished.

9.2.4 Proof of Lemma 9.2.6

Let

$$u^* = \overline{u} - \overline{\overline{u}}, \ v^* = \overline{v} - \overline{\overline{v}}. \tag{9.2.57}$$

By the definition (9.2.11)–(9.2.13) of M, it is easy to know that

$$\Box u^* - \sum_{i,j=1}^{n} b_{ij}(\overline{v}, D\overline{v})u^*_{x_i x_j} - 2\sum_{j=1}^{n} a_{0j}(\overline{v}, D\overline{v})u^*_{t x_j} = F^*, \tag{9.2.58}$$

$$t = 0 : u^* = u^*_t = 0, \tag{9.2.59}$$

where

$$F^* = \sum_{i,j=1}^{n} (b_{ij}(\overline{v}, D\overline{v}) - b_{ij}(\overline{\overline{v}}, D\overline{\overline{v}}))\overline{\overline{u}}_{x_i x_j} + 2\sum_{j=1}^{n}(a_{0j}(\overline{v}, D\overline{v}) - a_{0j}(\overline{\overline{v}}, D\overline{\overline{v}}))\overline{\overline{u}}_{t x_j} + F(\overline{v}, D\overline{v}) - F(\overline{\overline{v}}, D\overline{\overline{v}}). \tag{9.2.60}$$

We first estimate $\|u^*(t,\cdot)\|_{\Gamma,S-1,2}$.

Applying (4.5.17) (in which we take $N = S - 1$) in Chap. 4 to Cauchy problem (9.2.58)–(9.2.59), we get

$$\|u^*(t,\cdot)\|_{\Gamma,S-1,2} \le C\int_0^t \left(\|\hat{F}^*(\tau,\cdot)\|_{\Gamma,S-1,q,\chi_1} + (1+\tau)^{-\frac{n-2}{2}}\|\hat{F}^*(\tau,\cdot)\|_{\Gamma,S-1,1,2,\chi_2}\right)d\tau, \tag{9.2.61}$$

where

$$\hat{F}^* = \sum_{i,j=1}^{n} b_{ij}(\overline{v}, D\overline{v})u^*_{x_i x_j} + 2\sum_{j=1}^{n} a_{0j}(\overline{v}, D\overline{v})u^*_{t x_j} + F^*, \tag{9.2.62}$$

q satisfies (9.2.19), χ_1 is the characteristic function of set $\{(t,x)||x| \le \frac{1+t}{2}\}$, and $\chi_2 = 1 - \chi_1$.

Similarly to (9.2.35), we have

$$\left\|\left(\sum_{i,j=1}^{n} b_{ij}(\overline{v}, D\overline{v})u^*_{x_i x_j} + 2\sum_{j=1}^{n} a_{0j}(\overline{v}, D\overline{v})u^*_{t x_j}\right)(\tau,\cdot)\right\|_{\Gamma,S-1,q,\chi_1} \le C(1+\tau)^{-K}E^\alpha D_{S-1,T}(u^*), \quad \forall \tau \in [0,T]. \tag{9.2.63}$$

In addition, from (5.2.36) in Lemma 5.2.7 and (5.2.46) in Lemma 5.2.8 of Chap. 5 (in which we take $N = S - 1$, $r = q$ and $p = n$), and noting that $L^{q,2}(I\!\!R^n) \subset L^q(I\!\!R^n)$ is continuous embedding, it is easy to know that

$$\|F^*(\tau,\cdot)\|_{\Gamma,S-1,q,\chi_1} \le C(1+\tau)^{-K}E^\alpha D_{S-1,T}(v^*), \quad \forall \tau \in [0,T]. \tag{9.2.64}$$

Combining (9.2.63) and (9.2.64), we obtain

$$\|\hat{F}^*(\tau,\cdot)\|_{\Gamma,S-1,q,\chi_1} \le C(1+\tau)^{-K}E^\alpha (D_{S-1,T}(u^*) + D_{S-1,T}(v^*)), \quad \forall \tau \in [0,T]. \tag{9.2.65}$$

Moreover, from (5.2.35) in Lemma 5.2.7 and (5.2.45) in Lemma 5.2.8 of Chap. 5 (in which we take $N = S-1$, $r=1$ and $p=2$), similarly we have

$$(1+\tau)^{-\frac{n-2}{2}}\left\|\left(\sum_{i,j=1}^{n} b_{ij}(\overline{v}, D\overline{v})u^*_{x_i x_j} + 2\sum_{j=1}^{n} a_{0j}(\overline{v}, D\overline{v})u^*_{tx_j}\right)(\tau,\cdot)\right\|_{\Gamma,S-1,1,2,\chi_2}$$
$$\leq C(1+\tau)^{-K}E^{\alpha}D_{S-1,T}(u^*), \quad \forall \tau\in[0,T] \tag{9.2.66}$$

and

$$(1+\tau)^{-\frac{n-2}{2}}\|F^*(\tau,\cdot)\|_{\Gamma,S-1,1,2,\chi_2} \leq C(1+\tau)^{-K}E^{\alpha}D_{S-1,T}(v^*), \quad \forall \tau\in[0,T], \tag{9.2.67}$$

then we get

$$(1+\tau)^{-\frac{n-2}{2}}\|\hat{F}^*(\tau,\cdot)\|_{\Gamma,S-1,1,2,\chi_2} \leq C(1+\tau)^{-K}E^{\alpha}(D_{S-1,T}(u^*) + D_{S-1,T}(v^*)), \quad \forall \tau\in[0,T]. \tag{9.2.68}$$

Substituting (9.2.65) and (9.2.68) into (9.2.61), we have

$$\sup_{0\leq t\leq T}\|u^*(t,\cdot)\|_{\Gamma,S-1,2} \leq CR(E,T)(D_{S-1,T}(u^*) + D_{S-1,T}(v^*)). \tag{9.2.69}$$

Now we estimate $\|D^2u^*(t,\cdot)\|_{\Gamma,S-1,2}$.

Similarly to (9.2.41), for any given multi-index k ($|k|\leq S-1$), we have

$$\|(\Gamma^k Du^*(t,\cdot))_t\|^2_{L^2(I\!R^n)} + \sum_{i,j=1}^{n}\int_{I\!R^n} a_{ij}(\overline{v}, D\overline{v})(t,\cdot)(\Gamma^k Du^*(t,\cdot))_{x_i}(\Gamma^k Du^*(t,\cdot))_{x_j}dx$$
$$= \sum_{i,j=1}^{n}\int_0^t\int_{I\!R^n}\frac{\partial b_{ij}(\overline{v}, D\overline{v})(\tau,\cdot)}{\partial\tau}(\Gamma^k Du^*(\tau,\cdot))_{x_i}(\Gamma^k Du^*(\tau,\cdot))_{x_j}dxd\tau$$
$$-2\sum_{i,j=1}^{n}\int_0^t\int_{I\!R^n}\frac{\partial b_{ij}(\overline{v}, D\overline{v})(\tau,\cdot)}{\partial x_i}(\Gamma^k Du^*(\tau,\cdot))_{x_j}(\Gamma^k Du^*(\tau,\cdot))_{\tau}dxd\tau$$
$$-2\sum_{j=1}^{n}\int_0^t\int_{I\!R^n}\frac{\partial a_{0j}(\overline{v}, D\overline{v})(\tau,\cdot)}{\partial x_j}(\Gamma^k Du^*(\tau,\cdot))_{\tau}(\Gamma^k Du^*(\tau,\cdot))_{\tau}dxd\tau$$
$$+2\int_0^t\int_{I\!R^n}\widetilde{G}_k(\tau,\cdot)(\Gamma^k Du^*(\tau,\cdot))_{\tau}dxd\tau + 2\int_0^t\int_{I\!R^n}\tilde{g}_k(\tau,\cdot)(\Gamma^k Du^*(\tau,\cdot))_{\tau}dxd\tau$$
$$+2\int_0^t\int_{I\!R^n}\hat{g}_k(\tau,\cdot)(\Gamma^k Du^*(\tau,\cdot))_{\tau}dxd\tau$$
$$\stackrel{\text{def.}}{=} \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI}, \tag{9.2.70}$$

where

$$\widetilde{G}_k = \sum_{i,j=1}^{n} \Big\{ \Big(\Gamma^k D(b_{ij}(\overline{v}, D\overline{v}) u^*_{x_i x_j} - b_{ij}(\overline{v}, D\overline{v}) \Gamma^k D u^*_{x_i x_j} \Big) + b_{ij}(\overline{v}, D\overline{v}) \Big(\Gamma^k D u^*_{x_i x_j} - (\Gamma^k D u^*)_{x_i x_j} \Big) \Big\}$$
$$+ 2 \sum_{j=1}^{n} \Big\{ \Big(\Gamma^k D(a_{0j}(\overline{v}, D\overline{v})) u^*_{t x_j} - a_{0j}(\overline{v}, D\overline{v}) \Gamma^k D u^*_{t x_j} \Big) + a_{0j}(\overline{v}, D\overline{v}) \Big(\Gamma^k D u^*_{t x_j} - (\Gamma^k D u^*)_{t x_j} \Big) \Big\}, \tag{9.2.71}$$

$$\widetilde{g}_k = \sum_{|l| \le |k|} \widetilde{B}_{kl} \Gamma^l \left(\sum_{i,j=1}^{n} b_{ij}(\overline{v}, D\overline{v}) u^*_{x_i x_j} + 2 \sum_{j=1}^{n} a_{0j}(\overline{v}, D\overline{v}) u^*_{t x_j} \right), \tag{9.2.72}$$

$$\hat{g}_k = \Gamma^k D F^* + \sum_{|l| \le |k|} \widetilde{B}_{kl} \Gamma^l F^*. \tag{9.2.73}$$

Similarly to the proof of Lemma 9.2.5, we obtain

$$|\mathrm{I}| + |\mathrm{II}| + |\mathrm{III}| + |\mathrm{IV}| + |\mathrm{V}| \le C R(E, T) D^2_{S-1,T}(u^*). \tag{9.2.74}$$

It remains to estimate VI. By (9.2.60), we have

$$DF^* = \sum_{i,j=1}^{n} \big(Db_{ij}(\overline{v}, D\overline{v}) - Db_{ij}(\overline{\overline{v}}, D\overline{\overline{v}}) \big) \overline{\overline{u}}_{x_i x_j} + \sum_{i,j=1}^{n} \big(b_{ij}(\overline{v}, D\overline{v}) - b_{ij}(\overline{\overline{v}}, D\overline{\overline{v}}) \big) D\overline{\overline{u}}_{x_i x_j}$$
$$+ 2 \sum_{j=1}^{n} \big(Da_{0j}(\overline{v}, D\overline{v}) - Da_{0j}(\overline{\overline{v}}, D\overline{\overline{v}}) \big) \overline{\overline{u}}_{t x_j} + 2 \sum_{j=1}^{n} \big(a_{0j}(\overline{v}, D\overline{v}) - a_{0j}(\overline{\overline{v}}, D\overline{\overline{v}}) \big) D\overline{\overline{u}}_{t x_j}$$
$$+ DF(\overline{v}, D\overline{v}) - DF(\overline{\overline{v}}, D\overline{\overline{v}}). \tag{9.2.75}$$

Then, using (5.2.35) in Lemma 5.2.7 and (5.2.45) in Lemma 5.2.8 of Chap. 5 (in which we take $N = S - 1$, $r = 2$, then $p = +\infty$), we get

$$\|\hat{g}_k(\tau, \cdot)\|_{L^2(\mathbb{R}^n)} \le C \left(\|DF^*\|_{\Gamma, S-1,2} + \|F^*\|_{\Gamma, S-1,2} \right) \le C(1+\tau)^{-K} E^{\alpha} D_{S-1,T}(v^*), \quad \forall \tau \in [0, T], \tag{9.2.76}$$

so

$$|\mathrm{VI}| \le C R(E, T) D_{S-1,T}(u^*) D_{S-1,T}(v^*). \tag{9.2.77}$$

By (9.2.74) and (9.2.77), similarly to (9.2.50), we have

$$\sup_{0\leq t\leq T} \|D^2u^*(t,\cdot)\|^2_{\Gamma,S-1,2} \leq CR(E,T)(D^2_{S-1,T}(u^*) + D_{S-1,T}(u^*)D_{S-1,T}(v^*)), \tag{9.2.78}$$

then

$$\sup_{0\leq t\leq T} \|D^2u^*(t,\cdot)\|_{\Gamma,S-1,2} \leq C\sqrt{R(E,T)}(D_{S-1,T}(u^*) + D_{S-1,T}(v^*)). \tag{9.2.79}$$

Moreover, similarly to (9.2.52), we have

$$\sup_{0\leq t\leq T} \|Du^*(t,\cdot)\|_{\Gamma,S-1,2} \leq C\sqrt{R(E,T)}(D_{S-1,T}(u^*) + D_{S-1,T}(v^*)). \tag{9.2.80}$$

Combining (9.2.69) and (9.2.79)–(9.2.80), we get the desired (9.2.22). Lemma 9.2.6 is proved.

9.2.5 The Case that the Nonlinear Term on the Right-Hand Side Does not Depend on u Explicitly: $F = F(DU, D_x DU)$

In the special case that the nonlinear term on the right-hand side does not depend on u explicitly:

$$F = F(Du, D_x Du), \tag{9.2.81}$$

using similar but much simpler arguments, similarly as in Theorem 9.2.1, we can obtain the following complete results on the life-span $\widetilde{T}(\varepsilon)$ $(> T(\varepsilon))$ of classical solutions for the space dimension $n \geq 2$:

$$T(\varepsilon) = \begin{cases} +\infty, & \text{if } K_0 > 1, \\ \exp\{a\varepsilon^{-\alpha}\} - 1, & \text{if } K_0 = 1, \\ b\varepsilon^{-\frac{\alpha}{1-K_0}} - 1, & \text{if } K_0 < 1, \end{cases} \tag{9.2.82}$$

where

$$K_0 = \frac{n-1}{2}\alpha, \tag{9.2.83}$$

and a and b are positive constants depending only on α and n (see Li and Chen 1992 for details).

From (9.2.82), we have the lower bound estimates for life-span $\widetilde{T}(\varepsilon)$ as shown in the following table:

	$\widetilde{T}(\varepsilon) \geq$		
n $\alpha =$	1	2	$3, 4, \ldots$
2	$b\varepsilon^{-2}$	$\exp\{a\varepsilon^{-2}\}$	
3	$\exp\{a\varepsilon^{-1}\}$	$+\infty$	
$4, 5, \ldots$			

This gives not only the second table for $n \geq 3$ in Sect. 9.1, but also the sharp result for $n = 2$ (see Chaps. 13 and 14 for reference). Together with (8.1.12) in Chap. 8 for $n = 1$, we obtain, in this special case, the following full results on the lower bound estimates of life-span $\widetilde{T}(\varepsilon)$ for any given $n \geq 1$ and $\alpha \geq 1$:

	$\widetilde{T}(\varepsilon) \geq$				
n $\alpha =$	1	2	$\cdots$	α	$\cdots$
1	$b\varepsilon^{-1}$	$b\varepsilon^{-2}$	$\cdots$	$b\varepsilon^{-\alpha}$	$\cdots$
2	$b\varepsilon^{-2}$	$\exp\{a\varepsilon^{-2}\}$			
3	$\exp\{a\varepsilon^{-1}\}$		$+\infty$		
$4, 5, \ldots$					

To get (9.2.82)–(9.2.83), since the nonlinear term on the right-hand side does not depend on u explicitly, we only need to replace (9.2.3) by

$$D_{S,T}(v) = \sup_{0 \leq t \leq T} \|Dv(t, \cdot)\|_{\Gamma,S,2}, \tag{9.2.84}$$

and do not need to estimate the L^2 norm of the solution itself, thus, we do not need (4.5.17) in Chap. 4, which is only applicable when $n \geq 3$.

9.3 Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (9.1.11)–(9.1.12) (Continued)

In this section, under the assumption

$$\partial_u^2 F(0, 0) = 0 \tag{9.3.1}$$

(i.e., (9.1.20) holds), we will prove the lower bound estimates shown by the second table in Sect. 1, for the life-span of classical solutions to Cauchy problem (9.1.11)–(9.1.12) of n (≥ 3) dimensional second-order quasi-linear hyperbolic equation. For this, we only need to improve the corresponding results given in the first table of Sect. 9.1 when $\alpha = 1$ and $n = 3, 4$.

To facilitate the narrative, we assume that the support of initial data (9.1.12) satisfies

$$\mathrm{supp}\{\varphi, \psi\} \subseteq \{x||x| \leq \rho\}, \tag{9.3.2}$$

where $\rho > 0$ is a constant.

9.3.1 Metric Space $X_{S,E,T}$. Main results

By the Sobolev embedding theorem, there exists a suitably small $E_0 > 0$, such that

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq \nu_0, \quad \forall f \in H^{[\frac{n}{2}]+1}(\mathbb{R}^n), \ \|f\|_{H^{[\frac{n}{2}]+1}(\mathbb{R}^n)} \leq E_0. \tag{9.3.3}$$

For any given integer $S \geq 2n+4$, any given positive numbers $E\,(\leq E_0)$ and $T\ (0 < T \leq +\infty)$, we introduce the set of functions

$$X_{S,E,T} = \{v(t,x) | D_{S,T}(v) \leq E,\ \partial_t^l v(0,x) = u_l^{(0)}(x)\ \ (l = 0, 1, \ldots, S+1)\}, \tag{9.3.4}$$

where

$$\begin{aligned} D_{S,T}(v) = {} & \sum_{i=1}^{2} \sup_{0\leq t\leq T} \|D^i v(t,\cdot)\|_{\Gamma,S,2} + \sup_{0\leq t\leq T} f_n^{-1}(t) \|v(t,\cdot)\|_{\Gamma,S,2} \\ & + \sup_{0\leq t\leq T} (1+t)^{\frac{n-1}{2}} \|v(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \end{aligned} \tag{9.3.5}$$

with

$$f_n(t) = \begin{cases} (1+t)^{\frac{1}{2}}, & \text{if } n = 3; \\ \ln(2+t), & \text{if } n = 4, \end{cases} \tag{9.3.6}$$

the definitions of $\|\cdot\|_{\Gamma,S,2}$ etc. can be found in Sect. 3.1.3 of Chap. 3, moreover, if T is finite, the supremum is taken on the interval $[0, T]$; while, if $T = +\infty$, the supremum is taken in $[0, +\infty)$. For simplicity, we use the united notation $[0, T]$ to represent the corresponding interval. In addition, $u_0^{(0)} = \varepsilon\varphi(x)$, $u_1^{(0)} = \varepsilon\psi(x)$, while, for $l = 2, \ldots, S+1$, $u_l^{(0)}(x)$ are values of $\partial_t^l u(t,x)$ at $t = 0$, which are determined uniquely by Eq. (9.1.11) and initial condition (9.1.12). Obviously, $u_l^{(0)}(l = 0, 1, \ldots, S+1)$ are all smooth functions with compact support in $\{x||x| \leq \rho\}$.

Introduce the following metric on $X_{S,E,T}$:

$$\rho(\overline{v}, \overline{\overline{v}}) = D_{S,T}(\overline{v} - \overline{\overline{v}}), \quad \forall \overline{v}, \overline{\overline{v}} \in X_{S,E,T}. \tag{9.3.7}$$

Similarly to Lemma 9.2.1, we have

Lemma 9.3.1 *When $\varepsilon > 0$ is suitably small, $X_{S,E,T}$ is a non-empty complete metric space.*

Denote by $\widetilde{X}_{S,E,T}$ a subset of $X_{S,E,T}$, which is composed of all the elements in $X_{S,E,T}$, whose support with respect x is in $\{x | |x| \le t + \rho\}$ for any given $t \in [0, T]$.

Lemma 9.3.2 *When $S \ge 2n + 4$, for any given $v \in \widetilde{X}_{S,E,T}$, we have*

$$\|(v, Dv, D^2 v)(t, \cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le CE(1+t)^{-\frac{n-1}{2}}, \quad \forall t \in [0, T], \tag{9.3.8}$$

where C is positive constant.

Proof Noting $S \ge 2n + 4$, from (3.4.30) in Chap. 3 (in which we take $p = 2$, $N = [\frac{S}{2}] + 1$ and $s = [\frac{n}{2}] + 1$), we have

$$\|(Dv, D^2 v)(t, \cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le CE(1+t)^{-\frac{n-1}{2}}, \quad \forall t \in [0, T],$$

while, similar estimates for v follows immediately from the definition (9.3.5) of $D_{S,T}(v)$. □

The main result of this section is the following

Theorem 9.3.1 *Let $\alpha = 1$, and $n = 3, 4$. Under assumptions (9.1.15)–(9.1.19), we assume furthermore that (9.3.1) holds, then for any given integer $S \ge 2n + 4$, there exist positive constants ε_0 and C_0 with $C_0 \varepsilon_0 \le E_0$, such that for any given $\varepsilon \in (0, \varepsilon_0]$, there exists a positive number $T(\varepsilon)$ such that Cauchy problem (9.1.11)–(9.1.12) admits a unique classical solution $u \in \widetilde{X}_{S,C_0\varepsilon,T(\varepsilon)}$ on $[0, T(\varepsilon)]$, and $T(\varepsilon)$ can be taken as*

$$T(\varepsilon) = \begin{cases} \exp\{a\varepsilon^{-1}\} - 1, & \text{if } n = 3; \\ +\infty, & \text{if } n = 4, \end{cases} \tag{9.3.9}$$

where a is a positive constant independent of ε.

Moreover, after possible change of values for t on a zero-measure set of interval $[0, T(\varepsilon)]$, for any given finite T_0 satisfying $0 < T_0 \le T(\varepsilon)$, we have

$$u \in C\left([0, T_0]; H^{S+1}(\mathbb{R}^n)\right), \tag{9.3.10}$$

$$u_t \in C\left([0, T_0]; H^{S}(\mathbb{R}^n)\right), \tag{9.3.11}$$

$$u_{tt} \in C\left([0, T_0]; H^{S-1}(\mathbb{R}^n)\right). \tag{9.3.12}$$

9.3.2 Framework to Prove Theorem 9.3.1—The Global Iteration Method

To prove Theorem 9.3.1, for any given $v \in \widetilde{X}_{S,E,T}$, similarly, we define a mapping by solving Cauchy problem (9.2.11)–(9.2.12) of linear hyperbolic equation:

$$M : v \longrightarrow u = Mv. \tag{9.3.13}$$

We want to prove: when $\varepsilon > 0$ is suitable small, we can find a positive constant C_0 such that when $E = C_0\varepsilon$ and $T = T(\varepsilon)$ is defined by (9.3.9), M admits a unique fixed point in $\widetilde{X}_{S,E,T}$, which is exactly the classical solution to Cauchy problem (9.1.11)–(9.1.12) on $0 \leq t \leq T(\varepsilon)$.

Similarly to Lemma 9.2.3, we have

Lemma 9.3.3 *When $E > 0$ is suitable small, for any given $v \in \widetilde{X}_{S,E,T}$, after possible change of values on a zero-measure set of t, for any given finite T_0 satisfying $0 < T_0 \leq T$, we have*

$$u = Mv \in C\left([0, T_0]; H^{S+1}(I\!R^n)\right), \tag{9.3.14}$$

$$u_t \in C\left([0, T_0]; H^{S}(I\!R^n)\right), \tag{9.3.15}$$

$$u_{tt} \in L^\infty\left(0, T_0; H^{S-1}(I\!R^n)\right). \tag{9.3.16}$$

Moreover, for any given $t \in [0, T]$, the support of $u = u(t, x)$ with respect to x lies in $\{x \,|\, |x| \leq t + \rho\}$.

Similarly to Lemma 9.2.4, we have

Lemma 9.3.4 *For $u = u(t, x) = Mv$, the values of $\partial_t^l u(0, \cdot)(l = 0, 1, \cdots, S + 2)$ are independent of the choice of $v \in \widetilde{X}_{S,E,T}$, and*

$$\partial_t^l u(0, x) = u_l^{(0)}(x) \quad (l = 0, 1, \cdots, S + 1). \tag{9.3.17}$$

Moreover,

$$\|u(0, \cdot)\|_{\Gamma,S+2,p} + \|u_t(0, \cdot)\|_{\Gamma,S+1,p,q} \leq C\varepsilon, \tag{9.3.18}$$

where $1 \leq p, q \leq +\infty$, and C is a positive constant.

The following two Lemmas are crucial to the proof of Theorem 9.3.1.

Lemma 9.3.5 *Under the assumptions of Theorem 9.3.1, when $E > 0$ is suitably small, for any given $v \in \widetilde{X}_{S,E,T}$, $u = Mv$ satisfies*

$$D_{S,T}(u) \leq C_1\{\varepsilon + (R + \sqrt{R} + E)(E + D_{S,T}(u))\}, \tag{9.3.19}$$

wher C_1 is a positive constant, and

$$R = R(E, T) \stackrel{\text{def.}}{=} E \int_0^T (1 + t)^{-\frac{n-1}{2}} dt. \tag{9.3.20}$$

Lemma 9.3.6 *Under the assumptions of Lemma 9.3.5, for any given $\overline{v}, \overline{\overline{v}} \in \widetilde{X}_{S,E,T}$, if $\overline{u} = M\overline{v}$ and $\overline{\overline{u}} = M\overline{\overline{v}}$ also satisfy $\overline{u}, \overline{\overline{u}} \in \widetilde{X}_{S,E,T}$, then we have*

$$D_{S-1,T}(\overline{u}-\overline{\overline{u}}) \leq C_2\left(R+\sqrt{R}+E\right)\left(D_{S-1,T}(\overline{u}-\overline{\overline{u}})+D_{S-1,T}(\overline{v}-\overline{\overline{v}})\right), \tag{9.3.21}$$

where C_2 is a positive constant, and $R = R(E,T)$ is still defined by (9.3.20).

The proof of Lemmas 9.3.5 and 9.3.6 will be given later. Now we first utilize these two lemmas to prove Theorem 9.3.1.

Proof of Theorem 9.3.1 Take

$$C_0 = 3\max(C_1, C_2), \tag{9.3.22}$$

where C_1 and C_2 are positive constants appearing in Lemmas 9.3.5 and 9.3.6, respectively.

Similarly to Sect. 9.2, we can prove: if there exists a positive number ε_0 with $C_0\varepsilon_0 \leq E_0$, such that for any given $\varepsilon \in (0,\varepsilon_0]$, $E = E(\varepsilon) = C_0\varepsilon$ and $T = T(\varepsilon) > 0$ satisfy

$$R(E(\varepsilon),T(\varepsilon)) + \sqrt{R(E(\varepsilon),T(\varepsilon))} + E(\varepsilon) \leq \frac{1}{C_0}, \tag{9.3.23}$$

then the mapping M admits a unique fixed point in $\widetilde{X}_{S,E(\varepsilon),T(\varepsilon)}$, and this fixed point $u = u(t,x)$ is exactly the classical solution to Cauchy problem (9.1.11)–(9.1.12) on $0 \leq t \leq T(\varepsilon)$.

Now we determine $\varepsilon_0 > 0$ and $T(\varepsilon)(0 < \varepsilon \leq \varepsilon_0)$ such that (9.3.23) holds. We always assume that $\varepsilon_0 > 0$ is so small that (9.3.3) holds when $E_0 = C_0\varepsilon_0$.

When $n = 3$, from (9.3.20) we have

$$R = R(E,T) = E\ln(1+T).$$

Then, if we take $E = C_0\varepsilon$ and

$$T(\varepsilon) = \exp\{a\varepsilon^{-1}\} - 1,$$

where a is a positive number satisfying

$$C_0(aC_0 + \sqrt{aC_0}) < 1, \tag{9.3.24}$$

it is easy to verify (9.3.23) when $\varepsilon_0 > 0$ is sufficiently small. At this moment, we obtain an almost global solution.

When $n = 4$, from (9.3.20) we have

$$R = R(E,T) = E\int_0^T (1+t)^{-\frac{3}{2}}dt \leq \widetilde{C}E, \quad \forall T > 0,$$

where $\widetilde{C}$ is a positive constant independent of T. Then, if we take $E = C_0\varepsilon$ and

$$T(\varepsilon) = +\infty,$$

when $\varepsilon_0 > 0$ is sufficiently small, it is easy to verify (9.3.23). At this moment, we obtain a global solution. □

9.3.3 Proof of Lemma 9.3.5

We first estimate $\|D^2u(t,\cdot)\|_{\Gamma,S,2}$.

For this, we still use the energy integral formula (9.2.41), where G_k and g_k are given by (9.2.39) and (9.2.40), respectively, and $k(|k| \leq S)$ is an arbitrarily given multi-index.

Noting $\alpha = 1$, from Corollary 3.1.1 in Chap. 3 and Lemma 9.3.2, it is easy to obtain

$$\begin{aligned} |\mathrm{I}|, |\mathrm{II}|, |\mathrm{III}| &\leq CE\int_0^t (1+\tau)^{-\frac{n-1}{2}} \|D^2u(\tau,\cdot)\|^2_{\Gamma,S,2}d\tau \\ &\leq CR(E,T)D^2_{S,T}(u), \quad \forall t \in [0,T]. \end{aligned} \tag{9.3.25}$$

Now we estimate the L^2 norm of $G_k(\tau,\cdot)$.

Using the Taylor expansion of $b_{ij}(v,Dv)$, it is easy to show that

$$\begin{aligned} &\|(\Gamma^k D(b_{ij}(v,Dv)u_{x_ix_j}) - b_{ij}(v,Dv)\Gamma^k Du_{x_ix_j})(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \\ \leq\ & C\big\{\|\Gamma^k(Dvu_{x_ix_j})\|_{L^2(\mathbb{R}^n)} + \|\Gamma^k(D^2vu_{x_ix_j})\|_{L^2(\mathbb{R}^n)} \\ &+ \|\Gamma^k(vDu_{x_ix_j}) - v\Gamma^k Du_{x_ix_j}\|_{L^2(\mathbb{R}^n)} \\ &+ \|\Gamma^k(DvDu_{x_ix_j}) - Dv\Gamma^k Du_{x_ix_j}\|_{L^2(\mathbb{R}^n)} \\ &+ \|\Gamma^k(D\widetilde{b}_{ij}(v,Dv)u_{x_ix_j})\|_{L^2(\mathbb{R}^n)} \\ &+\|\Gamma^k(\widetilde{b}_{ij}(v,Dv)Du_{x_ix_j}) - \widetilde{b}_{ij}(v,Dv)\Gamma^k Du_{x_ix_j}\|_{L^2(\mathbb{R}^n)}\big\} \\ \overset{\text{def.}}{=}\ & I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned} \tag{9.3.26}$$

where $\widetilde{b}_{ij}(v,Dv)$ denote higher order terms of $b_{ij}(v,Dv)$.

By (5.1.19) in Remark 5.1.2 of Chap. 5 (in which we take $N = S$, $\chi(t,x) \equiv 1$, $p = q = p_2 = q_2 = p_3 = q_3 = 2$, $p_1 = q_1 = p_4 = q_4 = +\infty$), noticing Lemma 9.3.2 and the definition of $X_{S,E,T}$, it is easy to know that

$$I_1, I_2 \leq CE(1+\tau)^{-\frac{n-1}{2}} D_{S,T}(u), \quad \tau \in [0,T]. \tag{9.3.27}$$

By (5.1.32) in Lemma 5.1.4 of Chap. 5 (in which we take $N = S$, $p = q = 2$, $p_1 = q_1 = +\infty$), noticing Lemma 9.3.2 and the definition of $X_{S,E,T}$, and using (3.4.30) in Chap. 3 (in which we take $p = 2$, $N = [\frac{S}{2}] + 1$ and $s = [\frac{n}{2}] + 1$), we then obtain

$$I_3, I_4 \leq CE(1+\tau)^{-\frac{n-1}{2}} D_{S,T}(u), \quad \forall \tau \in [0,T]. \tag{9.3.28}$$

Noting (9.3.6), it is easy to know that for $n = 3, 4$ and any given integer $\beta \geq 2$, we always have

$$(1+\tau)^{-\frac{n-1}{2}(\beta-1)} f_n^{\beta}(\tau) \leq C, \quad \forall \tau \geq 0. \tag{9.3.29}$$

Then, by (5.2.32) in Lemma 5.2.6 of Chap. 5 (in which we take $N = S, \gamma = 2, p = +\infty$), and noting the definition of $X_{S,E,T}$, it is easy to get

$$I_6 \leq CE(1+\tau)^{-\frac{n-1}{2}} D_{S,T}(u), \quad \forall \tau \in [0, T]. \tag{9.3.30}$$

Finally, by (5.1.15) in Remark 5.1.1 of Chap. 5 (in which we take $N = S, r = q_1 = p_2 = 2, p_1 = q_2 = +\infty$), using (3.4.30) in Chap. 3 (in which we take $p = 2, N = [\frac{S}{2}] + 1$ and $s = [\frac{n}{2}] + 1$), (5.2.13) in Chap. 5 (in which we take $N = [\frac{S-1}{2}] + 1, p, q, p_i, q_i (i = 0, 1, \cdots, \beta)$ are all $+\infty$) and (5.2.32) in Chap. 5 (in which we take $N = S, r = 2, p = +\infty$), and noticing (9.3.29) and the definition of $X_{S,E,T}$, we have

$$I_5 \leq CE(1+\tau)^{-\frac{n-1}{2}} D_{S,T}(u), \quad \forall \tau \in [0, T]. \tag{9.3.31}$$

Hence, from (9.3.27)–(9.3.28) and (9.3.30)–(9.3.31) we obtain

$$\begin{aligned} &\|(\Gamma^k D(b_{ij}(v, Dv)u_{x_i x_j}) - b_{ij}(v, Dv)\Gamma^k D u_{x_i x_j})(\tau, \cdot)\|_{L^2(\mathbb{R}^n)} \\ &\leq CE(1+\tau)^{-\frac{n-1}{2}} D_{S,T}(u), \quad \forall \tau \in [0, T]. \end{aligned} \tag{9.3.32}$$

On the other hand, noticing Corollary 3.1.1 in Chap. 3 and (9.3.8), it is easy to get

$$\begin{aligned} &\|b_{ij}(v, Dv)(\Gamma^k D u_{x_i x_j} - (\Gamma^k D u)_{x_i x_j})(\tau, \cdot)\|_{L^2(\mathbb{R}^n)} \\ &\leq CE(1+\tau)^{-\frac{n-1}{2}} D_{S,T}(u), \quad \forall \tau \in [0, T]. \end{aligned} \tag{9.3.33}$$

For the terms in G_k involving $a_{0j}(v, Dv)$, similar estimates hold. Then,

$$\|G_k(\tau, \cdot)\|_{L^2(\mathbb{R}^n)} \leq CE(1+\tau)^{-\frac{n-1}{2}} D_{S,T}(u), \quad \forall \tau \in [0, T], \tag{9.3.34}$$

therefore

$$|\mathrm{IV}| \leq CR(E, T) D_{S,T}^2(u). \tag{9.3.35}$$

Now we estimate the L^2 norm of $g_k(\tau, \cdot)$.

Making Taylor expansion of $F(v, Dv)$, and noticing (9.3.1), we have

$$\Gamma^k DF(v, Dv) = \sum_{a=0}^{n} C_a \Gamma^k D(v \partial_a v) + \sum_{a,b=0}^{n} C_{ab} \Gamma^k D(\partial_a v \partial_b v) + \Gamma^k D\widetilde{F}(v, Dv)$$

$$= \sum_{a=0}^{n} C_a \Gamma^k (v D\partial_a v) + \sum_{a=0}^{n} C_a \Gamma^k (Dv\partial_a v) + \sum_{a,b=0}^{n} C_{ab} \Gamma^k D(\partial_a v \partial_b v) + \Gamma^k D\widetilde{F}(v, Dv), \tag{9.3.36}$$

where C_a, C_{ab} are some constants, and $\widetilde{F}(v, Dv)$ is the higher order term in $F(v, Dv)$. Then

$$\begin{aligned} &\|\Gamma^k DF(v, Dv)(\tau, \cdot)\|_{L^2(I\!R^n)} \\ \leq\ & C\left\{\|\Gamma^k(vD^2v)\|_{L^2(I\!R^n)} + \|\Gamma^k(DvDv)\|_{L^2(I\!R^n)} \right. \\ &\left. + \|\Gamma^k D(DvDv)\|_{L^2(I\!R^n)} + \|\Gamma^k D\widetilde{F}(v, Dv)\|_{L^2(I\!R^n)}\right\} \\ \stackrel{\text{def.}}{=}\ & J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{9.3.37}$$

Using (5.1.31) in Lemma 5.1.4 of Chap. 5 (in which we take $N = S$, $p = q = 2$, $p_1 = q_1 = +\infty$), and noting (9.3.8) and the definition of $X_{S,E,T}$, it is easy to have

$$J_1 \leq CE^2(1+\tau)^{-\frac{n-1}{2}}, \quad \forall \tau \in [0, T]. \tag{9.3.38}$$

Using (5.1.15) in Remark 5.1.1 of Chap. 5 (in which we take $N = S, r = q_1 = p_2 = 2$, $p_1 = q_2 = +\infty$), and noting the definition of $X_{S,E,T}$, it is easy to have

$$J_2, J_3 \leq CE^2(1+\tau)^{-\frac{n-1}{2}}, \quad \forall \tau \in [0, T]. \tag{9.3.39}$$

Using again (5.1.15) in Remark 5.1.1 of Chap. 5, using also (5.2.13) in Chap. 5 (in which we take $N = [\frac{S}{2}]$, $p, q, p_i, q_i (i = 0, 1, \cdots, \beta)$ are all $+\infty$) and (5.2.32) in Chap. 5 (in which we take $N = S, r = 2, p = \infty$), and noticing (9.3.29) and the definition of $X_{S,E,T}$, we have

$$J_4 \leq CE^2(1+\tau)^{-\frac{n-1}{2}}, \quad \forall \tau \in [0, T]. \tag{9.3.40}$$

Hence, we obtain

$$\|\Gamma^k DF(v, Dv)(\tau, \cdot)\|_{L^2(I\!R^n)} \leq CE^2(1+\tau)^{-\frac{n-1}{2}}, \quad \forall \tau \in [0, T]. \tag{9.3.41}$$

Similarly, we can prove: for any given multi-index $l (|l| \leq |k| \leq S)$ we have

$$\|\Gamma^l F(v, Dv)(\tau, \cdot)\|_{L^2(I\!R^n)} \leq CE^2(1+\tau)^{-\frac{n-1}{2}}, \quad \forall \tau \in [0, T]. \tag{9.3.42}$$

Moreover, using Taylor expansion, for any given multi-index $l (|l| \leq |k|)$, we have

$$
\begin{aligned}
&\|\Gamma^l(b_{ij}(v,Dv)u_{x_ix_j})(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\\
\le\ & C\left\{\|\Gamma^l(vu_{x_ix_j})\|_{L^2(\mathbb{R}^n)}+\|\Gamma^l(Dvu_{x_ix_j})\|_{L^2(\mathbb{R}^n)}+\|\Gamma^l(\widetilde{b}_{ij}(v,Dv)u_{x_ix_j})\|_{L^2(\mathbb{R}^n)}\right\},
\end{aligned}
\tag{9.3.43}
$$

where $\widetilde{b}_{ij}(v, Dv)$ still stand for the higher-order terms in $b_{ij}(v, Dv)$. Adopting similar method in estimating $J_i (i = 1, 2, 3, 4)$, we obtain

$$
\|\Gamma^l(b_{ij}(v,Dv)u_{x_ix_j})(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\le CE(1+\tau)^{-\frac{n-1}{2}}D_{S,T}(u),\quad \forall\tau\in[0,T].
\tag{9.3.44}
$$

For the terms involving $a_{0j}(v, Dv)$ in g_k, we have similar estimates. Hence, we obtain

$$
\|g_k(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}\le CE(1+\tau)^{-\frac{n-1}{2}}(E+D_{S,T}(u)),\quad \forall\tau\in[0,T],
\tag{9.3.45}
$$

therefore,

$$
|V|\le CR(E,T)(ED_{S,T}(u)+D^2_{S,T}(u)).
\tag{9.3.46}
$$

From (9.3.25), (9.3.35) and (9.3.46), noting (9.1.18) and Lemma 9.3.4, and using Corollary 3.1.1 in Chap. 3, it immediately yields

$$
\sup_{0\le t\le T}\|D^2u(t,\cdot)\|^2_{\Gamma,S,2}\le C\{\varepsilon^2+R(E,T)(ED_{S,T}(u)+D^2_{S,T}(u))\}.
\tag{9.3.47}
$$

Now we estimate $\|Du(t,\cdot)\|_{\Gamma,S,2}$.

For any given multi-index $k(|k| \le S)$, from (9.2.53) we have

$$
\Box\Gamma^k u=\Gamma^k\hat{F}(v,Dv,D_xDu)+\sum_{|l|\le|k|-1}B_{kl}\Gamma^l\hat{F}(v,Dv,D_xDu),
\tag{9.3.48}
$$

from this we obtain the following energy integral formula:

$$
\begin{aligned}
&\|(\Gamma^k u(t,\cdot))_t\|^2_{L^2(\mathbb{R}^n)}+\sum_{i=1}^n\|(\Gamma^k u(t,\cdot))_{x_i}\|^2_{L^2(\mathbb{R}^n)}\\
=\ &\|(\Gamma^k u(0,\cdot))_t\|^2_{L^2(\mathbb{R}^n)}+\sum_{i=1}^n\|(\Gamma^k u(0,\cdot))_{x_i}\|^2_{L^2(\mathbb{R}^n)}+2\int_0^t\int_{\mathbb{R}^n}g_k^*(\tau,\cdot)(\Gamma^k u(\tau,\cdot))_\tau dxd\tau,
\end{aligned}
\tag{9.3.49}
$$

where

$$
\begin{aligned}
g_k^*&=\sum_{|l|\le|k|}C^*_{kl}\Gamma^l\hat{F}(v,Dv,D_xDu)\\
&=\sum_{|l|\le|k|}C^*_{kl}\Gamma^l\left(\sum_{i,j=1}^n b_{ij}(v,Dv)u_{x_ix_j}+2\sum_{j=1}^n a_{0j}(v,Dv)u_{tx_j}+F(v,Dv)\right),
\end{aligned}
$$

(9.3.50)

and C^*_{kl} are some constants. From (9.3.42) and (9.3.44) we immediately have

$$\|g_k^*(\tau,\cdot)\|_{L^2(I\!R^n)} \leq CE(1+\tau)^{-\frac{n-1}{2}}(E+D_{S,T}(u)), \quad \forall \tau \in [0,T], \tag{9.3.51}$$

then it is easy to get

$$\sup_{0\leq t\leq T} \|Du(t,\cdot)\|^2_{\Gamma,S,2} \leq C\{\varepsilon^2 + R(E,T)(ED_{S,T}(u) + D^2_{S,T}(u))\}. \tag{9.3.52}$$

Now we estimate $\|u(t,\cdot)\|_{\Gamma,S,2}$.

To this end, noting (9.3.1), $\hat{F}(v, Dv, D_x Du)$ defined by (9.2.11) can be rewritten as

$$\hat{F}(v, Dv, D_x Du) = \sum_{a=0}^{n} c_a \partial_a(v^2) + \sum_{i=1}^{n} d_i \partial_i(vDu) + Q(Dv, Du, D_x Du) + P(v, Dv, D_x Du), \tag{9.3.53}$$

where c_a, d_i are some constants, $Q(Dv, Du, D_x Du)$ are the remaining second-order terms which are affine with respect to Du and $D_x Du$, and $P(v, Du, D_x Du)$, affine with respect to $D_x Du$, are the higher-order (≥ 3) terms in $\hat{F}(v, Dv, D_x Du)$. Then, by the superposition principle, we have

$$u = Mv = u_1 + u_2 + u_3 + u_4, \tag{9.3.54}$$

while, u_1, u_2, u_3, u_4 satisfy, respectively,

$$\Box u_1 = \sum_{a=0}^{n} c_a \partial_a(v^2), \tag{9.3.55}$$

$$\Box u_2 = \sum_{i=0}^{n} d_i \partial_i(vDu), \tag{9.3.56}$$

$$\Box u_3 = Q(Dv, Du, D_x Du) \tag{9.3.57}$$

and

$$\Box u_4 = P(v, Dv, D_x Du), \tag{9.3.58}$$

and u_1, u_2 and u_4 have the zero initial condition, while, u_3 has the following initial condition:

$$t = 0 : u_3 = \varepsilon\varphi(x), \; u_{3t} = \varepsilon\psi(x). \tag{9.3.59}$$

Suppose that $\overline{u}_1, \overline{\overline{u}}_1$ and $\overline{u}_2$ satisfy, respectively,

$$\Box \overline{u}_1 = v^2, \tag{9.3.60}$$

$$\Box \overline{\overline{u}}_1 = 0 \tag{9.3.61}$$

and

$$\Box \overline{u}_2 = vDu, \tag{9.3.62}$$

and $\overline{u}_1$ and $\overline{u}_2$ have the zero initial condition, while, $\overline{\overline{u}}_1$ has the following initial condition:

$$t = 0 : \overline{\overline{u}}_1 = 0, \ \overline{\overline{u}}_{1t} = v^2(0, \cdot), \tag{9.3.63}$$

it is easy to know that

$$u_1 = \sum_{a=0}^{n} c_a \partial_a \overline{u}_1 - c_0 \overline{\overline{u}}_1 \tag{9.3.64}$$

and

$$u_2 = \sum_{i=1}^{n} d_i \partial_i \overline{u}_2. \tag{9.3.65}$$

By (4.5.17) in Corollary 4.5.1 in Chap. 4 and Lemma 9.3.4, we have

$$\begin{aligned} \|u_1(t, \cdot)\|_{\Gamma,S,2} &\le C(\|D\overline{u}_1(t, \cdot)\|_{\Gamma,S,2} + \|\overline{\overline{u}}_1(t, \cdot)\|_{\Gamma,S,2}) \\ &\le C(\varepsilon + \|D\overline{u}_1(t, \cdot)\|_{\Gamma,S,2}). \end{aligned} \tag{9.3.66}$$

From the energy estimates of wave equation (see Lemma 4.5.2 in Chap. 4) and Lemma 3.1.5 in Chap. 3, noting (5.1.15) in Remark 5.1.1 of Chap. 5, Lemma 9.3.2 and the definition of $X_{S,E,T}$, we have

$$\begin{aligned} \|u_1(t, \cdot)\|_{\Gamma,S,2} &\le C\left(\varepsilon + \int_0^t \|v^2(\tau, \cdot)\|_{\Gamma,S,2} d\tau\right) \\ &\le C\left(\varepsilon + \int_0^t E^2 f_n(\tau)(1+\tau)^{-\frac{n-1}{2}} d\tau\right) \\ &\le C(\varepsilon + E^2 f_n(t)). \end{aligned} \tag{9.3.67}$$

Similarly, we have

$$\|u_2(t, \cdot)\|_{\Gamma,S,2} \le C(\varepsilon + E f_n(t) D_{S,T}(u)). \tag{9.3.68}$$

In addition, by (4.5.17) in Corollary 4.5.1 of Chap. 4 and Lemma 9.3.4, we have

$$\|u_3(t,\cdot)\|_{\Gamma,S,2} \leq C\left\{\varepsilon + \int_0^t \Big(\|Q(Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,q,\chi_1} + (1+\tau)^{-\frac{n-2}{2}}\|Q(Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,1,2}\Big)d\tau\right\}, \tag{9.3.69}$$

where q satisfies

$$\frac{1}{q} = \frac{1}{2} + \frac{1}{n}, \tag{9.3.70}$$

and $\chi_1(t,x)$ is the characteristic function of set $\left\{(t,x) | |x| \leq \frac{1+t}{2}\right\}$.

Using (5.2.24) in Lemma 5.2.5 of Chap. 5 (in which we take $N=S$, $r=q$, $p=n$ and $\beta=1$), noticing the definition of $X_{S,E,T}$, the fact that $L^{q,2}(I\!R^n) \subset L^q(I\!R^n)$ is a continuous embedding, and that $Q(Dv,Du,D_xDu)$ is affine with respect to Du and D_xDu, we then have

$$\|Q(Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,q,\chi_1} \leq C(1+\tau)^{-\frac{n-2}{2}}E(E+D_{S,T}(u)), \quad \forall\tau\in[0,T]. \tag{9.3.71}$$

Using (5.2.24) in Lemma 5.2.5 of Chap. 5 (in which we take $N=S$, $r=q$, $p=n$ and $\beta=1$), we can obtain a similar inequality for $(1+\tau)^{-\frac{n-2}{2}}\|Q(Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,1,2}$. Therefore, we have

$$\|u_3(t,\cdot)\|_{\Gamma,S,2} \leq C\left\{\varepsilon + Ef_n(t)(E+D_{S,T}(u))\right\}. \tag{9.3.72}$$

By Corollary 4.5.2 in Chap. 4, we have

$$\|u_4(t,\cdot)|_{\Gamma,S,2} \leq C\left\{\varepsilon + \int_0^t \|P(v,Dv,D_xDu)(\tau,\cdot)\|_{\Gamma,S,q}d\tau\right\}, \tag{9.3.73}$$

where q is still given by (9.3.70). Noting that the higher-order (≥ 3) term $P(v,Dv,D_xDu)$ is affine with respect to D_xDu, by (5.1.15) in Remark 5.1.1 and (5.2.13) in Lemma 5.2.2 of Chap. 5, we have

$$\begin{aligned}
&\|P(v,Dv,D_xDu)(\tau,\cdot)\|_{\Gamma,S,q}\\
&\leq C\Big\{\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],\infty}\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],n}\|D^2u\|_{\Gamma,S,2}\\
&\quad+\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],\infty}\|D^2u\|_{\Gamma,[\frac{S}{2}],n}\|(v,Dv)\|_{\Gamma,S,2}\\
&\quad+\|D^2u\|_{\Gamma,[\frac{S}{2}],\infty}\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],n}\|(v,Dv)\|_{\Gamma,S,2}\\
&\quad+\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],\infty}\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],n}\|(v,Dv)\|_{\Gamma,S,2}\Big\}\\
&\stackrel{\text{def.}}{=} K_1+K_2+K_3+K_4.
\end{aligned} \tag{9.3.74}$$

Noting Lemma 5.2.4 in Chap. 5, we have

$$\|\cdot\|_{\Gamma,[\frac{S}{2}],n} \leq C\|\cdot\|^{1-\frac{2}{n}}_{\Gamma,[\frac{S}{2}],\infty}\|\cdot\|^{\frac{2}{n}}_{\Gamma,[\frac{S}{2}],2}, \tag{9.3.75}$$

then, noting Lemma 3.3.2, Corollary 3.4.4 in Chap. 3 (in which we take $N=[\frac{S}{2}]$, $p=2$, $s=[\frac{n}{2}]+1$) and the definition of $X_{S,E,T}$, we obtain

$$K_1 \leq CE^2(1+\tau)^{-\frac{n-1}{2}}(1+\tau)^{-\frac{n-1}{2}(1-\frac{2}{n})}f_n^{\frac{2}{n}}(\tau)D_{S,T}(u), \tag{9.3.76}$$

$$K_2 \leq CE^2(1+\tau)^{-\frac{n-1}{2}}(1+\tau)^{-\frac{n-1}{2}(1-\frac{2}{n})}f_n(\tau)D_{S,T}(u), \tag{9.3.77}$$

$$K_3 \leq CE^2(1+\tau)^{-\frac{n-1}{2}}(1+\tau)^{-\frac{n-1}{2}(2-\frac{2}{n})}f_n^{1+\frac{2}{n}}(\tau)D_{S,T}(u) \tag{9.3.78}$$

and

$$K_4 \leq CE^3(1+\tau)^{-\frac{n-1}{2}}(1+\tau)^{-\frac{n-1}{2}(2-\frac{2}{n})}f_n^{1+\frac{2}{n}}(\tau). \tag{9.3.79}$$

Then, for $n=3,4$, noticing the definition (9.3.6) of $f_n(t)$, by (9.3.73) we get

$$\|u_4(t,\cdot)\|_{\Gamma,S,2} \leq C\{\varepsilon + Ef_n(t)(E+D_{S,T})\}. \tag{9.3.80}$$

Combining (9.3.67)–(9.3.68), (9.3.72) and (9.3.80), we obtain

$$\sup_{0\leq t\leq T} f_n^{-1}(t)\|u(t,\cdot)\|_{\Gamma,S,2} \leq C\{\varepsilon + E(E+D_{S,T}(u))\}. \tag{9.3.81}$$

Finally, we estimate $\|u(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}$.
From (9.3.54) we have

$$\begin{aligned}(1+t)^{\frac{n-1}{2}}\|u(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq & \sum_{i=1}^{4}(1+t)^{\frac{n-1}{2}}\|u_i(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \\ \stackrel{\text{def.}}{=} & \; L_1+L_2+L_3+L_4. \end{aligned}\tag{9.3.82}$$

Noticing (9.3.64)–(9.3.65) and that $[\frac{S}{2}]+n+2 \leq S$ for $S \geq 2n+4$, from (4.6.157) in Corollary 4.6.4 of Chap. 4, and using Lemma 9.3.4, we obtain

$$\begin{aligned}L_1+L_2 \leq C\Big\{&\varepsilon + \int_0^t (1+\tau)^{\frac{n-1}{2}}\|(v^2, vDu)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}d\tau \\ &+ \int_0^t (1+\tau)^{-\frac{n+1}{2}}\|(v^2, vDu)(\tau,\cdot)\|_{\Gamma,S,1}d\tau\Big\}. \end{aligned}\tag{9.3.83}$$

From (5.1.15) in Remark 5.1.1 of Chap. 5, using Corollary 3.4.4 in Chap. 3 (in which we take $N=[\frac{S}{2}]+1$, $s=\frac{n}{2}+1$, $p=2$) and Lemma 9.3.2, and noting the definition of $X_{S,E,T}$, we have

$$\|(v^2, vDu)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq CE(1+\tau)^{-(n+1)}(E+D_{S,T}(u)), \quad \forall\tau\in[0,T], \tag{9.3.84}$$

$$\|(v^2, vDu)(\tau,\cdot)\|_{\Gamma,S,1} \leq CEf_n^2(\tau)(E+D_{S,T}(u)), \quad \forall\tau\in[0,T], \tag{9.3.85}$$

then, noting (9.3.6) and (9.3.20), we get

$$L_1 + L_2 \leq C_\rho\{\varepsilon + R(E,T)(E+D_{S,T}(u))\}. \tag{9.3.86}$$

Similarly, by (4.6.156) in Corollary 4.6.3 of Chap. 4, we obtain

$$\begin{aligned} L_3 + L_4 \leq C \Big\{\varepsilon + \int_0^t (1+\tau)^{-\frac{n-1}{2}} (\|Q(Dv, Du, D_xDu)(\tau,\cdot)\|_{\Gamma,S,1} \\ + \|P(v, Dv, D_xDu)(\tau,\cdot)\|_{\Gamma,S,1})d\tau\Big\}. \end{aligned} \tag{9.3.87}$$

Using (5.1.15) in Remark 5.1.1 of Chap. 5, and noting Corollary 3.4.4 in Chap. 3 and Lemma 9.3.2, from the fact that $Q(Dv, Du, D_xDu)$ is affine with respect to Du and D_xDu, and that $P(v, Dv, D_xDu)$ is affine with respect to D_xDu, we obtain

$$\begin{aligned} &\|Q(Dv, Du, D_xDu)(\tau,\cdot)\|_{\Gamma,S,1} \\ &\leq C\|Dv\|_{\Gamma,S,2}(\|(Du, D^2u)\|_{\Gamma,S,2} + \|Dv\|_{\Gamma,S,2}) \\ &\leq CE(E+D_{S,T}(u)), \end{aligned}$$

$$\begin{aligned} &\|P(v, Dv, D_xDu)(\tau,\cdot)\|_{\Gamma,S,1} \\ &\leq C\Big\{\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],\infty}\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],2}\|D^2u\|_{\Gamma,S,2} \\ &\quad +\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],\infty}\|D^2u\|_{\Gamma,[\frac{S}{2}],2}\|(v,Dv)\|_{\Gamma,S,2} \\ &\quad +\|D^2u\|_{\Gamma,[\frac{S}{2}],\infty}\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],2}\|(v,Dv)\|_{\Gamma,S,2} \\ &\quad +\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],\infty}\|(v,Dv)\|_{\Gamma,[\frac{S}{2}],2}\|(v,Dv)\|_{\Gamma,S,2}\Big\} \\ &\leq CE^2(1+\tau)^{-\frac{n-1}{2}} f_n^2(\tau)(E+D_{S,T}(u)) \\ &\leq CE^2(E+D_{S,T}(u)), \end{aligned}$$

therefore,

$$L_3 + L_4 \leq C\{\varepsilon + R(E,T)(E+D_{S,T}(u))\}. \tag{9.3.88}$$

Thus, we obtain

$$\sup_{0\leq t\leq T}(1+t)^{\frac{n-1}{2}}\|u(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq C\{\varepsilon + R(E,T)(E+D_{S,T}(u))\}. \tag{9.3.89}$$

Combing (9.3.47), (9.3.52), (9.3.81) and (9.3.89), we get the desired (9.3.19). The proof of Lemma 9.3.5 is finished.

9.3.4 Proof of Lemma 9.3.6

Let

$$u^* = \overline{u} - \overline{\overline{u}}, \ v^* = \overline{v} - \overline{\overline{v}}. \tag{9.3.90}$$

At this moment we still have (9.2.58)–(9.2.60).

We first estimate $\|D^2 u^*(t,\cdot)\|_{\Gamma,S-1,2}$.

For any given multi-index $k(|k| \le S-1)$, we still have (9.2.70)–(9.2.73). Similarly to the proof of Lemma 9.3.5, we have

$$|\mathrm{I}|, |\mathrm{II}|, |\mathrm{III}|, |\mathrm{IV}|, |\mathrm{V}| \le CR(E,T) D^2_{S-1,T}(u^*). \tag{9.3.91}$$

It remains to estimate VI. Now we still have (9.2.75). By (5.1.31) in Lemma 5.1.4 (in which we take $p = q = 2$, $p_1 = q_1 = +\infty$) and (5.1.19) in Remark 5.1.2 of Chap. 5, using Corollary 5.4.4 in Chap. 5 and noting (9.3.53), we obtain

$$\begin{aligned} \|\hat{g}_k(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} &\le C\left(\|DF^*(\tau,\cdot)\|_{\Gamma,S-1,2} + \|F^*(\tau,\cdot)\|_{\Gamma,S-1,2}\right) \\ &\le CE(1+\tau)^{-\frac{n-1}{2}} D_{S-1,T}(v^*), \end{aligned} \tag{9.3.92}$$

then we have

$$|\mathrm{IV}| \le CR(E,T) D_{S-1,T}(u^*) D_{S-1,T}(v^*). \tag{9.3.93}$$

Hence, we obtain

$$\sup_{0\le t\le T} \|D^2 u^*(t,\cdot)\|^2_{\Gamma,S-1,2} \le CR(E,T)(D^2_{S-1,T}(u^*) + D_{S-1,T}(u^*) D_{S-1,T}(v^*)). \tag{9.3.94}$$

Similarly, we obtain

$$\sup_{0\le t\le T} \|Du^*(t,\cdot)\|^2_{\Gamma,S-1,2} \le CR(E,T)(D^2_{S-1,T}(u^*) + D_{S-1,T}(u^*) D_{S-1,T}(v^*)). \tag{9.3.95}$$

Finally, using arguments similar to the proof of (9.3.81) and (9.3.89), we obtain, respectively,

$$\sup_{0\le t\le T} f_n^{-1}(t)\|u^*(t,\cdot)\|_{\Gamma,S-1,2} \le CE(D_{S-1,T}(u^*) + D_{S-1,T}(v^*)) \tag{9.3.96}$$

and

$$\sup_{0\leq t\leq T}(1+t)^{\frac{n-1}{2}}\|u^*(t,\cdot)\|_{\Gamma,[\frac{S-1}{2}]+1,\infty}\leq CR(E,T)(D_{S-1,T}(u^*)+D_{S-1,T}(v^*)). \tag{9.3.97}$$

Combining (9.3.94)–(9.3.97) yields the desired (9.3.21).

The proof of Lemma 9.3.6 is finished.

Chapter 10
Cauchy Problem of Two-Dimensional Nonlinear Wave Equations

10.1 Introduction

In this chapter we consider the following Cauchy problem for two-dimensional nonlinear wave equations with small initial data:

$$\Box u = F(u, Du, D_x Du), \tag{10.1.1}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{10.1.2}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2} \tag{10.1.3}$$

is the two-dimensional wave operator,

$$D_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right),\ D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right), \tag{10.1.4}$$

φ and ψ are sufficiently smooth functions with compact support, we may assume that

$$\varphi, \psi \in C_0^\infty(I\!R^2) \tag{10.1.5}$$

with

$$\text{supp}\{\varphi, \psi\} \subseteq \{x||x| \le \rho\}\ (\rho > 0 \text{ is a constant}), \tag{10.1.6}$$

and $\varepsilon > 0$ is a small parameter.

Denote

$$\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, 2; (\lambda_{ij}), i, j = 0, 1, 2, i + j \ge 1). \tag{10.1.7}$$

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Assume that in a neighborhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \le \nu_0$ (ν_0 is a suitably small positive number), the nonlinear term $F(\hat{\lambda})$ is a sufficiently smooth function and satisfies

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}), \tag{10.1.8}$$

where $\alpha \ge 1$ is an integer.

This chapter is aimed at studying the life-span $\widetilde{T}(\varepsilon)$ of classical solution $u = u(t,x)$ to Cauchy problem (10.1.1)–(10.1.2) for any given integer $\alpha \ge 1$. For different values of α, we will use the global iteration method to prove the following results: there exists a suitably small positive number ε_0 such that for any given $\varepsilon \in (0, \varepsilon_0]$,
(1) When $\alpha = 1$, we have (see Li Tatsien and Zhou Yi 1994b)

$$\widetilde{T}(\varepsilon) \ge \begin{cases} be(\varepsilon); \\ b\varepsilon^{-1}, \ \text{if} \int_{I\!R^2} \psi(x)dx = 0; \\ b\varepsilon^{-2}, \ \text{if} \partial_u^2 F(0,0,0) = 0, \end{cases} \tag{10.1.9}$$

where b is a positive constant independent of ε, and $e(\varepsilon)$ is defined by

$$\varepsilon^2 e^2(\varepsilon) \ln(1 + e(\varepsilon)) = 1. \tag{10.1.10}$$

(2) When $\alpha = 2$, we have (see Li Tatsien and Zhou Yi 1993)

$$\widetilde{T}(\varepsilon) \ge \begin{cases} b\varepsilon^{-6}; \\ \exp\{a\varepsilon^{-2}\}, \ \text{if } \partial_u^\beta F(0,0,0) = 0 \ (\beta = 3, 4), \end{cases} \tag{10.1.11}$$

where a and b are positive constants independent of ε.
(3) When $\alpha \ge 3$, we have (see Li Tatsien and Zhou Yi 1994a)

$$\widetilde{T}(\varepsilon) = +\infty. \tag{10.1.12}$$

Remark 10.1.1 In the sequel, we will adopt a simpler way to present the above results instead of repeating the original proof given in Li and Zhou (1993, 1994a, b).

According to the above results, when $n = 2$, we have the following lower bound estimates for the life-span $\widetilde{T}(\varepsilon)$:

<table>
<tr><td>$\alpha =$</td><td>1</td><td>2</td><td>$3, 4, \cdots$</td></tr>
<tr><td rowspan="4">$\widetilde{T}(\varepsilon) \ge$</td><td>$be(\varepsilon)$</td><td rowspan="2">$b\varepsilon^{-6}$</td><td rowspan="4">$+\infty$</td></tr>
<tr><td rowspan="2">$b\varepsilon^{-1}$
(if $\int_{I\!R^2} \psi(x)dx = 0$)</td></tr>
<tr><td rowspan="2">$\exp\{a\varepsilon^{-2}\}$
(if $\partial_u^\beta F(0,0,0) = 0$,
$\beta = 3, 4$)</td></tr>
<tr><td>$b\varepsilon^{-2}$
(if $\partial_u^2 F(0,0,0) = 0$)</td></tr>
</table>

In particular, when the nonlinear term on the right-hand side does not depend on u explicitly:

$$F = F(Du, D_x Du), \tag{10.1.13}$$

from the above table we have

$\alpha=$	1	2	$3, 4, \cdots$
$\widetilde{T}(\varepsilon) \geq$	$b\varepsilon^{-2}$	$\exp\{a\varepsilon^{-2}\}$	$+\infty$

It yields immediately the related results given in Sect. 9.2.5 of Chap. 9 for $n = 2$.

Thanks to the results in Chaps. 13 and 14, the above lower bound estimates on the life-span are all sharp.

Due to Chap. 7, to prove the above results for Cauchy problem (10.1.1)–(10.1.2) of two-dimensional nonlinear wave equations, it suffices essentially to consider the following Cauchy problem of two-dimensional second-order quasi-linear hyperbolic equations:

$$\Box u = \sum_{i,j=1}^{2} b_{ij}(u, Du)u_{x_i x_j} + 2\sum_{j=1}^{2} a_{0j}(u, Du)u_{tx_j} + F(u, Du), \tag{10.1.14}$$

$$t = 0 : u = \varepsilon\varphi(x),\ \ u_t = \varepsilon\psi(x), \tag{10.1.15}$$

here, $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$ still satisfy condition (10.1.6), and $\varepsilon > 0$ is a small parameter.

Let

$$\tilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1, 2). \tag{10.1.16}$$

Assume that when $|\tilde{\lambda}| \leq \nu_0$, $b_{ij}(\tilde{\lambda}), a_{0j}(\tilde{\lambda})$ and $F(\tilde{\lambda})$ are all sufficiently smooth functions satisfying

$$b_{ij}(\tilde{\lambda}) = b_{ji}(\tilde{\lambda}) \quad (i, j = 1, 2), \tag{10.1.17}$$

$$b_{ij}(\tilde{\lambda}), a_{0j}(\tilde{\lambda}) = O(|\tilde{\lambda}|^\alpha)\ (i, j = 1, 2), \tag{10.1.18}$$

$$F(\tilde{\lambda}) = O(|\tilde{\lambda}|^{1+\alpha}) \tag{10.1.19}$$

and

$$\sum_{i,j=1}^{2} a_{ij}(\tilde{\lambda})\xi_i\xi_j \geq m_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^2, \tag{10.1.20}$$

where $\alpha \geq 1$ is an integer, m_0 is a positive constant, and

$$a_{ij}(\tilde{\lambda}) = \delta_{ij} + b_{ij}(\tilde{\lambda}) \quad (i, j = 1, 2), \tag{10.1.21}$$

in which δ_{ij} is the Kronecker symbol. In addition, condition $\partial_u^2 F(0,0,0) = 0$ for $\alpha = 1$ in (10.1.9) and condition $\partial_u^\beta F(0,0,0) = 0$ ($\beta = 3, 4$) for $\alpha = 2$ in (10.1.11) now reduce to, respectively,

$$\partial_u^2 F(0,0) = 0 \quad \text{for } \alpha = 1 \tag{10.1.22}$$

and

$$\partial_u^\beta F(0,0) = 0 \quad (\beta = 3, 4) \quad \text{for } \alpha = 2. \tag{10.1.23}$$

10.2 Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (10.1.14)–(10.1.15) (The Case $\alpha = 1$)

In this section, we will consider the life-span of classical solutions to Cauchy problem (10.1.14)–(10.1.15) of two-dimensional second-order quasi-linear hyperbolic equations, and prove the lower bound estimates given by the first two formulas in (10.1.9) when $\alpha = 1$, while, the lower bound estimate given by the last formula in (10.1.9) will be proved in Sect. 10.3.

10.2.1 Metric Space $X_{S,E,T}$. Main Results

From the Sobolev embedding theorem, there exists a suitably small $E_0 > 0$, such that

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le \nu_0, \quad \forall f \in H^2(\mathbb{R}^2), \ \|f\|_{H^2(\mathbb{R}^2)} \le E_0. \tag{10.2.1}$$

For any given integer $S \ge 6$, and any given positive numbers E ($\le E_0$) and T, we introduce the set of functions

$$X_{S,E,T} = \{v(t,x) | D_{S,T}(v) \le E, \ \partial_t^l v(0,x) = u_l^{(0)}(x) \quad (l = 0, 1, \dots, S+1)\}, \tag{10.2.2}$$

where

$$D_{S,T}(v) = \sum_{i=1}^{2} \sup_{0 \le t \le T} \|D^i v(t,\cdot)\|_{\Gamma,S,2} + \sup_{0 \le t \le T} g^{-1}(t) \|v(t,\cdot)\|_{\Gamma,S,2}, \tag{10.2.3}$$

in which

$$g(t) = \begin{cases} \sqrt{\ln(2+t)}, & \text{if } \int_{\mathbb{R}^2} \psi(x)dx \ne 0; \\ 1, & \text{if } \int_{\mathbb{R}^2} \psi(x)dx = 0, \end{cases} \tag{10.2.4}$$

and $u_0^{(0)} = \varepsilon\varphi(x)$, $u_1^{(0)} = \varepsilon\psi(x)$, and, when $l = 2, \ldots, S+1$, $u_l^{(0)}(x)$ are values of $\partial_t^l u(t, x)$ at $t = 0$, which are uniquely determined by Eq. (10.1.14) and initial condition (10.1.15). It is obvious that $u_l^{(0)}(l = 0, 1, \ldots, S+1)$ are all sufficiently smooth functions with compact support (see (10.1.6)).

Similarly to the previous two chapters, it is easy to prove

Lemma 10.2.1 *Introduce the following metric in $X_{S,E,T}$:*

$$\rho(\overline{v}, \overline{\overline{v}}) = D_{S,T}(\overline{v} - \overline{\overline{v}}), \quad \forall \overline{v}, \overline{\overline{v}} \in X_{S,E,T}. \tag{10.2.5}$$

Then, when $\varepsilon > 0$ is suitably small, $X_{S,E,T}$ is a non-empty complete metric space.

Lemma 10.2.2 *When $S \geq 6$, for any given $v \in X_{S,E,T}$, we have*

$$\|v(t, \cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq CEg(t)(1+t)^{-\frac{1}{2}}, \quad \forall t \in [0, T], \tag{10.2.6}$$

$$\|(Dv, D^2 v)(t, \cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq CE(1+t)^{-\frac{1}{2}}, \quad \forall t \in [0, T], \tag{10.2.7}$$

where C is a positive constant.

Proof Since $S \geq 6$, by (10.4.30) in Chap. 3 (in which we take $n = 2$, $p = 2$, $N = [\frac{S}{2}] + 1$, and $s = 2$), and noticing the definition of $X_{S,E,T}$, (10.2.6)–(10.2.7) follows immediately. □

The main result of this section is the following

Theorem 10.2.1 *Let $n = 2$ and $\alpha = 1$. Under assumptions (10.1.5)–(10.1.6) and (10.1.17)–(10.1.21), for any given integer $S \geq 6$, there exist positive constants ε_0 and C_0 with $C_0\varepsilon_0 \leq E_0$, and for any given $\varepsilon \in (0, \varepsilon_0]$, there exists a positive number $T(\varepsilon)$, such that Cauchy problem (10.1.14)–(10.1.15) admits a unique classical solution $u \in X_{S,C_0\varepsilon,T(\varepsilon)}$ on $[0, T(\varepsilon)]$, where $T(\varepsilon)$ can be taken as*

$$T(\varepsilon) = \begin{cases} be(\varepsilon) - 1, \\ b\varepsilon^{-1} - 1, \ \text{if} \ \int_{\mathrm{R}^2} \ (\mathrm{x})\mathrm{dx} = 0, \end{cases} \tag{10.2.8}$$

where $e(\varepsilon)$ is defined by (10.1.10), and b is a positive constant independent of ε.

Moreover, after a possible change of values for t on a zero-measure set of the interval $[0, T(\varepsilon)]$, we have

$$u \in C\Big([0, T(\varepsilon)];\, H^{S+1}(I\!R^2)\Big), \tag{10.2.9}$$

$$u_t \in C\Big([0, T(\varepsilon)];\, H^{S}(I\!R^2)\Big), \tag{10.2.10}$$

$$u_{tt} \in C\Big([0, T(\varepsilon)];\, H^{S-1}(I\!R^2)\Big). \tag{10.2.11}$$

10.2.2 Framework to Prove Theorem 10.2.1—The Global Iteration Method

To prove Theorem 10.2.1, for any given $v \in X_{S,E,T}$, similarly to the previous two chapters, by solving the following Cauchy problem of linear hyperbolic equations:

$$\begin{aligned} \Box u &= \hat{F}(v, Dv, D_x Du) \\ &\overset{\text{def.}}{=} \sum_{i,j=1}^{2} b_{ij}(v, Dv) u_{x_i x_j} + 2\sum_{j=1}^{2} a_{0j}(v, Dv) u_{t x_j} + F(v, Dv), \end{aligned} \tag{10.2.12}$$

$$t = 0: \ u = \varepsilon\varphi(x), \ u_t = \varepsilon\psi(x), \tag{10.2.13}$$

we define a mapping

$$M : v \longrightarrow u = Mv. \tag{10.2.14}$$

We want to prove that: when $\varepsilon > 0$ is suitably small, there exists a positive constant C_0 such that when $E = C_0\varepsilon$ and $T = T(\varepsilon)$ is defined by (10.2.8), the mapping M admits a unique fixed point in $X_{S,E,T}$, which is just the classical solution to Cauchy (10.1.14)–(10.1.15) on $0 \le t \le T(\varepsilon)$.

Similarly to the previous two chapters, it is easy to prove the following two lemmas.

Lemma 10.2.3 *When $E > 0$ is suitably small, for any given $v \in X_{S,E,T}$, after a possible change of values for t on a zero-measure set, we have*

$$u = Mv \in C\Big([0, T]; H^{S+1}(I\!R^2)\Big), \tag{10.2.15}$$

$$u_t \in C\Big([0, T]; H^{S}(I\!R^2)\Big), \tag{10.2.16}$$

$$u_{tt} \in L^\infty\Big(0, T; H^{S-1}(I\!R^2)\Big). \tag{10.2.17}$$

Moreover, for any given $t \in [0, T]$, the support of $u = u(t, x)$ with respect to x is included in $\{x \mid |x| \le t + \rho\}$.

Lemma 10.2.4 *For $u = u(t, x) = Mv$, the values of $\partial_t^l u(0, \cdot)$ $(l = 0, 1, \cdots, S+2)$ do not depend on the choice of $v \in X_{S,E,T}$, and*

$$\partial_t^l u(0, x) = u_l^{(0)}(x) \quad (l = 0, 1, \cdots, S+1). \tag{10.2.18}$$

Moreover,

$$\|u(0, \cdot)\|_{\Gamma, S+2, p} \le C\varepsilon, \tag{10.2.19}$$

where $1 \le p \le +\infty$, and C is a positive constant.

The following two lemmas are crucial to the proof of Theorem 10.2.1.

Lemma 10.2.5 *Under the assumptions of Theorem 10.2.1, if*

$$T \le \exp\{a\varepsilon^{-2}\} \ (a > 0 \text{ is a constant}), \tag{10.2.20}$$

then, when $E > 0$ is suitably small, for any given $v \in X_{S,E,T}$, $u = Mv$ satisfies

$$D_{S,T}(u) \le C_1\{\varepsilon + (R + \sqrt{R})(E + D_{S,T}(u))\}, \tag{10.2.21}$$

where C_1 is a positive constant,

$$R = R(E, T) \overset{\text{def.}}{=} E(1 + T)g(T), \tag{10.2.22}$$

and $g(T)$ is defined by (10.2.4).

Lemma 10.2.6 *Under the assumptions of Lemma 10.2.5, for any given $\overline{v}, \overline{\overline{v}} \in X_{S,E,T}$, if both $\overline{u} = M\overline{v}$ and $\overline{\overline{u}} = M\overline{\overline{v}}$ satisfy $\overline{u}, \overline{\overline{u}} \in X_{S,E,T}$, then we have*

$$D_{S-1,T}(\overline{u} - \overline{\overline{u}}) \le C_2\Big(R + \sqrt{R}\Big)\Big(D_{S-1,T}(\overline{u} - \overline{\overline{u}}) + D_{S-1,T}(\overline{v} - \overline{\overline{v}})\Big), \tag{10.2.23}$$

where C_2 is a positive constant, and $R = R(E, T)$ is still defined by (10.2.22).

The proof of Lemmas 10.2.5 and 10.2.6 will be given later. Now we first utilize these two lemmas to prove Theorem 10.2.1.

Proof of Theorem 10.2.1 Take

$$C_0 = 3\max(C_1, C_2), \tag{10.2.24}$$

where C_1 and C_2 are positive constants appearing in Lemmas 10.2.5 and 10.2.6, respectively.

Similarly to the previous two chapters, we can prove that: if there exists a positive number ε_0 satisfying $C_0\varepsilon_0 \le E_0$, and for any given $\varepsilon \in (0, \varepsilon_0]$, $E = E(\varepsilon) = C_0\varepsilon$ and $T = T(\varepsilon) > 0$ satisfying (10.2.20), we have

$$R(E(\varepsilon), T(\varepsilon)) + \sqrt{R(E(\varepsilon), T(\varepsilon))} \le \frac{1}{C_0}, \tag{10.2.25}$$

then the mapping M admits a unique fixed point $u = u(t, x) \in X_{S,E(\varepsilon),T(\varepsilon)}$, which is exactly the classical solution to Cauchy problem (10.1.14)–(10.1.15) on $0 \le t \le T(\varepsilon)$.

Now we determine $\varepsilon_0 > 0$ and $T(\varepsilon)(0 < \varepsilon \le \varepsilon_0)$, such that both (10.2.20) and (10.2.25) hold. We always assume that $\varepsilon_0 > 0$ is so small that (10.2.1) is satisfied when $E_0 = C_0\varepsilon_0$.

In the case that $\int_{I\!R^2} \psi(x)dx \neq 0$, from the first formula in (10.2.4) as well as (10.2.22), it follows that

$$R = R(E, T) = E(1 + T)\sqrt{\ln(2 + T)}.$$

Then, if we take $E = C_0\varepsilon$ and

$$T(\varepsilon) = be(\varepsilon) - 1,$$

where $e(\varepsilon)$ is defined by (10.1.10), and $b(\leq 1)$ is a positive constant satisfying

$$C_0(bC_0 + \sqrt{bC_0}) \leq 1, \tag{10.2.26}$$

it is easy to show that both (10.2.20) and (10.2.25) hold when $\varepsilon_0 > 0$ is small enough. This proves the first formula in (10.2.8).

In the case that $\int_{I\!R^2} \psi(x)dx = 0$,

$$R = R(E, T) = E(1 + T).$$

Then, if we take $E = C_0\varepsilon$ and

$$T(\varepsilon) = b\varepsilon^{-1} - 1,$$

where b is still a positive constant satisfying (10.2.26), then it is clear that both (10.2.20) and (10.2.25) hold when $\varepsilon_0 > 0$ is small enough. This proves the second formula in (10.2.8).

10.2.3 Proof of Lemmas 10.2.5 and 10.2.6

First, we prove Lemma 10.2.5.

We first estimate $\|u(t, \cdot)\|_{\Gamma,S,2}$.

For any given multi-index $k(|k| \leq S)$, from Lemma 3.1.5 in Chap. 3, by using (10.2.12) we have

$$\begin{aligned}\Box\Gamma^k u &= \Gamma^k \hat{F}(v, Dv, D_x Du) + \sum_{|l|\leq|k|-1} B_{kl}\Gamma^l \hat{F}(v, Dv, D_x Du)\\ &\overset{\text{def.}}{=} \sum_{|l|\leq|k|} C_{kl}\Gamma^l \hat{F}(v, Dv, D_x Du),\end{aligned} \tag{10.2.27}$$

where B_{kl} and C_{kl} are constants, and the initial value of $\Gamma^k u$ can be uniquely determined by $u_l^{(0)}$ $(l = 0, 1, \cdots, S + 1)$.

Set

$$\Gamma^k u = w_k^{(0)} + w_k^{(1)}, \tag{10.2.28}$$

where $w_k^{(0)}$ satisfies the linear homogeneous wave equation

$$\Box w_k^{(0)} = 0$$

and the same initial values as $\Gamma^k u$, while, $w_k^{(1)}$ satisfies

$$\Box w_k^{(1)} = \sum_{|l|\le|k|} C_{kl}\Gamma^l \hat{F}(v, Dv, D_x Du) \tag{10.2.29}$$

and the zero initial data.

Set

$$w_k^{(0)} = \overline{w}_k^{(0)} + \overline{\overline{w}}_k^{(0)}, \tag{10.2.30}$$

where

$$\overline{w}_k^{(0)} = \Gamma^k u^{(0)}, \tag{10.2.31}$$

and $u^{(0)}$ satisfies

$$\Box u^{(0)} = 0, \tag{10.2.32}$$

$$t = 0: \; u^{(0)} = \varepsilon\varphi(x), \; u_t^{(0)} = \varepsilon\psi(x). \tag{10.2.33}$$

It is obvious from Lemma 3.1.5 in Chap. 3 that

$$\Box\overline{w}_k^{(0)} = 0. \tag{10.2.34}$$

Then, from 1^o in Theorem 4.3.1 of Chap. 4, it is clear that

$$\|\overline{w}_k^{(0)}(t,\cdot)\|_{L^2(I\!R^2)} \le C\sqrt{\ln(2+t)}\varepsilon. \tag{10.2.35}$$

In the case that $\int_{I\!R^2}\psi(x)dx = 0$, the second initial value of $\overline{w}_k^{(0)}$ should satisfy a similar condition, namely,

$$\int_{I\!R^2} \overline{w}_{k,t}^{(0)}(0,x)dx = 0. \tag{10.2.36}$$

To prove this, it suffices to prove

$$\int_{I\!R^2} \frac{\partial}{\partial t}(\Gamma u^{(0)})(0,x)dx = 0. \tag{10.2.37}$$

In fact, by Sect. 3.1 of Chap. 3, we have $\Gamma = (\Omega_x, L, \partial)$, where

$$\Omega_x = (x_i\partial_j - x_j\partial_i)_{1\le i<j\le n}, \tag{10.2.38}$$

$$L = \Big(t\partial_t + \sum_{i=1}^{n} x_i\partial_i,\ t\partial_i - x_i\partial_t\ (i = 1, \cdots, n)\Big) \tag{10.2.39}$$

and

$$\partial = (-\partial_t, \partial_1, \cdots, \partial_n). \tag{10.2.40}$$

Noting that φ and ψ have compact support, and $\int_{I\!R^2} \psi(x)dx = 0$, using (10.2.32) and (10.2.33), and integrating by parts if necessary, the proof of (10.2.37) is straightforward, we do not go into details here.

Thus, when $\int_{I\!R^2} \psi(x)dx = 0$, from (10.2.36) and using 2° in Theorem 4.3.1 of Chap. 4, we have

$$\|\overline{w}_k^{(0)}(t,\cdot)\|_{L^2(I\!R^2)} \le C\varepsilon. \tag{10.2.41}$$

Combining (10.2.35) and (10.2.41), and noting (10.2.4), we obtain

$$\|\overline{w}_k^{(0)}(t,\cdot)\|_{L^2(I\!R^2)} \le Cg(t)\varepsilon. \tag{10.2.42}$$

$\overline{\overline{w}}_k^{(0)}$ in (10.2.30) is still the solution to the two-dimensional homogeneous linear wave equation, whose initial values are the difference of the initial values of $\Gamma^k u$ and $\Gamma^k u^{(0)}$. Noticing that the Cauchy problems satisfied by u and $u^{(0)}$ are (10.2.12)–(10.2.13) and (10.2.32)–(10.2.33), respectively, and noticing (10.1.18)–(10.1.19) (in which $\alpha = 1$), it is easy to know that the initial values of $\overline{\overline{w}}_k^{(0)}$ are of the order ε^2. Then, using again 1° in Theorem 4.3.1 of Chap. 4 and noting the first formula in (10.2.4), we have

$$\|\overline{\overline{w}}^{(0)}(t,\cdot)\|_{L^2(I\!R^2)} \le C\sqrt{\ln(2+t)}\varepsilon^2. \tag{10.2.43}$$

Thus, from (10.2.30) we get

$$\|w_k^{(0)}(t,\cdot)\|_{L^2(I\!R^2)} \le C(g(t)\varepsilon + \sqrt{\ln(2+t)}\varepsilon^2). \tag{10.2.44}$$

Now we estimate $\|w_k^{(1)}(t,\cdot)\|_{L^2(I\!R^2)}$.

By 2° in Theorem 4.5.1 of Chap. 4 (in which we take $\sigma = \frac{1}{3}$, and $q = (1 - \frac{\sigma}{2})^{-1} = \frac{6}{5}$), we obtain

$$\begin{aligned}\|w_k^{(1)}(t,\cdot)\|_{L^2(I\!R^2)} \le C(1+t)^{\frac{1}{3}} \int_0^t \Big(&\|\hat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \\ &+(1+\tau)^{-\frac{1}{3}}\|\hat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}\Big)d\tau.\end{aligned} \tag{10.2.45}$$

Noting $\alpha = 1$ and Lemma 10.2.2, using the estimates about product functions and composite functions in Chap. 5, it is clear that

$$\begin{aligned}\|\hat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C\Big\{&\|(v, Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\|(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,2}\\ &+\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\|(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2}\Big\},\end{aligned} \tag{10.2.46}$$

where χ_1 is the characteristic function of set $\left\{(t,x)\Big||x| \le \frac{1+t}{2}\right\}$, and $\chi_2 = 1 - \chi_1$. Using 2° in Corollary 3.4.1 of Chap. 3 (in which we take $n = 2$, $p = 2$, $q = 3$, $N = [\frac{S}{2}]$ and $s = 1$), we have

$$\|(v, Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1} \le C(1+t)^{-\frac{1}{3}}\|(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2}$$

and a similar estimate for $\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}$. Noticing furthermore the definition of $X_{S,E,T}$, it is easy to get

$$\begin{aligned}\|\hat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} &\le C(1+\tau)^{-\frac{1}{3}}\|(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2}\|(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,2}\\ &\le C(1+\tau)^{-\frac{1}{3}}g^2(\tau)(E^2 + ED_{S,T}(u)),\end{aligned} \tag{10.2.47}$$

where $g(t)(\ge 1)$ is defined by (10.2.4).

Similarly, noting $\alpha = 1$ and using the estimates about product functions and composite functions in Chap. 5, we have

$$\begin{aligned}\|\hat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} \le C\Big\{&\|(v, Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty}\|(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,2}\\ &+\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty}\|(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2}\Big\}.\end{aligned} \tag{10.2.48}$$

Using the embedding theorem on a sphere (i.e., 1° in Theorem 3.2.1 of Chap. 3, in which we take $n = 2$, $p = 2$, $s = 1$), we have

$$\|(v, Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty} \le C\|(v, Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,2} \le C\|(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2}$$

and a similar estimate for $\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty}$. By the definition of $X_{S,E,T}$, similarly to (10.2.47), we obtain

$$\begin{aligned}\|\hat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} &\le C\|(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2}\|(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,2}\\ &\le Cg^2(\tau)(E^2 + ED_{S,T}(u)).\end{aligned} \tag{10.2.49}$$

Plugging (10.2.47) and (10.2.49) in (10.2.45), we get

$$\begin{aligned}\|w_k^{(1)}(t,\cdot)\|_{L^2(I\!R^2)} &\le C(1+t)^{\frac{1}{3}}\int_0^t (1+\tau)^{-\frac{1}{3}} g^2(\tau)d\tau\cdot(E^2+ED_{S,T}(u))\\ &\le C(1+t)g^2(t)(E^2+ED_{S,T}(u))\\ &\le Cg(t)R(E,T)(E+D_{S,T}(u)),\end{aligned}\tag{10.2.50}$$

where $R(E,T)$ is given by (10.2.22).

Using (10.2.44) and (10.2.50), from (10.2.28) we get, when $\int_{I\!R^2}\psi(x)dx\neq 0$,

$$\|u(t,\cdot)\|_{\Gamma,S,2}\le Cg(t)\big(\varepsilon+R(E,T)(E+D_{S,T}(u))\big);\tag{10.2.51}$$

while, when $\int_{I\!R^2}\psi(x)dx=0$, (10.2.51) still holds as long as (10.2.20) is satisfied.

Hence, under condition (10.2.20) we always have

$$\sup_{0\le t\le T} g^{-1}(t)\|u(t,\cdot)\|_{\Gamma,S,2}\le C\big(\varepsilon+R(E,T)(E+D_{S,T}(u))\big).\tag{10.2.52}$$

Now we estimate $\|(Du,D^2u)(t,\cdot)\|_{\Gamma,S,2}$.

For any given multi-index $k(|k|\le S)$, by (9.2.41) in Chap. 9, we have the following energy integral formula:

$$\begin{aligned}&\|(\Gamma^k Du(t,\cdot))_t\|^2_{L^2(I\!R^2)}+\sum_{i,j=1}^2\int_{I\!R^2}a_{ij}(v,Dv)(t,\cdot)(\Gamma^k Du(t,\cdot))_{x_i}(\Gamma^k Du(t,\cdot))_{x_j}dx\\
=&\|(\Gamma^k Du(0,\cdot))_t\|^2_{L^2(I\!R^2)}+\sum_{i,j=1}^2\int_{I\!R^2}a_{ij}(v,Dv)(0,\cdot)(\Gamma^k Du(0,\cdot))_{x_i}(\Gamma^k Du(0,\cdot))_{x_j}dx\\
&+\sum_{i,j=1}^2\int_0^t\int_{I\!R^2}\frac{\partial b_{ij}(v,Dv)(\tau,\cdot)}{\partial\tau}(\Gamma^k Du(\tau,\cdot))_{x_i}(\Gamma^k Du(\tau,\cdot))_{x_j}dxd\tau\\
&-2\sum_{i,j=1}^2\int_0^t\int_{I\!R^2}\frac{\partial b_{ij}(v,Dv)(\tau,\cdot)}{\partial x_i}(\Gamma^k Du(\tau,\cdot))_{x_j}(\Gamma^k Du(\tau,\cdot))_{\tau}dxd\tau\\
&-2\sum_{j=1}^2\int_0^t\int_{I\!R^2}\frac{\partial a_{0j}(v,Dv)(\tau,\cdot)}{\partial x_j}(\Gamma^k Du(\tau,\cdot))_{\tau}(\Gamma^k Du(\tau,\cdot))_{\tau}dxd\tau\\
&+2\int_0^t\int_{I\!R^2}G_k(\tau,\cdot)(\Gamma^k Du(\tau,\cdot))_\tau dxd\tau+2\int_0^t\int_{I\!R^2}g_k(\tau,\cdot)(\Gamma^k Du(\tau,\cdot))_\tau dxd\tau\\
\stackrel{\text{def.}}{=}&\|(\Gamma^k Du(0,\cdot))_t\|^2_{L^2(I\!R^2)}+\sum_{i,j=1}^2\int_{I\!R^2}a_{ij}(v,Dv)(0,\cdot)(\Gamma^k Du(0,\cdot))_{x_i}(\Gamma^k Du(0,\cdot))_{x_j}dx\\
&+\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V},\end{aligned}\tag{10.2.53}$$

where

$$
\begin{aligned}
G_k = \sum_{i,j=1}^{2} \Big\{ & \big(\Gamma^k D(b_{ij}(v, Dv)u_{x_i x_j}) - b_{ij}(v, Dv)\Gamma^k Du_{x_i x_j}\big) \\
& + b_{ij}(v, Dv)\big(\Gamma^k Du_{x_i x_j} - (\Gamma^k Du)_{x_i x_j}\big)\Big\} \\
+2\sum_{j=1}^{2} \Big\{ & \big(\Gamma^k D(a_{0j}(v, Dv)u_{t x_j}) - a_{0j}(v, Dv)\Gamma^k Du_{t x_j}\big) \\
& + a_{0j}(v, Dv)\big(\Gamma^k Du_{t x_j} - (\Gamma^k Du)_{t x_j}\big)\Big\},
\end{aligned}
\tag{10.2.54}
$$

$$
g_k = \Gamma^k DF(v, Dv) + \sum_{|l|\le|k|} \widetilde{B}_{kl}\Gamma^l \widehat{F}(v, Dv, D_x Du), \tag{10.2.55}
$$

and $\widehat{F}(v, Dv, D_x Du)$ is given by (10.2.12), $\widetilde{B}_{kl}$ are some constants.

In what follows we only explain briefly those estimates similar to Sect. 9.2 in Chap. 9.

Noticing $\alpha = 1$ and Lemma 10.2.2, using the estimates about product functions and composite functions in Chap. 5, we get

$$
\begin{aligned}
|\mathrm{I}|, |\mathrm{II}|, |\mathrm{III}| & \le C\int_0^t \|(v, Dv, D^2v)(\tau, \cdot)\|_{L^\infty(\mathbb{R}^2)} d\tau \cdot D_{S,T}^2(u) \\
& \le CE\int_0^t g(\tau)(1+\tau)^{-\frac{1}{2}} d\tau \cdot D_{S,T}^2(u) \\
& \le CEg(t)(1+t)^{\frac{1}{2}} D_{S,T}^2(u) \le CR(E, T)D_{S,T}^2(u).
\end{aligned}
\tag{10.2.56}
$$

Now we estimate the L^2 norm of $G_k(\tau, \cdot)$.

First, we have

$$
\begin{aligned}
& \|b_{ij}(v, Dv)\big(\Gamma^k Du_{x_i x_j} - (\Gamma^k Du)_{x_i x_j}\big)(\tau, \cdot)\|_{L^2(\mathbb{R}^2)} \\
& \le C\|b_{ij}(v, Dv)(\tau, \cdot)\|_{L^\infty(\mathbb{R}^2)} \cdot \|\big(\Gamma^k Du_{x_i x_j} - (\Gamma^k Du)_{x_i x_j}\big)(\tau, \cdot)\|_{L^2(\mathbb{R}^2)} \\
& \le CEg(\tau)(1+\tau)^{-\frac{1}{2}}\|D^2u(\tau, \cdot)\|_{\Gamma,S,2} \\
& \le CEg(\tau)(1+\tau)^{-\frac{1}{2}} D_{S,T}(u).
\end{aligned}
\tag{10.2.57}
$$

Second, we have

$$
\begin{aligned}
& \|\big(\Gamma^k D(b_{ij}(v, Dv)u_{x_i x_j}) - b_{ij}(v, Dv)\Gamma^k Du_{x_i x_j}\big)(\tau, \cdot)\|_{L^2(\mathbb{R}^2)} \\
& \le \|\big(\Gamma^k(b_{ij}(v, Dv))Du_{x_i x_j} - b_{ij}(v, Dv)\Gamma^k Du_{x_i x_j}\big)(\tau, \cdot)\|_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\Gamma^k(Db_{ij}(v, Dv))u_{x_ix_j})(\tau,\cdot)\|_{L^2(I\!R^2)}\\
&\le C(1+\tau)^{-\frac{1}{2}}\|(v, Dv, D^2v)(\tau,\cdot)\|_{\Gamma,S,2}\|D^2u(\tau,\cdot)\|_{\Gamma,S,2}\\
&\le CEg(\tau)(1+\tau)^{-\frac{1}{2}}D_{S,T}(u).
\end{aligned}
\tag{10.2.58}
$$

We have similar estimates for the terms composed of a_{0j} in G_k. Therefore, we have

$$
\|G_k(\tau,\cdot)\|_{L^2(I\!R^2)} \le CEg(\tau)(1+\tau)^{-\frac{1}{2}}D_{S,T}(u), \quad \forall \tau \in [0, T], \tag{10.2.59}
$$

then we get

$$
|\mathrm{IV}| \le CR(E, T)D^2_{S,T}(u). \tag{10.2.60}
$$

Similarly, we have

$$
\begin{aligned}
\|g_k(\tau,\cdot)\|_{L^2(I\!R^2)} \le C\Big(&\|DF(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2} + \|F(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2}\\
&+\sum_{i,j=1}^{2}\|b_{ij}(v, Dv)u_{x_ix_j}(\tau,\cdot)\|_{\Gamma,S,2} + 2\sum_{j=1}^{2}\|a_{0j}(v, Dv)u_{tx_j}(\tau,\cdot)\|_{\Gamma,S,2}\Big)\\
\le\ & CEg(\tau)(1+\tau)^{-\frac{1}{2}}(E + D_{S,T}(u)), \quad \forall \tau \in [0, T],
\end{aligned}
\tag{10.2.61}
$$

then

$$
|\mathrm{V}| \le CR(E, T)(ED_{S,T}(u) + D^2_{S,T}(u)). \tag{10.2.62}
$$

Hence, by (10.2.53), and noting (10.1.20) and (10.1.15), we obtain

$$
\sup_{0\le t\le T}\sum_{|k|\le S}\|D\Gamma^kDu(t,\cdot)\|^2_{L^2(I\!R^2)} \le C\Big\{\varepsilon^2 + R(E, T)(ED_{S,T}(u) + D^2_{S,T}(u))\Big\}, \tag{10.2.63}
$$

then, by Corollary 3.1.1 in Chap. 3 we get

$$
\sup_{0\le t\le T}\|D^2u(t,\cdot)\|_{\Gamma,S,2} \le C\Big\{\varepsilon + \sqrt{R(E, T)}(E + D_{S,T}(u))\Big\}. \tag{10.2.64}
$$

Moreover, for any given multi-index $k(|k| \le S)$, from (9.2.56) in Chap. 9 we also have the following energy integral formula:

$$
\begin{aligned}
&\|(\Gamma^ku(t,\cdot))_t\|^2_{L^2(I\!R^2)} + \sum_{i,j=1}^{2}\int_{I\!R^2} a_{ij}(v, Dv)(t,\cdot)(\Gamma^ku(t,\cdot))_{x_i}(\Gamma^ku(t,\cdot))_{x_j}dx\\
=&\|(\Gamma^ku(0,\cdot))_t\|^2_{L^2(I\!R^2)} + \sum_{i,j=1}^{2}\int_{I\!R^2} a_{ij}(v, Dv)(0,\cdot)(\Gamma^ku(0,\cdot))_{x_i}(\Gamma^ku(0,\cdot))_{x_j}dx
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{2}\int_0^t\int_{I\!R^2}\frac{\partial b_{ij}(v,Dv)(\tau,\cdot)}{\partial\tau}(\Gamma^k u(\tau,\cdot))_{x_i}(\Gamma^k u(\tau,\cdot))_{x_j}dxd\tau\\
&-2\sum_{i,j=1}^{2}\int_0^t\int_{I\!R^2}\frac{\partial b_{ij}(v,Dv)(\tau,\cdot)}{\partial x_i}(\Gamma^k u(\tau,\cdot))_{x_j}(\Gamma^k u(\tau,\cdot))_{\tau}dxd\tau\\
&-2\sum_{j=1}^{2}\int_0^t\int_{I\!R^2}\frac{\partial a_{0j}(v,Dv)(\tau,\cdot)}{\partial x_j}(\Gamma^k u(\tau,\cdot))_{\tau}(\Gamma^k u(\tau,\cdot))_{\tau}dxd\tau\\
&+2\int_0^t\int_{I\!R^2}\overline{G}_k(\tau,\cdot)(\Gamma^k u(\tau,\cdot))_{\tau}dxd\tau+2\int_0^t\int_{I\!R^2}\overline{g}_k(\tau,\cdot)(\Gamma^k u(\tau,\cdot))_{\tau}dxd\tau,
\end{aligned}
\tag{10.2.65}
$$

where

$$
\begin{aligned}
\overline{G}_k=&\sum_{i,j=1}^{2}\Big\{\big(\Gamma^k(b_{ij}(v,Dv)u_{x_ix_j})-b_{ij}(v,Dv)\Gamma^k u_{x_ix_j}\big)\\
&\qquad+b_{ij}(v,Dv)\big(\Gamma^k u_{x_ix_j}-(\Gamma^k u)_{x_ix_j}\big)\Big\}\\
&+2\sum_{j=1}^{2}\Big\{\big(\Gamma^k(a_{0j}(v,Dv)u_{tx_j})-a_{0j}(v,Dv)\Gamma^k u_{tx_j}\big)\\
&\qquad+a_{0j}(v,Dv)\big(\Gamma^k u_{tx_j}-(\Gamma^k u)_{tx_j}\big)\Big\},
\end{aligned}
\tag{10.2.66}
$$

$$
\overline{g}_k=\Gamma^k F(v,Dv)+\sum_{|l|\le|k|-1}B_{kl}\Gamma^l\widehat{F}(v,Dv,D_xDu),
\tag{10.2.67}
$$

and $\widehat{F}(v,Dv,D_xDu)$ is given by (10.2.12), B_{kl} are some constants.

Using similar arguments as in the proof of (10.2.64), we obtain

$$
\sup_{0\le t\le T}\|Du(t,\cdot)\|_{\Gamma,S,2}\le C\Big\{\varepsilon+\sqrt{R(E,T)}(E+D_{S,T}(u))\Big\}.
\tag{10.2.68}
$$

Combining (10.2.52), (10.2.64) and (10.2.68), we obtain the desired (10.2.21). The proof of Lemma 10.2.5 is finished.

Lemma 10.2.6 can be proved similarly to Sect. 9.2.4 in Chap. 9, we do not go into details here.

10.3 Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (10.1.14)–(10.1.15) (The Case $\alpha \geq 2$)

In this section, we will consider the life-span of classical solutions to Cauchy problem (10.1.14)–(10.1.15) of two-dimensional second-order quasi-linear hyperbolic equations and prove the lower bound estimates given by the first formula in (10.1.11) when $\alpha = 2$ and formula (10.1.12) when $\alpha \geq 3$, while, the lower bound estimate given by the last formula in (10.1.11) will be proved in Sect. 10.3. For narrative simplicity, in what follows we emphasize only the difference with the proof in Sect. 10.2, and obviously it suffices to consider the two cases $\alpha = 2$ and $\alpha = 3$.

10.3.1 Metric Space $X_{S,E,T}$. Main Results

For any given integer $S \geq 6$, and any given real number $E(\leq E_0)$ 及 $T(0 < T \leq +\infty)$, we still introduce the set of functions, $X_{S,E,T}$, by (10.2.2), whereas

$$D_{S,T}(v) = \sum_{i=1}^{2} \sup_{0\leq t\leq T} \|D^i v(t,\cdot)\|_{\Gamma,S,2} + \widetilde{D}_{S,T}(v), \tag{10.3.1}$$

where, when $\alpha = 2$ and 3, we take

$$\widetilde{D}_{S,T}(v) = \sup_{0\leq t\leq T} (1+t)^{-(\frac{1}{2}-\frac{1}{1+\alpha})} \|v(t,\cdot)\|_{\Gamma,S,2,\chi_1} + \sup_{0\leq t\leq T} (1+t)^{\frac{1}{2}-\frac{1}{1+\alpha}} \|v(t,\cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2}, \tag{10.3.2}$$

and χ_1 is the characteristic function of set $\left\{(t,x)\Big||x| \leq \frac{1+t}{2}\right\}$, $\chi_2 = 1 - \chi_1$.

It is easy to prove

Lemma 10.3.1 *Introduce the following metric in $X_{S,E,T}$:*

$$\rho(\overline{v}, \overline{\overline{v}}) = D_{S,T}(\overline{v} - \overline{\overline{v}}), \quad \forall \overline{v}, \overline{\overline{v}} \in X_{S,E,T}. \tag{10.3.3}$$

Then, when $\varepsilon > 0$ is suitably small, $X_{S,E,T}$ is a non-empty complete metric space.

Lemma 10.3.2 *When $S \geq 6$, for any given $v \in X_{S,E,T}$, we have*

$$\|v(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq CE(1+t)^{-\frac{1}{2}}, \quad \forall t \in [0,T], \tag{10.3.4}$$

$$\|(Dv, D^2v)(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq CE(1+t)^{-\frac{1}{2}}, \quad \forall t \in [0,T], \tag{10.3.5}$$

where C is a positive constant.

Proof Noting $S \geq 6$, by (3.4.30) in Chap. 3, and noticing the definition of $X_{S,E,T}$, similarly to (10.2.7), (10.3.5) follows immediately.

Still by (3.4.30) in Chap. 3 (in which we take $n = 2$, $p = 1 + \alpha$, $N = [\frac{S}{2}] + 1$, and $s = 1$), noting $S \geq 6$, when $\alpha = 2$ and 3 we have

$$\|v(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq C(1+t)^{-\frac{1}{1+\alpha}}\|v(t,\cdot)\|_{\Gamma,S-1,1+\alpha}, \tag{10.3.6}$$

and

$$\|v(t,\cdot)\|_{\Gamma,S-1,1+\alpha} \leq \|v(t,\cdot)\|_{\Gamma,S-1,1+\alpha,\chi_1} + \|v(t,\cdot)\|_{\Gamma,S-1,1+\alpha,\chi_2}, \tag{10.3.7}$$

where χ_1 is the characteristic function of set $\left\{(t,x)\Big||x| \leq \frac{1+t}{2}\right\}$, and $\chi_2 = 1 - \chi_1$.

By (3.4.13) in Chap. 3 (in which we take $n = 2$, $p = 2$, $q = 1 + \alpha$, $N = S - 1$, and $s = 1$), we get

$$(1+t)^{-\frac{1}{1+\alpha}}\|v(t,\cdot)\|_{\Gamma,S-1,1+\alpha,\chi_1} \leq C(1+t)^{-\frac{\alpha}{1+\alpha}}\|v(t,\cdot)\|_{\Gamma,S,2,\chi_1}. \tag{10.3.8}$$

While, using the Sobolev embedding theorem on a sphere (i.e., 1° in Theorem 3.2.1 of Chap. 3, in which we take $n = 2$, $p = 2$, $s = 1$), for $\alpha = 2, 3$ it is easy to show that

$$\|v(t,\cdot)\|_{\Gamma,S-1,1+\alpha,\chi_2} \leq C\|v(t,\cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2}. \tag{10.3.9}$$

Thus, using (10.3.8) and (10.3.9), by (10.3.6) and noting (10.3.2), we have

$$\begin{aligned}\|v(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} &\leq \left\{(1+t)^{-\frac{\alpha}{1+\alpha}}\|v(t,\cdot)\|_{\Gamma,S,2,\chi_1} + (1+t)^{-\frac{1}{1+\alpha}}\|v(t,\cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2}\right\}\\ &\leq CE(1+t)^{-\frac{1}{2}}.\end{aligned} \tag{10.3.10}$$

This proves (10.3.4). □

The main result of this section is the following

Theorem 10.3.1 *Let $n = 2$ and $\alpha \geq 2$. Under assumptions (10.1.5)–(10.1.6) and (10.1.17)–(10.1.21), for any given integer $S \geq 6$, there exist positive constants ε_0 and C_0 with $C_0\varepsilon_0 \leq E_0$, and for any given $\varepsilon \in (0, \varepsilon_0]$, there exists a positive number $T(\varepsilon)$, such that Cauchy problem (10.1.14)–(10.1.15) admits a unique classical solution $u \in X_{S,C_0\varepsilon,T(\varepsilon)}$ on $[0, T(\varepsilon)]$, and $T(\varepsilon)$ can be taken as*

$$T(\varepsilon) = \begin{cases} b\varepsilon^{-b} - 1, & \alpha = 2, \\ +\infty, & \alpha \geq 3, \end{cases} \tag{10.3.11}$$

where b is a positive constant independent of ε. Moreover, after a possible change of values for t on a zero-measure set of $[0, T(\varepsilon)]$, we have (10.2.9)–(10.2.11).

10.3.2 Framework to Prove Theorem 10.3.1—The Global Iteration Method

Similarly to Sect. 10.2.2, the following two lemmas are crucial to the proof of Theorem 10.3.1.

Lemma 10.3.3 *Under the assumptions of Theorem 10.3.1, for $\alpha = 2$ and 3, when $E > 0$ is suitably small, for any given $v \in X_{S,E,T}$, $u = Mv$ satisfies*

$$D_{S,T}(u) \le C_1\{\varepsilon + (R + \sqrt{R})(E + D_{S,T}(u))\}, \tag{10.3.12}$$

where C_1 is a positive constant, and

$$R = R(E,T) \stackrel{\text{def.}}{=} \begin{cases} E^2(1+T)^{\frac{1}{3}}, & \alpha = 2, \\ E^3, & \alpha = 3. \end{cases} \tag{10.3.13}$$

Lemma 10.3.4 *Under the assumptions of Lemma 10.3.3, for any given $\bar{v}, \bar{\bar{v}} \in X_{S,E,T}$, if both $\bar{u} = M\bar{v}$ and $\bar{\bar{u}} = M\bar{\bar{v}}$ satisfy $\bar{u}, \bar{\bar{u}} \in X_{S,E,T}$, then we have*

$$D_{S-1,T}(\bar{u} - \bar{\bar{u}}) \le C_2(R + \sqrt{R})\big(D_{S-1,T}(\bar{u} - \bar{\bar{u}}) + D_{S-1,T}(\bar{v} - \bar{\bar{v}})\big), \tag{10.3.14}$$

where C_2 is a positive constant, and $R = R(E,T)$ is still defined by (10.3.13).

10.3.3 Proof of Lemmas 10.3.3 and 10.3.4

We first estimate $\widetilde{D}_{S,T}(u)$.

By (10.2.12)–(10.2.13), for $\alpha = 2$ and 3, using Corollary 4.5.3 in Chap. 4 (in which we tatk $n = 2$, $p = 1 + \alpha$, $N = S$, $s = \frac{1}{2} - \frac{1}{p} = \frac{1}{2} - \frac{1}{1+\alpha}$), we have

$$\begin{aligned} &(1+t)^{\frac{1}{2} - \frac{1}{1+\alpha}} \|u(t,\cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2} \\ &\le C\Big\{\varepsilon + \int_0^t \Big(\|\widehat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,\gamma,\chi_1} \\ &\qquad + (1+\tau)^{-(\frac{1}{2} - \frac{1}{1+\alpha})} \|\widehat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}\Big) \mathrm{d}\tau\Big\}, \end{aligned} \tag{10.3.15}$$

where

$$\frac{1}{\gamma} = \frac{1}{2} + \frac{1}{2}\Big(\frac{1}{2} + \frac{1}{1+\alpha}\Big) \stackrel{\text{def.}}{=} \frac{1}{2} + \frac{\alpha}{H}. \tag{10.3.16}$$

Noting (10.3.16), using the estimates about product functions and composite functions in Chap. 5, we have

$$\begin{aligned}
&\|\widehat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,\gamma,\chi_1}\\
&\le C\Big\{\|(v, Dv)(\tau,\cdot)\|^{\alpha}_{\Gamma,[\frac{S}{2}],H,\chi_1}\|(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,2,\chi_1}\\
&\quad+\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],H,\chi_1}\|(v, Dv)(\tau,\cdot)\|^{\alpha-1}_{\Gamma,[\frac{S}{2}],H,\chi_1}\|(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2,\chi_1}\Big\}.
\end{aligned} \tag{10.3.17}$$

Using 2° in Corollary 3.4.1 of Chap. 3 (in which we take $n = 2$, $N = [\frac{S}{2}]$, $p = 2$, $q = H$, and $s = 1$), and noticing the definition of $X_{S,E,T}$, from the above formula we easily have

$$\begin{aligned}
&\|\widehat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,\gamma,\chi_1}\\
&\le C(1+\tau)^{-\alpha+\frac{1}{2}+\frac{1}{1+\alpha}}\|(v, Dv)(\tau,\cdot)\|^{\alpha}_{\Gamma,S,2,\chi_1}\|(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,2,\chi_1}\\
&\le C(1+\tau)^{-\frac{\alpha}{2}+\frac{1}{1+\alpha}}E^{\alpha}\big(E + D_{S,T}(u)\big).
\end{aligned} \tag{10.3.18}$$

Moreover, noting Lemma 10.3.2 and using Hölder inequality, from the estimates about product functions and composite functions in Chap. 5, it is easy to know that, to estimate the norm $\|\widehat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,1,2,,\chi_2}$, it suffices to estimate the corresponding norms of $v^{1+\alpha}$, $(Dv)^{1+\alpha}$, $v^{\alpha}D_x Du$ and $(Dv)^{\alpha}D_x Du$.

First, from the estimates about product functions in Chap. 5, using the Sobolev embedding theorem on a sphere (see 1° in Theorem 3.2.1 of Chap. 3, in which we take $n = 2$, $p = 2$, and $s > \frac{1}{2}$), and noticing (10.3.2), we have

$$\begin{aligned}
\|v^{1+\alpha}(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} &\le C\|v(\tau,\cdot)\|^{\alpha}_{\Gamma,[\frac{S}{2}],1+\alpha,\infty,\chi_2}\|v(\tau,\cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2}\\
&\le C\|v(\tau,\cdot)\|^{1+\alpha}_{\Gamma,S,1+\alpha,2,\chi_2}\\
&\le C(1+\tau)^{-\frac{\alpha-1}{2}}E^{1+\alpha}.
\end{aligned} \tag{10.3.19}$$

Second, from the Sobolev embedding theorem on a sphere, and using Corollary 3.4.4 in Chap. 3 (in which we take $N = [\frac{S}{2}]$, $n = 2$, $p = 2$, and $s > 1$), noting (10.3.1), we have

$$\begin{aligned}
\|(Dv)^{1+\alpha}(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} &\le C\|Dv(\tau,\cdot)\|^{\alpha-1}_{\Gamma,[\frac{S}{2}],\infty}\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty}\|Dv(\tau,\cdot)\|_{\Gamma,S,2}\\
&\le C(1+\tau)^{-\frac{\alpha-1}{2}}\|Dv(\tau,\cdot)\|^{1+\alpha}_{\Gamma,S,2}\\
&\le C(1+\tau)^{-\frac{\alpha-1}{2}}E^{1+\alpha}.
\end{aligned} \tag{10.3.20}$$

Third, from the estimates about product functions in Chap. 5, it is clear that

$$\begin{aligned}
&\|v^\alpha D_x Du(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}\\
&\le C\Big\{\|v(\tau,\cdot)\|^{\alpha}_{\Gamma,[\frac{S}{2}],2\alpha,\infty,\chi_2}\|D_x Du(\tau,\cdot)\|_{\Gamma,S,2}\\
&\qquad +\|v(\tau,\cdot)\|^{\alpha-1}_{\Gamma,[\frac{S}{2}],1+\alpha,\infty,\chi_2}\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],1+\alpha,\infty}\|v(\tau,\cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2}\Big\}.
\end{aligned} \tag{10.3.21}$$

Using the obvious estimate

$$\|f\|_{L^{2\alpha}}=\left(\int|f|^{2\alpha}\right)^{\frac{1}{2\alpha}}=\left(\int|f|^{\alpha-1}|f|^{1+\alpha}\right)^{\frac{1}{2\alpha}}\le\|f\|^{\frac{\alpha-1}{2\alpha}}_{L^\infty}\|f\|^{\frac{\alpha+1}{2\alpha}}_{L^{1+\alpha}},$$

it is easy to show that

$$\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2\alpha,\infty,\chi_2}\le\|v(\tau,\cdot)\|^{\frac{\alpha-1}{2\alpha}}_{\Gamma,[\frac{S}{2}],\infty,\chi_2}\|v(\tau,\cdot)\|^{\frac{\alpha+1}{2\alpha}}_{\Gamma,[\frac{S}{2}],1+\alpha,\infty,\chi_2}. \tag{10.3.22}$$

Using Corollary 3.4.4 in Chap. 3 (in which we take $n=2$, $N=[\frac{S}{2}]$, $p=1+\alpha$, and $s=1$), we have

$$\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty,\chi_2}\le C(1+\tau)^{-\frac{1}{1+\alpha}}\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,1+\alpha,\chi_2}, \tag{10.3.23}$$

Then, noting furthermore that $L^{1+\alpha,\infty}(\mathbb{R}^n)\subset L^{1+\alpha}(\mathbb{R}^n)$ is a continuous embedding, we have

$$\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty,\chi_2}\le C(1+\tau)^{-\frac{1}{1+\alpha}}\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,1+\alpha,\infty,\chi_2}. \tag{10.3.24}$$

Using the Sobolev embedding theorem on a sphere (see 1° in Theorem 3.2.1 of Chap. 3, in which we take $n=2$, $p=2$, and $s=1$), we have

$$\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,1+\alpha,\infty,\chi_2}\le C\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+2,1+\alpha,2,\chi_2}. \tag{10.3.25}$$

Thus, by (10.3.22) and noting the definition of $X_{S,E,T}$, it is easy to get

$$\begin{aligned}
\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2\alpha,\infty,\chi_2}&\le C(1+\tau)^{-(\frac{1}{1+\alpha}-\frac{1}{2\alpha})}\|v(\tau,\cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2}\\
&\le C(1+\tau)^{-\frac{\alpha-1}{2\alpha}}E.
\end{aligned} \tag{10.3.26}$$

Similarly, we have

$$\begin{aligned}
\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],1+\alpha,\infty,\chi_2}&\le C\|v(\tau,\cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2}\\
&\le C(1+\tau)^{-(\frac{1}{2}-\frac{1}{1+\alpha})}E.
\end{aligned} \tag{10.3.27}$$

Moreover, using the obvious estimate

$$\|f\|_{1+\alpha} = \left(\int |f|^{1+\alpha}\right)^{\frac{1}{1+\alpha}} = \left(\int |f|^{\alpha-1}|f|^2\right)^{\frac{1}{1+\alpha}} \le \|f\|_{L^\infty}^{\frac{\alpha-1}{\alpha+1}} \|f\|_{L^2}^{\frac{2}{1+\alpha}},$$

we have

$$\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],1+\alpha,\infty} \le \|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}^{\frac{\alpha-1}{1+\alpha}} \|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty}^{\frac{2}{1+\alpha}}. \tag{10.3.28}$$

Using Corollary 3.4.4 in Chap. 3 (in which we take $n=2$, $N=[\frac{S}{2}]$, $p=2$, and $s=2$), we have

$$\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \le C(1+\tau)^{-\frac{1}{2}} \|D_x Du(\tau,\cdot)\|_{\Gamma,S,2}, \tag{10.3.29}$$

and using the Sobolev embedding theorem on a sphere, we have

$$\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty} \le C\|D_x Du(\tau,\cdot)\|_{\Gamma,S,2}. \tag{10.3.30}$$

Then, by (10.3.28) and noting (10.3.1), we get

$$\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],1+\alpha,\infty} \le C(1+\tau)^{-(\frac{1}{2}-\frac{1}{1+\alpha})} D_{S,T}(u). \tag{10.3.31}$$

Substituting (10.3.26)–(10.3.27) and (10.3.31) into (10.3.21) and noticing the definition of $X_{S,E,T}$, we obtain

$$\|v^\alpha D_x Du(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{\alpha-1}{2}} E^\alpha D_{S,T}(u). \tag{10.3.32}$$

Finally, we have

$$\begin{aligned}
&\|(Dv)^\alpha D_x Du(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}\\
&\le C\Big\{\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty}\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}^{\alpha-1}\|D_x Du(\tau,\cdot)\|_{\Gamma,S,2}\\
&\qquad + \|Dv(\tau,\cdot)\|_{\Gamma,S,2}\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}^{\alpha-1}\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty}\Big\}.
\end{aligned} \tag{10.3.33}$$

By Corollary 3.4.4 in Chap. 3 (in which we take $n=2$, $N=[\frac{S}{2}]$, $p=2$, and $s=2$), we have

$$\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \le C(1+\tau)^{-\frac{1}{2}} \|Dv(\tau,\cdot)\|_{\Gamma,S,2}, \tag{10.3.34}$$

and from the Sobolev embedding theorem on a sphere we have

$$\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty} \le C\|Dv(\tau,\cdot)\|_{\Gamma,S,2} \tag{10.3.35}$$

and a similar estimate for $\|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty}$. Then, from (10.3.33) we get

$$\|(Dv)^\alpha D_x Du(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{\alpha-1}{2}} E^\alpha D_{S,T}(u). \tag{10.3.36}$$

Thus, using (10.3.19)–(10.3.20), (10.3.32) and (10.3.36) we obtain

$$\|\widehat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{\alpha-1}{2}} E^\alpha (E + D_{S,T}(u)). \tag{10.3.37}$$

Plugging (10.3.18) and (10.3.37) in (10.3.15), we finally obtain

$$(1+t)^{\frac{1}{2}-\frac{1}{1+\alpha}} \|u(t,\cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2} \le C\Big\{\varepsilon + (1+t)^{1-\frac{\alpha}{2}+\frac{1}{1+\alpha}} E^\alpha \big(E + D_{S,T}(u)\big)\Big\}, \tag{10.3.38}$$

then, when $\alpha = 2$ and 3, noting (10.3.13), we get

$$(1+t)^{\frac{1}{2}-\frac{1}{1+\alpha}} \|u(t,\cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2} \le C\Big\{\varepsilon + R(E,T)\big(E + D_{S,T}(u)\big)\Big\}. \tag{10.3.39}$$

This finishes the estimates on the second term in $\widetilde{D}_{S,T}(u)$.

Now we estimate the first term in $\widetilde{D}_{S,T}(u)$.

From (4.5.18) in Corollary 4.5.1 of Chap. 4 (in which we take $N = S$, $\sigma = \frac{1}{2} - \frac{1}{1+\alpha}$, and q satisfies $\frac{1}{q} = 1 - \frac{\sigma}{2} = \frac{3}{4} + \frac{1}{2(1+\alpha)}$), it is easy to get

$$\begin{aligned}
&(1+t)^{-(\frac{1}{2}-\frac{1}{1+\alpha})} \|u(t,\cdot)\|_{\Gamma,S,2,\chi_1} \\
&\le C\Big\{\varepsilon + \int_0^t \Big(\|\widehat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,q,\chi_1} \\
&\qquad + (1+\tau)^{-(\frac{1}{2}-\frac{1}{1+\alpha})} \|\widehat{F}(v, Dv, D_x Du(\tau,\cdot))\|_{\Gamma,S,1,2,\chi_2}\Big) \mathrm{d}\tau\Big\}.
\end{aligned} \tag{10.3.40}$$

Since the estimate for the second term on the right-hand side of the above formula can be found in (10.3.37), it remains to estimate the first term.

Noticing Lemma 10.3.2 and using Hölder inequality, by the estimates on product functions and composite functions in Chap. 5, it is easy to show that

$$\begin{aligned}
&\|\widehat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,q,\chi_1} \\
&\le C\Big\{\|(v, Dv)(\tau,\cdot)\|^\alpha_{\Gamma,[\frac{S}{2}],\ell,\chi_1} \|(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S,2,\chi_1} \\
&\qquad + \|D_x Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\ell,\chi_1} \|(v, Dv)(\tau,\cdot)\|^{\alpha-1}_{\Gamma,[\frac{S}{2}],\ell,\chi_1} \|(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2,\chi_1}\Big\},
\end{aligned} \tag{10.3.41}$$

where l is determined by $\frac{1}{q} = \frac{1}{2} + \frac{\alpha}{l}$. Using 2° in Corollary 3.4.1 of Chap. 3 (in which we take $n = 2$, $N = [\frac{S}{2}]$, $p = 2$, $q = l$, and $s = 1$), and noting the definition of $X_{S,E,T}$, similarly to (10.3.18), we have

$$
\begin{aligned}
&\|\widehat{F}(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S,q,\chi_1} \\
&\leq C(1+\tau)^{-2\alpha(\frac{1}{2}-\frac{1}{l})}\|(v, Dv)(\tau, \cdot)\|^{\alpha}_{\Gamma,S,2,\chi_1}\|(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S,2,\chi_1} \\
&\leq C(1+\tau)^{-(\frac{\alpha}{2}-\frac{1}{1+\alpha})}E^{\alpha}\big(E + D_{S,T}(u)\big). \qquad (10.3.42)
\end{aligned}
$$

Plugging (10.3.37) and (10.3.42) in (10.3.40) and noticing the definition (10.3.13) of $R(E, T)$ when $\alpha = 2$ and 3, it follows immediately that

$$
(1+t)^{-(\frac{1}{2}-\frac{1}{1+\alpha})}\|u(t, \cdot)\|_{\Gamma,S,2,\chi_1} \leq C\Big\{\varepsilon + R(E, T)\big(E + D_{S,T}(u)\big)\Big\}. \qquad (10.3.43)
$$

Combining (10.3.39) and (10.3.43), we obtain

$$
\widetilde{D}_{S,T}(u) \leq C\Big\{\varepsilon + R(E, T)\big(E + D_{S,T}(u)\big)\Big\}. \qquad (10.3.44)
$$

Now we estimate $\|(Du, D^2u)(t, \cdot)\|_{\Gamma,S,2}$.

We still have (10.2.53)–(10.2.55). When $\alpha = 2$ and 3, using Lemma 10.3.2 and noting (10.3.13), it is obvious that

$$
\begin{aligned}
|\mathrm{I}|, |\mathrm{II}|, |\mathrm{III}| &\leq C\int_0^t \|(v, Dv, D^2v)(\tau, \cdot)\|^{\alpha}_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau \cdot D^2_{S,T}(u) \\
&\leq CR(E, T)D^2_{S,T}(u). \qquad (10.3.45)
\end{aligned}
$$

Now we estimate the L^2 norm of $G_k(\tau, \cdot)$.

Similarly to (10.2.57), from Lemma 10.3.2 we have

$$
\begin{aligned}
&\|b_{ij}(v, Dv)\big(\Gamma^k Du_{x_ix_j} - (\Gamma^k Du)_{x_ix_j}\big)(\tau, \cdot)\|_{L^2(\mathbb{R}^2)} \\
&\leq C\|b_{ij}(v, Dv)(\tau, \cdot)\|_{L^\infty(\mathbb{R}^2)}\|D^2u(\tau, \cdot)\|_{\Gamma,S,2} \\
&\leq C(1+\tau)^{-\frac{\alpha}{2}}E^{\alpha}D_{S,T}(u). \qquad (10.3.46)
\end{aligned}
$$

By estimate (5.1.19) about product functions in Chap. 5, using (3.4.30) in Chap. 3 (in which we take $N = [\frac{S}{2}]$, $p = 2$, $s = 2$), it is clear that

$$
\begin{aligned}
&\|(\Gamma^k D(b_{ij}(v, Dv)u_{x_ix_j}) - b_{ij}(v, Dv)\Gamma^k Du_{x_ix_j})(\tau, \cdot)\|_{L^2(\mathbb{R}^2)} \\
&\leq C\Big\{\|Db_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,S,2}\|D^2u(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \\
&\qquad + \|D^2u(\tau, \cdot)\|_{\Gamma,S,2}\|b_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\Big\} \\
&\leq C\Big\{(1+\tau)^{-\frac{1}{2}}\|Db_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,S,2} \\
&\qquad + \|b_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\Big\} \cdot \|D^2u(\tau, \cdot)\|_{\Gamma,S,2}. \qquad (10.3.47)
\end{aligned}
$$

From Lemma 10.3.2, it is easy to know that

$$\|b_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \le C(1+\tau)^{-\frac{\alpha}{2}} E^{\alpha}. \tag{10.3.48}$$

In addition, using Lemma 10.3.2, from the estimates about product functions in Chap. 5 we easily have

$$\begin{aligned}
&\|Db_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,S,2} \\
&\le C\Big\{\|(v, Dv, D^2v)(\tau, \cdot)\|^{\alpha-1}_{\Gamma,[\frac{S}{2}],\infty}\|(Dv, D^2v)(\tau, \cdot)\|_{\Gamma,S,2} \\
&\qquad + \|(v, Dv, D^2v)(\tau, \cdot)\|^{\alpha-1}_{\Gamma,[\frac{S}{2}],\infty,\chi_1}\|v(\tau, \cdot)\|_{\Gamma,S,2,\chi_1} \\
&\qquad + \|(v, Dv, D^2v)(\tau, \cdot)\|^{\alpha-1}_{\Gamma,[\frac{S}{2}],2(1+\alpha),\infty,\chi_2}\|v(\tau, \cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2}\Big\},
\end{aligned} \tag{10.3.49}$$

where χ_1 is the characteristic function of set $\{(t, x)\big||x| \le \frac{1+t}{2}\}$, and $\chi_2 = 1 - \chi_1$.

Using the obvious estimate

$$\|f\|_{L^{2(1+\alpha)}} \le \|f\|^{\frac{1}{2}}_{L^{\infty}}\|f\|^{\frac{1}{2}}_{L^{1+\alpha}},$$

similarly to the proof of (10.3.26), and noticing the definition of $X_{S,E,T}$, we have

$$\begin{aligned}
\|v(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2(1+\alpha),\infty,\chi_2} &\le \|v(\tau, \cdot)\|^{\frac{1}{2}}_{\Gamma,[\frac{S}{2}],\infty,\chi_2}\|v(\tau, \cdot)\|^{\frac{1}{2}}_{\Gamma,[\frac{S}{2}],1+\alpha,\infty,\chi_2} \\
&\le C(1+\tau)^{-\frac{1}{2(1+\alpha)}}\|v(\tau, \cdot)\|_{\Gamma,S,1+\alpha,2,\chi_2} \\
&\le C(1+\tau)^{-\frac{\alpha}{2(1+\alpha)}} E.
\end{aligned} \tag{10.3.50}$$

Meanwhile, similarly to the proof of (10.3.31), and noticing the definition of $X_{S,E,T}$, we also have

$$\begin{aligned}
\|(Dv, D^2v)(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2(1+\alpha),\infty} &\le \|(Dv, D^2v)(\tau, \cdot)\|^{\frac{\alpha}{1+\alpha}}_{\Gamma,[\frac{S}{2}],\infty}\|(Dv, D^2v)(\tau, \cdot)\|^{\frac{1}{1+\alpha}}_{\Gamma,[\frac{S}{2}],2,\infty} \\
&\le C(1+\tau)^{-\frac{\alpha}{2(1+\alpha)}}\|(Dv, D^2v)(\tau, \cdot)\|_{\Gamma,S,2} \\
&\le C(1+\tau)^{-\frac{\alpha}{2(1+\alpha)}} E.
\end{aligned} \tag{10.3.51}$$

Thus, using Lemma 10.3.2 and noticing the definition of $X_{S,E,T}$, from (10.3.49) we get

$$\begin{aligned}
\|Db_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,S,2} &\le C\Big\{(1+\tau)^{-\frac{\alpha-1}{2}} + (1+\tau)^{-\frac{\alpha-1}{2}+(\frac{1}{2}-\frac{1}{1+\alpha})}\Big\}E^{\alpha} \\
&\le C(1+\tau)^{-\frac{\alpha-1}{2}+(\frac{1}{2}-\frac{1}{1+\alpha})}E^{\alpha}.
\end{aligned} \tag{10.3.52}$$

Substituting (10.3.48) and (10.3.52) into (10.3.47), and using (3.4.30) in Chap. 3 (in which we take $N = [\frac{S}{2}]$, $p = 2$, $s = 2$), we obtain

$$\begin{aligned}
&\|(\Gamma^k D(b_{ij}(v, Dv)u_{x_i x_j}) - b_{ij}(v, Dv)\Gamma^k D u_{x_i x_j})(\tau, \cdot)\|_{L^2(\mathbb{R}^2)} \\
&\le C(1+\tau)^{-\frac{1}{2}}\left\{(1+\tau)^{-\frac{\alpha-1}{2}-\frac{1}{1+\alpha}} + (1+\tau)^{-\frac{\alpha}{2}}\right\} E^\alpha D_{S,T}(u) \\
&\le C(1+\tau)^{-\frac{\alpha}{2}-\frac{1}{1+\alpha}} E^\alpha D_{S,T}(u).
\end{aligned} \tag{10.3.53}$$

Combining (10.3.46) and (10.3.53), and using similar estimates about the terms involving $a_{0j}(v, Dv)$ in (10.2.54), it yields

$$\|G_k(\tau, \cdot)\|_{L^2(\mathbb{R}^2)} \le C(1+\tau)^{-\frac{\alpha}{2}} E^\alpha D_{S,T}(u), \tag{10.3.54}$$

then, noting (10.3.13), when $\alpha = 2$ and 3 we have

$$|\mathrm{IV}| \le CR(E, T) D^2_{S,T}(u). \tag{10.3.55}$$

Similarly, we can estimate $\|g_k(\tau, \cdot)\|_{L^2(\mathbb{R}^2)}$, then, when $\alpha = 2$ and 3 we have

$$|\mathrm{V}| \le CR(E, T)(E + D_{S,T}(u)) D_{S,T}(u). \tag{10.3.56}$$

Using (10.3.45) and (10.3.55)–(10.3.56), from (10.2.53) we easily get

$$\sup_{0\le t\le T} \|D^2 u(t, \cdot)\|_{\Gamma,S,2} \le C\{\varepsilon + \sqrt{R(E, T)}(E + D_{S,T}(u))\}. \tag{10.3.57}$$

By (10.2.65)–(10.2.67), similarly we obtain

$$\sup_{0\le t\le T} \|Du(t, \cdot)\|_{\Gamma,S,2} \le C\{\varepsilon + \sqrt{R(E, T)}(E + D_{S,T}(u))\}. \tag{10.3.58}$$

Combining (10.3.44) and (10.3.57)–(10.3.58), the desired (10.3.12) is proved. The proof of Lemma 10.3.3 is finished.

The proof of Lemma 10.3.4 is similar, we omit the details here.

10.4 Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (10.1.14)–(10.1.15) (The Cases $\alpha = 1$ and 2) (Continued)

In this section, we will prove, for the life-span of classical solutions to Cauchy problem (10.1.14)–(10.1.15) of two-dimensional second-order quasi-linear hyperbolic equations, the lower bounds estimates given by the last formula in (10.1.9) when $\alpha = 1$ and the last formula in (10.1.11) when $\alpha = 2$. At this moment, the conditions imposed on $F(u, Du)$ can be expressed by (10.1.22) and (10.1.23), respectively, or unified as follows: when $\alpha = 1$ and 2,

$$\partial_u^{1+\alpha} F(0,0) = \cdots = \partial_u^{2\alpha} F(0,0) = 0. \tag{10.4.1}$$

To simplify description, in what follows we emphasize only on the difference of the proof with that in Sects. 10.2–10.3.

10.4.1 Metric Space $X_{S,E,T}$. Main Results

For any given integer $S \geq 8$, for any given real numbers $E(\leq E_0)$ and $T(>0)$, we still introduce $X_{S,E,T}$, the set of functions, by (10.2.2), whereas

$$D_{S,T}(v) = \sum_{i=1}^{2} \sup_{0\leq t\leq T} \|D^i v(t,\cdot)\|_{\Gamma,S,2} + \overline{D}_{S,T}(v), \tag{10.4.2}$$

where

$$\overline{D}_{S,T}(v) = \sup_{0\leq t\leq T} (1+t)^{\frac{1}{2}} \|v(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} + \sup_{0\leq t\leq T} (1+t)^{-\frac{1}{2}} \|v(t,\cdot)\|_{\Gamma,S,2}. \tag{10.4.3}$$

It is easy to prove

Lemma 10.4.1 *Introduce the following metric in $X_{S,E,T}$:*

$$\rho(\overline{v}, \overline{\overline{v}}) = D_{S,T}(\overline{v} - \overline{\overline{v}}), \quad \forall \overline{v}, \overline{\overline{v}} \in X_{S,E,T}. \tag{10.4.4}$$

Then, when $\varepsilon > 0$ is suitably small, $X_{S,E,T}$ is a non-empty complete metric space.

Lemma 10.4.2 *When $S \geq 8$, for any given $v \in X_{S,E,T}$, we have*

$$\|(v, Dv, D^2 v)(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq CE(1+t)^{-\frac{1}{2}}, \quad \forall t \in [0,T]. \tag{10.4.5}$$

Denote by $\widetilde{X}_{S,E,T}$ a subset of $X_{S,E,T}$, which is composed of all the elements of $X_{S,E,T}$, having support with respect to x, included in $\left\{x \middle| |x| \leq t+\rho\right\}$ for any given $t \in [0,T]$, and $\rho > 0$ is the constant appearing in (10.1.6).

The main result of this section is the following

Theorem 10.4.1 *Let $n = 2$ and $\alpha = 1$ or 2. Under assumptions (10.1.5)–(10.1.6) and (10.1.17)–(10.1.21), we furthermore assume (10.1.22)–(10.1.23) (i.e., (10.4.1)), for any given integer $S \geq 8$, there exist positive constants ε_0 and C_0 with $C_0\varepsilon_0 \leq E_0$, and for any given $\varepsilon \in (0, \varepsilon_0]$, there exists a positive number $T(\varepsilon)$ such that Cauchy problem (10.1.14)–(10.1.15) admits a unique classical solution $u \in \widetilde{X}_{S,C_0\varepsilon,T(\varepsilon)}$ on $[0, T(\varepsilon)]$, and $T(\varepsilon)$ can be taken as*

$$T(\varepsilon) = \begin{cases} b\varepsilon^{-2} - 1, & \alpha = 1, \\ \exp\{a\varepsilon^{-2}\} - 1, & \alpha = 2, \end{cases} \tag{10.4.6}$$

where a and b are positive constants depending possibly on ρ but not on ε. Moreover, after a possible change of values for t on a zero-measure set of $[0, T(\varepsilon)]$, (10.2.9)–(10.2.11) hold.

10.4.2 Framework to Prove Theorem 10.4.1—The Global Iteration Method

Similarly to Sect. 10.2.2, the following two lemmas are crucial to the proof of Theorem 10.4.1.

Lemma 10.4.3 *Under the assumptions of Theorem 10.4.1, when $E > 0$ is suitably small, for any given $v \in \widetilde{X}_{S,E,T}$, $u = Mv$ satisfies*

$$D_{S,T}(u) \leq C_1\big\{\varepsilon + \big(R^2 + R + \sqrt{R}\big)\big(E + D_{S,T}(u)\big)\big\}, \tag{10.4.7}$$

where C_1 is a positive constant, and

$$R = R(E,T) \overset{\text{def.}}{=} \begin{cases} E\big(1+T\big)^{\frac{1}{2}}, & \alpha = 1; \\ E^2 \ln(1+T), & \alpha = 2. \end{cases} \tag{10.4.8}$$

Lemma 10.4.4 *Under the assumptions of Lemma 10.4.3, for any given $\overline{v}, \overline{\overline{v}} \in \widetilde{X}_{S,E,T}$, if both $\overline{u} = M\overline{v}$ and $\overline{\overline{u}} = M\overline{\overline{v}}$ satisfy $\overline{u}, \overline{\overline{u}} \in \widetilde{X}_{S,E,T}$, then we have*

$$D_{S-1,T}(\overline{u} - \overline{\overline{u}}) \leq C_2\big(R^2 + R + \sqrt{R}\big)\big(D_{S-1,T}(\overline{u} - \overline{\overline{u}}) + D_{S-1,T}(\overline{v} - \overline{\overline{v}})\big), \tag{10.4.9}$$

where C_2 is a positive constant, and $R = R(E,T)$ is still defined by (10.4.8).

10.4.3 Proof of Lemmas 10.4.3 and 10.4.4

In what follows we only give the key points in the proof of Lemma 10.4.3, and the proof of Lemma 10.4.4 is similar.

First, we estimate $\overline{D}_{S,T}(u)$.

For this we first estimate $\|u(t,\cdot)\|_{\Gamma,S,2}$.

Noting that

$$\begin{aligned} b_{ij}(v, Dv)u_{x_i x_j} &= b_{ij}(v,0)u_{x_i x_j} + (b_{ij}(v,Dv) - b_{ij}(v,0))u_{x_i x_j} \\ &= \frac{\partial}{\partial x_i}\big(b_{ij}(v,0)u_{x_j}\big) - \frac{\partial b_{ij}(v,0)}{\partial x_i}u_{x_j} + (b_{ij}(v,Dv) - b_{ij}(v,0))u_{x_i x_j} \end{aligned}$$

and

$$
\begin{aligned}
F(v, Dv) =& F(v, 0) + (F(v, Dv) - F(v, 0)) \\
=& F(v, 0) + \widetilde{F}(v, Dv)Dv \\
=& F(v, 0) + \widetilde{F}(v, 0)Dv + (\widetilde{F}(v, Dv) - \widetilde{F}(v, 0))Dv \\
=& F(v, 0) + \sum_{i=0}^{2} \widetilde{F}_i(v, 0)\partial_i v + (\widetilde{F}(v, Dv) - \widetilde{F}(v, 0))Dv \\
=& F(v, 0) + \sum_{i=0}^{2} \partial_i \widetilde{G}_i(v, 0) + (\widetilde{F}(v, Dv) - \widetilde{F}(v, 0))Dv,
\end{aligned}
$$

where $\widetilde{G}_i(v, 0)$ are primitive functions of $\widetilde{F}_i(v, 0)$ $(i = 0, 1, 2)$, using (10.1.18)–(10.1.19) and the additional assumption (10.4.1), $\widehat{F}(v, Dv, D_x Du)$ defined by (10.2.12) can be rewritten as

$$
\begin{aligned}
\widehat{F}(v, Dv, D_x Du) =& \sum_{i=0}^{2} \partial_i \widehat{G}_i(v, Du) + \sum_{i,j=0}^{2} \widehat{A}_{ij}(v) v_{x_i} u_{x_j} \\
&+ \sum_{\substack{i,j,m=0 \\ j+m\geq 1}}^{2} \widehat{B}_{ijm}(v, Dv) v_{x_i} u_{x_j x_m} + \sum_{i,j=0}^{2} \widehat{C}_{ij}(v, Dv) v_{x_i} v_{x_j} + F(v), \qquad (10.4.10)
\end{aligned}
$$

where, in a neighborhood of the origin, we have

$$
F(v) \overset{\text{def.}}{=} F(v, 0) = O(|v|^{2\alpha+1}), \qquad (10.4.11)
$$

$$
\widehat{G}_i(\overline{\lambda}) = O(|\overline{\lambda}|^{\alpha+1}), i = 0, 1, 2, \ \overline{\lambda} = (v, Du), \qquad (10.4.12)
$$

and $\widehat{G}_i(v, Du) (i = 0, 1, 2)$ are affine with respect to Du,

$$
\widehat{A}_{ij}(v) = O(|v|^{\alpha-1}), \quad i, j = 0, 1, 2 \qquad (10.4.13)
$$

and

$$
\widehat{B}_{ijm}(\widetilde{\lambda}), \widehat{C}_{ij}(\widetilde{\lambda}) = O(|\widetilde{\lambda}|^{\alpha-1}), i, j, m = 0, 1, 2, \widetilde{\lambda} = (v, Dv). \qquad (10.4.14)
$$

Thus, the solution $u = Mv$ to Cauchy problem (10.1.14)–(10.1.15) can be written as

$$
u = u_1 + u_2 + u_3, \qquad (10.4.15)
$$

where u_1 is the solution of equation

$$
\Box u_1 = \sum_{i=0}^{2} \partial_i \widehat{G}_i(v, Du) \qquad (10.4.16)
$$

with the zero initial condition, and u_2 is the solution of

$$\Box u_2 = Q(v, Dv, Du, D_x Du) \tag{10.4.17}$$

with the same initial condition (10.1.15) as u, where

$$\begin{aligned} &Q(v, Dv, Du, D_x Du)\\ &= \sum_{i,j=0}^{2} \widehat{A}_{ij}(v) v_{x_i} u_{x_j} + \sum_{\substack{i,j,m=0\\ j+m\geq 1}}^{2} \widehat{B}_{ijm}(v, Dv) v_{x_i} u_{x_j x_m} + \sum_{i,j=0}^{2} \widehat{C}_{ij}(v, Dv) v_{x_i} v_{x_j}, \end{aligned} \tag{10.4.18}$$

while, u_3 is the solution of equation

$$\Box u_3 = F(v) \tag{10.4.19}$$

with the zero initial condition.

In addition, it is easy to know that u_1 can be written as

$$u_1 = \sum_{i=0}^{2} \partial_i \overline{u}_i + \overline{\overline{u}}_1, \tag{10.4.20}$$

where, for $i = 0, 1, 2$, $\overline{u}_i$ is the solution of equation

$$\Box \overline{u}_i = \widehat{G}_i(v, Du) \tag{10.4.21}$$

with the zero initial condition, and $\overline{\overline{u}}_1$ is the solution of equation

$$\Box \overline{\overline{u}}_1 = 0 \tag{10.4.22}$$

with corresponding non-zero initial condition (of order $O(\varepsilon^2)$).

By 1° in Theorem 4.3.1 of Chap. 4, and noticing Lemma 3.1.5 in Chap. 3, it is easy to show that

$$\|\overline{\overline{u}}_1(t, \cdot)\|_{\Gamma,S,2} \leq C\varepsilon^2 \sqrt{\ln(2+t)}. \tag{10.4.23}$$

From the energy estimates of wave equation (see Lemma 4.5.2 in Chap. 4) and Lemma 3.1.5 in Chap. 3, and noting (10.4.12), we have

$$\|D\overline{u}_i(t, \cdot)\|_{\Gamma,S,2} \leq C\Big(\varepsilon^2 + \int_0^t \|\widehat{G}_i(v, Du)(\tau, \cdot)\|_{\Gamma,S,2} d\tau\Big), \quad i = 0, 1, 2. \tag{10.4.24}$$

Thus, by (10.4.20) we get

$$\|u_1(t,\cdot)\|_{\Gamma,S,2} \le C\Big(\varepsilon^2\sqrt{\ln(2+t)} + \sum_{i=0}^{2}\int_0^t \|\widehat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,S,2}d\tau\Big). \quad (10.4.25)$$

Noticing Lemma 10.4.2, from the estimates about composite functions in Chap. 5, we have

$$\sum_{i=0}^{2}\|\widehat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,S,2}$$
$$\le C\Big\{\|v(\tau,\cdot)\|^{\alpha}_{\Gamma,[\frac{S}{2}],\infty}\|(v,Du)(\tau,\cdot)\|_{\Gamma,S,2} + \|v(\tau,\cdot)\|^{\alpha-1}_{\Gamma,[\frac{S}{2}],\infty}\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|_{\Gamma,S,2}\Big\}. \quad (10.4.26)$$

From Corollary 3.4.4 in Chap. 3 (in which we take $n = 2$, $N = [\frac{S}{2}]$, $p = 2$, and $s = 2$), and noticing the definition of $X_{S,E,T}$, we have

$$\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \le C(1+\tau)^{-\frac{1}{2}}\|Du(\tau,\cdot)\|_{\Gamma,S,2} \le C(1+\tau)^{-\frac{1}{2}}D_{S,T}(u). \quad (10.4.27)$$

Using Lemma 10.4.2, and noting the definition of $X_{S,E,T}$, from (10.4.26) we get

$$\sum_{i=0}^{2}\|\widehat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,S,2} \le CE^{\alpha}(1+\tau)^{-\frac{\alpha}{2}}(E(1+\tau)^{\frac{1}{2}} + D_{S,T}(u)), \quad (10.4.28)$$

then, noticing (10.4.8), it is easy to know that

$$\sum_{i=0}^{2}\int_0^t \|\widehat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,S,2}d\tau \le C(1+t)^{\frac{1}{2}}R(E,T)(E + D_{S,T}(u)), \quad (10.4.29)$$

then from (10.4.25) we obtain

$$\|u_1(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\frac{1}{2}}\big\{\varepsilon^2 + R(E,T)(E + D_{S,T}(u))\big\}. \quad (10.4.30)$$

From (4.5.18) in Corollary 4.5.1 of Chap. 4 (in which we take $N = S$, $\sigma = \frac{1}{3}$, and q is determined by $\frac{1}{q} = 1 - \frac{\sigma}{2}$ as $\frac{6}{5}$), we easily have

$$\|u_2(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\frac{1}{3}}\Big\{\varepsilon + \int_0^t \big(\|Q(v,Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1}$$
$$+(1+\tau)^{-\frac{1}{3}}\|Q(v,Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}\big)d\tau\Big\}. \quad (10.4.31)$$

Utilizing the estimates about composite functions in Chap. 5, and noting (10.4.13), we can estimate $\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1}$.

In fact, when $\alpha = 1$ we have

$$\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} = C\Big\{\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\|Du(\tau,\cdot)\|_{\Gamma,S,2} + \|Dv(\tau,\cdot)\|_{\Gamma,S,2}\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\Big\}. \tag{10.4.32}$$

From 2° in Corollary 3.4.1 of Chap. 3 (in which we take $n = 2$, $N = [\frac{S}{2}]$, $q = 3$, $p = 2$, and $s = 1$), we have

$$\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1} \le C(1+\tau)^{-\frac{1}{3}}\|Dv(\tau,\cdot)\|_{\Gamma,S,2} \tag{10.4.33}$$

and similar estimates for Du. Noting also the definition of $X_{S,E,T}$, from (10.4.32) we get

$$\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{1}{3}}ED_{S,T}(u). \tag{10.4.34}$$

While, when $\alpha = 2$ we have

$$\begin{aligned}&\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1}\\ &\le C\Big\{\|vDv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\|Du(\tau,\cdot)\|_{\Gamma,S,2} + \|vDv(\tau,\cdot)\|_{\Gamma,S,2}\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\Big\}\\ &\le C\Big\{\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\|Du(\tau,\cdot)\|_{\Gamma,S,2} + \|vDv(\tau,\cdot)\|_{\Gamma,S,2}\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\Big\}.\end{aligned} \tag{10.4.35}$$

From (1.31) in Lemma 5.1.4 of Chap. 5, we obtain

$$\|vDv(\tau,\cdot)\|_{\Gamma,S,2} \le C_\rho\Big\{\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|Dv(\tau,\cdot)\|_{\Gamma,S,2} + \|Dv(\tau,\cdot)\|_{\Gamma,S,2}\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\Big\}, \tag{10.4.36}$$

where C_ρ is a positive constant depending on ρ (see (10.1.6)). Hence, by Lemma 10.4.2, (10.4.33) and similar estimates for Du, and noting the definition of $X_{S,E,T}$, from (10.4.35) and (10.4.36) we get

$$\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{1}{2}-\frac{1}{3}}E^2D_{S,T}(u). \tag{10.4.37}$$

When $\alpha = 1$ and 2, (10.4.34) and (10.4.37) can be combined as

$$\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{\alpha-1}{2}-\frac{1}{3}}E^\alpha D_{S,T}(u). \tag{10.4.38}$$

Similarly, utilizing the estimates about composite functions in Chap. 5, and noting (10.4.14), we can estimate $\|\widehat{B}_{ijm}(v,Dv)v_{x_i}u_{x_jx_m}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1}$ and $\|\widehat{C}_{ij}(v,Dv)v_{x_i}v_{x_j}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1}$, and obtain, when $\alpha = 1$ and 2,

$$\|\widehat{B}_{ijm}(v, Dv)v_{x_i}u_{x_jx_m}(\tau, \cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{\alpha-1}{2}-\frac{1}{3}}E^{\alpha}D_{S,T}(u) \tag{10.4.39}$$

and

$$\|\widehat{C}_{ij}(v, Dv)v_{x_i}v_{x_j}(\tau, \cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{\alpha-1}{2}-\frac{1}{3}}E^{\alpha+1}. \tag{10.4.40}$$

Combining (10.4.38)–(10.4.40), we get

$$\|Q(v, Dv, Du, D_xDu)(\tau, \cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{\alpha-1}{2}-\frac{1}{3}}E^{\alpha}(E+D_{S,T}(u)). \tag{10.4.41}$$

Now we estimate $\|Q(v, Dv, Du, D_xDu)(\tau, \cdot)\|_{\Gamma,S,1,2,\chi_2}$.

First, by using the estimates about composite functions in Chap. 5, and noting (10.4.13), we estimate $\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau, \cdot)\|_{\Gamma,S,1,2,\chi_2}$.

When $\alpha = 1$, using the Sobolev embedding theorem on a sphere (see 1° in Theorem 3.2.1 of Chap. 3, in which we take $n = 2$, $p = 2$, and $s = 1$), and noticing the definition of $X_{S,E,T}$, we have

$$\begin{aligned}
&\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau, \cdot)\|_{\Gamma,S,1,2,\chi_2} \\
&\le C\{\|Dv(\tau, \cdot)\|_{\Gamma,S,2}\|Du(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2} + \|Dv(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2}\|Du(\tau, \cdot)\|_{\Gamma,S,2}\} \\
&\le C\{\|Dv(\tau, \cdot)\|_{\Gamma,S,2}\|Du(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}]+1,2} + \|Dv(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}]+1,2}\|Du(\tau, \cdot)\|_{\Gamma,S,2}\} \\
&\le C\|Dv(\tau, \cdot)\|_{\Gamma,S,2}\|Du(\tau, \cdot)\|_{\Gamma,S,2} \le CED_{S,T}(u).
\end{aligned} \tag{10.4.42}$$

While, when $\alpha = 2$, we have

$$\begin{aligned}
&\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau, \cdot)\|_{\Gamma,S,1,2,\chi_2} \\
&\le C\{\|vDv(\tau, \cdot)\|_{\Gamma,S,2}\|Du(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2} + \|vDv(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2}\|Du(\tau, \cdot)\|_{\Gamma,S,2}\} \\
&\le C\{\|vDv(\tau, \cdot)\|_{\Gamma,S,2}\|Du(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2} + \|v(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|Dv(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2}\|Du(\tau, \cdot)\|_{\Gamma,S,2}\}.
\end{aligned} \tag{10.4.43}$$

Using the Sobolev inequality on a sphere and Lemma 10.4.2, and noticing (10.4.36) and the definition of $X_{S,E,T}$, we obtain

$$\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau, \cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{\frac{1}{2}}E^2D_{S,T}(u). \tag{10.4.44}$$

When $\alpha = 1$ and 2, (10.4.42) and (10.4.44) can be combined as

$$\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau, \cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{\alpha-1}{2}}E^{\alpha}D_{S,T}(u). \tag{10.4.45}$$

Similarly, we have

$$\|\widehat{B}_{ijm}(v, Dv)v_{x_i}u_{x_jx_m}(\tau, \cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{\alpha-1}{2}}E^{\alpha}D_{S,T}(u) \tag{10.4.46}$$

and

$$\|\widehat{C}_{ij}(v, Dv)v_{x_i}v_{x_j}(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{\alpha-1}{2}}E^{\alpha+1}. \qquad (10.4.47)$$

Combining (10.4.45)–(10.4.47), we get

$$\|Q(v, Dv, Du, D_x Du)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{\alpha-1}{2}}E^{\alpha}(E+D_{S,T}(u)). \qquad (10.4.48)$$

Plugging (10.4.41) and (10.4.48) in (10.4.31), and noting (10.4.8), we obtain

$$\|u_2(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\frac{1}{2}}\{\varepsilon + R(E,T)(E+D_{S,T}(u))\}. \qquad (10.4.49)$$

By 2° in Corollary 4.5.1 of Chap. 4 (in which we take $N = S$, $\sigma = \frac{1}{3}$, and $q = \frac{6}{5}$), from (10.4.19) we get

$$\|u_3(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\frac{1}{3}}\left\{\varepsilon + \int_0^t \left(\|F(v)(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} + (1+\tau)^{-\frac{1}{3}}\|F(v)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}\right)d\tau\right\}. \qquad (10.4.50)$$

Noting (10.4.12), from the estimates about composite functions in Chap. 5, and using Lemma 10.4.2, the estimates similar to (10.4.33) for v and the definition of $X_{S,E,T}$, when $\alpha = 1$ and $\alpha = 2$, we have

$$\begin{aligned}\|F(v)(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} &\le C\|v(\tau,\cdot)\|^{2\alpha-1}_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\|v(\tau,\cdot)\|_{\Gamma,S,2}\\ &\le C(1+\tau)^{-\frac{2\alpha-1}{2}+1-\frac{1}{3}}E^{2\alpha+1}.\end{aligned} \qquad (10.4.51)$$

Meanwhile, using the Sobolev estimates on a sphere, we have

$$\begin{aligned}\|F(v)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} &\le C\|v(\tau,\cdot)\|^{2\alpha-1}_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2}\|v(\tau,\cdot)\|_{\Gamma,S,2}\\ &\le C\|v(\tau,\cdot)\|^{2\alpha-1}_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|^2_{\Gamma,S,2}\\ &\le C(1+\tau)^{-\frac{2\alpha-1}{2}+1}E^{2\alpha+1}.\end{aligned} \qquad (10.4.52)$$

Substituting (10.4.51)–(10.4.52) into (10.4.50), we get

$$\begin{aligned}\|u_3(t,\cdot)\|_{\Gamma,S,2} &\le C(1+t)^{\frac{1}{3}}\{\varepsilon + E^{2\alpha+1}\int_0^t (1+\tau)^{-\frac{2\alpha-1}{2}+1-\frac{1}{3}}d\tau\}\\ &\le C(1+t)^{\frac{1}{2}}\{\varepsilon + E^{2\alpha+1}\int_0^t (1+\tau)^{-\alpha+1}d\tau\}.\end{aligned} \qquad (10.4.53)$$

By (10.4.8), when $\alpha = 1$ we have

$$E^{2\alpha}\int_0^t (1+\tau)^{-\alpha+1}d\tau \le E^2(1+t) \le R^2(E,T);$$

while, when $\alpha = 2$ we have

$$E^{2\alpha}\int_0^t (1+\tau)^{-\alpha+1}d\tau \le E^4 \ln(1+t) \le R^2(E,T).$$

Hence, from (10.4.53) we get

$$\|u_3(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\frac{1}{2}}(\varepsilon + R^2(E,T)E). \tag{10.4.54}$$

Combining (10.4.30), (10.4.49) and (10.4.54), it follows from (10.4.15) that

$$\|u(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\frac{1}{2}}\{\varepsilon + (R^2(E,T)+R(E,T))(E+D_{S,T}(u))\}. \tag{10.4.55}$$

Second, we estimate $\|u(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}$.

By Corollary 4.6.4 in Chap. 4 (in which we take $n=2$, $N=[\frac{S}{2}]+1$, thus when $S \ge 8$, $N+n+1 \le S$), u_1, as the solution of Eq. (10.4.16) with the zero initial condition, should satisfy

$$\begin{aligned}\|u_1(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le C(1+t)^{-\frac{1}{2}}\Big\{\varepsilon + \sum_{i=0}^{2}\int_0^t \Big((1+\tau)^{\frac{1}{2}}\|\widehat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \\ +(1+\tau)^{-\frac{3}{2}}\|\widehat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,S,1}\Big)d\tau\Big\}.\end{aligned} \tag{10.4.56}$$

Similarly to (10.4.26), noticing Lemma 10.4.2, the definition of $X_{S,E,T}$ and the estimates similar to (10.4.27), we have

$$\begin{aligned}\sum_{i=0}^{2}\|\widehat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} &\le C\|v(\tau,\cdot)\|^{\alpha}_{\Gamma,[\frac{S}{2}]+1,\infty}\|(v,Du)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \\ &\le C(1+\tau)^{-\frac{\alpha+1}{2}}E^{\alpha}(E+D_{S,T}(u))\end{aligned} \tag{10.4.57}$$

and

$$\begin{aligned}\sum_{i=0}^{2}\|\widehat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,S,1} &\le C\|v(\tau,\cdot)\|^{\alpha-1}_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|_{\Gamma,S,2}(\|Du(\tau,\cdot)\|_{\Gamma,S,2}+\|v(\tau,\cdot)\|_{\Gamma,S,2}) \\ &\le C(1+\tau)^{-\frac{\alpha-1}{2}+1}E^{\alpha}(E+D_{S,T}(u)).\end{aligned} \tag{10.4.58}$$

Thus, noticing (10.4.8), from (10.4.56) we obtain

$$\|u_1(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le C(1+t)^{-\frac{1}{2}}\{\varepsilon + R(E,T)(E+D_{S,T}(u))\}. \quad (10.4.59)$$

Similarly, according to Corollary 4.6.3 in Chap. 4, u_2, as the solution of Eq. (10.4.17) with the same initial value (10.1.15) as u, should satisfy

$$\|u_2(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le C(1+t)^{-\frac{1}{2}}\{\varepsilon + \int_0^t (1+\tau)^{-\frac{1}{2}}\|Q(v,Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,1}d\tau\}. \quad (10.4.60)$$

Utilizing the estimates about composite functions in Chap. 5, and noting (10.4.13), we can estimate $\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,1}$.

In fact, when $\alpha=1$, by the definition of $X_{S,E,T}$, it is easy to have

$$\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,1} \le C\|Dv(\tau,\cdot)\|_{\Gamma,S,2}\|Du(\tau,\cdot)\|_{\Gamma,S,2} \le CED_{S,T}(u); \quad (10.4.61)$$

while, when $\alpha=2$, noting (10.4.36) and the definition of $X_{S,E,T}$, and using Lemma 10.4.2, it is easy to obtain

$$\begin{aligned}
&\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,1}\\
&\le C\|vDv(\tau,\cdot)\|_{\Gamma,S,2}\|Du(\tau,\cdot)\|_{\Gamma,S,2}\\
&\le C_\rho\left(\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|Dv(\tau,\cdot)\|_{\Gamma,S,2} + \|Dv(\tau,\cdot)\|_{\Gamma,S,2}\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\right)\|Du(\tau,\cdot)\|_{\Gamma,S,2}\\
&\le CE^2(1+\tau)^{-\frac{1}{2}}D_{S,T}(u). \quad (10.4.62)
\end{aligned}$$

Combining (10.4.61) and (10.4.62), when $\alpha=1$ and 2, we have

$$\|\widehat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,1} \le C(1+\tau)^{-\frac{\alpha-1}{2}}E^\alpha D_{S,T}(u). \quad (10.4.63)$$

Utilizing the estimates about composite functions in Chap. 5, and noting (10.4.14), when $\alpha=1$ and 2, we obtain, similarly,

$$\|\widehat{B}_{ijm}(v,Dv)v_{x_i}u_{x_jx_m}(\tau,\cdot)\|_{\Gamma,S,1} \le C(1+\tau)^{-\frac{\alpha-1}{2}}E^\alpha D_{S,T}(u) \quad (10.4.64)$$

and

$$\|\widehat{C}_{ij}(v,Dv)v_{x_i}v_{x_j}(\tau,\cdot)\|_{\Gamma,S,1} \le C(1+\tau)^{-\frac{\alpha-1}{2}}E^{\alpha+1}. \quad (10.4.65)$$

Combining (10.4.63)–(10.4.65), we get

$$\|Q(v,Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,1} \le C(1+\tau)^{-\frac{\alpha-1}{2}}E^\alpha(E+D_{S,T}(u)). \quad (10.4.66)$$

Thus, from (10.4.60) and noting (10.4.8), it follows that

$$\|u_2(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le C(1+t)^{-\frac{1}{2}}\{\varepsilon + R(E,T)(E+D_{S,T}(u))\}. \quad (10.4.67)$$

Similarly to (10.4.60), we have

$$\|u_3(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le C(1+t)^{-\frac{1}{2}}\{\varepsilon + \int_0^t (1+\tau)^{-\frac{1}{2}}\|F(v)(\tau,\cdot)\|_{\Gamma,S,1}d\tau\}. \tag{10.4.68}$$

Utilizing the estimates about composite functions in Chap. 5, and noting (10.4.12), Lemma 10.4.2 and the definition of $X_{S,E,T}$, we get

$$\begin{aligned}\|F(v)(\tau,\cdot)\|_{\Gamma,S,1} &\le C\|v(\tau,\cdot)\|^{2\alpha-1}_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|^2_{\Gamma,S,2}\\ &\le C(1+\tau)^{-\frac{2\alpha-1}{2}+1}E^{2\alpha+1},\end{aligned} \tag{10.4.69}$$

thus, by (10.4.68) and noting (10.4.8), it is easy to deduce

$$\|u_3(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le C(1+t)^{-\frac{1}{2}}(\varepsilon + R^2(E,T)E). \tag{10.4.70}$$

Combining (10.4.59), (10.4.67) and (10.4.70), from (10.4.15) we get

$$\|u(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le C(1+t)^{-\frac{1}{2}}\{\varepsilon + (R^2(E,T)+R(E,T))(E+D_{S,T}(u))\}. \tag{10.4.71}$$

Finally, we estimate $\|D^2u(t,\cdot)\|_{\Gamma,S,2}$ and $\|Du(t,\cdot)\|_{\Gamma,S,2}$, respectively.

We still have (10.2.53)–(10.2.55). Similarly to (10.3.45), when $\alpha = 1$ and 2 we have

$$|\mathrm{I}|, |\mathrm{II}|, |\mathrm{III}| \le CR(E,T)D^2_{S,T}(u). \tag{10.4.72}$$

Now we estimate the L^2 norm of $G_k(\tau,\cdot)$.

Similarly to (10.3.46), we have

$$\|b_{ij}(v,Dv)(\Gamma^k Du_{x_ix_j} - (\Gamma^k Du)_{x_ix_j})(\tau,\cdot)\|_{L^2(I\!R^2)} \le C(1+\tau)^{-\frac{\alpha}{2}}E^\alpha D_{S,T}(u). \tag{10.4.73}$$

At this moment, (10.3.47) still holds, namely,

$$\begin{aligned}&\|(\Gamma^k D(b_{ij}(v,Dv)u_{x_ix_j}) - b_{ij}(v,Dv)\Gamma^k Du_{x_jx_i})(\tau,\cdot)\|_{L^2(I\!R^2)}\\ \le & C\left\{(1+\tau)^{-\frac{1}{2}}\|Db_{ij}(v,Dv)(\tau,\cdot)\|_{\Gamma,S,2} + \|b_{ij}(v,Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\right\}\|D^2u(\tau,\cdot)\|_{\Gamma,S,2}.\end{aligned} \tag{10.4.74}$$

It is easy to know, from Lemma 10.4.2, that

$$\|b_{ij}(v,Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \le C(1+\tau)^{-\frac{\alpha}{2}}E^\alpha. \tag{10.4.75}$$

By the estimates about composite functions in Chap. 5, and using Lemma 10.4.2 and the definition of $X_{S,E,T}$, when $\alpha = 1$ we have

$$\|Db_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,S,2} \le C\|(Dv, D^2v)(\tau, \cdot)\|_{\Gamma,S,2} \le CE; \tag{10.4.76}$$

while, when $\alpha = 2$ we have

$$\begin{aligned}&\|Db_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,S,2}\\ \le\ & C(\|v(Dv, D^2v)(\tau, \cdot)\|_{\Gamma,S,2} + \|Dv(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|(Dv, D^2v)(\tau, \cdot)\|_{\Gamma,S,2})\\ \le\ & C(\|v(Dv, D^2v)(\tau, \cdot)\|_{\Gamma,S,2} + (1+\tau)^{-\frac{1}{2}}E^2),\end{aligned} \tag{10.4.77}$$

then, noticing that, similarly to (10.4.36), we have

$$\begin{aligned}&\|v(Dv, D^2v)(\tau, \cdot)\|_{\Gamma,S,2}\\ \le\ & C_\rho\{\|v(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|(Dv, D^2v)(\tau, \cdot)\|_{\Gamma,S,2} + \|Dv(\tau, \cdot)\|_{\Gamma,S,2}\|(Dv, D^2v)(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\}\\ \le\ & C(1+\tau)^{-\frac{1}{2}}E^2,\end{aligned} \tag{10.4.78}$$

we finally get

$$\|Db_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,S,2} \le C(1+\tau)^{-\frac{1}{2}}E^2. \tag{10.4.79}$$

Combining (10.4.76) and (10.4.79), we then have

$$\|Db_{ij}(v, Dv)(\tau, \cdot)\|_{\Gamma,S,2} \le C(1+\tau)^{-\frac{\alpha-1}{2}}E^\alpha. \tag{10.4.80}$$

Hence, from (10.4.74) we get

$$\|(\Gamma^k D(b_{ij}(v, Dv)u_{x_ix_j}) - b_{ij}(v, Dv)\Gamma^k Du_{x_jx_i})(\tau, \cdot)\|_{L^2(\mathbb{R}^2)} \le C(1+\tau)^{-\frac{\alpha}{2}}E^\alpha D_{S,T}(u). \tag{10.4.81}$$

We can estimate the terms involving a_{0j} in G_k similarly. Then, we obtain

$$\|G_k(\tau, \cdot)\|_{L^2(\mathbb{R}^2)} \le C(1+\tau)^{-\frac{\alpha}{2}}E^\alpha D_{S,T}(u). \tag{10.4.82}$$

So, noting (10.4.8), we have

$$|\mathrm{IV}| \le CR(E, T)D_{S,T}(u). \tag{10.4.83}$$

Now we estimate the L^2 norm of $g_k(\tau, \cdot)$.
To this end, we write g_k, given by (10.2.55), as

$$g_k = g_k^{(1)} + g_k^{(2)}, \tag{10.4.84}$$

where

$$g_k^{(2)} = \sum_{|l|\le|k|} \widetilde{B}_{kl}\Gamma^l \widetilde{F}(v, 0, 0), \tag{10.4.85}$$

and $g_k^{(1)}$ stands for other terms.

Using similar arguments for dealing with $Db_{ij}(v, Dv)$ to estimating the L^2 norm of G_k, we obtain

$$\|g_k^{(1)}(\tau, \cdot)\|_{L^2(R^2)} \le C(1+\tau)^{-\frac{\alpha}{2}} E^\alpha (E + D_{S,T}(u)). \tag{10.4.86}$$

Noting $\widetilde{F}(v, 0, 0) = F(v, 0) = F(v)$ and (10.4.12), by Lemma 10.4.2 and the definition of $X_{S,E,T}$, it is easy to show that

$$\begin{aligned} \|g_k^{(2)}(\tau, \cdot)\|_{L^2(\mathbb{R}^2)} &\le C\|v(\tau, \cdot)\|^{2\alpha}_{\Gamma,[\frac{S}{2}],\infty} \|v(\tau, \cdot)\|_{\Gamma,S,2} \\ &\le C(1+\tau)^{-\alpha+\frac{1}{2}} E^{2\alpha+1}. \end{aligned} \tag{10.4.87}$$

From (10.4.86)–(10.4.87) and noting (10.4.8), we get

$$|V| \le C(R^2(E, T) + R(E, T))(E + D_{S,T}(u))D_{S,T}(u). \tag{10.4.88}$$

Thus, combining (10.4.72), (10.4.83) and (10.4.88), similarly to the arguments in Sects. 10.2 and 10.3, we then obtain

$$\sup_{0\le t\le T} \|D^2u(t, \cdot)\|_{\Gamma,S,2} \le C\{\varepsilon + (R(E, T) + \sqrt{R(E, T)})(E + D_{S,T}(u))\}. \tag{10.4.89}$$

By (10.2.65), similarly we have

$$\sup_{0\le t\le T} \|Du(t, \cdot)\|_{\Gamma,S,2} \le C\{\varepsilon + (R(E, T) + \sqrt{R(E, T)})(E + D_{S,T}(u))\}. \tag{10.4.90}$$

Combining (10.4.55), (10.4.71) and (10.4.89)–(10.4.90), and noting the definition of $X_{S,E,T}$, Lemma 10.4.3 is proved.

Chapter 11
Cauchy Problem of Four-Dimensional Nonlinear Wave Equations

11.1 Introduction

In this chapter we furthermore consider the following Cauchy problem of four-dimensional nonlinear wave equations with small initial data:

$$\Box u = F(u, Du, D_x Du), \tag{11.1.1}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{11.1.2}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2} \tag{11.1.3}$$

is the four-dimensional wave operator,

$$D_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_4}\right), D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_4}\right), \tag{11.1.4}$$

φ and ψ are sufficiently smooth functions with compact support, we may assume that

$$\varphi, \psi \in C_0^\infty(I\!R^4) \tag{11.1.5}$$

with

$$\text{supp}\,\{\varphi, \psi\} \subseteq \{x\ ||x| \le \rho\}\ \ (\rho > 0 \text{ is a constant}), \tag{11.1.6}$$

and $\varepsilon > 0$ is a small parameter.

Denote

$$\hat{\lambda} = \left(\lambda; (\lambda_i), i = 0, 1, \cdots, 4; (\lambda_{ij}), i, j = 0, 1, \cdots, 4, i + j \ge 1\right). \tag{11.1.7}$$

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Suppose that in a neighborhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \le \nu_0$, the nonlinear term $F(\hat{\lambda})$ is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^2). \tag{11.1.8}$$

In Chap. 9, we already proved that: there exists a suitably small positive number ε_0 such that for any given $\varepsilon \in (0, \varepsilon_0]$, the life-span $\widetilde{T}(\varepsilon)$ of the classical solution $u = u(t, x)$ to Cauchy problem (11.1.1)–(11.1.2) satisfies the following lower bound estimate:

$$\widetilde{T}(\varepsilon) \ge \exp\{a\varepsilon^{-1}\}, \tag{11.1.9}$$

where a is a positive constant independent of ε (see Hörmander 1991). But then we already pointed out that this result will be improved as

$$\widetilde{T}(\varepsilon) \ge \exp\{a\varepsilon^{-2}\}. \tag{11.1.10}$$

In fact, this improved result was first given by Li Tatsien and Zhou Yi (1995b, 1995c), then, Lindblad and Sogge (1996) simplified the related proof. In this chapter, we will give a much simpler proof to obtain estimate (11.1.10) by using a new L^2 estimate (see Sect. 4.4 in Chap. 4) based on results in Hidano et al. (2009), which was established for solutions of the wave equation.

Thanks to the counter-example given by H. Takamura and K. Wakasa in recent years (see Takamura and Wakasa 2011), (11.1.10), same as other results in previous chapters, is sharp. See Chap. 14 for details.

Due to Chap. 7, to prove (11.1.10) for Cauchy problem (11.1.1)–(11.1.2) of four-dimensional nonlinear wave equations, essentially it suffices to consider the following Cauchy problem of four-dimensional second-order quasi-linear hyperbolic equations:

$$\Box u = \sum_{i,j=1}^{4} b_{ij}(u, Du)u_{x_i x_j} + 2\sum_{j=1}^{4} a_{0j}(u, Du)u_{t x_j} + F(u, Du), \tag{11.1.11}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{11.1.12}$$

where, $\varphi, \psi \in C_0^\infty(I\!R^4)$ still satisfy condition (11.1.6), and $\varepsilon > 0$ is a small parameter.

Let

$$\widetilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \cdots, 4)\,. \tag{11.1.13}$$

Assume that when $|\widetilde{\lambda}| \le \nu_0$, $b_{ij}(\widetilde{\lambda}), a_{0j}(\widetilde{\lambda})$ and $F(\widetilde{\lambda})$ are all sufficiently smooth functions satisfying

$$b_{ij}(\widetilde{\lambda}) = b_{ji}(\widetilde{\lambda}) \quad (i, j = 1, \cdots, 4), \tag{11.1.14}$$
$$b_{ij}(\widetilde{\lambda}), a_{0j}(\widetilde{\lambda}) = O(|\widetilde{\lambda}|) \quad (i, j = 1, \cdots, 4), \tag{11.1.15}$$
$$F(\widetilde{\lambda}) = O(|\widetilde{\lambda}|^2) \tag{11.1.16}$$

and

$$\sum_{i,j=1}^{4} a_{ij}(\widetilde{\lambda})\xi_i\xi_j \geq m_0|\xi|^2, \quad \forall\, \xi \in I\!R^4, \tag{11.1.17}$$

where, m_0 is a positive constant, and

$$a_{ij}(\widetilde{\lambda}) = \delta_{ij} + b_{ij}(\widetilde{\lambda}) \quad (i, j = 1, \cdots, 4), \tag{11.1.18}$$

in which δ_{ij} is the Kronecker symbol.

11.2 Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (11.1.11)–(11.1.12)

11.2.1 Metric Space $X_{S,E,T}$. Main Results

According to the Sobolev embedding theorem, there exists a suitably small $E_0 > 0$, such that

$$\|f\|_{L^\infty(I\!R^4)} \leq \nu_0, \ \forall f \in H^3(I\!R^4), \|f\|_{H^3(I\!R^4)} \leq E_0. \tag{11.2.1}$$

For any given integer $S \geq 8$, and for any given positive numbers $E(\leq E_0)$ and T,introduce the set of functions

$$X_{S,E,T} = \left\{ v(t,x) \middle| \, D_{S,T}(v) \leq E, \partial_t^l v(0,x) = u_l^{(0)}(x) \quad (l = 0, 1, \cdots, S+1) \right\}, \tag{11.2.2}$$

where

$$D_{S,T}(v) = \sum_{i=0}^{2} \sup_{0 \leq t \leq T} \|D^i v(t,\cdot)\|_{\Gamma,S,2}, \tag{11.2.3}$$

while, $u_0^{(0)}(x) = \varepsilon\varphi(x)$, $u_1^{(0)}(x) = \varepsilon\psi(x)$, and when $l = 2, \cdots, S+1$, $u_l^{(0)}(x)$ are values of $\partial_t^l u(t,x)$ at $t = 0$, which are uniquely determined by Eq. (11.1.11) and initial condition (11.1.12). Obviously, $u_l^{(0)}(x) (l = 0, 1 \cdots, S+1)$ are all sufficiently smooth functions with compact support (see (11.1.6)).

Similarly to the previous three chapters, it is easy to prove

Lemma 11.2.1 *Introduce the following metric in $X_{S,E,T}$:*

$$\rho(\overline{v}, \overline{\overline{v}}) = D_{S,T}(\overline{v} - \overline{\overline{v}}), \quad \forall \overline{v}, \overline{\overline{v}} \in X_{S,E,T}, \tag{11.2.4}$$

then when $\varepsilon > 0$ is suitably small, $X_{S,E,T}$ is a non-empty complete metric space.

Lemma 11.2.2 *When $S \geq 8$, for any given $v \in X_{S,E,T}$, we have*

$$\|(v, Dv, D^2 v)(t, \cdot)\|_{\Gamma, [\frac{S}{2}]+1, \infty} \leq CE(1+t)^{-\frac{3}{2}}, \quad \forall t \in [0, T], \tag{11.2.5}$$

where C is a positive constant.

Proof When $S \geq 8$, from (3.4.30) in Chap. 3 (in which we take $n = 4$, $p = 2$, $N = \left[\frac{S}{2}\right] + 1$, and $s = 3$), and noting the definition of $X_{S,E,T}$, (11.2.5) follows immediately.

Denote by $\widetilde{X}_{S,E,T}$ a subset of $X_{S,E,T}$, which is composed of all the elements in $X_{S,E,T}$ with compact support with respect to x included in $\{x \mid |x| \leq t + \rho\}$ for any given $t \in [0, T]$, and $\rho > 0$ is the constant appearing in (11.1.6).

The main result of this chapter is the following

Theorem 11.2.1 *Let $n = 4$. Under assumptions (11.1.5)–(11.1.6) and (11.1.14)–(11.1.18), for any given integer $S \geq 8$, there exist positive constants ε_0 and C_0 satisfying $C_0\varepsilon_0 \leq E_0$, and for any given $\varepsilon \in (0, \varepsilon_0]$, there exists a positive number $T(\varepsilon)$ such that Cauchy problem (11.1.11)–(11.1.12) admits a unique classical solution $u \in \widetilde{X}_{S, C_0\varepsilon_0, T(\varepsilon)}$ on $[0, T(\varepsilon)]$, and*

$$T(\varepsilon) = \exp\{a\varepsilon^{-2}\} - 2, \tag{11.2.6}$$

where a is a positive constant independent of ε.

Moreover, after a possible change of values for t on a zero-measure set of $[0, T(\varepsilon)]$, we have

$$u \in C\left([0, T(\varepsilon)]; H^{S+1}(\mathbb{R}^4)\right), \tag{11.2.7}$$

$$u_t \in C\left([0, T(\varepsilon)]; H^{S}(\mathbb{R}^4)\right), \tag{11.2.8}$$

$$u_{tt} \in C\left([0, T(\varepsilon)]; H^{S-1}(\mathbb{R}^4)\right). \tag{11.2.9}$$

11.2.2 Framework to Prove Theorem 11.2.1—The Global Iteration Method

To prove Theorem 11.2.1, for any given $v \in \widetilde{X}_{S,E,T}$, same as in the previous three chapters, by solving the following Cauchy problem of linear hyperbolic equations:

$$\Box u = \hat{F}(v, Dv, D_x Du) \stackrel{\text{def.}}{=} \sum_{i,j=1}^{4} b_{ij}(v, Dv)u_{x_i x_j} + 2\sum_{j=1}^{4} a_{0j}(v, Dv)u_{tx_j} + F(v, Dv), \tag{11.2.10}$$

$$t = 0 : u = \varepsilon\varphi(x), u_t = \varepsilon\psi(x), \tag{11.2.11}$$

we define a mapping

$$M : \ v \to u = Mv. \tag{11.2.12}$$

We want to prove that: when $\varepsilon > 0$ is suitably small, we can find a positive constant C_0 such that when $E = C_0\varepsilon$ and $T = T(\varepsilon)$ is defined by (11.2.6), M admits a unique fixed point in $\widetilde{X}_{S,E,T}$, which is exactly the classical solution to Cauchy problem (11.1.11)–(11.1.12) on $0 \le t \le T(\varepsilon)$.

Similarly to the previous three chapters, it is easy to prove the following two lemmas.

Lemma 11.2.3 *When $E > 0$ is suitably small, for any given $v \in \widetilde{X}_{S,E,T}$, after a possible change of values for t on a zero-measure set, we have*

$$u = Mv \in C\left([0, T]; H^{S+1}(I\!R^4)\right), \tag{11.2.13}$$

$$u_t \in C\left([0, T]; H^{S}(I\!R^4)\right), \tag{11.2.14}$$

$$u_{tt} \in L^{\infty}\left(0, T; H^{S-1}(I\!R^4)\right). \tag{11.2.15}$$

Lemma 11.2.4 *For $u = u(t, x) = Mv$, the values of $\partial_t^l u(0, \cdot)$ $(l = 0, 1, \cdots, S + 2)$ do not depend on the choice of $v \in \widetilde{X}_{S,E,T}$, and*

$$\partial_t^l u(0, x) = u_l^{(0)}(x) \quad (l = 0, 1, \cdots, S + 1). \tag{11.2.16}$$

Moreover,

$$\|u(0, \cdot)\|_{\Gamma,S+2,p} \le C\varepsilon, \tag{11.2.17}$$

where $1 \le p \le +\infty$, and C is a positive constant.

Similarly as in the previous three chapters, the crucial points to prove Theorem 11.2.1 are the following two lemmas.

Lemma 11.2.5 *Under the assumptions of Theorem 11.2.1, when $E > 0$ is suitably small, for any given $v \in \widetilde{X}_{S,E,T}$, $u = Mv$ satisfies*

$$D_{S,T}(u) \le C_1\left\{\varepsilon + (R + \sqrt{R})(E + D_{S,T}(u))\right\}, \tag{11.2.18}$$

where C_1 is a positive constant, and

$$R = R(E, T) \stackrel{\text{def.}}{=} E\sqrt{\ln(2 + T)}. \tag{11.2.19}$$

Lemma 11.2.6 *Under the assumptions of Lemma 11.2.5, for any given $\overline{v}, \overline{\overline{v}} \in \widetilde{X}_{S,E,T}$, if both $\overline{u} = M\overline{v}$ and $\overline{\overline{u}} = M\overline{\overline{v}}$ satisfy $\overline{u}, \overline{\overline{u}} \in \widetilde{X}_{S,E,T}$, then we have*

$$D_{S-1,T}(\overline{u} - \overline{\overline{u}}) \le C_2(R + \sqrt{R})(D_{S-1,T}(\overline{u} - \overline{\overline{u}}) + D_{S-1,T}(\overline{v} - \overline{\overline{v}})), \quad (11.2.20)$$

where C_2 is a positive constant, and $R = R(E, T)$ is still defined by (11.2.19).

11.2.3 Proof of Lemmas 11.2.5 and 11.2.6

Now we prove Lemma 11.2.5. Lemma 11.2.6 can be proved similarly, we omit the details here.

The key point is to estimate $\|u(t, \cdot)\|_{\Gamma,S,2}$.

For any given multi-index $k(|k| \le S)$, similarly to (10.2.27) in Chap. 10, we have

$$\Box \Gamma^k u = \sum_{|l| \le |k|} C_{kl} \Gamma^l \hat{F}(v, Dv, D_x Du), \quad (11.2.21)$$

where C_{kl} are constants, and the initial values satisfied by $\Gamma^k u$ can be determined uniquely by $u_l^{(0)}(x)$ $(l = 0, 1, \cdots, S+1)$. Thus, according to Theorem 4.4.1 in Chap. 4 (in which we take $n = 4$, $s = \frac{3}{4}$, and then from $\frac{1}{q} = \frac{1}{2} + \frac{\frac{3}{2} - s}{n}$ we can determine $q = \frac{16}{11}$), it is easy to know that

$$\|u(t, \cdot)\|_{\Gamma,S,2} \le C_\rho \left\{ \varepsilon + \left(\int_0^t (1+\tau)^{\frac{3}{2}} \|\hat{F}(v, Dv, D_x Du)(\tau, \cdot)\|^2_{\Gamma, S, \frac{16}{11}, \chi_1} d\tau \right)^{\frac{1}{2}} \right.$$
$$\left. + \left(\int_0^t (1+\tau)^{-1} \|\hat{F}(v, Dv, D_x Du)(\tau, \cdot)\|^2_{\Gamma, S, 1, 2, \chi_2} d\tau \right)^{\frac{1}{2}} \right\}, \quad (11.2.22)$$

where $\chi_1(t, x)$ is the characteristic function of set $\{(t, x) \mid |x| \le \frac{1+t}{2}\}$, $\chi_2 = 1 - \chi_1$, and C_ρ is a positive constant possibly depending on ρ.

Noticing Lemma 11.2.2, using the estimates about product functions and composite functions in Chap. 5, it is easy to show that

$$\|\hat{F}(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma, S, \frac{16}{11}, \chi_1} \le C \left\{ \|(v, Dv)(\tau, \cdot)\|_{\Gamma, [\frac{S}{2}], \frac{16}{3}, \chi_1} \|(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma, S, 2} \right.$$
$$\left. + \|(v, Dv)(\tau, \cdot)\|_{\Gamma, S, 2} \|D_x Du(\tau, \cdot)\|_{\Gamma, [\frac{S}{2}], \frac{16}{3}, \chi_1} \right\}. \quad (11.2.23)$$

Due to 2° in Corollary 3.4.1 of Chap. 3 (in which we take $n = 4$, $p = 2$, $N = \left[\frac{S}{2}\right]$, $q = \frac{16}{3}$, and $s = \frac{n}{p} = 2$), it is clear that

$$\|(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],\frac{16}{3},\chi_1} \le C(1+\tau)^{-\frac{5}{4}}\|(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S,2}. \tag{11.2.24}$$

Hence, noticing the definition of $X_{S,E,T}$, from (11.2.23) we get

$$\|\hat{F}(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S,\frac{16}{11},\chi_1} \le C(1+\tau)^{-\frac{5}{4}} E(E + D_{S,T}(u)). \tag{11.2.25}$$

Similarly to (11.2.23), we have

$$\begin{aligned}\|\hat{F}(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S,1,2,\chi_2} \le C \Big\{&\|(v, Dv)(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2}\|(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S,2} \\ &+\|(v, Dv)(\tau, \cdot)\|_{\Gamma,S,2}\|D_x Du(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2}\Big\}.\end{aligned} \tag{11.2.26}$$

Due to the Sobolev embedding theorem on a sphere (see 1° in Theorem 3.2.1 of Chap. 3, in which we take $n = 4$, $p = 2$, $s = 2$), it is easy to know that

$$\begin{aligned}\|(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty} &\le C\|(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,[\frac{S}{2}]+2,2} \\ &\le C\|(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S,2}.\end{aligned} \tag{11.2.27}$$

Hence, noticing the definition of $X_{S,E,T}$, from (11.2.26) we get

$$\|\hat{F}(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S,1,2,\chi_2} \le CE(E + D_{S,T}(u)). \tag{11.2.28}$$

Plugging (11.2.25) and (11.2.28) in (11.2.22), we have

$$\begin{aligned}\|u(t, \cdot)\|_{\Gamma,S,2} &\le C\left\{\varepsilon + \left(\int_0^t (1+\tau)^{-1} d\tau\right)^{\frac{1}{2}} E(E + D_{S,T}(u))\right\} \\ &= C\left\{\varepsilon + \sqrt{\ln(1+t)}E(E + D_{S,T}(u))\right\},\end{aligned} \tag{11.2.29}$$

then, noting (11.2.19), we get

$$\sup_{0\le t\le T} \|u(t, \cdot)\|_{\Gamma,S,2} \le C\left\{\varepsilon + R(E, T)(E + D_{S,T}(u))\right\}. \tag{11.2.30}$$

Now we estimate $\|(Du, D^2u)(t, \cdot)\|_{\Gamma,S,2}$.

For any given multi-index $k(|k| \le S)$, the energy integral formula (9.2.41) in Chap. 9 still holds.

From Lemma 11.2.2, it is easy to show, for terms therein, that

$$
\begin{aligned}
|\mathrm{I}|, |\mathrm{II}|, |\mathrm{III}| &\le CE\int_0^t (1+\tau)^{-\frac{3}{2}}\mathrm{d}\tau \cdot D^2_{S,T}(u)\\
&\le CED^2_{S,T}(u) \le CR(E,T)D^2_{S,T}(u). \qquad (11.2.31)
\end{aligned}
$$

Now we estimate the L^2 norm of $G_k(\tau,\cdot)$ therein.

Similarly to (9.2.43) in Chap. 9, noting (11.1.15), Lemma 11.2.2 and the definition of $X_{S,E,T}$, from Lemmas 5.2.5 and 5.2.6 in Chap. 5 (in which we take $n=4$, and $r=2$, then $p=+\infty$) we get

$$
\begin{aligned}
&\|(\Gamma^k D(b_{ij}(v,Dv)u_{x_ix_j}) - b_{ij}(v,Dv)\Gamma^k Du_{x_ix_j})(\tau,\cdot)\|_{L^2(I\!R^4)}\\
&\le C(1+\tau)^{-\frac{3}{2}}\|(v,Dv,D^2v(\tau,\cdot)\|_{\Gamma,S,2}\|D^2u(\tau,\cdot)\|_{\Gamma,S,2}\\
&\le C(1+\tau)^{-\frac{3}{2}}ED_{S,T}(u), \quad \forall\tau\in[0,T]. \qquad (11.2.32)
\end{aligned}
$$

Moreover, similarly to (9.2.44) in Chap. 9, noticing (11.1.15), Lemma 11.2.2 and the definition of $X_{S,E,T}$, and using Corollary 3.1.1 in Chap. 3, we then have

$$
\|b_{ij}(v,Dv)(\Gamma^k Du_{x_ix_j} - (\Gamma_k Du)_{x_ix_j})(\tau,\cdot)\|_{L^2(I\!R^4)} \le C(1+\tau)^{-\frac{3}{2}}ED_{S,T}(u), \quad \forall\tau\in[0,T]. \qquad (11.2.33)
$$

For terms involving a_{0j} in G_k, we have similar estimates. Therefore, we have

$$
\|G_k(\tau,\cdot)\|_{L^2(I\!R^4)} \le C(1+\tau)^{-\frac{3}{2}}ED_{S,T}(u), \quad \forall\tau\in[0,T], \qquad (11.2.34)
$$

then we get

$$
|\mathrm{IV}| \le CR(E,T)D^2_{S,T}(u). \qquad (11.2.35)
$$

Similarly, we obtain

$$
|\mathrm{V}| \le CR(E,T)(E+D_{S,T}(u))D_{S,T}(u). \qquad (11.2.36)
$$

So, similarly to (9.2.51) in Chap. 9, we get

$$
\sup_{0\le t\le T}\|D^2u(t,\cdot)\|_{\Gamma,S,2} \le C\{\varepsilon + \sqrt{R(E,T)}(E+D_{S,T}(u))\}. \qquad (11.2.37)
$$

Moreover, similarly to (9.2.52) in Chap. 9, we also have

$$
\sup_{0\le t\le T}\|Du(t,\cdot)\|_{\Gamma,S,2} \le C\{\varepsilon + \sqrt{R(E,T)}(E+D_{S,T}(u))\}. \qquad (11.2.38)
$$

Combining (11.2.30) and (11.2.37)–(11.2.38), we obtain the desired (11.2.18). The proof of Lemma 11.2.5 is finished.

Chapter 12
Null Condition and Global Classical Solutions to the Cauchy Problem of Nonlinear Wave Equations

12.1 Introduction

In the previous chapters, we already studied systematically the classical solution and the lower bound estimates of its life-span for the Cauchy problem of nonlinear wave equations with small initial data. It is shown by the results therein that: when the space dimension n and the power $1 + \alpha$ (or α) of the nonlinear terms on the right-hand side are large enough, we can obtain the **global classical solution**; otherwise, we only obtain the local classical solution. In the latter case, when n and α are still quite large, as $\varepsilon \to 0$ (ε is the order of initial data), the life-span $\widetilde{T}(\varepsilon)$ of the classical solution will increase exponentially with respect to ε^{-1}, so it is called the **almost global classical solution**; while, when n and α are quite small, as $\varepsilon \to 0$, the life-span $\widetilde{T}(\varepsilon)$ of the classical solution will increase only in the power of ε^{-1}, which will result in a blow-up phenomenon in practical applications.

It is also shown in the previous chapters that: when the nonlinear term $F(u, Du, D_x Du)$ on the right-hand side satisfies some special conditions, especially in the case that the nonlinear term on the right-hand side does not depend on u explicitly, the estimate on the life-span $\widetilde{T}(\varepsilon)$ of the classical solution will be somewhat or even significantly improved, for instance, it can be improved to the exponential growth from the power growth of ε^{-1} (see the case that $n = 3$ and $\alpha = 1$, and the case that $n = 2$ and $\alpha = 2$), or, it can be improved to the global existence from the exponential growth (see the case that $n = 4$ and $\alpha = 1$); But in the case that $n = 1$ and $\alpha \geq 1$ and in the case that $n = 2$ and $\alpha = 1$, the improvement is not obvious (for example, the latter is only improved to the power ε^{-2} from the power ε^{-1}).

The above discussion is about the general nonlinear term $F(u, Du, D_x Du)$ on the right-hand side. Just as shown in Sect. 1.2 of Chap. 1, even if the global existence of classical solutions is not ensured for general nonlinear term $F(u, Du, D_x Du)$ on the right-hand side, it is still possible to obtain the global classical solution for some specially chosen nonlinear terms on the right-hand side, in particular, when there exists a certain consistency between the nonlinear term on the right-hand side and

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the wave operator. The null condition to be introduced in this chapter is a kind of important additional requirements which is put forward on the nonlinear term on the right-hand side to ensure the global existence of classical solutions, and it has a lot of significant applications.

The notion of null condition was first introduced by S. Klainerman when studying the following Cauchy problem of three-dimensional nonlinear wave equations with small initial data:

$$\Box u = Q(Du, D_x Du), \tag{12.1.1}$$

$$t = 0: \ u = \varepsilon\varphi(x), \ u_t = \varepsilon\psi(x), \tag{12.1.2}$$

where the nonlinear term Q on the right-hand side does not depend on u explicitly, and is a quadratic form of Du and $D_x Du$ (i.e., the corresponding $\alpha = 1$, and it does not include terms of higher degree). Thanks to the results in Chap. 9, the above Cauchy problem admits a unique almost global classical solution on $t \geq 0$ with the life-span $\widetilde{T}(\varepsilon)$ satisfying

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-1}\}, \tag{12.1.3}$$

where a is a positive constant independent of ε. In his talk (see Klainerman 1983) at the international congress of mathematicians held in Warsaw, Poland in 1983, S. Klainerman conjectured that: If Q satisfies the null condition, then Cauchy problem (12.1.1)–(12.1.2) admits a unique global classical solution. This conjecture was verified by Christodoulou (1986) and Klainerman (1986), respectively. We will present the details for this case in Sect. 12.2 of this chapter.

According to Chap. 7, in order to investigate the Cauchy problem of nonlinear wave equations with small initial data, it essentially suffices to consider the following Cauchy problem of second-order quasi-linear hyperbolic equations

$$\begin{aligned}\Box u &= \sum_{i,j=1}^{n} b_{ij}(u, Du)u_{x_i x_j} + 2\sum_{j=1}^{n} a_{0j}(u, Du)u_{t x_j} + F(u, Du) \\ &\stackrel{\text{def.}}{=} \hat{F}(u, Du, D_x Du)\end{aligned} \tag{12.1.4}$$

with small initial data (12.1.2). Therefore, in what follows the discussion regarding the null condition as well as the global existence of classical solutions when the null condition is satisfied will be focused on the quasi-linear wave equations of the form (12.1.4).

In this chapter, we will give the forms of null conditions in the case that $n = 3$ and $\alpha = 1$ and in the case that $n = 2$ and $\alpha = 2$, respectively, and prove accordingly the global existence of corresponding classical solutions. Regarding these two cases, since the global existence of classical solutions is already established for both the case that $n = 3$ and $\alpha \geq 2$ and the case that $n = 2$ and $\alpha \geq 3$, the null condition is only required on the lowest-degree term on the right-hand side (namely, the quadratic term in the case that $n = 3$ and $\alpha = 1$, and the cubic term in the case that $n = 2$ and $\alpha = 2$), while, no any other additional assumptions are needed for the higher-degree terms.

12.2 Null Condition and Global Existence of Classical Solutions to the Cauchy Problem of Three-Dimensional Nonlinear Wave Equations

12.2.1 Null Condition of Three-Dimensional Nonlinear Wave Equations

Consider the three-dimensional nonlinear wave equation

$$\Box u = \hat{F}(u, Du, D_x Du), \tag{12.2.1}$$

where $\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial {x_i}^2}$ is the three-dimensional wave operator, $D = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, $D_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, and

$$\hat{F}(u, Du, D_x Du) = \sum_{i,j=1}^{3} b_{ij}(u, Du) u_{x_i x_j} + 2 \sum_{j=1}^{3} a_{0j}(u, Du) u_{t x_j} + F(u, Du). \tag{12.2.2}$$

Denote

$$\tilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1, 2, 3). \tag{12.2.3}$$

Suppose that in a neighborhood of $\tilde{\lambda} = 0$, say, for $|\tilde{\lambda}| \le \nu_0$, $b_{ij}(\tilde{\lambda})$, $a_{0j}(\tilde{\lambda})$ and $F(\tilde{\lambda})$ are sufficiently smooth functions satisfying

$$b_{ij}(\tilde{\lambda}) = b_{ji}(\tilde{\lambda}) \quad (i, j = 1, 2, 3), \tag{12.2.4}$$

$$b_{ij}(\tilde{\lambda}), a_{0j}(\tilde{\lambda}) = O(|\tilde{\lambda}|) \quad (i, j = 1, 2, 3), \tag{12.2.5}$$

$$F(\tilde{\lambda}) = O(|\tilde{\lambda}|^2) \tag{12.2.6}$$

and

$$\sum_{i,j=1}^{3} a_{ij}(\tilde{\lambda}) \xi_i \xi_j \ge m_0 |\xi|^2, \quad \forall \xi \in I\!R^3, \tag{12.2.7}$$

where, m_0 is a positive constant, and

$$a_{ij}(\tilde{\lambda}) = \delta_{ij} + b_{ij}(\tilde{\lambda}) \quad (i, j = 1, 2, 3), \tag{12.2.8}$$

where δ_{ij} is the Kronecker symbol.

Let

$$\hat{F}(u, Du, D_x Du) = N(u, Du, D_x Du) + H(u, Du, D_x Du), \tag{12.2.9}$$

where $N(u, Du, D_x Du)$ is the quadratic form of involving variables and is affine with respect to $D_x Du$, while $H(u, Du, D_x Du)$ is the term of higher degree. Denote

$$\hat{\lambda} = (\lambda; (\lambda_i),\ i = 0, 1, 2, 3; (\lambda_{ij}),\ i, j = 0, 1, 2, 3, i + j \geq 1). \tag{12.2.10}$$

$H(u, Du, D_x Du)$ satisfies, in a neighborhood of $\hat{\lambda} = 0$,

$$H(\hat{\lambda}) = O(|\hat{\lambda}|^3). \tag{12.2.11}$$

The **null condition**, as an additional condition, will be imposed on the lowest-degree term (the quadratic term) $N(u, Du, D_x Du)$ in $\hat{F}(u, Du, D_x Du)$. It requires that: for three-dimensional homogeneous wave equation

$$\Box u = 0, \tag{12.2.12}$$

its any plane wave solution

$$u(t, x) = U(s), \tag{12.2.13}$$

where

$$s = y_0 t + \sum_{i=1}^{3} y_i x_i \tag{12.2.14}$$

and $y = (y_0, y_1, y_2, y_3)$ is a constant vector, and

$$U(0) = U'(0) = 0, \tag{12.2.15}$$

must be a solution to the three-dimensional nonlinear wave equation

$$\Box u = N(u, Du, D_x Du), \tag{12.2.16}$$

i.e., it satisfies

$$N(U, DU, D_x DU) \equiv 0. \tag{12.2.17}$$

Plugging (12.2.13)–(12.2.14) in (12.2.12), we have

$$(y_0^2 - y_1^2 - y_2^2 - y_3^2)U''(s) = 0. \tag{12.2.18}$$

Noting (12.2.15) we get: to make (12.2.13) to be a nontrivial plane wave solution of (12.2.12), it suffices that the vector $y = (y_0, y_1, y_2, y_3)$ satisfies

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 = 0. \tag{12.2.19}$$

Such a vector y is called the **null vector**. Now, it is easy to know that

$$N(U, DU, D_x DU) = N(U, yU', \tilde{y}yU''), \tag{12.2.20}$$

where $y = (y_0, y_1, y_2, y_3)$ is a null vector, and $\tilde{y} = (y_1, y_2, y_3)$. Therefore, it is said to satisfy the null condition if and only if: for any given null vector y satisfying (12.2.19) and for any given real numbers p, q and r, we have

$$N(p, qy, r\tilde{y}y) = 0. \tag{12.2.21}$$

Denote

$$N_0(f, g) = \partial_0 f \partial_0 g - \sum_{i=1}^{3} \partial_i f \partial_i g \tag{12.2.22}$$

and

$$N_{ab}(f, g) = \partial_a f \partial_b g - \partial_b f \partial_a g. \tag{12.2.23}$$

Hereinafter, $a, b, c = 0, 1, 2, 3$; $i, j, k = 1, 2, 3$, and $\partial_0 = -\frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$. It is easy to verify that: $N_{ab}(\partial_i u, u)$, $N_0(\partial_i u, u)$ and $N_0(u, u)$ (They all do not depend on u explicitly!) all satisfy the above null condition, collectively called the **null form**. Meanwhile, we can prove

Lemma 12.2.1 *If the quadratic form $N(u, Du, D_x Du)$ of u, Du and $D_x Du$ in (12.2.9) is affine with respect to $D_x Du$ and satisfies the null condition, then N must be a linear combination of the null forms $N_{ab}(\partial_i u, u)$, $N_0(\partial_i u, u)$ and $N_0(u, u)$, namely, we have*

$$N(u, Du, D_x u) = \sum_{i,a,b} c_{iab} N_{ab}(\partial_i u, u) + \sum_i c_i N_0(\partial_i u, u) + c N_0(u, u), \tag{12.2.24}$$

where c_{iab}, c_i and c $(a, b = 0, 1, 2, 3; i = 1, 2, 3)$ are all constants.

Proof We first prove that: if the null condition is satisfied, then $N(u, Du, D_x u)$ does not depend on u explicitly.

Taking specially in (12.2.21) $y = 0$, it is clear that $N(u, Du, D_x u)$ does not have terms of u^2. Then we can assume that

$$N(u, Du, D_x Du) = \sum_a d_a u \partial_a u + \sum_{a,i} d_{ai} u \partial_a \partial_i u + \overline{N}(Du, D_x Du), \quad (12.2.25)$$

where d_a and d_{ai} ($a = 0, 1, 2, 3; i = 1, 2, 3$) are constants, and $\overline{N}(Du, D_x Du)$ is a quadratic form of its variables.

Taking specially in (12.2.21) $r = 0$, and p and q are not zero, but $|q|$ is small enough such that tending to zero finally, from (12.2.25) we easily get

$$\sum_a d_a y_a = 0. \quad (12.2.26)$$

Taking the null vector $y = (1, \pm 1, 0, 0)$ in the above formula, we obtain

$$d_0 \pm d_1 = 0,$$

then $d_0 = d_1 = 0$. Similarly we have $d_2 = d_3 = 0$. Then, (12.2.25) can be rewritten as

$$N(u, Du, D_x Du) = \sum_{a,i} d_{ai} u \partial_a \partial_i u + \overline{N}(Du, D_x Du), \quad (12.2.27)$$

and we may assume that

$$d_{ij} = d_{ji} \quad (i, j = 1, 2, 3). \quad (12.2.28)$$

Taking specially in (12.2.21) $q = 0$, and p and r are not zero, from (12.2.27) we immediately have

$$\sum_{a,i} d_{ai} y_a y_i = 0. \quad (12.2.29)$$

Taking the null vector $y = (1, \pm 1, 0, 0)$ in the above formula, we obtain

$$d_{01} \pm d_{11} = 0,$$

then $d_{01} = d_{11} = 0$. Similarly we have $d_{0i} = d_{ii} = 0$ $(i = 1, 2, 3)$. Thus, noting (12.2.28), (12.2.29) can be reduced to

$$\sum_{i<j} d_{ij} y_i y_j = 0.$$

Taking the null vector $y = (\sqrt{2}, 1, 1, 0)$ in the above formula, we get

$$d_{12} = 0;$$

similarly we have $d_{ij} = 0$ $(i, j = 1, 2, 3; i < j)$. Then, finally (12.2.27) can be written as

$$N(u, Du, D_x Du) = \overline{N}(Du, D_x Du), \tag{12.2.30}$$

and $\overline{N}(Du, D_x Du)$ is a quadratic form of Du and $D_x Du$. This proves that N does not depend on u explicitly: $N = N(Du, D_x Du)$.

Since $N(Du, D_x Du)$ is affine with respect to $D_x Du$, we can set

$$N(Du, D_x Du) = N_1(Du, D_x Du) + N_2(Du), \tag{12.2.31}$$

where

$$N_1(Du, D_x Du) = \sum_{i,a,b} e_{iab} \partial_i \partial_a u \partial_b u \tag{12.2.32}$$

and

$$N_2(Du) = \sum_{a,b} e_{ab} \partial_a u \partial_b u, \tag{12.2.33}$$

where e_{iab} and e_{ab} $(a, b = 0, 1, 2, 3; i = 1, 2, 3)$ are constants, and, without loss of generality, we may assume that

$$e_{jib} = e_{ijb} \tag{12.2.34}$$

and

$$e_{ab} = e_{ba}. \tag{12.2.35}$$

Noticing that q and r in (12.2.21) are two independent real numbers, from the fact that N satisfies the null condition, it is obvious that both N_1 and N_2 satisfy the null condition.

Since N_2 satisfies the null condition, by the corresponding (12.2.21) we obtain that: for any given null vector $y = (y_0, y_1, y_2, y_3)$ satisfying (12.2.19), we always have

$$\sum_{a,b} e_{ab} y_a y_b = 0. \tag{12.2.36}$$

Since for any given null vector $y = (y_0, y_1, y_2, y_3)$, $\hat{y} = (-y_0, y_1, y_2, y_3)$ is also a null vector, if we substitute y and $\hat{y}$ into (12.2.36), respectively, the sign-changing terms must vanish. Then, noting (12.2.35), we have

$$\sum_i e_{0i} y_0 y_i = 0,$$

then, taking $y_0 \neq 0$, we get: for any given non-null vector $y = (y_1, y_2, y_3)$, we have

$$\sum_i e_{0i} y_i = 0.$$

Therefore, we have

$$e_{0i} = 0 \quad (i = 1, 2, 3).$$

Similarly, we get

$$e_{ab} = 0 \quad (a, b = 0, 1, 2, 3; a \neq b).$$

Thus, we have

$$N_2(Du) = \sum_a e_{aa} (\partial_a u)^2, \tag{12.2.37}$$

and the corresponding null condition is

$$\sum_a e_{aa} y_a^2 = 0.$$

Taking specially $y = (1, 1, 0, 0)$ in the above formula, we obtain

$$e_{00} + e_{11} = 0;$$

furthermore, taking specially $y = (\sqrt{2}, 1, 1, 0)$, we obtain

$$2e_{00} + e_{11} + e_{22} = 0.$$

Thus, we have

$$e_{11} = e_{22} = -e_{00}.$$

Generally speaking, we obtain

$$e_{ii} = -e_{00} \quad (i = 1, 2, 3).$$

Then from (12.2.37) we obtain

$$N_2(Du) = e_{00} N_0(u, u). \tag{12.2.38}$$

On the other hand, it is easy to rewrite N_1 as

$$\begin{aligned} N_1(Du, D_x Du) &= \sum_{i,a,b} \frac{1}{2}(e_{iab} + e_{iba})\partial_i \partial_a u \partial_b u + \sum_{i,a,b} \frac{1}{2}(e_{iab} - e_{iba})\partial_i \partial_a u \partial_b u \\ &= \sum_{i,a,b} \tilde{e}_{iab} \partial_i \partial_a u \partial_b u + \sum_{i,a,b} \hat{e}_{iab} N_{ab}(\partial_i u, u), \end{aligned} \tag{12.2.39}$$

where $\tilde{e}_{abi}$ and $\hat{e}_{abi}$ are some constants, and

$$\tilde{e}_{iab} = \tilde{e}_{iba} \tag{12.2.40}$$

and

$$\tilde{e}_{ijb} = \tilde{e}_{jib}. \tag{12.2.41}$$

Noting that $N_{ab}(\partial_i u, u)$ already satisfies the null condition, since $N_1(Du, D_x Du)$ satisfies the null condition,

$$N_3(Du, D_x Du) = \sum_{i,a,b} \tilde{e}_{iab} \partial_i \partial_a u \partial_b u \tag{12.2.42}$$

should also satisfy the null condition. From the corresponding (12.2.21), similarly, we obtain that: for any given null vector $y = (y_0, y_1, y_2, y_3)$ satisfying (12.2.19), we always have

$$\sum_{i,a,b} \tilde{e}_{iab} y_i y_a y_b = 0. \tag{12.2.43}$$

Since for any given null vector $y = (y_0, y_1, y_2, y_3)$, $\hat{y} = (-y_0, y_1, y_2, y_3)$ is also a null vector, if we substitute y and $\hat{y}$ into (12.2.43), respectively, then its sign-changing terms must vanish. Then, noting (12.2.40), we have

$$\sum_{i,j} \tilde{e}_{ij0} y_i y_j y_0 = 0.$$

Taking $y_0 \neq 0$, we obtain from the above formula that: for any given non-null vector $\tilde{y} = (y_1, y_2, y_3)$, we have

$$\sum_{i,j} \tilde{e}_{ij0} y_i y_j = 0.$$

Noting (12.2.41), from this we immediately have

$$\tilde{e}_{ij0} = 0 \quad (i, j = 1, 2, 3). \tag{12.2.44}$$

Similarly, since for any given null vector $y = (y_0, y_1, y_2, y_3)$, $\hat{\hat{y}} = (y_0, -y_1, y_2, y_3)$ is also a null vector, if we substitute y and $\hat{\hat{y}}$ into (12.2.43), then its sign-changing terms must vanish. Then, noting (12.2.40) and (12.2.44), it is clear that

$$\tilde{e}_{111} y_1^3 + \sum_{a,b \neq 1} \tilde{e}_{1ab} y_1 y_a y_b + 2 \sum_{i,j \neq 1} \tilde{e}_{ij1} y_1 y_i y_j = 0,$$

then we have

$$\tilde{e}_{111} y_1^2 + \sum_{a,b \neq 1} \tilde{e}_{1ab} y_a y_b + 2 \sum_{i,j \neq 1} \tilde{e}_{ij1} y_i y_j = 0.$$

Inserting the null vectors $y = (y_0, y_1, y_2, y_3)$ and $(y_0, y_1, -y_2, y_3)$ in the above formula, respectively, the corresponding sign-changing terms should vanish. Then, noting (12.2.40) and (12.2.41), we obtain

$$\tilde{e}_{123} + \tilde{e}_{231} + \tilde{e}_{312} = 0.$$

Generally speaking, we obtain that: for any given i, j, k which are not equal to each other, we have

$$\tilde{e}_{ijk} + \tilde{e}_{jki} + \tilde{e}_{kij} = 0.$$

Noting furthermore (12.2.40) and (12.2.41), we get

$$\tilde{e}_{ijk} = 0 \quad (i, j, k = 1, 2, 3; \ \ i, j, k \ \text{ are distinct from each other}). \tag{12.2.45}$$

From (12.2.44) and (12.2.45), we obtain

$$N_3(Du, D_x Du) = \sum_{i,a} \tilde{e}_{iaa} \partial_i \partial_a u \partial_a u, \tag{12.2.46}$$

and the corresponding null condition becomes

$$\sum_{i,a} \tilde{e}_{iaa} y_i y_a^2 = 0.$$

Inserting the null vectors $y = (y_0, y_1, y_2, y_3)$ and $\hat{\hat{y}} = (y_0, -y_1, y_2, y_3)$ into the above formula, respectively, the corresponding sign-changing terms must vanish. Then, for any given null vector y, we have

$$\sum_a \tilde{e}_{1aa} y_a^2 = 0.$$

Similarly, we have

$$\sum_a \tilde{e}_{iaa} y_a^2 = 0 \quad (i = 1, 2, 3).$$

Thus, completely similar to the derivation of (12.2.38) from (12.2.37), from the above formula we get

$$N_3(Du, D_x Du) = \sum_i \tilde{e}_{i00} N_0(\partial_i u, u). \tag{12.2.47}$$

Combining (12.2.31), (12.2.38)–(12.2.39), (12.2.42) and (12.2.47), the proof of Lemma 12.2.1 is finished. □

Remark 12.2.1 $N_{ij}(\partial_0 u, u) = \partial_i \partial_0 u \partial_j u - \partial_j \partial_0 u \partial_i u$ also satisfy the null condition. It apparently is not of the form of $N_{ab}(\partial_i u, u)$, but since

$$\begin{aligned} N_{ij}(\partial_0 u, u) &= (\partial_i \partial_0 u \partial_j u - \partial_i \partial_j u \partial_0 u) + (\partial_i \partial_j u \partial_0 u - \partial_j \partial_0 u \partial_i u) \\ &= N_{0j}(\partial_i u, u) - N_{0i}(\partial_j u, u), \end{aligned}$$

$N_{ij}(\partial_0 u, u)$ is in fact a linear combination of the null form $N_{ab}(\partial_i u, u)$.

12.2.2 Some Properties of the Null Form

Now, for functions $N_0(f, g)$ and $N_{ab}(f, g)$, defined by (12.2.22) and (12.2.23) and used to generate the null form, we list some important properties which will be used in what follows.

Lemma 12.2.2 *For any given functions $f = f(t, x)$ and $g = g(t, x)$, we have*

$$\begin{aligned} &|N_0(f, g)(t, x)|, \ |N_{ab}(f, g)(t, x)| \\ &\le C(1+t)^{-1}(|Df(t, x)||\Gamma g(t, x)| + |\Gamma f(t, x)||Dg(t, x)|), \\ &\qquad \forall t \in I\!R^+, x \in I\!R^n, \end{aligned} \tag{12.2.48}$$

where C is a positive constant, and Γ is defined by (3.1.18) in Chap. 3.

Proof Due to (12.2.22) and (12.2.23), it is obvious that

$$|N_0(f, g)(t, x)|, \ |N_{ab}(f, g)(t, x)| \le C_1 |Df(t, x)||Dg(t, x)|, \tag{12.2.49}$$

where C_1 is a positive constant. Thus, to prove (12.2.48), it suffices to prove that

$$\begin{aligned} &|N_0(f,g)(t,x)|,\ \ |N_{ab}(f,g)(t,x)| \\ &\le C_2 t^{-1}(|Df(t,x)||\Gamma g(t,x)| + |\Gamma f(t,x)||Dg(t,x)|), \quad \forall t > 0, \end{aligned} \tag{12.2.50}$$

where C_2 is a positive constant.

Noting (3.1.8) and (3.1.11)–(3.1.12) in Chap. 3, we have

$$\begin{aligned} L_0 &= t\partial_t + \sum_i x_i\partial_i, \\ \Omega_{ij} &= x_i\partial_j - x_j\partial_i \end{aligned}$$

and

$$L_i = \Omega_{0i} = t\partial_i + x_i\partial_t,$$

then we have

$$\begin{aligned} tN_0(f,g) &= t(\partial_t f\partial_t g - \sum_i \partial_i f\partial_i g) \\ &= L_0 f\partial_t g - \sum_i (x_i\partial_i f\partial_t g + t\partial_i f\partial_i g) \\ &= L_0 f\partial_t g - \sum_i (\partial_i f L_i g), \end{aligned} \tag{12.2.51}$$

$$\begin{aligned} tN_{ij}(f,g) &= t(\partial_i f\partial_j g - \partial_j f\partial_i g) \\ &= L_i f\partial_j g - x_i\partial_t f\partial_j g - L_j f\partial_i g + x_j\partial_t f\partial_i g \\ &= L_i f\partial_j g - L_j f\partial_i g - \partial_t f\Omega_{ij} g \end{aligned} \tag{12.2.52}$$

and

$$tN_{0i}(f,g) = -t\partial_t f\partial_i g + t\partial_i f\partial_t g = -\partial_t f L_i g + L_i f\partial_t g. \tag{12.2.53}$$

By (12.2.51)–(12.2.53), and noting the definition of Γ (see (3.1.18) in Chap. 3), we immediately obtain the desired (12.2.50). The proof is finished. □

Denoting by $N(f,g)$, for convenience, the null forms $N_0(f,g)$ and $N_{ab}(f,g)$, we define

$$\{\Gamma, N(f,g)\} = \Gamma N(f,g) - N(\Gamma f, g) - N(f, \Gamma g). \tag{12.2.54}$$

We have

Lemma 12.2.3 *We have*

$$\{\partial_a, N_0(f,g)\} = 0, \tag{12.2.55}$$

$$\{\Omega_{ab}, N_0(f,g)\} = 0, \tag{12.2.56}$$

$$\{L_0, N_0(f,g)\} = -2N_0(f,g), \tag{12.2.57}$$

$$\{\partial_c, N_{ab}(f,g)\} = 0, \tag{12.2.58}$$

$$\{\Omega_{cd}, N_{ab}(f,g)\} = \eta^{ac}N_{bd}(f,g) - \eta^{ad}N_{bc}(f,g) - \eta^{bc}N_{ad}(f,g) + \eta^{bd}N_{ac}(f,g), \tag{12.2.59}$$

$$\{L_0, N_{ab}(f,g)\} = -2N_{ab}(f,g), \tag{12.2.60}$$

where $(\eta^{ab})_{4\times 4} = diag\{-1, 1, 1, 1\}$ *is the Lorentz metric.*

Proof Due to (12.2.22) and (12.2.23), it is easy to show (12.2.55) and (12.2.58).

Now we first prove the seemingly most complicated (12.2.59).

We have

$$\begin{aligned}
&\Omega_{cd}N_{ab}(f,g)\\
=&\Omega_{cd}(\partial_a f\partial_b g - \partial_b f\partial_a g)\\
=&\Omega_{cd}\partial_a f\cdot\partial_b g + \partial_a f\cdot\Omega_{cd}\partial_b g - (a|b)\\
=&[\Omega_{cd},\partial_a]f\cdot\partial_b g + \partial_a f\cdot[\Omega_{cd},\partial_b]g + \partial_a\Omega_{cd}f\cdot\partial_b g + \partial_a f\cdot\partial_b\Omega_{cd}g - (a|b)\\
=&[\Omega_{cd},\partial_a]f\cdot\partial_b g + \partial_a f\cdot[\Omega_{cd},\partial_b]g - [\Omega_{cd},\partial_b]f\cdot\partial_a g - \partial_b f\cdot[\Omega_{cd},\partial_a]g\\
&+ N_{ab}(\Omega_{cd}f, g) + N_{ab}(f, \Omega_{cd}g),
\end{aligned}$$

hereinafter we denote by $(a|b)$, for convenience, the result by exchanging a and b in the previous terms. Using Lemma 3.1.1 in Chap. 3, and noting (12.2.54), we obtain

$$\begin{aligned}
&\{\Omega_{cd}, N_{ab}(f,g)\}\\
=&[\Omega_{cd},\partial_a]f\cdot\partial_b g + \partial_a f\cdot[\Omega_{cd},\partial_b]g - [\Omega_{cd},\partial_b]f\cdot\partial_a g - \partial_b f\cdot[\Omega_{cd},\partial_a]g\\
=&\eta^{da}\partial_c f\partial_b g - \eta^{ca}\partial_d f\partial_b g + \eta^{db}\partial_a f\partial_c g - \eta^{cb}\partial_a f\partial_d g\\
&- \eta^{db}\partial_c f\partial_a g + \eta^{cb}\partial_d f\partial_a g - \eta^{da}\partial_b f\partial_c g + \eta^{ca}\partial_b f\partial_d g\\
=&\eta^{ac}N_{bd}(f,g) - \eta^{ad}N_{bc}(f,g) - \eta^{bc}N_{ad}(f,g) + \eta^{bd}N_{ac}(f,g).
\end{aligned}$$

This is exactly (12.2.59).

Now we prove (12.2.60).

We have

$$\begin{aligned}
L_0N_{ab}(f,g) =& L_0(\partial_a f\partial_b g - \partial_b f\partial_a g)\\
=& L_0\partial_a f\cdot\partial_b g + \partial_a f\cdot L_0\partial_b g - (a|b)\\
=& [L_0,\partial_a]f\cdot\partial_b g + \partial_a f\cdot[L_0,\partial_b]g + \partial_a L_0 f\cdot\partial_b g + \partial_a f\cdot\partial_b L_0 g - (a|b)\\
=& [L_0,\partial_a]f\cdot\partial_b g + \partial_a f\cdot[L_0,\partial_b]g - [L_0,\partial_b]f\cdot\partial_a g - \partial_b f\cdot[L_0,\partial_a]g\\
&+ N_{ab}(L_0 f, g) + N_{ab}(f, L_0 g).
\end{aligned}$$

Similarly, using Lemma 3.1.1 in Chap. 3, we get

$$\begin{aligned}\{L_0, N_{ab}(f,g)\} &= [L_0, \partial_a] f \cdot \partial_b g + \partial_a f \cdot [L_0, \partial_b] g - [L_0, \partial_b] f \cdot \partial_a g - \partial_b f \cdot [L_0, \partial_a] g \\ &= -\partial_a f \partial_b g - \partial_a f \partial_b g + \partial_b f \partial_a g + \partial_b f \partial_a g \\ &= -2N_{ab}(f,g).\end{aligned}$$

This is just (12.2.60).

Now we prove (12.2.56).

Noticing that

$$\begin{aligned}&\Omega_{ab} N_0(f,g) \\ =& \Omega_{ab}(\partial_0 f \partial_0 g - \sum_i \partial_i f \partial_i g) \\ =& \Omega_{ab}\partial_0 f \cdot \partial_0 g - \sum_i \Omega_{ab}\partial_i f \cdot \partial_i g + \partial_0 f \cdot \Omega_{ab}\partial_0 g - \sum_i \partial_i f \cdot \Omega_{ab}\partial_i g \\ =& [\Omega_{ab}, \partial_0] f \cdot \partial_0 g - \sum_i [\Omega_{ab}, \partial_i] f \cdot \partial_i g + \partial_0 f \cdot [\Omega_{ab}, \partial_0] g - \sum_i \partial_i f \cdot [\Omega_{ab}, \partial_i] g \\ &+ \partial_0 \Omega_{ab} f \cdot \partial_0 g - \sum_i \partial_i \Omega_{ab} f \cdot \partial_i g + \partial_0 f \cdot \partial_0 \Omega_{ab} g - \sum_i \partial_i f \cdot \partial_i \Omega_{ab} g \\ =& [\Omega_{ab}, \partial_0] f \cdot \partial_0 g - \sum_i [\Omega_{ab}, \partial_i] f \cdot \partial_i g + \partial_0 f \cdot [\Omega_{ab}, \partial_0] g - \sum_i \partial_i f \cdot [\Omega_{ab}, \partial_i] g \\ &+ N_0(\Omega_{ab} f, g) + N_0(f, \Omega_{ab} g),\end{aligned}$$

similarly we have

$$\begin{aligned}&\{\Omega_{ab}, N_0(f,g)\} \\ =& [\Omega_{ab}, \partial_0] f \cdot \partial_0 g - \sum_i [\Omega_{ab}, \partial_i] f \cdot \partial_i g + \partial_0 f \cdot [\Omega_{ab}, \partial_0] g - \sum_i \partial_i f \cdot [\Omega_{ab}, \partial_i] g \\ =& (\eta^{b0}\partial_a f \partial_0 g - \eta^{a0}\partial_b f \partial_0 g) - \sum_i (\eta^{bi}\partial_a f \partial_i g - \eta^{ai}\partial_b f \partial_i g) \\ &+ (\eta^{b0}\partial_0 f \partial_a g - \eta^{a0}\partial_0 f \partial_b g) - \sum_i (\eta^{bi}\partial_i f \partial_a g - \eta^{ai}\partial_i f \partial_b g).\end{aligned}$$

It is easy to show that the right-hand side of the above formula always vanishes for all the possible cases like $a, b = 0, 1, \ldots, n$ and $a \neq b$. This proves (12.2.56).

Finally we prove (12.2.57).

Noting that

$$\begin{aligned}L_0 N_0(f,g) &= L_0(\partial_0 f \partial_0 g - \sum_i \partial_i f \partial_i g) \\ &= L_0 \partial_0 f \cdot \partial_0 g - \sum_i L_0 \partial_i f \cdot \partial_i g + \partial_0 f \cdot L_0 \partial_0 g - \sum_i \partial_i f \cdot L_0 \partial_i g\end{aligned}$$

$$
\begin{aligned}
&= [L_0, \partial_0] f \cdot \partial_0 g - \sum_i [L_0, \partial_i] f \cdot \partial_i g + \partial_0 f \cdot [L_0, \partial_0] g - \sum_i \partial_i f \cdot [L_0, \partial_i] g \\
&\quad + \partial_0 L_0 f \cdot \partial_0 g - \sum_i \partial_i L_0 f \cdot \partial_i g + \partial_0 f \cdot \partial_0 L_0 g - \sum_i \partial_i f \cdot \partial_i L_0 g \\
&= [L_0, \partial_0] f \cdot \partial_0 g - \sum_i [L_0, \partial_i] f \cdot \partial_i g + \partial_0 f \cdot [L_0, \partial_0] g - \sum_i \partial_i f \cdot [L_0, \partial_i] g \\
&\quad + N_0(L_0 f, g) + N_0(f, L_0 g),
\end{aligned}
$$

similarly we have

$$
\begin{aligned}
\{L_0, N_0(f, g)\} &= [L_0, \partial_0] f \cdot \partial_0 g - \sum_i [L_0, \partial_i] f \cdot \partial_i g + \partial_0 f \cdot [L_0, \partial_0] g - \sum_i \partial_i f \cdot [L_0, \partial_i] g \\
&= -2(\partial_0 f \partial_0 g - \sum_i \partial_i f \partial_i g) \\
&= -2N_0(f, g).
\end{aligned}
$$

This is exactly (12.2.57). □

12.2.3 Metric Space $X_{S,E}$. Main Results

Consider the Cauchy problem of the three-dimensional quasi-linear wave equation

$$
\begin{aligned}
\Box u &= \hat{F}(u, Du, D_x Du) \\
&\stackrel{\text{def.}}{=} \sum_{i,j=1}^{3} b_{ij}(u, Du) u_{x_i x_j} + 2 \sum_{j=1}^{3} a_{0j}(u, Du) u_{t x_j} + F(u, Du)
\end{aligned}
\tag{12.2.61}
$$

with the initial data

$$
t = 0 : u = \varepsilon \varphi(x), \ \ u_t = \varepsilon \psi(x). \tag{12.2.62}
$$

Suppose that (12.2.4)–(12.2.8) hold, and

$$
\varphi, \psi \in C_0^\infty(I\!R^3) \tag{12.2.63}
$$

with

$$
\text{supp}\{\varphi, \psi\} \subseteq \{x \,|\, |x| \le \rho\} \ (\rho > 0 \ \text{ is a constant}), \tag{12.2.64}
$$

where $\varepsilon > 0$ is a small parameter.

From the results in Chap. 9, for Cauchy problem (12.2.61)–(12.2.62), in general, the life-span of its classical solution has only the following lower bound estimate:

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-2}, \tag{12.2.65}$$

where b is a positive constant independent of ε; even if the nonlinear term on the right-hand side does not depend on u explicitly:

$$\hat{F} = \hat{F}(Du, D_x Du), \tag{12.2.66}$$

the life-span of its classical solution has only the following exponential lower bound estimate:

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-1}\}, \tag{12.2.67}$$

where a is a positive constant independent of ε. However, next we will prove that: if the quadratic nonlinear part $N(u, Du, D_x Du)$ of the term $\hat{F}(u, Du, D_x Du)$ on the right-hand side satisfies the above-mentioned null condition, then the global existence of classical solutions of Cauchy problem (12.2.61)–(12.2.62) will be guaranteed.

By Sobolev embedding theorem, there exists a suitably small $E_0 > 0$, such that

$$\|f\|_{L^\infty(I\!R^3)} \leq \nu_0, \quad \forall f \in H^2(I\!R^3), \ \|f\|_{H^2(I\!R^3)} \leq E_0. \tag{12.2.68}$$

For any given integer $S \geq 14$ and any given positive number E $(\leq E_0)$, introduce the set of functions

$$X_{S,E} = \{v(t,x) | D_S(v) \leq E, \ \partial_t^l v(0,x) = u_l^{(0)}(x) \quad (l = 0, 1, \ldots, S+1)\}, \tag{12.2.69}$$

where

$$D_S(v) = \sup_{t \geq 0}(1+t)^{-\sigma}\|(Dv, D^2 v)(t, \cdot)\|_{\Gamma,S,2} + \sup_{t\geq 0}\|v(t,\cdot)\|_{\Gamma,S-1,2}, \tag{12.2.70}$$

in which σ is a suitably small positive number (say, we can take $\sigma = \frac{1}{100}$), $u_0^{(0)}(x) = \varepsilon\varphi(x)$, $u_1^{(0)}(x) = \varepsilon\psi(x)$, and when $l = 2, \ldots, S+1$, $u_l^{(0)}(x)$ are values of $\partial_t^l u(t,x)$ at $t = 0$, which are determined uniquely by Eq. (12.2.61) and initial condition (12.2.62). Obviously, $u_l^{(0)}$ $(l = 0, 1, \ldots, S+1)$ are all sufficiently smooth functions with compact support.

Remark 12.2.2 By Theorem 3.4.2 in Chap. 3 (in which we take $n = 3$, $p = 2$ and $s = 2$), it is easy to show that

$$\sum_{|k| \leq S-4} \sup_{t\geq 0}(1+t)\|(1+|t-|\cdot||)^{\frac{1}{2}}\Gamma^k v(t,\cdot)\|_{L^\infty(I\!R^3)} \leq C D_S(v), \tag{12.2.71}$$

then we have

$$(1+t)\|v(t,\cdot)\|_{\Gamma,S-4,\infty} \leq CD_S(v), \quad \forall t \geq 0. \tag{12.2.72}$$

Introduce the following metric in $X_{S,E}$:

$$\rho(\bar{v},\bar{\bar{v}}) = D_S(\bar{v}-\bar{\bar{v}}), \quad \forall \bar{v},\bar{\bar{v}} \in X_{S,E}. \tag{12.2.73}$$

When $\varepsilon > 0$ is suitably small, $X_{S,E}$ is a non-empty complete metric space. Denote by $\widetilde{X}_{S,E}$ a subset of $X_{S,E}$, which is composed of all the elements in $X_{S,E}$ with support, with respect to x, not surpassing $\{x \,|\, |x| \leq t+\rho\}$ for any given $t \geq 0$.

The main result of this section is the following

Theorem 12.2.1 *Under the above assumptions, suppose furthermore that $\hat{F}(u, Du, D_x Du)$ satisfies the above-mentioned null condition, i.e., the quadratic form $N(u, Du, D_x Du)$ therein (see (12.2.9)) is a linear combination of null forms, expressed by (12.2.24). Then for any given integer $S \geq 14$, there exist positive constants ε_0 and C_0, depending on $\rho > 0$, such that $C_0\varepsilon_0 \leq E_0$, and for any given $\varepsilon \in (0,\varepsilon_0]$, Cauchy problem (12.2.61)–(12.2.62) admits a unique global classical solution $u = u(t,x) \in \widetilde{X}_{S,C_0\varepsilon}$ on $t \geq 0$. Moreover, after a possible change of values for t on a zero-measure set of $[0,+\infty)$, we have*

$$u \in C\Big([0,+\infty); H^{S+1}(I\!R^3)\Big), \tag{12.2.74}$$

$$u_t \in C\Big([0,+\infty); H^{S}(I\!R^3)\Big), \tag{12.2.75}$$

$$u_{tt} \in C\Big([0,+\infty); H^{S-1}(I\!R^3)\Big). \tag{12.2.76}$$

In order to prove Theorem 12.2.1 by using the global iteration method, for any given $v \in \widetilde{X}_{S,E}$, by solving the Cauchy problem of the following linear hyperbolic equation

$$\begin{aligned}\Box u &= \hat{F}(v, Dv, D_x Du)\\ &\overset{\text{def.}}{=} \sum_{i,j=1}^{3} b_{ij}(v,Dv)u_{x_i x_j} + 2\sum_{j=1}^{3} a_{0j}(v,Dv)u_{tx_j} + F(v,Dv)\end{aligned} \tag{12.2.77}$$

with initial condition (12.2.62), we define a mapping

$$M : v \longrightarrow u = Mv. \tag{12.2.78}$$

We want to prove that: when $\varepsilon > 0$ is suitably small, we can find a positive constant C_0 such that when $E = C_0\varepsilon$, M admits a unique fixed point in $\widetilde{X}_{S,E}$, which is exactly the global classical solution of Cauchy problem (12.2.61)–(12.2.62) on $t \geq 0$.

To prove the above conclusion, it is crucial to prove the following two lemmas.

Lemma 12.2.4 *Under the assumptions of Theorem 12.2.1, when $E > 0$ is suitably small, for any given $v \in \widetilde{X}_{S,E}$, $u = Mv$ satisfies*

$$D_S(u) \le C_1\{\varepsilon + \sqrt{E}(E + D_S(u))\}, \tag{12.2.79}$$

where C_1 is a positive constant independent of $\varepsilon > 0$ but possibly depending on $\rho > 0$.

Lemma 12.2.5 *Under the assumptions of Lemma 12.2.4, for any given $\bar{v}, \bar{\bar{v}} \in \widetilde{X}_{S,E}$, if both $\bar{u} = M\bar{v}$ and $\bar{\bar{u}} = M\bar{\bar{v}}$ satisfy $\bar{u}, \bar{\bar{u}} \in \widetilde{X}_{S,E}$, then we have*

$$D_{S-1}(\bar{u} - \bar{\bar{u}}) \le C_2\sqrt{E}(D_{S-1}(\bar{u} - \bar{\bar{u}}) + D_{S-1}(\bar{v} - \bar{\bar{v}})), \tag{12.2.80}$$

where C_2 is a positive constant independent of $\varepsilon > 0$ but possibly depending on $\rho > 0$.

12.2.4 Proof of Lemmas 12.2.4 and 12.2.5

Here we only prove Lemma 12.2.4. Lemma 12.2.5 can be proved similarly.

We first estimate $\|u(t, \cdot)\|_{\Gamma,S-1,2}$.

From Von Wahl inequality (see (4.5.8) in Chap. 4), it is easy to know that

$$\|u(t, \cdot)\|_{\Gamma,S-1,2} \le C\varepsilon + \int_0^t \|\hat{F}(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S-1,\frac{6}{5}} d\tau. \tag{12.2.81}$$

To estimate $\|\hat{F}(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S-1,\frac{6}{5}}$, we first estimate the quadratic term $N(Dv, D_x Du)$ in $\hat{F}$, which does not depend on v explicitly due to the null condition. Noting that $N(Dv, D_x Du)$ is affine with respect to $D_x Du$, and involves the first-order partial derivatives of v, by Lemma 12.2.1, it should be a linear combination of terms $N_0(\partial_i u, v)$, $N_{ab}(\partial_i u, v)$ and $N_0(v, v)$. Therefore, we only need to estimate $\|N_0(\partial_i u, v)(\tau, \cdot)\|_{\Gamma,S-1,\frac{6}{5}}$, $\|N_{ab}(\partial_i u, v)(\tau, \cdot)\|_{\Gamma,S-1,\frac{6}{5}}$ and $\|N_0(v, v)(\tau, \cdot)\|_{\Gamma,S-1,\frac{6}{5}}$.

Denote by $N(\partial_i u, v)$, unitedly, $N_0(\partial_i u, v)$ and $N_{ab}(\partial_i u, v)$. From Lemma 12.2.3, it is easy to show that

$$\|N(\partial_i u, v)\|_{\Gamma,S-1,\frac{6}{5}} \le C \sum_{|k_1|+|k_2| \le S-1} \|N(\Gamma^{k_1}\partial_i u, \Gamma^{k_2} v)\|_{L^{\frac{6}{5}}(I\!R^3)}, \tag{12.2.82}$$

and from Lemma 12.2.2 we have

$$\begin{aligned} &|N(\Gamma^{k_1}\partial_i u, \Gamma^{k_2} v)(\tau, x)| \\ &\le C(1+\tau)^{-1}\{|\Gamma\Gamma^{k_1}\partial_i u(\tau, x)||D\Gamma^{k_2} v(\tau, x)| + |D\Gamma^{k_1}\partial_i u(\tau, x)||\Gamma\Gamma^{k_2} v(\tau, x)|\}. \end{aligned} \tag{12.2.83}$$

When $|k_1| \geq |k_2|$, we have $|k_2| \leq [\frac{S-1}{2}]$, then by Hölder inequality we have

$$\begin{aligned}
\mathrm{I} &\overset{\text{def.}}{=} \| |\Gamma\Gamma^{k_1}\partial_i u||D\Gamma^{k_2} v|(\tau,\cdot)\|_{L^{\frac{6}{5}}(I\!R^3)} + \| |D\Gamma^{k_1}\partial_i u||\Gamma\Gamma^{k_2} v|(\tau,\cdot)\|_{L^{\frac{6}{5}}(I\!R^3)} \\
&\leq C\|Du\|_{\Gamma,S,2}\|v\|_{\Gamma,[\frac{S-1}{2}]+1,3}.
\end{aligned}$$

Using the interpolation inequality (Lemma 5.2.4 in Chap. 5), we then have

$$\mathrm{I} \leq C\|Du\|_{\Gamma,S,2}\|v\|^{\frac{1}{3}}_{\Gamma,[\frac{S-1}{2}]+1,\infty}\|v\|^{\frac{2}{3}}_{\Gamma,[\frac{S-1}{2}]+1,2}.$$

Thus, using Corollary 3.4.4 in Chap. 3 (in which we take $n = 3$, $N = [\frac{S-1}{2}] + 1$, $p = 2$, $s = 2$), and noting that when $S \geq 14$, $[\frac{S-1}{2}] + 3 \leq S - 1$, we have

$$\begin{aligned}
\mathrm{I} &\leq C(1+\tau)^{-\frac{1}{3}}\|Du\|_{\Gamma,S,2}\|v\|_{\Gamma,[\frac{S-1}{2}]+3,2} \\
&\leq C(1+\tau)^{-\frac{1}{3}}\|Du\|_{\Gamma,S,2}\|v\|_{\Gamma,S-1,2} \\
&\leq C(1+\tau)^{-\frac{1}{3}+\sigma} ED_S(u). \qquad (12.2.84)
\end{aligned}$$

When $|k_1| \leq |k_2|$, we have $|k_1| \leq [\frac{S-1}{2}]$. Noting (3.1.26) in Chap. 3, and using again Corollary 3.4.4 in Chap. 3, we obtain

$$\begin{aligned}
\mathrm{II} &\overset{\text{def.}}{=} \| |\Gamma\Gamma^{k_1}\partial_i u||D\Gamma^{k_2} v|(\tau,\cdot)\|_{L^{\frac{6}{5}}(I\!R^3)} \\
&\leq C\|\Gamma\Gamma^{k_1}\partial_i u\|_{L^3(I\!R^3)}\|D\Gamma^{k_2} v\|_{L^2(I\!R^3)} \\
&\leq C\|Du\|_{\Gamma,[\frac{S-1}{2}]+1,3}\|Dv\|_{\Gamma,S,2} \\
&\leq C\|Du\|^{\frac{1}{3}}_{\Gamma,[\frac{S-1}{2}]+1,\infty}\|Du\|^{\frac{2}{3}}_{\Gamma,[\frac{S-1}{2}]+1,2}\|Dv\|_{\Gamma,S,2} \\
&\leq C(1+\tau)^{-\frac{1}{3}}\|Du\|_{\Gamma,[\frac{S-1}{2}]+3,2}\|Dv\|_{\Gamma,S,2} \\
&\leq C(1+\tau)^{-\frac{1}{3}}\|Du\|_{\Gamma,S,2}\|Dv\|_{\Gamma,S,2} \\
&\leq C(1+\tau)^{-\frac{1}{3}+2\sigma} ED_S(u). \qquad (12.2.85)
\end{aligned}$$

On the other hand, when $|k_1| \leq |k_2|$, from Lemma 5.1.3 in Chap. 5, and noting (3.1.26) in Chap. 3 and the interpolation inequality (see Lemma 5.2.4 in Chap. 5), it is easy to get

$$\begin{aligned}
\mathrm{III} &\overset{\text{def.}}{=} \| |D\Gamma^{k_1}\partial_i u||\Gamma\Gamma^{k_2} v|(\tau,\cdot)\|_{L^{\frac{6}{5}}(I\!R^3)} \\
&\leq C\|\Gamma\Gamma^{k_1}\partial_i u\|_{L^3(I\!R^3)}\|D\Gamma\Gamma^{k_2} v\|_{L^2(I\!R^3)} \\
&\leq C\|Du\|^{\frac{1}{3}}_{\Gamma,[\frac{S-1}{2}]+1,\infty}\|Du\|^{\frac{2}{3}}_{\Gamma,[\frac{S-1}{2}]+1,2}\|Dv\|_{\Gamma,S,2}.
\end{aligned}$$

Using again Corollary 3.4.4 in Chap. 3, when $S \geq 14$ we obtain

$$\text{III} \le C(1+\tau)^{-\frac{1}{3}}\|Du\|_{\Gamma,S,2}\|Dv\|_{\Gamma,S,2} \le C(1+\tau)^{-\frac{1}{3}+2\sigma}ED_S(u). \tag{12.2.86}$$

Thus, from (12.2.82)–(12.2.83) we get

$$\|N_0(\partial_i u, v)(\tau,\cdot)\|_{\Gamma,S-1,\frac{6}{5}},\ \|N_{ab}(\partial_i u, v)(\tau,\cdot)\|_{\Gamma,S-1,\frac{6}{5}} \le C(1+\tau)^{-\frac{4}{3}+2\sigma}ED_S(u). \tag{12.2.87}$$

Similarly, we obtain

$$\|N_0(v, v)(\tau,\cdot)\|_{\Gamma,S-1,\frac{6}{5}} \le C(1+\tau)^{-\frac{4}{3}+2\sigma}E^2. \tag{12.2.88}$$

Combining (12.2.87) and (12.2.88), we have

$$\|N(Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S-1,\frac{6}{5}} \le C(1+\tau)^{-\frac{4}{3}+2\sigma}E(E+D_S(u)).\square \tag{12.2.89}$$

Now we denote

$$H(v, Dv, D_x Du) = \hat{F}(v, Dv, D_x Du) - N(Dv, D_x Du), \tag{12.2.90}$$

which contains all the terms higher than quadratic. Noting (12.2.77), it can be specifically written as

$$H(v, Dv, D_x Du) = \sum_{i,j=1}^{3}\bar{b}_{ij}(v, Dv)u_{x_i x_j} + 2\sum_{j=1}^{3}\bar{a}_{0j}(v, Dv)u_{tx_j} + \overline{F}(v, Dv), \tag{12.2.91}$$

where $\bar{b}_{ij}(\tilde{\lambda})$, $\bar{a}_{0j}(\tilde{\lambda})$ and $\overline{F}(\tilde{\lambda})$ are sufficiently smooth in a neighborhood of $\tilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1, 2, 3) = 0$, and satisfy

$$\bar{b}_{ij}(\tilde{\lambda}) = \bar{b}_{ji}(\tilde{\lambda}) \quad (i, j = 1, 2, 3 \tag{12.2.92}$$

$$\bar{b}_{ij}(\tilde{\lambda}), \bar{a}_{0j}(\tilde{\lambda}) = O(|\tilde{\lambda}|^2) \quad (i, j = 1, 2, 3), \tag{12.2.93}$$

$$\overline{F}(\tilde{\lambda}) = O(|\tilde{\lambda}|^3). \tag{12.2.94}$$

Utilizing the estimates for product functions and composite functions in Chap. 5, we have

$$\begin{aligned}
&\|H(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S-1,\frac{6}{5}}\\
&\le C\|(v, Dv)\|_{\Gamma,[\frac{S-1}{2}],6}\big(\|D^2u\|_{\Gamma,S-1,2}\\
&\|(v, Dv)\|_{\Gamma,[\frac{S-1}{2}],6} + \|(v, Dv)\|_{\Gamma,S-1,2}\| \quad (v, Dv, D^2u)\|_{\Gamma,[\frac{S-1}{2}],6}\big).
\end{aligned} \tag{12.2.95}$$

Using the interpolation inequality (see Lemma 5.2.4 in Chap. 5) and Corollary 3.4.4 in Chap. 3, and noting $S \geq 14$, we have

$$\begin{aligned}\|(v, Dv)(\tau, \cdot)\|_{\Gamma, [\frac{S-1}{2}], 6} &\leq C\|(v, Dv)\|^{\frac{2}{3}}_{\Gamma, [\frac{S-1}{2}], \infty}\|(v, Dv)\|^{\frac{1}{3}}_{\Gamma, [\frac{S-1}{2}], 2} \\ &\leq C(1+\tau)^{-\frac{2}{3}}\|(v, Dv)\|_{\Gamma, [\frac{S-1}{2}], 2} \\ &\leq C(1+\tau)^{-\frac{2}{3}}\|(v, Dv)(\tau, \cdot)\|_{\Gamma, S-1, 2} \\ &\leq C(1+\tau)^{-\frac{2}{3}+\sigma}E,\end{aligned}$$

and, similarly, we have

$$\|D^2u(\tau, \cdot)\|_{\Gamma, [\frac{S-1}{2}], 6} \leq C(1+\tau)^{-\frac{2}{3}+\sigma}D_S(u).$$

Thus, (12.2.95) easily yields

$$\|H(v, Dv, D_xDu)(\tau, \cdot)\|_{\Gamma, S-1, \frac{6}{5}} \leq C(1+\tau)^{-\frac{4}{3}+3\sigma}E^2(E + D_S(u)). \tag{12.2.96}$$

From (12.2.88) and (12.2.95), we finally obtain

$$\|\hat{F}(v, Dv, D_xDu)(\tau, \cdot)\|_{\Gamma, S-1, \frac{6}{5}} \leq C(1+\tau)^{-\frac{4}{3}+3\sigma}E(E + D_S(u)), \tag{12.2.97}$$

then from (12.2.80) we get

$$\|u(t, \cdot)\|_{\Gamma, S-1, 2} \leq C\{\varepsilon + E(E + D_S(u))\}. \quad \square \tag{12.2.98}$$

Finally, we estimate $\|(Du, D_xDu)(t, \cdot)\|_{\Gamma, S, 2}$.

For any given multi-index k ($|k| \leq S$), we have the energy integral formula (9.2.41) in Chap. 9, in which G_k and g_k are given by (9.2.39) and (9.2.40) in Chap. 9, respectively (taking $n = 3$ in (12.2.39)–(12.2.41)).

Noting (3.1.26) in Chap. 3 and $S \geq 14$, it is clear that

$$\begin{aligned}|\mathrm{I}|, |\mathrm{II}|, |\mathrm{III}| &\leq C\int_0^t \|(Dv, D^2v)(\tau, \cdot)\|_{L^\infty(\mathbb{R}^3)}\|D^2u(\tau, \cdot)\|^2_{\Gamma, S, 2}d\tau \\ &\leq C\int_0^t \|v(\tau, \cdot)\|_{\Gamma, 2, \infty}\|D^2u(\tau, \cdot)\|^2_{\Gamma, S, 2}d\tau \\ &\leq C\int_0^t (1+\tau)^{-1}\|v(\tau, \cdot)\|_{\Gamma, S-1, 2}\|D^2u(\tau, \cdot)\|^2_{\Gamma, S, 2}d\tau \\ &\leq C\int_0^t (1+\tau)^{-1+2\sigma}ED_S^2(u)d\tau \\ &\leq C(1+t)^{2\sigma}ED_S^2(u).\end{aligned} \tag{12.2.99}$$

Now we estimate the L^2 norm of $G_k(\tau, \cdot)$.

From the estimates about product functions and composite functions in Chap. 5, using (12.2.72) in Remark 12.2.2 and noting that $[\frac{S}{2}] + 3 \le S - 4$ when $S \ge 14$, it is easy to show that

$$
\begin{aligned}
&\|(\Gamma^k D(b_{ij}(v, Dv)u_{x_i x_j}) - b_{ij}(v, Dv)\Gamma^k Du_{x_i x_j})(\tau, \cdot)\|_{L^2(I\!R^3)} \\
&\le C(\|(Dv, D^2 v)\|_{\Gamma,[\frac{S}{2}],\infty}\|D^2 u\|_{\Gamma,S,2} + \|(Dv, D^2 v)\|_{\Gamma,S,2}\|D^2 u\|_{\Gamma,[\frac{S}{2}]+1,\infty}) \\
&\le C(\|v\|_{\Gamma,[\frac{S}{2}]+2,\infty}\|D^2 u\|_{\Gamma,S,2} + \|(Dv, D^2 v)\|_{\Gamma,S,2}\|u\|_{\Gamma,[\frac{S}{2}]+3,\infty}) \\
&\le C(1+\tau)^{-1+\sigma} E D_S(u). \qquad (12.2.100)
\end{aligned}
$$

In addition, similarly we have

$$
\begin{aligned}
&\|(b_{ij}(v, Dv)(\Gamma^k Du_{x_i x_j}) - (\Gamma^k Du)_{x_i x_j})(\tau, \cdot)\|_{L^2(I\!R^3)} \\
&\le C\|b_{ij}(v, Dv)\|_{L^\infty(I\!R^3)}\|D^2 u\|_{\Gamma,S,2} \\
&\le C\|(v, Dv)\|_{L^\infty(I\!R^3)}\|D^2 u\|_{\Gamma,S,2} \\
&\le C(1+\tau)^{-1+\sigma} E D_S(u). \qquad (12.2.101)
\end{aligned}
$$

We can similarly estimate those terms involving a_{0j} in G_k to have

$$\|G_k(\tau, \cdot)\|_{L^2(I\!R^3)} \le C(1+\tau)^{-1+\sigma} E D_S(u), \qquad (12.2.102)$$

then

$$|\mathrm{IV}| \le C(1+t)^{2\sigma} E D_S^2(u). \qquad (12.2.103)$$

Finally, we estimate the L^2 norm of $g_k(\tau, \cdot)$.
For this, we rewrite (9.2.40) in Chap. 9 as

$$
\begin{aligned}
g_k &= \Gamma^k DF(v, Dv) + \sum_{|l|\le|k|} \widetilde{B}_{kl}\Gamma^l(\hat{F}(v, Dv, D_x Du) - F(v, 0)) + \sum_{|l|\le|k|} \widetilde{B}_{kl}\Gamma^l F(v, 0) \\
&= \Gamma^k DF(v, Dv) + \sum_{|l|\le|k|} \widetilde{B}_{kl}\Gamma^l \Big\{ \sum_{i,j=1}^{3} b_{ij}(v, Dv)u_{x_i x_j} + 2\sum_{j=1}^{3} a_{0j}(v, Dv)u_{t x_j} \\
&\qquad + (F(v, Dv) - F(v, 0)) \Big\} \\
&\quad + \sum_{|l|\le|k|} \widetilde{B}_{kl}\Gamma^l F(v, 0) \\
&\stackrel{\text{def.}}{=} \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \qquad (12.2.104)
\end{aligned}
$$

Using the estimates about product functions and composite functions, especially Lemma 5.1.4, in Chap. 5, we have

$$\begin{aligned}\|\mathrm{I}_1\|_{L^2(I\!R^3)} &\le C\|DF(v, Dv)(\tau,\cdot)\|_{\Gamma,S,2}\\ &\le C\big(\|(v, Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|(Dv, D^2v)(\tau,\cdot)\|_{\Gamma,S,2}\\ &\qquad + \|(Dv, D^2v)(\tau,\cdot)\|_{\Gamma,S,2}\|(v, Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\big). \quad (12.2.105)\end{aligned}$$

By (12.2.72) in Remark 12.2.2, and noticing $S \ge 14$, we have

$$\|(v, Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le C\|v\|_{\Gamma,[\frac{S}{2}]+2,\infty} \le C\|v\|_{\Gamma,S-4,\infty} \le C(1+\tau)^{-1}E,$$

then we get

$$\|\mathrm{I}_1\|_{L^2(I\!R^3)} \le C(1+\tau)^{-1+\sigma}E^2. \quad (12.2.106)$$

Similarly, we have

$$\begin{aligned}&\|\mathrm{I}_2\|_{L^2(I\!R^3)}\\ &\le C\big\{\|(v, Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}(\|D^2u(\tau,\cdot)\|_{\Gamma,S,2} + \|Dv(\tau,\cdot)\|_{\Gamma,S,2})\\ &\qquad + \|(Dv, D^2v)(\tau,\cdot)\|_{\Gamma,S,2}(\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} + \|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty})\big\}\\ &\le C(1+\tau)^{-1+\sigma}E(E + D_S(u)). \quad (12.2.107)\end{aligned}$$

Moreover, by the null condition (see Lemma 12.2.1) it is obvious that

$$F(v, 0) = O(|v|^3), \quad (12.2.108)$$

therefore

$$\|\mathrm{I}_3\|_{L^2(I\!R^3)} \le C\|F(v,0)(\tau,\cdot)\|_{\Gamma,S,2} \le C \sum_{\substack{|k_1|+|k_2|+|k_3|\le S\\ |k_1|\le|k_2|\le|k_3|}} \|\Gamma^{k_1}v\cdot\Gamma^{k_2}v\cdot\Gamma^{k_3}v(\tau,\cdot)\|_{L^2(I\!R^3)}. \quad (12.2.109)$$

By (12.2.71) in Remark 12.2.2, and noticing that $|k_1| + |k_2| \le [\frac{S}{2}] \le S - 4$ when $S \ge 14$, we have

$$\begin{aligned}|\Gamma^k v(t,x)| &\le C(1+t)^{-1}(1+|t-|x||)^{-\frac{1}{2}}D_S(v)\\ &\le C(1+t)^{-1}(1+|t-|x||)^{-\frac{1}{2}}E \quad (\text{ in which } k = k_1, k_2),\end{aligned}$$

then, using (5.1.28) in Lemma 5.1.3 of Chap. 5, we have

$$\begin{aligned}\|\mathrm{I}_3\|_{L^2(I\!R^3)} &\le C(1+\tau)^{-2}E^2\left\|\frac{\Gamma^{k_3}v(\tau,\cdot)}{1+|\tau-|\cdot||}\right\|_{L^2(I\!R^3)}\\ &\le C(1+\tau)^{-2}E^2\|Dv(\tau,\cdot)\|_{\Gamma,S,2}\\ &\le C(1+\tau)^{-2+\sigma}E^3. \quad (12.2.110)\end{aligned}$$

Combining (12.2.106)–(12.2.107) and (12.2.110), it follows from (12.2.104) that

$$\|g_k(\tau,\cdot)\|_{L^2(I\!R^3)} \le C(1+\tau)^{-1+\sigma} E(E + D_S(u)), \tag{12.2.111}$$

thus

$$|\mathrm{V}| \le C(1+t)^{2\sigma} E D_S(u)(E + D_S(u)). \tag{12.2.112}$$

Similarly to Chap. 9, using (12.2.99), (12.2.103) and (12.2.112), we get

$$\|D^2u(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\sigma}\{\varepsilon + \sqrt{E}(E + D_S(u))\}. \tag{12.2.113}$$

Moreover, using similar arguments as in Chap. 9, we obtain

$$\|Du(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\sigma}\{\varepsilon + \sqrt{E}(E + D_S(u))\}. \tag{12.2.114}$$

Combining (12.2.98) and (12.2.113)–(12.2.114), we get the desired (12.2.79). The proof of Lemma 12.2.4 is finished.

12.3 Null Condition and Global Existence of Classical Solutions to the Cauchy Problem of Two-Dimensional Nonlinear Wave Equations

12.3.1 Introduction

Consider the two-dimensional nonlinear wave equation

$$\Box u = \hat{F}(u, Du, D_x Du), \tag{12.3.1}$$

where $\Box = \frac{\partial^2}{\partial t^2} - \sum\limits_{i=1}^{2} \frac{\partial^2}{\partial {x_i}^2}$ is the two-dimensional wave operator, $D = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $D_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, and

$$\hat{F}(u, Du, D_x Du) = \sum_{i,j=1}^{2} b_{ij}(u, Du) u_{x_i x_j} + 2\sum_{j=1}^{2} a_{0j}(u, Du) u_{t x_j} + F(u, Du). \tag{12.3.2}$$

Denote

$$\tilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1, 2). \tag{12.3.3}$$

Assume that in a neighborhood of $\tilde{\lambda} = 0$, say, for $|\tilde{\lambda}| \leq \nu_0$, $b_{ij}(\tilde{\lambda})$, $a_{0j}(\tilde{\lambda})$ and $F(\tilde{\lambda})$ are sufficiently smooth functions and satisfy

$$b_{ij}(\tilde{\lambda}) = b_{ji}(\tilde{\lambda}) \quad (i, j = 1, 2), \tag{12.3.4}$$

$$b_{ij}(\tilde{\lambda}), a_{0j}(\tilde{\lambda}) = O(|\tilde{\lambda}|^2) \quad (i, j = 1, 2), \tag{12.3.5}$$

$$F(\tilde{\lambda}) = O(|\tilde{\lambda}|^3) \tag{12.3.6}$$

and

$$\sum_{i,j=1}^{2} a_{ij}(\tilde{\lambda})\xi_i\xi_j \geq m_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^2, \tag{12.3.7}$$

where, m_0 is a positive constant, and

$$a_{ij}(\tilde{\lambda}) = \delta_{ij} + b_{ij}(\tilde{\lambda}) \quad (i, j = 1, 2), \tag{12.3.8}$$

where δ_{ij} is the Kronecker symbol.

Under the above assumptions, (12.3.1) is a two-dimensional quasi-linear wave equation with nonlinearity of (at least) third order (correspondingly, $\alpha = 2$). Denote

$$\hat{F}(u, Du, D_x Du) = C(u, Du, D_x Du) + H(u, Du, D_x Du), \tag{12.3.9}$$

where, $C(u, Du, D_x Du)$ denotes a cubic function of its variables and is affine with respect to $D_x Du$, while, $H(u, Du, D_x Du)$ is composed of higher order terms. Noting (12.3.2), $H(u, Du, D_x Du)$ can be written as

$$H(u, Du, D_x Du) = \sum_{i,j=1}^{2} \bar{b}_{ij}(u, Du)u_{x_i x_j} + 2\sum_{j=1}^{2} \bar{a}_{0j}(u, Du)u_{tx_j} + \overline{F}(u, Du), \tag{12.3.10}$$

where $\bar{b}_{ij}(\tilde{\lambda})$, $\bar{a}_{0j}(\tilde{\lambda})$ and $\overline{F}(\tilde{\lambda})$ are sufficiently smooth in a neighborhood of $\tilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1, 2) = 0$ and satisfy

$$\bar{b}_{ij}(\tilde{\lambda}) = \bar{b}_{ji}(\tilde{\lambda}) \quad (i, j = 1, 2), \tag{12.3.11}$$

$$\bar{b}_{ij}(\tilde{\lambda}), \bar{a}_{0j}(\tilde{\lambda}) = O(|\tilde{\lambda}|^3) \quad (i, j = 1, 2), \tag{12.3.12}$$

$$\overline{F}(\tilde{\lambda}) = O(|\tilde{\lambda}|^4). \tag{12.3.13}$$

(12.3.10) can be rewritten as

$$H(u, Du, D_x Du) = H_1(u, Du)D_x Du + H_2(u, Du)Du + H_3(u), \tag{12.3.14}$$

where

$$H_1(u, Du)D_x Du = \sum_{i,j=1}^{2} \bar{b}_{ij}(u, Du)u_{x_i x_j} + 2\sum_{j=1}^{2} \bar{a}_{0j}(u, Du)u_{tx_j}, \tag{12.3.15}$$

$$H_2(u, Du)Du = \overline{F}(u, Du) - \overline{F}(u, 0) \tag{12.3.16}$$

and

$$H_3(u) = \overline{F}(u, 0). \tag{12.3.17}$$

From (12.3.12)–(12.3.13) we have

$$H_1(\hat{\lambda}) = O(|\hat{\lambda}|^3),\ \ H_2(\tilde{\lambda}) = O(|\tilde{\lambda}|^3) \tag{12.3.18}$$

and

$$H_3(\lambda) = O(|\lambda|^4), \tag{12.3.19}$$

in which $\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, 2; (\lambda_{ij}), i, j = 0, 1, 2, i + j \geq 1)$ and $\tilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1, 2)$.

For the two-dimensional quasi-linear wave equations (12.3.1) with nonlinearity of (at least) third order ($\alpha = 2$), consider its Cauchy problem with the small initial data

$$t = 0:\ \ u = \varepsilon\varphi(x),\ \ u_t = \varepsilon\psi(x), \tag{12.3.20}$$

where

$$\varphi, \psi \in C_0^\infty(I\!R^2) \tag{12.3.21}$$

with

$$\text{supp}\{\varphi, \psi\} \subseteq \{x||x| \leq \rho\}\ (\rho > 0\ \text{ is a constant}), \tag{12.3.22}$$

and $\varepsilon > 0$ is a small parameter.

Following the results in Chap. 10, when $n = 2$ and $\alpha = 2$, the life-span of the classical solution to Cauchy problem (12.3.1) and (12.3.20) has only the following lower bound estimate in general:

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-6}, \tag{12.3.23}$$

where b is a positive constant independent of ε; even when the nonlinear term on the right-hand side does not depend on u explicitly:

$$\hat{F} = \hat{F}(Du, D_x Du), \tag{12.3.24}$$

the life-span of the classical solution has only the following exponential lower bound estimate:

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-2}\}, \tag{12.3.25}$$

where a is a positive constant independent of ε. However, in what follows we will prove that: as long as the cubic nonlinear part $C(u, Du, D_x Du)$ of the term $\hat{F}(u, Du, D_x Du)$ on the right-hand side satisfies a suitable null condition, Cauchy problem (12.3.1) and (12.3.20) admits a global classical solution. This result first appeared in the Ph.D dissertation of Zhou Yi in 1992, but was not published. Later, A. Hoshiga gave a corresponding proof in 1995 (see Hoshiga 1995).

According to (12.2.24) in Lemma 12.2.1, the null condition satisfied by the cubic term $C(u, Du, D_x Du)$ is given as follows:

$$\begin{aligned} C(u, Du, D_x u) = &\sum_{i,a,b} c_{iab}(u, Du) N_{ab}(\partial_i u, u) + \sum_i c_i(u, Du) N_0(\partial_i u, u) \\ &+ c(u, Du, D_x Du) N_0(u, u), \end{aligned} \tag{12.3.26}$$

hereinafter $a, b, \cdots = 0, 1, 2$; $i, j, \cdots = 1, 2$, N_0 and N_{ab} are defined by (12.2.22) and (12.2.23), respectively, and c_{iab}, c_i and c are all linear homogeneous functions of its variables. Moreover, for the expression (12.3.14) of higher order term $H(u, Du, D_x Du)$, on this occasion we have

$$H_3(u) = \overline{F}(u, 0) = F(u, 0) \overset{\text{def.}}{=} F(u), \tag{12.3.27}$$

and we furthermore assume that

$$H_1(\lambda, 0),\ H_2(\lambda, 0) = O(|\lambda|^4) \tag{12.3.28}$$

and

$$F(\lambda) = H_3(\lambda) = O(|\lambda|^6). \tag{12.3.29}$$

Remark 12.3.1 For Cauchy problem (12.3.1) and (12.3.20) of the two-dimensional quasi-linear wave equation with (at least) quadratic nonlinearity (correspondingly, $\alpha = 1$) and small initial data, it follows from the results in Chap. 10 that, even under special condition (12.3.24), the life-span of its classical solution has only the following lower bound estimate:

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-2}, \tag{12.3.30}$$

where b is a positive constant independent of ε. On this occasion, it is more difficult to study the corresponding null condition, and the only result up to now belongs to S. Alinhac in a special case, see Sect. 15.2.1 in Chap. 15 (cf. Alinhac 2001).

12.3.2 Metric Space $X_{S,E}$. Main Results

Due to the Sobolev embedding theorem, there exists a suitably small $E_0 > 0$, such that

$$\|f\|_{L^\infty(I\!R^2)} \le \nu_0, \quad \forall f \in H^2(I\!R^2), \ \|f\|_{H^2(I\!R^2)} \le E_0. \tag{12.3.31}$$

For any given integer $S \ge 8$ and any given positive number E ($\le E_0$), introduce the set of functions

$$X_{S,E} = \{v(t,x) | D_S(v) \le E, \ \partial_t^l v(0,x) = u_l^{(0)}(x) \quad (l = 0, 1, \dots, S+1)\}, \tag{12.3.32}$$

where

$$\begin{aligned} D_S(v) = & \sup_{t\ge 0}(1+t)^{-\sigma}\|(Dv, D^2v)(t,\cdot)\|_{\Gamma,S,2} \\ & + \sup_{t\ge 0}(1+t)^{-\frac{1}{2}-2\sigma}\|v(t,\cdot)\|_{\Gamma,S,2} + \sup_{t\ge 0}(1+t)^{\frac{1}{2}}\|v(t,\cdot)\|_{\Gamma,S-2,\infty}, \end{aligned} \tag{12.3.33}$$

in which σ is a suitably small positive number (we can take, say, $\sigma = \frac{1}{100}$), $u_l^{(0)}(x)$ ($l = 0, 1, \dots, S+1$) are defined as in Sect. 12.2.3.

Introduce the following metric in $X_{S,E}$:

$$\rho(\bar{v}, \bar{\bar{v}}) = D_S(\bar{v} - \bar{\bar{v}}), \quad \forall \bar{v}, \bar{\bar{v}} \in X_{S,E}. \tag{12.3.34}$$

When $\varepsilon > 0$ is suitably small, $X_{S,E}$ is a non-empty complete metric space. Denote by $\widetilde{X}_{S,E}$ a subset of $X_{S,E}$, composed of all the elements in $X_{S,E}$ with support with respect to x, contained in $\{x | |x| \le t + \rho\}$ for any given $t \ge 0$.

The main result of this section is the following

Theorem 12.3.1 *Under assumptions (12.3.4)–(12.3.8) and (12.3.21)–(12.3.22), assume furthermore that the nonlinear term $\hat{F}(u, Du, D_x Du)$ on the right-hand side satisfies the above-mentioned null condition, that is, the cubic term $C(u, Du, D_x Du)$ is given by (12.3.26), and the higher order term $H(u, Du, D_x Du)$ satisfies (12.3.28)–(12.3.29). Then for any given integer $S \ge 8$, there exist positive constants ε_0 and C_0 depending on $\rho > 0$ and satisfying $C_0\varepsilon_0 \le E_0$, such that for any given $\varepsilon \in (0, \varepsilon_0]$, Cauchy problem (12.3.1) and (12.3.20) admits a unique global classical solution*

$u = u(t, x) \in \widetilde{X}_{S,C_0\varepsilon}$ *on* $t \geq 0$. *Moreover, after possible change of values for* t *on a zero-measure set, for any given* $T > 0$, *we have*

$$u \in C\Big([0, T]; H^{S+1}(I\!R^2)\Big), \tag{12.3.35}$$

$$u_t \in C\Big([0, T]; H^{S}(I\!R^2)\Big), \tag{12.3.36}$$

$$u_{tt} \in C\Big([0, T]; H^{S-1}(I\!R^2)\Big). \quad \Box \tag{12.3.37}$$

In order to adopt the global iteration method to prove Theorem 12.3.1, for any given $v \in \widetilde{X}_{S,E}$, by solving the Cauchy problem of the following linear hyperbolic equation

$$\Box u = \hat{F}(v, Dv, D_x Du) \stackrel{\text{def.}}{=} \sum_{i,j=1}^{2} b_{ij}(v, Dv)u_{x_i x_j} + 2\sum_{j=1}^{2} a_{0j}(v, Dv)u_{tx_j} + F(v, Dv) \tag{12.3.38}$$

with initial condition (12.3.20), we define a mapping

$$M : v \longrightarrow u = Mv. \tag{12.3.39}$$

We want to prove that: when $\varepsilon > 0$ is suitably small, we can find a positive constant C_0, such that when $E = C_0\varepsilon$, M admits a unique fixed point in $\widetilde{X}_{S,E}$, which is just the global classical solution to the Cauchy problem (12.3.1) and (12.3.20) on $t \geq 0$.

To prove the above conclusion, it is crucial to prove the following two lemmas.

Lemma 12.3.1 *Under the assumptions of Theorem 12.3.1, when* $E > 0$ *is suitably small, for any given* $v \in \widetilde{X}_{S,E}$, $u = Mv$ *satisfies*

$$D_S(u) \leq C_1\{\varepsilon + E(E + D_S(u))\}, \tag{12.3.40}$$

where C_1 *is a positive constant independent of* ε *but possibly depending on* $\rho > 0$.

Lemma 12.3.2 *Under the assumptions of Lemma 12.3.1, for any given* $\bar{v}, \bar{\bar{v}} \in \widetilde{X}_{S,E}$, *if both* $\bar{u} = M\bar{v}$ *and* $\bar{\bar{u}} = M\bar{\bar{v}}$ *satisfy* $\bar{u}, \bar{\bar{u}} \in \widetilde{X}_{S,E}$, *then we have*

$$D_{S-1}(\bar{u} - \bar{\bar{u}}) \leq C_2 E(D_{S-1}(\bar{u} - \bar{\bar{u}}) + D_{S-1}(\bar{v} - \bar{\bar{v}})), \tag{12.3.41}$$

where C_2 *is a positive constant independent of* ε *but possibly depending on* $\rho > 0$.

12.3.3 Proof of Lemmas 12.3.1 and 12.3.2

Now we prove Lemma 12.3.1 only. The proof of Lemma 12.3.2 can be conducted similarly.

We first estimate $\|u(t,\cdot)\|_{\Gamma,S-2,\infty}$.

From the L^1-L^∞ estimates for solutions to the wave equation (see Theorem 4.6.1 and Theorem 4.6.2 in Chap. 4, in which we take $n=2, l=0$), it is easy to show that

$$\|u(t,\cdot)\|_{\Gamma,S-2,\infty} \le C(1+t)^{-\frac{1}{2}}\Big(\varepsilon + \int_0^t (1+\tau)^{-\frac{1}{2}}\|\hat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S-1,1}d\tau\Big). \tag{12.3.42}$$

To estimate $\|\hat{F}(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S-1,1}$, we first estimate $\|C(v, Dv, D_x Du)(\tau,\cdot)\|_{\Gamma,S-1,1}$. By (12.3.26), it suffices to estimate

$$\|(v, Dv)N(\partial_i u, v)(\tau,\cdot)\|_{\Gamma,S-1,1} \quad \text{and} \quad \|(v, Dv, D_x Du)N_0(v, v)(\tau,\cdot)\|_{\Gamma,S-1,1},$$

in which we still denote by $N(\partial_i u, v)$ the null forms $N_0(\partial_i u, v)$ and $N_{ab}(\partial_i u, v)$.

From the estimates about product functions and composite functions in Chap. 5, we have

$$\begin{aligned}&\|(v, Dv)N(\partial_i u, v)(\tau,\cdot)\|_{\Gamma,S-1,1}\\ &\le C(\|(v, Dv)\|_{\Gamma,[\frac{S-1}{2}],\infty}\|N(\partial_i u, v)\|_{\Gamma,S-1,1} + \|(v, Dv)\|_{\Gamma,S-1,2}\|N(\partial_i u, v)\|_{\Gamma,S-1,2}).\end{aligned} \tag{12.3.43}$$

By Lemma 12.2.3, we have

$$\|N(\partial_i u, v)\|_{\Gamma,S-1,1} \le C \sum_{|k_1|+|k_2|\le S-1} \|N(\Gamma^{k_1}\partial_i u, \Gamma^{k_2} v)\|_{L^1(\mathbb{R}^2)}, \tag{12.3.44}$$

and due to Lemma 12.2.2, we still have (12.2.81). Then, similarly to the proof of (12.2.86), when $S \ge 8$ we get

$$\|N(\partial_i u, v)(\tau,\cdot)\|_{\Gamma,S-1,1} \le C(1+\tau)^{-1+2\sigma} E D_S(u). \tag{12.3.45}$$

Similarly, we can prove

$$\|N(\partial_i u, v)(\tau,\cdot)\|_{\Gamma,S-1,2} \le C(1+\tau)^{-\frac{3}{2}+2\sigma} E D_S(u). \tag{12.3.46}$$

Hence, by (12.3.43), and noting that when $S \ge 8$ we have

$$\|(v, Dv)(\tau,\cdot)\|_{\Gamma,[\frac{S-1}{2}],\infty} \le C\|v\|_{\Gamma,[\frac{S-1}{2}]+1,\infty} \le C\|v\|_{\Gamma,S-2,\infty} \le C(1+\tau)^{-\frac{1}{2}}E, \tag{12.3.47}$$

we obtain

$$\|(v, Dv)N(\partial_i u, v)(\tau,\cdot)\|_{\Gamma,S-1,1} \le C(1+\tau)^{-1+4\sigma}E^2 D_S(u). \tag{12.3.48}$$

Similarly we have

$$\|(v, Dv, D_x Du)N_0(v, v)(\tau, \cdot)\|_{\Gamma,S-1,1} \le C(1+\tau)^{-1+4\sigma}E^2(E + D_S(u)). \tag{12.3.49}$$

Thus, we have

$$\|C(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S-1,1} \le C(1+\tau)^{-1+4\sigma}E^2(E + D_S(u)). \quad \square \tag{12.3.50}$$

Now we estimate $\|H(v, Dv, D_x Du)(\tau, \cdot)\|_{\Gamma,S-1,1}$.
By (12.3.14) and noting (12.3.27), we have

$$\begin{aligned}
&H(v, Dv, D_x Du)\\
=\,&H_1(v, Dv)D_x Du + H_2(v, Dv)Dv + F(v)\\
=\,&(H_1(v, Dv) - H_1(v, 0))D_x Du + (H_2(v, Dv) - H_2(v, 0))Dv\\
&+ H_1(v, 0)D_x Du + H_2(v, 0)Dv + F(v)\\
\overset{\text{def.}}{=}\,&\overline{H}_1(v, Dv)DvD_x Du + \overline{H}_2(v, Dv)(Dv)^2 + H_1(v, 0)D_x Du + H_2(v, 0)Dv + F(v),
\end{aligned} \tag{12.3.51}$$

where

$$\overline{H}_1(\tilde{\lambda}),\ \overline{H}_2(\tilde{\lambda}) = O(|\tilde{\lambda}|^2). \tag{12.3.52}$$

Using the estimates about product functions and composite functions, especially Lemma 5.1.4, in Chap. 5, we have

$$\begin{aligned}
&\|\overline{H}_1(v, Dv)DvD_x Du\|_{\Gamma,S-1,1}\\
\le\,&C\|(v, Dv)^2 Dv\|_{\Gamma,S-1,2}\|D^2u\|_{\Gamma,S-1,2}\\
\le\,&C(\|(v, Dv)^2\|_{\Gamma,[\frac{S-1}{2}],\infty}\|Dv\|_{\Gamma,S-1,2} + \|(v, Dv)(Dv, D^2v)\|_{\Gamma,S-1,2}\|v\|_{\Gamma,[\frac{S-1}{2}]+1,\infty})\|D^2u\|_{\Gamma,S,2},
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\|(v, Dv)(Dv, D^2v)\|_{\Gamma,S-1,2}\\
\le\,&C\big(\|(v, Dv)\|_{\Gamma,[\frac{S-1}{2}],\infty}\|(Dv, D^2v)\|_{\Gamma,S-1,2} + \|(Dv, D^2v)\|_{\Gamma,S-1,2}\|(v, Dv)\|_{\Gamma,[\frac{S-1}{2}]+1,\infty}\big).
\end{aligned}$$

Noting that $\|(v, Dv)^2\|_{\Gamma,[\frac{S-1}{2}],\infty} \le C\|(v, Dv)\|^2_{\Gamma,[\frac{S-1}{2}],\infty}$, and that $[\frac{S-1}{2}] + 2 \le S - 2$ when $S \ge 8$, we get

$$\|\overline{H}_1(v, Dv)DvD_x Du(\tau, \cdot)\|_{\Gamma,S-1,1} \le C(1+\tau)^{-1+2\sigma}E^3 D_S(u). \tag{12.3.53}$$

Similarly, we have

$$\|\overline{H}_2(v, Dv)(Dv)^2(\tau, \cdot)\|_{\Gamma,S-1,1} \le C(1+\tau)^{-1+2\sigma}E^4. \tag{12.3.54}$$

Moreover, using Lemma 3.1.4 in Chap. 3, and noting assumption (12.3.28), we have

$$\|H_1(v,0)D_xDu(\tau,\cdot)\|_{\Gamma,S-1,1} \le C(\|v^4\|_{\Gamma,[\frac{S-1}{2}],2}\|D^2u\|_{\Gamma,S-1,2} + \|v^3Dv\|_{\Gamma,S-1,2}\|Du\|_{\Gamma,[\frac{S-1}{2}]+1,2}).$$

Repeatedly using Lemma 3.1.4 in Chap. 3, we have

$$\|v^3Dv\|_{\Gamma,S-1,2} \le C\|v\|^3_{\Gamma,[\frac{S-1}{2}]+1,\infty}\|Dv\|_{\Gamma,S-1,2};$$

moreover, it is obvious that

$$\|v^4\|_{\Gamma,[\frac{S-1}{2}],2} \le C\|v\|^3_{\Gamma,[\frac{S-1}{2}],\infty}\|v\|_{\Gamma,[\frac{S-1}{2}],2}.$$

Then, noting $S \ge 8$, it is easy to get

$$\begin{aligned}\|H_1(v,0)D_xDu(\tau,\cdot)\|_{\Gamma,S-1,1} &\le C\|v\|^3_{\Gamma,S-2,\infty}(\|v\|_{\Gamma,S,2} + \|Dv\|_{\Gamma,S,2})\|Du\|_{\Gamma,S,2}\\ &\le C(1+\tau)^{-1+3\sigma}E^4D_S(u).\end{aligned} \tag{12.3.55}$$

Similarly, we have

$$\|H_2(v,0)Dv(\tau,\cdot)\|_{\Gamma,S-1,1} \le C(1+\tau)^{-1+3\sigma}E^5. \tag{12.3.56}$$

Finally, by assumption (12.3.29), we have

$$\|F(v)(\tau,\cdot)\|_{\Gamma,S-1,1} \le C\|v\|^4_{\Gamma,[\frac{S-1}{2}],\infty}\|v\|^2_{\Gamma,S,2} \le C(1+\tau)^{-1+4\sigma}E^6. \tag{12.3.57}$$

Combining (12.3.53)–(12.3.57), we get

$$\|H(v,Dv,D_xDu)(\tau,\cdot)\|_{\Gamma,S-1,1} \le C(1+\tau)^{-1+4\sigma}E^3(E+D_S(u)). \tag{12.3.58}$$

Noting (12.3.50), we have

$$\|\hat{F}(v,Dv,D_xDu)(\tau,\cdot)\|_{\Gamma,S-1,1} \le C(1+\tau)^{-1+4\sigma}E^2(E+D_S(u)),$$

then from (12.3.42) we obtain

$$\|u(t,\cdot)\|_{\Gamma,S-2,\infty} \le C(1+t)^{-\frac{1}{2}}\{\varepsilon + E^2(E+D_S(u))\}. \quad \square \tag{12.3.59}$$

Now we estimate $\|u(t,\cdot)\|_{\Gamma,S,2}$.

Noting $\alpha = 2$, and that assumption (12.3.29) leads to (10.1.23) in Chap. 10 easily, $\hat{F}(v,Dv,D_xDu)$ can be written in the form of (10.4.10) in Chap. 10, namely,

$$\hat{F}(v, Dv, D_x Du) = \sum_{i=0}^{2} \partial_i \hat{G}_i(v, Du) + \sum_{i,j=0}^{2} \hat{A}_{ij}(v) v_{x_i} u_{x_j} + \sum_{\substack{i,j,m=0 \\ j+m\ge 1}}^{2} \hat{B}_{ijm}(v, Dv) v_{x_i} u_{x_j x_m} + \sum_{i,j=0}^{2} \hat{C}_{ij}(v, Dv) v_{x_i} v_{x_j} + F(v), \tag{12.3.60}$$

and we also have the corresponding (10.4.12)–(10.4.14) in Chap. 10, that is

$$\hat{G}_i(\bar{\lambda}) = O(|\bar{\lambda}|^3), \ i = 0, 1, 2; \ \bar{\lambda} = (v, Du), \tag{12.3.61}$$

and $\hat{G}_i(v, Du)(i = 0, 1, 2)$ are affine with respect to Du,

$$\hat{A}_{ij}(v) = O(|v|), \quad i, j = 0, 1, 2 \tag{12.3.62}$$

and

$$\hat{B}_{ijm}(\tilde{\lambda}), \hat{C}_{ij}(\tilde{\lambda}) = O(|\tilde{\lambda}|), \ i, j, m = 0, 1, 2; \ \tilde{\lambda} = (v, Dv), \tag{12.3.63}$$

but (10.4.11) in Chap. 10 should be replaced by (12.3.29) in this section.

Thus, similarly to Sect. 10.4 in Chap. 10, the solution $u = Mv$ to Cauchy problem (12.3.38) and (12.3.20) can be written as

$$u = u_1 + u_2 + u_3, \tag{12.3.64}$$

where u_1 is the solution to the equation

$$\Box u_1 = \sum_{i=0}^{2} \partial_i \hat{G}_i(v, Du) \tag{12.3.65}$$

with the zero initial condition, u_2 is the solution to the equation

$$\Box u_2 = Q(v, Dv, Du, D_x Du) \tag{12.3.66}$$

with the same initial value (12.3.20) as u, where

$$Q(v, Dv, Du, D_x Du) = \sum_{i,j=0}^{2} \hat{A}_{ij}(v) v_{x_i} u_{x_j} + \sum_{\substack{i,j,m=0 \\ j+m\ge 1}}^{2} \hat{B}_{ijm}(v, Dv) v_{x_i} u_{x_j x_m} + \sum_{i,j=0}^{2} \hat{C}_{ij}(v, Dv) v_{x_i} v_{x_j}, \tag{12.3.67}$$

while, u_3 is the solution to the equation

$$\Box u_3 = F(v) \tag{12.3.68}$$

with the zero initial condition.

Same as (10.4.25)–(10.4.26) in Chap. 10, we get

$$\|u_1(t,\cdot)\|_{\Gamma,S,2} \le C\Big(\varepsilon^2\sqrt{\ln(2+t)} + \sum_{i=0}^{2}\int_0^t \|\hat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,S,2}d\tau\Big) \tag{12.3.69}$$

and

$$\begin{aligned}&\sum_{i=0}^{2}\|\hat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,S,2}\\ &\le C(\|v(\tau,\cdot)\|^2_{\Gamma,[\frac{S}{2}],\infty}\|(v,Du)(\tau,\cdot)\|_{\Gamma,S,2} + \|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|_{\Gamma,S,2}).\end{aligned} \tag{12.3.70}$$

Noticing that when $S \ge 8$,

$$\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \le C\|v(\tau,\cdot)\|_{\Gamma,S-2,\infty} \le C(1+\tau)^{-\frac{1}{2}}E$$

and

$$\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \le C\|u(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le C\|u(\tau,\cdot)\|_{\Gamma,S-2,\infty} \le C(1+\tau)^{-\frac{1}{2}}D_S(u),$$

we have

$$\sum_{i=0}^{2}\|\hat{G}_i(v,Du)(\tau,\cdot)\|_{\Gamma,S,2} \le C(1+\tau)^{-\frac{1}{2}+2\sigma}E^2(E+D_S(u)), \tag{12.3.71}$$

then from (12.3.69) we obtain

$$\|u_1(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\frac{1}{2}+2\sigma}\{\varepsilon + E^2(E+D_S(u))\}. \quad \Box \tag{12.3.72}$$

Moreover, by (10.4.31) in Chap. 10, we have

$$\begin{aligned}\|u_2(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\frac{1}{3}}\Big\{\varepsilon + \int_0^t \big(\|Q(v,Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1}\\ + (1+\tau)^{-\frac{1}{3}}\|Q(v,Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}\big)d\tau\Big\},\end{aligned} \tag{12.3.73}$$

where χ_1 is the characteristic function of the set $\{(t,x)\big||x| \le \frac{1+t}{2}\}$, $\chi_2 = 1-\chi_1$, and $Q(v,Dv,Du,D_xDu)$ is given by (10.4.18) in Chap. 10.

By (10.4.35) in Chap. 10, we have

$$
\begin{aligned}
&\|\hat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1}\\
&\le C\{\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\|Du(\tau,\cdot)\|_{\Gamma,S,2}+\|vDv(\tau,\cdot)\|_{\Gamma,S,2}\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\}.
\end{aligned}
\tag{12.3.74}
$$

Using the interpolation inequality (see Lemma 5.2.4 in Chap. 5), and 1° in Corollary 3.4.1 of Chap. 3 (in which we take $n=2$, $p=2$, $N=[\frac{S}{2}]$, $s=2$), we have

$$
\begin{aligned}
\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1} &\le C\|Dv(\tau,\cdot)\|^{\frac{1}{3}}_{\Gamma,[\frac{S}{2}],\infty,\chi_1}\|Dv(\tau,\cdot)\|^{\frac{2}{3}}_{\Gamma,[\frac{S}{2}],2,\chi_1}\\
&\le C(1+\tau)^{-\frac{1}{3}}\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+2,2,\chi_1}\\
&\le C(1+\tau)^{-\frac{1}{3}}\|Dv(\tau,\cdot)\|_{\Gamma,S,2}.
\end{aligned}
$$

Similarly, we have

$$
\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1} \le C(1+\tau)^{-\frac{1}{3}}\|Du(\tau,\cdot)\|_{\Gamma,S,2}.
$$

Noting (10.4.36) in Chap. 10, from (12.3.74) we finally obtain

$$
\|\hat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{5}{6}+2\sigma}E^2D_S(u). \tag{12.3.75}
$$

Similarly, we have

$$
\|\hat{B}_{ijm}(v,Dv)v_{x_i}u_{x_jx_m}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{5}{6}+2\sigma}E^2D_S(u) \tag{12.3.76}
$$

and

$$
\|\hat{C}_{ij}(v,Dv)v_{x_i}v_{x_j}(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{5}{6}+2\sigma}E^3. \tag{12.3.77}
$$

Thus, we have

$$
\|Q(v,Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{5}{6}+2\sigma}E^2(E+D_S(u)). \tag{12.3.78}
$$

From (10.4.43) in Chap. 10, we have

$$
\begin{aligned}
&\|\hat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2}\\
&\le C\{\|vDv(\tau,\cdot)\|_{\Gamma,S,2}\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2}\\
&+\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2}\|Du(\tau,\cdot)\|_{\Gamma,S,2}\}.
\end{aligned}
\tag{12.3.79}
$$

Using the Sobolev estimates on sphere (see 1° in Theorem 3.2.1 of Chap. 3, in which we take $n = 2$, $p = 2$, $s = 1$), we have

$$\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2} \le C\|Dv(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,2} \le C\|Dv(\tau,\cdot)\|_{\Gamma,S,2} \le C(1+\tau)^{\sigma}E$$

and

$$\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2} \le C\|Du(\tau,\cdot)\|_{\Gamma,S,2} \le C(1+\tau)^{\sigma}D_S(u).$$

Noting (10.4.36) in Chap. 10, from (12.3.79) we then have

$$\|\hat{A}_{ij}(v)v_{x_i}u_{x_j}(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{1}{2}+2\sigma}E^2D_S(u). \tag{12.3.80}$$

Similarly, we have

$$\|\hat{B}_{ijm}(v,Dv)v_{x_i}u_{x_jx_m}(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{1}{2}+2\sigma}E^2D_S(u) \tag{12.3.81}$$

and

$$\|\hat{C}_{ij}(v,Dv)v_{x_i}v_{x_j}(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{1}{2}+2\sigma}E^3. \tag{12.3.82}$$

Thus, we have

$$\|Q(v,Dv,Du,D_xDu)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} \le C(1+\tau)^{-\frac{1}{2}+2\sigma}E^2(E+D_S(u)). \tag{12.3.83}$$

Plugging (12.3.78) and (12.3.83) in (12.3.73), it is easy to get

$$\|u_2(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\frac{1}{2}+2\sigma}\{\varepsilon + E^2(E+D_S(u))\}. \quad \square \tag{12.3.84}$$

Finally, we estimate $\|u_3(t,\cdot)\|_{\Gamma,S,2}$.
By (10.4.50) in Chap. 10 we have

$$\|u_3(t,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{\frac{1}{3}}\Big\{\varepsilon + \int_0^t (\|F(v)(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} + (1+\tau)^{-\frac{1}{3}}\|F(v)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2})d\tau\Big\}. \tag{12.3.85}$$

Using the estimates about product functions and composite functions in Chap. 5, the interpolation inequality and 1° in Corollary 3.4.1 of Chap. 3 (in which we take $n = 2$, $p = 2$, $s = 2$), we obtain

$$\begin{aligned}\|F(v)(\tau,\cdot)\|_{\Gamma,S,\frac{6}{5},\chi_1} &\le C\|v(\tau,\cdot)\|^4_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],3,\chi_1}\|v(\tau,\cdot)\|_{\Gamma,S,2}\\ &\le C\|v(\tau,\cdot)\|^4_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|^{\frac{1}{3}}_{\Gamma,[\frac{S}{2}],\infty,\chi_1}\|v(\tau,\cdot)\|^{\frac{5}{3}}_{\Gamma,S,2}\end{aligned}$$

$$\leq C(1+\tau)^{-\frac{1}{3}}\|v(\tau,\cdot)\|^4_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|^2_{\Gamma,S,2}$$
$$\leq C(1+\tau)^{-\frac{4}{3}+4\sigma}E^6. \tag{12.3.86}$$

Meanwhile, using again the Sobolev estimates on sphere, similarly we have

$$\begin{aligned}\|F(v)(\tau,\cdot)\|_{\Gamma,S,1,2,\chi_2} &\leq C\|v(\tau,\cdot)\|^4_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],2,\infty,\chi_2}\|v(\tau,\cdot)\|_{\Gamma,S,2}\\ &\leq C\|v(\tau,\cdot)\|^4_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|^2_{\Gamma,S,2}\\ &\leq C(1+\tau)^{-1+4\sigma}E^6.\end{aligned} \tag{12.3.87}$$

Plugging (12.3.86) and (12.3.87) in (12.3.85), it is easy to get

$$\|u_3(t,\cdot)\|_{\Gamma,S,2}\leq C(1+t)^{\frac{1}{2}+2\sigma}(\varepsilon+E^6). \tag{12.3.88}$$

Combining (12.3.72), (12.3.84) and (12.3.88), we obtain

$$\|u(t,\cdot)\|_{\Gamma,S,2}\leq C(1+t)^{\frac{1}{2}+2\sigma}\{\varepsilon+E^2(E+D_S(u))\}.\quad\square \tag{12.3.89}$$

Finally, we estimate $\|(Du, D_xDu)(t,\cdot)\|_{\Gamma,S,2}$.

For any given multi-index k ($|k|\leq S$), we still have the energy integral formula (10.2.53) in Chap. 10, where G_k and g_k are given by (10.2.54) and (10.2.55) in Chap. 10, respectively.

When $\alpha=2$, it is easy to show that

$$\begin{aligned}|\mathrm{I}|,|\mathrm{II}|,|\mathrm{III}| &\leq C\int_0^t\|(v,Dv,D^2v)(\tau,\cdot)\|^2_{L^\infty(I\!R^2)}\|D^2u(\tau,\cdot)\|^2_{\Gamma,S,2}d\tau\\ &\leq C\int_0^t\|v(\tau,\cdot)\|^2_{\Gamma,S,\infty}\|D^2u(\tau,\cdot)\|^2_{\Gamma,S,2}d\tau\\ &\leq C(1+\tau)^{2\sigma}E^2D_S^2(u).\end{aligned} \tag{12.3.90}$$

Now we estimate the L^2 norm of $G_k(\tau,\cdot)$.

By the estimates about product functions and composite functions in Chap. 5, when $S\geq 8$ it is clear that

$$\begin{aligned}&\|(\Gamma^kD(b_{ij}(v,Dv)u_{x_ix_j})-b_{ij}(v,Dv)\Gamma^kDu_{x_ix_j})(\tau,\cdot)\|_{L^2(I\!R^2)}\\ &\leq C\|(Dv,D^2v)\|_{\Gamma,[\frac{S}{2}],\infty}(\|(Dv,D^2v)\|_{\Gamma,[\frac{S}{2}],\infty}\|D^2u\|_{\Gamma,S,2}+\|(Dv,D^2v)\|_{\Gamma,S,2}\|D^2u\|_{\Gamma,[\frac{S}{2}],\infty})\\ &\leq C\|v\|_{\Gamma,[\frac{S}{2}]+2,\infty}(\|v\|_{\Gamma,[\frac{S}{2}]+2,\infty}\|D^2u\|_{\Gamma,S,2}+\|(Dv,D^2v)\|_{\Gamma,S,2}\|u\|_{\Gamma,[\frac{S}{2}]+2,\infty})\\ &\leq C\|v\|_{\Gamma,S-2,\infty}(\|v\|_{\Gamma,S-2,\infty}\|D^2u\|_{\Gamma,S,2}+\|(Dv,D^2v)\|_{\Gamma,S,2}\|u\|_{\Gamma,S-2,\infty})\\ &\leq C(1+\tau)^{-1+\sigma}E^2D_S(u).\end{aligned} \tag{12.3.91}$$

Similarly, we have

$$\|b_{ij}(v, Dv)(\Gamma^k Du_{x_i x_j} - (\Gamma^k Du)_{x_i x_j})(\tau, \cdot)\|_{L^2(I\!R^2)} \le C(1+\tau)^{-1+\sigma} E^2 D_S(u). \tag{12.3.92}$$

All the terms involving a_{0j} in G_k can be estimated similarly, then we have

$$\|G_k(\tau, \cdot)\|_{L^2(I\!R^2)} \le C(1+\tau)^{-1+\sigma} E^2 D_S(u), \tag{12.3.93}$$

therefore

$$|\text{IV}| \le C(1+\tau)^{2\sigma} E^2 D_S^2(u). \tag{12.3.94}$$

At last, we estimate the L^2 norm of $g_k(\tau, \cdot)$.

To this end, we still rewrite (10.2.55) of Chap. 10 in the form of (10.2.104).

Using the estimates about product functions and composite functions, especially Lemma 5.1.4, in Chap. 5, and noting $\alpha = 2$ and $S \ge 8$, we have

$$\begin{aligned}
\|\text{I}_1\|_{L^2(I\!R^2)} &\le C\|DF(v, Dv)(\tau, \cdot)\|_{\Gamma,S,2} \le C\|(v, Dv)^2 \cdot (Dv, D^2v)(\tau, \cdot)\|_{\Gamma,S,2} \\
&\le C(\|(v, Dv)^2\|_{\Gamma,[\frac{S}{2}],\infty}\|(Dv, D^2v)\|_{\Gamma,S,2} + \|(v, Dv)(Dv, D^2v)\|_{\Gamma,S,2}\|(v, Dv)\|_{\Gamma,[\frac{S}{2}]+1,\infty}) \\
&\le C\|(v, Dv)\|^2_{\Gamma,[\frac{S}{2}]+1,\infty}\|(Dv, D^2v)\|_{\Gamma,S,2} \le C\|v\|^2_{\Gamma,S-2,\infty}\|(Dv, D^2v)\|_{\Gamma,S,2} \\
&\le C(1+\tau)^{-1+\sigma}E^3.
\end{aligned} \tag{12.3.95}$$

Similarly, we have

$$\|\text{I}_2\|_{L^2(I\!R^2)} \le C(1+\tau)^{-1+\sigma} E^2(E + D_S(u)). \tag{12.3.96}$$

Moreover, from assumption (12.3.29) we have

$$\begin{aligned}
\|\text{I}_3\|_{L^2(I\!R^2)} &\le C\|F(v, 0)(\tau, \cdot)\|_{\Gamma,S,2} \le C\|v\|^5_{\Gamma,[\frac{S}{2}],\infty}\|v\|_{\Gamma,S,2} \\
&\le C\|v\|^5_{\Gamma,S-2,\infty}\|v\|_{\Gamma,S,2} \le C(1+\tau)^{-2+2\sigma}E^6.
\end{aligned} \tag{12.3.97}$$

Thus, we obtain

$$\|g_k(\tau, \cdot)\|_{L^2(I\!R^2)} \le C(1+\tau)^{-1+\sigma} E^2(E + D_S(u)), \tag{12.3.98}$$

then

$$|\text{V}| \le C(1+\tau)^{2\sigma} E^2 D_S(u)(E + D_S(u)). \tag{12.3.99}$$

Similarly to Chap. 10, using (12.3.90), (12.3.94) and (12.3.99), we get

$$\|D^2u(t,\cdot)\|_{\Gamma,S,2} \leq C(1+t)^{\sigma}\{\varepsilon + E(E + D_S(u))\}. \tag{12.3.100}$$

By similar arguments we obtain

$$\|Du(t,\cdot)\|_{\Gamma,S,2} \leq C(1+t)^{\sigma}\{\varepsilon + E(E + D_S(u))\}. \tag{12.3.101}$$

Combining (12.3.59), (12.3.89) and (12.3.100)–(12.3.101), we get the desired (12.3.40).

The proof of Lemma 12.3.1 is finished.

Chapter 13
Sharpness of Lower Bound Estimates on the Life-Span of Classical Solutions to the Cauchy Problem—The Case that the Nonlinear Term $F = F(Du, D_x Du)$ on the Right-Hand Side Does not Depend on u Explicitly

13.1 Introduction

Consider the following Cauchy problem of nonlinear wave equations with small initial data:

$$\Box u = F(Du, D_x Du), \tag{13.1.1}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{13.1.2}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{13.1.3}$$

is the $n-$dimensional wave operator,

$$D_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right), D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right), \tag{13.1.4}$$

φ and ψ are sufficiently smooth functions with compact support, without loss of generality, we assume that

$$\varphi, \psi \in C_0^\infty(I\!R^n) \tag{13.1.5}$$

with

$$\text{supp}\{\varphi, \psi\} \subseteq \{x||x| \le \rho\}\ (\rho > 0 \text{ is a constant}), \tag{13.1.6}$$

and $\varepsilon > 0$ is a small parameter.

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Denote

$$\hat{\lambda} = ((\lambda_i), i = 0, 1, \ldots, n; (\lambda_{ij}), i, j = 0, 1, \ldots, n, i + j \geq 1). \tag{13.1.7}$$

Assume that in a neighborhood of $\hat{\lambda} = 0$, the nonlinear term $F(\hat{\lambda})$ on the right-hand side is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}), \tag{13.1.8}$$

where $\alpha \geq 1$ is an integer.

In Chaps. 8–10, we established lower bound estimates for the life-span $\widetilde{T}(\varepsilon)$ of classical solutions to Cauchy problem (13.1.1)–(13.1.2). In addition to proving the global existence of classical solutions (namely, $\widetilde{T}(\varepsilon) = +\infty$), lower bound estimates for the life-span of classical solutions are listed below, respectively:

(1) When $n = 1$, for any given integer $\alpha \geq 1$, we have

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-\alpha}. \tag{13.1.9}$$

(2) When $n = 2$, for $\alpha = 1$, we have

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-2}; \tag{13.1.10}$$

while, for $\alpha = 2$, we have

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-2}\}. \tag{13.1.11}$$

(3) When $n = 3$, for $\alpha = 1$, we have

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-1}\}. \tag{13.1.12}$$

In (13.1.9)–(13.1.12), a and b are both positive constants independent of ε.

In this chapter we will show the sharpness of the above lower bound estimates on the life-span, i.e., the estimates cannot be improved in general. For this, it suffices to prove that for some specially chosen nonlinear term $F(Du, D_x u)$ on the right-hand side and initial functions $\varphi(x)$ and $\psi(x)$, the life-span of corresponding classical solutions has the upper bound estimates of the same type. Except the case that $n = 2$ and $\alpha = 2$, the sharpness of these lower bound estimates on the life-span was already established earlier, see Lax (1964) (for the case that $n = 1$ and $\alpha = 1$), John (1984) (for the case that $n = 2, 3$ and $\alpha = 1$), Kong (1992) (for the case $n = 1$ and $\alpha \geq 1$) and Zhou (2001) (for the case that $n \geq 1$, and $\alpha \geq 1$ is odd), while, when $n = 2$ and $\alpha = 2$, the corresponding result is due to the recent work of Zhou and Han (2011).

In this chapter, we will consider, as an example, the Cauchy problem of the semi-linear wave equation

$$\Box u = u_t^{1+\alpha} \tag{13.1.13}$$

with the initial data (13.1.2), and prove in a unified way that: for some initial functions $\varphi(x)$ and $\psi(x)$ satisfying certain conditions, the life-span of the corresponding classical solution has the following upper bound estimates:

(1) When $n = 1$, for any given $\alpha \geq 1$ we have

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-\alpha}. \tag{13.1.14}$$

(2) When $n = 2$, for $\alpha = 1$ we have

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-2}; \tag{13.1.15}$$

while, for $\alpha = 2$ we have

$$\widetilde{T}(\varepsilon) \leq \exp\{\bar{a}\varepsilon^{-2}\}. \tag{13.1.16}$$

(3) When $n = 3$, for $\alpha = 1$ we have

$$\widetilde{T}(\varepsilon) \leq \exp\{\bar{a}\varepsilon^{-1}\}. \tag{13.1.17}$$

In (13.1.14)–(13.1.17), $\bar{a}$ and $\bar{b}$ are both positive constants independent of ε.

For this, first, in Sect. 13.2 we will give the upper bound estimates on the life-span of classical solutions to the Cauchy problem of the semi-linear wave equation

$$\Box u = |u_t|^{1+\beta} \tag{13.1.18}$$

with initial value (13.1.2), where β is a positive number. Then, following the results in Sect. 13.2, we prove the desired conclusions (13.1.14)–(13.1.17) in Sect. 13.3, in which some special skills will be adopted in the proof of conclusion (13.1.16).

13.2 Upper Bound Estimates on the Life-Span of Classical Solutions to the Cauchy Problem of a Kind Of Semi-linear Wave Equations

In this section, we consider the following Cauchy problem of semi-linear wave equation with small initial data:

$$\Box u = |u_t|^{1+\beta}, \tag{13.2.1}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{13.2.2}$$

where β is a positive number, $\varepsilon > 0$ is a small parameter, other assumptions are the same as given in (13.1.3) and (13.1.5)–(13.1.6), moreover, we assume that

$$\varphi(x) \geq 0, \ \ \psi(x) \geq 0, \ \ \text{and} \ \ \psi(x) \not\equiv 0. \tag{13.2.3}$$

We want to prove

Lemma 13.2.1 *Suppose that Cauchy problem (13.2.1)–(13.2.2) admits a solution $u = u(t, x)$ on $0 \leq t < \widetilde{T}(\varepsilon)$, such that all the derivations in the proof of this lemma are valid, for instance,*

$$u \in C([0, \widetilde{T}(\varepsilon)); H^1(I\!R^n)), \tag{13.2.4}$$

$$u_t \in C([0, \widetilde{T}(\varepsilon)); L^q(I\!R^n)) \tag{13.2.5}$$

with

$$q = \max(2, 1 + \beta), \tag{13.2.6}$$

and

$$supp\{u\} \subseteq \{(t, x)||x| \leq t + \rho\}. \tag{13.2.7}$$

Then when the initial functions $\varphi(x)$ and $\psi(x)$ satisfy (13.1.5)–(13.1.6) and (13.2.3), we have the following conclusions:

(1) When $\beta < \frac{2}{n-1}$, there exists a positive constant $\bar{b}$ independent of ε, such that

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-\frac{\beta}{1-(n-1)\beta/2}}. \tag{13.2.8}$$

(2) When $\beta = \frac{2}{n-1}$, there exists a positive constant $\bar{a}$ independent of ε, such that

$$\widetilde{T}(\varepsilon) \leq \exp\{\bar{a}\varepsilon^{-\beta}\}. \tag{13.2.9}$$

Remark 13.2.1 Another result similar to Lemma 13.2.1 and a different proof can be found in Zhou (2001).

Proof of Lemma 13.2.1 Introduce the function $F(x)$ in the following way: when $n = 1$, we take

$$F(x) = e^x + e^{-x}; \tag{13.2.10}$$

while, when $n \geq 2$, we take

$$F(x) = \int_{S^{n-1}} e^{x\cdot\omega} d\omega, \tag{13.2.11}$$

where $\omega = (\omega_1, \ldots, \omega_n)$ and $|\omega| = 1$.

Obviously, we have

$$F(x) > 0 \tag{13.2.12}$$

and

$$\Delta F(x) = F(x), \tag{13.2.13}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the $n-$dimensional Laplace operator.

When $n \geq 2$, denoting $\tilde{\omega} = (\omega_2, \ldots, \omega_n)$, from the rotational invariance we have

$$F(x) = \int_{S^{n-1}} e^{|x|\omega_1} d\omega = \int_{\omega_1^2 + \tilde{\omega}^2 = 1} e^{|x|\omega_1} d\omega.$$

Cutting the unit sphere S^{n-1} by planes perpendicular to the ω_1 axis, the above integral can be reduced to the superposition of the integrals over strip sphere element with height $d\omega_1$ and radius $\sqrt{1-\omega_1^2}$ along the ω_1 direction, then it is easy to show that

$$\begin{aligned} F(x) &= C \int_{-1}^{1} e^{|x|\omega_1} (1-\omega_1^2)^{\frac{n-3}{2}} d\omega_1 \\ &= C\left(\int_0^1 e^{|x|\omega_1} (1-\omega_1^2)^{\frac{n-3}{2}} d\omega_1 + \int_{-1}^{0} e^{|x|\omega_1} (1-\omega_1^2)^{\frac{n-3}{2}} d\omega_1 \right) \\ &= C\left(\int_0^1 e^{|x|\omega_1} (1-\omega_1^2)^{\frac{n-3}{2}} d\omega_1 + \int_0^1 e^{-|x|\omega_1} (1-\omega_1^2)^{\frac{n-3}{2}} d\omega_1 \right). \end{aligned}$$

Noting $n \geq 2$, we have

$$\begin{aligned} F(x) &\leq C \int_0^1 e^{|x|\omega_1} (1-\omega_1^2)^{\frac{n-3}{2}} d\omega_1 + C_0 \text{ } (C_0 \text{ is a certain positive constant}) \\ &\leq Ce^{|x|} \int_0^1 e^{-|x|(1-\omega_1)} (1-\omega_1^2)^{\frac{n-3}{2}} d\omega_1 + C_0 \\ &= Ce^{|x|} |x|^{-\frac{n-1}{2}} \int_0^{|x|} e^{-\lambda} \lambda^{\frac{n-3}{2}} d\lambda + C_0 \text{ } (\text{let } \lambda = |x|(1-\omega_1)) \\ &\leq Ce^{|x|} |x|^{-\frac{n-1}{2}} \int_0^{\infty} e^{-\lambda} \lambda^{\frac{n-3}{2}} d\lambda + C_0 \\ &= C_1 e^{|x|} |x|^{-\frac{n-1}{2}}, \end{aligned} \tag{13.2.14}$$

where C_1 is a positive constant. On the other hand, it is obvious that

$$F(x) \le e^{|x|} \int_{S^{n-1}} d\omega = C_2 e^{|x|}, \tag{13.2.15}$$

where C_2 is a positive constant.

Combining (13.2.14) and (13.2.15), and noting (13.2.12), when $n \ge 2$, we have

$$0 < F(x) \le \widetilde{C} e^{|x|}(1 + |x|)^{-\frac{n-1}{2}}, \tag{13.2.16}$$

where $\widetilde{C}$ is a positive constant. While, when $n = 1$, the above formula follows obviously from (13.2.10).

Now let

$$G(t, x) = e^{-t} F(x). \tag{13.2.17}$$

From (13.2.13) it is easy to show that

$$\Delta_x G(t, x) = G(t, x) \tag{13.2.18}$$

and

$$G_t(t, x) = -G(t, x). \tag{13.2.19}$$

Multiplying both sides of Eq. (13.2.1) by $G(t, x)$, and integrating with respect to x, we get

$$\int_{I\!R^n} G(u_{tt} - \Delta u)dx = \int_{I\!R^n} G|u_t|^{1+\beta} dx. \tag{13.2.20}$$

Noticing (13.2.18), we have

$$\int_{I\!R^n} G\Delta u dx = \int_{I\!R^n} \Delta G \cdot u dx = \int_{I\!R^n} G u dx,$$

then (13.2.20) can be rewritten as

$$\int_{I\!R^n} G(u_{tt} - u)dx = \int_{I\!R^n} G|u_t|^{1+\beta} dx. \tag{13.2.21}$$

Noting (13.2.19), we have

$$\frac{d}{dt} \int_{I\!R^n} G u_t dx = \int_{I\!R^n} G(u_{tt} - u_t)dx \tag{13.2.22}$$

and

$$\frac{d}{dt}\int_{I\!R^n} Gudx = \int_{I\!R^n} G(u_t - u)dx, \tag{13.2.23}$$

then form (13.2.21) we get

$$\frac{d}{dt}\int_{I\!R^n} G(u_t + u)dx = \int_{I\!R^n} G|u_t|^{1+\beta}dx. \tag{13.2.24}$$

Thus, integrating with respect to t and using the initial value (13.2.2), we obtain

$$\int_{I\!R^n} G(u_t + u)dx = \varepsilon\int_{I\!R^n} F(x)(\varphi(x) + \psi(x))dx + \int_0^t\int_{I\!R^n} G|u_\tau|^{1+\beta}dxd\tau. \tag{13.2.25}$$

Adding (13.2.21) and (13.2.25), we get

$$\begin{aligned}\int_{I\!R^n} G(u_{tt} + u_t)dx = \varepsilon\int_{I\!R^n} F(x)(\varphi(x) + \psi(x))dx + \int_{I\!R^n} G|u_t|^{1+\beta}dx \\ + \int_0^t\int_{I\!R^n} G|u_\tau|^{1+\beta}dxd\tau.\end{aligned}$$

Noting (13.2.19), the above formula can be rewritten as

$$\begin{aligned}&\frac{d}{dt}\int_{I\!R^n} Gu_tdx + 2\int_{I\!R^n} Gu_tdx \\ &= \varepsilon\int_{I\!R^n} F(x)(\varphi(x) + \psi(x))dx + \int_{I\!R^n} G|u_t|^{1+\beta}dx + \int_0^t\int_{I\!R^n} G|u_\tau|^{1+\beta}dxd\tau.\end{aligned} \tag{13.2.26}$$

Denote

$$H(t) = \int_{I\!R^n} Gu_tdx - \frac{1}{2}\int_0^t\int_{I\!R^n} G|u_\tau|^{1+\beta}dxd\tau - \frac{\varepsilon}{2}\int_{I\!R^n} F(x)\psi(x)dx. \tag{13.2.27}$$

From (13.2.3) and (13.2.12) we have

$$H(0) = \frac{\varepsilon}{2}\int_{I\!R^n} F(x)\psi(x)dx > 0. \tag{13.2.28}$$

By (13.2.26), and noticing (13.2.3) and (13.2.12), we have

$$\frac{d}{dt}H(t)+2H(t)=\frac{d}{dt}\int_{I\!R^n}Gu_t dx+2\int_{I\!R^n}Gu_t dx-\frac{1}{2}\int_{I\!R^n}G|u_t|^{1+\beta}dx$$
$$-\int_0^t\int_{I\!R^n}G|u_\tau|^{1+\beta}dxd\tau-\varepsilon\int_{I\!R^n}F(x)\psi(x)dx$$
$$=\frac{1}{2}\int_{I\!R^n}G|u_t|^{1+\beta}dx+\varepsilon\int_{I\!R^n}F(x)\varphi(x)dx\geq 0,$$

then

$$\frac{d}{dt}\Big(e^{2t}H(t)\Big)\geq 0.$$

Therefore, from (13.2.28) we have

$$H(t)>0,$$

then

$$\int_{I\!R^n}Gu_t dx\geq\frac{\varepsilon}{2}\int_{I\!R^n}F(x)\psi(x)dx+\frac{1}{2}\int_0^t\int_{I\!R^n}G|u_\tau|^{1+\beta}dxd\tau. \tag{13.2.29}$$

Let

$$I(t)=\frac{\varepsilon}{2}\int_{I\!R^n}F(x)\psi(x)dx+\frac{1}{2}\int_0^t\int_{I\!R^n}G|u_\tau|^{1+\beta}dxd\tau. \tag{13.2.30}$$

It is clear that $I(t)>0$, and from (13.2.29) we have

$$I(t)\leq\int_{I\!R^n}Gu_t dx. \tag{13.2.31}$$

Using Hölder inequality, and noting (13.2.7), it follows from the above formula that

$$\begin{aligned}I(t)&\leq\int_{I\!R^n}G^{\frac{\beta}{1+\beta}}\left(G^{\frac{1}{1+\beta}}u_t\right)dx\\&\leq\left(\int_{I\!R^n}Gdx\right)^{\frac{\beta}{1+\beta}}\left(\int_{I\!R^n}G|u_t|^{1+\beta}dx\right)^{\frac{1}{1+\beta}}\\&=\left(\int_{|x|\leq t+\rho}Gdx\right)^{\frac{\beta}{1+\beta}}\left(\int_{|x|\leq t+\rho}G|u_t|^{1+\beta}dx\right)^{\frac{1}{1+\beta}}.\end{aligned} \tag{13.2.32}$$

From (13.2.17) we have

$$\int_{|x|\leq t+\rho}Gdx=e^{-t}\int_{|x|\leq t+\rho}F(x)dx.$$

Using (13.2.16) we have

$$\begin{aligned}\int_{|x|\le t+\rho} F(x)dx &\le C\int_0^{t+\rho} e^r(1+r)^{-\frac{n-1}{2}} r^{n-1}dr\\ &\le C(1+t)^{\frac{n-1}{2}}\int_0^{t+\rho} e^r dr\\ &\le C(1+t)^{\frac{n-1}{2}} e^{t+\rho},\end{aligned}$$

then

$$\int_{|x|\le t+\rho} G dx \le C(1+t)^{\frac{n-1}{2}}. \tag{13.2.33}$$

In addition, from (13.2.30) we have

$$I'(t) = \frac{1}{2}\int_{\mathbb{R}^n} G|u_t|^{1+\beta}dx. \tag{13.2.34}$$

Then, noting (13.2.33), from (13.2.32) we get

$$I'(t) \ge C\frac{I^{1+\beta}(t)}{(1+t)^{\frac{(n-1)\beta}{2}}},$$

that is

$$-\frac{d}{dt}(I^{-\beta}(t)) \ge \widetilde{C}(1+t)^{-\frac{(n-1)\beta}{2}},$$

then it is easy to get

$$I(t) \ge (I^{-\beta}(0) - \widetilde{C}\int_0^t (1+\tau)^{-\frac{(n-1)\beta}{2}}d\tau)^{-\frac{1}{\beta}}.$$

Thus, noting (13.2.30) we have

$$I(t) \ge (\varepsilon^{-\beta} - \widetilde{\widetilde{C}}\int_0^t (1+\tau)^{-\frac{(n-1)\beta}{2}}d\tau)^{-\frac{1}{\beta}}, \tag{13.2.35}$$

where $\widetilde{C}$ and $\widetilde{\widetilde{C}}$ are some positive constants. The conclusion of Lemma 13.2.1 can be drawn easily from (13.2.35). The proof is finished.

13.3 Proof of the Main Results

In this section, we consider the following Cauchy problem of semi-linear wave equation with small initial data:

$$\Box u = u_t^{1+\alpha}, \tag{13.3.1}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{13.3.2}$$

where $\alpha \geq 1$ is an integer, and $\varepsilon > 0$ is a small parameter. Assume that

$$\frac{(n-1)\alpha}{2} \leq 1, \tag{13.3.3}$$

in other words, the values of n and α correspond to the following cases of the main results (13.1.14)–(13.1.17):

$$\begin{aligned} &n = 1,\ \alpha \geq 1 \text{ is any given integer;} \\ &n = 2,\ \alpha = 1 \text{ or } \alpha = 2; \\ &n = 3,\ \alpha = 1. \end{aligned}$$

Theorem 13.3.1 *Let $n = 1$ and $\alpha \geq 1$ be any given integer. Suppose that the initial functions $\varphi(x)$ and $\psi(x)$ satisfy not only (13.1.5)–(13.1.6) and (13.2.3), but also, when α is even,*

$$\varphi(x) \equiv 0. \tag{13.3.4}$$

Then there exists a positive constant $\bar{b}$ independent of ε, such that there is the following upper bound estimate on the life-span $\widetilde{T}(\varepsilon)$ of the classical solution $u = u(t, x)$ to Cauchy problem (13.3.1)–(13.3.2):

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-\alpha}, \tag{13.3.5}$$

i.e., (13.1.14) holds.

Proof Consider the Cauchy problem of the following one-dimensional semi-linear wave equation

$$u_{tt} - u_{xx} = |u_t|^{1+\alpha} \tag{13.3.6}$$

with the same initial data (13.3.2). By (13.2.8) in Lemma 13.2.1 (in which we take $n = 1$ and $\beta = \alpha$), the life-span of its classical solution satisfies (13.3.5).

If α is odd, $|u_t|^{1+\alpha} = u_t^{1+\alpha}$, the desired conclusion for Cauchy problem (13.3.1)–(13.3.2) follows immediately.

If α is even, noting assumption (13.3.4), by D'Alembert formula, the solution to Cauchy problem (13.3.6) and (13.3.2) can be written as

$$u(t,x) = \frac{\varepsilon}{2}\int_{x-t}^{x+t}\psi(\xi)d\xi + \frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}|u_\tau(\tau,y)|^{1+\alpha}dyd\tau.$$

Differentiating with respect to t and noting (13.2.3), we get

$$u_t(t,x) = \frac{\varepsilon}{2}(\psi(x+t)+\psi(x-t)) + \frac{1}{2}\int_0^t(|u_\tau(\tau,x+t-\tau)|^{1+\alpha} \\ + |u_\tau(\tau,x-t+\tau)|^{1+\alpha})d\tau \geq 0,$$

then we still have $|u_t|^{1+\alpha} = u_t^{1+\alpha}$, the desired conclusion can be drawn similarly. The proof is finished. □

Theorem 13.3.2 *Let $n = 2$ and 3, and $\alpha = 1$. Suppose that the initial functions $\varphi(x)$ and $\psi(x)$ satisfy (13.1.5)–(13.1.6) and (13.2.3). Then we have the following upper bound estimates on the life-span $\widetilde{T}(\varepsilon)$ of the classical solution $u = u(t,x)$ to Cauchy problem (13.3.1)–(13.3.2):*

(1) When $n = 2$, there exists a positive constant $\bar{b}$ independent of ε, such that

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-2}, \tag{13.3.7}$$

i.e., (13.1.15) holds.

(2) When $n = 3$, there exists a positive constant $\bar{a}$ independent of ε, such that

$$\widetilde{T}(\varepsilon) \leq \exp\{\bar{a}\varepsilon^{-1}\}, \tag{13.3.8}$$

i.e., (13.1.17) holds.

Proof In (13.2.8) of Lemma 13.2.1, specially taking $n = 2$ and $\beta = \alpha = 1$, we get (13.3.7); while, in (13.2.9) of Lemma 13.2.1, specially taking $n = 3$ and $\beta = \alpha = 1$, we get (13.3.8). Since when $\alpha = 1$, $|u_t|^{1+\alpha} = u_t^{1+\alpha} = u_t^2$, Theorem 13.3.2 follows immediately from Lemma 13.2.1. □

Theorem 13.3.3 (See Zhou and Han 2011) *Let $n = 2$ and $\alpha = 2$. Suppose that the initial functions $\varphi(x)$ and $\psi(x)$ satisfy not only (13.1.5)–(13.1.6) and (13.3.4), but also*

$$\psi(x) = \psi(|x|) \geq 0, \quad and \quad \psi(x) \not\equiv 0. \tag{13.3.9}$$

Then there exists a positive constant $\bar{a}$ independent of ε, such that the life-span $\widetilde{T}(\varepsilon)$ of the classical solution $u = u(t,x)$ to Cauchy problem (13.3.1)–(13.3.2) has the following upper bound estimate:

$$\widetilde{T}(\varepsilon) \leq \exp\{\bar{a}\varepsilon^{-2}\}, \tag{13.3.10}$$

i.e., (13.1.16) holds.

Proof Now Eq. (13.3.1) is the two-dimensional semi-linear wave equation

$$\Box u = u_t^3. \tag{13.3.11}$$

Nest we will prove that: the solution $u = u(t, x)$ to Cauchy problem (13.3.1)–(13.3.2) will always satisfy, on the domain $|x| \geq t$,

$$u \geq 0, \ u_t \geq 0. \tag{13.3.12}$$

From the existence and uniqueness of local solution, the solution $u = u(t, x)$ to Cauchy problem (13.3.1)–(13.3.2) can be obtained by the following Picard iteration:

$$u^{(0)}(t, x) \equiv 0, \tag{13.3.13}$$

and

$$\Box u^{(m)} = \left(u_t^{(m-1)}\right)^3, \tag{13.3.14}$$

$$t = 0 : u^{(m)} = 0, \ u_t^{(m)} = \varepsilon\psi(|x|). \tag{13.3.15}$$

Using the mathematical induction we can prove that: on the domain $|x| \geq t$ we have

$$u^{(m)} \geq 0, \ u_t^{(m)} \geq 0. \tag{13.3.16}$$

In fact, (13.3.16) is obvious when $m = 0$. Now we suppose that $u^{(m-1)}$ satisfies (13.3.16), then, from the positivity of the fundamental solution of the two-dimensional wave equation (see Sect. 2.1.1 and Remark 2.2.2 in Chap. 2), and noting (13.3.9), it follows immediately that the first formula of (13.3.16) holds on the domain $|x| \geq t$. To prove the second formula of (13.3.16) on the domain $|x| \geq t$, noting $\psi(x) = \psi(|x|)$, from the radial symmetry we have

$$u^{(m)}(t, x) = u^{(m)}(t, r), \tag{13.3.17}$$

where $r = |x|$. Thus, Cauchy problem (13.3.14)–(13.3.15) can be rewritten as

$$u_{tt}^{(m)} - u_{rr}^{(m)} - \frac{1}{r} u_r^{(m)} = \left(u_t^{(m-1)}\right)^3, \tag{13.3.18}$$

$$t = 0 : u = 0, \ u_t = \varepsilon\psi(r). \tag{13.3.19}$$

From this it is easy to get

$$\left(\partial_t^2-\partial_r^2\right)\left(r^{\frac{1}{2}}u^{(m)}(t,r)\right)=\frac{1}{4}r^{-\frac{3}{2}}u^{(m)}+r^{\frac{1}{2}}\left(u_t^{(m-1)}\right)^3, \tag{13.3.20}$$

$$t=0: r^{\frac{1}{2}}u^{(m)}=0,\ \left(r^{\frac{1}{2}}u^{(m)}\right)_t=\varepsilon r^{\frac{1}{2}}\psi(r). \tag{13.3.21}$$

(13.3.20)–(13.3.21) can be regarded as a Cauchy problem with respect to $r^{\frac{1}{2}}u^{(m)}(t,r)$ of a one-dimensional wave equation with a non-negative right-hand side. By D'Alembert formula, similarly to the proof in Theorem 13.3.1, we have, on the domain $r \geq t$,

$$r^{\frac{1}{2}}u_t^{(m)} \geq 0,$$

then the second formula in (13.3.16) is proved. This proves (13.3.12).

Similarly to (13.3.20)–(13.3.21), rewriting the Cauchy problem of the two-dimensional semi-linear wave equation (13.3.11) with the initial data

$$t=0: u=0,\ u_t=\varepsilon\psi(|x|) \tag{13.3.22}$$

as

$$(\partial_t^2-\partial_r^2)(r^{\frac{1}{2}}u)=\frac{1}{4}r^{-\frac{3}{2}}u+r^{\frac{1}{2}}u_t^3, \tag{13.3.23}$$

$$t=0: r^{\frac{1}{2}}u=0,\ (r^{\frac{1}{2}}u)_t=\varepsilon r^{\frac{1}{2}}\psi(r), \tag{13.3.24}$$

where $u=u(t,r)$. By d'Alembert formula, on the domain $r \geq t$ we have

$$r^{\frac{1}{2}}u(t,r)=\frac{\varepsilon}{2}\int_{r-t}^{r+t}\Psi(\xi)d\xi+\frac{1}{2}\int_0^t\int_{r-(t-\tau)}^{r+(t-\tau)}\left(\frac{1}{4}\lambda^{-\frac{3}{2}}u(\tau,\lambda)+\lambda^{\frac{1}{2}}u_\tau^3(\tau,\lambda)\right)d\lambda d\tau, \tag{13.3.25}$$

where we denote

$$\Psi(r)=r^{\frac{1}{2}}\psi(r), \tag{13.3.26}$$

and from (13.3.9) we have

$$\Psi \geq 0, \quad \text{and} \quad \Psi \not\equiv 0. \tag{13.3.27}$$

Differentiating (13.3.25) with respect to t, on the domain $r \geq t$ we obtain

$$\begin{aligned}
r^{\frac{1}{2}}u_t(t,r) = &\frac{\varepsilon}{2}(\Psi(r+t)+\Psi(r-t))\\
&+\frac{1}{8}\int_0^t\left[\left(\lambda^{-\frac{3}{2}}u(\tau,\lambda)\right)\Big|_{\lambda=r+t-\tau}+\left(\lambda^{-\frac{3}{2}}u(\tau,\lambda)\right)\Big|_{\lambda=r-t+\tau}\right]d\tau\\
&+\frac{1}{2}\int_0^t\left[\left(\lambda^{\frac{1}{2}}u_\tau^3(\tau,\lambda)\right)\Big|_{\lambda=r+t-\tau}+\left(\lambda^{\frac{1}{2}}u_\tau^3(\tau,\lambda)\right)\Big|_{\lambda=r-t+\tau}\right]d\tau,
\end{aligned}$$

then, noting (13.3.12) and (13.3.27), we have

$$r^{\frac{1}{2}}u_t(t,r) \geq \frac{\varepsilon}{2}\Psi(r-t)+\frac{1}{2}\int_0^t\left(\lambda^{\frac{1}{2}}u_\tau^3(\tau,\lambda)\right)\Big|_{\lambda=r-t+\tau}d\tau. \tag{13.3.28}$$

Due to (13.3.27), there exists a point $\sigma_0 > 0$, such that

$$\Psi(\sigma_0) > 0. \tag{13.3.29}$$

Let

$$v(t) = (t+\sigma_0)^{\frac{1}{2}}u_t(t,t+\sigma_0). \tag{13.3.30}$$

Checking (13.3.28) on $r = t+\sigma_0$, we get

$$v(t) \geq \frac{\varepsilon}{2}\Psi(\sigma_0)+\frac{1}{2}\int_0^t(\tau+\sigma_0)^{-1}v^3(\tau)d\tau. \tag{13.3.31}$$

Let

$$w(t) = \frac{\varepsilon}{2}\Psi(\sigma_0)+\frac{1}{2}\int_0^t(\tau+\sigma_0)^{-1}v^3(\tau)d\tau. \tag{13.3.32}$$

It is obvious that

$$v(t) \geq w(t). \tag{13.3.33}$$

From (13.3.32), and noticing (13.3.33), we have

$$w'(t) = \frac{1}{2}(t+\sigma_0)^{-1}v^3(t) \geq \frac{1}{2}(t+\sigma_0)^{-1}w^3(t) \tag{13.3.34}$$

and

$$w(0) = \frac{\varepsilon}{2}\Psi(\sigma_0) > 0. \tag{13.3.35}$$

From this it is easy to show that

$$w(t) \geq \left[\left(\frac{\varepsilon}{2} \Psi(\sigma_0) \right)^{-2} - \ln \left(\frac{t + \sigma_0}{\sigma_0} \right) \right]^{-\frac{1}{2}},$$

then we get the desired (13.3.10). The proof is finished. □

Chapter 14
Sharpness of Lower Bound Estimates on the Life-Span of Classical Solutions to the Cauchy Problem—The Case that the Nonlinear Term $F = F(u, Du, D_x Du)$ on the Right-Hand Side Depends on u Explicitly

14.1 Introduction

We consider the following Cauchy problem of nonlinear wave equations with small initial data:

$$\Box u = F(u, Du, D_x Du), \tag{14.1.1}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{14.1.2}$$

where $x = (x_1, \ldots, x_n)$,

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{14.1.3}$$

is the n-dimensional wave operator,

$$D_x = \Big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\Big),\ D = \Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\Big), \tag{14.1.4}$$

φ and ψ are sufficiently smooth functions with compact support, without loss of generality, we assume that

$$\varphi, \psi \in C_0^\infty(\mathbb{R}^n) \tag{14.1.5}$$

with

$$\mathrm{supp}\{\varphi, \psi\} \subseteq \{x \,|\, |x| \le \rho\}\ (\rho > 0 \text{ is a constant}), \tag{14.1.6}$$

and $\varepsilon > 0$ is a small parameter.

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Denote

$$\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \ldots, n; (\lambda_{ij}), i, j = 0, 1, \ldots, n, i + j \geq 1). \tag{14.1.7}$$

Suppose that in a neighborhood of $\hat{\lambda} = 0$, the nonlinear term $F(\hat{\lambda})$ on the right-hand side is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}), \tag{14.1.8}$$

where $\alpha \geq 1$ is an integer.

In Chaps. 8–11, we established lower bound estimates on the life-span $\widetilde{T}(\varepsilon)$ of the classical solution $u = u(t, x)$ to Cauchy problem (14.1.1)–(14.1.2). In addition to proving the global existence of classical solutions (namely, $\widetilde{T}(\varepsilon) = +\infty$), the related lower bound estimates on the life-span of classical solutions are listed below, respectively:

(1) When $n = 1$, for any given integer $\alpha \geq 1$, we have

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-\frac{\alpha}{2}}; \tag{14.1.9}$$

when

$$\int_{I\!R} \psi(x)\,dx = 0, \tag{14.1.10}$$

we have

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-\frac{\alpha(1+\alpha)}{2+\alpha}}; \tag{14.1.11}$$

while, when

$$\partial_u^\beta F(0, 0, 0) = 0, \ \forall 1 + \alpha \leq \beta \leq 2\alpha, \tag{14.1.12}$$

we have

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-\alpha}. \tag{14.1.13}$$

(2) When $n = 2$ and $\alpha = 1$, we have

$$\widetilde{T}(\varepsilon) \geq be(\varepsilon), \tag{14.1.14}$$

where $e(\varepsilon)$ is defined by

$$\varepsilon^2 e^2(\varepsilon) \ln\big(1 + e(\varepsilon)\big) = 1; \tag{14.1.15}$$

when

$$\int_{\mathbb{R}^2} \psi(x)\,dx = 0, \tag{14.1.16}$$

we have

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-1}; \tag{14.1.17}$$

while, when

$$\partial_u^2 F(0,0,0) = 0, \tag{14.1.18}$$

we have

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-2}. \tag{14.1.19}$$

(3) When $n = 2$ and $\alpha = 2$, we have

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-6}; \tag{14.1.20}$$

while, when

$$\partial_u^\beta F(0,0,0) = 0, \ \beta = 3, 4, \tag{14.1.21}$$

we have

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-2}\}. \tag{14.1.22}$$

(4) When $n = 3$ and $\alpha = 1$, we have

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-2}; \tag{14.1.23}$$

while, when (14.1.18) is satisfied, we have

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-1}\}. \tag{14.1.24}$$

(5) When $n = 4$ and $\alpha = 1$, we have

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-2}\}. \tag{14.1.25}$$

Here both a and b are positive constants independent of ε.

In this chapter, we are going to prove the sharpness of the above lower bound estimates on the life-span, that is, the estimates cannot be improved in general. For this, it suffices to prove that: for some specially chosen nonlinear term $F(u, Du, D_x Du)$

on the right-hand side and some specially chosen initial functions $\varphi(x)$ and $\psi(x)$, the corresponding life-span of the classical solution has the upper bound estimate of the same type. Due to the results of the previous chapter, among the above lower bound estimates of the life-span, we do not have to worry about the sharpness of (14.1.13), (14.1.19), (14.1.22) and (14.1.24), and we only need to show the sharpness of the lower bound estimates (14.1.9), (14.1.11), (14.1.14), (14.1.17), (14.1.20), (14.1.23) and (14.1.25). Except the case that $n = 4$ and $\alpha = 1$, the sharpness of these lower bound estimates was already obtained earlier, see John (1979), Lindblad (1990a) and Zhou (1992, 1993, 1992), while, the sharpness of (14.1.25) when $n = 4$ and $\alpha = 1$ was recently obtained (see Takamura and Wakasa 2011 and the simplified proof of Zhou and Han 2014).

In this chapter, we will consider, as an example, the Cauchy problem of the following semi-linear wave equation

$$\Box u = u^{1+\alpha} \ (\alpha \geq 1 \text{ is an integer}) \tag{14.1.26}$$

with the initial value (14.1.2), and prove, in a unified way, that: for the initial functions $\varphi(x)$ and $\psi(x)$ satisfying a certain conditions, the life-span of the corresponding classical solution has the following upper bound estimates:

(1) When $n = 1$, for any given integer $\alpha \geq 1$, we have

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-\frac{\alpha}{2}}; \tag{14.1.27}$$

while, when (14.1.10) is satisfied, we have

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-\frac{\alpha(1+\alpha)}{2+\alpha}}. \tag{14.1.28}$$

(2) When $n = 2$ and $\alpha = 1$, we have

$$\widetilde{T}(\varepsilon) \leq \bar{b}e(\varepsilon), \tag{14.1.29}$$

where $e(\varepsilon)$ is defined by (14.1.15); while, when (14.1.16) is satisfied, we have

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-1}. \tag{14.1.30}$$

(3) When $n = 2$ and $\alpha = 2$, we have

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-6}. \tag{14.1.31}$$

(4) When $n = 3$ and $\alpha = 1$, we have

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-2}. \tag{14.1.32}$$

(5) When $n = 4$ and $\alpha = 1$, we have

$$\widetilde{T}(\varepsilon) \le \exp\{\bar{a}\varepsilon^{-2}\}. \tag{14.1.33}$$

Here both $\bar{a}$ and $\bar{b}$ are positive constants independent of ε.

To this end, in Sects. 14.3 and 14.4, we first give the upper bound estimates on the life-span of classical solutions to the Cauchy problem of the semi-linear wave equation of the form

$$\Box u = |u|^p \tag{14.1.34}$$

with the initial value (14.1.2), where $p > 1$ is a real number. Then in Sect. 14.5, we first use the results in Sect. 14.3 to prove (14.1.27)–(14.1.32), then use the results in Sect. 14.4 to prove (14.1.33). To obtain the results in Sects. 14.3 and 14.4, as preliminaries, we first give some lemmas on differential inequalities in Sect. 14.2. In addition, for the need of Sect. 14.4, we give an appendix about Fuchs-type differential equations in Sect. 14.6.

14.2 Some Lemmas on Differential Inequalities

In this section, for the needs of the coming parts, we will give two lemmas about differential inequalities.

Lemma 14.2.1 (see Sideris 1984) *Suppose that the function $I = I(t)$ satisfies the following differential inequalities:*

$$I(t) \ge \delta(1+t)^a, \tag{14.2.1}$$

$$I''(t) \ge C(1+t)^{-b} I^p(t), \tag{14.2.2}$$

where $p > 1$, $a \ge 1$ and $b \ge 0$ are all real numbers, and satisfy

$$(p-1)a > b-2, \tag{14.2.3}$$

and $\delta > 0$ is a small parameter, C is a positive constant. Then $I = I(t)$ must blow up in a finite time, and its life-span satisfies

$$\widetilde{T}(\delta) \le C_0 \delta^{-\mathcal{K}}, \tag{14.2.4}$$

where

$$\mathcal{K} = \frac{p-1}{(p-1)a-b+2}, \tag{14.2.5}$$

and C_0 is a positive constant independent of δ.

Proof We first prove that $I = I(t)$ must blow up in a finite time.

Substituting (14.2.1) into (14.2.2), we get

$$I''(t) \geq C_1(1+t)^{pa-b}, \tag{14.2.6}$$

hereinafter, C_i $(i = 1, 2, \ldots)$ all stand for positive constants.

Noting that from (14.2.3) and $a \geq 1$, we have

$$pa - b > a - 2 \geq -1,$$

integrating (14.2.6) we get

$$I'(t) \geq C_2(1+t)^{pa-b+1} - |I'(0)|,$$

then there exists a $T_1 > 0$, such that when $t \geq T_1$,

$$I'(t) \geq 0. \tag{14.2.7}$$

Thus, when $t \geq T_1$, multiplying both sides of (14.2.2) by $I'(t)$, and noting $b \geq 0$, it is easy to get

$$\begin{aligned}\big(I'^2(t)\big)' &\geq C_3(1+t)^{-b}\big(I^{p+1}(t)\big)' \\ &= C_3\big((1+t)^{-b}I^{p+1}(t)\big)' + C_3 b(1+t)^{-b-1}I^{p+1}(t) \\ &\geq C_3\big((1+t)^{-b}I^{p+1}(t)\big)',\end{aligned}$$

then, noticing (14.2.1), we have

$$I'^2(t) \geq C_3(1+t)^{-b}I^{p+1}(t) - C_4 \geq \frac{C_3}{2}(1+t)^{-b}I^{b+1}(t) + C_5(1+t)^{(p+1)a-b} - C_4.$$

Noticing that, from $a \geq 1$ and (14.2.3) we have $(p+1)a - b > 0$, then, there exists a $T_2 \geq T_1$, such that when $t \geq T_2$, we have

$$I'^2(t) \geq C_6(1+t)^{-b}I^{p+1}(t),$$

so, when $t \geq T_2$ we have

$$I'(t) \geq C_7(1+t)^{-\frac{b}{2}}I^{\frac{p+1}{2}}(t). \tag{14.2.8}$$

Let

$$I(t) = (1+t)^a J(t). \tag{14.2.9}$$

From (14.2.1) we have

$$J(t) \geq \delta. \tag{14.2.10}$$

Plugging (14.2.9) in (14.2.8), and noting (14.2.10), we get

$$\begin{aligned} J'(t) &\geq C_7(1+t)^{-\frac{b}{2}+\frac{(p-1)}{2}a} J^{\frac{p+1}{2}}(t) - \frac{a}{1+t} J(t) \\ &= J(t)\left[C_7(1+t)^{-\frac{b}{2}+\frac{(p-1)}{2}a} J^{\frac{p-1}{2}}(t) - a(1+t)^{-1}\right] \\ &\geq J(t)\left[\frac{C_7}{2}(1+t)^{-\frac{b}{2}+\frac{(p-1)}{2}a} J^{\frac{p-1}{2}}(t) + \frac{C_7}{2}\delta^{\frac{p-1}{2}}(1+t)^{-\frac{b}{2}+\frac{(p-1)}{2}a} - a(1+t)^{-1}\right]. \end{aligned}$$

Noticing that from (14.2.3) we have $-\frac{b}{2} + \frac{(p-1)}{2}a > -1$, then there exists a $T_3 \geq T_2$, such that when $t \geq T_3$ we have

$$J'(t) \geq C_8(1+t)^{-\frac{b}{2}+\frac{(p-1)}{2}a} J^{\frac{p+1}{2}}(t). \tag{14.2.11}$$

Noting $p > 1$, it is clear that $J(t)$ and then $I(t)$ must blow up in a finite time.

Now we prove the upper bound estimate (14.2.4) on the life-span.

Let

$$1 + \tau = \delta^{\frac{p-1}{(p-1)a-b+2}}(1+t) \tag{14.2.12}$$

and

$$H(\tau) = \delta^{\frac{b-2}{(p-1)a-b+2}} I(t). \tag{14.2.13}$$

By (14.2.1)–(14.2.2), it is easy to know that

$$H(\tau) \geq (1+\tau)^a, \tag{14.2.14}$$

$$H''(\tau) \geq C(1+\tau)^{-b} H^p(\tau). \tag{14.2.15}$$

From the above discussion, $H(\tau)$ must blow up in a finite time, then (14.2.4) follows immediately from (14.2.12). The proof is finished. □

Lemma 14.2.2 (See Zhou and Han 2014) *Suppose that the C^2 functions $h = h(t)$ and $k = k(t)$ satisfy, on $0 \leq t < T$,*

$$a(t)h''(t) + h'(t) \leq b(t)h^{1+\alpha}(t), \tag{14.2.16}$$

$$a(t)k''(t) + k'(t) \geq b(t)k^{1+\alpha}(t), \tag{14.2.17}$$

where $\alpha \geq 0$ is a real number, and

$$a(t),\ b(t) > 0,\ 0 \leq t < T. \tag{14.2.18}$$

If

$$k(0) > h(0), \tag{14.2.19}$$
$$k'(0) \geq h'(0), \tag{14.2.20}$$

then we have

$$k'(t) > h'(t),\ 0 < t < T, \tag{14.2.21}$$

hence

$$k(t) > h(t),\ 0 \leq t < T. \tag{14.2.22}$$

Proof Without loss of generality, we may assume that

$$k'(0) > h'(0). \tag{14.2.23}$$

If not, assume $k'(0) = h'(0)$, then from (14.2.16)–(14.2.17) and noticing (14.2.18)–(14.2.19), it is clear that

$$k^{''}(0) > h^{''}(0),$$

then there exists a $\delta_0 > 0$, such that

$$k'(t) > h'(t),\ \forall 0 < t \leq \delta_0,$$

then, noting (14.2.19), we have

$$k(t) > h(t),\ \forall 0 \leq t \leq \delta.$$

Thus, we only need to take $t = \delta_0$ as the initial time in later discussion.

We prove by contradiction. If (14.2.21) does not hold, then noticing (14.2.23), by continuity, there exists a $t^* > 0$, such that

$$k'(t) > h'(t),\ 0 \leq t < t^*, \tag{14.2.24}$$

and

$$k'(t^*) = h'(t^*), \tag{14.2.25}$$

then we have

$$k''(t^*) \le h''(t^*). \tag{14.2.26}$$

On the other hand, noting (14.2.19), by (14.2.24)–(14.2.25) we get

$$k(t) > h(t), \ 0 \le t \le t^*,$$

in particular, we have

$$k(t^*) > h(t^*). \tag{14.2.27}$$

Thus, from (14.2.16)–(14.2.17) and (14.2.18), and noting (14.2.25) and (14.2.27), it is easy to show that

$$k''(t^*) > h''(t^*). \tag{14.2.28}$$

This contradicts (14.2.26). The proof is finished. □

14.3 Upper Bound Estimates on the Life-Span of Classical Solutions to the Cauchy Problem of a Kind of Semi-linear Wave Equations—The Subcritical Case

In this section, we consider the following Cauchy problem of semi-linear wave equations with small initial data:

$$\Box u = |u|^p, \tag{14.3.1}$$

$$t = 0 : u = \varepsilon\varphi(x), \ u_t = \varepsilon\psi(x), \tag{14.3.2}$$

where $p > 1$ is a real number, $\varepsilon > 0$ is a small parameter, other assumptions are the same as given in (14.1.3) and (14.1.5)–(14.1.7).

We first give the following

Lemma 14.3.1 (See Yordanov and Zhang 2006) *Suppose that Cauchy problem (14.3.1)–(14.3.2) admits a solution $u = u(t, x)$ on $0 \le t < \widetilde{T}(\varepsilon)$, such that all the derivations in the proof of this lemma are valid, for instance,*

$$u \in C([0, \widetilde{T}(\varepsilon)); H^1(I\!R^n) \cap L^p(I\!R^n)), \tag{14.3.3}$$

$$u_t \in C([0, \widetilde{T}(\varepsilon)); L^2(I\!R^n)) \tag{14.3.4}$$

and

$$supp\{u\} \subseteq \{(t,x) | |x| \le t + \rho\}. \tag{14.3.5}$$

Suppose furthermore that the initial functions $\varphi(x)$ *and* $\psi(x)$ *satisfy*

$$\int_{I\!R^n} F(x)\varphi(x)dx > 0, \int_{I\!R^n} F(x)\psi(x)dx \ge 0, \tag{14.3.6}$$

where $F(x)$ *is defined by (13.2.10)–(13.2.11) in Chap. 13, i.e.,*

$$F(x) = \begin{cases} e^x + e^{-x}, & n = 1, \\ \int_{S^n} e^{x\cdot\omega} d\omega, & n \ge 2. \end{cases} \tag{14.3.7}$$

Then, when $0 \le t < \widetilde{T}(\varepsilon)$*, we have*

$$\int_{I\!R^n} |u(t,x)|^p dx \ge C_0 \varepsilon^p (1+t)^{n-1-\frac{n-1}{2}p}, \tag{14.3.8}$$

where C_0 *is a positive constant.*

Proof From Sect. 13.2 in Chap. 13, $F(x)$ satisfies

$$\Delta F(x) = F(x) \tag{14.3.9}$$

and

$$0 < F(x) \le \widetilde{C} e^{|x|} (1+|x|)^{-\frac{n-1}{2}}, \tag{14.3.10}$$

where $\widetilde{C}$ is a positive constant.

Similarly to Sect. 13.2 in Chap. 13, let

$$G(t,x) = e^{-t} F(x). \tag{14.3.11}$$

We have

$$G_t(t,x) = -G(t,x), \ G_{tt}(t,x) = G(t,x) \tag{14.3.12}$$

and

$$\Delta_x G(t,x) = G_{tt}(t,x). \tag{14.3.13}$$

Multiplying both sides of Eq. (14.3.1) by $G(t,x)$, and integrating with respect to x, we have

$$\int_{I\!R^n} G(t,x)(u_{tt} - \Delta u)(t,x)dx = \int_{I\!R^n} G(t,x)|u(t,x)|^p dx.$$

Noting (14.3.13), by Green formula we get

$$\int_{I\!R^n} G\Delta u dx = \int_{I\!R^n} \Delta G \cdot u dx = \int_{I\!R^n} G_{tt} u dx,$$

then we get

$$\int_{I\!R^n} (G u_{tt} - G_{tt} u)dx = \int_{I\!R^n} G|u|^p dx,$$

i.e.,

$$\frac{d}{dt}\int_{I\!R^n} (G u_t - G_t u)dx = \int_{I\!R^n} G|u|^p dx.$$

Noticing the first formula in (14.3.12), from the above formula we have

$$\frac{d}{dt}\int_{I\!R^n} (G u_t + G u)dx = \int_{I\!R^n} G|u|^p dx. \tag{14.3.14}$$

Noting that $G > 0$, integrating the above formula with respect to t and using the initial value (14.3.2), we have

$$\int_{I\!R^n} (G u_t + G u)dx \geq \varepsilon \int_{I\!R^n} F(x)(\varphi(x) + \psi(x))dx,$$

then, using again the first formula in (14.3.12), we obtain

$$\frac{d}{dt}\int_{I\!R^n} G u dx + 2\int_{I\!R^n} G u dx \geq \varepsilon \int_{I\!R^n} F(x)(\varphi(x) + \psi(x))dx,$$

i.e.,

$$\frac{d}{dt}\left(e^{2t}\int_{I\!R^n} G u dx\right) \geq \varepsilon e^{2t}\int_{I\!R^n} F(x)(\varphi(x) + \psi(x))dx.$$

Then it yields

$$\int_{I\!R^n} G u dx \geq \varepsilon e^{-2t}\int_{I\!R^n} F(x)\varphi(x)dx + \frac{\varepsilon}{2}(1 - e^{-2t})\int_{I\!R^n} F(x)(\varphi(x) + \psi(x))dx.$$

Hence, it is easy to show by assumption (14.3.6) that

$$\int_{I\!R^n} Gudx \geq C\varepsilon,\ 0 \leq t < \widetilde{T}(\varepsilon), \tag{14.3.15}$$

hereinafter, C stands for a certain positive constant which may take different values at different places.

On the other hand, from Hölder inequality, and noting (14.3.5), we have

$$\int_{I\!R^n} Gudx \leq \left(\int_{I\!R^n} |u|^p dx\right)^{\frac{1}{p}} \left(\int_{|x|\leq t+\rho} G^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}. \tag{14.3.16}$$

But from (14.3.10), it is clear that

$$\int_{|x|\leq t+\rho} G^{\frac{p}{p-1}} dx \leq C \int_0^{t+\rho} e^{-\frac{p}{p-1}(t-r)}(1+r)^{n-1-\frac{n-1}{2}\frac{p}{p-1}} dr. \tag{14.3.17}$$

□

To estimate the integral on the right-hand side of (14.3.17), we will use the following

Remark 14.3.1 For any given positive number q_1 and real number q_2, we have the following estimate:

$$\int_0^{t+\rho} e^{-q_1(t-r)}(1+r)^{q_2} dr \leq C_0(1+t)^{q_2}, \tag{14.3.18}$$

where $\rho > 0$ is a given constant, and C_0 is positive constant.

Proof of Remark 14.3.1

$$\begin{aligned}
&\text{The left-hand side of (14.3.18)}\\
&= \int_0^{\frac{t+\rho}{2}} e^{-q_1(t-r)}(1+r)^{q_2} dr + \int_{\frac{t+\rho}{2}}^{t} e^{-q_1(t-r)}(1+r)^{q_2} dr\\
&\leq C\left(e^{-\frac{q_1}{2}t} \int_0^{\frac{t+\rho}{2}} (1+r)^{q_2} dr + (1+t)^{q_2} \int_{\frac{t+\rho}{2}}^{t} e^{-q_1(t-r)} dr\right)\\
&\leq C_0(1+t)^{q_2}.
\end{aligned}$$

Using Remark 14.3.1, from (14.3.17) we obtain

$$\int_{|x|\leq t+\rho} G^{\frac{p}{p-1}} dx \leq C(1+t)^{n-1-\frac{n-1}{2}\frac{p}{p-1}}. \tag{14.3.19}$$

Thus, from (14.3.16) and noting (14.3.15), we obtain

$$\int_{\mathbb{R}^n} |u(t,x)|^p dx \geq \frac{(\int_{\mathbb{R}^n} Gudx)^p}{(\int_{|x|\leq t+\rho} G^{\frac{p}{p-1}} dx)^{p-1}} \geq C\varepsilon^p (1+t)^{n-1-\frac{n-1}{2}p}, \quad 0 \leq t < \widetilde{T}(\varepsilon).$$

The proof of Lemma 14.3.1 is finished. □

Remark 14.3.2 In Lemma 14.3.1, if condition (14.3.6) is weakened to

$$\int_{\mathbb{R}^n} F(x)\varphi(x)dx \geq 0, \int_{\mathbb{R}^n} F(x)\psi(x)dx \geq 0,$$

and these two are not simultaneously zero, then when $1 \leq t < \widetilde{T}(\varepsilon)$, (14.3.8) holds.

In this section, for $p > 1$, we consider only the subcritical case, i.e., we assume that

$$p < p_0(n), \tag{14.3.20}$$

where $p_0(n)$ is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0. \tag{14.3.21}$$

While, the critical case

$$p = p_0(n) \tag{14.3.22}$$

will be discussed in the next section.

Remark 14.3.3 When $n = 1$, Eq. (14.3.21) has no positive root, therefore, any given real number $p > 1$ belongs to the subcritical case.

Remark 14.3.4 When $n > 1$, it is clear that when $1 < p < p_0(n)$,

$$(n-1)p^2 - (n+1)p - 2 < 0; \tag{14.3.23}$$

while, when $n = 1$, for any given $p > 1$, obviously, the above formula is true as well.

Lemma 14.3.2 (See Sideris 1984) *When $p > 1$ satisfies the subcritical condition (14.3.20), suppose that Cauchy problem (14.3.1)–(14.3.2) admits on $0 \leq t < \widetilde{T}(\varepsilon)$ a solution $u = u(t,x)$ satisfying (14.3.3)–(14.3.5), and that the initial functions $\varphi(x)$ and $\psi(x)$ satisfy not only the requirements given in Remark 14.3.2, but also*

$$\int_{\mathbb{R}^n} \varphi(x)dx \geq 0, \int_{\mathbb{R}^n} \psi(x)dx \geq 0. \tag{14.3.24}$$

Then there exists a positive constant $\bar{b}$ independent of ε, such that

$$\widetilde{T}(\varepsilon) \le \bar{b}\varepsilon^{-\gamma}, \tag{14.3.25}$$

where

$$\gamma = \frac{2p(p-1)}{2+(n+1)p-(n-1)p^2}. \tag{14.3.26}$$

Proof Let

$$I(t) = \int_{I\!R^n} u(t,x)dx. \tag{14.3.27}$$

Integrating Eq. (14.3.1) with respect to x, it is easy to get

$$I''(t) = \int_{I\!R^n} |u(t,x)|^p dx. \tag{14.3.28}$$

From Hölder inequality, and noting (14.3.5), we have

$$|I(t)| \le \left(\int_{I\!R^n} |u(t,x)|^p dx\right)^{\frac{1}{p}} \left(\int_{|x|\le t+\rho} dx\right)^{\frac{p-1}{p}} \le C(1+t)^{\frac{n(p-1)}{p}} \left(\int_{I\!R^n} |u(t,x)|^p dx\right)^{\frac{1}{p}},$$

then from (14.3.28) we get

$$I''(t) \ge C\frac{|I(t)|^p}{(1+t)^{n(p-1)}}. \tag{14.3.29}$$

On the other hand, from Remark 14.3.2 and noting (14.3.28), when $1 \le t < \widetilde{T}(\varepsilon)$ we have

$$I''(t) \ge C\varepsilon^p(1+t)^{n-1-\frac{n-1}{2}p},$$

then, noting (14.3.28), we have

$$I''(t) \ge \begin{cases} 0, & 0 \le t < 1, \\ C\varepsilon^p(1+t)^{n-1-\frac{n-1}{2}p}, & 1 \le t < \widetilde{T}(\varepsilon). \end{cases} \tag{14.3.30}$$

It is easy to prove that: when $n \ge 1$ and $1 < p < p_0(n)$, we always have

$$n-1-\frac{n-1}{2}p > -1.$$

Integrating (14.3.30) with respect to t starting from 0, and using (14.3.24), we obtain: when $1 \leq t < \widetilde{T}(\varepsilon)$, we have

$$I(t) \geq \widetilde{C}\varepsilon^p (1+t)^{n+1-\frac{n-1}{2}p}, \tag{14.3.31}$$

where $\widetilde{C}$ is a positive constant.

Taking $\delta = \widetilde{C}\varepsilon^p$ and $a = n+1-\frac{n-1}{2}p > 1, b = n(p-1) > 0$ in Lemma 14.2.1, and noting (14.3.23), it is easy to verify that

$$(p-1)a - (b-2) > 0,$$

then the desired (14.3.25) follows from (14.3.29) and (14.3.31). □

Lemma 14.3.3 *Let $n = 1$, and $p > 1$ be any given real number. Suppose that Cauchy problem (14.3.1)–(14.3.2) has a solution $u = u(t, x)$ on $0 \leq t < \widetilde{T}(\varepsilon)$, such that all the derivations in the proof of this lemma are valid, for instance,*

$$u \in C([0, \widetilde{T}(\varepsilon)); H^1(\mathbb{R})) \tag{14.3.32}$$

and (14.3.4)–(14.3.5) hold. If the initial function $\psi(x)$ satisfies

$$\int_{\mathbb{R}} \psi(x)dx > 0, \tag{14.3.33}$$

then there must exist a positive constant $\bar{b}$ independent of ε, such that

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-\frac{p-1}{2}}. \tag{14.3.34}$$

Proof We still denote

$$I(t) = \int_{-\infty}^{\infty} u(t, x)dx. \tag{14.3.35}$$

By (14.3.29) we have

$$I''(t) \geq C\frac{|I(t)|^p}{(1+t)^{p-1}}, \tag{14.3.36}$$

in particular,

$$I''(t) \geq 0.$$

Integrating the above formula twice, we get

$$I(t) \geq \varepsilon \left[\left(\int_{-\infty}^{\infty} \psi(x)dx \right) t + \int_{-\infty}^{\infty} \varphi(x)dx \right].$$

Due to assumption (14.3.33), there must exist a $t_0 \geq 0$ depending only on $\int_{-\infty}^{\infty} \varphi(x)dx$ and $\int_{-\infty}^{\infty} \psi(x)dx$, such that

$$I(t) \geq \frac{\varepsilon}{2} \left(\int_{-\infty}^{\infty} \varphi(x)dx \right) t, \ \ t \geq t_0. \tag{14.3.37}$$

Taking $\delta = \bar{C}\varepsilon$ ($\bar{C}$ is a certain positive constant) and $a = 1$, $b = p - 1$ in Lemma 14.2.1, it is easy to verify that

$$(p-1)a - b + 2 = 2 > 0,$$

then the desired estimate (14.3.34) follows immediately from (14.3.36) and (14.3.37). □

Lemma 14.3.4 *Let $n = 2$ and $p = 2$. Suppose that the initial functions $\varphi(x)$ of $\psi(x)$ in Cauchy problem (14.3.1)–(14.3.2) satisfy not only (14.1.5)–(14.1.7) but also*

$$\varphi(x) \equiv 0, \ \ \psi(x) \geq 0 \ \ \text{and} \ \ \psi(x) \not\equiv 0. \tag{14.3.38}$$

Then there exists a positive constant $\bar{b}$ independent of ε, such that the life-span $\widetilde{T}(\varepsilon)$ of the classical solution $u = u(t, x)$ to the Cauchy problem satisfies

$$\widetilde{T}(\varepsilon) \leq \bar{b}e(\varepsilon), \tag{14.3.39}$$

where $e(\varepsilon)$ is defined by

$$\varepsilon^2 e^2(\varepsilon) \ln(1 + e(\varepsilon)) = 1. \tag{14.3.40}$$

Remark 14.3.5 When $n = 2$, $p = 2 < p_0(2)$ belongs to the subcritical case.

Proof of Lemma 14.3.4 From the positiveness of the fundamental solution to the wave equation for $n = 2$ (see Sect. 2.1.1 and Remark 2.2.2 in Chap. 2), it is easy to know that

$$u(t, x) \geq \varepsilon u_0(t, x), \ \ 0 \leq t < \widetilde{T}(\varepsilon), \ \ x \in I\!R^2, \tag{14.3.41}$$

and $u_0(t, x)$ on the right-hand side satisfies

$$\Box u_0(t, x) = 0, \tag{14.3.42}$$

$$t = 0 : u_0 = 0, \ \ \partial_t u_0 = \psi(x). \tag{14.3.43}$$

From the expression of the solution to the Cauchy problem of the wave equation for $n = 2$ (see (2.1.62) and (2.1.64) in Chap. 2), we have

$$u_0(t, x) = C \int_{|y-x|\le t} \frac{\psi(y)}{\sqrt{t^2 - |y-x|^2}} dy. \tag{14.3.44}$$

From the compact support assumption of $\psi(x)$, we can assume in the above formula that $|y| \le \rho$. Thus, when $t - |x| \ge 2\rho$, for $|y - x| \le t$ we have

$$\begin{aligned} t^2 - |y-x|^2 &= (t - |y-x|)(t + |y-x|) \\ &\le 2t(t - |y-x|) \\ &\le 2t(t - |x| + |y|) \\ &\le 2t(t - |x| + \rho). \end{aligned}$$

Then, from (14.3.44) and noting (14.3.38), we obtain: when $t - |x| \ge 2\rho$, we have

$$u_0(t, x) \ge C t^{-\frac{1}{2}} (t - |x| + \rho)^{-\frac{1}{2}}, \tag{14.3.45}$$

where C is a positive constant depending only on $\psi(x)$.

Thus, noticing (14.3.41), we have

$$\begin{aligned} \int_{\mathbb{R}^n} u^2(t, x)dx &\ge \int_{t-|x|\ge 2\rho} u^2(t, x)dx \\ &\ge \varepsilon^2 \int_{t-|x|\ge 2\rho} u_0^2(t, x)dx \\ &\ge C\varepsilon^2 \int_{t-|x|\ge 2\rho} t^{-1}(t - |x| + \rho)^{-1} dx \\ &\ge C\varepsilon^2 t^{-1} \int_0^{t-2\rho} (t - r + \rho)^{-1} r dr. \end{aligned} \tag{14.3.46}$$

Noting that

$$\begin{aligned} \int_0^{t-2\rho} \frac{r}{t - r + \rho} dr &= -(t - 2\rho) + (t + \rho) \int_0^{t-2\rho} \frac{1}{t - r + \rho} dr \\ &= -(t - 2\rho) + (t + \rho) \ln \frac{t + \rho}{3\rho}, \end{aligned}$$

when $t \ge 2\rho$, it yields from (14.3.46) that

$$\int_{\mathbb{R}^2} u^2(t, x)dx \ge C\varepsilon^2 \ln t. \tag{14.3.47}$$

Let

$$I(t) = \int_{\mathbb{R}^2} u(t,x)dx. \tag{14.3.48}$$

From (14.3.28), and noticing that now $p = 2$ and (14.3.47) holds, we obtain: when $t \geq 2\rho$, we have

$$I''(t) \geq C\varepsilon^2 \ln(1+t).$$

By (14.3.28), from $t \geq 0$ we always have $I''(t) \geq 0$. Using similar arguments as in obtaining (14.3.31), integrating twice with respect to t starting from zero, and noticing (14.3.38), it is easy to get: when $t \geq 2\rho$, we have

$$I(t) \geq \widetilde{C}\varepsilon^2(1+t)^2 \ln(1+t). \tag{14.3.49}$$

On the other hand, from (14.3.29), and noticing that now $n = 2$ and $p = 2$, we have

$$I''(t) \geq \widetilde{\widetilde{C}} \frac{|I(t)|^2}{(1+t)^2}. \tag{14.3.50}$$

Here, $\widetilde{C}$ and $\widetilde{\widetilde{C}}$ are some positive constants.

Introducing a new variable τ from

$$1 + t = e(\varepsilon)\tau \tag{14.3.51}$$

where $e(\varepsilon)$ is defined by (14.3.40). Thus, when $\varepsilon > 0$ is suitably small, from (14.3.49) we obtain: when $\tau \geq 2$, we have

$$I(\tau) \geq \widetilde{C}\varepsilon^2 e^2(\varepsilon)(\ln e(\varepsilon) + \ln \tau)\tau^2, \tag{14.3.52}$$

then noticing (14.3.40), it is easy to know that when $\tau \geq 2$, we have

$$I(\tau) \geq \widetilde{C} \frac{\ln e(\varepsilon)}{\ln(1+e(\varepsilon))}\tau^2 \geq \widetilde{C}_1(1+\tau)^2. \tag{14.3.53}$$

Moreover, when $\tau \geq 2$, (14.3.50) can be written as

$$I''(\tau) \geq \widetilde{\widetilde{C}} \frac{|I(\tau)|^2}{\tau^2} \geq \widetilde{\widetilde{C}}_2 \frac{|I(\tau)|^2}{(1+\tau)^2}. \tag{14.3.54}$$

Here, $\widetilde{C}_1$ and $\widetilde{\widetilde{C}}_2$ are some positive constants.

From Lemma 14.2.1 (in which we take $\delta = \widetilde{C}_1$, $p = a = b = 2$), it is clear that the life-span of $I(\tau)$ is finite, then it follows the desired (14.3.39) from (14.3.51). The proof is finished.

14.4 Upper Bound Estimates on the Life-Span of Classical Solutions to the Cauchy Problem of a Kind of Semi-linear Wave Equations—The Critical Case

In this section, according to Zhou and Han (2014), we continue to consider Cauchy problem (14.3.1)–(14.3.2) mentioned in Sect. 14.3 with emphasis only on the critical case of the exponent p, in other words, we consider only the case that $n \geq 2$ and $p = p_0(n)$, where $p_0(n)$ is the positive root of the quadratic equation (14.3.21).

For this purpose, we first consider, on the domain $\{(t, x) | t \geq 0, |x| \leq t\}$, the n-dimensional wave equation

$$\Box \Phi = 0 \tag{14.4.1}$$

and find its solution of the following form:

$$\Phi = \Phi_q = (t + |x|)^{-q} h_q \left(\frac{2|x|}{t + |x|} \right), \tag{14.4.2}$$

where $q > 0$.

Denoting $r = |x|$, and noticing that for the radial function $R = R(r)$, the n-dimensional Laplace operator can be written as

$$\Delta_x R = \frac{n-1}{r} R' + R'', \tag{14.4.3}$$

by plugging (14.4.2) in (14.4.1), it is not hard to prove that $h = h_q(z)$ ($z = \frac{2|x|}{t+|x|}$) satisfies the following ordinary differential equation

$$z(1-z)h''(z) + \left[n - 1 - \left(q + \frac{n+1}{2} \right) z \right] h'(z) - \frac{n-1}{2} q h(z) = 0. \tag{14.4.4}$$

That is to say, $h = h_q(z)$ satisfies the hypergeometric equation

$$z(1-z)h''(z) + [\gamma - (\alpha + \beta + 1)z]h'(z) - \alpha\beta h(z) = 0. \tag{14.4.5}$$

in the case that $\alpha = q$, $\beta = \frac{n-1}{2}$ and $\gamma = n - 1$.

It is known that (see Wang and Guo 1979), the hypergeometric series

$$h = F(\alpha, \beta, \gamma; z) \stackrel{\text{def.}}{=} \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{k! (\gamma)_k} z^k, \tag{14.4.6}$$

which is convergent when $|z| < 1$, is a solution of (14.4.5), where

$$\begin{cases} (\lambda)_0 = 1, \\ (\lambda)_k = \lambda(\lambda + 1) \cdots (\lambda + k - 1) = \dfrac{\Gamma(\lambda + k)}{\Gamma(\lambda)} \quad (k \geq 1). \end{cases} \tag{14.4.7}$$

Then we take, in (14.4.2),

$$h = h_q(z) = F\Big(q, \frac{n-1}{2}, n-1; z\Big). \tag{14.4.8}$$

Proposition 14.4.1 *When* $\gamma > \beta > 0$*, we have*

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt \ (|z| < 1). \tag{14.4.9}$$

Proof When $\gamma > \beta > 0$, noting (2.4.7) with (2.4.5)–(2.4.6) in Chap. 2, it yields from (14.4.6) that

$$\begin{aligned}
F(\alpha, \beta, \gamma; z) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \sum_{k=0}^{\infty} \frac{(\alpha)_k \Gamma(\beta+k)\Gamma(\gamma-\beta)}{k!\Gamma(\gamma+k)} z^k \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} B(\beta+k, \gamma-\beta) z^k \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \int_0^t t^{\beta+k-1}(1-t)^{\gamma-\beta-1} dt \cdot z^k \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} (zt)^k dt \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt.
\end{aligned}$$

□

From proposition 14.4.1, we have

$$h_q(z) = \frac{\Gamma(n-1)}{\Gamma^2(\frac{n-1}{2})} \int_0^1 t^{\frac{n-3}{2}}(1-t)^{\frac{n-3}{2}}(1-zt)^{-q} dt, \tag{14.4.10}$$

then

$$h_q(z) > 0, \ 0 \le z < 1. \tag{14.4.11}$$

Proposition 14.4.2 *When*

$$0 < q < \frac{n-1}{2}, \tag{14.4.12}$$

we have

$$\widetilde{C}_1 \le h_q(z) \le C_1,\ 0 \le z \le 1; \tag{14.4.13}$$

while, when

$$q > \frac{n-1}{2}, \tag{14.4.14}$$

we have

$$\widetilde{C}_2(1-z)^{\frac{n-1}{2}-q} \le h_q(z) \le C_2(1-z)^{\frac{n-1}{2}-q},\ 0 \le z \le 1, \tag{14.4.15}$$

where C_1, $\widetilde{C}_1$, C_2 and $\widetilde{C}_2$ are positive constants.

Proof The hypergeometric equation (14.4.5) is of the standard form of Fuchs-type differential equation with three regular singular points $z = 0$, $z = 1$ and $z = \infty$ (see Sect. 14.6).

Near the singular point $z = 0$, the solution $h = h(z)$ can be written in the from

$$h(z) = z^\rho \sum_{n=0}^{\infty} c_n z^n, \tag{14.4.16}$$

where $c_0 \neq 0$, and ρ is called the index of $h(z)$ at $z = 0$. Plugging (14.4.16) in (14.4.5), noting that

$$h'(z) = \rho z^{\rho-1} \sum_{n=0}^{\infty} c_n z^n + z^\rho \sum_{n=1}^{\infty} n c_n z^{n-1},$$

$$h''(z) = \rho(\rho-1) z^{\rho-2} \sum_{n=0}^{\infty} c_n z^n + 2\rho z^{\rho-1} \sum_{n=1}^{\infty} n c_n z^{n-1} + z^\rho \sum_{n=2}^{\infty} n(n-1) c_n z^{n-2},$$

and comparing the coefficients of the leading term $z^{\rho-1}$, we obtain the index equation which can be used to determine ρ:

$$\rho(\rho-1) + \gamma\rho = 0. \tag{14.4.17}$$

It has two roots

$$\rho = 0 \ \text{ and } \ \rho = 1 - \gamma. \tag{14.4.18}$$

Similarly, near the singular point $z = 1$, the solution $h = h(z)$ can be written to the form of

$$h(z) = (z-1)^\rho \sum_{n=0}^{\infty} c_n (z-1)^n, \tag{14.4.19}$$

where $c_0 \neq 0$, and ρ is called the index of $h(z)$ at $z = 1$. Plugging (14.4.19) in (14.4.5), comparing the coefficients of the leading term $(z-1)^{\rho-1}$, we obtain that the index equation to determine ρ is

$$\rho(\rho-1)-(\gamma-(\alpha+\beta+1))\rho=0. \tag{14.4.20}$$

It has two roots

$$\rho=0 \ \text{ and } \ \rho=\gamma-\alpha-\beta. \tag{14.4.21}$$

Now we specifically study the hypergeometric equation (14.4.4), in which $\alpha=q$, $\beta=\frac{n-1}{2}$ and $\gamma=n-1$. At this moment, the hypergeometric series solution (14.4.8) is the solution corresponding to the index $\rho=0$ near $z=0$.

When q satisfies (14.4.12), due to (14.4.10), $h_q(z)$ is convergent at $z=1$, and $h_q(1)>0$, then, noting that $\gamma-\alpha-\beta=\frac{n-1}{2}-q>0$, its index ρ at $z=1$ is also 0. Thus, noting (14.4.11), $h_q(z)$, for real z, is continuous and positive on $0\le z\le 1$ so (14.4.13) follows.

When q satisfies (14.4.14), from (14.4.11) we know that $h_q(z)$, for real z, is continuous and positive on $0\le z<1$ and $h_q(z)$ is divergent at $z=1$, so its index at $z=1$ is impossible to be zero, and must be

$$\gamma-\alpha-\beta=\frac{n-1}{2}-q<0.$$

This proves (14.4.15). □

Proposition 14.4.3 *For the function $\Phi_q(t,x)$ defined by (14.4.2), we have*

$$\frac{\partial\Phi_q(t,x)}{\partial t}=-q\Phi_{q+1}(t,x). \tag{14.4.22}$$

Proof It is easy to know from (14.4.2) that, to prove (14.4.22) it suffices to prove

$$qh_q(z)+zh_q'(z)=qh_{q+1}(z). \tag{14.4.23}$$

From (14.4.6) and noting $\alpha=q$, $\beta=\frac{n-1}{2}$ and $\gamma=n-1$, we have

$$\begin{aligned} qh_q(z)+zh_q'(z) &= q\sum_{k=0}^{\infty}\frac{(q)_k(\frac{n-1}{2})_k}{k!(n-1)_k}z^k+\sum_{k=1}^{\infty}\frac{(q)_k(\frac{n-1}{2})_k}{(k-1)!(n-1)_k}z^k \\ &= \sum_{k=0}^{\infty}\frac{(q+k)(q)_k(\frac{n-1}{2})_k}{k!(n-1)_k}z^k=q\sum_{k=0}^{\infty}\frac{(q+1)_k(\frac{n-1}{2})_k}{k!(n-1)_k}=qh_{q+1}(z). \end{aligned}$$

The proof is finished. □

Now we continue to consider Cauchy problem (14.3.1)–(14.3.2), where $\varepsilon>0$ is a small parameter, and suppose that (14.1.5)–(14.1.7) still hold.

Lemma 14.4.1 *Assume that* $n \geq 2,\ p = p_0(n)$*, and the initial functions satisfy*

$$\varphi(x) \geq 0,\ \ \psi(x) \geq 0\ \ \textit{and}\ \ \psi(x) \not\equiv 0. \tag{14.4.24}$$

Suppose that Cauchy problem (14.3.1)–(14.3.2) has a solution $u = u(t,x)$ *on* $0 \leq t < \widetilde{T}(\varepsilon)$*, such that all the derivations in the proof of this lemma are valid, and*

$$supp\{u\} \subseteq \{(t,x) \| x| \leq t + \rho\}. \tag{14.4.25}$$

Denote

$$G(t) = \int_0^t (t-\tau)(1+\tau)\int_{I\!R^n} \widetilde{\Phi}_q(\tau,x)|u(\tau,x)|^p dx d\tau, \tag{14.4.26}$$

where

$$q = \frac{n-1}{2} - \frac{1}{p} \tag{14.4.27}$$

and

$$\widetilde{\Phi}_q(t,x) = \Phi_q(t+\rho+1,x). \tag{14.4.28}$$

Then we have

$$G'(t) \geq \mathcal{K}_0(2+t)(\ln(2+t))^{-(p-1)}\Big(\int_0^t (2+\tau)^{-3}G(\tau)d\tau\Big)^p,\quad 1 \leq t < \widetilde{T}(\varepsilon), \tag{14.4.29}$$

where $\mathcal{K}_0$ *is a positive constant independent of* ε*.*

Remark 14.4.1 For $n \geq 2$ and $p = p_0(n)$, it is easy to show that, for the q defined by (14.4.27), we have $q > 0$.

Remark 14.4.2 From the definition of $\widetilde{\Phi}_q(t,x)$, its domain of definition is $\{(t,x) \| x| \leq t+\rho+1\}$. Then, noting (14.4.25), all the integrals over the whole space $I\!R^n$ in the following proof make sense.

Proof of Lemma 14.4.1 From (14.4.26), we have

$$G'(t) = \int_0^t (1+\tau)\int_{I\!R^n} \widetilde{\Phi}_q(\tau,x)|u(\tau,x)|^p dx d\tau \tag{14.4.30}$$

and

$$G''(t) = (1+t)\int_{I\!R^n} \widetilde{\Phi}_q(t,x)|u(t,x)|^p dx. \tag{14.4.31}$$

Multiplying both sides of Eq. (14.3.1) by $\widetilde{\Phi}_q(t,x)$ and integrating with respect to x, we have

$$\int_{\mathbb{R}^n} \widetilde{\Phi}_q (u_{tt} - \Delta u) dx = \int_{\mathbb{R}^n} \widetilde{\Phi}_q |u|^p dx. \tag{14.4.32}$$

Noting that $\widetilde{\Phi}_q$ satisfies the wave equation (14.4.1), using Green formula, we have

$$\int_{\mathbb{R}^n} \widetilde{\Phi}_q \Delta u dx = \int_{\mathbb{R}^n} \Delta \widetilde{\Phi}_q \cdot u dx = \int_{\mathbb{R}^n} \widetilde{\Phi}_{qtt} u dx,$$

then

$$\int_{\mathbb{R}^n} \widetilde{\Phi}_q (u_{tt} - \Delta u) dx = \int_{\mathbb{R}^n} (\widetilde{\Phi}_q u_{tt} - \widetilde{\Phi}_{qtt} u) dx = \frac{d}{dt} \int_{\mathbb{R}^n} (\widetilde{\Phi}_q u_t - \widetilde{\Phi}_{qt} u) dx. \tag{14.4.33}$$

But using Proposition 14.4.3, we have

$$\begin{aligned}\int_{\mathbb{R}^n} (\widetilde{\Phi}_q u_t - \widetilde{\Phi}_{qt} u) dx =& \frac{d}{dt} \int_{\mathbb{R}^n} \widetilde{\Phi}_q u dx - 2 \int_{\mathbb{R}^n} \widetilde{\Phi}_{qt} u dx \\ =& \frac{d}{dt} \int_{\mathbb{R}^n} \widetilde{\Phi}_q u dx + 2q \int_{\mathbb{R}^n} \widetilde{\Phi}_{q+1} u dx.\end{aligned} \tag{14.4.34}$$

Plugging (14.4.33) and (14.4.34) in (14.4.32), we get

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^n} \widetilde{\Phi}_q u dx + 2q \frac{d}{dt} \int_{\mathbb{R}^n} \widetilde{\Phi}_{q+1} u dx = \int_{\mathbb{R}^n} \widetilde{\Phi}_q |u|^p dx. \tag{14.4.35}$$

Integrating the above formula with respect to t starting from 0, and noticing that the value of

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^n} \widetilde{\Phi}_q u dx + 2q \int_{\mathbb{R}^n} \widetilde{\Phi}_{q+1} u dx &= \int_{\mathbb{R}^n} (\widetilde{\Phi}_{qt} u + \widetilde{\Phi}_q u_t) dx + 2q \int_{\mathbb{R}^n} \widetilde{\Phi}_{q+1} u dx \\ &= \int_{\mathbb{R}^n} (q \widetilde{\Phi}_{q+1} u + \widetilde{\Phi}_q u_t) dx\end{aligned}$$

at $t = 0$ is

$$\varepsilon \int_{\mathbb{R}^n} (q \widetilde{\Phi}_{q+1}(0, x) \varphi(x) + \widetilde{\Phi}_q(0, x) \psi(x)) dx,$$

we have

$$\begin{aligned}&\frac{d}{dt} \int_{\mathbb{R}^n} \widetilde{\Phi}_q u dx + 2q \int_{\mathbb{R}^n} \widetilde{\Phi}_{q+1} u dx \\ &= \varepsilon \int_{\mathbb{R}^n} (q \widetilde{\Phi}_{q+1}(0, x) \varphi(x) + \widetilde{\Phi}_q(0, x) \psi(x)) dx + \int_0^t \int_{\mathbb{R}^n} \widetilde{\Phi}_q |u|^p dx d\tau.\end{aligned} \tag{14.4.36}$$

Integrating the above formula with respect to t starting from 0, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}\widetilde{\Phi}_q u dx + 2q\int_0^t\int_{\mathbb{R}^n}\widetilde{\Phi}_{q+1}u dx d\tau\\
=&\varepsilon\int_{\mathbb{R}^n}\widetilde{\Phi}_q(0,x)\varphi(x)dx + \varepsilon t\int_{\mathbb{R}^n}(q\widetilde{\Phi}_{q+1}(0,x)\varphi(x)+\widetilde{\Phi}_q(0,x)\psi(x))dx\\
&+\int_0^t(t-\tau)\int_{\mathbb{R}^n}\widetilde{\Phi}_q|u|^p dx d\tau.
\end{aligned} \tag{14.4.37}$$

Integrating again the above formula with respect to t starting from 0, we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^n}\widetilde{\Phi}_q u dx d\tau + 2q\int_0^t(t-\tau)\int_{\mathbb{R}^n}\widetilde{\Phi}_{q+1}u dx d\tau\\
=&\varepsilon t\int_{\mathbb{R}^n}\widetilde{\Phi}_q(0,x)\varphi(x)dx + \frac{1}{2}\int_0^t(t-\tau)^2\int_{\mathbb{R}^n}\widetilde{\Phi}_q|u|^p dx d\tau\\
&+\frac{\varepsilon}{2}t^2\int_{\mathbb{R}^n}(q\widetilde{\Phi}_{q+1}(0,x)\varphi(x)+\widetilde{\Phi}_q(0,x)\psi(x))dx.
\end{aligned} \tag{14.4.38}$$

From this and noticing (14.4.11) and assumption (14.4.24), we get

$$\int_0^t\int_{\mathbb{R}^n}\widetilde{\Phi}_q u dx d\tau + 2q\int_0^t(t-\tau)\int_{\mathbb{R}^n}\widetilde{\Phi}_{q+1}u dx d\tau \geq \frac{1}{2}\int_0^t(t-\tau)^2\int_{\mathbb{R}^n}\widetilde{\Phi}_q|u|^p dx d\tau. \tag{14.4.39}$$

Using Hölder inequality and the expression (14.4.30) of $G'(t)$, and noting (14.4.25), we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^n}\widetilde{\Phi}_q u dx d\tau\\
=&\int_0^t\int_{\mathbb{R}^n}\Big((1+\tau)^{\frac{1}{p}}\widetilde{\Phi}_q^{\frac{1}{p}}u\Big)\cdot\Big((1+\tau)^{-\frac{1}{p}}\widetilde{\Phi}_q^{\frac{1}{p'}}\Big)dx d\tau\\
\leq&\Big(\int_0^t\int_{\mathbb{R}^n}(1+\tau)\widetilde{\Phi}_q|u|^p dx d\tau\Big)^{\frac{1}{p}}\Big(\int_0^t\int_{|x|\leq\tau+\rho}(1+\tau)^{-\frac{p'}{p}}\widetilde{\Phi}_q dx d\tau\Big)^{\frac{1}{p'}}\\
=&(G'(t))^{\frac{1}{p}}\Big(\int_0^t\int_{|x|\leq\tau+\rho}(1+\tau)^{-\frac{p'}{p}}\widetilde{\Phi}_q dx d\tau\Big)^{\frac{1}{p'}},
\end{aligned} \tag{14.4.40}$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

From (14.4.27), it is obvious that $0 < q < \frac{n-1}{2}$, then it is easy to know from Proposition 14.4.2 that

$$C_1(1+\tau)^{-q} \leq \widetilde{\Phi}_q(\tau,x) \leq C_2(1+\tau)^{-q}, \tag{14.4.41}$$

where C_1 and C_2 are positive constants. Thus, we have

$$\int_0^t \int_{|x|\le\tau+\rho} (1+\tau)^{-\frac{p'}{p}} \widetilde{\Phi}_q dx d\tau \le C \int_0^t (1+\tau)^{n-q-\frac{p'}{p}} d\tau.$$

But from $p = p_0(n)$ and the definition (14.4.27) of q, it is easy to know that

$$n - q - \frac{p'}{p} = 1 + \frac{p'}{p},$$

then from (14.4.40) we get

$$\int_0^t \int_{I\!R^n} \widetilde{\Phi}_q u dx d\tau \le C(G'(t))^{\frac{1}{p}} (1+t)^{2-\frac{1}{p}}. \tag{14.4.42}$$

Moreover, from (14.4.27) and noting $p > 1$, it is obvious that $q + 1 > \frac{n-1}{2}$, then it is easy to know from Proposition 14.4.2 that

$$\begin{aligned} &C_3(1+\tau)^{-\frac{n-1}{2}}(1+\rho+\tau-|x|)^{-(q+1-\frac{n-1}{2})} \\ &\le \widetilde{\Phi}_{q+1}(\tau, x) \le C_4(1+\tau)^{-\frac{n-1}{2}}(1+\rho+\tau-|x|)^{-(q+1-\frac{n-1}{2})}, \end{aligned} \tag{14.4.43}$$

where C_3 and C_4 are positive constants. Then, using Hölder inequality and the expression (14.4.30) of $G'(t)$, and noting (14.4.25), we obtain

$$\begin{aligned} &\int_0^t (t-\tau) \int_{I\!R^n} \widetilde{\Phi}_{q+1} u dx d\tau \\ &= \int_0^t \int_{I\!R^n} \Big((1+\tau)^{\frac{1}{p}} \widetilde{\Phi}_q^{\frac{1}{p}} u\Big)\Big((t-\tau)\widetilde{\Phi}_q^{\frac{1}{p'}} \Big(\frac{\widetilde{\Phi}_{q+1}}{\widetilde{\Phi}_q}\Big)(1+\tau)^{-\frac{1}{p}}\Big) dx d\tau \\ &\le (G'(t))^{\frac{1}{p}} \Big(\int_0^t (t-\tau)^{p'} \int_{|x|\le\tau+\rho} \widetilde{\Phi}_q \Big(\frac{\widetilde{\Phi}_{q+1}}{\widetilde{\Phi}_q}\Big)^{p'} (1+\tau)^{-\frac{p'}{p}} dx d\tau\Big)^{\frac{1}{p'}}. \end{aligned} \tag{14.4.44}$$

While, from Propositions 14.4.2 and 14.4.3, it is easy to show that

$$\begin{aligned} &\int_{|x|\le\tau+\rho} \widetilde{\Phi}_q \Big(\frac{\widetilde{\Phi}_{q+1}}{\widetilde{\Phi}_q}\Big)^{p'} (1+\tau)^{-\frac{p'}{p}} dx \\ &\le C(1+\tau)^{n-1+q(p'-1)-\frac{n-1}{2}p'-\frac{p'}{p}} \int_0^{\tau+\rho} (1+\rho+\tau-r)^{-p'(q+1-\frac{n-1}{2})} dr. \end{aligned} \tag{14.4.45}$$

From (14.4.27) and noting $p = p_0(n)$, it is easy to get

$$p'\Big(q + 1 - \frac{n-1}{2}\Big) = 1$$

and

$$n-1+q(p'-1)-\frac{n-1}{2}p'-\frac{p'}{p}=0,$$

then

$$\int_{|x|\le\tau+\rho}\widetilde{\Phi}_q\left(\frac{\widetilde{\Phi}_{q+1}}{\widetilde{\Phi}_q}\right)^{p'}(1+\tau)^{-\frac{p'}{p}}dx\le C\int_0^{\tau+\rho}(1+\rho+\tau-r)^{-1}dr\le C\ln(2+\tau). \tag{14.4.46}$$

Then it follows easily from (14.4.44) that

$$\int_0^t(t-\tau)\int_{I\!R^n}\widetilde{\Phi}_{q+1}u\,dx\,d\tau\le C(G'(t))^{\frac{1}{p}}(1+t)^{2-\frac{1}{p}}(\ln(2+t))^{\frac{1}{p'}}. \tag{14.4.47}$$

Plugging (14.4.42) and (14.4.47) in (14.4.39), and noting (14.4.31), we get

$$(G'(t))^{\frac{1}{p}}(1+t)^{2-\frac{1}{p}}(\ln(2+t))^{1-\frac{1}{p}}\ge C\int_0^t(t-\tau)^2(1+\tau)^{-1}G''(\tau)d\tau. \tag{14.4.48}$$

Integrating by parts, and noting that $G'(0)=G(0)=0$, we have

$$\begin{aligned}\int_0^t(t-\tau)^2(1+\tau)^{-1}G''(\tau)d\tau &= -\int_0^t\partial_\tau[(t-\tau)^2(1+\tau)^{-1}]G'(\tau)d\tau\\ &=\int_0^t\partial_\tau^2[(t-\tau)^2(1+\tau)^{-1}]G(\tau)d\tau.\end{aligned}$$

Since

$$\partial_\tau^2[(t-\tau)^2(1+\tau)^{-1}]=2(1+t)^2(1+\tau)^{-3},$$

from (14.4.48) it follows

$$(G'(t))^{\frac{1}{p}}(1+t)^{2-\frac{1}{p}}(\ln(2+t))^{1-\frac{1}{p}}\ge C(1+t)^2\int_0^t(1+\tau)^{-3}G(\tau)d\tau,$$

that is,

$$G'(t)\ge C(1+t)(\ln(2+t))^{-(p-1)}\left(\int_0^t(1+\tau)^{-3}G(\tau)d\tau\right)^p.$$

From this we immediately get the desired (14.4.29). The proof of Lemma 14.4.1 is finished.

Lemma 14.4.2 *Under the assumptions of Lemma 14.4.1, we assume furthermore that $\varphi(x)\not\equiv 0$, then for Cauchy problem (14.3.1)–(14.3.2), there exists a positive constant $\bar{a}$ independent of ε, such that*

$$\widetilde{T}(\varepsilon) \le \exp\{\bar{a}\varepsilon^{-p(p-1)}\}. \tag{14.4.49}$$

Proof Let

$$H(t) = \int_0^t (2+\tau)^{-3} G(\tau) d\tau, \tag{14.4.50}$$

where $G(t)$ is defined by (14.4.26). We have

$$H'(t) = (2+t)^{-3} G(t), \tag{14.4.51}$$

i.e.,

$$G(t) = (2+t)^3 H'(t). \tag{14.4.52}$$

Then, (14.4.29) can be rewritten as

$$((2+t)^3 H'(t))' \ge \mathcal{K}_0 (2+t)(\ln(2+t))^{-(p-1)} H^p(t). \tag{14.4.53}$$

From the definition (14.4.26) of $G(t)$, noting (14.4.41) and using Lemma 14.3.1, we have

$$\begin{aligned} G(t) &= \int_0^t (t-\tau)(1+\tau) \int_{\mathbb{R}^n} \widetilde{\Phi}_q(\tau,x) |u(\tau,x)|^p dx d\tau \\ &\ge C \int_0^t (t-\tau)(1+\tau)^{1-q} \int_{\mathbb{R}^n} |u(\tau,x)|^p dx d\tau \\ &\ge C\varepsilon^p \int_0^t (t-\tau)(1+\tau)^{1-q+n-1-\frac{n-1}{2}p} d\tau. \end{aligned} \tag{14.4.54}$$

From (14.4.27) and $p = p_0(n)$, it is clear that

$$1 - q + n - 1 - \frac{n-1}{2} p = 0,$$

then from the above formula we get

$$G(t) \ge C\varepsilon^p t^2.$$

So, from (14.4.50) and (14.4.51), when $t \ge 1$ we obtain

$$\begin{aligned} H(t) &\ge C\varepsilon^p \int_0^t (2+\tau)^{-3} \tau^2 d\tau \ge C\varepsilon^p \int_1^t (2+\tau)^{-3}\tau^2 d\tau \\ &\ge C\varepsilon^p \int_1^t (2+\tau)^{-1} d\tau \ge C_0 \varepsilon^p \ln(2+t) \end{aligned} \tag{14.4.55}$$

and

$$H'(t) \geq C\varepsilon^p(2+t)^{-3}t^2 \geq C_0\varepsilon^p(2+t)^{-1}, \tag{14.4.56}$$

where C_0 is a certain positive constant.

From (14.4.53) we have

$$(2+t)^2H''(t) + 3(2+t)H'(t) \geq \mathcal{K}_0(\ln(2+t))^{-(p-1)}H^p(t). \tag{14.4.57}$$

Taking the change of variables

$$\tau = \ln(2+t), \tag{14.4.58}$$

and denoting

$$H_0(\tau) = H(t) = H(e^\tau - 2), \tag{14.4.59}$$

we have

$$H_0'(\tau) = (2+t)H'(t), \tag{14.4.60}$$

$$H_0''(\tau) = (2+t)^2H''(t) + (2+t)H'(t). \tag{14.4.61}$$

Then, (14.4.57) and (14.4.55)–(14.4.56) can be rewritten, respectively, as

$$H_0''(\tau) + 2H_0'(\tau) \geq \mathcal{K}_0\tau^{-(p-1)}H_0^p(\tau), \tag{14.4.62}$$

$$H_0(\tau) \geq C_0\varepsilon^p\tau, \tag{14.4.63}$$

$$H_0'(\tau) \geq C_0\varepsilon^p. \tag{14.4.64}$$

Denoting

$$H_1(s) = \varepsilon^{p(p-2)}H_0(\varepsilon^{-p(p-1)}s), \tag{14.4.65}$$

we obtain correspondingly that

$$\varepsilon^{p(p-1)}H_1''(s) + 2H_1'(s) \geq \mathcal{K}_0 s^{-(p-1)}H_1^p(s), \tag{14.4.66}$$

$$H_1(s) \geq C_0 s, \tag{14.4.67}$$

$$H_1'(s) \geq C_0. \tag{14.4.68}$$

Now we take positive constants s_0 and δ independent of ε, such that the positive constants $\mathcal{K}_0$ and C_0 appearing in (14.4.66)–(14.4.68) satisfy

$$\mathcal{K}_0, C_0 \ll s_0 \ll \frac{1}{\delta}. \tag{14.4.69}$$

Let

$$H_2(s) = sH_3(s), \tag{14.4.70}$$

and $H_3(s)$ be determined by solving the following Cauchy problem of ordinary differential equation:

$$H_3'(s) = \delta H_3^{\frac{p+1}{2}}(s), \ \ s \geq s_0, \tag{14.4.71}$$

$$H_3(s_0) = \frac{C_0}{4}. \tag{14.4.72}$$

Then, when $s \geq s_0$ it is easy to know that

$$H_2'(s) = H_3(s) + \delta s H_3^{\frac{p+1}{2}}(s), \tag{14.4.73}$$

$$H_2''(s) = 2\delta H_3^{\frac{p+1}{2}}(s) + \frac{1}{2}(p+1)\delta^2 s H_3^p(s), \tag{14.4.74}$$

so, when $s \geq s_0$ we have

$$\begin{aligned}&\varepsilon^{p(p-1)} H_2''(s) + 2H_2'(s)\\ =&\frac{1}{2}(p+1)\delta^2\varepsilon^{p(p-1)}s^{-(p-1)}H_2^p(s) + 2\delta\varepsilon^{p(p-1)}H_3^{\frac{p+1}{2}}(s) + 2\delta s H_3^{\frac{p+1}{2}}(s) + 2H_3(s).\end{aligned} \tag{14.4.75}$$

Noting that from (14.4.71)–(14.4.72) we have

$$H_3(s) \geq \frac{C_0}{4}, \ \ s \geq s_0, \tag{14.4.76}$$

when $s \geq s_0$ we have

$$\frac{1}{4}\mathcal{K}_0 s^{-(p-1)} H_2^p(s) = \frac{1}{4}\mathcal{K}_0 s H_3^p(s) \geq \frac{1}{4}\mathcal{K}_0 s_0 \left(\frac{C_0}{4}\right)^{p-1} H_3(s).$$

Therefore, as long as s_0 is large enough, we have $\frac{1}{4}\mathcal{K}_0 s_0\left(\frac{C_0}{4}\right)^{p-1} > 1$, so, when $s \geq s_0$,

$$H_3(s) \leq \frac{1}{4}\mathcal{K}_0 s^{-(p-1)} H_2^p(s). \tag{14.4.77}$$

Moreover, noting (14.4.76), as long as $\delta > 0$ is small enough, when $s \geq s_0$, we have

$$2\delta\varepsilon^{p(p-1)} H_3^{\frac{p+1}{2}}(s) + 2\delta s H_3^{\frac{p+1}{2}}(s) \leq \frac{1}{4}\mathcal{K}_0 s H_3^p(s) = \frac{1}{4}\mathcal{K}_0 s^{-(p-1)} H_2^p(s), \tag{14.4.78}$$

and it is obvious that

$$\frac{1}{2}(p+1)\delta^2\varepsilon^{p(p-1)} \leq \frac{\mathcal{K}_0}{4}. \tag{14.4.79}$$

Thus, from (14.4.75) we get

$$\varepsilon^{p(p-1)}H_2''(s) + 2H_2'(s) \leq \mathcal{K}_0 s^{-(p-1)}H_2^p(s). \tag{14.4.80}$$

Moreover, from (14.4.72)–(14.4.73) it is easy to know that, when $\delta > 0$ is small enough, we have

$$H_2(s_0) \leq C_0 s_0, \tag{14.4.81}$$

$$H_2'(s_0) \leq C_0. \tag{14.4.82}$$

Hence, using Lemma 14.2.2, from (14.4.66)–(14.4.68) and (14.4.80)–(14.4.82) we immediately have: when $s \geq s_0$,

$$H_1(s) \geq H_2(s) = sH_3(s). \tag{14.4.83}$$

Noting that $H_3(s)$ is the solution to the Riccati equation (14.4.71), there must exists a value $s_1(> 0)$ independent of ε, such that when $s = s_1$, $H_3(s)$ and then $H_2(s)$ tend to the infinity, then, from (14.4.83), the life-span of $H_1(s)$ has an upper bound s_1. By (14.4.65), the life-span of $H_0(s)$ has an upper bound $\varepsilon^{-p(p-1)}s_1$, then with (14.4.59), the life-span of $H(t)$ has an upper bound $\exp\{\varepsilon^{-p(p-1)}s_1\}$. This proves the conclusion in Lemma 14.4.2. □

14.5 Proof of the Main Results

In this section, we consider the following Cauchy problem of semi-linear wave equation with small initial data:

$$\Box u = u^{1+\alpha}, \tag{14.5.1}$$

$$t = 0 : u = \varepsilon\varphi(x),\ u_t = \varepsilon\psi(x), \tag{14.5.2}$$

where $\alpha \geq 1$ is an integer, $\varepsilon > 0$ is a small parameter, and the initial functions $\varphi(x)$ and $\psi(x)$ satisfy (14.1.5) and (14.1.7).

We first look at the case $n = 1$. We have

Theorem 14.5.1 *Let $n = 1$, and $\alpha \geq 1$ be any give integer. Denote by $\widetilde{T}(\varepsilon)$ the life-span of the classical solution $u = u(t, x)$ to Cauchy problem (14.5.1)–(14.5.2).*

(1) *If*

$$\varphi(x) \geq 0, \quad \psi(x) \geq 0 \tag{14.5.3}$$

and

$$\int_{I\!R} \psi(x)dx > 0, \tag{14.5.4}$$

then

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-\frac{\alpha}{2}}. \tag{14.5.5}$$

(2) *If*

$$\varphi(x) \geq 0 \ \ with \ \ \varphi(x) \not\equiv 0, \quad and \ \ \psi(x) \equiv 0, \tag{14.5.6}$$

then

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-\frac{\alpha(1+\alpha)}{2+\alpha}}. \tag{14.5.7}$$

In (14.5.5) and (14.5.7), $\bar{b}$ is a positive constant independent of ε. This proves the desired (14.1.27) and (14.1.28), respectively.

Proof We consider the Cauchy problem of the following one-dimensional semi-linear wave equation

$$u_{tt} - u_{xx} = |u|^{1+\alpha} \tag{14.5.8}$$

with the same initial value (14.5.2).

Due to D'Alembert formula, the solution to Cauchy problem (14.5.8) and (14.5.2) can be expressed by

$$u(t,x) = \frac{\varepsilon}{2}(\varphi(x+t) + \varphi(x-t)) + \frac{\varepsilon}{2}\int_{x-t}^{x+t} \psi(\xi)d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} |u(\tau, y)|^{1+\alpha} dy d\tau.$$

Under assumption (14.5.3), it is clear from the above formula that

$$u(t,x) \geq 0,$$

then the solution to Cauchy problem (14.5.8) and (14.5.2) is exactly the solution to Cauchy problem (14.5.1)–(14.5.2).

Taking $p = 1 + \alpha$ in Lemma 14.3.1, and noting (14.5.4), we get (14.5.5) immediately, which proves the conclusion in (1).

Moreover, taking $n = 1$ and $p = 1 + \alpha$ in Lemma 14.3.2, and noting (14.5.6), we get (14.5.7) immediately, which proves the conclusion in (2). □

Now we look at the case $n = 2$. We have

Theorem 14.5.2 *Let $n = 2$ and $\alpha = 1$. Denoting by $\widetilde{T}(\varepsilon)$ the life-span of the classical solution $u = u(t, x)$ to Cauchy problem (14.5.1)–(14.5.2).*

(1) *If*

$$\varphi(x) \equiv 0, \ \psi(x) \geq 0 \ \text{and} \ \psi(x) \not\equiv 0, \tag{14.5.9}$$

then

$$\widetilde{T}(\varepsilon) \leq \bar{b}e(\varepsilon), \tag{14.5.10}$$

where $e(\varepsilon)$ is defined by

$$\varepsilon^2 e^2(\varepsilon) \ln(1 + e(\varepsilon)) = 1. \tag{14.5.11}$$

(2) *If*

$$\int_{I\!R^2} \varphi(x)dx > 0, \ \int_{I\!R^2} \psi(x)dx = 0, \tag{14.5.12}$$

then

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-1}. \tag{14.5.13}$$

In (14.5.10) and (14.5.13), $\bar{b}$ is a positive constant independent of ε. This proves the desired (14.1.29) and (14.1.30), respectively.

Proof When $\alpha = 1$, Cauchy problem (14.5.1)–(14.5.2) is exactly the Cauchy problem of the semi-linear wave equation

$$\Box u = |u|^{1+\alpha} \tag{14.5.14}$$

with the same initial value (14.5.2).

Thus, the desired (14.5.10) in (1) follows immediately from Lemma 14.3.4; while, taking $n = 2$ and $p = 1 + \alpha = 2$ in Lemma 14.3.1, and noting that $p < p_0(2) = \frac{3+\sqrt{17}}{2}$, which belongs to the subcritical case, the desired (14.5.13) in (2) follows immediately. □

Theorem 14.5.3 *Let $n = 2$ and $\alpha = 2$. If (14.5.9) holds, then there exists a positive constant $\bar{b}$ independent of ε, such that the life-span $\widetilde{T}(\varepsilon)$ of the classical solution $u = u(t, x)$ to Cauchy problem (14.5.1)–(14.5.2) has the following upper bound estimate:*

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-6}. \tag{14.5.15}$$

This proves the desired (14.1.31).

Proof Consider the Cauchy problem of the two-dimensional semi-linear wave equation

$$\Box u = |u|^3 \tag{14.5.16}$$

with the initial value (14.5.2). From the positiveness of the fundamental solution to the wave equation for $n = 2$ (see Sect. 2.1.1 and Remark 2.2.2 in Chap. 2), under assumption (14.5.9), the solution $u = u(t, x)$ to the Cauchy problem must satisfy

$$u(t, x) \geq 0,$$

then it is also the solution to the corresponding Cauchy problem (14.5.1)–(14.5.2).

Taking $n = 2$ and $p = 1 + \alpha = 3$ in Lemma 14.3.1, and noticing that $p < p_0(2) = \frac{3+\sqrt{17}}{2}$, which belongs to the subcritical case, the desired (14.5.15) follows immediately. □

Now we look at the case $n = 3$. We have

Theorem 14.5.4 *Let $n = 3$ and $\alpha = 1$. If*

$$\varphi(x) \geq 0, \quad \psi(x) \geq 0, \tag{14.5.17}$$

and $\varphi(x)$ and $\psi(x)$ are not identically equal to zero simultaneously, then there exists a positive constant $\bar{b}$ independent of ε, such that the life-span of the classical solution $u = u(t, x)$ to Cauchy problem (14.5.1)–(14.5.2) satisfies

$$\widetilde{T}(\varepsilon) \leq \bar{b}\varepsilon^{-2}. \tag{14.5.18}$$

This proves the desired (14.1.32).

Proof Due to $\alpha = 1$, Cauchy problem (14.5.1)–(14.5.2) is exactly Cauchy problem (14.5.14) and (14.5.2). Taking $n = 3$ and $p = 1 + \alpha = 2$ in Lemma 14.3.2, and noting that $p < p_0(3) = 1 + \sqrt{2}$, which belongs to the subcritical case, the desired (14.5.18) follows immediately. □

Finally we look at the case that $n = 4$ and $\alpha = 1$. Since now $p = 1 + \alpha = p_0(n) = 2$, which belongs to the critical case, we need to use the result in Sect. 14.4. We have

Theorem 14.5.5 *Let $n = 4$ and $\alpha = 1$. If (14.5.17) holds, and $\varphi(x) \not\equiv 0$, then there exists a positive constant $\bar{a}$ independent of ε, such that the life-span of the classical solution $u = u(t, x)$ to Cauchy problem (14.5.1)–(14.5.2) satisfies*

$$\widetilde{T}(\varepsilon) \leq \exp\{\bar{a}\varepsilon^{-2}\}. \tag{14.5.19}$$

This proves (14.1.33).

Proof Due to $\alpha = 1$, Cauchy problem (14.5.1)–(14.5.2) is exactly Cauchy problem (14.5.14) and (14.5.2). Taking $n = 4$ and $p = p_0(n) = 2$ in Lemma 14.4.2, we immediately obtain the desired estimate. □

14.6 Appendix—Fuchs-Type Differential Equations and Hypergeometric Equations

14.6.1 Regular Singular Points of Second-Order Linear Ordinary Differential Equations

We consider the following second-order linear ordinary differential equation

$$w'' + p(z)w' + q(z)w = 0, \tag{14.6.1}$$

where $w = w(z)$ is the unknown function, and the coefficients $p(z)$ and $q(z)$, except for finite number of isolated singular points, are single-valued analytic functions of z.

Let $z = z_0$ be a singular point of p and q. If

$$(z - z_0)p(z) \text{ and } (z - z_0)^2 q(z) \tag{14.6.2}$$

are analytic in a neighborhood of $z = z_0$, i.e., $z = z_0$ is no worse than a first-order pole of $p(z)$ and a second-order pole of $q(z)$, then $z = z_0$ is called a **regular singular point** of Eq. (14.6.1). At this moment, we can find a solution (called a **regular solution**) of the following form, to Eq. (14.6.1) in a neighborhood of $z = z_0$:

$$w(z) = (z - z_0)^\rho \sum_{n=0}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^{\rho+n}, \tag{14.6.3}$$

where ρ and the coefficients c_n $(n = 0, 1, 2, \ldots)$ are all constants to be determined, and $c_0 \neq 0$.

Equation (14.6.1) can be rewritten as

$$(z - z_0)^2 w'' + (z - z_0) p_1(z) w' + q_1(z) w = 0, \tag{14.6.4}$$

where

$$\begin{cases} p_1(z) \overset{\text{def.}}{=} (z-z_0)p(z) = \sum\limits_{k=0}^{\infty} a_k(z-z_0)^k, \\ q_1(z) \overset{\text{def.}}{=} (z-z_0)^2 q(z) = \sum\limits_{k=0}^{\infty} b_k(z-z_0)^k. \end{cases} \tag{14.6.5}$$

Plugging (14.6.3) in (14.6.4), removing the common factor $(z-z_0)^\rho$, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} c_n(\rho+n)(\rho+n-1)(z-z_0)^n \\ &+\sum_{k=0}^{\infty} a_k(z-z_0)^k \cdot \sum_{n=0}^{\infty} c_n(\rho+n)(z-z_0)^n \\ &+\sum_{k=0}^{\infty} b_k(z-z_0)^k \cdot \sum_{n=0}^{\infty} c_n(z-z_0)^n = 0. \end{aligned} \tag{14.6.6}$$

Setting the lowest-order term (i.e., the term not containing $z-z_0$) in the above formula to be 0, and noticing $c_0 \neq 0$, we obtain

$$\rho(\rho-1) + a_0\rho + b_0 = 0,$$

that is,

$$\rho^2 + (a_0-1)\rho + b_0 = 0. \tag{14.6.7}$$

This is the equation to determine ρ, called the **index equation**.

Now setting the coefficients of $(z-z_0)^n$ $(n \geq 1)$ in (14.6.6) to be 0, respectively, we obtain the following recursive relations:

$$[(\rho+n)(\rho+n-1) + a_0(\rho+n) + b_0]c_n + \sum_{k=1}^{n} [a_k(\rho+n-k) + b_k]c_{n-k} = 0 \quad (n = 1, 2, \quad \cdots). \tag{14.6.8}$$

Suppose that ρ is a root of the index equation (14.6.7), and for any given integer $n \geq 1$, $\rho + n$ is no longer a root of the index equation (14.6.7), in other words,

$$(\rho+n)(\rho+n-1) + a_0(\rho+n) + b_0 \neq 0 \quad (n = 1, 2, \ldots), \tag{14.6.9}$$

then using the recursive relations (14.6.8), all the c_n $(n = 1, 2, \ldots)$ can be determined in turn by c_0. Since (14.6.1) is a linear equation, according to the superposition principle, we can always take $c_0 = 1$ in advance, therefore, all the coefficients c_n $(n = 0, 1, 2, \ldots)$ can be determined in turn. Thus, we obtain a regular solution of the form (14.6.3) to Eq. (14.6.1).

Hence, if the difference between two roots ρ_1 and ρ_2 of the index equation (14.6.7) is not an integer, we can use the above method to find two linearly independent regular solutions of the form (14.6.3) to Eq. (14.6.1) in a neighborhood of z_0. Their linear combination constitutes the general solution to Eq. (14.6.1).

If the difference between two roots ρ_1 and ρ_2 of the index equation (14.6.7) is an integer(including the case of multiple roots), we can only find one regular solution of the form (14.6.3) by using the above method. But we can prove that: if there exists a $m \in \{0, 1, 2, \ldots\}$, such that

$$\rho_1 - \rho_2 = m, \tag{14.6.10}$$

then in addition to the regular solution

$$w_1 = (z - z_0)^{\rho_1} \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad (c_0 \neq 0) \tag{14.6.11}$$

of the form (14.6.3) by using the above method for ρ_1, we can also obtain another regular solution of the form

$$w_2 = (z - z_0)^{\rho_2} \sum_{n=0}^{\infty} d_n (z - z_0)^n + \gamma w_1 \ln(z - z_0) \; (d_0 \neq 0), \tag{14.6.12}$$

where γ is a constant, which may be 0 in some special cases. The linear combination of these two solutions constitutes the general solution to Eq. (14.6.1).

The two roots ρ_1 and ρ_2 of the index equation (14.6.7) are called the **index** of the regular singular point $z = z_0$, denoted by (ρ_1, ρ_2).

What we investigate up to now is the case that $z = z_0$ is a finite regular singular point. Whether the infinity is a regular singular point will be determined by whether $t = 0$ is a regular singular point of the equation under the transform

$$z = \frac{1}{t}. \tag{14.6.13}$$

It is easy to know that Eq. (14.6.1) is reduced to the following equation under the transform (14.6.13):

$$t^4 \frac{d^2 w}{dt^2} + \left[2t^3 - t^2 p\left(\frac{1}{t}\right)\right] \frac{dw}{dt} + q\left(\frac{1}{t}\right) w = 0. \tag{14.6.14}$$

Thus, $t = 0$ is the regular singular point of Eq. (14.6.14) provided that $t\bar{p}(t)$ and $t^2\bar{q}(t)$ are analytic in a neighborhood of $t = 0$, where

$$\bar{p}(t) = \frac{2}{t} - \frac{1}{t^2} p\left(\frac{1}{t}\right), \quad \bar{q}(t) = \frac{1}{t^4} q\left(\frac{1}{t}\right). \tag{14.6.15}$$

Then, $p\left(\frac{1}{t}\right)$ and $q\left(\frac{1}{t}\right)$ should have the following expansions:

$$p\left(\frac{1}{t}\right) = d_1 t + d_2 t^2 + \cdots ,$$
$$q\left(\frac{1}{t}\right) = d_2' t + d_3' t^2 + \cdots ,$$

so, $p(z)$ and $q(z)$ should have the following expansions near $z = \infty$:

$$\begin{cases} p(z) = \dfrac{d_1}{z} + \dfrac{d_2}{z^2} + \cdots , \\ q(z) = \dfrac{d_2'}{z^2} + \dfrac{d_3'}{z^3} + \cdots , \end{cases} \tag{14.6.16}$$

that is, $zp(z)$ and $z^2 q(z)$ are analytic near $z = \infty$, in other words, $z = \infty$ is at least a first-order zero of $p(z)$ and a second-order zero of $q(z)$.

Noting that, now for Eq. (14.6.14), we have $a_0 = 2 - d_1$ and $b_0 = d_2'$ in the index equation (14.6.7) at the point $t = 0$, so when $z = \infty$ is a regular singular point, the corresponding index equation is

$$\rho^2 + (1 - d_1)\rho + d_2' = 0. \tag{14.6.17}$$

14.6.2 Fuchs-Type Differential Equations

Equation (14.6.1) whose singular points (the total number is supposed to be finite) are all regular singular points is called a **Fuchs-type differential equation**. For later use, here we always assume that $z = \infty$ is a regular singular point, and that all the finite regular singular points of the equation are $\alpha_1, \ldots, \alpha_n$.

From the definition of regular singular points, $p(z)$ is at most first-order for poles $z = \alpha_i$ $(i = 1, \ldots, n)$ and is zero at $z = \infty$, and then it can be written in the form of rational fraction

$$p(z) = \frac{\bar{p}(z)}{(z - \alpha_1) \cdots (z - \alpha_n)}, \tag{14.6.18}$$

where $\bar{p}(z)$ is a polynomial of order $(n - 1)$ at most. Similarly, $q(z)$ is at most second-order for poles $z = \alpha_i$ $(i = 1, \ldots, n)$, and is zero of at least second-order at $z = \infty$, so

$$q(z) = \frac{\bar{q}(z)}{(z - \alpha_1)^2 \cdots (z - \alpha_n)^2}, \tag{14.6.19}$$

where $\bar{q}(z)$ is a polynomial of order $(2n-2)$ at most. Decomposing the rational fractions (14.6.18)–(14.6.19) into the simplest fractions, we obtain the general expression for the coefficients of the Fuchs-type differential equation as

$$\begin{cases} p(z) = \sum_{k=1}^{n} \frac{A_k}{z-\alpha_k}, \\ q(z) = \sum_{k=1}^{n} \left[\frac{B_k}{(z-\alpha_k)^2} + \frac{C_k}{z-\alpha_k} \right], \end{cases} \tag{14.6.20}$$

where $A_k,\ B_k$ and C_k $(k = 1, \ldots, n)$ are constants, and since $q(z)$ is zero of at least second-order at $z = \infty$, we have

$$\sum_{k=1}^{n} C_k = 0. \tag{14.6.21}$$

From the above discussion, it is easy to know that the index equation at $z = \alpha_k$ is

$$\rho^2 + (A_k - 1)\rho + B_k = 0 \ \ (k = 1, \ldots, n). \tag{14.6.22}$$

Moreover, noting that

$$\frac{1}{z-\alpha_k} = \frac{1}{z} \frac{1}{1 - \frac{\alpha_k}{z}} = \frac{\alpha_k}{z^2} + \frac{\alpha_k^2}{z^3} + \cdots ,$$

it is easy to know that the index equation at $z = \infty$ is

$$\rho^2 + \Big(1 - \sum_{k=1}^{n} A_k\Big)\rho + \sum_{k=1}^{n} (B_k + \alpha_k C_k) = 0. \tag{14.6.23}$$

From (14.6.22) and (14.6.23) we know that the sum of the indexes of all the regular singular points is equal to

$$n - \sum_{k=1}^{n} A_k + \sum_{k=1}^{n} A_k - 1 = n - 1, \tag{14.6.24}$$

i.e., the number of finite regular singular points minus 1.

14.6.3 Hypergeometric Equations

Now we consider specifically the Fuchs-type differential equation with three regular singular points $z = a, b$ and ∞. The corresponding indexes of these three points are denoted by (α_1, α_2), (β_1, β_2) and (γ_1, γ_2), respectively. From (14.6.24) we have

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 1, \tag{14.6.25}$$

i.e., the sum of all the indexes is 1.

Due to (14.6.20), now the coefficients of the equation can be written as

$$\begin{cases} p(z) = \dfrac{A_1}{z-a} + \dfrac{A_2}{z-b}, \\ q(z) = \dfrac{B_1}{(z-a)^2} + \dfrac{C_1}{z-a} + \dfrac{B_2}{(z-b)^2} + \dfrac{C_2}{z-b}, \end{cases} \tag{14.6.26}$$

and

$$C_1 + C_2 = 0. \tag{14.6.27}$$

By (14.6.22)–(14.6.23), the corresponding index equations can be written as

$$\begin{cases} \rho^2 + (A_1 - 1)\rho + B_1 = 0, \\ \rho^2 + (A_2 - 1)\rho + B_2 = 0, \\ \rho^2 + (1 - A_1 - A_2)\rho + (B_1 + B_2 + aC_1 + bC_2) = 0. \end{cases} \tag{14.6.28}$$

By Vièta theorem, we have

$$\begin{cases} \alpha_1 + \alpha_2 = 1 - A_1, & \alpha_1\alpha_2 = B_1, \\ \beta_1 + \beta_2 = 1 - A_2, & \beta_1\beta_2 = B_2, \\ \gamma_1 + \gamma_2 = A_1 + A_2 - 1, & \gamma_1\gamma_2 = B_1 + B_2 + aC_1 + bC_2. \end{cases} \tag{14.6.29}$$

Noting (14.6.27), from this we obtain that

$$\begin{cases} A_1 = 1 - \alpha_1 - \alpha_2, \ A_2 = 1 - \beta_1 - \beta_2, \\ B_1 = \alpha_1\alpha_2, \ B_2 = \beta_1\beta_2, \\ C_1 = -C_2 = \dfrac{\gamma_1\gamma_2 - \alpha_1\alpha_2 - \beta_1\beta_2}{a-b}, \end{cases} \tag{14.6.30}$$

then the corresponding Fuchs-type differential equations can be written as

$$\begin{aligned} &w'' + \left\{ \frac{1-\alpha_1-\alpha_2}{z-a} + \frac{1-\beta_1-\beta_2}{z-b} \right\} w' \\ &\quad + \frac{1}{(z-a)(z-b)} \left\{ \frac{\alpha_1\alpha_2(a-b)}{z-a} + \frac{\beta_1\beta_2(b-a)}{z-b} + \gamma_1\gamma_2 \right\} w = 0. \end{aligned} \tag{14.6.31}$$

Thus, the form of the Fuchs-type differential equation under consideration can be determined completely by its regular singular points a, b, ∞ and their corresponding indexes. Then, all the solutions to Eq. (14.6.31) can be denoted by

$$w = P\left\{\begin{array}{l} a, \quad b, \quad \infty \\ \alpha_1, \ \beta_1, \ \gamma_1 \ ; z \\ \alpha_2, \ \beta_2, \ \gamma_2 \end{array}\right\}. \tag{14.6.32}$$

This notation was first introduced by Riemann.

Now we explain why we can always assume, without loss of generality, that

$$a = 0, \ b = 1 \tag{14.6.33}$$

and

$$\alpha_1 = \beta_1 = 0. \tag{14.6.34}$$

At this moment, due to (14.6.34), and noting (14.6.25), we can take

$$\begin{cases} \alpha_1 = 0, \ \alpha_2 = 1 - \gamma, \\ \beta_1 = 0, \ \beta_2 = \gamma - \alpha - \beta, \\ \gamma_1 = \alpha, \ \gamma_2 = \beta. \end{cases} \tag{14.6.35}$$

Thus, Eq. (14.6.31) can be simplified to

$$w'' + \left(\frac{\gamma}{z} + \frac{1 - \gamma + \alpha + \beta}{z - 1}\right) w' + \frac{\alpha\beta w}{z(z-1)} = 0$$

or

$$z(z-1)w'' + [\gamma - (\alpha + \beta + 1)z]w' - \alpha\beta w = 0, \tag{14.6.36}$$

and its solutions can be expressed by

$$w = P\left\{\begin{array}{ccc} 0, & 1, & \infty \\ 0, & 0, & \alpha \ ; z \\ 1-\gamma, & \gamma - \alpha - \beta, & \beta \end{array}\right\}. \tag{14.6.37}$$

We first claim that we can always assume (14.6.33).

Under suitable fractional linear transformation of the independent variables

$$\zeta = \frac{Az + B}{Cz + D},$$

we can always reduce the three singular points to $\zeta = 0$, 1 and ∞. In the original case that the singular points are $z = a, b$ and ∞, this fractional linear transformation can be taken as, say,

$$\zeta = \frac{b-a}{z-a}. \tag{14.6.38}$$

It turns $z = a$ into $\zeta = \infty$, $z = b$ into $\zeta = 1$, and $z = \infty$ into $\zeta = 0$. Under this transformation, it is easy to prove that Eq. (14.6.31) is turned into

$$w'' + \left\{ \frac{1-\gamma_1-\gamma_2}{\zeta} + \frac{1-\beta_1-\beta_2}{\zeta-1} \right\} w' + \frac{1}{\zeta(\zeta-1)} \left\{ -\frac{\gamma_1\gamma_2}{\zeta} + \frac{\beta_1\beta_2}{\zeta-1} + \alpha_1\alpha_2 \right\} w = 0. \tag{14.6.39}$$

It has three singular points $\zeta = 0, 1$ and ∞, and is a Fuchs-type differential equation with index (γ_1, γ_2) at $\zeta = 0$, index (β_1, β_2) at $\zeta = 1$ and index (α_1, α_2) at $\zeta = \infty$. This explains why we can assume (14.6.33). This also states that the corresponding indices are invariant under the fractional linear transformation (14.6.38).

Now we claim that we can always assume (14.6.34).

When (14.6.33) holds, Eq. (14.6.31) can be written as

$$w'' + \left\{ \frac{1-\alpha_1-\alpha_2}{z} + \frac{1-\beta_1-\beta_2}{z-1} \right\} w' + \frac{1}{z(z-1)} \left\{ -\frac{\alpha_1\alpha_2}{z} + \frac{\beta_1\beta_2}{z-1} + \gamma_1\gamma_2 \right\} w = 0. \tag{14.6.40}$$

Perform the transformation of unknown functions

$$w = z^p (z-1)^q u. \tag{14.6.41}$$

It is easy to verify directly that: the equation of the unknown function u still has three regular singular points $z = 0, 1$ and ∞, but the index at $z = 0$ is turned from (α_1, α_2) into $(\alpha_1 - p, \alpha_2 - p)$, the index at $z = 1$ is turned from (β_1, β_2) into $(\beta_1 - q, \beta_2 - q)$, and correspondingly, the index at $z = \infty$ is turned from (γ_1, γ_2) into $(\gamma_1 + p + q, \gamma_2 + p + q)$. Therefore, taking specially $p = \alpha_1$ and $q = \beta_1$ allows (14.6.34) to hold for the equation of u.

Hence, for the Fuchs-type differential equation with three regular singular points, it suffices to investigate the equation of the form (14.6.36). It is called the **hypergeometric equation** or the **Gauss equation**, whose solutions are expressed by (14.6.37). An analytic solution to hypergeometric equation (14.6.36) in a neighborhood of $z = 0$ can be expressed by a hypergeometric series as

$$\begin{aligned} w =& F(\alpha, \beta, \gamma; z) \\ \overset{\text{def.}}{=}& 1 + \frac{\alpha\beta}{1!\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} z^2 + \cdots + \\ &+ \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n \\ &+ \cdots , \end{aligned} \tag{14.6.42}$$

or denoted, for convenience, as

$$w = F(\alpha, \beta, \gamma; z) \overset{\text{def.}}{=} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!(\gamma)_n} z^n, \tag{14.6.43}$$

where

$$\begin{cases} (\lambda)_0 = 1, \\ (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) = \dfrac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \ (n \geq 1). \end{cases} \tag{14.6.44}$$

The above series is convergent when $|z| < 1$, and it is obvious that

$$F(\alpha, \beta, \gamma; z) = F(\beta, \alpha, \gamma; z). \tag{14.6.45}$$

Chapter 15
Applications and Developments

15.1 Applications

The results obtained above in this book can be widely applied, a few illustrative examples are given here.

15.1.1 Potential Solutions to Compressible Euler Equations

Under the isentropic hypothesis, the compressible Euler equations are composed of conservation laws of mass and momentum, with the following form (see Chap. 2 in Li and Qin 2012):

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0, \tag{15.1.1}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u} + pI) = 0, \tag{15.1.2}$$

where $\rho > 0$ is the density, $\mathbf{u} = (u_1, \ldots, u_n)$ is the velocy, $n = 2$ or 3 is the space dimension, $\mathbf{u} \otimes \mathbf{u}$ is the tensor product expressed by $(u_i u_j)$, and $p = p(\rho)$ is the pressure given by the equation of state of the fluid, and we usually have

$$p'(\rho) > 0, \ \forall \rho > 0. \tag{15.1.3}$$

Writing (15.1.1)–(15.1.2) in the form of components, we have

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \frac{\partial (\rho u_i)}{\partial x_i} = 0, \tag{15.1.4}$$

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$$\frac{\partial(\rho u_i)}{\partial t}+\sum_{k=1}^{n}\frac{\partial(\rho u_i u_k)}{\partial x_k}+\frac{\partial p(\rho)}{\partial x_i}=0,\quad i=1,\ldots,n. \tag{15.1.5}$$

By (15.1.4), (15.1.5) can be rewritten as

$$\frac{\partial u_i}{\partial t}+\sum_{k=1}^{n}u_k\frac{\partial u_i}{\partial x_k}+\frac{1}{\rho}\frac{\partial p}{\partial x_i}=0,\quad i=1,\ldots,n. \tag{15.1.6}$$

Letting $f=f(\rho)$ satisfy

$$f'(\rho)=\frac{p'(\rho)}{\rho}, \tag{15.1.7}$$

(15.1.6) can be written as

$$\frac{\partial u_i}{\partial t}+\sum_{k=1}^{n}u_k\frac{\partial u_i}{\partial x_k}+\frac{\partial f}{\partial x_i}=0,\quad i=1,\ldots,n. \tag{15.1.8}$$

Consider the Cauchy problem of Euler equations (15.1.4) and (15.1.8) with the initial data

$$t=0:\ \rho=\rho_0(x),\ \mathbf{u}=\mathbf{u}_0(x), \tag{15.1.9}$$

where both $\rho_0(x)$ and $\mathbf{u}_0(x)$ are sufficiently smooth functions.

Proposition 15.1.1 *If there is no vacuum at the initial time $t=0$, i.e.,*

$$\rho_0(x)>0,\quad x\in I\!R^n, \tag{15.1.10}$$

then the vacuum will never occur in the whole domain of existence for classical solutions to the Cauchy problem of Euler equations (15.1.4) and (15.1.8) with initial value (15.1.9), that is,

$$\rho(t,x)>0,\quad t\geq 0,\ x\in I\!R^n. \tag{15.1.11}$$

Proof Rewriting Eq. (15.1.4) as

$$\frac{\partial\rho}{\partial t}+\mathbf{u}\cdot\mathrm{grad}\rho+(\mathrm{div}\mathbf{u})\rho=0,$$

i.e.,

$$\frac{d\rho}{dt}+(\mathrm{div}\mathbf{u})\rho=0, \tag{15.1.12}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial}{\partial x_k} \tag{15.1.13}$$

stands for the derivative with respective to t when fixing the fluid particle. Therefore, along the motion law $x_k = x_k(t)$ $(k = 1, \ldots, n)$ of any fixed fluid particle, ρ satisfies a homogeneous ordinary differential equation, which yields the conclusion of Proposition 15.1.1 immediately. □

Proposition 15.1.2 *If the initial velocity field* $\mathbf{u}_0(x)$ *is irrotational, i.e.,*

$$rot\mathbf{u}_0(x) \equiv 0, \quad x \in I\!R^n, \tag{15.1.14}$$

then, the whole velocity field $\mathbf{u}(t, x)$ *keeps irrotational in the whole domain of existence of classical solutions to the Cauchy problem of Euler equations (15.1.4) and (15.1.8) with initial value (15.1.9):*

$$rot\mathbf{u}(t, x) \equiv 0, \quad t \geq 0, \ x \in I\!R^n. \tag{15.1.15}$$

Proof When $n = 2$, $\mathbf{u} = (u_1, u_2)$ and

$$\mathrm{rot}\mathbf{u} = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \stackrel{\text{def.}}{=} r. \tag{15.1.16}$$

Differentiating the first formula in (15.1.8) with respect to x_2, and differentiating the second formula in (15.1.8) with respect to x_1, then substracting from each other, it is easy to get the equation satisfied by $r = \mathrm{rot}\mathbf{u}$:

$$\frac{\partial r}{\partial t} + \mathbf{u} \cdot \mathrm{grad} r + (\mathrm{div}\mathbf{u})r = 0,$$

i.e.,

$$\frac{dr}{dt} + (\mathrm{div}\mathbf{u})r = 0, \tag{15.1.17}$$

where $\frac{d}{dt}$ is defined by (15.1.13). Therefore, along the motion law $x_k = x_k(t)$ $(k = 1, 2)$ of any fixed fluid particle, r satisfies a homogeneous linear ordinary differential equation, from this the conclusion of Proposition 15.1.2 for $n = 2$ follows immediately.

When $n = 3$, $\mathbf{u} = (u_1, u_2, u_3)$ and

$$\mathrm{rot}\mathbf{u} = \left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}, \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) \stackrel{\text{def.}}{=} (r_1, r_2, r_3). \tag{15.1.18}$$

Differentiating the first formula in (15.1.8) with respect to x_2, differentiating the second formula in (15.1.8) with respect to x_1, then substracting from each other, it is easy to obtain that

$$\frac{\partial r_3}{\partial t} + \mathbf{u} \cdot \mathrm{grad} r_3 + (\mathrm{div}\mathbf{u}) r_3 - \frac{\partial \mathbf{u}}{\partial x_3} \cdot \mathrm{rot}\mathbf{u} = 0.$$

Similar formulas can be obtained for r_1 and r_2. Combining these three formulas we obtain the following system of differential equations satisfied by rot$\mathbf{u}$:

$$\frac{\partial(\mathrm{rot}\mathbf{u})}{\partial t} + \mathbf{u} \cdot \mathrm{grad}\,\mathrm{rot}\mathbf{u} + (\mathrm{div}\mathbf{u})\mathrm{rot}\mathbf{u} - \mathrm{grad}\mathbf{u} \cdot \mathrm{rot}\mathbf{u} = 0,$$

i.e.,

$$\frac{d(\mathrm{rot}\mathbf{u})}{dt} + (\mathrm{div}\mathbf{u})\mathrm{rot}\mathbf{u} - \mathrm{grad}\mathbf{u} \cdot \mathrm{rot}\mathbf{u} = 0, \tag{15.1.19}$$

where $\frac{d}{dt}$ is still defined by (15.1.13). This shows that along the motion law $x_k = x_k(t)$ $(k = 1, 2, 3)$ of any fixed fluid particle, rot$\mathbf{u}$ satisfies a homogeneous linear system of ordinary differential equations, from this the conclusion of Proposition 15.1.2 for $n = 3$ follows immediately. □

From Propositions 15.1.1 and 15.1.2, we can always assume that $\rho(t, x) > 0$ (no vacuum occurs), and that the velocity field $\mathbf{u}(t, x)$ is irrotational, that is, there exists a potential function $\phi(t, x)$ such that

$$\mathbf{u} = -\mathrm{grad}\,\phi, \tag{15.1.20}$$

where grad stands for the gradient with respect to $\mathbf{x} = (x_1, \ldots, x_n)^T$, i.e.,

$$u_i(t, x) = -\frac{\partial \phi(t, x)}{\partial x_i}, \quad i = 1, \ldots, n. \tag{15.1.21}$$

Then, from Eq. (15.1.4) we get

$$\frac{\partial \rho}{\partial t} - \sum_{i=1}^{n} \frac{\partial(\rho \phi_{x_i})}{\partial x_i} = 0, \tag{15.1.22}$$

while, it yields easily, from Eq. (15.1.8), the following Bernoulli law:

$$-\phi_t + \frac{1}{2}|\mathrm{grad}\phi|^2 + f(\rho) = C, \tag{15.1.23}$$

where C is a constant.

Noting that the $f(\rho)$ defined by (15.1.7) can be different up to an arbitrary constant, we can absorb the constant C into the definition of $f(\rho)$, then (15.1.23) can be simplified as

$$f(\rho) = \phi_t - \frac{1}{2}|\mathrm{grad}\phi|^2. \tag{15.1.24}$$

Denoting by H the inverse function of f, we obtain

$$\rho = H\left(\phi_t - \frac{1}{2}|\mathrm{grad}\phi|^2\right), \tag{15.1.25}$$

and noting $\rho > 0$ and (15.1.3), it is easy to get

$$H(0) > 0, \;\; H'(0) > 0. \tag{15.1.26}$$

Plugging (15.1.27) in (15.1.22), we obtain the partial differential equation satisfied by $\phi = \phi(t, x)$:

$$\left(H(\phi_t - \frac{1}{2}|\mathrm{grad}\phi|^2)\right)_t - \sum_{i=1}^{n}\left(H(\phi_t - \frac{1}{2}|\mathrm{grad}\phi|^2)\phi_{x_i}\right)_{x_i} = 0. \tag{15.1.27}$$

It is easy to know that, near $\phi = 0$, this equation is a nonlinear wave equation which does not depend on ϕ explicitly, and the value of α corresponding to the term $F(D\phi, D_x D\phi)$ on the right-hand side is $\alpha = 1$. Considering its Cauchy problem with small initial data

$$t = 0: \;\; \phi = \varepsilon\varphi(x), \;\; \phi_t = \varepsilon\psi(x), \tag{15.1.28}$$

where $\varepsilon > 0$ is a small parameter, and $\varphi(x), \psi(x) \in C_0^\infty(I\!R^n)$, we can give the lower bound estimates on the life-span $\widetilde{T}(\varepsilon)$ of its classical solutions by using previous results.

Specifically speaking, when $n = 2$, from the results in Sect. 10.4 of Chap. 10, we have

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-2}, \tag{15.1.29}$$

where b is a positive constant independent of ε; while,when $n = 3$, form the results in Sect. 15.3, we have

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-1}\}, \tag{15.1.30}$$

where a is a positive constant independent of ε.

Remark 15.1.1 For the above result, we refer the reader to Sideris (1985, 1992) and Alinhac (1993).

15.1.2 Time-Like Minimal Hypersurface in Minkowski Space

We consider the following functional in Minkowski space

$$\mathcal{L}(\phi)=\iint\sqrt{1-\phi_t^2+\sum_{k=1}^{n}\phi_{x_k}^2}\,dxdt, \tag{15.1.31}$$

its corresponding Euler-Lagrange equation is

$$\begin{aligned}&\left(\phi_t\left(1-\phi_t^2+\sum_{k=1}^{n}\phi_{x_k}^2\right)^{-1/2}\right)_t\\ &\qquad-\sum_{i=1}^{n}\left(\phi_{x_i}\left(1-\phi_t^2+\sum_{k=1}^{n}\phi_{x_k}^2\right)^{-1/2}\right)_{x_i}=0,\end{aligned} \tag{15.1.32}$$

where $\phi=\phi(t,x_1,\ldots,x_n)$. The solution $\phi=\phi(t,x_1,\ldots,x_n)$ of (15.1.32) is called the **time-like minimal hypersurface**.

It is clear that (15.1.32) is a nonlinear wave equation with the term on the right-hand side not depending on ϕ:

$$\Box\phi=F(D\phi,D_xD\phi), \tag{15.1.33}$$

and in a neighborhood of $\phi=0$, the value of α corresponding to the term F on the right-hand side is $\alpha=2$. In addition, neglecting the higher order terms, the corresponding term on the right-hand side can be written as

$$\widetilde{F}(D\phi,D_xD\phi)=-\phi_tQ_0(\phi,\phi_t)+\sum_{i=1}^{n}\phi_{x_i}Q_0(\phi,\phi_{x_i}), \tag{15.1.34}$$

where

$$Q_0(f,g)=f_tg_t-\sum_{k=1}^{n}f_{x_k}g_{x_k}. \tag{15.1.35}$$

We consider the Cauchy problem of Eq. (15.1.32) with the initial data

$$t=0:\ \phi=\varepsilon\phi_0(x_1,\ldots,x_n),\ \phi_t=\varepsilon\phi_1(x_1,\ldots,x_n), \tag{15.1.36}$$

where $\varepsilon>0$ is a small parameter, and $\phi_0,\phi_1\in C_0^\infty(\mathbb{R}^n)$.

From Chap. 9, when $n\geq 3$, Cauchy (15.1.32) and (15.1.36) must have global classical solutions. When $n=2$, noting (15.1.34), from Sect. 12.3 in Chap. 12, Eq. (15.1.32) satisfies the corresponding null condition, then its Cauchy problem with the initial value (15.1.36) must have global classical solutions as well. While, when $n = 1$, from Chap. 8, the life-span $\widetilde{T}(\varepsilon)$ of classical solutions to the Cauchy problem (15.1.32) and (15.1.36) has the following lower bound estimate:

$$\widetilde{T}(\varepsilon) \geq a\varepsilon^{-2}, \tag{15.1.37}$$

where a is a positive constant independent of ε.

Remark 15.1.2 For the above results, we refer the reader to Lindblad (2004).

15.2 Some Further Results

15.2.1 Further Results When n = 2

When $n = 2$ and $\alpha = 2$, from Chap. 10, if we assume that

$$\partial_u^\beta F(0, 0, 0) = 0 \quad (\beta = 3, 4), \tag{15.2.1}$$

then for the corresponding Cauchy problem with small initial data, the life-span $\widetilde{T}(\varepsilon)$ of classical solutions has the following lower bound estimate:

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-2}\}, \tag{15.2.2}$$

where a is a positive constant independent of ε, that is to say, now we have the almost global classical solution.

Instead of (15.2.1), if we only assume that

$$\partial_u^3 F(0, 0, 0) = 0, \tag{15.2.3}$$

Katayama (2001) already proved that the corresponding life-span of classical solutions has the lower bound estimate

$$\widetilde{T}(\varepsilon) \geq b\varepsilon^{-18}, \tag{15.2.4}$$

where b is a positive constant independent of ε. The sharpness of this lower bound estimate was already proved by Han and Zhou (2014). Moreover, when $n = 2$ and $\alpha = 2$, in Sect. 12.3 of Chap. 12, under the additional assumption that the lowest order term (cubic term) on the right-hand side satisfies the null condition, the global existence of classical solutions has been proved for the Cauchy problem with small initial data. But when $n = 2$ and $\alpha = 1$, if the corresponding lowest order term (quadratic term) on the right-hand side satisfies the null condition, can the original lower bound estimate (see (10.1.9) in Chap. 10) on the Life-span of classical solutions life-span of classical solutions be obviously improved? Focusing on this point, S. Alinhac considered the following Cauchy problem of quasi-linear wave equation with small initial data (see Alinhac 2001):

$$\Box u + \sum_{\mu,\nu=0}^{2} g_{\mu\nu}(Du)\partial_{\mu\nu}u = 0, \tag{15.2.5}$$

$$t = 0: \ u = \varepsilon\varphi(x), \ u_t = \varepsilon\psi(x), \tag{15.2.6}$$

where $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$, $g_{\mu\nu}(0) = 0$ $(\mu, \nu = 0, 1, 2)$, and $\varepsilon > 0$ is a small parameter. For this kind of second-order quasi-linear wave equation with special form (correspondingly, $\alpha = 1$), he proved that: when the quadratic term of the equation satisfies the null condition, the life-span of its classical solution has the same lower bound estimate (15.2.2) as in the case $\alpha = 2$, while, when both the quadratic and cubic terms of the equation satisfy the null conditions, the corresponding Cauchy problem admits a global classical solution. The extension of this result to the general case is still open.

15.2.2 Further Results When $n = 3$

When $n = 3$ and $\alpha = 1$, from the discussion in Sect. 12.2 of Chap. 12, if the quasi-linear wave equation under consideration satisfies the null condition, then the corresponding Cauchy problem with small initial data must have global classical solutions. This shows that the null condition is a sufficient condition to ensure the global existence of classical solutions, but this condition is not always necessary. Sometime even the null condition is not satisfied, the corresponding Cauchy problem with small initial data may still have global classical solutions. Lindblad (1992, 2004), Alinhac (2003) considered the Cauchy problem of the following quasi-linear wave equation

$$\sum_{\mu,\nu=1}^{3} g_{\mu\nu}(u)\partial^{\mu\nu}u = 0 \tag{15.2.7}$$

with small initial value (15.2.6), where

$$(g_{\mu\nu}(0)) = \text{diag}\{-1, 1, 1, 1\}, \tag{15.2.8}$$

and proved the global existence of classical solutions. How to fit this result under special occasions in a general framework is an interesting problem (see Sect. 15.3.2).

15.3 Some Important Developments

The ideas and methods for solving the Cauchy problem of nonlinear wave equations with small initial data can also be applied to some other important equations or systems in physics, such as three-dimensional nonlinear elastodynamics equations and Einstein equation in a vacuum, and so on. Even if these applications are beyond the scope of this book, we can learn and then continue in-depth study of related references and contents on a very good basis as long as we master the contents and methods in this book. In this section, we only give a brief description on these developments and applications.

15.3.1 Three-Dimensional Nonlinear Elastodynamics Equations

Suppose that the elastic body is in the state of nature before deformation (at a certain moment, say, $t = 0$), and has a unit density, with the position coordinate $\mathbf{x} = (x_1, x_2, x_3)^T$ of any given particle. Suppose that after this moment, the elastic body deforms with the motion law depicted by

$$\mathbf{y}(t, x) = (y_1(t, x), y_2(t, x), y_3(t, x))^T,$$

where $\mathbf{y} = \mathbf{y}(t, x)$ stands for the position coordinate of the particle at time t, which is located at position $\mathbf{x}$ at time $t = 0$. The deformation of the elastic body at time t is described by the **deformation gradient tensor** $\mathbf{F} = \left(\frac{\partial y_i}{\partial x_j}\right)$.

For small deformation, we can assume that

$$\mathbf{y} = \mathbf{x} + \mathbf{u}, \tag{15.3.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ is a sufficiently small vector. Thus, we have

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}. \tag{15.3.2}$$

Due to the conservation law of momentum, in the case without external force, the corresponding nonlinear elastodynamics equations can be written as (see (5.74) in Chap. 5 of Li and Qin 2012)

$$\frac{\partial^2 u_i}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial p_{ij}(\nabla u)}{\partial x_j} = 0, \quad i = 1, 2, 3, \tag{15.3.3}$$

where $\mathbf{P} = (p_{ij})$ is the **Piola stress tensor**.

Denote

$$\Sigma = \mathbf{F}^{-1}\mathbf{P}. \tag{15.3.4}$$

It is called the **second Piola stress tensor**, and is a symmetric tensor. It is known that (see Theorem 5.5 in Chap. 5 of Li and Qin 2012)

$$\Sigma = \lambda(\mathrm{tr}\widetilde{\mathbf{E}})\mathbf{I} + 2\mu\widetilde{\mathbf{E}} + \mathrm{o}(|\widetilde{\mathbf{E}}|), \tag{15.3.5}$$

where λ and μ are the **Lamé constants**, $\mathrm{o}(|\widetilde{\mathbf{E}}|)$ stands for the higher order term of $|\widetilde{\mathbf{E}}|$, and noting (15.3.2),

$$\widetilde{\mathbf{E}} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) + \mathrm{o}(|\nabla\mathbf{u}|) = \mathbf{E} + \mathrm{o}(|\nabla\mathbf{u}|), \tag{15.3.6}$$

in which

$$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \tag{15.3.7}$$

is the **Cauchy strain tensor** in the linear elastic case. Thus, from (15.3.5) we have

$$\Sigma = \lambda(\mathrm{tr}\mathbf{E})\mathbf{I} + 2\mu\mathbf{E} + \mathrm{o}(|\nabla\mathbf{u}|), \tag{15.3.8}$$

then from (15.3.4) and noting (15.3.2), we have

$$\mathbf{P} = \lambda(\mathrm{tr}\mathbf{E})\mathbf{I} + 2\mu\mathbf{E} + \mathrm{o}(|\nabla\mathbf{u}|). \tag{15.3.9}$$

Plugging (15.3.9) in (15.3.3), it is easy to get

$$\frac{\partial^2\mathbf{u}}{\partial t^2} - a_2^2\Delta\mathbf{u} - (a_1^2 - a_2^2)\nabla\mathrm{div}\mathbf{u} = \mathbf{F}(\nabla\mathbf{u}, \nabla^2\mathbf{u}), \tag{15.3.10}$$

where a_1^2 and a_2^2 are determined by

$$\lambda + \mu = a_1^2 - a_2^2, \quad \mu = a_2^2, \tag{15.3.11}$$

and $\mathbf{F}(\nabla\mathbf{u}, \nabla^2\mathbf{u})$ is the term of second or above order, which is linear with respect to $\nabla^2\mathbf{u}$. a_1 and a_2 are the propagation speed of the longitudinal wave and the transverse wave, respectively, and it can always be assumed that $a_1 > a_2 > 0$.

As a quasi-linear hyperbolic system with double wave speeds, the nonlinear elastodynamics equations can be studied using arguments similar to those for the wave equation, although it cannot be dealt with by being reduced directly to the corresponding wave equation.

For the Cauchy problem of system (15.3.10) with small initial data

$$t = 0: \ \mathbf{u} = \varepsilon \boldsymbol{\varphi}(\mathbf{x}), \ \mathbf{u}_t = \varepsilon \boldsymbol{\psi}(\mathbf{x}), \tag{15.3.12}$$

where $\boldsymbol{\varphi}(\mathbf{x}), \boldsymbol{\psi}(\mathbf{x}) \in (C_0^\infty(I\!R^3))^3$, and $\varepsilon > 0$ is a small parameter, the almost global existence of classical solutions can be proved, in other words, its life-span $\widetilde{T}(\varepsilon)$ satisfies

$$\widetilde{T}(\varepsilon) \geq \exp\{a\varepsilon^{-1}\}, \tag{15.3.13}$$

where a is a positive constant independent of ε, see John (1988), Klainerman and Sideris (1996).

To obtain the global existence of classical solutions to Cauchy problem (15.3.10) and (15.3.12), it is necessary to prescribe some appropriate null conditions to the term $\mathbf{F}(\nabla \mathbf{u}, \nabla^2 \mathbf{u})$ on the right-hand side of system (15.3.10).

To specify this, we furthermore assume that the material under consideration is the isotropic **hyperelastic material**. By the hyperelastic assumption of the material (see Definition 5.3 in Chap. 5 of Li and Qin 2012), there exists a **stored-energy function** $W = \widehat{W}(\mathbf{F}) = W(\nabla \mathbf{u})$ such that the Piola stress tensor

$$p_{ij} = \frac{\partial W(\nabla \mathbf{u})}{\partial u_{ij}}, \tag{15.3.14}$$

where we denote

$$u_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{15.3.15}$$

Then system (15.3.3) can be written as

$$\frac{\partial^2 u_i}{\partial t^2} - \sum_{j,k,l=1}^{3} a_{ijkl}(\nabla \mathbf{u}) \frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0, \quad i = 1, 2, 3, \tag{15.3.16}$$

where

$$a_{ijkl} = \frac{\partial^2 W}{\partial u_{ij} \partial u_{kl}}. \tag{15.3.17}$$

Using Taylor expansion, the coefficients a_{ijkl} in system (15.3.16) can be written as

$$a_{ijkl}(\nabla \mathbf{u}) = a_{ijkl}(0) + \sum_{m,n=1}^{3} b_{ijklmn} u_{mn} + \mathrm{o}(|\nabla \mathbf{u}|), \tag{15.3.18}$$

where

$$b_{ijklmn} = \frac{\partial^3 W}{\partial u_{ij} \partial u_{kl} \partial u_{mn}}(0). \tag{15.3.19}$$

Thus, system (15.3.16) can be written as

$$\frac{\partial^2 u_i}{\partial t^2} - \sum_{j,k,l=1}^{3} a_{ijkl}(0) \frac{\partial^2 u_k}{\partial x_j \partial x_l} = \sum_{j,k,l,m,n=1}^{3} b_{ijklmn} u_{mn} \frac{\partial^2 u_k}{\partial x_j \partial x_l} + \widetilde{f}_i(\nabla \mathbf{u}, \nabla^2 \mathbf{u}), \quad i = 1, 2, 3, \tag{15.3.20}$$

where $\widetilde{f}_i(\nabla \mathbf{u}, \nabla^2 \mathbf{u})$ $(i = 1, 2, 3)$ are terms of third or above order, which are linear with respect to $\nabla^2 \mathbf{u}$. Obviously, the left-hand side of (15.3.20) should be the same as that of (15.3.10).

From the isotropic assumption of the material, the stored-energy function W is a function of the principal values k_1, k_2 and k_3 of the matrix $\mathbf{F}^T \mathbf{F} - \mathbf{I}$ (see Sect. 5.4.3 in Chap. 5 of Li and Qin 2012), and these principal values k_1, k_2 and k_3 can all be explicitly given by (u_{ij}), therefore, the dependence of W on $\nabla \mathbf{u} = (u_{ij})$ can be realized by the dependence on these principal values, then the derivatives of W with respect to u_{ij} can be expressed by

$$\frac{\partial W}{\partial u_{ij}} = \sum_{l=1}^{3} \frac{\partial W}{\partial k_l} \frac{\partial k_l}{\partial u_{ij}}$$

and so on. Thus, the first term on the right-hand side of (15.3.20) can be written as

$$\sum_{j,k,l,m,n=1}^{3} b_{ijklmn} u_{mn} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = 2(2W_{111}(0) + 3W_{11}(0)) \nabla (\mathrm{div} \mathbf{u})^2 + \dots, \tag{15.3.21}$$

where, except the first term on the right-hand side, all the other unwritten terms can be properly treated when doing energy estimates. Assuming that

$$2W_{111}(0) + 3W_{11}(0) = 0, \tag{15.3.22}$$

where

$$W_{11}(0) = \frac{\partial^2 W}{\partial k_1^2}(0), \;\; W_{111}(0) = \frac{\partial^3 W}{\partial k_1^3}(0), \tag{15.3.23}$$

the global existence of classical solutions can be proved for Cauchy problem (15.3.20) and (15.3.12) with small initial data, see Sideris (2000), Agemi (2000), Xin (2002).

Equation (15.3.22) is just the null condition for the nonlinear elastodynamics equations in the case of isotropic hyperelastic material.

15.3.2 Einstein Equation in a Vacuum

According to the general relativity, the space-time is a four-dimensional pseudo-Riemannian manifold with the metric

$$ds^2 = g_{\mu\nu}(x)dx^{\mu}dx^{\nu}, \tag{15.3.24}$$

where $x = (x^0, x^1, x^2, x^3)$, $\mu, \nu = 0, 1, 2, 3$, which is also the case for values of other Greek indices, the summation convention is adopted for the same superscript and subscript, and $g = (g_{\mu\nu})$ is a second-order covariant symmetric tensor with the symbol $(-1, 1, 1, 1)$.

Introduce Christoffel notation

$$\Gamma_{\mu\delta\nu} = \frac{1}{2}\left(\frac{\partial g_{\delta\nu}}{\partial x^{\mu}} + \frac{\partial g_{\delta\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\delta}}\right) = \Gamma_{\nu\delta\mu} \tag{15.3.25}$$

and

$$\Gamma_{\mu\ \nu}^{\ \lambda} = g^{\lambda\delta}\Gamma_{\mu\delta\nu} = \Gamma_{\nu\ \mu}^{\ \lambda}, \tag{15.3.26}$$

where $(g^{\lambda\delta})$, the inverse matrix of $(g_{\mu\nu})$, is a second-order contravariant symmetric tensor. The corresponding **Riemann curvature tensor** is given by:

$$R_{\mu\ \nu\beta}^{\ \lambda} = \frac{\partial\Gamma_{\mu\ \nu}^{\ \lambda}}{\partial x^{\beta}} - \frac{\partial\Gamma_{\mu\ \beta}^{\ \lambda}}{\partial x^{\nu}} + \Gamma_{\rho\ \beta}^{\ \lambda}\Gamma_{\mu\ \nu}^{\ \rho} - \Gamma_{\rho\ \nu}^{\ \lambda}\Gamma_{\mu\ \beta}^{\ \rho} \tag{15.3.27}$$

and

$$R_{\mu\alpha\nu\beta} = g_{\alpha\lambda}R_{\mu\ \nu\beta}^{\ \lambda}, \tag{15.3.28}$$

and the **Ricci curvature tensor** is the contraction of the Riemann curvature tensor:

$$R_{\mu\nu} = R_{\mu\ \nu\alpha}^{\ \alpha}, \tag{15.3.29}$$

which is a second-order covariant tensor. Contracting once more the Ricci curvature tensor, we get the **curvature scalar**:

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{15.3.30}$$

The *Einstein tensor* which is very useful for the general relativity is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{15.3.31}$$

and the Einstein equation in a vacuum can be written as

$$G_{\mu\nu} = 0, \quad \mu, \nu = 0, 1, 2, 3. \tag{15.3.32}$$

Noting (15.3.31), by contracting (15.3.32) we immediately obtain

$$R = 0, \tag{15.3.33}$$

so the Einstein equation in a vacuum can be written as

$$R_{\mu\nu} = 0, \quad \mu, \nu = 0, 1, 2, 3. \tag{15.3.34}$$

It is a system of second-order partial differential equations satisfied by the metric tensor $(g_{\mu\nu})$.

Obviously, Eq. (15.3.34) has a solution given by the flat metric

$$(m_{\mu\nu}) = \text{diag}\{-1, 1, 1, 1\}, \tag{15.3.35}$$

called the **Minkowski space-time**. The stability of the Minkowski space-time (15.3.35) is a highly significant and challenging problem, that is, when the initial value is a small perturbation of the Minkowski metric (15.3.35) in some sense, whether the corresponding Cauchy problem of the Einstein equation (15.3.34) in a vacuum exists a global classical solution close to the Minkowski space-time (15.3.35) in some sense. D. Christodoulou and S. Klainerman proved at considerable length in 1993 the stability of the Minkowski space-time, see Christodoulou and Klainerman (1993). Later, H. Lindblad and I. Rodnianski gave a simplified proof of this result in 2005, see Lindblad and Rodnianski (2005, 2010). Here we briefly sketch the proof of the latter.

We first point out that, since $(R_{\mu\nu})$ is a tensor, Eq. (15.3.34) is invariant under any given reversible coordinate transformation, therefore, solutions to Eq. (15.3.34) (even if under given initial conditions) are not unique. To ensure the uniqueness of solutions to equation (15.3.34), we need to find a special coordinate system and to discuss the subject under this system. For Einstein equation, we usually take the so-called **harmonic coordinates** (now called the **wave coordinates**), namely, the coordinates x^μ $(\mu = 0, 1, 2, 3)$ are required to satisfy

$$\Box_g x^\mu = 0, \quad \mu = 0, 1, 2, 3, \tag{15.3.36}$$

where $\Box_g$ is the Laplace–Beltrami operator corresponding to $g = (g_{\mu\nu})$. Under the local coordinates,

$$\Box_g = \frac{1}{\sqrt{|g|}}\partial_\mu(g^{\mu\nu}\sqrt{|g|}\partial_\nu), \tag{15.3.37}$$

where $|g| = \det(g_{\mu\nu})$.

By the definition of the determinant, it is easy to know that

$$\frac{\partial |g|}{\partial g_{\mu\nu}} = |g| g^{\mu\nu},$$

then we have

$$\frac{\partial |g|}{\partial x^{\mu}} = \frac{\partial |g|}{\partial g_{\nu\gamma}} \frac{\partial g_{\nu\gamma}}{\partial x^{\mu}} = |g| g^{\nu\gamma} \frac{\partial g_{\nu\gamma}}{\partial x^{\mu}}. \tag{15.3.38}$$

Since

$$g_{\lambda\delta} g^{\delta\nu} = \delta_{\lambda}^{\nu},$$

where δ_{λ}^{ν} is the Kronecker symbol, it is easy to get

$$\frac{\partial g^{\mu\nu}}{\partial x^{\gamma}} = -g^{\mu\lambda} \frac{\partial g_{\lambda\delta}}{\partial x^{\gamma}} g^{\delta\nu}, \tag{15.3.39}$$

then

$$\frac{\partial g^{\mu\nu}}{\partial x^{\mu}} = -g^{\mu\lambda} \frac{\partial g_{\lambda\delta}}{\partial x^{\mu}} g^{\delta\nu}. \tag{15.3.40}$$

Using (15.3.38) and (15.3.40), (15.3.37) can be rewritten as

$$\Box_g = g^{\mu\nu} \frac{\partial^2}{\partial x^{\mu} \partial x^{\nu}} - g^{\mu\nu} \Gamma_{\mu\ \nu}^{\ \delta} \frac{\partial}{\partial x^{\delta}}, \tag{15.3.41}$$

where $\Gamma_{\mu\ \nu}^{\ \delta}$ is the Christoffel symbol defined by (15.3.26).

It follows immediately from (15.3.36) and (15.3.41) that under the wave coordinates we always have

$$g^{\mu\nu} \Gamma_{\mu\ \nu}^{\ \alpha} = 0, \quad \alpha = 0, 1, 2, 3, \tag{15.3.42}$$

then under the wave coordinates, the Laplace–Beltrami operator is reduced to

$$\Box_g = g^{\mu\nu} \frac{\partial^2}{\partial x^{\mu} \partial x^{\nu}}. \tag{15.3.43}$$

In addition, noting (15.3.25)–(15.3.26), from (15.3.42) we obtain

$$g^{\mu\nu} \Gamma_{\mu\gamma\nu} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\gamma\mu}}{\partial x^{\nu}} + \frac{\partial g_{\gamma\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\gamma}} \right) = 0, \quad \gamma = 0, 1, 2, 3,$$

that is,

$$g^{\mu\nu}\frac{\partial g_{\mu\gamma}}{\partial x^{\nu}} = \frac{1}{2}g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^{\gamma}}, \quad \gamma = 0, 1, 2, 3, \tag{15.3.44}$$

or by (15.3.39) we equivalently have

$$\frac{\partial g^{\mu\nu}}{\partial x^{\nu}} = \frac{1}{2}g_{\nu\gamma}g^{\mu\lambda}\frac{\partial g^{\nu\gamma}}{\partial x^{\lambda}}. \tag{15.3.45}$$

Now we write specifically the Einstein equation (15.3.34) in a vacuum under the wave coordinates.

First, from (15.3.25) it is clear that

$$\frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} = \Gamma_{\beta\alpha\mu} + \Gamma_{\beta\mu\alpha}, \tag{15.3.46}$$

then differentiating the following formula equivalent to (15.3.26):

$$\Gamma_{\mu\alpha\nu} = g_{\alpha\lambda}\Gamma_{\mu\ \nu}^{\ \lambda}$$

once with respect to x^{β}, we get

$$g_{\alpha\lambda}\frac{\partial \Gamma_{\mu\ \nu}^{\ \lambda}}{\partial x^{\beta}} = \frac{\partial \Gamma_{\mu\alpha\nu}}{\partial x^{\beta}} - \left(\Gamma_{\beta\alpha\lambda} + \Gamma_{\beta\lambda\alpha}\right)\Gamma_{\mu\ \nu}^{\ \lambda}. \tag{15.3.47}$$

Thus, noticing $\Gamma_{\alpha\lambda\beta} = \Gamma_{\beta\lambda\alpha}$, from (15.3.27)–(15.3.28) we get

$$\begin{aligned} R_{\mu\alpha\nu\beta} &= g_{\alpha\lambda}\left(\frac{\partial \Gamma_{\mu\ \nu}^{\ \lambda}}{\partial x^{\beta}} - \frac{\partial \Gamma_{\mu\ \beta}^{\ \lambda}}{\partial x^{\nu}} + \Gamma_{\rho\ \beta}^{\ \lambda}\Gamma_{\mu\ \nu}^{\ \rho} - \Gamma_{\rho\ \nu}^{\ \lambda}\Gamma_{\mu\ \beta}^{\ \rho}\right) \\ &= \frac{\partial \Gamma_{\mu\alpha\nu}}{\partial x^{\beta}} - \frac{\partial \Gamma_{\mu\alpha\beta}}{\partial x^{\nu}} + \Gamma_{\nu\lambda\alpha}\Gamma_{\mu\ \beta}^{\ \lambda} - \Gamma_{\alpha\lambda\beta}\Gamma_{\mu\ \nu}^{\ \lambda}, \end{aligned} \tag{15.3.48}$$

then

$$\begin{aligned} R_{\mu\nu} &= g^{\alpha\beta}\left(\frac{\partial \Gamma_{\mu\alpha\nu}}{\partial x^{\beta}} - \frac{\partial \Gamma_{\mu\alpha\beta}}{\partial x^{\nu}} + \Gamma_{\nu\lambda\alpha}\Gamma_{\mu\ \beta}^{\ \lambda} - \Gamma_{\alpha\lambda\beta}\Gamma_{\mu\ \nu}^{\ \lambda}\right) \\ &= g^{\alpha\beta}\left(\frac{\partial \Gamma_{\mu\alpha\nu}}{\partial x^{\beta}} - \frac{\partial \Gamma_{\mu\alpha\beta}}{\partial x^{\nu}} + \Gamma_{\nu\lambda\alpha}\Gamma_{\mu\ \beta}^{\ \lambda}\right), \end{aligned} \tag{15.3.49}$$

in which we used (15.3.42) under the wave coordinates to obtain the last formula.

Differentiating (15.3.44) once, and using (15.3.39), we obtain

$$
\begin{aligned}
g^{\alpha\beta}\left(\frac{\partial^2 g_{\beta\nu}}{\partial x^\mu \partial x^\alpha} - \frac{1}{2}\frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu}\right) &= -\frac{\partial g^{\alpha\beta}}{\partial x^\mu}\left(\frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right) \\
&= g^{\alpha\alpha'} g^{\beta\beta'} \frac{\partial g_{\alpha'\beta'}}{\partial x^\mu}\left(\frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right), \qquad (15.3.50)
\end{aligned}
$$

then from (15.3.25) it is easy to show that

$$
\begin{aligned}
& g^{\alpha\beta}\left(\frac{\partial \Gamma_{\mu\alpha\nu}}{\partial x^\beta} - \frac{\partial \Gamma_{\mu\alpha\beta}}{\partial x^\nu}\right) \\
&= -\frac{1}{2} g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} + \frac{1}{2} g^{\alpha\beta}\left(\frac{\partial^2 g_{\beta\nu}}{\partial x^\alpha \partial x^\mu} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^\nu \partial x^\beta} - \frac{\partial^2 g_{\beta\alpha}}{\partial x^\nu \partial x^\mu}\right) \\
&= -\frac{1}{2} g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} + \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'}\left(\frac{\partial g_{\alpha'\beta'}}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial x^\alpha} + \frac{\partial g_{\alpha'\beta'}}{\partial x^\nu}\frac{\partial g_{\beta\mu}}{\partial x^\alpha} - \frac{\partial g_{\alpha'\beta'}}{\partial x^\nu}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right).
\end{aligned}
\qquad (15.3.51)
$$

Noting (15.3.44), we have

$$
\begin{aligned}
& g^{\alpha\alpha'} g^{\beta\beta'} \frac{\partial g_{\alpha'\beta'}}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial x^\alpha} \\
=& g^{\alpha\alpha'} g^{\beta\beta'} \frac{\partial g_{\alpha'\beta'}}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\mu} + g^{\alpha\alpha'} g^{\beta\beta'}\left(\frac{\partial g_{\alpha'\beta'}}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha'\beta'}}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\mu}\right) \\
=& \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \frac{\partial g_{\alpha'\alpha}}{\partial x^{\beta'}}\frac{\partial g_{\beta\nu}}{\partial x^\mu} + g^{\alpha\alpha'} g^{\beta\beta'}\left(\frac{\partial g_{\alpha'\beta'}}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha'\beta'}}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\mu}\right) \\
=& \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \frac{\partial g_{\alpha'\alpha}}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial x^{\beta'}} \\
&+ g^{\alpha\alpha'} g^{\beta\beta'}\left[\frac{1}{2}\left(\frac{\partial g_{\alpha'\alpha}}{\partial x^{\beta'}}\frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\alpha'\alpha}}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial x^{\beta'}}\right) + \left(\frac{\partial g_{\alpha'\beta'}}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha'\beta'}}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\mu}\right)\right] \\
=& \frac{1}{4} g^{\alpha\alpha'} g^{\beta\beta'} \frac{\partial g_{\alpha\alpha'}}{\partial x^\mu}\frac{\partial g_{\beta\beta'}}{\partial x^\nu} \\
&+ g^{\alpha\alpha'} g^{\beta\beta'}\left[\frac{1}{2}\left(\frac{\partial g_{\alpha'\alpha}}{\partial x^{\beta'}}\frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\alpha'\alpha}}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial x^{\beta'}}\right) + \left(\frac{\partial g_{\alpha'\beta'}}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha'\beta'}}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\mu}\right)\right].
\end{aligned}
\qquad (15.3.52)
$$

Thus, the second term on the right-hand side of (15.3.51) can be written as

$$
\begin{aligned}
& \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'}\left(\frac{\partial g_{\alpha'\beta'}}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial x^\alpha} + \frac{\partial g_{\alpha'\beta'}}{\partial x^\nu}\frac{\partial g_{\beta\mu}}{\partial x^\alpha} - \frac{\partial g_{\alpha'\beta'}}{\partial x^\nu}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right) \\
=& g^{\alpha\alpha'} g^{\beta\beta'}\left(\frac{1}{4}\frac{\partial g_{\alpha\alpha'}}{\partial x^\mu}\frac{\partial g_{\beta\beta'}}{\partial x^\nu} - \frac{1}{2}\frac{\partial g_{\alpha'\beta'}}{\partial x^\nu}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right)
\end{aligned}
$$

$$+\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\left[\left(\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial x^{\alpha}}-\frac{\partial g_{\alpha'\beta'}}{\partial x^{\alpha}}\frac{\partial g_{\beta\nu}}{\partial x^{\mu}}\right)+\left(\frac{\partial g_{\alpha'\beta'}}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}-\frac{\partial g_{\alpha'\beta'}}{\partial x^{\alpha}}\frac{\partial g_{\beta\mu}}{\partial x^{\nu}}\right)\right]$$
$$+\frac{1}{4}g^{\alpha\alpha'}g^{\beta\beta'}\left[\left(\frac{\partial g_{\alpha'\alpha}}{\partial x^{\beta'}}\frac{\partial g_{\beta\nu}}{\partial x^{\mu}}-\frac{\partial g_{\alpha'\alpha}}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial x^{\beta'}}\right)+\left(\frac{\partial g_{\alpha'\alpha}}{\partial x^{\beta'}}\frac{\partial g_{\beta\mu}}{\partial x^{\nu}}-\frac{\partial g_{\alpha'\alpha}}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial x^{\beta'}}\right)\right]. \tag{15.3.53}$$

On the other hand, we have

$$\begin{aligned}
&g^{\alpha\beta}\Gamma_{\nu\lambda\alpha}\Gamma_{\mu\ \beta}^{\ \lambda}\\
=&\frac{1}{4}g^{\alpha\beta}\left(\frac{\partial g_{\lambda\nu}}{\partial x^{\alpha}}+\frac{\partial g_{\lambda\alpha}}{\partial x^{\nu}}-\frac{\partial g_{\nu\alpha}}{\partial x^{\lambda}}\right)g^{\lambda\gamma}\left(\frac{\partial g_{\gamma\mu}}{\partial x^{\beta}}+\frac{\partial g_{\gamma\beta}}{\partial x^{\mu}}-\frac{\partial g_{\mu\beta}}{\partial x^{\gamma}}\right)\\
=&\frac{1}{4}g^{\alpha\alpha'}g^{\beta\beta'}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}+\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}}-\frac{\partial g_{\beta\nu}}{\partial x^{\alpha}}\right)\left(\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu}}+\frac{\partial g_{\alpha'\mu}}{\partial x^{\beta'}}-\frac{\partial g_{\beta'\mu}}{\partial x^{\alpha'}}\right)\\
=&\frac{1}{4}g^{\alpha\alpha'}g^{\beta\beta'}\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu}}+\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\left(\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\frac{\partial g_{\beta'\nu}}{\partial x^{\alpha'}}-\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\frac{\partial g_{\alpha'\nu}}{\partial x^{\beta'}}\right)\\
=&g^{\alpha\alpha'}g^{\beta\beta'}\left(\frac{1}{4}\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu}}+\frac{1}{2}\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\frac{\partial g_{\beta'\nu}}{\partial x^{\alpha'}}-\frac{1}{2}\frac{\partial g_{\beta\mu}}{\partial x^{\beta'}}\frac{\partial g_{\alpha'\nu}}{\partial x^{\alpha}}\right)\\
&-\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\left(\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\frac{\partial g_{\alpha'\nu}}{\partial x^{\beta'}}-\frac{\partial g_{\beta\mu}}{\partial x^{\beta'}}\frac{\partial g_{\alpha'\nu}}{\partial x^{\alpha}}\right)\\
=&g^{\alpha\alpha'}g^{\beta\beta'}\left(\frac{1}{4}\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu}}-\frac{1}{8}\frac{\partial g_{\beta\beta'}}{\partial x^{\mu}}\frac{\partial g_{\alpha\alpha'}}{\partial x^{\nu}}+\frac{1}{2}\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\frac{\partial g_{\beta'\nu}}{\partial x^{\alpha'}}\right)\\
&-\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\left(\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\frac{\partial g_{\alpha'\nu}}{\partial x^{\beta'}}-\frac{\partial g_{\beta\mu}}{\partial x^{\beta'}}\frac{\partial g_{\alpha'\nu}}{\partial x^{\alpha}}\right),
\end{aligned} \tag{15.3.54}$$

in which we used (15.3.44) under the wave coordinates to obtain the last formula.

Plugging (15.3.51)–(15.3.54) in (15.3.49), we obtain

$$\begin{aligned}
R_{\mu\nu}=&-\frac{1}{2}g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\partial x^{\beta}}+g^{\alpha\alpha'}g^{\beta\beta'}\left(-\frac{1}{4}\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu}}+\frac{1}{8}\frac{\partial g_{\beta\beta'}}{\partial x^{\mu}}\frac{\partial g_{\alpha\alpha'}}{\partial x^{\nu}}\right)\\
&+\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\frac{\partial g_{\beta'\nu}}{\partial x^{\alpha'}}-\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\left(\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}\frac{\partial g_{\alpha'\nu}}{\partial x^{\beta'}}-\frac{\partial g_{\beta\mu}}{\partial x^{\beta'}}\frac{\partial g_{\alpha'\nu}}{\partial x^{\alpha}}\right)\\
&+\frac{1}{2}g^{\alpha\alpha'}g^{\beta\beta'}\left[\left(\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial x^{\alpha}}-\frac{\partial g_{\alpha'\beta'}}{\partial x^{\alpha}}\frac{\partial g_{\beta\nu}}{\partial x^{\mu}}\right)+\left(\frac{\partial g_{\alpha'\beta'}}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}}-\frac{\partial g_{\alpha'\beta'}}{\partial x^{\alpha}}\frac{\partial g_{\beta\mu}}{\partial x^{\nu}}\right)\right]\\
&+\frac{1}{4}g^{\alpha\alpha'}g^{\beta\beta'}\left[\left(\frac{\partial g_{\alpha'\alpha}}{\partial x^{\beta'}}\frac{\partial g_{\beta\nu}}{\partial x^{\mu}}-\frac{\partial g_{\alpha'\alpha}}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial x^{\beta'}}\right)+\left(\frac{\partial g_{\alpha'\alpha}}{\partial x^{\beta'}}\frac{\partial g_{\beta\mu}}{\partial x^{\nu}}-\frac{\partial g_{\alpha'\alpha}}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial x^{\beta'}}\right)\right].
\end{aligned} \tag{15.3.55}$$

Thus, under the wave coordinates, the Einstein equation (15.3.34) in a vacuum is finally written as

$$\Box_g g_{\mu\nu}=P(\partial_\mu g,\partial_\nu g)+Q_{\mu\nu}(\partial g,\partial g), \tag{15.3.56}$$

where $\Box_g$ is the coupled wave operator given by (15.3.43), and

$$P(\partial_\mu g, \partial_\nu g) = \frac{1}{4} g^{\alpha\alpha'} \frac{\partial g_{\alpha\alpha'}}{\partial x^\mu} g^{\beta\beta'} \frac{\partial g_{\beta\beta'}}{\partial x^\nu} - \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \frac{\partial g_{\alpha'\beta'}}{\partial x^\nu}, \tag{15.3.57}$$

$$\begin{aligned} Q_{\mu\nu}(\partial g, \partial g) = & \frac{\partial g_{\beta\mu}}{\partial x^\alpha} g^{\alpha\alpha'} g^{\beta\beta'} \frac{\partial g_{\beta'\nu}}{\partial x^{\alpha'}} - g^{\alpha\alpha'} g^{\beta\beta'} \left(\frac{\partial g_{\beta\mu}}{\partial x^\alpha} \frac{\partial g_{\alpha'\nu}}{\partial x^{\beta'}} - \frac{\partial g_{\beta\mu}}{\partial x^{\beta'}} \frac{\partial g_{\alpha'\nu}}{\partial x^\alpha} \right) \\ & + g^{\alpha\alpha'} g^{\beta\beta'} \left(\frac{\partial g_{\alpha'\beta'}}{\partial x^\mu} \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha'\beta'}}{\partial x^\alpha} \frac{\partial g_{\beta\nu}}{\partial x^\mu} \right) + g^{\alpha\alpha'} g^{\beta\beta'} \left(\frac{\partial g_{\alpha'\beta'}}{\partial x^\nu} \frac{\partial g_{\beta\mu}}{\partial x^\alpha} - \frac{\partial g_{\alpha'\beta'}}{\partial x^\alpha} \frac{\partial g_{\beta\mu}}{\partial x^\nu} \right) \\ & + \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \left(\frac{\partial g_{\alpha'\alpha}}{\partial x^{\beta'}} \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\alpha'\alpha}}{\partial x^\mu} \frac{\partial g_{\beta\nu}}{\partial x^{\beta'}} \right) + \frac{1}{2} g^{\alpha\alpha'} g^{\beta\beta'} \left(\frac{\partial g_{\alpha'\alpha}}{\partial x^{\beta'}} \frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial g_{\alpha'\alpha}}{\partial x^\nu} \frac{\partial g_{\beta\mu}}{\partial x^{\beta'}} \right). \end{aligned} \tag{15.3.58}$$

It is clear that the right-hand side of Eq. (15.3.56) is a quadratic term with respect to ∂g, whose coefficients are smooth functions of g.

To consider the small perturbation of the Minkowski space-time (15.3.35), we write the unknown metric g as

$$g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3, \tag{15.3.59}$$

then from (15.3.56) we know that under the wave coordinates, $h = (h_{\mu\nu})$ should satisfy the equation

$$\Box_{m+h} h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h), \tag{15.3.60}$$

where $F_{\mu\nu}(h)(\partial h, \partial h)$ is a quadratic term with respect to ∂h, and its coefficients are smooth functions of h. Specifically, we have

$$F_{\mu\nu}(h)(\partial h, \partial h) = F(\partial_\mu h, \partial_\nu h) + G_{\mu\nu}(\partial h, \partial h) + H_{\mu\nu}(h)(\partial h, \partial h), \tag{15.3.61}$$

where

$$F(\partial_\mu h, \partial_\nu h) = \frac{1}{4} m^{\alpha\alpha'} \frac{\partial h_{\alpha\alpha'}}{\partial x^\mu} m^{\beta\beta'} \frac{\partial h_{\beta\beta'}}{\partial x^\nu} - \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \frac{\partial h_{\alpha\beta}}{\partial x^\mu} \frac{\partial h_{\alpha'\beta'}}{\partial x^\nu}, \tag{15.3.62}$$

$$\begin{aligned} & G_{\mu\nu}(\partial h, \partial h) \\ & = \frac{\partial h_{\beta\mu}}{\partial x^\alpha} m^{\alpha\alpha'} m^{\beta\beta'} \frac{\partial h_{\beta'\nu}}{\partial x^{\alpha'}} - m^{\alpha\alpha'} m^{\beta\beta'} \left(\frac{\partial h_{\beta\mu}}{\partial x^\alpha} \frac{\partial h_{\alpha'\nu}}{\partial x^{\beta'}} - \frac{\partial h_{\beta\mu}}{\partial x^{\beta'}} \frac{\partial h_{\alpha'\nu}}{\partial x^\alpha} \right) \\ & + m^{\alpha\alpha'} m^{\beta\beta'} \left(\frac{\partial h_{\alpha'\beta'}}{\partial x^\mu} \frac{\partial h_{\beta\nu}}{\partial x^\alpha} - \frac{\partial h_{\alpha'\beta'}}{\partial x^\alpha} \frac{\partial h_{\beta\nu}}{\partial x^\mu} \right) + m^{\alpha\alpha'} m^{\beta\beta'} \left(\frac{\partial h_{\alpha'\beta'}}{\partial x^\nu} \frac{\partial h_{\beta\mu}}{\partial x^\alpha} - \frac{\partial h_{\alpha'\beta'}}{\partial x^\alpha} \frac{\partial h_{\beta\mu}}{\partial x^\nu} \right) \\ & + \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \left(\frac{\partial h_{\alpha'\alpha}}{\partial x^{\beta'}} \frac{\partial h_{\beta\nu}}{\partial x^\mu} - \frac{\partial h_{\alpha'\alpha}}{\partial x^\mu} \frac{\partial h_{\beta\nu}}{\partial x^{\beta'}} \right) + \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \left(\frac{\partial h_{\alpha'\alpha}}{\partial x^{\beta'}} \frac{\partial h_{\beta\mu}}{\partial x^\nu} - \frac{\partial h_{\alpha'\alpha}}{\partial x^\nu} \frac{\partial h_{\beta\mu}}{\partial x^{\beta'}} \right), \end{aligned} \tag{15.3.63}$$

and $H_{\mu\nu}(h)(\partial h, \partial h)$, being the higher order term, is a quadratic term with respect to ∂h, whose coefficients are smooth functions of h, and $H_{\mu\nu}(h)(\partial h, \partial h)$ vanishes when $h = 0$: $H_{\mu\nu}(0)(\partial h, \partial h) = 0$.

$G_{\mu\nu}(\partial h, \partial h)$ given by (15.3.63) satisfies the null condition, however, $P(\partial_\mu h, \partial_\nu h)$ given by (15.3.62) does not satisfy the null condition. Let

$$\bar{h} = m^{\mu\nu} h_{\mu\nu}. \tag{15.3.64}$$

From (15.3.60) we know that $\bar{h}$ is the solution to the following equation satisfying the null condition:

$$g^{\alpha\beta} \frac{\partial^2 \bar{h}}{\partial x^\alpha \partial x^\beta} = G(\partial h, \partial h) + H(h)(\partial h, \partial h), \tag{15.3.65}$$

where $G(\partial h, \partial h)$ is a quadratic term with respect to ∂h, and $H(h)(\partial h, \partial h)$, being the higher order term, is a quadratic term with respect to ∂h, whose coefficients are smooth functions of h, and $H(h)(\partial h, \partial h)$ vanishes when $h = 0$: $H(0)(\partial h, \partial h) = 0$. We can call $\bar{h}$ the good component. By (15.3.62), the first term on the right-hand side of $F(\partial_\mu h, \partial_\nu h)$ is a quadratic term with respect to the good component $\bar{h}$, while, the second term needs further analysis which is omitted here. As for $\Box_{m+h} h_{\mu\nu}$, it can be treated by using the same arguments dealing with Eq. (15.2.7). Combining all these factors, we can obtain the global existence of $h = (h_{\mu\nu})$ with small initial data, and then the stability of the Minkowski space-time (15.3.35) (see Lindblad and Rodnianski 2005 for details).

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