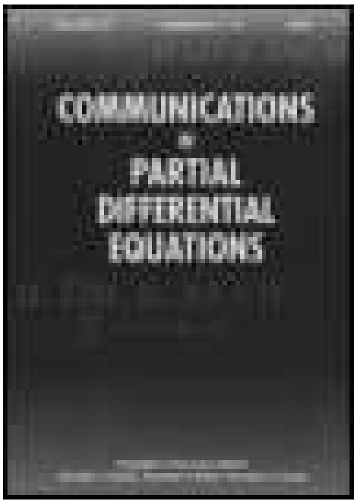

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Long-time behaviour of solutions of a system of nonlinear wave equations

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Long-time behaviour of solutions of a system of nonlinear wave equations

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In this paper, we will study long-time behaviour of solutions of the following initial-value problem:

$$c_i^2\partial_t^2 u_i - \Delta u_i = F_i(u', u_i''), \quad 1 \le i \le i_0, \quad x \in R^2 \tag{1a}$$

$$u(0,x) = \varepsilon f(x), \quad u_t(0,x) = \varepsilon g(x) \tag{1b}$$

where $u = (u_1, \ldots, u_{i_0})$ is a vector-function of $t > 0$ and $x \in R^2$; u', u'', u_i', u_i'' denote correspondingly all the first and second derivatives of u or its i-th scalar component and the F_i are smooth functions of (u', u_i'') independent of u_{tt} and linear in higher derivatives of u. Moreover, we assume that the F_i vanish at zero with all their derivatives of order less than p. The initial data is C_0^∞ and ε is a small parameter.

We chose $\mathbb{R}^2$ because it seems to be the most difficult case. The same method with less technicalities can be used for any $\mathbb{R}^n$.

Given f, g and F, we define the life-span $T_*(\varepsilon)$ to be the supremum over all $T \ge 0$ such that a C^∞-solution of (1a) exists for all $x \in R^n$, $0 \le t \le T$ and satisfies the initial conditions (lb). The following theorem asserts that $T_* > 0$.

THEOREM 1. *(Local existence).* [**10**]

Assume that the initial data is C_0^∞ and satisfies the condition $\sum_{i=0}^{n} |\partial_i u(0,x)| \cdot$ 1. Then there exists a number $T > 0$ and a unique vector-function $u(x,t) \in C^\infty$ for all $x \in R^n$ and $0 \le t \le T$ which satisfies (1a) for $x \in R^n$, $0 \le t \le T$ and the initial conditions (1b).

Elementary analysis of the proof shows that $T_* \ge \frac{A}{\varepsilon}$, where A is a constant depending on f, g and F.

A similar scalar problem (i.e. $i_0 = 1$) for one space dimension was considered by P. Lax [**19**] and for three and more space dimensions by S. Klainerman and F. John [**7**], [**6**]. Their results are summarized in the following table for ε sufficiently small.

value of p	# of space dimensions	lower bound for T_*	upper bound for T_*
2	1	$\frac{A}{\varepsilon}$	$\frac{A_*}{\varepsilon}$
2	3	$A \exp \frac{A}{\varepsilon}$	$A_* \exp \frac{A_*}{\varepsilon}$
2	≥ 4	solution exists globally	
≥ 3	≥ 3		

The case of two space dimensions was left open. Here we derive lower bounds for T_* for the 2-dimensional case; moreover, we derive them for nonscalar case i.e. $i_0 > 1$.[1] More elaborate analysis and history of the problem are given in [**18**].

[1]Recently L. Hörmander rederived estimates of Klainerman and John and obtained estimates similar to ours for the scalar case.

In this paper, we will use the following notations:

$$\partial_0 = \frac{\partial}{\partial t};\ \partial_i = \frac{\partial}{\partial x_i} \quad \text{for} \quad i = 1,2; \quad \Omega = x^1\partial_2 - x^2\partial_1 \tag{2a}$$

$$\|u\|_m = \sum_{|a|+b\le m} \sum_{i=1}^{i_0} \|\partial^a \Omega^b u_i\|_{L^2(R^2)} \tag{2b}$$

$$|u|_m = \sum_{|a|+b\le m} \sum_{i=1}^{i_0} \|\partial^a \Omega^b u_i\|_{L^\infty(R^2)} \tag{2c}$$

$$w(t,r,p) = \left\{ \frac{1}{(r+1)^{p/2}} \sum_{|a|=p} \sqrt{\prod_{i=1}^{i_0} \frac{1}{(|c_i t - r| + 1)^{a_i}}} \right\}^{-1} \tag{2d}$$

$$|||u|||_{m,p} = \sum_{|a|+b\le m} \sum_{i=1}^{i_o} \|w(t,|x|,p)\partial^a \Omega^b u_i(t,x)\|_{L^2(R^2)} \tag{2e}$$

$$[u]_{m,p} = \sum_{|a|+b\le m} \sum_{i=1}^{i_0} \|(w(t,|x|,p)^{\frac{1}{p}} \partial^a \Omega^b u_i(t,x)\|_{L^\infty(R^2)} \tag{2f}$$

where $a = (a_0, a_1, a_2)$ is a multi-index. Moreover we will assume that

(1) If u depends on a parameter t , then putting t at the end of the row of indices means taking the supremum of the corresponding norm on the interval $[0,t]$; for example,

$$|u|_{k,t} = \sup_{0\le\tau\le t} |u(\tau,x)|_k \tag{2g}$$

(2) Omitting index of a vector means summation over that index; for example,

$$|\partial u|_k = \sum_{i,j} |\partial_i u_j|_k \tag{2h}$$

(3) For two vectors $x_1,\ldots,x_n$ and $a_1,\ldots,a_n$

$$x^a = \prod_{(i)} x_i^{a_i} \tag{2i}$$

(4) C will stand for a constant, which may vary from step to step. (2j)

Without loss of generality, we may assume that each F_i has the following form

$$F_i(u', u_i'') = \sum_{a+b>0} g_i^{ab}(u')\partial_a\partial_b u_i + \sum_{a,b\geq 0} g_{bi}^{a}(u')\partial_a u_b \tag{3a}$$

with g_i^{ab}, $g_{bi}^a \in C^\infty$ and satisfying

$$\left|\frac{\partial^j}{(\partial u')^j} g_i^{ab}\right| + \left|\frac{\partial^j}{(\partial u')^j} g_{bi}^{a}\right| \leq C|\partial u|^{p-1-j} \tag{3b}$$

$$\sum_{a,b,i} |g_i^{ab}(u')| \leq \frac{1}{2} \tag{3c}$$

for $|\partial u| \leq 1$ and $0 \leq j \leq p-1$

Our main results are given in Theorems 2 and 3.

THEOREM 2. *(Decay estimates)*

Let w be as defined in (2) and F as defined in (3). Then for any $0 < \gamma \leq \frac{1}{2}$, there is a constant A depending only on f, g, F, c_i and γ such that the corresponding solution of (1) verifies the following decay estimates on the interval of existence:

$$|\partial u_i(t,x)| \leq \frac{A\{\varepsilon + |||F|||_{3,p,t}\}}{(c_i t + |x| + 1)^{0.5-\gamma}(|x|+1)^{\gamma}(|c_i t - |x|| + 1)^{0.5}} \quad \text{if} \quad p \geq 3 \tag{4a}$$

$$|\partial u_i(t,x)| \leq \frac{A\{\varepsilon + \ell n(2+t)\sqrt{1+t}|||F|||_{3,p,t}\}}{(c_i t + |x| + 1)^{0.5-\gamma}(|x|+1)^{\gamma}(|c_i t - |x|| + 1)^{0.5}} \quad \text{if} \quad p = 2 \tag{4b}$$

We will prove this theorem in section 2.

THEOREM 3. *(Long-time existence).*

Let the F_i be as determined by (3). Then there exists an ε_0 and a constant A depending only on f, g and F such that for all $0 \le \varepsilon \le \varepsilon_0$, the life-span T_* of the corresponding solution of (1)

$$a) \text{ exceeds the number } \frac{A}{\varepsilon^2(\ell n\frac{1}{\varepsilon})^2}, \quad \text{if} \quad p = 2 \tag{5a}$$

$$b) \text{ exceeds the number } A \exp \frac{A}{\varepsilon^2}, \quad \text{if} \quad p = 3 \tag{5b}$$

$$c) \text{ equals to } \infty, \text{i.e., the solution exists globally, if } p > 3 \tag{5c}$$

Remark 1: Similar decay estimates and long-time existence results can be proved if we allow the F_i to depend on u and require that each of the F_i could be written in the divergence form, i.e. $F_i = \sum_a \partial_a f_{ia}(u, u') + f_i$ where the f_i vanish at 0 along with their first p derivatives and the f_{ia} vanish at 0 along with their first $p-1$ derivatives.

We will need the following theorem and lemma which will be proved in section 1.

THEOREM 4. *(Energy estimates).*

There exist constants B_N for all integers $N \ge 0$, depending only on $F(u', u'')$ with the following property: Whenever $u(t,x)$ is a C^∞-solution of (1) and $\int_0^t |\partial u|^{p-1} ds \le 1$:

$$||\partial u(t,x)||_N \le B_N ||\partial u(0,x)||_N \tag{6}$$

LEMMA 1. *Let $u \in C_0^\infty$ and the F_i be as defined in (3). Then*

$$|||F|||_{m,p} \le C[\partial u]_{0,p}^{p-1} ||\partial u||_{m+1} \tag{7}$$

Proof of Theorem 3: (Part a).

Combining (4b) and (7), we obtain with $p = 1$

$$[\partial u]_{1,1,t} \leq C\{\varepsilon + \ell n(2+t)\sqrt{1+t}[\partial u]_{1,1,t}\|\partial u\|_{5,t}\} \tag{8}$$

Let T be the largest number $0 \leq T \leq T_*$ such that $\|\partial u\|_{5,T} \leq k\varepsilon$ and $|\partial u|_{0,T} < 1$ for some number k which will be determined later. Then either $\sqrt{1+T}\ell n(2+T) \geq \frac{1}{2k\varepsilon C}$ in which case $\sqrt{1+T_*}\ell n(2+T_*) \geq \frac{1}{2k\varepsilon C}$ for all sufficiently small ε and part a) of the Theorem 3 follows or $\sqrt{1+T}\ell n(2+T) < \frac{1}{2k\varepsilon C}$. In the latter case, (8) yields

$$[\partial u]_{1,1,t} \leq C\varepsilon \tag{9}$$

which implies that

$$\int_0^T |\partial u(s)|_1 \, ds \leq [\partial u]_{1,1,T} \int_0^T \frac{ds}{\sqrt{1+s}} \leq \frac{1}{2k}.$$

Then (6) and (9) imply with possibly a different constant C,

$$\|\partial u\|_{5,T} \leq C\varepsilon$$

and

$$|\partial u|_{0,T} \leq C\varepsilon$$

If k is chosen such that $C \leq \frac{k}{2}$ and $C\varepsilon < \frac{1}{2}$, we have $\|u\|_{5,T} \leq \frac{1}{2}k\varepsilon$ and $|\partial u|_{0,T} \leq \frac{1}{2}$ which contradict the choice of T and thus $\sqrt{1+T}\ell n(2+T) \geq \frac{1}{2k\varepsilon C}$.

Parts b and c:

Combining (4a) and (7), we obtain with $p > 2$

$$[\partial u]_{1,p,t} \leq C\{\varepsilon + [\partial u]_{1,p,t}^{p-1} \|\partial u\|_{5,t}\} \tag{10}$$

By (6) $\|\partial u\|_{5,t} \leq C\varepsilon$ provided that

$$\int_0^t |\partial u|_1^{p-1} ds \leq C[\partial u]_{1,p,t}^{p-1} \int_0^t (1+s)^{1-p} ds \leq 1 \tag{11}$$

Let T be the largest number such that

$$C[\partial u]_{1,p,t}^{p-1} \int_0^T (1+s)^{1-p} ds \leq 1 \quad \text{and} \quad |\partial u|_{1,T} \leq 1$$

Substituting (11) into (10) we obtain:

$$[\partial u]_{1,T} \leq C\varepsilon\{1 + [\partial u]_{1,T}^{p-1}\} \tag{12}$$

Using that $|\partial u|_{1,T} \leq 1$, we obtain for $\varepsilon < \frac{1}{2}C$

$$[\partial u]_{1,p,T} \leq 2C\varepsilon$$

Then either

$$C[\partial u]_{1,p,T}^{p-1} \int_0^T (1+s)^{1-p} ds \leq C\varepsilon^{p-1} \int_0^T (1+s)^{1-p} ds < \frac{1}{2} \tag{13}$$

or

$$C\varepsilon^{p-1} \int_0^T (1+s)^{1-p} ds > \frac{1}{2}. \tag{14}$$

For ε sufficiently small and $p \geq 4$, (13) always holds and thus $|\partial u|_{1,T} \leq \frac{1}{2}$ which contradicts the choice of T, unless $T = \infty$. For ε sufficiently small and $p = 3$, either (14) holds and part b) follows or (13) holds and $|\partial u|_{1,T} \leq \frac{1}{2}$ and thus, we can continue the solution beyond T which contradicts the choice of T and therefore, (14) must hold.

1. Energy Estimates

In this section, we will prove theorems 4 and lemma 1.

We will use the following lemma which is proved in [**10**].

Lemma 2. *Consider the equation*

$$v_{tt} = \Delta v - \sum_{i+j>0}^{n} b_{ij}(t,x)\partial_i\partial_j v = h(t,x) \tag{1.1}$$

such that for $a_{ij} = \delta_{ij} + b_{ij}$; $1 \le i,\ j \le n$ *and a constant* m

$$\frac{1}{m}|\xi|^2 \le \sum_{i,j=1}^{n} \alpha_{ij}(t,x)\xi_i\xi_j \le m|\xi|^2 \tag{1.2}$$

holds uniformly in $t \ge 0$, $x \in R^2$. *Then the following inequality is true.*

$$\|\partial v(t,x)\|_0 \le C\left\{\|\partial v(0,x)\|_0 + \int_0^t \|h(s,x)\|_0 ds\right\} \exp C\int_0^t |\partial b(s,x)|_0 dx \tag{1.3}$$

Applying lemma 2 to (1) with F in the form (3) we obtain for $v = \partial^\alpha\Omega^\beta u_1$:

$$c_k^2\partial_t^2 v - \Delta v = \sum_{a+b>0} g_i^{ab}\partial_a\partial_b v + h_{\alpha\beta k} \tag{1.4}$$

where $1 \le k \le i_0$, $0 \le a,b \le 2$, $\sum_{a,b,i}|g_i^{ab}| \le \frac{1}{2}$ and

$$h_{\alpha\beta k} = h^1_{\alpha\beta k} + h^2_{\alpha\beta k} + h^3_{\alpha\beta k} \tag{1.5a}$$

$$h^1_{\alpha\beta k} = \sum_{a,b}[\partial^\alpha\Omega^\beta(g^{ab}_k\partial_a\partial_b u_k) - g^{ab}_k\partial^\alpha\Omega^\beta\partial_a\partial_b u_k] \tag{1.5b}$$

$$h^2_{\alpha\beta k} = \sum_{a,b}[g^{ab}_k(\partial^\alpha\Omega^\beta\partial_a\partial_b u_k - \partial_a\partial_b\partial^\alpha\Omega^\beta u_k)] \tag{1.5c}$$

$$h^3_{\alpha\beta k} = \sum_{a,b}\partial^\alpha\Omega^\beta(g^a_{bk}\partial_b u_a)^1 \tag{1.5d}$$

By virtue of lemma 2 applied to (1.4)

$$\|\partial v(t,x)\|_0 \le C(\|\partial v(0,x)\|_0 + \int_0^t \|h_{\alpha\beta k}\|_{L^2}dx \ \exp C\int_0^t |\partial g|_0 ds$$

and hence for some positive constants C_N:

$$\|\partial u(t,x)\|_N \le C_N(\|\partial u(0,x)\|_N + \int_0^t \|h(s)\|_0 ds)\exp C_N\int_0^t |\partial u|_1^{p-1}ds \tag{1.6}$$

where

$$\|h\|_0 = \sum_{\alpha+\beta\le N}\ \sum_{1\le k\le i_0}\ \sum_{1\le\gamma\le 3}\|h^\gamma_{\alpha\beta k}\|_0 \tag{1.5e}$$

Now we use the following lemma which will be proved later on in this section.

LEMMA 3. *Let $\|h\|_0$ be as defined in (1.5). Then*

$$\|h\|_0 \le C|\partial u|_1^{p-1}\|\partial u\|_N \tag{1.7}$$

Combining (1.3), (1.7) and $\int_0^t |\partial u|_1^{p-1}ds \le 1$, we obtain

$$\|\partial u(t,x)\|_N \le C(\|\partial u(0,x)\|_N + \int_0^t |\partial u(s,x)|_1^{p-1}\|\partial u(s,x)\|_N ds)$$

which is equivalent to

[1] In order to avoid ambiguity we again clarify the difference between ∂_a and ∂^α: $\partial_a = \frac{\partial}{\partial x^\alpha}$, $\partial^\alpha = \prod_k \partial^{\alpha_k}_{a_k}$.

$$\frac{d}{dt}\int_0^t |\partial u(s)|_1^{p-1}\|\partial u(s)\|_N ds \ \exp(-C\int_0^t |\partial u(\tau)|^{p-1}d\tau)$$
$$\leq \|\partial u(0)\|_N |\partial u(t)|_1^{p-1}\exp(C\int_0^t |\partial u(\tau)|_1^{p-1}d\tau) \tag{1.8}$$

Integrating this, we obtain

$$\int_0^t |\partial u(s)|_1^{p-1}\|\partial u(s)\|_N ds \leq$$
$$\leq \|\partial u(0)\|_N \int_0^t |\partial u(s)|_1^{p-1}ds \ \exp C\int_0^t |\partial u(s)|_1^{p-1}ds \tag{1.9}$$

(1.9) and the fact that $\int_0^t |\partial u(s)|_1^{p-1}ds \leq 1$ imply

$$\|\partial u(t)\|_N \leq C\|\partial u(0)\|_N(1+\exp C\int_0^t |\partial u(s)|_1^{p-1}ds).$$

This yields (6).

Now we want to prove lemmas 1 and 3. Their proofs are based on the following propositions which will be discussed later on in this section.

We will use the following notation in the propositions:

$\mathcal{D}$ means either ∂_k or $\Omega_{m\ell} = x^m\partial_\ell - x^\ell\partial_m, k, m, \ell > 0$;

$$\mathcal{D}^\alpha = \prod_{k\geq 1}\partial_k^{\alpha_k}\prod_{\ell>m\geq 1}\Omega_{m\ell}^{\alpha_{m\ell}}, \text{ where } \sum_{k\geq 1}\alpha_k + \Sigma\alpha_{m\ell} = |\alpha|.$$

Sometimes we write $\mathcal{D}^N$ instead of $\mathcal{D}^\alpha$, $|\alpha| = N$.

PROPOSITION A1. *For a function $u \in C_0^\infty(R^n)$ and $|\alpha| = i \leq k$, we have*

$$\|\mathcal{D}u\|_{0,L^p(R^n)} \leq C\|u\|_{0,L^\infty(R^n)}^{1-a}\|u\|_{k,L^r(R^n)}^a$$

where $\frac{i}{k} \leq a = \frac{i-\frac{n}{p}}{k-\frac{n}{r}} \leq 1$ *and* $\|u\|_{k,L^r(R^n)} = \sum_{|\beta|\leq k}\|\mathcal{D}^\beta u\|_{L^r(R^n)}$

PROPOSITION A2. *Let $f_1, f_2, \ldots, f_r \in C^\infty(R^n)$ and be such that all norms appearing below are bounded. Moreover, let $\omega = \omega(f)$ be a C^q function satisfying*

$$\left|\frac{\partial^i\omega}{(\partial f)^i}\right| \leq B|f|^{q-1}$$

for $0 \le i \le q$, $|f| \le 1$ *and some constant* B. *Then there exists a constant* C *depending only on* ω *such that*

$$\|\mathcal{D}^{\beta}(\omega \circ f)\|_{L^p(R^n)} \le C|f|^{q-1}_{L^\infty(R^n)} \sum_{|\alpha|=N} \|\mathcal{D}^{\alpha} f\|_{L^p(R^n)}$$

PROPOSITION A3. *Let* $f, g \in C_0^\infty$, *then*

$$\|\mathcal{D}^{\alpha}(fg) - f\mathcal{D}^{\alpha} g\|_{L^2(R^n)} \le C(|f|_1\|g\|_{k-1} + |g|_0\|f\|_k)$$

PROPOSITION A4. *Let* $\varphi(r)$ *be a positive function which is* C^0 *for all* $r \ge 0$ *and* C^1 *for all* $r \ge 0$ *except possibly a finite number of points. Moreover, let* $\varphi(r)$ *satisfy the following inequality for all* $r \ge 0$ *and* $0 \le r' \le 1$ *and some constant* M:

$$\frac{1}{M}\varphi(r + r') \le \varphi(r) \le M\varphi(r + r').$$

If f *is as in proposition A2, then for* $|\beta| = N$

$$\|\mathcal{D}^{\beta}(\omega \circ f)\|_{L^p,\varphi} \le C|f|^{q-1}_{0,\varphi^{1/q-1}} \sum_{|\alpha|=N} \|\mathcal{D}^{\alpha} f\|_{L^p} \tag{1.10}$$

Here $|\cdot|_{0,\varphi}$ and $\|\cdot\|_{L^p,\varphi}$ are correspondingly the weighted L^∞ and L^p-norm with weight-function φ.

Proof of Proposition A1, A3 exactly repeats proof of similar propositions given in [**18**], [**7**], proof of proposition A4 is by application of proposition A2 and appropriate partition of unity. Proposition A2 is generalization of Moser's lemma. So we prove it here.

$$\mathcal{D}^N(\omega \circ f) = \sum_{0 \le s \le N} \frac{\partial^s \omega}{(\partial f)^s} \sum_{|\alpha|=s} C_{s\alpha} \prod_{\ell=1}^{N} (\mathcal{D}^{\ell} f)^{\alpha_\ell}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_N = s$, $1 \cdot \alpha_1 + 2\alpha_2 + \cdots + N\alpha_N = N$

Choosing $p_\ell = \dfrac{pN}{\ell \alpha_N}$, we obtain by means of generalized Hölder inequality:

$$\|\mathcal{D}^\beta(\omega \circ f)\|_{L^p} \leq C \sum_{0 \leq s \leq N} \left| \frac{\partial^s \omega}{(\partial f)^s} \right|_{L^\infty} \sum_{|\alpha|=s} \prod_{|\ell|=1} \|\mathcal{D}^\ell f)^{\alpha_\ell}\|_{L^{p_\ell}}$$

Using (1.9) and $\|(\mathcal{D}^\ell f)^{\alpha_i}\|_{L^{p_i}} \leq \|\mathcal{D}^\ell f\|^{\alpha_i}_{L^{pN/\ell}}$ we conclude:

$$\|\mathcal{D}^N(\omega \circ f)\|_{L^p} \leq C \sum_{0 \leq s \leq N} |f|^{q-s}_{L^\infty} \sum_{|\alpha|=s} \prod_{\ell=1}^{N} \|\mathcal{D}^\ell f\|^{\alpha_\ell}_{L^{pN/\ell}} \tag{1.11}$$

By proposition A1:

$$\|\mathcal{D}^\ell f\|_{L^{pN\ell}} \leq |f|^{1-\frac{\ell}{N}}_{L^\infty} \|\mathcal{D}^N f\|^{\frac{\ell}{N}}_{L^p}$$

Substituting this into (1.11) yields:

$$\|\mathcal{D}^N(\omega \circ f)\|_{L^p} \leq C \sum_{0 \leq s \leq q} |f|^{q-s}_{L^\infty} |f|^{\Sigma(1-\frac{\ell}{N})\alpha_\ell}_{L^\infty} \|\mathcal{D}^N f\|^{\Sigma \frac{\ell}{N}\alpha_\ell}_{L^p} =$$
$$= C|f|^{q-1}_{L^\infty} \|\mathcal{D}^N f\|_{L^p}$$

Proof of lemma 1: now follows immediately from proposition A4 since the w defined by (2d) satisfies all the required properties

Proof of Lemma 3: By proposition A3

$$\|h^1_{\alpha\beta k}\|_{L^2} \leq C(|g|_1 \|\partial u\|_N + |\partial u|_0 \|g\|_N)$$

which by proposition A2 implies

$$\|h^1_{\alpha\beta k}\|_{L^2} \leq C|\partial u|^{p-1}_1 \|\partial u\|_N.$$

We also have

$$\|h^2_{\alpha\beta k} \le C|g|_0\|\partial u\|_N \le C|\partial u|_1\|\partial u\|_N$$

and

$$\|h^3_{\alpha\beta k}\|_{L^2} \le C|\partial u|_1\|\partial u\|_N$$

Since $\|h\|_{L^2} \le \sum_{\gamma=1}^{3} \sum_{\substack{\alpha+\beta\le N \\ i\le K\le i_0}} \|h^\gamma_{\alpha\beta k}\|_{L^2(R^2)}$ we obtain (1.7).

2. Decay estimates

Proof of Theorem 2: The theorem is a direct consequence of the following two lemmas.

LEMMA 4. *Let u be a solution of*

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ \quad u(0,x) = 0, \ u_t(0,x) = \varepsilon g(x) \end{cases} \tag{2.1}$$

with $g \in C_0^\infty$. Then there exists a constant C depending only on g such that

$$|u(t,x)| \le \frac{C\varepsilon}{\sqrt{(t+|x|1)(|t-|x||+1)}} \tag{2.2}$$

LEMMA 5. *Let u be a solution of*

$$\begin{cases} c_i\partial_t^2 u_i - \Delta u_i = F_i(x,t), \quad x \in R^2, \quad 1 \le i \le i_0 \\ u(0,x) = u_t(0,x) = 0 \end{cases} \tag{2.3}$$

Moreover, let $w(p,\alpha,t,r)$ be defined as

$$w(p,\alpha,t,r) = (r+1)^{\frac{p-1}{2}} \prod_{i=1}^{i_0} (|c_i t - r| + 1)^{\frac{\alpha_i}{2}} \tag{2.4a}$$

where $\alpha = (\alpha_1, \dots, \alpha_{i_0})$ is a multi-index, $|\alpha| = \Sigma\alpha_i = p-1$.
We will suppress everywhere the dependence of w on i, p and α and write symbolically as

$$w(t,r) = (r+1)^{\frac{p-1}{2}} (|ct-r|+1)^{\frac{\alpha}{2}}, |\alpha| = p-1 \tag{2.4b}$$

Then,

a. For any $0 < \gamma \leq \frac{1}{2}$, there exists a constant C depending only on F, c_i and γ such that the corresponding solution of (2.3) verifies the following estimate for $p \geq 3$ and any multi-index $\alpha : |\alpha| = p-1$

$$|\partial u_i| \leq \frac{C}{(c_it+r+1)^{\frac{1}{2}-\gamma}(r+1)^{\gamma}(|c_it-r|+1)^{\frac{1}{2}}}|||F|||_{3,p,t} \tag{2.5a}$$

b. There exists a constant C depending only on F and c_i such that the corresponding solution of (2.3) verifies the following estimate for $p = 2$ and any multi-index $\alpha : |\alpha| = 1$

$$|\partial u_i| \leq \frac{c\ell n(2+t)}{\sqrt{1+t}}\frac{1+t}{1+|c_it-r|}|||F|||_{3,p,t} \tag{2.5b}$$

with $|||\cdot|||$ defined with weight function w given by (2.4) instead of (2d).

In order to simplify the notation, we will prove these estimates only for u_1. Moreover, we will assume that $c_1 = 1$. The proof of these lemmas is based essentially on the properties of the functional K defined by the following definition.

DEFINITION 2.1. *Let $g(y)$ be a $C^0(R^2)$ function and r, $\theta+\psi$ be polar coordinates of y. Define $K(t,a,r,\theta,g) =$*

$$\begin{cases} \displaystyle\int_{-\pi}^{\pi}\frac{g(r,\theta+\psi)dx}{\sqrt{t^2-a^2-r^2+2\arccos\psi}}, & \text{if} \quad \left|\dfrac{a^2+r^2-t^2}{2ar}\right| \geq 1 \\ \displaystyle\int_{-\varphi}^{\varphi}\frac{g(r,\theta+\psi)dx}{\sqrt{t^2-a^2-r^2+2\arccos\psi}}, & \text{if} \quad \left|\dfrac{a^2+r^2-t^2}{2ar}\right| \leq 1 \\ & \text{and} \quad \varphi = \arccos\dfrac{a^2+r^2-t^2}{2ar} \end{cases} \quad 3 \tag{2.6a}$$

Also define

$$K(t,a,r) = K(t,a,r,\theta,1) \tag{2.6b}$$

The properties of $K(t,a,r)$ are given in the proposition below.

PROPOSITION 2.1.

If $t \geq a + r$ *and* $\left| \frac{a^2 + r^2 - t^2}{2ar} \right| \geq 1$, *then* $K(t,a,r)$ *satisfies*

$$A) \quad |K(t,a,r)| \leq C \frac{\ell n[2 + \frac{ar}{t^2-(a+r)^2}]}{\sqrt{t^2 - a^2 - r^2}} \leq \frac{C}{\sqrt{t^2 - (a+r)^2}} \tag{2.7a}$$

$$B) \quad \left| \frac{\partial K(t,a,r)}{\partial t} \right| + \left| \frac{\partial K(t,a,r)}{\partial r} \right| \leq \frac{Ct}{(t-a-r)(t+a+r)\sqrt{t^2 - a^2 - r^2}} \tag{2.7b}$$

If $t \leq a + r$ *and* $\left| \frac{t^2 - a^2 - r^2}{2ar} \right| \leq 1$, *then*

$$A^*) \qquad K(t,a,r) = \sqrt{\frac{2}{ar}} \int_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)(2 + P\tau - \tau)}} \tag{2.7c}$$

with $P = \frac{a^2 + r^2 - t^2}{2ar}$

$$B^*) \qquad K(t,a,r) \leq \frac{C}{\sqrt{ar}} \ell n[2 + \frac{ar}{(a+r)^2 - t^2} \chi(t-a)] \tag{2.7d}$$

where χ *is the characteristic function of positive numbers.*

$$C^*) \quad \left| \frac{\partial K(t,a,r)}{\partial t} \right| + \left| \frac{\partial K(t,a,r)}{\partial r} \right| \leq \frac{C(t+a)}{(a+r-t)(a+r+t)\sqrt{ar}} \tag{2.7e}$$

We will prove the proposition later on in this section.

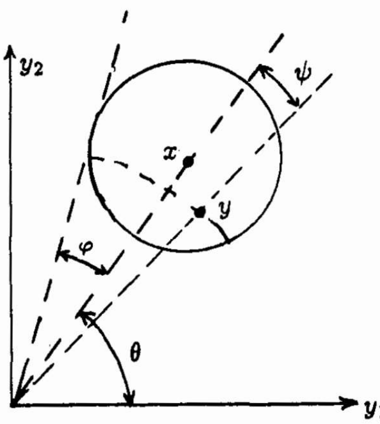
Proof of Lemma 4: As well known

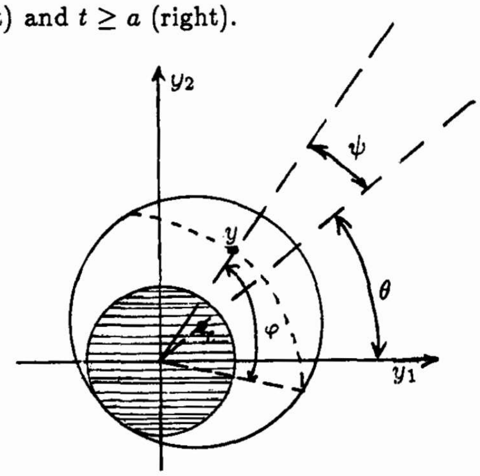
$$u(t,x) = \frac{1}{2\pi} \int\limits_{|x-y| \leq t} \frac{g(y) dy_1 dy_2}{\sqrt{t^2 - |x-y|^2}}$$

In polar coordinates, the formula becomes:

$$u(x,t) = u(a,\theta,t) = \int_{|t-a|}^{t+a} r dr \int_{-\varphi}^{\varphi} \frac{g(r,\theta+\psi)dx}{\sqrt{t^2-a^2-r^2+2\arccos\psi}} + \\ + \chi(t-a)\int_0^{t-a} r dr \int_{-\pi}^{\pi} \frac{g(r,\theta+\psi)dx}{\sqrt{t^2-a^2-r^2+2\arccos\psi}} \tag{2.8}$$

where a, θ are polar coordinates of x, and r, $\theta+\psi$ are polar coordinates of y. χ is the characteristic function of positive numbers. The domain of integration is shown below for $t \le a$ (left) and $t \ge a$ (right).





The unshaded part corresponds to the first integral and is always present. The shaded region corresponds to the second term and is present only if $t \ge a$, what we have expressed by introducing $\chi(t-a)$ as a factor. Using functional $K(t,a,r,\theta,g)$, we can rewrite (2.8) in a more concise form.

$$u(x,t) = u(a,\theta,t) = \int_{|t-a|}^{t+a} rK(t,a,r,\theta,g)dr \\ + \chi(t-a)\int_0^{t-a} rK(t,a,r,\theta,g)dr \tag{2.9}$$

Using the notation

$$|g|_\psi = \sup_{0\le\psi\le 2\pi} |g(r,\psi)|$$

we can estimate $|u|$ in the following manner

$$|u(x,t)| \leq \int_{\max(a-t,0)}^{a+b} |g|_\psi K(t,a,r) r\,dr$$

$$\leq C\left\{\int_{|t-a|}^{t+a} |g|_\psi \sqrt{\frac{r}{a}} \ell n[2+\frac{ar\,\chi(t-a)}{|(a+r)^2-t^2|}]dr\right.$$

$$\left. +\chi(t-a)\int_0^{t-a} |g|_\psi \frac{r}{\sqrt{t^2-a^2-r^2}} \ell n[2+\frac{ar\,\chi(t-a)}{|(a+r)^2-t^2|}]dr\right\}$$

$$\leq C\left\{\frac{1}{\sqrt{a}|t-a|}\int_{|t-a|}^{t+a} |g|_\psi r^{3/2} \ell n[2+\frac{ar\,\chi(t-a)}{|(a+r)^2-t^2|}]dr\right.$$

$$\left. +\frac{\chi(t-a)}{\sqrt{t(t-a)}}\int_0^{|t-a|} |g|_\psi \ell n[2+\frac{ar\,\chi(t-a)}{|(a+r)^2-t^2|}]r\,dr\right\}$$

$$\leq \frac{c}{\sqrt{|t-a|(t+a}}\int_0^{t+a} (1+r)^{3/2} \ell n[2+\frac{ar\,\chi(t-a)}{|(a+r)^2-t^2|}]|g|_\psi\, dr \tag{2.10}$$

If $t \geq a+1$ then

$$\int_0^{t+a} |g|_\psi \ell n[2+\frac{ar\chi(t-a)}{t^2(a+r)^2|}](r+1)^{3/2}$$

$$\leq C\int_{[0,t-a-\frac{1}{2}]\cup[t-a+\frac{1}{2},t+a]} |g|_\psi \ell n[2+\frac{ar}{|t^2-(a+r)^2|}](r+1)^{3/2}dr$$

$$+\int_{t-a-\frac{1}{2}}^{t-a+\frac{1}{2}} |g|_\psi \ell n[2+\frac{ar}{t^2-(a+r)^2}](r+1)^{3/2}dr$$

$$\leq C[\int_0^\infty |g|_\psi (1+r)^{3/2} \ell n(2+r)dr$$

$$+ |g(x)(|x|+1)^{3/2}\ell n(2+|x|)|_{L^\infty(R^2)} \int_{t-a-\frac{1}{2}}^{t-a+\frac{1}{2}} \ell n[2+\frac{1}{|t-a-r|}]dr$$

$$\leq C\varepsilon \tag{2.11}$$

Similarly, we prove that if $t \leq a-1$ the integral in (2.10) is also less than $C\varepsilon$. Thus, we obtain for $|t-a| \geq 1$

$$|u(x,t)| \leq \frac{C\varepsilon}{\sqrt{(t+a)|t-a|}} \tag{2.12}$$

Combining this with the estimate

$$|u(x,t)| \leq \frac{C\varepsilon}{\sqrt{t+1}}$$

proved by Klainerman in [**10**], we obtain (2.2). In order to prove lemma 5, we need the following definition.

PROPOSITION 2.2. *Let u be a solution of*

$$u = \partial_t^2 u - \Delta u = F(x,t)$$
$$u(0,x) = u_t(0,x) = 0$$

Then

$$u(t,x) = \int_{\max(0,t-a)}^{t} \int_{a-t+s}^{a+t-s} rdr \int_{-\varphi}^{\varphi} \frac{F(r,\theta+\psi,s)dx}{\sqrt{(t-s)^2-a^2-r^2+2\arccos\psi}}$$
$$+ \chi(t-a)\int_0^{t-a} ds \int_{t-a-s}^{t+a-s} rdr \int_{-\varphi}^{\varphi} \frac{F(r,\theta-\psi,s)dx}{\sqrt{(t-s)^2-a^2-r^2+2\arccos\psi}}$$
$$+ \chi(t-a)\int_0^{t-a} ds \int_0^{t-a-s} rdr \int_{-\pi}^{\pi} \frac{F(r,\theta+\psi,s)dx}{\sqrt{(t-s)^2-a^2-r^2+2\arccos\psi}} \tag{2.13a}$$

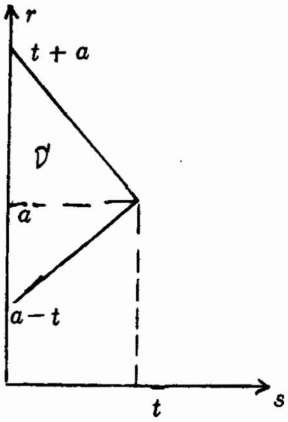
where a, θ are polar coordinates of x, and

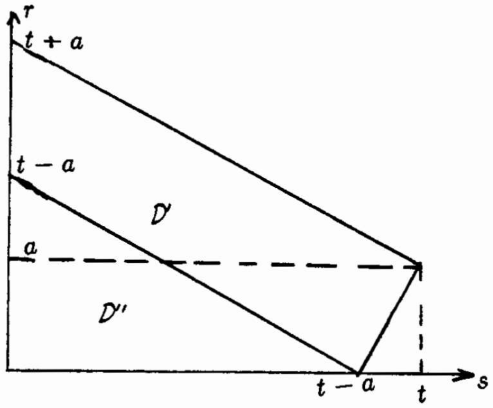
$$\varphi = \arccos \frac{a^2 + r^2 - (t-s)^2}{2ar}$$

or

$$u(t,x) = \int_{\mathcal{D}'} K(t-s,a,r,\theta,F(s))r\,dr\,ds + \chi(t-a)\int_{\mathcal{D}''} K(t-s,a,r,\theta,F(s))r\,dr\,ds \tag{2.13b}$$

where the domains $\mathcal{D}'$ and $\mathcal{D}''$ are shown in the pictures below.





Proof: Apply Duhamel's principal with the solution of (2.1) in form (2.8) and (2.9).

Proof of Lemma 5: By proposition 2.2

$$\begin{aligned} \partial u_i(t,x) = &\int_{\mathcal{D}'} K(t-s,a,r,\theta,\partial F(s))r\,dr\,ds \\ &+ \chi(t-a)\int_{\mathcal{D}'} K(t-s,a,r,\theta,\partial F(s))r\,dr\,ds \end{aligned} \tag{2.14}$$

where a, θ are polar coordinates of x.

We derive estimates (2.5) for the case $t \geq a$. If $t \leq a$, then the second term disappears and the derivation is the same as for the first term.

We split our region of integration $\mathcal{D}' \cup \mathcal{D}''$ into subregions according to the following scheme

Blue region $= \{(r, s) \in \mathcal{D}' : t - a \leq r + s \leq t - a + \min(\delta, 2a)\}$

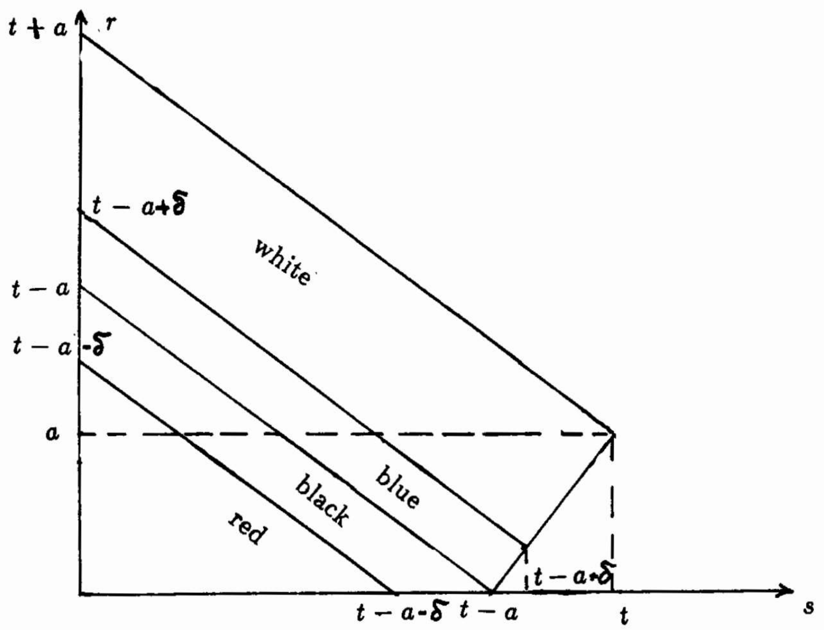
Black region $= \{(r, s) \in \mathcal{D} : \max(0, t - a - \delta) \leq r + s \leq t - a\}$

White region $= \mathcal{D}' \backslash$Blue region

Red region $= \mathcal{D}'' \backslash$Black region

δ is chosen sufficiently small, say 0.1

For example, if $t - a \geq 1$, $a \geq 1$. The break-up of the domain of integration looks like this:



If $t - a$ is small the red ration is swallowed by the black region and similarly, if a is small the white region is swallowed by the blue region.

Using that in the white and blue regions

$$K(t-s,a,r,\theta,\partial F(s)) = \int_{-\varphi}^{\varphi} \frac{\partial F(s,r,\theta+\psi)dx}{\sqrt{(t-s)^2-a^2-r^2+2\arccos\psi}}$$
$$= \sum_{i=0}^{1} \frac{1}{\sqrt{2ar}} \int_0^2 \frac{\partial F(s,r,\theta+(-1)^i\psi)d\tau}{\sqrt{\tau(1-\tau)(2+P\tau-\tau)}}$$

with $\tau = \dfrac{1-\cos\psi}{1-\cos\varphi}$, $\cos\varphi = \dfrac{a^2+r^2-(t-s)^2}{2ar} = P$ we can rewrite (2.14) as

$$\partial u_1 = \int_{\text{blue}\cup\text{black}} K(s,a,r,\theta,\partial F(s))r\,ds\,dr +$$
$$+ \sum_{i=0}^{1} \int_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)}} \int_{\text{white}} \sqrt{\frac{r}{2a(2+p\tau-\tau)}}\, \partial F(s,r,\theta+(-1)^i\psi)ds\,dr$$
$$+ \int_{-\pi}^{\pi} d\psi \int_{\text{red}} \frac{r}{\sqrt{(t-s)^2-a^2-r^2+2\arccos\psi}} \partial F(t-s,r,\theta+\psi)ds\,dr$$

If $\partial = \partial_1$ or ∂_2, we can write out $\partial_k = \alpha_k(x)\partial_r + \dfrac{\beta_k(x)}{r}\partial_\psi$ and integrate by parts the terms containing $\alpha_k\partial_r$ in $\int_{\text{white}}$ and $\int_{\text{red}}$ obtaining thus:

$$|\partial_1 u_1| + |\partial_2 u_1| \leq \int_{\text{black}} K(t-s,a,r,\theta,\partial F)|r\,ds\,dr$$

$$+ \int_{\text{blue}} K(t-s,a,r,\theta,\partial F) r\,ds\,dr$$

$$+ \int_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)}} \left\{ \int_{t-a+\delta}^{t+a} \frac{\sqrt{r}|F(t,r,\theta \pm \psi)|dr}{\sqrt{2a(2+P\tau-\tau)}} \right|_{s=0}$$

$$+ \int_0^t \frac{\sqrt{r}|F(s,r,\theta \pm \psi)|ds}{\sqrt{2a(2+P\tau-\tau)}} \Bigg|_{r=t+a-s}$$

$$+ \int_{t-a+\frac{\delta}{2}}^t \frac{\sqrt{r}|F(t,r,\theta \pm \psi)|ds}{\sqrt{2a(2+P\tau-\tau)}} \Bigg|_{r=s-t+a}$$

$$+ \int_0^{t-a+\delta/2} \frac{\sqrt{r}|F(s,r,\theta \pm \psi)|}{\sqrt{2a(2+P\tau-\tau)}} \Bigg|_{r=t-a-s+\delta}$$

$$+ \int_{\text{white}} ds\,dr \left(\left| \frac{\partial}{\partial s} \frac{\sqrt{r}}{\sqrt{2a(2+P\tau-\tau)}} \right| \right.$$

$$\left. + \left| \frac{\partial}{\partial r} \frac{\sqrt{r}}{\sqrt{2a(2+P\tau-\tau)}} \right| \right) |F(s,r,\theta,\pm\psi)| \Bigg\}$$

$$+ \int_{-\pi}^{\pi} dx \left\{ \int_0^{t-a-\delta} \frac{r|F(s,r,\theta+\psi)|ds}{\sqrt{(t^2-a^2-r^2+2\arccos\psi}} \right|_{s=0}$$

$$+ \int_0^{t-a-\delta} \frac{r|F(s,r,\theta \pm \psi)|ds}{\sqrt{(t-s)^2-a^2-r^2+2\arccos\psi}} \Bigg|_{r=t-a-\delta-s}$$

$$+ \int_{\text{red}} ds\,dr \left(\left| \frac{\partial}{\partial s} \frac{1}{\sqrt{(t-s)^2-a^2-r^2+2\arccos\psi}} \right| \right. \tag{2.15}$$

$$\left. + \left| \frac{\partial}{\partial r} \frac{1}{\sqrt{(t-s)^2-a^2-r^2+2\arccos\psi}} \right| \right) |F(s,r,\theta+\psi)| \Bigg\}$$

$$+ \int_{\text{red}\cup\text{white}} K\left(s, a, r, \theta, \frac{\beta_k}{r}\frac{\partial F}{\partial \psi}\right) ds dr$$

If $\partial = \partial_t$ integrating by parts again, we obtain that $|\partial_t u_1|$ satisfies (2.15) as well as $|\partial_1 u_1|$ and $|\partial_2 u_1|$. In order to continue, we need

LEMMA 6. *Let $w(r)$ be a positive function which is C^0 for all $r \geq 0$ and C^1 for all $r \geq 0$ except possibly a finite number of points and assume that for some constant A : $\left|\frac{\partial u}{\partial r}\right| \leq Aw(r)$. Then there exists a constant B depending only on w such that for all $C^3(R^2)$-functions*

$$|x|f^2(x)w^2(|x|) \leq B \sum_{a_1+a_2+a_3\leq 2} \|w\partial_1^{a_1}\partial_2^{a_2}\Omega^{a_3}f\|^2_{L^2(R^2)}.$$

We will prove the lemma later on in this section.

By Lemma 6:

$$|\sqrt{r}F(s,r,\psi)w(p,s,r)| \leq C|||F|||_{3,p,t}$$

Applying it for F, ∂F and $\frac{\partial F}{\partial \psi}$, we obtain:

$$|\partial u_1| \leq C|||F|||_{3,p,t} \left\{ \int_{\text{blue}} \frac{K(t-s,a,r)\sqrt{r}ds\ dr}{w(p,s,r)} \right.$$

$$+\int_{t-a+\delta}^{t+a}\frac{K(t,a,r)\sqrt{r}dr}{w(p,t,r)}\bigg|_{s=0}+\int_0^t\frac{K(t-s,a,r)\sqrt{r}ds}{w(p,s,r)}\bigg|_{r=t+a-s}$$

$$+\int_{t-a+\frac{\delta}{2}}^{t}\frac{K(t-s,a,r)\sqrt{r}ds}{w(p,s,r)}\bigg|_{r=s-t+a}+\int_0^{t-a+\delta/2}\frac{K(t-s,a,r)\sqrt{r}ds}{w(p,s,r)}\bigg|_{r=t-s+\delta-a}$$

$$+\int_{\text{white}}\left(\left|\frac{\partial K(t-s,a,r)}{\partial t}\right|+\left|\frac{\partial K(t-s,a,r)}{\partial r}\right|+\frac{K(t-s,a,r)}{r}\right)\frac{\sqrt{r}dsdr}{w(p,s,r)}$$

$$+\int_{\text{black}}\frac{K(t-s,a,r)\sqrt{r}dsdr}{w(p,s,r)}+\int_0^{t-a-\delta}\frac{K(t,a,r)\sqrt{r}dr}{w(p,s,r)}\bigg|_{s=0}$$

$$+\int_0^{t-a-\delta}\frac{K(t-s,a,r)\sqrt{r}ds}{w(p,s,r)}\ |_{r=t-a-\delta-s}$$

$$+\int_{\text{red}}\left|\frac{\partial K(t-s,a,r)}{\partial t}\right|+\left|\frac{\partial K(t-s,a,r)}{\partial r}\right|+\frac{K(t-s,a,r)}{r}\bigg)\frac{\sqrt{r}dsdr}{w(p,s,r)}\ \}\tag{2.16}$$

Using estimates (2.7) of proposition 2.1 and the following symbolic notation

$$(|ct-r|+1)^{\frac{\alpha}{2}}=\sqrt{\prod_{i=1}^{i_0}(|c_it-r|+1)^{\alpha_i}}$$

we can rewrite (2.16) as:

$$|\partial u|\le C|||F|||_{3,p,t}(I_1+\cdots+I_{12})\tag{2.17}$$

where

$$I_1 = \frac{1}{\sqrt{a}} \int_{\text{blue}} \frac{\ell n[2 + \frac{ar}{(a+r)^2-(t-s)^2}]dsdr}{(r+1)^{\frac{p-1}{2}}(|cs-r|+1)^{\frac{\alpha}{2}}}$$

$$I_2 = \frac{1}{\sqrt{a}} \int_{t-a+\delta}^{t+a} \frac{\ell n[2 + \frac{ar}{(a+r)^2-t^2}]dr}{(r+1)^{p-1}}$$

$$I_3 = \frac{1}{\sqrt{a}} \int_0^t \frac{\ell n[2 + \frac{ar}{(a+r)^2-(t-s)^2}]ds}{(r+1)^{\frac{p-1}{2}}(|cs-r|+1)^{\frac{\alpha}{2}}} \Big|_{r=t+a-s}$$

$$I_4 = \frac{1}{\sqrt{a}} \int_{t-a+\frac{\delta}{2}}^{t} \frac{\ell n[2 + \frac{ar}{(a+r)^2-(t-s)^2}]ds}{(r+1)^{\frac{p-1}{2}}(|cs-r|+1)^{\frac{\alpha}{2}}} \Big|_{r=s-t+a}$$

$$I_5 = \frac{1}{\sqrt{a}} \int_0^{t-a+\delta/2} \frac{\ell n[2 + \frac{ar}{(a+r)^2-(t-s)^2}]}{(r+1)^{\frac{p-1}{2}}(|cs-r|+1)^{\frac{\alpha}{2}}} \Big|_{r=t-a-s+\delta}$$

$$I_6 = \frac{1}{\sqrt{a}} \int_{\text{white}} \frac{dsdr}{(r+1)^{\frac{p-1}{2}}(|cs-r|+1)^{\frac{\alpha}{2}}(a+r-s-t)}$$

$$I_7 = \frac{1}{\sqrt{a}} \int_{\text{white}} \frac{\ell n[2 + \frac{ar}{(a+r)^2-(t-s)^2}]ds\,dr}{(r+1)^{\frac{p+1}{2}}(|cs-r|+1)^{\frac{\alpha}{2}}}$$

$$I_8 = \int_{\text{black}} \frac{\ell n[2 + \frac{ar}{(t-s)^2-(a+r)^2}]ds\,dr}{(r+1)^{\frac{p-1}{2}}(|cs-r|+1)^{\frac{\alpha}{2}}\sqrt{(t-s)^2-a^2-r^2}}$$

$$I_9 = \int_0^{t-a-\delta} \frac{\ell n[2 + \frac{ar}{(t-s)^2-(a+r)^2}]dr}{(r+1)^{\frac{p-3}{2}}\sqrt{t^2-a^2-r^2}}$$

$$I_{10} = \int_0^{t-a-\gamma} \frac{\ell n[2 + \frac{ar}{(t-s)^2-(a+r)^2}]ds}{(r+1)^{\frac{p-1}{2}}(|cs-r|+1)^{\frac{\alpha}{2}}\sqrt{(t-s)^2-a^2-r^2}} \Big|_{r=t-a-\delta-s}$$

$$I_{11} = \int_{\text{red}} \frac{\ell n[2 + \frac{ar}{(t-s)^2-(a+r)^2}]ds\,dr}{(r+1)^{\frac{p-1}{2}}(|cs-r|+1)^{\frac{\alpha}{2}}(t-s-a-r)\sqrt{(t-s)^2-r^2}}$$

$$I_{12} = \int_{\text{red}} \frac{\ell n[2 + \frac{ar}{(t-s)^2-(a+r)^2}]ds\,dr}{(r+1)^{\frac{p}{2}}(|cs-r|+1)^{\frac{\alpha}{2}}\sqrt{(t-s)^2-(a+r)^2}}$$

All the integrals satisfy the following inequality

$$I_k \le \begin{cases} C[ln(2+t)]\sqrt{\frac{1+t}{(1+a)(|t-a|+1)}}, & \text{if } p=2 \\ \dfrac{C}{(t+a+1)^{\frac{1}{2}-\gamma}(a+1)^{\gamma}(|t-a|+1)^{\frac{1}{2}}}, & \text{if } p \ge 3 \end{cases}$$

where $0 < \gamma \le \dfrac{1}{2}$ and C depends on γ for $p \ge 3$.

Substituting these estimates into (2.17), we obtain estimates (2.5) of lemma 5.

Proof of proposition 2.1: Proof of A and B. Let $b(r) = \dfrac{t^2-a^2-r^2}{2ar}$. $b(r)$ is a monotonically decreasing function, $b(0) = \infty$, $b(t-a) = 1$. Thus,

$$\begin{aligned}
\int_0^{\pi} \frac{d\psi}{(t^2-a^2-r^2+2\arccos\psi)^{\alpha}} &= \frac{1}{(t^2-a^2-r^2)^{\alpha}} \int_0^{\pi} \frac{d\psi}{(1+\frac{1}{b}\cos\psi)} \\
&\le \frac{1}{(t^2-a^2-r^2)^{\alpha}} \left\{ \int_0^{2\pi/4} \frac{d\psi}{(1+\frac{\cos\psi}{b})^{\alpha}} \right. \\
&\quad \left. + \int_{3\pi/4}^{\pi} \frac{d\psi}{((1-\cos\psi)+(1-\frac{1}{b})\cos\psi)^{\alpha}} \right\} \\
&\le \frac{C}{(t^2-a^2-r^2)^{\alpha}} \left\{ 1 + \int_0^{\pi/4} \frac{d\psi}{((1-\cos\psi)+(1-\frac{1}{b}))^{\alpha}} \right\}
\end{aligned}$$

$$\leq \frac{C}{(t^2-a^2-r^2)^\alpha}\left\{1+\int_0^{\pi/4}\frac{d\psi}{(\psi^4+\frac{b-1}{b})^\alpha}\right\} \tag{2.18}$$

$$K(t,a,r) = 2\int_0^\pi \frac{d\psi}{\sqrt{t^2-a^2-r^2+2\arccos\psi}}$$

and thus, by (2.18)

$$K(t,a,r) \leq \frac{c}{\sqrt{t^2-a^2-r^2}}\left\{1+\int_0^{\pi/4}\frac{d\psi}{\sqrt{x^2+1-\frac{1}{b}}}\right\}$$

$$\leq \frac{Cln[2+\frac{ar}{t^2-(a+r)^2}}{\sqrt{t^2-a^2-r^2}}$$

In order to prove part B, we observe that

$$\left|\frac{\partial K}{\partial r}\right| + \left|\frac{\partial K}{\partial t}\right| \leq Ct\int_0^\pi \frac{d\psi}{(t^2-a^2-r^2+2\arccos\psi)^{3/2}}$$

and by (2.18) this is less than

$$\frac{Ct}{(t^2-a^2-r^2)^{3/2}}\left\{1+\int_0^{\pi/4}\frac{d\psi}{(\psi^2+\frac{b-1}{b})^{3/2}}\right\} \leq \frac{Ct}{\sqrt{t^2-a^2-r^2}(t^2-(a+r)^2)}$$

Proof of A^*, B^* and C^*: We have

$$K(t,a,r) = \int_{-\varphi}^{\varphi}\frac{dx}{\sqrt{t^2-a^2-r^2+2\arccos\psi}} = \sqrt{\frac{2}{ar}}\int_0^\varphi \frac{dx}{\sqrt{\cos\psi-\cos\varphi}}$$

where $\cos\varphi = \dfrac{a^2+r^2-t^2}{2ar} = P.$

Changing variable of integration for $\tau = \dfrac{1-\cos\psi}{1-\cos\varphi}$, we obtain

$$K(t,a,r) = \sqrt{\frac{2}{ar}}\int_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)[2+(p-1)\tau]}}$$

and the integral is differentiable for any $P > -1$, which implies part A^*.

In order to prove part B, it's sufficient to prove it for $-1 \le P \le -0.9$. In this case,

$$\int_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)[2+(P-1)\tau]}}$$
$$= \int_0^{\frac{1}{1-P}} \frac{d\tau}{\sqrt{\tau(1-\tau)[2+(P-1)\tau]}} + \int_{\frac{1}{1-P}}^1 \frac{d\tau}{\sqrt{\tau(1-\tau)[2+(P-1)\tau]}}$$
$$\le C\left\{1 + \int_{\frac{1}{1-P}}^1 \frac{d\tau}{\sqrt{(\tau-1)(\tau-\frac{2}{1-P})}}\right\} \le C\{1 - \ell n(1+P)\}$$
$$\le C ln[2 + \frac{ar}{(a+r)^2 - t^2}]$$

Combining this with (2.7c), we obtain the desired estimate. Straightforward computations show that

$$\left|\frac{\partial K(t,a,r)}{\partial t}\right| + \left|\frac{\partial K(t,a,r)}{\partial r}\right| \le 2\left|\frac{K}{P}\right| \frac{t+a}{ar}$$

with

$$\left|\frac{\partial K}{\partial P}\right| = \frac{1}{\sqrt{2ar}} \int_0^1 \sqrt{\frac{\tau}{1-\tau}} \frac{d\tau}{[2+(P-1)\tau]^{3/2}}$$
$$\le \frac{C}{\sqrt{ar}}\left\{1 + \int_0^{\frac{1}{1-P}} \sqrt{\frac{1}{1-\tau}} \frac{d\tau}{[2+(P-1)\tau]^{3/2}}\right\}$$
$$\le \frac{C}{\sqrt{2ra}}\left\{1 + \frac{1-P}{1+P}\right\} \le \frac{C}{\sqrt{ar}(1+P)}$$

Therefore:

$$\left|\frac{\partial K}{\partial t}\right| + \left|\frac{\partial K}{\partial r}\right| \le \frac{C(t+a)}{ar}\left|\frac{\partial K}{\partial P}\right| \le \frac{C(t+a)}{(ar)^{3/2}(1+P)} \le \frac{C(t+a)}{\sqrt{ar}[(a+r)^2 - t^2]}$$

what implies (2.7d).

Proof of Lemma 6: Let r, ψ be polar coordinates of x. We have

$$r^{n-1}f^2(r,\psi)w^2(r) = 2r^{n-1}\left(\int_r f\frac{\partial f}{\partial r}w^2 dr + \int_r^\infty f^2 w\frac{\partial w}{\partial r}dr\right)$$

and thus

$$|r^{n-1}f^2w^2| \leq C\left(\int_0^\infty r^{n-1}f^2w^2dr + \int_0^\infty |\partial f|^2 w^2 dr\right).$$

Using that

$$|f(r,\psi)|^2 + |\partial f(r,\psi)|^2 \leq C\int_0^2 (|f|^2 + |\Omega f|^2 + |\partial f|^2 + |\Omega\partial f|^2)d\psi$$

we obtain Lemma 6.

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