Global Existence for Three-Dimensional Incompressible Isotropic Elastodynamics via the Incompressible Limit

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Abstract

The existence of global-in-time classical solutions to the Cauchy problem for incompressible nonlinear isotropic elastodynamics for small initial displacements is proved. Solutions are constructed via approximation by slightly compressible materials. The energy for the approximate solutions remains uniformly bounded on a time scale that goes to infinity as the material approaches incompressibility. A necessary component to the long-time existence of the approximating solution is a null or linear degeneracy condition, inherent in the isotropic case, which limits the quadratic interaction of the shear waves. The proof combines energy and decay estimates based on commuting vector fields and a compactness argument.

1 Introduction

The motion of isotropic elastic materials occupying all of $\mathbb{R}^3$ features the nonlinear interaction of fast pressure waves and slow shear waves, concentrated along their respective characteristic cones. Under appropriate constitutive assumptions, sending the speed of the pressure waves to infinity penalizes volume changes and drives the motion toward incompressibility. This article presents two results in support of this picture.

First, for sufficiently small initial displacements, classical local solutions of the equations of motion are shown to exist with uniform stability estimates yielding a lifespan proportional to the speed of the pressure waves, substantially improving upon [15]. This result depends on the careful assessment of nonlinear wave interactions through the inherent null structure of shear waves in isotropic materials, the small amplitude of pressure waves caused by their rapid dispersion in the incompressible limit, and the separation of the individual wave families.

Second, the uniform stability of the local existence family allows for convergence to a global solution of the limiting incompressible equations by means of compactness arguments. The strength of this convergence improves with the degree of incompressibility satisfied by the initial conditions.

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Instead of the classical second-order Lagrangian formulation, the problem will be analyzed within the framework of first-order symmetric hyperbolic systems because, with Eulerian coordinates and the proper choice of dependent variables, the singular terms are linear. The transformed system can be viewed as a natural extension of the compressible Euler equations of fluid dynamics in which the inverse deformation gradient is now coupled with the density and velocity. It also shares common features with various relativistic theories of elasticity [3, 7, 21] and viscoelastic theories [13, 14]. With the addition of new variables, certain natural constraints need to be taken into account. In particular, as noticed by John [6], a so-called null Lagrangian must be introduced to restore the positivity of the system, and moreover, nearly all of our estimates hold only for the constrained system.

In his pioneering study of the incompressible limit for elasticity, Schochet [15] already discovered the advantages of this basic approach (although his choice of dependent variables differs slightly from ours). With a first-order formulation, he was able to apply the energy methods of Klainerman and Majda [10, 11] originating in the study of the incompressible limit for the equations of fluid dynamics. However, energy estimates alone can only give uniform stability estimates on a bounded time interval and convergence in the incompressible limit to a local solution of the limiting equations.

Long-time stability of solutions to the elastodynamics equations depends on strong dispersive estimates. For the wave equation, the generalized energy method, based on Lorentz invariance and global Sobolev inequalities, provides an elegant and efficient means of combining energy and decay estimates; see [8, 9], for example. The equations of elasticity, being merely Galilean and scaling invariant, require an additional intermediate series of weighted L^2 estimates to compensate for the smaller symmetry group, an approach that has been developed in [12, 17, 19, 20]. In particular, it was shown in [17, 19] that the initial value problem for the Lagrangian equations of motion for compressible elastodynamics have global small solutions for certain isotropic materials satisfying an additional null condition for the pressure waves; see also [1, 2]. In this respect, the PDEs of elastodynamics are better behaved than the equations of fluids for which shear waves do not disperse, on the linear level. Nevertheless, the use of decay estimates also allows for some improvements in the study of the incompressible limit for fluids [16, 18, 22].

The present work relies on the methods of [19], translated into the first-order context, but in addition it requires new weighted estimates that are uniform in the speed of the pressure waves. Another way to summarize the difficulty is that time derivatives do not automatically have uniform estimates and must be treated separately. Such bounds are essential in order to pass to the incompressible limit. They can be achieved either by preparing the initial data appropriately or, in weaker form, by using scaling invariance.

Global solutions to the approximating compressible equations are not constructed here. This is the price of Eulerian coordinates. Convective derivative terms are inconsistent with the null condition.

The global existence of small solutions to the three-dimensional incompressible and isotropic elasticity equations was announced by Ebin in [5]. His direct argument relies on the Lorentz invariance of the wave equation, the linearized operator in the incompressible case; however, in our view insufficient attention is paid to the incompressibility constraint that is incompatible with the Lorentz rotations. The special case of incompressible neo-Hookean materials was studied in [4].

Complete statements of the main results can be found in Section 2.6 after a reformulation of the problem and the introduction of required notation. The steps of the proof of the long-time stability estimates are broken up into a series of propositions in the following sections. The final two sections then complete the proofs of the main theorems.

2 Preliminaries

2.1 Equations of Motion

The motion of an elastic body is classically described by a time-dependent family of orientation preserving diffeomorphisms $x(t,\cdot)$, $0 \le t < T$. Material points X in the reference configuration are deformed to the spatial position $x(t,X)$ at time t.

The equations of motion for homogeneous, hyperelastic, isotropic materials can be derived from the formal variational problem

$$\delta \iint \left[\frac{1}{2}|D_t x|^2 - W(Dx)\right] dX\, dt = 0\,,$$

in which the strain energy function $W(F) \in C^\infty(\mathrm{GL}^3_+, \mathbb{R})$ depends on F through the principal invariants of the strain matrix $F^{\mathsf{T}}F$. We use the notation GL^3_+ for the group of invertible 3×3 matrices over $\mathbb{R}$ with positive determinant, $\mathbb{M}^3$ for the set of all 3×3 matrices over $\mathbb{R}$, and (D_t, D) for derivatives with respect to the material coordinates (t, X). The density in the undeformed reference configuration has been set equal to 1. The equations of motion take the form

$$D_t^2 x^i - D_\ell\big[S_i^\ell(Dx)\big] = 0\,, \tag{2.1}$$

where

$$S_i^\ell(F) = \frac{\partial W}{\partial F_\ell^i}(F)$$

is the (first) Piola-Kirchhoff stress tensor. Summation over repeated indices will always be understood.

2.2 Reformulation as a First-Order System

Our analysis relies on the reformulation of the second-order equations of motion in material coordinates to a first-order system in spatial coordinates. Spatial coordinates allow for a compact and tractable expression for the singular terms. It will be convenient to work with the family of inverse transformations $X(t, x)$ whose gradient $H(t, x)$ satisfies a particularly simple constraint (see (2.6b) below). Derivatives with respect to the spatial coordinates (t, x) will be denoted by $\partial = (\partial_t, \nabla)$.

The following series of simple results establishes the equivalence.

LEMMA 2.1 *Given a family of deformations $x(t, X)$ with inverse $X(t, x)$, define the velocity, inverse deformation gradient, and density as follows*:

$$v(t, x) = D_t x(t, X(t, x)), \tag{2.2a}$$

$$H(t, x) = \nabla X(t, x), \tag{2.2b}$$

$$\rho(t, x) = \det H(t, x). \tag{2.2c}$$

Then for $(t, x) \in [0, T) \times \mathbb{R}^3$,

$$\partial_t H + \nabla(Hv) = \partial_t H + v \cdot \nabla H + H \nabla v = 0, \tag{2.3a}$$

$$\partial_t \rho + v \cdot \nabla \rho + \rho \nabla \cdot v = 0. \tag{2.3b}$$

PROOF: Since $X(t, x(t, X)) = X$, we see that $X(t, x)$ is constant along particle trajectories. This means that

$$\partial_t X + v \cdot \nabla X = 0.$$

Taking the gradient with respect to x yields (2.3a). Equation (2.3b) follows from (2.3a) because it is simply the evolution equation for the Jacobian $\det H(t, x(t, X))$. □

We remark that as soon as H satisfies (2.2b), it follows that

$$\partial_j H^i_k(t, x) = \partial_k H^i_j(t, x).$$

LEMMA 2.2 *Suppose that $(H(t, x), v(t, x), \rho(t, x))$ is bounded and continuously differentiable from $[0, T) \times \mathbb{R}^3$ to $\mathrm{GL}^3_+ \times \mathbb{R}^3 \times \mathbb{R}$. Suppose that there exists an orientation-preserving diffeomorphism $x(X)$ of $\mathbb{R}^3$ with inverse $X(x)$ such that $H(0, x) = \nabla X(x)$ and $\rho(0, x) = \det X(x)$. Finally, assume that* (2.3a) *and* (2.3b) *are satisfied on $[0, T) \times \mathbb{R}^3$. Then there exists a one-parameter family of diffeomorphisms $x(t, X)$ with $x(0, X) = x(X)$ such that* (2.2a), (2.2b), *and* (2.2c) *hold.*

PROOF: Given the bounded vector field $v(t, x)$ on $[0, T) \times \mathbb{R}^3$, construct the flow

$$D_t x(t, X) = v(t, x(t, X)), \quad x(0, X) = x(X).$$

For $0 \le t < T$, $x(t, X)$ defines a one-parameter family of diffeomorphisms on $\mathbb{R}^3$ since $x(X)$ is a diffeomorphism. It now follows by Lemma 2.1 that $\bar{H}(t, x) \equiv$

$\nabla X(t,x)$ and $\bar\rho(t,x) \equiv \det \bar H(t,x)$ satisfy (2.3a) and (2.3b) on $[0,T)\times\mathbb{R}^3$. Finally, we conclude that $H = \bar H$ and $\rho = \bar\rho$, by uniqueness, since this holds at $t = 0$. □

We have already noted that, in material coordinates, the equations of motion for the deformation $x(t,X)$ are given by (2.1). We will now show that in spatial coordinates, the corresponding equations of motion are given by the first-order system

$$D_t H + H\nabla v = 0\,, \tag{2.4a}$$

$$D_t v + D\cdot \hat S(H) = 0\,,\quad \hat S(H) = -S(H^{-1})\,, \tag{2.4b}$$

$$D_t \rho + \rho\nabla\cdot v = 0\,. \tag{2.4c}$$

Here we are using the abbreviations

$$D_t = \partial_t + v\cdot\nabla \quad\text{and}\quad D_\ell = (H^{-1})^k_\ell \partial_k\,, \tag{2.5}$$

consistent with chain rule.

PROPOSITION 2.3 *Suppose that $x(X)$ is an orientation-preserving diffeomorphism of $\mathbb{R}^3$ with inverse $X(x)$.*

Let $x(t,X)$ be a C^2 solution of (2.1) *on $[0,T)\times\mathbb{R}^3$ with $x(0,X) = x(X)$. Then (H,v,ρ) as defined in Lemma* 2.1 *solves the first-order system* (2.4a)–(2.4c)*, together with the constraints*

$$\det H = \rho\,, \tag{2.6a}$$

$$\partial_\ell H^i_m = \partial_m H^i_\ell\,. \tag{2.6b}$$

Conversely, suppose that (H,v,ρ) is a bounded C^1 solution of (2.4a)–(2.4c) *on $[0,T)\times\mathbb{R}^3$ such that $H(0,x) = \nabla X(x)$ and $\rho(0,x) = \det\nabla X(x)$. Then $x(t,X)$ as given by Lemma* 2.2 *is a one-parameter family of diffeomorphisms that solves* (2.1)*. Consequently, (H,v,ρ) satisfies the constraints* (2.6a) *and* (2.6b)*.*

PROOF: If $x(t,X)$ is a one-parameter family of orientation-preserving diffeomorphisms that solves (2.1), then define (H,v,ρ) by means of (2.2a), (2.2b), and (2.2c). By Lemma 2.1, equations (2.4a) and (2.4c) are satisfied, and equation (2.4b) follows from (2.1). The constraints hold by the definitions of H and ρ.

On the other hand, suppose that (H,v,ρ) solves (2.4a)–(2.4c), and that the initial data $H(0,x)$ and $\rho(0,x)$ satisfy the assumptions. Then by Lemma 2.2, there is a one-parameter family of diffeomorphisms $x(t,X)$ with $x(0,X) = x(X)$ that satisfies (2.2a), (2.2b), and (2.2c). But then (2.4b) implies that $x(t,X)$ satisfies (2.1), since $H^{-1}(t,x(t,X)) = Dx(t,X)$. □

2.3 Constitutive Assumptions

We will consider isotropic strain energy functions of the form

$$W^\lambda(F) = W(F) + \lambda^2 h(\rho)\,, \quad \rho = \det F^{-1},\ \lambda \in \mathbb{R}^+\,,$$

where W is independent of λ and depends on F through the principal invariants of the strain matrix $F^\mathsf{T} F$. The last term should be regarded as a penalization term that drives the motion toward incompressibility in the limit as the parameter λ becomes large.

In this case, the Piola-Kirchhoff stress has the form

$$S^\lambda(F) = \frac{\partial}{\partial F}[W(F) + \lambda^2 h(\rho)] = S(F) - \lambda^2 \rho h'(\rho) F^{-\mathsf{T}}\,.$$

We assume that $S(I) = 0$ and $h'(1) = 0$. This implies that the reference configuration is a stress-free state: $S^\lambda(I) = 0$.

Since $D \cdot (\det F) F^{-\mathsf{T}} = 0$, for $F = Dx$, the penalization term adds the expression

$$D \cdot \lambda^2 \rho h'(\rho) F^{-\mathsf{T}} = \lambda^2 \rho^{-1} \nabla[\rho^2 h'(\rho)] = \lambda^2 [\rho h(\rho)]'' \nabla \rho$$

to the equations. Because we are ultimately interested in the incompressible limit, we shall choose the function h so as to make this term as simple as possible while still being physically meaningful. Therefore, we set

$$h(\rho) = \frac{(\rho - 1)^2 (\rho + 2)}{6\rho} \tag{2.7}$$

so that $[\rho h(\rho)]'' = \rho$. Notice that $h(1) = h'(1) = 0$, h is convex and nonnegative, and $h(\rho) \to \infty$ as $\rho \to 0$ and ∞, so this choice is a physically reasonable correction to the strain energy. With this choice, we now have

$$D \cdot S^\lambda(Dx) = D \cdot S(Dx) - \lambda^2 \rho \nabla \rho\,. \tag{2.8}$$

Next, define the elasticity tensor

$$A_{ij}^{\ell m}(F) = \frac{\partial S_i^\ell}{\partial F_m^j}(F) = \frac{\partial^2 W}{\partial F_\ell^i \partial F_m^j}(F)\,. \tag{2.9a}$$

We impose the usual Legendre-Hadamard ellipticity condition upon the linearized elasticity tensor which, in the isotropic case, takes the form

$$A_{ij}^{\ell m}(I) = (\alpha^2 - 2\beta^2)\delta_i^\ell \delta_j^m + \beta^2 \big(\delta^{\ell m}\delta_{ij} + \delta_j^\ell \delta_i^m\big) \quad \text{with } \alpha > \beta > 0\,. \tag{2.9b}$$

The parameters α and β depend only on W, and they represent the speeds of propagation of pressure and shear waves, respectively (in the case where $\lambda = 0$). With the additional penalization term, the propagation speed of the pressure waves becomes $\bar{\lambda} = (\alpha^2 + \lambda^2)^{1/2}$, as will become clearer in the next section. Note that the hydrodynamical case $W \equiv 0$ is ruled out by the condition (2.9b).

We now proceed to express our system in terms of H, the inverse of the deformation gradient F, and to examine the structure of the nonlinear terms. In reformulating the problem as a first-order system, the associated energy density is no longer positive definite. This is repaired by the addition of a so-called null Lagrangian. This entails no real change to the equations as long as we only consider solutions that correspond to the original second-order problem and therefore satisfy the constraint (2.6b).

LEMMA 2.4 *For $H \in \mathrm{GL}^3_+$, define*

$$\hat{S}(H) = -S(F)\big|_{F=H^{-1}},$$

$$\hat{A}^{\ell m}_{ij}(H) = A^{LM}_{IJ}(F)F^I_i F^J_j F^\ell_L F^m_M\big|_{F=H^{-1}} + \beta^2\big(\delta^\ell_i\delta^m_j - \delta^\ell_j\delta^m_i\big). \tag{2.10}$$

For all $H \in C^1(\mathbb{R}^3, \mathrm{GL}^3_+)$ satisfying (2.6b)*, we have*

$$D_\ell \hat{S}^\ell_i(H) = \hat{A}^{\ell m}_{pj}(H)H^p_i\partial_\ell H^j_m, \tag{2.11a}$$

using the notation in (2.5)*. The coefficients have the properties*

$$\hat{A}^{\ell m}_{ij}(H) = \hat{A}^{m\ell}_{ji}(H), \tag{2.11b}$$

$$\hat{A}^{\ell m}_{ij}(I) = (\alpha^2 - \beta^2)\delta^\ell_i\delta^m_j + \beta^2\delta^{\ell m}\delta_{ij}, \tag{2.11c}$$

and for all $H \in \mathrm{GL}^3_+$ with $|H - I| < \delta$ sufficiently small and $\dot{H} \in \mathbb{M}^3$, we have

$$\hat{A}^{\ell m}_{ij}(H)\dot{H}^i_\ell\dot{H}^j_m \geq \frac{\beta^2}{2}|\dot{H}|^2. \tag{2.11d}$$

PROOF: Property (2.11b) is clear from the definitions (2.9a) and (2.10), while (2.11c) follows from (2.10) and (2.9b). The lower bound (2.11d) follows by Taylor expansion and (2.11c).

We differentiate $FH = I$ to obtain

$$D_L F^J_M = -F^J_j F^m_M D_L H^j_m = -F^J_j F^\ell_L F^m_M \partial_\ell H^j_m.$$

According to our definitions, with $F = H^{-1}$, we have

$$\begin{aligned}
D \cdot \hat{S}(H) &= D_L \hat{S}^L_i(H) \\
&= -D_L S^L_i(F) \\
&= -A^{LM}_{iJ}(F) D_L F^J_M \\
&= A^{LM}_{iJ}(F) F^J_j F^\ell_L F^m_M \partial_\ell H^j_m \\
&= A^{LM}_{IJ}(F) \delta^I_i F^J_j F^\ell_L F^m_M \partial_\ell H^j_m \\
&= A^{LM}_{IJ}(F) F^I_p F^J_j F^\ell_L F^m_M H^p_i \partial_\ell H^j_m.
\end{aligned}$$

To conclude the proof of (2.11a), notice that under the constraint (2.6b), we have

$$\big(\delta^\ell_p\delta^m_j - \delta^\ell_j\delta^m_p\big)H^p_i\partial_\ell H^j_m = 0.$$

□

As a consequence of (2.8) and (2.11a), the momentum equation (2.4b) can be updated as

$$\partial_t v^i + v \cdot \nabla v^i + \hat{A}^{\ell m}_{pj}(H) H_i^{\,p} \partial_\ell H^j_m + \lambda^2 \rho \partial_i \rho = 0 \,. \tag{2.12}$$

2.4 Vector Fields

Before defining the λ-dependent energy norm associated with the first-order system, we must first introduce the vector fields on which the norm will depend. Notice that the vector fields are defined using Eulerian derivatives, instead of the Lagrangian derivatives as in the second-order case [19]. The scaling operator is

$$S = t\partial_t + r\partial_r \quad \text{with } r = |x|,\ \partial_r = \frac{x}{r} \cdot \nabla \,, \tag{2.13}$$

and the angular momentum operator is defined by

$$\Omega = x \wedge \nabla \,. \tag{2.14}$$

Since the angular momentum operators defined act on scalars, we need to modify them to act on maps $U = (H, v, \rho)$ valued in $\mathbb{M}^3 \times \mathbb{R}^3 \times \mathbb{R}$. First define

$$V^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},\ V^{(2)} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ V^{(3)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.15}$$

Given $U = (H, v, \rho) : \mathbb{R}^3 \to \mathbb{M}^3 \times \mathbb{R}^3 \times \mathbb{R}$, define

$$\tilde{\Omega} U = (\Omega H + [V, H], \Omega v + V v, \Omega \rho) \,. \tag{2.16}$$

We will occasionally write $\tilde{\Omega} H = \Omega H + [V, H]$ and $\tilde{\Omega} v = \Omega v + V v$ for the indicated components of $\tilde{\Omega} U$.

A fact we will often use is that we can decompose the gradient into its radial and angular components,

$$\nabla = \omega \partial_r - \frac{1}{r}(\omega \wedge \Omega) \quad \text{where } \omega = \frac{x}{r} \,. \tag{2.17}$$

Our vector fields will be written succinctly as Γ. We let

$$\Gamma = (\Gamma_1, \dots, \Gamma_7) = (\nabla, \tilde{\Omega}, S) \,.$$

Hence by ΓU we mean any one of $\Gamma_i U$. By Γ^a, $a = (a_1, \dots, a_\kappa)$, we denote an ordered product of $\kappa = |a|$ vector fields $\Gamma_{a_1}, \dots, \Gamma_{a_\kappa}$. We note that the commutator of any two Γ's is again a Γ. Notice that the vector fields Γ are time homogeneous. The time derivatives $\partial_t U$ will be handled separately.

In order to characterize the initial data, we introduce the time-independent analogue of Γ. The only difference will be in the scaling operator. Set

$$\Lambda = (\Lambda_1, \dots, \Lambda_7) = (\nabla, \tilde{\Omega}, x \cdot \nabla) \,.$$

Then the commutator of any two Λ's is again a Λ.

2.5 Spaces and Norms

In the following, $\|\cdot\|$ and $\|\cdot\|_\infty$ will denote the norms in $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$, respectively.

Define

$$H^\kappa_\Lambda = \{U = (H, v, \rho) : \mathbb{R}^3 \to \mathbb{M}^3 \times \mathbb{R}^3 \times \mathbb{R} : \Lambda^a U \in L^2(\mathbb{R}^3),\ |a| \le \kappa\}.$$

Solutions will be constructed in the space

$$H^\kappa_\Gamma(T) \equiv \Big\{U = (H, v, \rho) : [0,T) \times \mathbb{R}^3 \to \mathrm{GL}^3_+ \times \mathbb{R}^3 \times \mathbb{R} \mid \dot U = (\dot H, \dot v, \dot\rho) \equiv (H - I, v, \lambda(\rho - 1)) \in \bigcap_{j=0}^{\kappa} C^j\big([0,T), H^{\kappa-j}_\Lambda\big)\Big\}.$$

We now define the energy norm associated with the first-order system. Given $U = (H, v, \rho) \in \mathrm{GL}^3_+ \times \mathbb{R}^3 \times \mathbb{R}$ and $\dot U = (\dot H, \dot v, \dot\rho) \in \mathbb{M}^3 \times \mathbb{R}^3 \times \mathbb{R}$, define

$$e_U(\dot U) = \frac{1}{2}\big[\hat A^{\ell m}_{ij}(H)\dot H^i_\ell \dot H^j_m + |\dot v|^2 + \dot\rho^2\big]. \tag{2.18}$$

Given $U \in H^\kappa_\Gamma(T)$, define

$$E_\kappa[U(t)] = \sum_{|a|\le\kappa} \int e_{U(t)}(\Gamma^a \dot U(t))dx\,.$$

By (2.11d), for $|\dot U(t)| < \delta$,

$$E^{1/2}_\kappa[U(t)] \sim \sum_{|a|\le\kappa} \|\Gamma^a \dot U(t)\|\,. \tag{2.19}$$

We caution the reader that $\dot U$ denotes a perturbation from the background state and not a derivative. We also point out that the parameter λ is *hidden* in the definition of $E_\kappa[U(t)]$ through its dependence on $\dot\rho = \lambda(\rho - 1)$.

2.6 Main Results

Taking into account the constitutive assumptions, the revised equations of motion (2.4a), (2.4c), and (2.12) are

$$\partial_t H^i_\ell + v\cdot\nabla H^i_\ell + H^i_p \partial_\ell v^p = 0\,, \tag{2.20a}$$

$$\partial_t v_i + v\cdot\nabla v_i + \hat A^{\ell m}_{pj}(H) H^p_i \partial_\ell H^j_m + \lambda^2 \rho\partial_i\rho = 0\,, \tag{2.20b}$$

$$\partial_t\rho + v\cdot\nabla\rho + \rho\nabla\cdot v = 0\,, \tag{2.20c}$$

together with the constraints

$$\det H = \rho\,, \tag{2.20d}$$

$$\partial_\ell H^i_m = \partial_m H^i_\ell\,. \tag{2.20e}$$

We emphasize that solutions will depend on the value of the parameter λ; however, with the exception of the statements of the main theorems, this dependence will not be displayed for reasons of notational convenience.

In what follows we use the notation $\langle f \rangle = (1 + |f|^2)^{1/2}$.

THEOREM 2.5 *Assume that the isotropic strain energy function W has the form*

$$W^\lambda(F) = W(F) + \lambda^2 h(\rho), \quad \lambda \in \mathbb{R}^+,$$

where W is independent of λ and satisfies condition (2.9b) *and h is given by* (2.7).

Let $X_0^\lambda(x)$ be an orientation-preserving diffeomorphism on $\mathbb{R}^3$, and let $v_0^\lambda(x)$ be a vector field on $\mathbb{R}^3$. Define

$$U_0^\lambda = \left(H_0^\lambda, v_0^\lambda, \rho_0^\lambda\right) = \left(\nabla X_0^\lambda, v_0^\lambda, \det \nabla X_0^\lambda\right),$$
$$\dot{U}_0^\lambda = \left(\dot{H}_0^\lambda, \dot{v}_0^\lambda, \dot{\rho}_0^\lambda\right) = \left(H_0^\lambda - I, v_0^\lambda, \lambda(\rho_0^\lambda - 1)\right).$$

Suppose that $\dot{U}_0^\lambda \in H_\Lambda^\kappa$, with $\kappa \geq 8$, and that

$$E_\kappa^{1/2}\left[U_0^\lambda\right] < C, \quad E_{\kappa-2}^{1/2}\left[U_0^\lambda\right] < \varepsilon, \quad \textit{and} \quad \left\|\dot{U}_0^\lambda\right\|_\infty < \delta \tag{2.21}$$

for uniform constants C, ε, and δ.

If ε and δ are sufficiently small, then the initial value problem for (2.20a), (2.20b), *and* (2.20c) *with initial data $U^\lambda(0) = U_0^\lambda$ has a unique solution $U^\lambda(t) \in H_\Gamma^\kappa(T^\lambda)$ with $T^\lambda \geq \lambda$, which satisfies the constraints* (2.20d) *and* (2.20e) *and the estimates*

$$E_{\kappa-2}^{1/2}[U^\lambda(t)] \leq C' E_{\kappa-2}^{1/2}\left[U_0^\lambda\right] \leq C'\varepsilon, \tag{2.22a}$$

$$E_\kappa^{1/2}[U^\lambda(t)] \leq C' E_\kappa^{1/2}\left[U_0^\lambda\right]\langle t\rangle^{C'\varepsilon}, \tag{2.22b}$$

$$E_{\kappa-1}^{1/2}\left[\partial_t U^\lambda(t)\right] \leq E_{\kappa-1}^{1/2}\left[\partial_t U^\lambda(0)\right]\exp(C'\langle t\rangle^{1+C'\varepsilon}), \tag{2.22c}$$

for all $t \in [0, T^\lambda)$, where C' is a uniform constant.

We point out that the uniform bound for the initial energy in (2.21) implies, in particular, the statement $\|\Gamma^a \dot{\rho}_0^\lambda\| \leq C$, $|a| \leq \kappa$, and so according to our definitions, $\|\Gamma^a(\rho_0^\lambda - 1)\| \leq C\lambda^{-1}$. Thus, in the limit as $\lambda \to \infty$, the initial deformation is driven toward incompressibility.

Since the bounds on the energy from Theorem 2.5 are uniform in λ, we will be able to take the limit as λ goes to infinity to obtain a global solution to the incompressible elasticity equations given below:

$$\partial_t H_\ell^i + v \cdot \nabla H_\ell^i + H_p^i \partial_\ell v^p = 0, \tag{2.23a}$$

$$\partial_t v^i + v \cdot \nabla v^i + \hat{A}_{pj}^{\ell m}(H) H_i^p \partial_\ell H_m^j + \partial_i q = 0, \tag{2.23b}$$

with the constraints

$$\nabla \cdot v = 0, \tag{2.23c}$$

$$\det H = 1, \tag{2.23d}$$

$$\partial_\ell H_m^i = \partial_m H_\ell^i, \tag{2.23e}$$

where ∇q in (2.23b) represents the limit of the singular term from (2.20b). This convergence will be discussed in the proof of Theorem 2.6.

THEOREM 2.6 *Suppose that the initial data* U_0^λ *satisfies the assumptions of Theorem* 2.5, *and in particular* (2.23e) *holds.*

(i) *The solution family* U^λ *has a subsequence* U^{λ_k}, $\lambda_k \nearrow \infty$, *such that*

$$U^{\lambda_k} \to U^\infty = (H^\infty, v^\infty, 1) \quad \text{in } C^{\kappa-3}_{\text{loc}}((0,\infty), \mathbb{R}^3)\,,$$

where (H^∞, v^∞) *is a global solution of the incompressible equations* (2.23a)–(2.23e) *with*

$$E_\kappa[U^\infty(t)] < \infty \quad \text{and} \quad E_{\kappa-2}[U^\infty(t)] \le C$$

for $0 < t < \infty$.

(ii) *If, in addition, the initial data is independent of* λ *and* (2.23d) *holds at time* $t = 0$, *then the full sequence* U^λ *satisfies*

$$U^\lambda \to U^\infty \quad \text{in } C^{\kappa-3}_{\text{loc}}((0,\infty), \mathbb{R}^3)$$

and

$$(H^\lambda, \pi v^\lambda) \to (H^\infty, v^\infty) \quad \text{in } C^0_{\text{loc}}([0,\infty) \times \mathbb{R}^3)\,,$$

where π *is the* L^2 *projection onto divergence-free vector fields. Moreover,* (H^∞, v^∞) *is the unique solution of* (2.23a)–(2.23e) *in* $C([0,\infty), W^{\kappa-1,2})$ *with initial data* $(H_0, \pi v_0)$.

(iii) *Finally, if the initial is independent of* λ *and incompressible, i.e.,* (2.23c) *and* (2.23d) *hold at time* $t = 0$, *then the full sequence* U^λ *satisfies*

$$U^\lambda \to U^\infty \quad \text{in } C^{\kappa-3}_{\text{loc}}((0,\infty), \mathbb{R}^3) \cap C^0([0,\infty) \times \mathbb{R}^3)\,,$$

and the limit (H^∞, v^∞) *is the unique solution of* (2.23a)–(2.23e) *in*$C([0,\infty), W^{\kappa-1,2})$ *with initial data* (H_0, v_0).

2.7 Galilean and Scaling Invariance

The vector fields defined above are closely related with the Galilean and scaling invariance of the system.

Consider the one-parameter family of rotations generated by the $V^{(j)}$ defined in (2.15):

$$Q_j{}'(s) = V^{(j)} Q_j(s)\,, \quad Q_j(0) = I\,.$$

If $U(t,x) : [0,T) \times \mathbb{R}^3 \to \mathbb{M}^3 \times \mathbb{R}^3 \times \mathbb{R}$ and Q is any rotation, we define the simultaneous rotation of U by

$$T_Q U(t,x) = (Q H(t, Q^{\mathsf{T}} x) Q^{\mathsf{T}}, Q v(t, Q^{\mathsf{T}} x), \rho(t, Q^{\mathsf{T}} x))\,.$$

The operators $\tilde{\Omega}_j$ defined in (2.16) are generated by $T_{Q_j(s)}$ in the sense that

$$\tilde{\Omega}_j U = \frac{d}{ds} T_{Q_j(s)} U \Big|_{s=0}\,.$$

Next, define the one-parameter family of dilations

$$R(s)U(t,x) = U((s+1)t, (s+1)x).$$

The operator S defined in (2.13) is generated by $R(s)$ through

$$SU(t,x) = \frac{d}{ds}R(s)U(t,x)\bigg|_{s=0}.$$

Finally, define the one-parameter family of translations by

$$\tau_j(s)U(t,x) = U(t, x+se_j)\,,$$

where $j = 1, 2, 3$ and e_j, $j = 1, 2, 3$, is the standard basis on $\mathbb{R}^3$. The operators ∂_j are generated by $\tau_j(s)$ as

$$\partial_j U = \frac{d}{ds}\tau_j(s)U\bigg|_{s=0}.$$

We do not include time translations here because the ensuing energy estimates require special treatment of time derivatives in the singular limit.

Suppose that $\Delta(s)$ is any of these families. The PDEs (2.20a)—(2.20c) and the constraints (2.20d) and (2.20e) have the following invariance property: if $U(t,x)$ is a solution, then so is $\Delta(s)U(t,x)$. More generally, let $\Delta^a(s) = \Delta_{a_1}(s_1)\cdots\Delta_{a_q}(s_q)$ be the product of $q \leq \kappa$ such transformations. Again, if $U(t,x)$ satisfies (2.20a)–(2.20e), then so does $\Delta^a(s)U(t,x)$. Notice that

$$\frac{d^q}{ds_1\cdots ds_q}\Delta^a(s)U\bigg|_{(s_1,\ldots,s_q)=(0,\ldots,0)} = (\Gamma^a\dot{H}, \Gamma^a\dot{v}, \lambda^{-1}\Gamma^a\dot{\rho})\,.$$

This notation allows us to define the differentiation of nonlinear quantities in U with respect to the vector fields. Suppose that

$$f : \mathbb{M}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^d$$

is a smooth mapping for some d. Given

$$U : [0,T) \times \mathbb{R}^3 \to \mathbb{M}^3 \times \mathbb{R}^3 \times \mathbb{R}\,,$$

we define

$$\Gamma^a f(U) = \frac{d^q}{ds_1\cdots ds_q} f(\Delta^a(s)U)\bigg|_{(s_1,\ldots,s_q)=(0,\ldots,0)}. \tag{2.24}$$

Invariance immediately implies the following commutation result for the first-order system.

PROPOSITION 2.7 *For any solution* $U = (H, v, \rho) \in H^\kappa_\Gamma(T)$ *of the PDEs* (2.20a)–(2.20c) *and the constraints* (2.20d)–(2.20e)*, we have*

$$\begin{aligned}\partial_t\Gamma^a\dot{H} + v\cdot\nabla\Gamma^a\dot{H} + H\nabla\Gamma^a\dot{v}&\\ + \sum_{\substack{b+c=a\\ c\neq a}}[\Gamma^b\dot{v}\cdot\nabla\Gamma^c\dot{H} + \Gamma^b\dot{H}\nabla\Gamma^c\dot{v}] = 0\,,&\end{aligned} \tag{2.25a}$$

$$\begin{aligned}\text{(2.25b)}\qquad &\partial_t(\Gamma^a\dot v)^i + v\cdot\nabla(\Gamma^a\dot v)^i + \hat A^{\ell m}_{pj}(H)H^p_i\partial_\ell(\Gamma^a\dot H)^j_m + \lambda\rho\partial_i\Gamma^a\dot\rho \\ &+\sum_{\substack{b+c=a\\ c\neq a}}\big\{\Gamma^b\dot v\cdot\nabla(\Gamma^c\dot v)^i + \Gamma^b[\hat A(H)H]^{\ell m}_{ij}\partial_\ell(\Gamma^c\dot H)^j_m \\ &\qquad + \Gamma^b\dot\rho\partial_i\Gamma^c\dot\rho\big\} = 0\,,\end{aligned}$$

$$\begin{aligned}\text{(2.25c)}\qquad &\partial_t\Gamma^a\dot\rho + v\cdot\nabla\Gamma^a\dot\rho + \lambda\rho\nabla\cdot\Gamma^a\dot v \\ &+\sum_{\substack{b+c=a\\ c\neq a}}[\Gamma^b\dot v\cdot\nabla\Gamma^c\dot\rho + \Gamma^b\dot\rho\nabla\cdot\Gamma^c\dot v] = 0\,,\end{aligned}$$

in which the sums extend over all ordered partitions of the sequence a, *with* $|a|\le\kappa$. *In addition, the following constraints hold*:

$$\text{(2.25d)}\qquad \partial_j(\Gamma^a H)^i_k = \partial_k(\Gamma^a H)^i_j\,,$$

$$\text{(2.25e)}\quad \Gamma^a\dot\rho = \lambda(\operatorname{tr}\Gamma^a\dot H + \Gamma^a\theta(\dot H))\,,\quad \theta(\dot H) = \frac12[(\operatorname{tr}\dot H)^2 - \operatorname{tr}\dot H^2] + \det\dot H\,.$$

Note that the notation (2.24) has been used in (2.25b) and (2.25e).

PROOF: Suppose that Γ^a is generated by $\Delta^a(s)$ in the sense described above. Starting with a solution U of (2.20a)–(2.20e), we use invariance to obtain that $\Delta^a(s)U$ satisfies the same equations. The results (2.25a)–(2.25d) follow by differentiation in s and then putting $s=0$.

Using a simple linear algebra fact, we have that

$$\det H = 1 + \operatorname{tr}\dot H + \frac12[(\operatorname{tr}\dot H)^2 - \operatorname{tr}\dot H^2] + \det\dot H.$$

Thus, we can rewrite the constraint (2.20d) as

$$\dot\rho = \lambda(\operatorname{tr}\dot H + \theta(\dot H))\,.$$

This constraint is also invariant, and so (2.25e) follows in the same manner. □

This proposition also holds for the case when Γ^a includes time derivatives, and later, in Section 6, we will need this with exactly one time derivative.

2.8 Projection Operators

Motivated by the second-order case [19], we consider the decomposition of solutions onto their longitudinal and transverse components. This allows us to approximately separate shear and pressure waves away from the origin. Given

$$U(t,x) = (H(t,x), v(t,x), \rho(t,x))\,,$$

we define

$$P_1U(t,x) = (\omega\otimes\omega H(t,x), \omega\otimes\omega v(t,x), \rho(t,x))\quad\text{with } \omega = \frac{x}{|x|}\,.$$

Then set $P_2 = I - P_1$. Note that $P_i^2 = P_i$ and P_i is self-adjoint for $i = 1, 2$ so that the operators P_1 and P_2 are projections onto orthogonal subspaces of $\mathrm{GL}_+^3 \times \mathbb{R}^3 \times \mathbb{R}$.

Occasionally, we will abuse notation slightly by writing P_1 and P_2 for the matrices $\omega \otimes \omega$ and $I - \omega \otimes \omega$, respectively.

2.9 The Null Condition

Here we formulate the condition for shear waves. In this context, a general 6-tensor $\mathcal{B}_{ijk}^{\ell mn}$ will be said to satisfy the null condition if

$$\mathcal{B}_{ijk}^{\ell mn}(\omega_\ell \eta_{(1)}^i)(\omega_m \eta_{(2)}^j)(\omega_n \eta_{(3)}^k) = 0 \tag{2.26a}$$

for all vectors $\omega, \eta_{(\alpha)} \in \mathbb{R}^3$, with $\langle \omega, \eta_{(\alpha)} \rangle = 0$. In terms of the projection matrices $P_1 = \omega \otimes \omega$ and $P_2 = I - P_2$, this implies that

$$\mathcal{B}_{ijk}^{\ell mn}(P_1)_\ell^L (P_1)_m^M (P_1)_n^N (P_2)_I^i (P_2)_J^j (P_2)_K^k = 0 \tag{2.26b}$$

for all $\omega \in \mathbb{R}^3$ and all I, J, K, L, M, and N.

It was shown in [19] that, for isotropic materials, the shear waves satisfy the null condition at the reference configuration. That is, the coefficients

$$B_{ijk}^{\ell mn} = \frac{\partial^3 W}{\partial F_\ell^i \partial F_m^j \partial F_n^k}(I) = \frac{\partial A_{ij}^{\ell m}}{\partial F_n^k}(I)$$

satisfy (2.26a). This is equivalent to the fact that the shear waves are linearly degenerate at the identity.

Since we have changed coordinates, we will actually encounter two sets of modified coefficients. Define

$$\hat{B}_{ijk}^{\ell mn}(H) = \frac{\partial \hat{A}_{ij}^{\ell m}}{\partial H_n^k}(H) \tag{2.27a}$$

and

$$\tilde{B}_{ijk}^{\ell mn} = \hat{A}_{kj}^{\ell m} \delta_i^n . \tag{2.27b}$$

A straightforward calculation based on the definition (2.10) shows that $\hat{B}(I) = B$, and so $\hat{B}(I)$ satisfies (2.26a). From (2.11c), it can be easily seen that $\tilde{B}$ also satisfies (2.26a).

3 Weighted L^2 Estimates

In this section we will derive the main estimates for our result. Define the weights

$$\mathcal{W}_1 = \left\langle t - \frac{r}{\bar{\lambda}} \right\rangle, \quad \bar{\lambda}^2 = \lambda^2 + \alpha^2 , \quad \text{and} \quad \mathcal{W}_2 = \left\langle t - \frac{r}{\beta} \right\rangle .$$

We remark that $\lambda \sim \bar{\lambda}$ for λ large. Estimation of the weighted L^2 quantity

$$\mathcal{X}_\kappa[U(t)] = \sum_{\substack{|a|\le\kappa-1\\ i=1,2,3\\ \gamma=1,2}} \left\| \mathcal{W}_\gamma P_\gamma \partial_i \Gamma^a \dot{U}(t) \right\| + \sum_{|a|\le\kappa-1} \left\| \mathcal{W}_1 \nabla \cdot \Gamma^a \dot{v} \right\| \tag{3.1}$$

is the key to our results. By bounding the quantity $\mathcal{X}_\kappa$, we see that the energy of $P_1 U$ is concentrated along the fast cone, whereas $P_2 U$ concentrates along the slow cone. In the incompressible limit, the fast component is swept to infinity.

The heart of the matter is contained in the following proposition, which, combined with generalized Sobolev inequalities, will lead to decay of solutions. Here and in what follows $\mathcal{Q}_\kappa(\dot{U})$ will represent any generic nonlinear term of degree two or higher at the origin with total derivatives summing at most to κ. Because of the form of the equations these terms will always contains at least one spatial derivative, and they will always be bounded *independently of the parameter* λ. Hence we will write somewhat schematically

$$\mathcal{Q}_\kappa(\dot{U}) = \sum_{|a|+|b|\le\kappa-1} \Gamma^a f(\dot{U}) \nabla \Gamma^b \dot{U} \tag{3.2}$$

where $f(0) = 0$.

PROPOSITION 3.1 *Let* $U \in H^\kappa_\Gamma(T)$ *solve equations* (2.20a)–(2.20c) *and the constraints* (2.20d) *and* (2.20e). *Then we have*

$$\mathcal{X}_\kappa[U(t)] \le C\left[E_\kappa^{1/2}[U(t)] + \|\langle t+r\rangle \mathcal{Q}_\kappa(\dot{U})\|\right]. \tag{3.3}$$

The proof of this proposition depends only on the linearized equations, which follow from (2.25a)–(2.25c). For $|a| \le \kappa - 1$, we have

$$\partial_t \Gamma^a \dot{H} + \nabla \Gamma^a \dot{v} = \mathcal{Q}^H_\kappa(\dot{U}), \tag{3.4a}$$

$$\partial_t (\Gamma^a \dot{v})_i + \hat{A}^{\ell m}_{ij}(I) \partial_\ell (\Gamma^a \dot{H})^j_m + \lambda \partial_i \Gamma^a \dot{\rho} = \mathcal{Q}^v_\kappa(\dot{U}), \tag{3.4b}$$

$$\partial_t \Gamma^a \dot{\rho} + \lambda \nabla \cdot \Gamma^a \dot{v} = \mathcal{Q}^\rho_\kappa(\dot{U}). \tag{3.4c}$$

Here it should be noted that, as a consequence of our choice of variable $\dot{U}$, the singular parameter λ appears only in the linear part of equations (3.4a)–(3.4c).

Before proceeding, the following algebraic lemma extracts the essential information from (3.4a) and (3.4c), and identifies the useful properties of the resulting special combinations of derivatives.

LEMMA 3.2 *Let* $U \in H^\kappa_\Gamma(T)$ *be a solution to equations* (3.4a) *and* (3.4c). *Then for each* $(t, x) = (t, r\omega)$, $r = |x|$, *and* $|a| \le \kappa - 1$, *we have*

$$r \nabla \Gamma^a \dot{\rho} - \lambda t \omega \nabla \cdot \Gamma^a \dot{v} = O(\Gamma^\kappa \dot{U}) - t \mathcal{Q}^\rho_\kappa(\dot{U}), \tag{3.5a}$$

$$r \nabla \cdot (\Gamma^a \dot{H} - \Gamma^a \dot{H}^\top) - t(\partial_r \Gamma^a \dot{v} - \omega^\top \nabla \Gamma^a \dot{v}) = O(\Gamma^\kappa \dot{U}) - t \mathcal{Q}^H_\kappa(\dot{U}). \tag{3.5b}$$

Also, we have that

$$\langle r\omega, \nabla \cdot (\Gamma^a \dot{H} - \Gamma^a \dot{H}^\top)\rangle = \Omega \Gamma^a \dot{H}, \tag{3.6a}$$

$$\langle r\omega, \partial_r \Gamma^a \dot{v} - \omega \nabla \cdot \Gamma^a \dot{v} \rangle = \Omega \Gamma^a \dot{v}\,, \tag{3.6b}$$

$$\langle r\omega, \partial_r \Gamma^a \dot{v} - \omega^{\mathsf{T}} \nabla \Gamma^a \dot{v} \rangle = 0\,, \tag{3.6c}$$

$$r(\omega^{\mathsf{T}} \nabla \Gamma^a \dot{v} - \omega \nabla \cdot \Gamma^a \dot{v}) = \Omega \Gamma^a \dot{v}\,. \tag{3.6d}$$

PROOF: To show (3.5a) we will multiply equation (3.4c) by t and use the scaling operator (2.13) to rewrite the equation as

$$r\partial_r \Gamma^a \dot{\rho} - \lambda t \nabla \cdot \Gamma^a \dot{v} = S\Gamma^a \dot{\rho} - t\mathcal{Q}_\kappa^\rho\,.$$

After multiplying by ω, we then use the angular derivatives (2.14) to get the result. We derive (3.5b) similarly from (3.4a).

For (3.6a)–(3.6d), simply write the expression and use the definition of Ω. For example, upon switching the indices of summation we have

$$\langle r\omega, \nabla \cdot (\dot{H} - \dot{H}^{\mathsf{T}}) \rangle = r\omega_i \partial_j (\dot{H}_j^i - \dot{H}_i^j) = r(\omega_i \partial_j - \omega^j \partial_i)\dot{H}_j^i = \Omega \dot{H}\,.$$

The proofs of the other statements are similar. □

PROOF OF PROPOSITION 3.1: Equation (3.4b) can be reorganized to make the estimates simpler. Using the constraints from Proposition 2.7 and the linear expansion of $\hat{A}(I)$ given in (2.11c), we see that (3.4b) has the following form:

$$\partial_t(\Gamma^a \dot{v}) + \beta^2 \nabla \cdot (\Gamma^a \dot{H} - \Gamma^a \dot{H}^{\mathsf{T}}) + \left(\frac{\bar{\lambda}^2}{\lambda}\right) \nabla \Gamma^a \dot{\rho} = \mathcal{Q}_\kappa^v + \alpha^2 \nabla \Gamma^a \theta(\dot{H})\,, \tag{3.7}$$

in which $|a| \le \kappa - 1$, $\bar{\lambda}^2 = \alpha^2 + \lambda^2$, and θ was defined in (2.25e). We note that the last term on the right is of the form $\mathcal{Q}_k(\dot{U})$.

Let $U(t,x) \in H_\Gamma^\kappa(T)$ solve the elasticity equations (2.20a)–(2.20c). In view of the fact that (3.4a)–(3.4c) are linear in $\Gamma^a \dot{U}$, we shall perform the estimates for $\dot{U}$. This gives the result for $\mathcal{X}_1[U]$. The final result will follow from the corresponding estimates for $\Gamma^a \dot{U}$ after summation over $|a| \le \kappa - 1$.

With this strategy in mind, we specialize (3.5a)–(3.5b) to the case $\kappa = 1$:

$$r \nabla \dot{\rho} - \lambda t \omega \nabla \cdot \dot{v} = A_1\,, \tag{3.8a}$$

$$r \nabla \cdot (\dot{H} - \dot{H}^{\mathsf{T}}) - t(\partial_r \dot{v} - \omega^{\mathsf{T}} \nabla \dot{v}) = A_2\,, \tag{3.8b}$$

in which $A_i = O(\Gamma \dot{U}) + t\mathcal{Q}_1(\dot{U})$.

Before getting to the terms in $\mathcal{X}_1[U]$, however, we must first derive some preliminary estimates for the singular terms containing the parameter λ.

Step 1. Estimates for $\|(r - \bar{\lambda}t)\nabla \cdot \dot{v}(t)\|$ and $\|\mathcal{W}_1 \nabla \cdot \dot{v}(t)\|$.

Multiply equation (3.7) by t and use the scaling operator (2.13) to obtain

$$r\partial_r \dot{v} - t\left[\left(\frac{\bar{\lambda}^2}{\lambda}\right) \nabla \dot{\rho} + \beta^2 \nabla \cdot (\dot{H} - \dot{H}^{\mathsf{T}})\right] = A_0\,, \tag{3.9}$$

where $A_0 = O(\Gamma \dot{U}) + t\mathcal{Q}_1(\dot{U})$. Using (3.8a) and (3.8b) to eliminate ρ and H from (3.9), we derive

$$rA_0 + \left(\frac{\bar\lambda^2}{\lambda}\right) tA_1 + \beta^2 tA_2 = \\ (r^2 - \bar\lambda^2 t^2)\omega\nabla\cdot\dot v + (r^2-\beta^2t^2)(\partial_r \dot v - \omega^{\mathsf T}\nabla\dot v) + r^2(\omega^{\mathsf T}\nabla\dot v - \omega\nabla\cdot\dot v)\,. \tag{3.10}$$

If we take the inner product of (3.10) with ω and use (3.6c) and (3.6d), we find that

$$(r-\bar\lambda t)\nabla\cdot\dot v = (r+\bar\lambda t)^{-1}\left[r\Omega\dot v + rA_0 + \left(\frac{\bar\lambda^2}{\lambda}\right)tA_1 + \beta^2 tA_2\right].$$

Take the L^2 norm to conclude with

$$\|(r-\bar\lambda t)\nabla\cdot v\| \le C\big[E_1^{1/2}[U(t)] + \langle t\rangle \|\mathcal Q_1(\dot U)\|\big]\,. \tag{3.11}$$

Before we move on, note that in the final step we can bound $\|\Omega v\|$ by the energy $E_1^{1/2}[U(t)]$, since by definition of $\tilde\Omega v$, we have that $\|\Omega v\|^2 \le \|\tilde\Omega v\|^2 + \|v\|^2$. The same holds for ΩH. These facts will be used throughout the rest of the proof without further mention.

Finally, since

$$\|\mathcal W_1\nabla\cdot v\| \le \|\nabla\cdot v\| + \bar\lambda^{-1}\|(\bar\lambda t - r)\nabla\cdot v\|\,,$$

the estimate

$$\|\mathcal W_1\nabla\cdot v\| \le C\big[E_1^{1/2}[U(t)] + \langle t\rangle\|\mathcal Q_1(\dot U)\|\big]$$

follows immediately.

Step 2. Estimates for $\|(r-\bar\lambda t)\nabla\dot\rho(t)\|$ and $\|\mathcal W_1\nabla\dot\rho(t)\|$.

Performing the indicated algebra with (3.9) and (3.8a), we obtain

$$-\lambda tA_0 - rA_1 = \\ (\bar\lambda^2t^2 - r^2)\nabla\dot\rho + \lambda\beta^2t^2\nabla\cdot(\dot H - \dot H^{\mathsf T}) - \lambda rt(\partial_r\dot v - \omega\nabla\cdot\dot v)\,. \tag{3.12}$$

The estimate in this case is more subtle than in step 1 because by (3.6a) and (3.6b) the local projection of the terms $\nabla\cdot(\dot H - \dot H^{\mathsf T})$ and $\partial_r\dot v - \omega\nabla\cdot\dot v$ is not zero. We will need to use the L^2 orthogonality of these terms with $\nabla\dot\rho$ in order to obtain the estimate. The details of this technicality are now displayed.

Starting with (3.12), divide by $(\bar\lambda t + r)$ and take the norm in L^2 to obtain

$$\begin{aligned}&\big\|(\bar\lambda t - r)\nabla\dot\rho \\ &\quad + (\bar\lambda t + r)^{-1}\lambda\big[\beta^2t^2\nabla\cdot(\dot H - \dot H^{\mathsf T}) - rt(\partial_r\dot v - \omega\nabla\cdot\dot v)\big]\big\|^2 \\ &\quad \le C\big[E_1[U(t)] + \langle t\rangle^2\|\mathcal Q_1(U)\|^2\big]\,.\end{aligned} \tag{3.13}$$

To get the desired bound we will first estimate the cross terms:

$$\int \lambda\left(\frac{\bar\lambda t - r}{\bar\lambda t + r}\right)\langle\nabla\dot\rho, [\beta^2t^2\nabla\cdot(\dot H - \dot H^{\mathsf T}) - rt(\partial_r\dot v - \omega\nabla\cdot\dot v)]\rangle dx = \\ \int \lambda\partial_i\dot\rho\big[\phi(r)\partial_j Z^i_j + \psi(r)\big(x^j\partial_j\dot v^i - x^i\partial_j\dot v^j\big)\big]dx\,,$$

where we denote

$$\phi(r) = t^2\left(\frac{\bar\lambda t - r}{\bar\lambda t + r}\right), \quad \psi(r) = -t\left(\frac{\bar\lambda t - r}{\bar\lambda t + r}\right), \quad \text{and} \quad Z^i_j = \beta^2(\dot H^i_j - \dot H^j_i)\,.$$

We proceed using integration by parts.

The first term exploits the antisymmetry of Z^i_j and the properties of the angular derivatives (2.17) to get

$$\begin{aligned}
&\int \lambda\phi(r)\partial_i\dot\rho\partial_j Z^i_j\,dx\\
&\quad = -\int \lambda\phi(r)\partial_i\partial_j\dot\rho Z^i_j\,dx - \int \lambda\phi'(r)\omega^j\partial_i\dot\rho Z^i_j\,dx\\
&\quad = \int \frac{2\lambda\bar\lambda t^3}{(\bar\lambda t + r)^2}\omega^j\partial_i\dot\rho Z^i_j\,dx\\
&\quad = \int \frac{2\lambda t^2}{(\bar\lambda t + r)^2}(\bar\lambda t - r)\omega^j\partial_i\dot\rho Z^i_j\,dx + \int \frac{2\lambda t^2 r}{(\bar\lambda t + r)^2}\partial_i\dot\rho\omega^j Z^i_j\,dx\\
&\quad = \int (\bar\lambda t - r)\partial_i\dot\rho\frac{2\lambda t^2}{(\bar\lambda t + r)^2}\omega^j Z^i_j\,dx - \int \frac{2\lambda t^2}{(\bar\lambda t + r)^2}\omega^j[(\omega\wedge\Omega)_i\dot\rho]Z^i_j\,dx\,.
\end{aligned}$$

The second piece of the cross term is treated as follows:

$$\begin{aligned}
&\int \lambda\psi(r)\partial_i\dot\rho(x^j\partial_j\dot v^i - x^i\partial_j\dot v^j)dx\\
&\quad = -\int \lambda\partial_j(\psi(r)\partial_i\dot\rho x^j)\dot v^i\,dx + \int \lambda\partial_j(\psi(r)\partial_i\dot\rho x^i)\dot v^j\,dx\\
&\quad = -\int \lambda\left[\psi(r)x^j\partial_j\partial_i\dot\rho\dot v^i + 3\psi(r)\partial_i\dot\rho\dot v^i + \psi'(r)\omega_j x^j\partial_i\dot\rho\dot v^i\right]dx\\
&\qquad + \int \lambda\left[\psi(r)x^i\partial_j\partial_i\dot\rho\dot v^j + \psi(r)\delta^i_j\partial_i\dot\rho\dot v^j + \psi'(r)\omega_j x^i\partial_i\dot\rho\dot v^j\right]dx\\
&\quad = -2\int \lambda\psi(r)\partial_i\dot\rho\dot v^i\,dx - \int \lambda\psi'(r)r\big(\partial_i\dot\rho\dot v^i - \partial_r\dot\rho\langle\omega,\dot v\rangle\big)dx\\
&\quad = 2\int (\bar\lambda t - r)\partial_i\dot\rho\,\frac{\lambda t}{\bar\lambda t + r}\,\dot v^i - 2\int \frac{\lambda\bar\lambda t^2}{(\bar\lambda t + r)^2}[(\omega\wedge\Omega)_i\dot\rho]\dot v^i\,dx\,.
\end{aligned}$$

Recalling that $\bar\lambda \sim \lambda$, we see that the cross terms are bounded below by

$$-\frac{1}{2}\|(\bar\lambda t - r)\nabla\dot\rho\|^2 - C[\|\dot H\|^2 + \|\dot v\|^2 + \|\Omega\dot\rho\|^2]\,.$$

From (3.13) this gives us the result

$$\|(\bar\lambda t - r)\nabla\dot\rho\| \le C\big[E_1^{1/2}[U(t)] + \langle t\rangle\|\mathcal{Q}_1(\dot U)\|\big]\,.$$

As in step 1, since

$$\|\mathcal{W}_1\nabla\dot\rho\| \le \|\nabla\dot\rho\| + \bar\lambda^{-1}\|(\bar\lambda t - r)\nabla\dot\rho\|\,,$$

the estimate

$$\|\mathcal{W}_1\nabla\dot\rho\| \le C\big[E_1^{1/2}[U(t)] + \langle t\rangle\|\mathcal{Q}_1(\dot U)\|\big] \tag{3.14}$$

is an immediate consequence of the preceding.

Step 3. Estimates for $\|\mathcal{W}_2(\partial_r\dot v - \omega^{\mathsf{T}}\nabla\dot v)\|$ and $\|\mathcal{W}_2\nabla\cdot(\dot H - \dot H^{\mathsf{T}})\|$.

First, we will add another useful identity to our list. Notice that returning to (3.8a), we can write

$$\begin{aligned}\left(\frac{\bar\lambda^2}{\lambda}\right)t\nabla\dot\rho - r\omega\nabla\cdot\dot v &= \left(\frac{\bar\lambda}{\lambda}\right)(\bar\lambda t - r)\nabla\dot\rho + (\bar\lambda t - r)\omega\nabla\cdot\dot v + \left(\frac{\bar\lambda}{\lambda}\right)A_1 \\ &\equiv B_0\,.\end{aligned} \tag{3.15}$$

By steps 1 and 2, B_0 is bounded in L^2 by the appropriate quantities.

If we go back to (3.9) and add our new identity (3.15), we get

$$r(\partial_r\dot v - \omega\nabla\cdot\dot v) - \beta^2 t\nabla\cdot(\dot H - \dot H^{\mathsf{T}}) = A_0 + B_0\,. \tag{3.16}$$

By using (3.6d) we can transform (3.16) slightly and pair it with (3.8b) to get the following linear system of equations:

$$\begin{aligned}r(\partial_r\dot v - \omega^{\mathsf{T}}\nabla\dot v) - \beta^2 t\nabla\cdot(\dot H - \dot H^{\mathsf{T}}) &= A_0 + B_0 - \Omega v\,,\\ t(\partial_r\dot v - \omega^{\mathsf{T}}\nabla\dot v) - r\nabla\cdot(\dot H - \dot H^{\mathsf{T}}) &= -A_2\,.\end{aligned}$$

If we multiply by the matrix

$$(\beta t + r)^{-1}\begin{bmatrix}-r & \beta^2 t\\ -t & r\end{bmatrix},$$

we find that the quantities $(\beta t - r)(\partial_r\dot v - \omega^{\mathsf{T}}\nabla\dot v)$ and $(\beta t - r)\nabla\cdot(\dot H - \dot H^{\mathsf{T}})$ have the desired bound in L^2, and this leads immediately to the estimates with the full weight:

$$\|\mathcal{W}_2(\partial_r\dot v - \omega^{\mathsf{T}}\nabla\dot v)\| \le C\big[E_1^{1/2}[U(t)] + \langle t\rangle\|\mathcal{Q}_1(\dot U)\|\big]\,, \tag{3.17a}$$

$$\|\mathcal{W}_2\nabla\cdot(\dot H - \dot H^{\mathsf{T}})\| \le C\big[E_1^{1/2}[U(t)] + \langle t\rangle\|\mathcal{Q}_1(\dot U)\|\big]\,. \tag{3.17b}$$

Step 4. Estimate for $\|\mathcal{W}_\alpha P_\alpha\nabla\dot H\|$.

To extend (3.17b) to an estimate for the full gradient of $\dot H$, we will have to consider two cases. These cases consist of splitting $\mathbb{R}^3$ into regions $D = \{r \le \langle\beta t\rangle/2\}$ and $D^c = \{r > \langle\beta t\rangle/2\}$.

Define a smooth cutoff function

$$\zeta(s) = \begin{cases}1 & \text{if } s \le \frac12\\ 0 & \text{if } s \ge \frac34,\end{cases}$$

and set

$$\Psi(t,r) = \zeta\left(\frac{r}{\langle\beta t\rangle}\right).$$

Then

$$\|\nabla\dot H\|_{L^2(D)} \le \|\Psi\nabla\dot H\|_{L^2}\,.$$

With the aid of the constraint (2.25d), integration by parts yields

$$\begin{aligned}\|\Psi\nabla\dot{H}\|^2 &= \sum_{i,j,k}\int \Psi^2(\partial_k \dot{H}^i_j)^2\,dx\\ &= \sum_{i,j,k}\int \Psi^2\partial_j\dot{H}^i_k\partial_k\dot{H}^i_j\,dx\\ &\le \int \Psi^2|\nabla\cdot\dot{H}|^2\,dx + C\int|\Psi\nabla\Psi\dot{H}\nabla\dot{H}|dx\\ &\le \|\Psi\nabla\cdot\dot{H}\|^2 + \langle\beta t\rangle^{-1}\|\dot{H}\|\,\|\Psi\nabla\dot{H}\|\\ &\le \|\Psi\nabla\cdot\dot{H}\|^2 + \frac{1}{2}\|\Psi\nabla\dot{H}\|^2 + C\langle\beta t\rangle^{-2}\|\dot{H}\|^2.\end{aligned}$$

This implies that

$$\|\Psi\nabla\dot{H}\|^2 \le C[\|\Psi\nabla\cdot\dot{H}\|^2 + \langle t\rangle^{-2}\|\dot{H}\|^2]\,. \tag{3.18a}$$

But we can bound

$$\|\Psi\nabla\cdot\dot{H}\| \le \|\Psi\nabla\cdot(\dot{H}-\dot{H}^{\mathsf{T}})\| + \|\Psi\nabla\cdot\dot{H}^{\mathsf{T}}\|\,, \tag{3.18b}$$

and then using (2.25d) and (2.25e), we have

$$\begin{aligned}\|\Psi\nabla\cdot\dot{H}^{\mathsf{T}}\| &= \|\Psi\nabla\,\mathrm{tr}\,\dot{H}\|\\ &= \left\|\Psi\nabla\left(\frac{\dot{\rho}}{\lambda}-\theta(\dot{H})\right)\right\|\\ &\le \left\|\Psi\nabla\frac{\dot{\rho}}{\lambda}\right\| + \|\nabla\theta(\dot{H})\|\,.\end{aligned} \tag{3.18c}$$

Hence all together, (3.18a)–(3.18c) imply

$$\|\Psi\nabla\dot{H}\| \le C\left[\left\|\Psi\nabla\frac{\dot{\rho}}{\lambda}\right\| + \|\Psi\nabla\cdot(\dot{H}-\dot{H}^{\mathsf{T}})\| + \langle t\rangle^{-1}\|\dot{H}\| + \|\nabla\theta(\dot{H})\|\right]. \tag{3.19}$$

On D we have that $\mathcal{W}_1 \sim \mathcal{W}_2 \sim \langle t\rangle$ since $r \le \langle\beta t\rangle/2$. Thus using (3.19), (3.17b), and the bound (3.14) from step 2, we have

$$\begin{aligned}&\sum_\alpha \|\mathcal{W}_\alpha P_\alpha\nabla\dot{H}\|_{L^2(D)}\\ &\quad\le C\langle t\rangle\|\Psi\nabla\dot{H}\|\\ &\quad\le C\left[\langle t\rangle\left\|\Psi\nabla\frac{\dot{\rho}}{\lambda}\right\| + \langle t\rangle\|\Psi\nabla\cdot(\dot{H}-\dot{H}^{\mathsf{T}})\| + \|\dot{H}\| + \langle t\rangle\|\nabla\theta(\dot{H})\|\right]\\ &\quad\le C\left[\left\|\mathcal{W}_1\nabla\frac{\dot{\rho}}{\lambda}\right\| + \|\mathcal{W}_2\nabla\cdot(\dot{H}-\dot{H}^{\mathsf{T}})\| + E_1^{1/2}[U(t)] + \langle t\rangle\|\nabla\theta(\dot{H})\|\right]\\ &\quad\le C\big[E_1^{1/2}[U(t)] + \langle t\rangle\|\mathcal{Q}_1(\dot{U})\|\big]\,.\end{aligned} \tag{3.20}$$

Now we consider the region $D^c = \{r > \langle \beta t\rangle/2\}$. Using (2.17), (2.25d), and (2.25e) we can write

$$\begin{aligned}(P_1\partial_k\dot{H})^i_j &= \omega^i\omega_l\partial_k\dot{H}^l_j\\ &= \omega^i\omega_k\partial_l\dot{H}^l_j + r^{-1}\Omega\dot{H}\\ &= \omega_k\omega^i\partial_j \operatorname{tr}\dot{H} + r^{-1}\Omega\dot{H}\\ &= \omega_k\omega^i\partial_j\left(\frac{\dot{\rho}}{\lambda} - \theta(\dot{H})\right) + r^{-1}\Omega\dot{H}\,.\end{aligned}$$

Hence, we have

$$\|\mathcal{W}_1 P_1\nabla\dot{H}\|_{L^2(D^c)} \le \left\|\mathcal{W}_1\nabla\frac{\dot{\rho}}{\lambda}\right\| + \|\mathcal{W}_1\nabla\theta(\dot{H})\| + \|\mathcal{W}_1 r^{-1}\Omega\dot{H}\|_{L^2(D^c)}\,.$$

On D^c, $\mathcal{W}_1 r^{-1}$ is bounded and so using (3.14) we have that

$$\|\mathcal{W}_1 P_1\nabla\dot{H}\|_{L^2(D^c)} \le C\big[E_1^{1/2}[U(t)] + \|\langle t+r\rangle\mathcal{Q}_1(\dot{U})\|\big]. \tag{3.21}$$

Recall that

$$(P_2\partial_k\dot{H})^i_j = \partial_k\dot{H}^i_j - \omega^i\omega_l\partial_k\dot{H}^l_j\,.$$

We use (2.17) and (2.25d) to write

$$\begin{aligned}\partial_k\dot{H}^i_j = \omega^l\omega_l\partial_k\dot{H}^i_j &= \omega^l\omega_k\partial_l\dot{H}^i_j + r^{-1}\Omega\dot{H}\\ &= \omega^l\omega_k\partial_j\dot{H}^i_l + r^{-1}\Omega\dot{H}\\ &= \omega_j\omega_k\partial_l\dot{H}^i_l + r^{-1}\Omega\dot{H}\,.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\omega_i\omega_l\partial_k\dot{H}^l_j &= \omega_k\omega_l\partial_i\dot{H}^l_j + r^{-1}\Omega H\\ &= \omega_k\omega_l\partial_j\dot{H}^l_i + r^{-1}\Omega\dot{H}\\ &= \omega_j\omega_k\partial_l\dot{H}^l_i + r^{-1}\Omega\dot{H}\,.\end{aligned}$$

It follows that

$$(P_2\partial_k\dot{H})^i_j = \omega_j\omega_k\nabla\cdot(\dot{H} - \dot{H}^{\mathsf{T}})^i + r^{-1}\Omega\dot{H}\,.$$

Thus using (3.17b) and the fact that $\mathcal{W}_2 r^{-1}$ is bounded on D^c, we have that

$$\begin{aligned}\|\mathcal{W}_2 P_2\nabla\dot{H}\|_{L^2(D^c)} &\le \|\mathcal{W}_2\nabla\cdot(\dot{H} - \dot{H}^{\mathsf{T}})\| + \|\mathcal{W}_2 r^{-1}\Omega\dot{H}\|_{L^2(D^c)}\\ &\le C\big[E_1^{1/2}[U(t)] + \langle t\rangle\|\mathcal{Q}_1(U\dot{U})\|\big].\end{aligned} \tag{3.22}$$

Combining (3.20)–(3.22) we obtain

$$\sum_\alpha\|\mathcal{W}_\alpha P_\alpha\nabla\dot{H}\| \le C\big[E_1^{1/2}[U(t)] + \|\langle t+r\rangle\mathcal{Q}_1(\dot{U})\|\big].$$

Step 5. Estimate for $\|\mathcal{W}_\alpha P_\alpha \nabla \dot{v}\|$.

On D we have $\mathcal{W}_\alpha \le C\langle t\rangle$ for $\alpha = 1, 2$; hence we can use equation (3.4a) and the vector field S from (2.13) to say that

$$\begin{aligned}\sum_\alpha \|\mathcal{W}_\alpha P_\alpha \nabla \dot{v}\|_{L^2(D)} &\le C\langle t\rangle \|\nabla \dot{v}\|_{L^2(D)}\\ &\le C\big[t\|\nabla \dot{v}\|_{L^2(D)} + \|\nabla \dot{v}\|\big]\\ &\le C\big[\big\|r\partial_r \dot{H}\big\|_{L^2(D)} + \big\|\nabla \dot{v}\big\| + \big\|\mathcal{Q}_1^H\big\| + \big\|S\dot{H}\big\|\big]\\ &\le C\Big[\sum_k \big\|r\partial_k \dot{H}\big\|_{L^2(D)} + \big\|\nabla \dot{v}\big\| + \big\|\mathcal{Q}_1^H\big\| + \big\|S\dot{H}\big\|\Big]\\ &\le C\Big[\sum_{k,\alpha} \big\|rP_\alpha\partial_k \dot{H}\big\|_{L^2(D)} + \big\|\nabla \dot{v}\big\| + \big\|\mathcal{Q}_1^H\big\| + \big\|S\dot{H}\big\|\Big]\\ &\le C\Big[\sum_\alpha \big\|\mathcal{W}_\alpha P_\alpha \nabla \dot{H}\big\|_{L^2(D)} + \big\|\nabla \dot{v}\big\| + \big\|\mathcal{Q}_1^H\big\| + \big\|S\dot{H}\big\|\Big].\end{aligned}$$

Hence using (3.20) we can bound

$$\sum_\alpha \big\|\mathcal{W}_\alpha P_\alpha \nabla \dot{v}\big\|_{L^2(D)} \le C\big[E_1^{1/2}U(t)) + \langle t\rangle \|\mathcal{Q}_1(\dot{U})\|\big]. \tag{3.23}$$

On D^c we will again make use of the fact that $\mathcal{W}_\alpha r^{-1}$ is bounded for $\alpha = 1, 2$. First we can use (2.17) to write

$$\begin{aligned}(P_1\partial_k \dot{v})^i &= \omega^i\omega_j\partial_k \dot{v}^j\\ &= \omega^i\omega_k\partial_j \dot{v}^j + r^{-1}\Omega\dot{v}\,,\end{aligned}$$

which implies that

$$\|\mathcal{W}_1 P_1 \nabla \dot{v}\|_{L^2(D^c)} \le \|\mathcal{W}_1 \nabla\cdot\dot{v}\| + \big\|\mathcal{W}_1 r^{-1}\Omega\dot{v}\big\|_{L^2(D^c)}\,. \tag{3.24}$$

Next we can use (2.17) to write

$$\begin{aligned}(P_2\partial_k \dot{v})^i &= \partial_k \dot{v}^i - \omega^i\omega_j\partial_k \dot{v}^j\\ &= \omega_k\partial_r \dot{v}^i - \omega_j\omega_k\partial_i \dot{v}^j + r^{-1}\Omega\dot{v}\\ &= \omega_k\big(\partial_r \dot{v} - \omega^{\mathsf{T}}\nabla\dot{v}\big)^i + r^{-1}\Omega\dot{v}\,,\end{aligned}$$

which implies that

$$\|\mathcal{W}_2 P_2 \nabla \dot{v}\|_{L^2(D^c)} \le \big\|\mathcal{W}_2(\partial_r \dot{v} - \omega^{\mathsf{T}}\nabla\dot{v})\big\| + \big\|\mathcal{W}_2 r^{-1}\Omega\dot{v}\big\|_{L^2(D^c)}\,. \tag{3.25}$$

Combining (3.24) and (3.25) with the previous results (3.11) and (3.17a), we see that

$$\sum_\alpha \|\mathcal{W}_\alpha P_\alpha \nabla \dot{v}\|_{L^2(D^c)} \le C\big[E_1^{1/2}[U(t)] + \langle t\rangle\big\|\mathcal{Q}_1(\dot{U})\big\|\big]. \tag{3.26}$$

Together, (3.23) and (3.26) imply the result

$$\sum_{\alpha} \|\mathcal{W}_\alpha P_\alpha \nabla \dot{v}\| \le C\big[E_1^{1/2}[U(t)] + \big\|\langle t+r\rangle \mathcal{Q}_1(\dot{U})\big\|\big].$$

This completes the proof.

□

4 Sobolev Estimates

The following weighted Sobolev-type inequalities appeared in [19]. The only important thing to note in our case is that even though $\mathcal{W}_1$ depends on λ, we still have uniform estimates because $|\partial_r \mathcal{W}_1| \le 1$.

LEMMA 4.1 *For $U \in C_0^\infty(\mathbb{R}^3)$, $r = |x|$, and $\alpha = 1, 2$ we have*

$$\text{(4.1a)} \quad r^{\frac{1}{2}}|U(x)| \le C \sum_{|a|\le 1} \|\nabla \tilde{\Omega}^a U\|,$$

$$\text{(4.1b)} \quad r|U(x)| \le C \sum_{|a|\le 1} \big\|\partial_r \tilde{\Omega}^a U\big\|_{L^2(|y|\ge r)}^{1/2} \cdot \sum_{|a|\le 2} \big\|\tilde{\Omega}^a U\big\|_{L^2(|y|\ge r)}^{1/2},$$

$$\text{(4.1c)} \quad r\mathcal{W}_\alpha^{1/2}|U(x)| \le C \sum_{|a|\le 1} \big\|\mathcal{W}_\alpha \partial_r \tilde{\Omega}^a U\big\|_{L^2(|y|\ge r)} + C \sum_{|a|\le 2} \big\|\tilde{\Omega}^a U\big\|_{L^2(|y|\ge r)},$$

$$\text{(4.1d)} \quad r\mathcal{W}_\alpha|U(x)| \le C \sum_{|a|\le 1} \big\|\mathcal{W}_\alpha \partial_r \tilde{\Omega}^a U\big\|_{L^2(|y|\ge r)} + C \sum_{|a|\le 2} \big\|\mathcal{W}_\alpha \tilde{\Omega}^a U\big\|_{L^2(|y|\ge r)}.$$

The next result will apply the above inequalities to higher derivatives and remove the singularity at the origin in order to be of use in the proof of Theorem 2.5. Again, this result appeared in [19] and although the proof is very similar, we will give it here because there are slight differences in the first-order case.

PROPOSITION 4.2 *Let $\dot{U} \in H_\Gamma^\kappa(T)$ with $\mathcal{X}_\kappa[U(t)] < \infty$ and $|\dot{U}| < \delta$ small. Then for $\alpha = 1, 2$,*

$$\text{(4.2a)} \quad \langle r\rangle|\Gamma^a \dot{U}(t,x)| \le C E_\kappa^{1/2}[U(t)], \qquad |a| + 2 \le \kappa,$$

$$\text{(4.2b)} \quad \langle r\rangle \mathcal{W}_\alpha^{1/2}\big|P_\alpha \Gamma^a \dot{U}(t,x)\big| \le C\big[E_\kappa^{1/2}[U(t)] + \mathcal{X}_\kappa[U(t)]\big], \qquad |a| + 2 \le \kappa,$$

$$\text{(4.2c)} \quad \langle r\rangle \mathcal{W}_\alpha|P_\alpha \partial_i \Gamma^a \dot{U}(t,x)| \le C\mathcal{X}_\kappa[U(t)], \qquad |a| + 3 \le \kappa.$$

PROOF: We choose δ small enough so that the energy E_κ can be used to dominate generalized derivatives to order κ in L^2.

To prove (4.2a) for $r \ge 1$, we apply (4.1b) to $\Gamma^a \dot{U}(t,x)$. To prove (4.2b) and (4.2c) for $r \ge 1$, we apply (4.1c) and (4.1d) to $P_\alpha \Gamma^a \dot{U}(t,x)$ and $P_\alpha \partial_i \Gamma^a \dot{U}(t,x)$, respectively. We use the fact that P_α commutes with ∂_r and $\tilde{\Omega}$. The latter fact is most easily seen from the commutation of P_α with the generators of $\tilde{\Omega}$, $T_{Q_j(s)}$.

For $r \le 1$, (4.2a) is a consequence of the Sobolev embedding

$$\text{(4.3)} \qquad W^{2,2}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3).$$

To obtain the other inequalities for $r \leq 1$, we will define a smooth cutoff function

$$\xi(r) = \begin{cases} 1 & \text{if } r < 1 \\ 0 & \text{if } r > 2. \end{cases} \tag{4.4}$$

To show (4.2b), first note that

$$\mathcal{W}_\alpha \sim \langle t \rangle \quad \text{when } r \leq 2. \tag{4.5}$$

Using (4.3) and (4.5), we can get (4.2b) as follows for $r < 1$,

$$\begin{aligned} \mathcal{W}_\alpha^{1/2} \left| P_\alpha \Gamma^a \dot{U}(t,x) \right| &\leq C \langle t \rangle^{\frac{1}{2}} \xi(r) |\Gamma^a \dot{U}(t,x)| \\ &\leq C \langle t \rangle^{\frac{1}{2}} \sum_{|b| \leq 2} \| \nabla^b (\xi \Gamma^a \dot{U}) \| \\ &\leq C \langle t \rangle^{\frac{1}{2}} \sum_{|b| \leq 2} \| \nabla^b \Gamma^a \dot{U} \|_{L^2(|y| \leq 2)} \\ &\leq C \sum_\beta \sum_{|b| \leq 2} \left\| \mathcal{W}_\beta^{1/2} P_\beta \nabla^b \Gamma^a \dot{U} \right\|_{L^2(|y| \leq 2)} \\ &\leq C \mathcal{X}_\kappa [U(t)] + C \sum_\beta \left\| \mathcal{W}_\beta^{1/2} P_\beta \Gamma^a \dot{U} \right\|_{L^2(|y| \leq 2)}. \end{aligned}$$

To complete the proof of (4.2b), we still have to deal with the last term above. We will now use (4.1c) to get

$$\begin{aligned} \left\| \mathcal{W}_\beta^{1/2} P_\beta \Gamma^a \dot{U} \right\|_{L^2(|y| \leq 2)} &\leq \left\| |y| \mathcal{W}_\beta^{1/2} P_\beta \Gamma^a \dot{U} \right\|_{L^\infty} \| |y|^{-1} \|_{L^2(|y| \leq 2)} \\ &\leq C \left[E_\kappa^{1/2} [U(t)] + \mathcal{X}_\kappa [U(t)] \right]. \end{aligned}$$

The proof of (4.2c) is very similar to that for (4.2b), but since we are applying it to $P_\alpha \nabla \Gamma^a \dot{U}(t,x)$, which already has one spatial derivative (as is needed for the $\mathcal{X}$ norm), the last step is no longer needed. □

The following result will be used in the final stages of the energy estimates.

LEMMA 4.3 *Suppose that* $f : \mathbb{R}^3 \to \mathbb{R}^3$, $\sum_{|a| \leq 2} \| \Omega^a f \|_{W^{2-|a|,2}} < \infty$, *and* $\nabla \wedge f = 0$. *Then*

$$|x|^{\frac{3}{2}} |P_2 f(x)| \leq C \sum_{|a| \leq 2} \| \Omega^a f \|. \tag{4.6}$$

PROOF: Writing $\omega = \frac{x}{|x|}$, notice that

$$\begin{aligned} |x| (P_2 f)_j(x) &= |x| (f - P_1 f)_j (x) \\ &= |x| (f_j(x) - \omega_j \omega_k f_k(x)) \\ &= \omega^k (x_k f_j(x) - x_j f_k(x)), \end{aligned}$$

and therefore it is enough to estimate $|x|^{1/2} (x \wedge f(x))$.

Fix $x \in \mathbb{R}^3$, and choose $R > |x|$. Let $\varphi_R(x) = \xi(|x|/R)$, where ξ is given in (4.4). We may now apply (4.1a) with $U(x) = \varphi_R(x)(x \wedge f(x))$:

$$\begin{aligned} |x|^{\frac{3}{2}}|P_2 f(x)| &\le |x|^{\frac{1}{2}}|x \wedge f| \\ &= |x|^{\frac{1}{2}}|\varphi_R(x)(x \wedge f)| \\ &\le C \sum_{|a|\le 1} \left\| \nabla \Omega^a[\varphi_R(y)(y \wedge f)] \right\| \\ &\le C \sum_{|a|\le 1} \left\| \Omega^a \nabla[\varphi_R(y)(y \wedge f)] \right\|, \end{aligned}$$

where in the last step we use the fact that the commutator of ∇ and Ω is in the span of ∇. Note that $(|x|\nabla)^k \varphi_R(x)$ is bounded independently of R and also that, thanks to the constraint $\nabla \wedge f = 0$, we have that $\partial_i(x_j f_k - x_k f_j) = (x_j \partial_k - x_k \partial_j) f_i + O(f)$. The result is thus a consequence of this inequality. □

This result is easily understood by expressing f as the gradient of a scalar potential σ. Then $P_2 f \sim \Omega\sigma$, and formally we can apply (4.1a). Our argument, as given, is meant to avoid unnecessary discussion of the properties of this potential.

5 Bootstrapping the Nonlinearity

In this section we will estimate the nonlinear terms on the right-hand side of (3.3) in Proposition 3.1. The factor $\langle t + r\rangle$ will be absorbed with the aid of the inequalities of Proposition 4.2 together with a few simple properties of the weights $\mathcal{W}_1 = \langle t - r/\bar{\lambda}\rangle$ and $\mathcal{W}_2 = \langle t - r/\beta\rangle$, which we collect below. As usual, all constants are independent of $\bar{\lambda}$.

Define the following neighborhoods of the characteristic cones:

$$\mathcal{C}_1 = \left\{ \left| t - \frac{r}{\bar{\lambda}} \right| < \frac{t}{2} \right\}, \quad \mathcal{C}_2 = \left\{ \left| t - \frac{r}{\beta} \right| < \frac{t}{2} \right\}.$$

Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint for $\bar{\lambda}$ large, and

$$\langle r\rangle \sim \langle \bar{\lambda} t\rangle \text{ on } \mathcal{C}_1 \quad \text{and} \quad \langle r\rangle \sim \langle \beta t\rangle \text{ on } \mathcal{C}_2.$$

Moreover, in addition to being bounded below by 1, the weights satisfy

$$\mathcal{W}_\alpha \ge C\langle t\rangle \quad \text{on } \mathcal{C}_\alpha^c,\ \alpha = 1, 2,$$

and so, in particular,

$$\mathcal{W}_\alpha \ge C\langle t\rangle \quad \text{on } (\mathcal{C}_1 \cup \mathcal{C}_2)^c,\ \alpha = 1, 2. \tag{5.1a}$$

Now by considering the regions $\mathcal{C}_1$, $\mathcal{C}_2$, and $(\mathcal{C}_1 \cup \mathcal{C}_2)^c$ in turn, we find that

$$C\langle t\rangle^{-3/2}\langle r\rangle \mathcal{W}_\alpha^{1/2}\mathcal{W}_\beta \ge 1, \quad \alpha \ne \beta. \tag{5.1b}$$

Also, by taking first $\mathcal{C}_1$ and then its complement, we have

$$C[\langle t\rangle^{-\frac{3}{2}} + \langle\bar{\lambda}t\rangle^{-1}]\langle r\rangle\mathcal{W}_1^{3/2} \geq 1\,. \tag{5.1c}$$

For the cubic and higher-order nonlinear terms, we will use the fact that

$$1 \leq C\langle t\rangle^{-\frac{3}{2}}\langle r\rangle^2\mathcal{W}_\alpha^{1/2}\mathcal{W}_\beta\,. \tag{5.1d}$$

Finally, since

$$\langle t+r\rangle \leq C\langle r\rangle\mathcal{W}_\alpha \quad \text{for } \alpha = 1, 2,$$

we have that

$$\begin{aligned}\langle t+r\rangle|U| &= \langle t+r\rangle|P_1U + P_2U|\\ &\leq \langle t+r\rangle(|P_1U| + |P_2U|)\\ &\leq C\langle r\rangle(\mathcal{W}_1|P_1U| + \mathcal{W}_2|P_2U|)\end{aligned} \tag{5.1e}$$

for all $U \in \mathbb{M}^3 \times \mathbb{R}^3 \times \mathbb{R}$.

The following technical result will be needed several times below.

LEMMA 5.1 *Suppose that* $U \in H_\Gamma^\kappa(T)$ *with* $\kappa \geq 3$. *Set* $\kappa' = [\frac{\kappa}{2}] + 2$ *(so that* $\kappa' \leq \kappa$*). Suppose that* $E_{\kappa'}[U(t)] < 1$ *and* $|\dot{U}(t)| \leq \delta$, $0 \leq t < T$, *with* δ *sufficiently small. Consider a smooth mapping* $f : \mathbb{M}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^d$ *for any* d. *If* f *vanishes to order* p *at the origin, then we have the pointwise estimate*

$$|\Gamma^b f(\dot{U}(t,x))| \leq C \sum_{|b_1|+\cdots+|b_p|\leq|b|} |\Gamma^{b_1}\dot{U}(t,x)|\cdots|\Gamma^{b_p}\dot{U}(t,x)|\,, \quad |b| \leq \kappa\,.$$

PROOF: Using the chain rule, we write

$$\Gamma^b f(\dot{U})(t,x) = \sum_{j\leq|b|}\sum_{b_1+\cdots+b_j=b} f^{(j)}(\dot{U}(t,x))\Gamma^{b_1}\dot{U}(t,x)\cdots\Gamma^{b_j}\dot{U}(t,x). \tag{5.2}$$

At most one derivative above can exceed order $[\frac{\kappa}{2}]$, since $|b| \leq \kappa$. Since $E_{\kappa'}[U(t)] < 1$ and $\dot{U}$ is small, we have by the Sobolev lemma and (2.19) that

$$|\Gamma^c\dot{U}(t,x)| \leq CE_{\kappa'}^{1/2}[U(t)] \leq C, \quad |c| \leq \left[\frac{\kappa}{2}\right].$$

The result now follows from (5.2) since by the mean value theorem

$$|f^{(j)}(\dot{U})| \leq C|\dot{U}|^{p-j}\,, \quad j \leq p\,,$$

for $|\dot{U}| \leq 1$. □

We are now ready to move to the main results of this section.

LEMMA 5.2 *Let* $U \in H_\Gamma^\mu(T)$, $\mu \geq 3$, *be a solution of the PDEs* (2.20a)–(2.20c) *and the constraints* (2.20d) *and* (2.20e). *Set* $\mu' = [\frac{\mu}{2}] + 2$, *and let* $E_{\mu'}[U(t)] < 1$ *and* $|\dot{U}(t)| < \delta$ *throughout* $[0, T)$ *with* δ *sufficiently small. Then we have*

$$\mathcal{X}_\mu[U(t)] \leq C\big[E_\mu^{1/2}[U(t)] + \mathcal{X}_{\mu'}[U(t)]E_\mu^{1/2}[U(t)] + \mathcal{X}_\mu[U(t)]E_{\mu'}^{1/2}[U(t)]\big]\,.$$

PROOF: Using Proposition 3.1 we have

$$\mathcal{X}_\mu[U(t)] \le C\big[E_\mu^{1/2}[U(t)] + \|\langle t+r\rangle \mathcal{Q}_\mu(\dot{U}(t))\|\big],$$

where the form of $\mathcal{Q}_\mu$ was displayed in (3.2):

$$\sum_{|b|+|c|\le\mu-1} \Gamma^b f(\dot{U}) \nabla \Gamma^c \dot{U}.$$

Here f vanishes to order $p = 1$. Applying Lemma 5.1, we have the pointwise estimate

$$\big|\mathcal{Q}_\mu(\dot{U}(t,x))\big| \le C \sum_{|b|+|c|\le\mu-1} |\Gamma^b \dot{U}(t,x)|\,|\nabla\Gamma^c\dot{U}(t,x)|\,.$$

With this and (5.1e), we obtain

$$\begin{aligned}\|\langle r+t\rangle \mathcal{Q}_\mu(\dot{U})\| &\le C \sum_{|b|+|c|\le\mu-1} \|\langle r+t\rangle |\Gamma^b\dot{U}|\,|\nabla\Gamma^c\dot{U}|\| \\ &\le C \sum_{\substack{\alpha=1,2\\ i=1,2,3\\ |b|+|c|\le\mu-1}} \|\langle r\rangle |\Gamma^b\dot{U}|\,\mathcal{W}_\alpha|P_\alpha\partial_i\Gamma^c\dot{U}|\|\,.\end{aligned}$$

In the sum, either $|b| \le [\frac{\mu}{2}]$ or $|c|+1 \le [\frac{\mu}{2}]$, according to which we estimate as follows:

$$\|\langle r\rangle|\Gamma^b\dot{U}|\,\mathcal{W}_\alpha|P_\alpha\partial_i\Gamma^c\dot{U}|\| \le \\ C\begin{cases}\|\langle r\rangle\mathcal{W}_\alpha P_\alpha\partial_i\Gamma^c\dot{U}\|_\infty\,\|\Gamma^b\dot{U}\| & \text{if } |c|+1\le[\frac{\mu}{2}]\\ \big\|\mathcal{W}_\alpha P_\alpha\partial_i\Gamma^c\dot{U}\big\|\,\|\langle r\rangle\Gamma^b\dot{U}\|_\infty & \text{if } |b|\le[\frac{\mu}{2}]\,.\end{cases}$$

In the first case, using (4.2c) we get the upper bound

$$C\mathcal{X}_{\mu'}[U(t)]E_\mu^{1/2}[U(t)]\,,$$

and in the second case, using (4.2a) we get the upper bound

$$C\mathcal{X}_{\mu}[U(t)]E_{\mu'}^{1/2}[U(t)]\,.$$

These estimates for the nonlinear terms yield the result. □

The next step is to bootstrap the preceding result to bound $\mathcal{X}$ by the energy.

PROPOSITION 5.3 *Let $U \in H_\Gamma^\kappa(T)$, $\kappa \ge 8$, be a solution of* (2.20a)–(2.20c). *If $E_\mu[U(t)] < \varepsilon'$, $\mu = \kappa - 2$, and $|\dot{U}(t)| < \delta$ remain sufficiently small on* $[0,T)$, *then*

$$\mathcal{X}_\mu[U(t)] \le CE_\mu^{1/2}[U(t)]\,, \tag{5.3a}$$

$$\mathcal{X}_\kappa[U(t)] \le CE_\kappa^{1/2}[U(t)]\,. \tag{5.3b}$$

PROOF: Since we have $\mu \geq 6$, it follows that $\mu' = [\frac{\mu}{2}] + 2 \leq \mu$. Thus, since $E_\mu[U(t)] < \varepsilon' < 1$, by Lemma 5.2, we have

$$\mathcal{X}_\mu[U(t)] \leq C\big[E_\mu^{1/2}[U(t)] + \mathcal{X}_\mu[U(t)]E_\mu^{1/2}[U(t)]\big],$$

and so we see that, since $E_\mu^{1/2}[U(t)] < \varepsilon'$, we have for ε' small enough that the bound (5.3a) holds.

Since $\kappa \geq 8$, we have that $\kappa' = [\frac{\kappa}{2}] + 2 \leq \kappa - 2 = \mu$. So again by Lemma 5.2, we may write

$$\mathcal{X}_\kappa[U(t)] \leq C\big[E_\kappa^{1/2}[U(t)] + \mathcal{X}_\mu[U(t)]E_\kappa^{1/2}[U(t)] + \mathcal{X}_\kappa[U(t)]E_\mu^{1/2}[U(t)]\big].$$

If $E_\mu^{1/2}[U(t)] < \varepsilon'$ is small, then this implies that

$$\mathcal{X}_\kappa[U(t)] \leq CE_\kappa^{1/2}[U(t)][1 + \mathcal{X}_\mu[U(t)]].$$

Thus we obtain (5.3b) from this, (5.3a), and the fact that $E_\mu[U(t)]$ is small for $t \in [0, T)$. □

6 Energy Estimates

This section ties everything together.

PROPOSITION 6.1 *Let $U \in H_\Gamma^\kappa(T)$ be a solution of* (2.20a)–(2.20c) *and the constraints* (2.20d) *and* (2.20e). *Suppose that $E_\mu[U(t)] < \varepsilon'$, $\mu = \kappa - 2$, and $|\dot{U}(t)| < \delta$, for $0 \leq t < T$, where ε' and δ are sufficiently small. Then we have the inequalities*

$$\frac{d}{dt}E_\kappa[U(t)] \leq C\langle t\rangle^{-1}E_\mu^{1/2}[U(t)]E_\kappa[U(t)], \tag{6.1a}$$

$$\frac{d}{dt}E_{\kappa-1}[\partial_t U(t)] \leq CE_\kappa^{1/2}[U(t)]E_{\kappa-1}[\partial_t U(t)], \tag{6.1b}$$

$$\frac{d}{dt}E_\mu[U(t)] \leq C[\langle\bar{\lambda}t\rangle^{-1} + \langle t\rangle^{-3/2}]E_\kappa^{1/2}[U(t)]E_\mu[U(t)]. \tag{6.1c}$$

PROOF: The size of $\varepsilon' < 1$ and δ are determined by Proposition 5.3.

Assume that $U(t) \in H_\Gamma^\kappa(T)$ is a local solution of (2.20a)–(2.20c). We will use the so-called generalized energy method. Start by applying the derivative Γ^a, $|a| \leq \kappa$, to the system (2.20a)–(2.20c), according to Proposition 2.7. We then symmetrize the system by multiplying (2.25a) by the tensor $\hat{A}$. This results in

$$\hat{A}_{pj}^{\ell m}(H)\big[\partial_t(\Gamma^a\dot{H})_\ell^p + v\cdot\nabla\Gamma^a\dot{H}_\ell^p + H_i^p\partial_\ell\Gamma^a\dot{v}^i\big] = \hat{Q}_a^H, \tag{6.2a}$$

$$\partial_t(\Gamma^a\dot{v})_i + v\cdot\nabla(\Gamma^a\dot{v})_i + H_i^p\hat{A}_{pj}^{\ell m}(H)\partial_\ell(\Gamma^a\dot{H})_m^j + \lambda\rho\partial_i\Gamma^a\dot{\rho} = \hat{Q}_a^v, \tag{6.2b}$$

$$\partial_t\Gamma^a\dot{\rho} + v\cdot\nabla\Gamma^a\dot{\rho} + \lambda\rho\nabla\cdot\Gamma^a\dot{v} = \hat{Q}_a^\rho. \tag{6.2c}$$

From (2.25a)–(2.25c) we have

$$\hat{Q}_a(\dot{U}) = (\hat{Q}_a^H, \hat{Q}_a^v, \hat{Q}_a^\rho)$$

defined as follows:

$$\text{(6.3a)}\quad \hat{Q}_a^H = -\hat{A}_{pj}^{\ell m}(H) \sum_{\substack{b+c=a \\ c\neq a}} \left[\Gamma^b \dot{v} \cdot \nabla(\Gamma^c \dot{H})_\ell^p + (\Gamma^b \dot{H})_i^p \partial_\ell (\Gamma^c \dot{v})^i\right],$$

$$\text{(6.3b)}\quad \hat{Q}_a^v = -\sum_{\substack{b+c=a \\ c\neq a}} \left\{\Gamma^b \dot{v} \cdot \nabla(\Gamma^c \dot{v})_i + \Gamma^b [\hat{A}(H)H]_{ij}^{\ell m} \partial_\ell (\Gamma^c \dot{H})_m^j + \Gamma^b \dot{\rho} \partial_i \Gamma^c \dot{\rho}\right\},$$

$$\text{(6.3c)}\quad \hat{Q}_a^\rho = -\sum_{\substack{b+c=a \\ c\neq a}} [\Gamma^b \dot{v} \cdot \nabla \Gamma^c \dot{\rho} + \Gamma^b \dot{\rho} \nabla \cdot \Gamma^c \dot{v}].$$

It is important to notice that $\hat{Q}_a(\dot{U})$ will never have more than κ derivatives falling on a single term.

Next we proceed with the energy method by taking the L^2 inner product of (6.2a)–(6.2c) with $\Gamma^a \dot{U}$. Because the system has been symmetrized, after integrating by parts we obtain (using the notation (2.18))

$$\text{(6.4)}\quad \begin{aligned} &\partial_t \int e_U(\Gamma^a \dot{U}) dx - \frac{1}{2} \int \partial_t \hat{A}_{pj}^{\ell m}(H) (\Gamma^a \dot{H})_\ell^p (\Gamma^a \dot{H})_m^j \, dx \\ &- \int \nabla \cdot v e_U(\Gamma^a \dot{U}) dx - \int \partial_k \hat{A}_{pj}^{\ell m}(H) v^k (\Gamma^a \dot{H})_\ell^p (\Gamma^a \dot{H})_m^j \, dx \\ &- \int \partial_\ell \big(H_i^p \hat{A}_{pj}^{\ell m}(H)\big) (\Gamma^a \dot{H})_m^j (\Gamma^a \dot{v})^i \, dx - \int \partial_i \dot{\rho} \Gamma^a \dot{\rho} (\Gamma^a \dot{v}^i) dx = \\ &\qquad\qquad \int \left\langle \hat{Q}_a(\dot{U}), \Gamma^a \dot{U} \right\rangle dx . \end{aligned}$$

Now, in the statement of Theorem 2.5 we do not assume any uniform bounds for time derivatives initially. Therefore we must consider the term in (6.4) that involves ∂_t. By (2.27a) and (2.20a), we have that

$$\begin{aligned} \partial_t \hat{A}_{pj}^{\ell m}(H) &= \hat{B}_{pjk}^{\ell mn}(H) \partial_t \dot{H}_n^k \\ &= -\hat{B}_{pjk}^{\ell mn}(H) \big[v \cdot \nabla H_n^k + H_q^k \partial_n v^q\big]. \end{aligned}$$

We substitute this into (6.4) and sum over $|a| \le \nu$, resulting in the energy identity

$$
\begin{aligned}
&\frac{d}{dt}E_\nu[U(t)] = \\
&\quad \sum_{|a|\le\nu}\Big[-\frac{1}{2}\int \hat{B}^{\ell mn}_{pjk}(H)\big[\dot{v}\cdot\nabla\dot{H}^k_n + H^k_q\partial_n\dot{v}^q\big](\Gamma^a\dot{H})^p_\ell(\Gamma^a\dot{H})^j_m\,dx \\
&\qquad + \int \nabla\cdot\dot{v}\,e_U(\Gamma^a\dot{U}(t))dx + \int \partial_k\hat{A}^{\ell m}_{pj}(H)\dot{v}^k(\Gamma^a\dot{H})^p_\ell(\Gamma^a\dot{H})^j_m\,dx \\
&\qquad + \int \partial_\ell\big(H^p_i\hat{A}^{\ell m}_{pj}(H)\big)(\Gamma^a\dot{H})^j_m(\Gamma^a\dot{v})^i\,dx + \int \partial_i\dot{\rho}\Gamma^a\dot{\rho}(\Gamma^a\dot{v})^i\,dx \\
&\qquad + \int\langle\hat{Q}_a(\dot{U}),\Gamma^a\dot{U}\rangle\,dx\Big].
\end{aligned}
\tag{6.5}
$$

Since $\kappa \ge 8$, notice that we have $[\frac{\kappa}{2}] + 2 \le \mu$. Thanks to our smallness conditions, we may apply (6.5), with $\nu = \kappa$, and Lemma 5.1 to write

$$
\frac{d}{dt}E_\kappa[U(t)] \le C \sum_{\substack{|b|+|c|\le|a|\\ c\ne a\\ |a|\le\kappa}} \||\Gamma^b\dot{U}|\,|\nabla\Gamma^c\dot{U}|\|\,\|\Gamma^a\dot{U}\|\,. \tag{6.6a}
$$

Set $m = [\frac{\kappa+1}{2}]$. Using property (5.1e) for the weights and the Sobolev inequalities (4.2a) and (4.2c), we have the following bound for the norms on the right:

$$
\begin{aligned}
&\||\Gamma^b\dot{U}|\,|\nabla\Gamma^c\dot{U}|\| \\
&\quad\le C\langle t\rangle^{-1}\sum_{\alpha,i}\big\|\langle r\rangle|\Gamma^b\dot{U}|\mathcal{W}_\alpha|P_\alpha\partial_i\Gamma^c\dot{U}|\big\| \\
&\quad\le C\langle t\rangle^{-1}\begin{cases}\sum_{\alpha,i}\|\langle r\rangle\Gamma^b\dot{U}\|_\infty\|\mathcal{W}_\alpha P_\alpha\partial_i\Gamma^c\dot{U}\|, & |b|\le m,\\ \sum_{\alpha,i}\|\Gamma^b\dot{U}\|\,\|\langle r\rangle\mathcal{W}_\alpha P_\alpha\partial_i\Gamma^c\dot{U}\|_\infty, & |c|\le m-1,\end{cases} \\
&\quad\le C\langle t\rangle^{-1}\begin{cases}E^{1/2}_{|b|+2}[U(t)]\mathcal{X}_{|c|+1}[U(t)], & |b|\le m,\\ E^{1/2}_{|b|}[U(t)]\mathcal{X}_{|c|+3}[U(t)], & |c|\le m-1,\end{cases}
\end{aligned}
$$

$$
\le C\langle t\rangle^{-1}\big(E^{1/2}_{m+2}[U(t)]E^{1/2}_\kappa[U(t)] + E^{1/2}_\kappa[U(t)]E^{1/2}_{m+2}[U(t)]\big)\,. \tag{6.6b}
$$

Now $\kappa \ge 8$, so $m + 2 \le \kappa - 2 = \mu$. Therefore, inequality (6.1a) follows from (6.6a) and (6.6b).

The identity (6.5) holds equally for derivatives of the form $\partial_t\Gamma^a$, with $|a| \le \kappa - 1$, and precisely one time derivative will appear in each of the terms Q_a on the right. Retracing the steps leading to (6.6a), we find

$$\frac{d}{dt}E_{\kappa-1}[\partial_t U(t)] \le C \sum_{\substack{|b|+|c|\le|a|\\ |a|\le\kappa-1}} \big\| |\Gamma^b\partial_t\dot U|\,|\Gamma^c\dot U|\big\| \, \big\|\partial_t\Gamma^a\dot U\big\| .$$

The inequality (6.1b) now follows just by interpolation. For this rough estimate, we do not use the decay.

In order to obtain the sharp estimate (6.1c), it is necessary at this stage to separate the quadratic portion of the nonlinear terms in (6.5). Recall that $H = I + \dot H$ and $\mu = \kappa - 2$. Referring to (6.3a)–(6.3c), we use $\nu = \kappa - 2 = \mu$ in (6.5) to obtain

$$\text{(6.7a)}\qquad \frac{d}{dt}E_\mu[U(t)] = \sum_{|a|\le\mu}\Big[\int\big\langle \bar Q_a(\dot U),\Gamma^a\dot U\big\rangle\,dx + \int\big\langle C_a(\dot U),\Gamma^a\dot U\big\rangle\,dx\Big],$$

in which $\bar Q_a(\dot U)$ and $C_a(\dot U)$ represent quadratic and higher-order terms, respectively. The precise form of the quadratic terms in (6.7a) is

$$\text{(6.7b)}\qquad \begin{aligned} &\big\langle \bar Q_a(\dot U),\Gamma^a\dot U\big\rangle = \\ &-\frac12 \hat B^{\ell mn}_{pjk}(I)\partial_n\dot v^k(\Gamma^a\dot H)^p_\ell(\Gamma^a\dot H)^j_m + \hat B^{\ell mn}_{ijk}(I)\partial_\ell\dot H^k_n(\Gamma^a\dot H)^j_m(\Gamma^a\dot v)^i\\ &+\nabla\cdot\dot v\,\frac12\big[\hat A^{\ell m}_{ij}(I)(\Gamma^a\dot H)^i_\ell(\Gamma^a\dot H)^j_m + |\Gamma^a\dot v|^2 + (\Gamma^a\dot\rho)^2\big]\\ &+\hat A^{\ell m}_{pj}(I)\partial_\ell\dot H^p_i(\Gamma^a\dot H)^j_m(\Gamma^a\dot v)^i + \partial_i\dot\rho\,\Gamma^a\dot\rho(\Gamma^a\dot v)^i\\ &-\sum_{\substack{b+c=a\\ c\ne a}}\big\{\hat A^{\ell m}_{pj}(I)\big[\Gamma^b\dot v\cdot\nabla(\Gamma^c\dot H)^p_\ell + (\Gamma^b\dot H)^p_i\partial_\ell(\Gamma^c\dot v)^i\big](\Gamma^a\dot H)^j_m\\ &\qquad + \big[\Gamma^b\dot v\cdot\nabla(\Gamma^c\dot v)^i + \hat A^{\ell m}_{pj}(I)(\Gamma^b\dot H)^p_i\partial_\ell(\Gamma^c\dot H)^j_m\\ &\qquad\quad + \hat B^{\ell mn}_{ijk}(I)(\Gamma^b\dot H)^k_n\partial_\ell(\Gamma^c\dot H)^j_m + \Gamma^b\dot\rho\,\partial_i\Gamma^c\dot\rho\big](\Gamma^a\dot v)^i\\ &\qquad + [\Gamma^b\dot v\cdot\nabla\Gamma^c\dot\rho + \Gamma^b\dot\rho\,\nabla\cdot\Gamma^c\dot v]\Gamma^a\dot\rho\big\}. \end{aligned}$$

But before confronting these crucial terms, let us first examine the highest-order terms in (6.7a). Using Lemma 5.1, we have

$$\begin{aligned}\int\langle C_a(\dot U),\Gamma^a\dot U\rangle\,dx &\le \|C_a(\dot U)\|\,\|\Gamma^a\dot U\|\\ &\le C\sum_{\substack{|b_1|+|b_2|+|b_3|\le|a|\\ |b_3|\ne|a|}} \||\Gamma^{b_1}\dot U|\,|\Gamma^{b_2}\dot U|\,|\nabla\Gamma^{b_3}\dot U|\|\,\|\Gamma^a\dot U\|.\end{aligned}$$

Without loss of generality, assume that $|b_1| \geq |b_2|$. Introducing the weights via (5.1d), we have that

$$
\begin{aligned}
&\| |\Gamma^{b_1}\dot U|\, |\Gamma^{b_2}\dot U|\, |\nabla\Gamma^{b_3}\dot U| \| \\
&\quad \leq C\langle t\rangle^{-\frac{3}{2}} \sum_{\alpha,\beta,i} \big\| \langle r\rangle^2 |\Gamma^{b_1}\dot U|\, \big|\mathcal{W}_\alpha^{1/2} P_\alpha \Gamma^{b_2}\dot U\big|\, \big|W_\beta P_\beta \partial_i \Gamma^{b_3}\dot U\big| \big\| \\
&\quad \leq C\langle t\rangle^{-\frac{3}{2}} \sum_{\alpha,\beta,i} \|\langle r\rangle \Gamma^{b_1}\dot U\|_\infty \big\|\langle r\rangle \mathcal{W}_\alpha^{1/2} P_\alpha \Gamma^{b_2}\dot U\big\|_\infty \big\| W_\beta P_\beta \partial_i \Gamma^{b_3}\dot U\big\| \,.
\end{aligned}
$$

With the aid of (4.2a) and (4.2b), this in turn is bounded by

$$
C\langle t\rangle^{-\frac{3}{2}} E^{1/2}_{|b_1|+2}[U(t)]\big(E^{1/2}_{|b_2|+2}[U(t)] + \mathcal{X}_{|b_2|+2}[U(t)]\big)\mathcal{X}_{|b_3|}[U(t)]\,.
$$

Now $2|b_2| \leq |b_1| + |b_2| \leq |a| \leq \mu$. Thus, $|b_2| + 2 \leq [\frac{\mu}{2}] + 2 \leq \mu$, since $\mu \geq 6$. We also have $|b_1| + 2 \leq \kappa$. Therefore, by the smallness assumption and Proposition 5.3, all of the higher-order terms on the right-hand side of (6.7a) are bounded by

$$
C\langle t\rangle^{-\frac{3}{2}} E_\kappa^{1/2}[U(t)] E_\mu[U(t)]\,,
$$

as required for (6.1c).

It remains to bound the terms in (6.7a) arising from the quadratic part of the nonlinearity. They appear explicitly in (6.7b). These must be grouped carefully in order to exploit the special cancellation properties of the nonlinear terms. In particular, the null condition for the shear waves and the rapid dispersion of the pressure waves enters the argument at this stage.

The most straightforward estimates occur for the terms in (6.7b) containing $\nabla\Gamma^c\dot\rho$ or $\nabla\cdot\Gamma^c\dot v$. Recall that by definition (3.1), the quantities $\|\mathcal{W}_1\nabla\Gamma^c\dot\rho\|$ and $\|\mathcal{W}_1\nabla\cdot\Gamma^c\dot v\|$ are bounded by $\mathcal{X}_\mu[U(t)]$ for $|c| \leq \mu - 1$. In the first case, for example, we have, using (5.1b) and (5.1c),

$$
\begin{aligned}
&\int |\Gamma^b\dot U|\, |\nabla\Gamma^c\dot\rho|\, |\Gamma^a\dot U| dx \\
&\quad \leq \| |\Gamma^b\dot U|\, |\nabla\Gamma^c\dot\rho| \|\, \|\Gamma^a\dot U\| \\
&\quad \leq \sum_\alpha \big\| |P_\alpha\Gamma^b\dot U|\, |\nabla\Gamma^c\dot\rho| \big\|\, \|\Gamma^a\dot U\| \\
&\quad \leq C[\langle\bar\lambda t\rangle^{-1} + \langle t\rangle^{-\frac{3}{2}}] \sum_\alpha \big\|\langle r\rangle \mathcal{W}_\alpha^{1/2} |P_\alpha\Gamma^b\dot U| \mathcal{W}_1 |\nabla\Gamma^c\dot\rho| \big\|\, \|\Gamma^a\dot U\| \\
&\quad \leq C\big[\langle\bar\lambda t\rangle^{-1} + \langle t\rangle^{-\frac{3}{2}}\big] \sum_\alpha \big\|\langle r\rangle \mathcal{W}_\alpha^{1/2} P_\alpha\Gamma^b\dot U\big\|_\infty \|\mathcal{W}_1\nabla\Gamma^c\dot\rho\|\, \|\Gamma^a\dot U\|\,.
\end{aligned}
$$

Keeping in mind that $b+c=a$, $c\neq a$, and $|a|\leq\mu$, we can use (4.2b) to bound the last expression by

$$C[\langle\bar{\lambda}t\rangle^{-1}+\langle t\rangle^{-\frac{3}{2}}]\left[E_\kappa^{1/2}[U(t)]+\mathcal{X}_\kappa[U(t)]\right]\mathcal{X}_\mu[U(t)]E_\mu^{1/2}[U(t)]\,.$$

By Proposition 5.3, this in turn is bounded by

$$C[\langle\bar{\lambda}t\rangle^{-1}+\langle t\rangle^{-\frac{3}{2}}]E_\kappa^{1/2}[U(t)]E_\mu[U(t)]\,,$$

as sought. Terms with $\nabla\cdot\Gamma^c\dot{v}$ are handled in the same way.

Next, we consider terms in (6.7b) containing a convective derivative, $\Gamma^b v\cdot\nabla$. We start out by writing

$$\int|\Gamma^b v\cdot\nabla\Gamma^c U|\,|\Gamma^a U|dx\leq\sum_\alpha\left\|P_\alpha\Gamma^b v\cdot\nabla\Gamma^c U\right\|\,\left\|\Gamma^a U\right\|. \tag{6.8a}$$

If $\alpha=1$, then similarly to the previous case, we have

$$\begin{aligned}&\left\|P_1\Gamma^b v\cdot\nabla\Gamma^c U\right\|\\&\quad\leq C[\langle\bar{\lambda}t\rangle^{-1}+\langle t\rangle^{-\frac{3}{2}}]\sum_{\beta,i}\left\|\langle r\rangle\mathcal{W}_1^{1/2}\left|P_1\Gamma^b v\right|\mathcal{W}_\beta\left|P_\beta\partial_i\Gamma^c U\right|\right\|\\&\quad\leq C[\langle\bar{\lambda}t\rangle^{-1}+\langle t\rangle^{-\frac{3}{2}}]\left[E_\kappa^{1/2}[U(t)]+\mathcal{X}_\kappa[U(t)]\right]\mathcal{X}_\mu[U(t)]\,.\end{aligned}\tag{6.8b}$$

On the other hand, when $\alpha=2$ in (6.8a), we partition the domain of integration into two components: $\mathcal{R}\equiv\{r\leq\langle\frac{\beta t}{2}\rangle\}$ and its complement. Since $\mathcal{R}\subset(\mathcal{C}_1\cup\mathcal{C}_2)^c$, up to a compact set, we have by (5.1a)

$$\begin{aligned}&\left\|P_2\Gamma^b v\cdot\nabla\Gamma^c U\right\|_{L^2(\mathcal{R})}\\&\quad\leq C\langle t\rangle^{-\frac{3}{2}}\sum_{\beta,i}\left\|\langle r\rangle\mathcal{W}_2^{1/2}\left|P_2\Gamma^b v\right|\mathcal{W}_\beta\left|P_\beta\partial_i\Gamma^c U\right|\right\|_{L^2(\mathcal{R})}\\&\quad\leq C\langle t\rangle^{-\frac{3}{2}}\left[E_\kappa^{1/2}[U(t)]+\mathcal{X}_\kappa[U(t)]\right]\mathcal{X}_\mu[U(t)]\,.\end{aligned}\tag{6.8c}$$

For the exterior region $\mathcal{R}^c$, we use the formula (2.17) to obtain

$$P_2\Gamma^b v\cdot\nabla=-r^{-1}P_2\Gamma^b v\cdot(w\wedge\Omega)\,,$$

and so by (4.2b) we have

$$\begin{aligned}\left\|P_2\Gamma^b v\cdot\nabla\Gamma^c U\right\|_{L^2(\mathcal{R}^c)}&\leq\left\|r^{-1}\left|P_2\Gamma^b v\right|\left|\Gamma^{c+1}U\right|\right\|_{L^2(\mathcal{R}^c)}\\&\leq C\langle t\rangle^{-2}\left\|\langle r\rangle\left|P_2\Gamma^b v\right|\left|\Gamma^{c+1}U\right|\right\|_{L^2(\mathcal{R}^c)}\\&\leq C\langle t\rangle^{-2}\left\|\langle r\rangle P_2\Gamma^b v\right\|_\infty\left\|\Gamma^{c+1}U\right\|\\&\leq C\langle t\rangle^{-2}\left[E_\kappa^{1/2}[U(t)]+\mathcal{X}_\kappa[U(t)]\right]E_\mu^{1/2}[U(t)]\,.\end{aligned}\tag{6.8d}$$

Together with Proposition 5.3, (6.8a)–(6.8d) gives the estimates for the terms with convective derivatives.

The remaining terms in (6.7b) all have the form

$$\int \mathcal{B}_{ijk}^{\ell mn}(\Gamma^b\dot{H})_m^j\partial_\ell(\Gamma^c\dot{H})_n^k(\Gamma^a\dot{v})^i\,dx \tag{6.9}$$

or

$$\int \mathcal{B}_{ijk}^{\ell mn}(\Gamma^a\dot{H})_m^j(\Gamma^b\dot{H})_n^k\partial_\ell(\Gamma^c\dot{v})^i\,dx\,,$$

in which $\mathcal{B}_{ijk}^{\ell mn}$ is either $\hat{B}(I)_{ijk}^{\ell mn}$, $\tilde{B}_{ijk}^{\ell mn}$, or $\tilde{B}_{ikj}^{\ell nm}$, as defined in (2.27a) and (2.27b). Thus, the coefficients $\mathcal{B}$ satisfy the null condition for shear waves (2.26b). As usual, the derivatives are constrained by the relations $b+c=a$, $c\neq a$, $|a|\leq\mu$. Both types of terms can be handled in the same manner, and so we will outline the procedure only for the first group (6.9).

Begin by writing (6.9) as

$$\sum_{\alpha,\beta,\gamma}\int \mathcal{B}_{ijk}^{\ell mn}\big(P_\alpha\Gamma^b\dot{H}\big)_m^j\partial_\ell\big(P_\beta\Gamma^c\dot{H}\big)_n^k\big(P_\gamma\Gamma^a\dot{v}\big)^i\,dx\,.$$

For those interactions involving at least one fast wave, we can proceed as above. For example, if $\beta=1$, then we can absorb the weight $\langle r\rangle\mathcal{W}_\alpha^{1/2}\mathcal{W}_1$. Otherwise, if $\alpha=1$ or $\gamma=1$, then the weight $\langle r\rangle\mathcal{W}_1^{1/2}\mathcal{W}_\beta$ is used. This results in the same decay as before.

Thus, we may restrict ourselves to the case of shear wave interactions $(\alpha,\beta,\gamma)=(2,2,2)$. We can also eliminate the region $\mathcal{R}$ using (5.1b), as was done for the terms with convective derivatives. Thus, we are faced with estimating

$$\int_{\mathcal{R}^c}\mathcal{B}_{ijk}^{\ell mn}(P_2)_I^i(P_2)_J^j(P_2)_K^k(\Gamma^b\dot{H})_m^J\partial_\ell(\Gamma^c\dot{H})_n^K(\Gamma^a\dot{v})^I\,dx. \tag{6.10}$$

We can further introduce projections in the remaining indices,

$$\mathcal{B}_{ijk}^{\ell mn}(P_2)_I^i(P_2)_J^j(P_2)_K^k=\sum_{\alpha,\beta,\gamma}\mathcal{B}_{ijk}^{\ell mn}(P_\alpha)_\ell^L(P_\beta)_m^M(P_\gamma)_n^N(P_2)_I^i(P_2)_J^j(P_2)_K^k\,.$$

Thanks to the null condition (2.26b), we can rule out $(\alpha,\beta,\gamma)=(1,1,1)$ in the sum, and so we need only consider the three possibilities that α, β, or γ is equal to 2.

Now if $\alpha=2$, then we use (2.17) to write $(P_2)_\ell^L\partial_L=-r^{-1}(\omega\wedge\Omega)_\ell$. Thus, this piece of our integral (6.10) is controlled by

$$\int_{\mathcal{R}^c}r^{-1}|\Gamma^b\dot{H}|\,|\Gamma^{c+1}\dot{H}|\,|\Gamma^a\dot{v}|dx\,.$$

Recall that on $\mathcal{R}^c$ we have $r\geq C\langle t\rangle$. Hence, using (4.2a), we find the upper bound

$$\langle t\rangle^{-2}E_\kappa^{1/2}[U(t)]E_\mu[U(t)]\,.$$

If $\beta=2$, then thanks to the constraint (2.20e), we can use Lemma 4.3 to see that

$$\big\|r^{\frac{3}{2}}(P_2)_m^M(\Gamma^b\dot{H})_M^j\big\|_\infty\leq CE_\kappa^{1/2}[U(t)]\,.$$

Also, when $\gamma = 2$ we get

$$\left\| r^{\frac{3}{2}} (P_2)_n^N \partial_\ell (\Gamma^c \dot{H})_N^k \right\|_\infty \le C E_\kappa^{1/2}[U(t)] \,.$$

In either case, this again leads to the bound

$$\langle t \rangle^{-\frac{3}{2}} E_\kappa^{1/2}[U(t)] E_\mu[U(t)]$$

for the remainder of (6.10). □

7 Long-Time Existence: Proof of Theorem 2.5

Under our assumptions, the construction of a local solution of the symmetric system in $H_\Gamma^\kappa(T)$ is routine. We therefore concentrate on the extension of this solution for large times via the priori estimates for the norm in $H_\Gamma^\kappa(T)$. We remark that it is enough to control the quantity $\sum_{|a| \le \kappa} \|\Gamma^a \dot{U}(t)\|$, for then by using the equations, the time derivatives $\sum_{j+|a| \le \kappa} \|\partial_t^j \Gamma^a \dot{U}(t)\|$ can be bounded inductively in j.

Recall that as long as $|\dot{U}(t)| < \delta$, we have that $\sum_{|a| \le \nu} \|\Gamma^a \dot{U}(t)\| \sim E_\nu^{1/2}[U(t)]$ for all $\nu \le \kappa$. On the other hand, since $\mu > 2$, the size of $|\dot{U}(t)|$ is controlled by $\sum_{|a| \le \mu} \|\Gamma^a \dot{U}(t)\|$. Thus, we ensure that

$$\sum_{|a| \le \kappa} \|\Gamma^a \dot{U}(t)\| < \infty \quad \text{and} \quad |\dot{U}(t)| < \delta$$

by establishing that

$$E_\kappa^{1/2}[U(t)] < \infty \quad \text{and} \quad E_\mu^{1/2}[U(t)] < \varepsilon'$$

for ε' sufficiently small. We also assume that ε' is small enough to apply Proposition 6.1.

Fix a large constant K, to be defined precisely below, and assume that (2.21) holds with $K\varepsilon < \varepsilon'$. Now, let us suppose we have a solution in $H_\Gamma^\kappa(T)$ with

$$E_\mu^{1/2}[U(t)] < K\varepsilon < \varepsilon' \quad \text{for } t \in [0, T) \,.$$

Using (6.1a), we see that

$$E_\kappa[U(t)] \le E_\kappa[U(0)] \langle t \rangle^{CK\varepsilon} \,.$$

Plugging this into (6.1c), we have

$$\frac{d}{dt} E_\mu[U(t)] \le C E_\kappa^{1/2}[U(0)] [\langle \bar{\lambda} t \rangle^{-1} + \langle t \rangle^{-\frac{3}{2}}] \langle t \rangle^{CK\varepsilon} E_\mu[U(t)] \,.$$

Integrating from 0 to τ we have

$$E_\mu(U(\tau)) \le E_\mu[U(0)] \exp\big[C E_\kappa^{1/2}[U(0)](I_1 + I_2)\big] \,, \tag{7.1}$$

where

$$I_1 = \int_0^\tau \langle \bar{\lambda} t \rangle^{CK\varepsilon - 1} dt \quad \text{and} \quad I_2 = \int_0^\infty \langle t \rangle^{CK\varepsilon - \frac{3}{2}} \, dt \,.$$

It is clear that I_2 is is bounded by a fixed constant as long as we further restrict ε' so that $CK\varepsilon < C\varepsilon' < \frac{1}{2}$. Now with this restriction on ε', we have that

$$I_1 < \int_0^\tau \langle \bar{\lambda} t \rangle^{-\frac{1}{2}} \, dt < 1 \,,$$

provided that $\bar{\lambda} > 1$ and $\tau < \bar{\lambda}$.

So using (7.1) and (2.21) we have that for $t < \bar{\lambda}$,

$$\begin{aligned} E_\mu^{1/2}[U(t)] &\le E_\mu^{1/2}[U(0)] \exp C E_\kappa^{1/2}[U(0)] \\ &< \varepsilon C \exp C E_\kappa^{1/2}[U(0)] \,. \end{aligned}$$

We define

$$K = C \exp C E_\kappa^{1/2}[U(0)] \,.$$

Thus if $T^\lambda = \lambda$, then the solution exists for $t \in [0, T^\lambda)$ as claimed in Theorem 2.5. Moreover, the preceding establishes the stated estimates (2.22a) and (2.22b) for E_κ and $E_{\kappa-2}$. The bound (2.22c) for the first time derivative now follows from (2.22b), (6.1b), and Gronwall's inequality. Higher-order time derivatives can be estimated successively. They remain finite but are not small.

8 Incompressible Limit: Proof of Theorem 2.6

Let $U^\lambda \in H_\Gamma^\kappa(T^\lambda)$ be the solution family constructed in Theorem 2.5. Recall that $T^\lambda \ge \lambda$ and by (2.22a) and (2.22b)

$$E_\kappa[U^\lambda(t)] \le C \langle t \rangle^p \quad \text{and} \quad E_{\kappa-2}[U^\lambda(t)] \le C \,, \quad 0 \le t < T^\lambda \,, \tag{8.1}$$

where C and p are independent of λ. Fix $T > 0$. Then for all $\lambda > T$, we have $T^\lambda > T$ and

$$E_\kappa[U^\lambda(t)] \le C(T) \,, \quad 0 \le t \le T \,,$$

with $C(T)$ independent of λ. Hence, by (2.19), we have

$$\sum_{|a| \le \kappa} \| \Gamma^a \dot{U}^\lambda(t) \| \le C(T) \,, \quad 0 \le t \le T \,,$$

provided that $\lambda \ge T$. In particular, $\dot{U}^\lambda$, $\lambda \ge T$, is uniformly bounded in the standard Sobolev class $L^\infty([0,T], W^{\kappa,2})$, and we can extract a subsequence $\lambda_k \nearrow \infty$, with $\lambda_k \ge k$, so that $\dot{U}^{\lambda_k} \to \dot{U}^\infty$ weak-star in $L^\infty([0,T], W^{\kappa,2})$.

Taking the sequence of times $T_k = k$ and using a diagonalization argument, we obtain a subsequence, also written as $\dot{U}^{\lambda_k}$, $\lambda_k \ge k$, converging to a limit function $\dot{U}^\infty = (\dot{H}^\infty, \dot{v}^\infty, \dot{\rho}^\infty)$ defined on $(0,\infty) \times \mathbb{R}^3$ that lies in $L^\infty([0,T], W^{\kappa,2})$ for every $T > 0$.

Recall the definition $U^\lambda = (H^\lambda, v^\lambda, \rho^\lambda) = (I + \dot{H}^\lambda, \dot{v}^\lambda, 1 + \lambda^{-1}\dot{\rho}^\lambda)$. Since $\rho^\lambda \to \dot{\rho}^\infty$, we have $\rho^\lambda - 1 = \lambda^{-1}\dot{\rho}^\lambda \to 0$. Thus, if we define $U^\infty = (I + \dot{H}^\infty, \dot{v}^\infty, 1)$, then $U^\lambda - U^\infty \to 0$.

In addition, we have $\Gamma^a \dot{U}^{\lambda_k} \to \dot{\Gamma}^a \dot{U}^\infty$ for all $|a| \le \kappa$ in the sense of distributions. However, thanks to the estimates (8.1), it follows that $U^\infty \in H_\Gamma^\kappa(T)$ for every $T > 0$. Moreover, the norm of U^∞ in $H_\Gamma^{\kappa-2}(T)$ is uniformly bounded with respect to T.

Fix a positive integer $\ell > 0$. Since

$$\|\Gamma^a \dot{U}^{\lambda_k}(t)\| \le C_\ell \quad \text{for } 0 \le t \le \ell \text{ and } |a| \le \kappa\,,$$

we have by the Sobolev lemma that

$$\|\Gamma^a \dot{U}^{\lambda_k}(t)\|_\infty \le C_\ell \quad \text{for } 0 \le t \le \ell \text{ and } |a| \le \kappa - 2.$$

In particular, we have

$$\|\nabla^a S^j \dot{U}^{\lambda_k}(t)\|_\infty \le C_\ell \quad \text{for } 0 \le t \le \ell \text{ and } |a| + j \le \kappa - 2.$$

Now let $\mathcal{R}_\ell$ be the compact space-time domain $[\ell^{-1}, \ell] \times B_\ell(0)$. Then since $S = t\partial_t + x \cdot \nabla$, we get that

$$\left|\nabla^a \partial_t^j \dot{U}^{\lambda_k}(t, x)\right| \le C_\ell \quad \text{for } (t, x) \in \mathcal{R}_\ell \text{ and } |a| + j \le \kappa - 2\,.$$

Thus, the derivatives $\nabla^a \partial_t^j \dot{U}^{\lambda_k}$, $|a| + j \le \kappa - 3$, are bounded and Lipschitz on $\mathcal{R}_\ell$. By the Arzela-Ascoli theorem, there is a further subsequence $\dot{U}^{\lambda_k}$ converging to $\dot{U}^\infty$ in $C^{\kappa-3}(\mathcal{R}_\ell)$. By another diagonalization argument, we obtain a subsequence $\dot{U}^{\lambda_k} \to \dot{U}^\infty$ in $C^{\kappa-3}_{\text{loc}}((0, \infty) \times \mathbb{R}^3)$, and then finally, we also have $U^{\lambda_k} \to U^\infty$ in $C^{\kappa-3}_{\text{loc}}((0, \infty) \times \mathbb{R}^3)$. Notice that since $\rho^\lambda - 1 \to 0$, we have that $\det H^\infty = 1$.

Armed with locally uniform convergence, we can now pass to the limit in the PDEs (2.20a)$_\lambda$–(2.20c)$_\lambda$. From (2.20a), we immediately see that H^∞ and v^∞ solve (2.23a). Substituting $\rho^\lambda = 1 + \lambda^{-1}\dot{\rho}^\lambda$ into (2.20c) and then taking the limit as $\lambda_k \to \infty$, we find that v^∞ satisfies the incompressibility condition (2.23c). Since the nonsingular terms in (2.20b) converge in $C^{\kappa-3}_{\text{loc}}$, we must have that

$$\lambda_k^2 \rho^{\lambda_k} \nabla \rho^{\lambda_k} = \lambda_k \nabla \dot{\rho}^{\lambda_k} + \dot{\rho}^{\lambda_k} \nabla \dot{\rho}^{\lambda_k} \longrightarrow \mathcal{L}^\infty$$

for some $\mathcal{L}^\infty \in C^{\kappa-3}$. Clearly, $\mathcal{L}^\infty$ must be the gradient of some function q^∞. This shows that $\nabla \dot{\rho}^\infty = 0$, but since $\dot{\rho}^\infty(t, \cdot) \in L^2$, we must have $\dot{\rho}^\infty = 0$. Thus, in the incompressible limit, we obtain a classical solution of (2.23a)–(2.23e) on $(0, \infty) \times \mathbb{R}^3$, with generalized derivatives to order κ satisfying the previously stated bounds. This completes the proof of part (i) of the theorem.

To prove (ii), assume that the initial data is independent of λ. Apply π, the L^2 projection onto divergence-free vectors, to (2.20b). This eliminates the singular term $\lambda\rho^\lambda \nabla \rho^\lambda$ from this equation. We can estimate the time derivatives $\partial_t H^\lambda$ and $\pi\partial_t v^\lambda$ in $W^{\kappa-1,2}$ directly from (2.20a) and (2.20b) by isolating them on one side of their equations to get

$$\left\|\partial_t H^\lambda(t)\right\|_{W^{\kappa-1,2}} + \left\|\partial_t \pi v^\lambda(t)\right\|_{W^{\kappa-1,2}} \le C(T) \quad \text{for } 0 \le t \le T. \tag{8.2}$$

Here we use the boundedness of π in $W^{\kappa-1,2}$ as well as the energy estimates.

Using the Sobolev lemma again, we see that the derivatives $(\nabla^\alpha \dot{H}^\lambda, \nabla^\alpha \dot{v}^\lambda)$, $|\alpha| \le \kappa - 3$, are locally Lipschitz on $[0, T] \times \mathbb{R}^3$. So after a further refinement of our subsequence, we obtain

$$(H^{\lambda_k}, \pi v^{\lambda_k}) \to (H^\infty, \pi v^\infty) \quad \text{in } C^0_{\text{loc}}([0, T] \times \mathbb{R}^3).$$

Since v^∞ is divergence free, we have that $\pi v^\infty = v^\infty$.

Using the uniform bound (8.2) and weak lower semicontinuity, we have

$$\begin{aligned}
&\|H^\infty(t) - H^\infty(s)\|_{W^{\kappa-1,2}} + \|v^\infty(t) - v^\infty(s)\|_{W^{\kappa-1,2}} \\
&\quad \leq \liminf_{k\to\infty} \|H^{\lambda_k}(t) - H^{\lambda_k}(s)\|_{W^{\kappa-1,2}} + \|v^{\lambda_k}(t) - v^{\lambda_k}(s)\|_{W^{\kappa-1,2}} \\
&\quad \leq C|t-s|
\end{aligned}$$

for all $t, s \in [0,T]$, showing that (H^∞, v^∞) lies in $C([0,T], W^{\kappa-1,2})$. This is a uniqueness class. Thus, (H^∞, v^∞) is the unique solution of (2.23a)–(2.23e) with initial data $(H_0, \pi v_0)$. Given the uniqueness of the limit, it follows that the full sequence U^λ converges to U^∞, proving part (ii).

Under the assumptions in part (iii), we have that $E_{\kappa-1}[\partial_t U(0)]$ is uniformly bounded. From (2.22c), we obtain

$$\left\|\partial_t U^\lambda(t)\right\|_{W^{\kappa-1,2}} \leq C(T) \quad \text{for } 0 \leq t \leq T.$$

This implies that the derivatives $\nabla^\alpha U^\lambda$, $|\alpha| \leq \kappa - 3$, are locally Lipschitz on $[0,T] \times \mathbb{R}^3$. The remainder of the argument is the same as in part (ii).

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