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me I *Integrals of Cauchy type on the ball*

me II *Lectures on nonlinear wave equations*

International Press Incorporated, Boston
P.O. Box 2872
Cambridge, MA 02238-2872

Library of Congress Catalog Card Number: 95-082049

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Lectures on Nonlinear Wave Equations

ISBN 1-57146-032-2

Typeset using Latex
Printed on acid free paper, in the United States of America

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PREFACE

These notes are based on a course I gave at UCLA in the fall of 1994. I tried to make the course self-contained, presenting as background basic facts about the solution of the linear wave equation as well as the basic tools from harmonic analysis that were used. The heart of the course concerned three types of problems in the theory of nonlinear wave equations, that, to varying degrees, have non-trivial overlap with analysis.

I first presented results concerning existence for certain quasilinear wave equations, usually with small Cauchy data. The global results relied on energy estimates, Sobolev's theorem, as well as Klainerman's generalized Sobolev inequalities which make use of vector fields preserving the equation $\Box u = 0$. As a preview of things to come, we also presented a recent low-regularity local existence theorem of Klainerman and Machedon which is based on a variation of Strichartz's restriction theorem for the light cone.

The next topic we covered involved various results concerning semilinear wave equations with small data. The first one presented was a remarkable theorem of John which says that in $\mathbb{R}^{1+3}_+$ the equation $\Box u = |u|^\kappa$ always has a global solution for small smooth compactly data if $\kappa > 1 + \sqrt{2}$, while, conversely, if $\kappa < 1 + \sqrt{2}$ there can be blow-up even for arbitrarily small data. We followed John's argument for the blow-up part of the theorem, but for the positive part we used a somewhat different argument which relies on the Hardy-Littlewood maximal theorem. After this, we presented some local and global existence theorems involving sharp regularity assumptions on the data. The proof uses mixed-norm variants of the Strichartz estimate mentioned before.

The last topic covered involves global existence results for arbitrary smooth data for equations of the form $\Box u = -|u|^{\kappa-1}u$ in $\mathbb{R}^{1+3}_+$. The sign of the nonlinearity is easily seen to be crucial. Using the earlier mixed-norm estimates we can prove a classical theorem of Jörgens saying that there is global existence for the "subcritical" range of $\kappa < 5$. This argument also gives a result of Rauch saying that there is global existence for the "critical" case $\kappa = 5$ if the data has small energy. Removing this assumption for this case is delicate, and we shall do this using an argument of Struwe based on a Morawetz-Pohožaev identity. This general result for the critical wave equation in $\mathbb{R}_+^{1+3}$ is due to Grillakis.

As I pointed out before, my main goal in preparing these informal notes was to try to provide the students with a self-contained treatment of certain problems in nonlinear wave equations. Because of focusing on this (and my ignorance), I am afraid that my treatment of historical background may be at best inadequate. For this reason, I refer the reader to the excellent notes of Hörmander [5], John [8] and Strauss [4]. These works also supplement mine since they were used while preparing the course. For background concerning the literature about generalized Strichartz inequalities the reader should consult the excellent survey article of Ginibre and Velo [4].

Finally, it is a great pleasure to thank the many people who have helped me in this endeavor. Most of all, I would like to thank everyone who participated in the course and offered many useful comments and criticisms, including A. Chang, I. Laba, G. Simonett and W. Wang. I also would like to thank S. Klainerman for his suggestions, and H. Lindblad, M. Machedon and H. Smith for going through portions of the notes. I am especially grateful to S. Cuccagna for thoroughly reading through the entire manuscript. Lastly, I would like to thank D.H. Phong for encouraging me to undertake this project.

This work was prepared using $\mathcal{AMS}$-TEX and was supported in part by the NSF.

Los Angeles C. D. Sogge

CHAPTER I

BACKGROUND AND GROUNDWORK

In this chapter we present much of the background that will be needed for the course. In the first section we go over the solution of the linear Cauchy problem for the d'Alembertian. Treating the most important case of three spatial dimensions first, we derive a classical formula which expresses the solution in terms of spherical means involving the data. This explicit formula will be used many times. Using it we can read off domain of dependence and decay properties of the solution, both of which will play an important role in most of the nonlinear problems we shall consider. In §1 we also review the Fourier integral representation of the solution and the concept of weak solutions. In §2 we go over basic energy estimates for the d'Alembertian and linear perturbations. We then apply these estimates in the next section to solve linear equations with data in a given L^2-Sobolev space. The rest of the chapter is devoted to proving local existence theorems that will be used throughout. We first treat the quasilinear equations we shall study and then turn to semilinear equations in three spatial dimensions. In both cases, we shall see that, for sufficiently smooth compactly supported data, there is either a global solution or else one of finitely many derivatives, depending on the dimension, must blow up pointwise. Thus, later in the course, when we wish to show that a given equation has a smooth global solution, we need only show that, if u is a local solution in, say, $[0, T_*) \times \mathbb{R}^n$, then, for some fixed N, $\sum_{|\alpha| \le N} |\partial^\alpha u(t, x)| \in L^\infty([0, T_*) \times \mathbb{R}^n)$. In the case of semilinear equations in $\mathbb{R}^{1+3}$, we shall see that we need only show that u is bounded in such a strip.

§1. Linear wave equation: a review

This course will concern various perturbations of the linear Cauchy problem

$$\begin{cases} \Box u(t, x) = 0 \\ u(0, x) = f(x), \quad \partial_t u(0, x) = g(x). \end{cases} \tag{1.1}$$

Here $(t, x) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{1+n}$ and

$$\Box = \partial_t^2 - \sum_{j=1}^n \partial_j^2 = \partial_t^2 - \Delta_x$$

denotes the d'Alembertian. We are using the notation that $\partial_j = \partial/\partial x_j$ and $\partial_t = \partial/\partial t$. At times we shall call $t = x_0$, in which case $\partial_t = \partial_0$.

Before turning to nonlinear problems it is of course important to understand the solution of the linear equation (1.1). Let us start by reviewing what happens in the $(1+3)$-dimensional case since this is the simplest one. For related reasons many of the most important nonlinear results to follow take place in the physically significant case where $n = 3$.

As we shall see later, the quickest way of solving (1.1) involves the use of the Fourier transform. However, for many nonlinear applications it can be useful to have a direct formula for the fundamental solution. In the $(1+3)$-dimensional case this involves the spherical means of a function $h(x)$, $x \in \mathbb{R}^3$:

$$(A_r h)(x) = \frac{1}{4\pi} \int_{S^2} h(x + ry)\, d\sigma(y)\,, \tag{1.2}$$

with $d\sigma(y)$ denoting Lebesgue measure on the unit sphere $S^2 \subset \mathbb{R}^3$, whose area, we recall, is 4π.

Using the divergence theorem we get

$$\begin{aligned}\partial_r (A_r h)(x) &= \frac{1}{4\pi} \int_{S^2} \langle \nabla_x h(x + ry), \, y \rangle \, d\sigma(y) \\ &= \frac{r}{4\pi} \int_{|y|<1} \Delta_x h(x + ry)\, dy \\ &= \frac{r^{-2}}{4\pi} \Delta_x \int_{|x-y|<r} h(y)\, dy\,.\end{aligned}$$

If we use polar coordinates we can rewrite the last integral

$$\frac{1}{4\pi} \int_{|y-x|<r} h(y)\, dy = \int_0^r \rho^2 A_\rho h(x)\, d\rho\,,$$

and so

$$\partial_r (A_r h(x)) = r^{-2} \Delta_x \int_0^r \rho^2 A_\rho h(x)\, d\rho\,.$$

From this we get

$$\frac{\partial}{\partial r} \big(r^2 \frac{\partial}{\partial r} A_r h(x) \big) = \Delta_x r^2 A_r h(x)\,,$$

or, equivalently, we find that $H(r, x) = A_r h(x)$ solves Darboux's equation

$$\big(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \big) H(r, x) = \Delta_x H(r, x)\,. \tag{1.3}$$

Notice that $r \to A_r h(x)$ is even, and so the initial values are

$$H(0,x) = h(x)\,, \quad \partial_r H(0,x) = 0\,. \tag{1.4}$$

Next, let us suppose that $u(t,x)$ is C^2 and that it solves (1.1) in $\mathbb{R}^{1+3}$. Let us set

$$U(r;t,x) = \big(\,A_r u(t,\,\cdot\,)\,\big)(x) = \frac{1}{4\pi}\int_{S^2} u(t, x+ry)\,d\sigma(y)\,.$$

Then, by (1.3),

$$\Delta_x U = \Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\Big)U = r^{-1}\frac{\partial^2}{\partial r^2}(rU)\,.$$

Also, since $\partial_t^2 u(t,x) = \Delta_x u(t,x)$

$$\begin{aligned}\Delta_x U &= \frac{1}{4\pi}\int_{S^2} \Delta_x u(t,x+ry)\,d\sigma(y)\\ &= \frac{1}{4\pi}\frac{\partial^2}{\partial t^2}\int_{S^2} u(t,x+ry)\,d\sigma(y) = \frac{\partial^2}{\partial t^2}U\,.\end{aligned}$$

Thus

$$v(t,r) = rU(r;t,x)$$

solves the $(1+1)$-dimensional wave equation

$$\begin{cases}\partial_t^2 v = \partial_r^2 v\,,\\ v(0,x) = rA_r f(x)\,, \quad \partial_t v(0,x) = rA_r g(x)\,.\end{cases}$$

Thus, if we recall the elementary solution of the $(1+1)$-dimensional wave equation, we get

$$v = \frac{1}{2}\big[(r+t)A_{r+t}f(x) + (r-t)A_{r-t}f(x)\big] + \frac{1}{2}\int_{r-t}^{r+t} \rho A_\rho g(x)\,d\rho\,.$$

Since $A_r f$ and $A_r g$ are even functions of r, and since $v = rU$, we get from this that

$$U = \frac{1}{2r}\big[(t+r)A_{t+r}f(x) - (t-r)A_{t-r}f(x)\big] + \frac{1}{2r}\int_{t-r}^{t+r} \rho A_\rho g(x)\,d\rho\,.$$

But $u(t,x) = U(0;t,x)$, and so after letting $r \to 0$, we finally obtain that

$$\begin{aligned}u(t,x) &= \partial_t(tA_t f(x)) + tA_t g(x)\\ &= \frac{1}{4\pi t^2}\int_{|x-y|=t} \big[tg(y) + f(y) - \langle \nabla_y f(y)\,,\, x-y\rangle\big]\,d\sigma(y)\,.\end{aligned} \tag{1.5}$$

This shows that any C^2 solution of the Cauchy problem in $\mathbb{R}^{1+3}_+ = \{(t,x) \in \mathbb{R}^{1+3} : t \geq 0\}$ must be given by formula (1.5), and therefore must be unique. Conversely, if $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$, then u given by (1.5) solves the Cauchy problem (1.1). Notice also that the *strong Huygen's* principle holds here, by which we mean that, for a given $(t,x) \in \mathbb{R}^{1+3}_+$, $u(t,x)$ just depends on the values of f and g (and derivatives) on the sphere of radius t centered at x.

If $n > 3$ is odd this construction can also be used to derive a formula for the solution of (1.1) depending on the spherical means in n-dimensions:

$$A_r h(x) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} h(x+ty)\, d\sigma(y)\,,$$

where ω_{n-1} denotes the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. If $n = 2k+1$, and if we now let

$$v = \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{k-1} r^{2k-1} A_r u(t,x)$$

then it turns out that $\partial_t^2 v = \partial_r^2 v$. Also, the initial values now are

$$v(0,r) = \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{k-1} r^{2k-1} A_r f(x)\,, \quad \partial_t v(0,r) = \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{k-1} r^{2k-1} A_r g(x)\,.$$

Let us call the first part of the data $\phi(r)$ and the second one $\psi(r)$. Then, if as before we use the fact that the spherical means are even functions of r, we get

$$v(r,t) = \frac{1}{2}\big[\phi(r+t) - \phi(r-t)\big] + \frac{1}{2}\int_{r-t}^{r+t} \psi(s)\, ds\,.$$

One can show that there are constants c_j, with

$$c_0 = 1\cdot 3\cdot 5\cdots(2k-1) = 1\cdot 3\cdot 5\cdots(n-2)$$

so that

$$\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{k-1}\big(r^{2k-1}\phi(r)\big) = \sum_{j=0}^{k-1} c_j r^{j+1} \frac{\partial^j}{\partial r^j}\phi(r)\,.$$

Therefore, the arguments for $n = 3$ yield

$$\begin{aligned} u(t,x) &= \lim_{r\to 0} A_r u(t,x) = \lim_{r\to 0} \frac{1}{c_0 r} v(r,t) \\ &= \frac{1}{c_0}\partial_r \phi|_{r=t} + \frac{1}{c_0}\psi(t)\,. \end{aligned}$$

Combining this with the formulas for ψ and ψ finally gives the following formula for *odd* n:

(1.6) $u(t,x) =$

$$\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\Big[\frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}} t^{n-2}A_t f(x) + \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}} t^{n-2}A_t g(x)\Big].$$

This somewhat involved formula shows that the strong form of Huygen's principle holds in arbitrary odd dimensions. It also shows of course that we get a C^2 solution of (1.1) if $f \in C^{(n+3)/2}(\mathbb{R}^n)$ and $g \in C^{(n+1)/2}(\mathbb{R}^n)$. We also see from (1.6) that this much regularity is needed to ensure that $u \in C^2$.

One can also obtain explicit formulas for even n using Hadamard's method of descent. The idea here is that if u solves the wave equation in $\mathbb{R}^{1+n}$, then it is also a solution of the wave equation in $\mathbb{R}\times\mathbb{R}^{n+1}$ which happens to be independent of the last variable x_{n+1}. Therefore, if n is even, we can obtain a formula for $u(t,x)$ by using the formula for the solution in $\mathbb{R}\times\mathbb{R}^{n+1}$ and then integrating out the redundant variable. Using this we are led to the following formula for *even* n:

(1.7)

$$u(t,x) = \frac{2}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\Big[\frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}} t^{n-1}\int_{|y|<1}\frac{f(x+ty)}{\sqrt{1-|y|^2}}\,dy + \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}} t^{n-1}\int_{|y|<1}\frac{g(x+ty)}{\sqrt{1-|y|^2}}\,dy\Big].$$

To prove this, one uses (1.6) to see that $u(t,x)$ equals

$$\frac{1}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\Big[\frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}} t^{n-1}\int_{|y|^2+y_{n+1}^2=1} f(x+ty)\,d\sigma(y,y_{n+1}) + \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}} t^{n-1}\int_{|y|^2+y_{n+1}^2=1} g(x+ty)\,d\sigma(y,y_{n+1})\Big].$$

This yields (1.7) since if we project the upper and lower hemispheres of S^n onto $|y|<1$, then on each $dy = \sqrt{1-|y|^2}\,d\sigma(y,y_{n+1})$.[1]

Notice that (1.7) implies that for even n one only has a weak form of Huygen's principle. Specifically, here, $u(t,x)$ depends on the values of f and g inside the solid ball of radius t centered at x.

Using (1.6) and (1.7) one can generalize the results stated before for the $(1+3)$-dimensional case and say something about the decay of u for compactly supported data:

[1] Here we have used the fact that if $S \subset \mathbb{R}^{n+1}$ is a graph of the form $S = \{(x,h(x))\}$, then $d\sigma = \sqrt{1+|\nabla h(x)|^2}dx$.

Theorem 1.1. *If $k = 2, 3, \ldots,$*

$$f \in C^{[n/2]+k}(\mathbb{R}^n), \quad \text{and } g \in C^{[n/2]+k-1}(\mathbb{R}^n),$$

then the Cauchy problem (1.1) has a unique solution $u \in C^k(\mathbb{R}^{1+n}_+)$. Also, if f and g are supported in $\{x : |x| < R\}$ and if n is odd then $u(t,x) = 0$, unless $|\,t - |x|\,| < R$, and $u(t,x) = O(\,(1+t)^{-\frac{n-1}{2}}\,)$. For such data and even n, $|x| \le t + R$, in the support of u, and $u(t,x) = O(\,(1+t)^{-\frac{n-1}{2}}\,(1+|\,t-|x|\,|)^{-\frac{n-1}{2}}\,)$.

Here $[n/2]$ denotes the integer part of $n/2$. Also, when we say that $h = O(\alpha)$, we mean that $|h| \le C\alpha$, for some constant C. Everything except the statement about the decay of u for compactly supported data follows easily from (1.6) and (1.7). This we leave as a good exercise for the reader. The proof for odd dimensions is not very hard, while for even n one can prove the result either directly from (1.7), or else from (1.6) via the method of descent. Notice that even though only a weak version of Huygen's principle holds for even n, the decay result somewhat compensates for this, saying that, for data with compact support, u decays more and more rapidly as one goes away from the light cone $\{(t,x) : t = |x|\}$. It turns out that the decay rate stated above is sharp. It will play a very important role in the global and long time existence results for nonlinear wave equations which are to follow. In the next chapter we shall prove decay estimates using another method.

The inhomogeneous wave equation.

Let us now consider the inhomogeneous wave equation with zero Cauchy data at $t = 0$:

$$\begin{cases} \Box w(t,x) = F(t,x)\,, & t > 0 \\ w(0,x) = \partial_t w(0,x) = 0\,. \end{cases} \tag{1.8}$$

If $F \in C^{1+[n/2]}$ we can obtain a C^2 solution of this equation using our solution of (1.1) and *Duhamel's principle.* Specifically, if for a given $s > 0$, $v(s;t,x)$ is the solution of the Cauchy problem

$$\begin{cases} (\partial_t^2 - \Delta_x)v = 0 \\ v(s;0,x) = 0, \quad \partial_t v(s;0,x) = F(s,x)\,, \end{cases}$$

then the solution of (1.8) is given by

$$w(t,x) = \int_0^t v(s; t-s, x)\, ds\,. \tag{1.9}$$

To verify this, we first note that if w is given by (1.9), then $w \in C^2$ by Theorem 1.1. Also, clearly $w(0,x) = 0$. The other boundary condition in (1.8) is also met since

$$\partial_t w(t,x) = v(t;0,x) + \int_0^t \partial_t v(s;t-s,x)\,ds = \int_0^t \partial_t v(s;t-s,x)\,ds\,.$$

Finally, $\Box w = F$, since differentiating the last formula gives

$$\begin{aligned}\partial_t^2 w(t,x) - \Delta_x w(t,x) &= \partial_t v(t;0,x) + \int_0^t (\partial_t^2 - \Delta_x) v(s;t-s,x)\,ds \\ &= \partial_t v(t;0,x) = F(t,x)\,.\end{aligned}$$

Notice that (1.9) implies that a form of Huygen's principle also holds for the inhomogeneous wave equation. It says that for even spatial dimensions $w(t,x)$ depends on the values of F in the solid backward light cone through (t,x):

$$\Lambda_{t,x}^- = \{\,(s,y) : 0 \le s < t\,,\ |x-y| \le t-s\,\}\,,$$

while for odd n a stronger version holds, saying that $w(t,x)$ depends only on the values of F on the boundary of the solid backward light cone, that is,

$$\Gamma_{t,x}^- = \partial\Lambda_{t,x}^- = \{\,(s,y) : 0 \le s < t\,,\ |x-y| = t-s\,\}\,.$$

In the $(1+3)$-dimensional case, one can use the formulas for the solution of the Cauchy problem obtained earlier to get a simple formula for the solution of (1.8) when $n=3$:

$$(1.10)\qquad \begin{aligned} w(t,x) &= \frac{1}{4\pi}\int_0^t\int_{S^2} (t-s)\,F(s,x-(t-s)y)\,d\sigma(y)ds \\ &= \frac{1}{4\pi}\int_0^t \frac{1}{t-s}\int_{|y|=t-s} F(s,x-y)\,d\sigma(y)ds \\ &= \frac{1}{4\pi}\int_{|y|<t} F(t-|y|,x-y)\,\frac{dy}{|y|} \\ &= \frac{1}{8\pi}\int_{\Gamma_{t,x}^-} F(s,y)\,\frac{d\sigma(s,y)}{|(t-s,x-y)|}\,. \end{aligned}$$

Thus, when $n=3$, a fundamental solution for $\Box$ is $E_+ = d\sigma(t,x)/8\pi|(t,x)|$, where $d\sigma$ here denotes Lebesgue measure on the forward light cone through the origin in $\mathbb{R}^{1+3}$. This is called the forward, or advanced, fundamental solution. Although it is not the unique fundamental solution, it is the unique one with this support. See Hörmander [2, chpt. 6].

Notice also that (1.10) implies the following useful comparison theorem, which is valid in 3 spatial dimensions. Namely, *if w_1 and w_2 have vanishing Cauchy data at $t = 0$ and if $\Box w_j = F_j$ then $|w_1| \le w_2$, if $|F_1| \le F_2$.* The reader can verify that this comparison theorem also holds when $n = 2$, but, unfortunately, not when $n > 3$. Formula (1.10) also shows that when $n = 3$, the mapping from F to w sends bounded functions to (locally in time) bounded functions. This too breaks down in higher dimensions, due to the fact that the solution to the Cauchy problem becomes increasingly singular as n increases.

Note that if $F(t, x)$ is spherically symmetric in the spatial variables then so is w. Abusing notation a bit and writing then $w(t, r) = w(t, x)$ if $|x| = r$, we can derive a simpler formula for the solution under the assumption of spherical symmetry when $n = 3$, namely,

$$rw(t,r) = \frac{1}{2}\int_0^t \int_{|r-(t-s)|}^{r+t-s} F(s,\rho)\,\rho d\rho ds\,. \tag{1.11}$$

To prove this, one uses the second equality in (1.10) and the fact that if h is spherically symmetric

$$\int_{|x-y|=t} h(|y|)\,d\sigma(y) = \frac{2\pi t}{r}\int_{|r-t|}^{r+t} h(\rho)\,\rho d\rho\,, \quad r = |x|\,.$$

To prove this, write $y = x + t\omega$, with $\omega = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$, and recall that with this parameterization $d\sigma(\omega) = \sin\theta d\phi d\theta$ on S^2. Thus, assuming as we may that $x = (0, 0, r)$, the left side of our identity is

$$\begin{aligned} t^2\int_{S^2} h(|x+t\omega|)\,d\sigma(\omega) &= 2\pi t^2\int_0^\pi h(\sqrt{t^2+r^2+2rt\cos\theta})\,\sin\theta\,d\theta \\ &= 2\pi t^2\int_{-1}^1 h(\sqrt{t^2+r^2+2rt\lambda})\,d\lambda = \frac{2\pi t}{r}\int_{|r-t|}^{r+t} h(\rho)\,\rho d\rho\,, \end{aligned}$$

where in the last step we used the fact that

$$\rho d\rho = rtd\lambda\,, \quad \text{if } \rho = \sqrt{t^2+r^2+2rt\lambda}\,.$$

When we study semilinear wave equations in $\mathbb{R}^{1+3}$ we shall use the comparison theorem mentioned above to reduce matters to proving estimates under the assumption of spherical symmetry, where we can make use of the simple formula (1.11). A related formula holds in other dimensions as well, but the $(1+3)$-dimensional case is the most straightforward.

Fourier transform.

Let $\hat{f}$ denote the Fourier transform of f:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx\,.$$

Also, let[2]

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : |\partial^\alpha f(x)| \le C_{\alpha,N}(1+|x|)^{-N}\,, \forall \alpha,\, N\}\,.$$

Then recall that Fourier's inversion formula holds for $f \in \mathcal{S}(\mathbb{R}^n)$:

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi\,. \tag{1.12}$$

Using this we can quickly write down a formula for the solution of

$$\begin{cases} \Box u(t,x) = F(t,x)\,,\ t > 0 \\ u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)\,, \end{cases} \tag{1.13}$$

if, say, $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $F \in C^2(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))$. Specifically, u is given by

$$\begin{aligned} u(t,x) = (2\pi)^{-n} &\int_{\mathbb{R}^n} e^{ix\cdot\xi} \cos t|\xi|\, \hat{f}(\xi)\, d\xi \\ &+ (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin t|\xi|}{|\xi|}\, \hat{g}(\xi)\, d\xi \\ &+ (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin(t-s)|\xi|}{|\xi|}\, \hat{F}(s,\xi)\, d\xi ds\,, \end{aligned} \tag{1.14}$$

if $\hat{F}(x,\xi)$ denotes the Fourier transform of $x \to F(s,x)$. We leave the proof of (1.14) as a good exercise. One just uses (1.12) and Duhamel's principle.

Weak solutions.

So far we have just dealt with *classical solutions* of equations like (1.13). By this we mean that $u \in C^2$ so that the derivatives involved make sense pointwise. At times we shall want to deal with weak, or distributional,

[2] Here we are using the short-hand notation that, for a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

solutions of (1.13). To define this notice that if $u \in C^2$ and if (1.13) holds, then, if $\psi(t,x) \in C_0^\infty(\mathbb{R}^{1+n})$, integration by parts gives

$$\begin{aligned}
\iint_{t>0} \psi F\, dtdx &= \iint_{t>0} \psi \Box u\, dtdx = \iint_{t>0} (\psi\partial_t^2 u - \Delta_x \psi u)\, dtdx \\
&= -\iint_{t>0} \partial_t\psi\partial_t u\, dtdx - \int_{\mathbb{R}^n} \psi(0,x)\partial_t u(0,x)\, dx - \iint_{t>0} \Delta_x\psi u\, dtdx \\
&= \iint_{t>0} \partial_t^2\psi u\, dtdx + \int_{\mathbb{R}^n} \partial_t\psi(0,x)u(0,x)\, dx - \int_{\mathbb{R}^n} \psi(0,x)\partial_t u(0,x)\, dx \\
&\qquad - \iint_{t>0} \Delta_x\psi u\, dtdx \\
&= \iint_{t>0} (\partial_t^2 - \Delta_x)\psi u\, dtdx - \int_{\mathbb{R}^n} \psi(0,x)\partial_t u(0,x)\, dx \\
&\qquad + \int_{\mathbb{R}^n} \partial_t\psi(0,x)u(0,x)\, dx\,.
\end{aligned}$$

Hence, we have arrived at the desired conclusion that

$$\begin{aligned}
(1.15)\qquad \iint_{t>0} \psi F\, dtdx &= \iint_{t>0} \Box\psi u\, dtdx \\
&- \int_{\mathbb{R}^n} \psi(0,x)g(x)\, dx + \int_{\mathbb{R}^n} \partial_t\psi(0,x)f(x)\, dx\,,\ \forall\psi \in C_0^\infty(\mathbb{R}^{1+n})\,.
\end{aligned}$$

Let $\mathcal{E}'$ denote the dual of C_0^∞. Then if f and g are in $\mathcal{E}'(\mathbb{R}^n)$ and if u and F are in $\mathcal{E}'(\mathbb{R}^{1+n})$ this equation makes perfect sense. If this happens, we say that u is a weak (or distributional) solution of (1.13).

One can also talk about weak solutions of variable coefficient equations. Assume $T > 0$ and that $a(t,x)$, $b^j(t,x)$ and $g^{jk}(t,x)$ are real and in $C^2([0,T) \times \mathbb{R}^n)$, where $0 \le j,k \le n$ and g^{jk} is symmetric. Let us, for convenience, put $\partial_0 = \partial_t$, and set

$$(1.16)\qquad Lu = \sum_{j,k=0}^{n} g^{jk}(t,x)\partial_j\partial_k u + \sum_{j=0}^{n} b^j(t,x)\partial_j u + a(t,x)u\,.$$

Then suppose that $u \in C^2([0,T) \times \mathbb{R}^n)$ solves

$$(1.17)\qquad \begin{cases} Lu = F\,,\ 0 < t < T \\ u(0,x) = f(x)\,, \quad \partial_0 u(0,x) = g(x)\,. \end{cases}$$

If $L^* = \sum \partial_j \partial_k g^{jk} - \sum \partial_j b^j + a$ denotes the adjoint operator, then integrating by parts as above shows that, for $\psi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$:

$$\iint\limits_{0<t<T} \psi F \, dtdx$$

$$= \iint\limits_{0<t<T} L^* \psi u \, dtdx - \int_{\mathbb{R}^n} \psi(0,x) g^{00}(0,x) \, g(x) \, dx$$

$$+ \int_{\mathbb{R}^n} \Big[\big(\partial_0(g^{00}\psi)\big)(0,x) - (b^0\psi)(0,x) + 2\sum_{j=1}^{n} \big(\partial_j(\psi g^{j0})\big)(0,x) \Big] \, f(x) \, dx \, .$$

Here too we could say that, if f, g, u, and F are distributions as above, then u is a weak solution of (1.17) if the above formula holds for all $\psi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$. One could also of course define the notion of weak solutions involving coefficients which are only appropriate types of distributions.

Often we will have a sequence of functions u_m each of which is a classical solution of equations of the form (1.13) with, say, Cauchy data and right sides $F = F_m$ also depending on m. Then, if $u_m \to u$ in $\mathcal{E}'(\mathbb{R}^{1+n})$, and if we wish to show that the distributional limit is a solution of the same type of equation, we must check that F_m converges to a limit F in $\mathcal{E}'(\mathbb{R}^{1+n})$ and that the Cauchy data $u_m(0, \cdot)$ and $\partial_0 u_m(0, \cdot)$ converge in $\mathcal{E}'(\mathbb{R}^n)$ to say f and g, respectively. If this is the case then it follows of course that u is a weak solution of (1.13).

The same considerations apply to variable coefficient problems as well. Here the coefficients g^{jk} and b^j may also depend on m, in which case we need to check that the coefficients converge appropriately.

One can use such an approximation argument to see that Theorem 1.1 extends to the case $k = 0$ and 1, if one allows weak solutions. The weak solution of (1.1) with data $(f, g) \in C^{[(n-3)/2]+k+1} \times C^{[(n-3)/2]+k}$ will still be given by (1.6) and (1.7), and so one gets time decay as before. The uniqueness part of the theorem for $k = 0, 1$, would require a different proof from the one corresponding to $k \geq 2$; however, it follows from results in §3. Also, if, say, $F \in C(\mathbb{R}_+^{1+3})$, then the weak solution of $\Box w = F$ with zero data is given by (1.10), and (1.11) also remains valid if F is spherically symmetric.

§2. Energy Inequality: a first version

The most basic estimate for the d'Alembertian comes from an application of the identity

$$\partial_0 u \Box u = \tfrac{1}{2}\partial_0 |u'|^2 - \sum_{j=1}^{n} \partial_j(\partial_0 u \partial_j u) \tag{2.1}$$
$$= \operatorname{div}\left(\partial_0 u \, g_0 u' - \tfrac{1}{2} g_0(u', u')\, \mathbf{1}\right).$$

Here $u' = \nabla u = (\partial_0 u, \dots, \partial_n u)$, $\mathbf{1} = (1, 0, \dots, 0)$, and

$$g_0 = g_0^{jk} = \operatorname{diag}(1, -1, \dots, -1)$$

is the Lorentzian matrix. The second formula in (2.1) will be useful when dealing with variable coefficients.

If $u \in C^2$ vanishes for large x we can integrate with respect to these spatial variables to get

$$\partial_0 \|u'(t,\,\cdot\,)\|_{L^2}^2 = \int \partial_0 |u'|^2 \, dx = \int \partial_0 |u'|^2 \, dx - 2\sum_{j=1}^{n} \int \partial_j\left(\partial_0 u \partial_j u\right) dx$$
$$= \int \operatorname{div}\left(|u'|^2, \, -2\partial_0 u \, \partial_x u\right) dx$$
$$= 2\int \partial_0 u \Box u \, dx \le 2\|u'(t,\,\cdot\,)\|_{L^2} \|\Box u(t,\,\cdot\,)\|_{L^2} .$$

But this implies

$$\partial_0 \|u'(t,\,\cdot\,)\|_{L^2} \le \|\Box u(t\,\cdot\,)\|_{L^2} ,$$

and, hence, we obtain the *energy inequality* for the d'Alembertian:

$$\|u'(t,\,\cdot\,)\|_{L^2} \le \|u'(0\,,\cdot\,)\|_{L^2} + \int_0^t \|\Box u(s,\,\cdot\,)\|_{L^2} \, ds . \tag{2.2}$$

If $\Box u$ vanished identically, this inequality would be replaced by an equality, meaning that u would have constant *kinetic energy* $\frac{1}{2}\|u'(t,\,\cdot\,)\|_{L^2}^2$. Thus, if u solved the homogeneous Cauchy problem (1.1), we would have

$$\int |u'(t,x)|^2 \, dx = \int |g(x)|^2 \, dx + \int |(\partial_x f)(x)|^2 \, dx .$$

Handling the L^2 norm of u itself is not as natural and the estimates are much less favorable. For instance, if u solved the Cauchy problem

$$\Box u = 0 , \quad u(0,x) = 0 , \quad \partial_t u(0,x) = g(x) ,$$

then by Plancherel's theorem and (1.14)

$$\int |u(t,x)|^2\,dx = (2\pi)^{-n}\int \big|\,\hat{g}(\xi)\,\sin(t|\xi|)/|\xi|\,\big|^2\,d\xi\,.$$

Based on this one sees that a sharp estimate for the L^2 norm of u must contain constants which blow-up as $t \to \infty$ if the right side involves Sobolev norms of the data:

$$\|u(t,\,\cdot\,)\|_{L^2} \le \|f\|_{L^2} + (1+t^2)^{1/2}\|g\|_{H^{-1}}\,.$$

We could of course also have proven (2.2) using Plancherel's formula and the Fourier transform formula (1.14) for u. However, unlike Fourier integral methods, the energy method is stable under perturbations and requires much less regularity on the coefficients.

To illustrate this, let us see that the above argument applies to variable coefficient operators which are sufficiently close to the d'Alembertian.

Proposition 2.1. *Let $u \in C^2([0,T]\times\mathbb{R}^n)$ vanish for large x and satisfy*

$$\sum g^{jk}(t,x)\partial_j\partial_k u = F, \quad 0 \le t < T\,. \tag{2.3}$$

If $g_0^{jk} = \mathrm{diag}\,(1,-1,\dots,-1)$ are the coefficients of the d'Alembertian set

$$r^{jk}(t,x) = g^{jk}(t,x) - g_0^{jk}\,.$$

Then if

$$\sum |r^{jk}(t,x)| \le \tfrac{1}{2}, \quad 0 \le t < T\,, \tag{2.4}$$

it follows that for $0 < t < T$

$$(2.5)\quad \|u'(t,\,\cdot\,)\|_{L^2}$$
$$\le 2\Big(\|u'(0,\,\cdot\,)\|_{L^2} + \int_0^t \|F(s,\,\cdot\,)\|_{L^2}\,ds\Big)\exp\Big(\int_0^t 2\sum \|\partial_i g^{jk}(s,\,\cdot\,)\|_{L^\infty}\,ds\Big).$$

Proof. Let $\Box_g u$ denote the left side of (2.3). Then we shall need the following variation on (2.1):

$$\partial_0 u \Box_g u = \mathrm{div}\,\Big(\partial_0 u \sum g^{jk}\partial_k u - \tfrac{1}{2}\sum g^{jk}\partial_j u\partial_k u\,\mathbf{1}\Big) - R\,,$$

where

$$R = \sum_{j,k=0}^{n} \partial_j g^{jk} \partial_k u \partial_0 u - \tfrac{1}{2} \sum_{j,k=0}^{n} \partial_0 g^{jk} \partial_j u \partial_k u$$
$$= \tfrac{1}{2}\partial_0 g^{00} (\partial_0 u)^2 + \sum_{j=1}^{n} \sum_{k=0}^{n} \partial_j g^{jk} \partial_j u \partial_0 u - \tfrac{1}{2} \sum_{j=1}^{n} \sum_{k=0}^{n} \partial_0 g^{jk} \partial_j u \partial_k u \,.$$

Let $e(u)$ be the energy density associated with $\Box_g$, that is, the 0-component of the divergence:

$$e(u) = \sum g^{0k} \partial_0 u \partial_k u - \tfrac{1}{2} \sum g^{jk} \partial_j u \partial_k u$$
$$= \tfrac{1}{2}|u'|^2 + \tfrac{1}{2} \sum_{k=0}^{n} r^{0k} \partial_0 u \partial_k u - \tfrac{1}{2} \sum_{j=1}^{n} \sum_{k=0}^{n} r^{jk} \partial_j u \partial_k u \,.$$

Thus (2.4) implies that for fixed t

$$\tfrac{1}{4}|u'|^2 \le e(u) \le |u'|^2 \,. \tag{2.4'}$$

We can use this to estimate R:

$$|R| \le |u'|^2 \sum |\partial_i g^{jk}| \le 4e(u) \sum |\partial_i g^{jk}| \,.$$

Let

$$E(t) = \int e(u)(t,x)\, dx \,.$$

Then

$$\partial_0 E(t) = \int \partial_0 e(u)(t,x)\, dx$$
$$= \int \operatorname{div} \left(\partial_0 u \sum g^{jk} \partial_k u - \tfrac{1}{2} \sum g^{jk} \partial_j u \partial_k u \,\mathbf{1} \right) dx$$
$$= \int \partial_0 u \Box_g u \, dx - \int R\, dx$$
$$= \int \partial_0 u(t,x)\, F(t,x)\, dx - \int R\, dx \,.$$

Consequently,

$$\partial_0 E(t) \le \|F(t,\cdot)\|_{L^2} \|\partial_0 u(t,\cdot)\|_{L^2} + \int |R|\, dx$$
$$\le 2\|F(t,\cdot)\|_{L^2} E(t)^{1/2} + 4 \sum \|\partial_i g^{jk}(t,\cdot)\|_{L^\infty} E(t) \,.$$

This gives

$$\partial_0 E(t)^{1/2} \le \|F(t,\,\cdot\,)\|_{L^2} + 2\sum \|\partial_i g^{jk}(t,\,\cdot\,)\|_{L^\infty} E(t)^{1/2}\,,$$

which in turn implies that

$$\begin{aligned}\partial_0\Big[\, E(t)^{1/2}\exp\Big(-\int_0^t 2\sum \|\partial_i g^{jk}(s,\,\cdot\,)\|_{L^\infty}\, ds\Big)\Big] \\ \le \|F(t,\,\cdot\,)\|_{L^2}\exp\Big(-\int_0^t 2\sum \|\partial_i g^{jk}(s,\,\cdot\,)\|_{L^\infty}\, ds\Big) \le \|F(t,\,\cdot\,)\|_{L^2}\,.\end{aligned}$$

If we integrate, we conclude that

$$\begin{aligned}&E(t)^{1/2} \\ &\le \Big(E(0)^{1/2} + \int_0^t \|F(s,\,\cdot\,)\|_{L^2}\, ds\Big)\exp\Big(\int_0^t 2\sum \|\partial_i g^{jk}(s,\,\cdot\,)\|_{L^\infty}\, ds\Big)\,.\end{aligned}$$

Since this estimate and (2.4′) imply (2.5), we are done. □

Let us end this section by seeing how the proof of (2.2) can be used to prove a useful uniqueness theorem of John [4], [8] for nonlinear equations.

Theorem 2.2. *Let u be a C^2 solution of $\Box u = F(u, u', u'')$ in the backward light cone through (t_0, x_0):*

$$\Lambda^-_{(t_0,x_0)} = \{\,(t,x) \in [0,t_0) \times \mathbb{R}^n : |x - x_0| < t_0 - t\,\}\,.$$

Assume that

$$F(0,0,u'') \equiv 0\,. \tag{2.6}$$

Then u vanishes in $\Lambda^-_{(t_0,x_0)}$ if

$$0 = u(0,x) = \partial_t u(0,x) \quad \text{when } |x - x_0| < t_0\,. \tag{2.7}$$

From this, one immediately sees that a form of Huygen's principle holds for this type of nonlinear equation:

Corollary 2.3. *Let (2.6) hold. Then if $u(t,x)$ is a C^2 solution of $\Box u = F(u,u',u'')$ in $[0,T) \times \mathbb{R}^n$ and if $u(0,x) = \partial_t u(0,x) = 0$ for $|x| > R$, then $u(t,x) = 0$ if $|x| > R + t$ and $0 \le t < T$.*

Proof of Theorem 2.2. We shall use the energy method. To do this we need to write $\Lambda^-_{(t_0,x_0)}$ as the union of a one parameter family of hyperboloids, on each of which we can control the integral of $|u'|^2$ using a variation of the argument used to prove (2.2). In the end, we shall see that u has zero energy in $\Lambda^-_{(t_0,x_0)}$ which implies that it must vanish there because of (2.7).

To be more specific, for $0 \le s < t_0$ let

$$\phi(s,x) = t_0 - \big[\,(t_0-s)^2 + t_0^{-2}(2t_0 s - s^2)|x-x_0|^2\,\big]^{1/2}.$$

Then

$$\phi(0,x) = 0\,, \quad \text{and} \ \lim_{s\to t_0} \phi(s,x) = t_0 - |x-x_0|\,.$$

Hence, if

$$R_s = \{\,(t,x) : \, 0 \le t \le \phi(s,x),\ |x-x_0| < t_0 - t\,\}\,,$$

then

$$\Lambda^-_{(t_0,x_0)} = \bigcup_{0\le s<t_0} R_s\,.$$

Notice also that in $\Lambda^-_{(t_0,x_0)}$

$$|\nabla_x \phi(s,x)| = \frac{t_0^{-2}(2t_0 s - s^2)|x-x_0|}{\big[\,(t_0-s)^2 + t_0^{-2}(2t_0 s - s^2)|x-x_0|^2\,\big]^{1/2}} \le \theta(s_0) < 1\,,$$

if $0 \le s \le s_0 < t_0$.

To use this let

$$\Lambda_s = \{\,(t,x) : \, t = \phi(s,x),\ |x-x_0| < t_0\,\}\,.$$

Note that the outward unit normal at $(\phi(s,x),x) \in \Lambda_s$ is

$$(1, -\nabla_x\phi)/\sqrt{1+|\nabla_x\phi|^2}\,.$$

Thus, by (2.1), (2.7) and the divergence theorem,

$$\begin{aligned} \int_{R_s} 2\partial_t u F\, dtdx &= \int_{R_s} 2\partial_t u\, \Box u\, dtdx \\ &= \int_{\Lambda_s} \big(\,|u'|^2 + 2\partial_t u\, \nabla_x u \cdot \nabla_x \phi\,\big) \frac{d\sigma}{\sqrt{1+|\nabla_x\phi|^2}} \qquad (2.8) \\ &\ge (1-\theta(s_0)) \int_{\Lambda_s} |u'|^2\, d\sigma/\sqrt{1+|\nabla_x\phi|^2}\,, \end{aligned}$$

if $0 \le s \le s_0$.

To get a bound in the other direction, let us first notice that, by (2.7),

$$\int_0^{\phi(s,x)} |u(t,x)|^2\,dt \le t_0^2 \int_0^{\phi(s,x)} |\partial_t u(t,x)|^2\,dt\,.$$

Also, by (2.6),

$$2|\partial_t uF| \le C(\,|u|^2 + |u'|^2\,)\,.$$

Thus,

$$\begin{aligned}\int_{R_s} 2\partial_t uF\,dtdx &\le C(1+t_0^2)\int_{R_s} |u'|^2\,dtdx \\ &= C(1+t_0^2)\int_0^s \int_{\Lambda_t} \partial_0\phi\,|u'|^2 \frac{d\sigma dt}{\sqrt{1+|\nabla_x\phi|^2}}\,.\end{aligned} \tag{2.9}$$

If we set

$$I(s) = \int_{\Lambda_s} |u'|^2\,d\sigma/\sqrt{1+|\nabla_x\phi|^2}\,,$$

then (2.8) and (2.9) imply that for $0 \le s \le s_0 < t_0$

$$(1-\theta(s_0))I(s) \le C(1+t_0^2) \sup_{\substack{0\le t\le s_0 \\ |x-x_0|<t_0}} |\partial_t\phi(t,x)| \int_0^s I(t)\,dt\,.$$

Applying Gronwall's inequality shows that $I(s) = 0$ for all $0 \le s < t_0$. Hence $u' = 0$ in $\Lambda^-_{(t_0,x_0)}$, and since this implies that u must also vanish here, on account of (2.7), we are done. □

§3. Existence and uniqueness for linear equations

Let us now assume that L is as in (1.16), and that all of its coefficients are C^∞ with uniform bounds on each derivative if $(t,x) \in [0,T] \times \mathbb{R}^n$. As before, we shall also assume that (2.4) holds. In a moment we shall see that Proposition 2.1 can be used to prove the following useful result.

Theorem 3.1. *Let $s \in \mathbb{Z}$, $0 < T < \infty$, and assume that L is as above. If*

$$u \in C([0,T];\, H^{s+1}) \cap C^1([0,T];\, H^s)\,,$$

and if

$$Lu \in L^1([0,T]; H^s)\,,$$

then for $0 < t < T$ we have

$$\sum_{|\alpha|\le 1} \|\partial^\alpha u(t,\,\cdot\,)\|_{H^s} \le C_{s,T}\,\Big(\sum_{|\alpha|\le 1} \|\partial^\alpha u(0,\,\cdot\,)\|_{H^s} + \int_0^t \|Lu(\tau,\,\cdot\,)\|_{H^s}\,d\tau\,\Big)\,. \tag{3.1}$$

Here H^s denotes the L^2-Sobolev space with norm

$$\|f\|_{H^s} = \|(I-\Delta)^{s/2} f\|_{L^2} = \left((2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{f}(\xi)(1+|\xi|^2)^{s/2}|^2 \, d\xi\right)^{1/2}.$$

And so, if s is a nonnegative integer there is a constant C_s so that

$$C_s^{-1} \|f\|_{H^s} \leq \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2} \leq C_s \|f\|_{H^s}.$$

Also, if X is a function space, then the notation $F(t,x) \in C^\alpha([0,T];X)$ means that, on $[0,T]$, $F(t,\cdot)$ is a C^α function with values in X.

Let us postpone the straightforward proof for a moment and see now how Theorem 3.1 implies the following useful result for linear hyperbolic equations.

Theorem 3.2. *Let $s \in \mathbb{Z}$. Then for every $f \in H^{s+1}(\mathbb{R}^n)$, $g \in H^s(\mathbb{R}^n)$ and $F \in L^1([0,T];H^s(\mathbb{R}^n))$ there is a unique*

$$u \in C([0,T];\, H^{s+1}) \cap C^1([0,T];\, H^s) \tag{3.2}$$

solving

$$\begin{cases} Lu = F, & 0 < t < T \\ u|_{t=0} = f, & \partial_t u|_{t=0} = g \end{cases} \tag{3.3}$$

Proof of Theorem 3.2. The uniqueness is easy. If u_1 and u_2 are two solutions with the same data, then $u_1 - u_2$ has zero Cauchy data at $t=0$ and $L(u_1 - u_2) = 0$. Therefore, if each satisfies (3.2), we may apply (3.1) to see that $u_1 - u_2 = 0$, giving the result.

For the existence part, let us assume for the moment that the Cauchy data is zero, that is, $f = g = 0$ in (3.3).

If $\psi \in C_0^\infty((-\infty,T) \times \mathbb{R}^n)$ then applying (3.1) to L^*, with t replaced by $T-t$, yields

$$\|\psi(t,\cdot)\|_{H^{-s}} \leq C \int_0^T \|L^*\psi(\tau,\cdot)\|_{H^{-s-1}} \, d\tau.$$

Hence, since H^s and H^{-s} are dual spaces, for fixed $F \in L^1([0,T];H^s)$ we have

$$|\langle F,\psi\rangle| = \left| \int_0^T \langle F(t,\cdot),\psi(t,\cdot)\rangle \, dt \right| \leq C' \int_0^T \|L^*\psi(t,\cdot)\|_{H^{-s-1}} \, dt.$$

So, by the Hahn-Banach Theorem, there is a $u \in L^\infty([0,T]; H^{s+1})$ satisfying $u = 0$ when $t < 0$ and, moreover,

$$\langle F, \psi \rangle = \langle u, L^*\psi \rangle, \quad \forall \psi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n).$$

Consequently, $Lu = F$ in $(0,T) \times \mathbb{R}^n$ in the sense of distributions. We have to check that u satisfies (3.2) and hence has vanishing Cauchy data.

Assume for the moment that $F \in C_0^\infty$. Then if $v = \partial_t u$ (3.3) gives

$$\partial_t v + 2\sum_{j=1}^n g^{j0}(t,x)\partial_j v + b^0 v = -\sum_{j,k\geq 1} g^{jk}(t,x)\partial_j\partial_k u - \sum_{j=1}^n b^j(t,x)\partial_j u$$
$$- a(t,x)u + F(t,x) \in L^\infty([0,T]; H^{s-1}).$$

By elliptic regularity, $\partial_t u = v \in L^\infty([0,T]; H^{s-1})$. Using the equation again we see that $\partial_t^2 u \in L^\infty([0,T]; H^{s-2})$. Thus, $u \in C([0,T]; H^{s-1}) \cap C^1([0,T]; H^{s-2})$. This is weaker than (3.2); however, since we are assuming that $F \in C_0^\infty$ and $f = g = 0$, we may replace s by $s+2$ and consequently obtain a solution satisfying (3.2) to which we may apply (3.1).

To remove the assumption about F, choose a sequence $F_m \in C_0^\infty$ so that if F is as in the theorem

$$\int_0^T \|F(t,\cdot) - F_m(t,\cdot)\|_{H^s}\, dt \to 0.$$

Then if $u_m \in C([0,T]; H^{s+1}) \cap C^1([0,T]; H^s)$ solves $Lu_m = F_m$ with vanishing Cauchy data, it follows from (3.1) that u_m is a Cauchy sequence in $C([0,T]; H^{s+1}) \cap C^1([0,T]; H^s)$ whose limit solves the equation and satisfies (3.2).

To solve the equation with the given Cauchy data, assume first that the data belong to C_0^∞. We then let $u_0(t,x) = f(x) + tg(x)$, so that u_0 has the right Cauchy data. Then, if v solves $Lv = F - Lu_0$ with zero data at $t = 0$, it follows that $u = v + u_0$ will solve (3.3) and satisfy (3.2). We assumed that the data were smooth to ensure that $Lu_0 \in L^1([0,T]; H^s)$. This assumption can be removed, using (3.1), by the above approximation argument. □

We still need to prove Theorem 3.1. We shall require the following result which will be used many times in the future.

Lemma 3.3. (Gronwall's Inequality) *Suppose that A, E and r are bounded nonnegative functions on $[0,T]$ and that E is increasing there. If*

$$A(t) \leq E(t) + \int_0^t r(s)A(s)\, ds, \quad 0 \leq t \leq T,$$

it follows that

$$A(t) \le E(t)\exp\Big(\int_0^t r(s)\,ds \Big). \tag{3.4}$$

Proof of Gronwall's inequality. It suffices to prove the estimate when $t = T$, in which case $E(t)$ may be replaced by the constant $E = E(T) \ge E(t)$. Then, if we set for $0 < t < T$

$$B(t) = E + \int_0^t r(s)A(s)\,ds\,,$$

it follows that

$$B'(t) = r(t)A(t) \le r(t)B(t)\,.$$

Thus,

$$\partial_t \Big(B(t)\exp\Big(-\int_0^t r(s)\,ds \Big)\Big) \le 0\,.$$

This implies that $B(t)\exp\big(-\int_0^t r(s)\,ds \big) \le B(0) = E$, and so

$$A(t) \le B(t) \le E\exp\Big(\int_0^t r(s)\,ds \Big)\,,$$

which finishes the proof. □

To proceed, let us first see that (3.1) holds when $s = 0$. Since the coefficients of L are bounded

$$\| \sum (g^{jk}\partial_j\partial_k u)(\tau,\,\cdot\,)\|_{L^2} \le \|Lu(\tau,\,\cdot\,)\|_{L^2} + C\sum_{|\alpha|\le 1} \|\partial^\alpha u(\tau,\,\cdot\,)\|_{L^2}\,.$$

Hence, (2.5) gives

$$\begin{aligned}
&\|u'(t,\,\cdot\,)\|_{L^2} \\
&\quad \le C_T \Big(\|u'(0,\,\cdot\,)\|_{L^2} + \int_0^t \Big(\|Lu(\tau,\,\cdot\,)\|_{L^2} + \sum_{|\alpha|\le 1} \|\partial^\alpha u(\tau,\,\cdot\,)\|_{L^2} \Big)\,d\tau \Big)\,.
\end{aligned}$$

To estimate the L^2 norm of $u(t,\,\cdot\,)$ itself, we just use the fundamental theorem of calculus and Minkowski's integral inequality to see that

$$\|u(t,\,\cdot\,)\|_{L^2} \le \|u(0,\,\cdot\,)\|_{L^2} + \int_0^t \|\partial_t u(\tau,\,\cdot\,)\|_{L^2}\,d\tau\,.$$

Combining the last two inequalities then leads to

$$\sum_{|\alpha|\le 1} \|\partial^\alpha u(t,\cdot)\|_{L^2}$$
$$\le C_T\Big(\sum_{|\alpha|\le 1} \|\partial^\alpha u(0,\cdot)\|_{L^2} + \int_0^t \Big(\sum_{|\alpha|\le 1} \|\partial^\alpha u(\tau,\cdot)\|_{L^2} + \|Lu(\tau,\cdot)\|_{L^2}\Big)\, d\tau\Big),$$

and hence, by (3.4),

$$\sum_{|\alpha|\le 1} \|\partial^\alpha u(t,\cdot)\|_{L^2} \le C_T'\Big(\sum_{|\alpha|\le 1} \|\partial^\alpha u(0,\cdot)\|_{L^2} + \int_0^t \|Lu(\tau,\cdot)\|_{L^2}\, d\tau\Big). \tag{3.5}$$

This proves Theorem 3.1 when $s = 0$. To handle $s \in \mathbb{N}$, note that, after dividing by g^{00}, we may assume without loss of generality that the coefficient of ∂_t^2 is one. Then

$$L\partial_x^\alpha u = \partial_x^\alpha Lu + [L, \partial_x^\alpha]u,$$

where $[L,\partial_x^\alpha] = L\partial_x^\alpha - \partial_x^\alpha L$ denotes the commutator, which is of order $\le |\alpha|+1$, and, because of our assumption about g^{00}, involves no t-derivatives of order > 1. Therefore, if we apply (3.5) to $\partial_x^\beta u$ with $|\beta| \le s$, we find that

$$\|\partial_x^\beta u(t,\cdot)\|_{L^2} + \|\partial_x^\beta u'(t,\cdot)\|_{L^2}$$
$$\le C\Big(\|\partial_x^\beta u(0,\cdot)\|_{L^2} + \|\partial_x^\beta u'(0,\cdot)\|_{L^2} + \int_0^t \big(\| [L,\partial_x^\beta]u(\tau,\cdot) \|_{L^2} + \|\partial_x^\beta Lu(\tau,\cdot)\|_{L^2}\big)\, d\tau\Big)$$
$$\le C\Big(\|\partial_x^\beta u(0,\cdot)\|_{L^2} + \|\partial_x^\beta u'(0,\cdot)\|_{L^2} + \int_0^t \Big(\sum_{|\alpha|\le s} \big(\|\partial_x^\alpha u(\tau,\cdot)\|_{L^2} + \|\partial_x^\alpha u'(\tau,\cdot)\|_{L^2}\big) + \|\partial_x^\beta Lu(\tau,\cdot)\|_{L^2}\Big)\, d\tau\Big).$$

If we sum over $|\beta| \le s$ and then apply Gronwall's inequality as before, we conclude that for $s = 0, 1, 2, \dots$

$$\sum_{|\alpha|\le s} \big(\|\partial_x^\alpha u(t,\cdot)\|_{L^2} + \|\partial_x^\alpha u'(t,\cdot)\|_{L^2}\big) \le C \sum_{|\alpha|\le s} \Big(\|\partial_x^\alpha u(0,\cdot)\|_{L^2} + \|\partial_x^\alpha u'(0,\cdot)\|_{L^2} + \int_0^t \|\partial_x^\alpha Lu(\tau,\cdot)\|_{L^2}\, d\tau\Big), \tag{3.6}$$

which is equivalent to (3.1) for such s.

To handle the case where $s \in -\mathbb{N}$, we shall apply the last inequality to

$$v(t,\,\cdot\,) = (I-\Delta)^s u(t,\,\cdot\,),$$

where $\Delta = \Delta_x$. First, though, note that

$$\|v(t,\,\cdot\,)\|_{H^{-s}} = \|(I-\Delta)^{s/2}u(t,\,\cdot\,)\|_{L^2} = \|u(t,\,\cdot\,)\|_{H^s}\,. \tag{3.7}$$

Since we are now assuming $-s \in \mathbb{N}$, we may apply (3.6) to obtain

$$\begin{aligned}\sum_{|\alpha|\le -s} \big(\|\partial_x^\alpha v(t,\,\cdot\,)\|_{L^2} + \|\partial_x^\alpha v'(t,\,\cdot\,)\|_{L^2}\big) \\ \le C\sum_{|\alpha|\le -s}\Big(\|\partial_x^\alpha v(0,\,\cdot\,)\|_{L^2} + \|\partial_x^\alpha v'(0,\,\cdot\,)\|_{L^2} + \int_0^t \|\partial_x^\alpha Lv(\tau,\,\cdot\,)\|_{L^2}\,d\tau\Big)\,.\end{aligned} \tag{3.8}$$

Note that

$$(I-\Delta)^{-s}Lv = Lu + [\,(I-\Delta)^{-s},\,L]v\,.$$

Hence multiplication by $(I-\Delta)^{s/2}$ gives that the last term in (3.8) can be estimated as follows:

$$\begin{aligned}\sum_{|\alpha|\le -s} \|\partial_x^\alpha Lv(\tau,\,\cdot\,)\|_{L^2} &\approx \|Lv(\tau,\,\cdot\,)\|_{H^{-s}} \\ &\le \|Lu(\tau,\,\cdot\,)\|_{H^s} + \|\,[\,(I-\Delta)^{-s},\,L]v(\tau,\,\cdot\,)\,\|_{H^s}\,.\end{aligned} \tag{3.9}$$

Now $[\,(I-\Delta)^{-s},\,L]$ is a differential operator of order $\le -2s+1$ involving no t-derivative of order > 1. Hence, we can write it as

$$[\,(I-\Delta)^{-s},\,L] = \sum_{\substack{|\alpha|,\,|\beta|\le -s\\ |\gamma|\le 1}} \partial_x^\alpha a_{\alpha\beta\gamma}\partial_x^\beta\partial^\gamma\,,$$

where, because of our assumptions on the coefficients of L, $a_{\alpha\beta\gamma} = O(1)$. Since $(I-\Delta)^{s/2}\partial_x^\alpha$ is bounded on L^2 for $|\alpha| \le -s$, we conclude that

$$\begin{aligned}&\|\,[\,(I-\Delta)^{-s},\,L]v(\tau,\,\cdot\,)\,\|_{H^s} \\ &\qquad\le C\sum_{\substack{|\beta|\le -s\\ |\gamma|\le 1}} \|\partial_x^\beta\partial^\gamma v(\tau,\,\cdot\,)\|_{L^2} \\ &\qquad\le C'\sum_{|\alpha|\le -s}\big(\|\partial_x^\alpha v(\tau,\,\cdot\,)\|_{L^2} + \|\partial_x^\alpha v'(\tau,\,\cdot\,)\|_{L^2}\big)\,.\end{aligned} \tag{3.10}$$

Using (3.9) and (3.10), we conclude that the left side of (3.8) is dominated by

$$\sum_{|\alpha|\le -s}\Big(\|\partial_x^\alpha v(0,\,\cdot\,)\|_{L^2}+\|\partial_x^\alpha v'(0,\,\cdot\,)\|_{L^2}$$
$$+\int_0^t\big(\|Lu(\tau,\,\cdot\,)\|_{H^s}+\|\partial_x^\alpha v(\tau,\,\cdot\,)\|_{L^2}+\|\partial_x^\alpha v'(\tau,\,\cdot\,)\|_{L^2}\big)\,d\tau\Big).$$

Since, by (3.7), the left side of (3.8) is comparable to

$$\|u(t,\,\cdot\,)\|_{H^s}+\|u'(t,\,\cdot\,)\|_{H^s},$$

an application of Gronwall's inequality now gives

$$\sum_{|\alpha|\le 1}\|\partial^\alpha u(t,\,\cdot\,)\|_{H^s}\le C_{T,s}\Big(\sum_{|\alpha|\le 1}\|\partial^\alpha u(0,\,\cdot\,)\|_{H^s}+\int_0^t\|Lu(\tau,\,\cdot\,)\|_{H^s}\Big).$$

Since we have now established (3.1) for negative s, the proof of Theorem 3.1 is complete.

Remark. For simplicity we only stated Theorem 3.1 for $s=\mathbb{Z}_+$ since that is all that we shall require. However, if one is willing to use pseudo-differential operators it is easy to extend Theorem 3.1, and hence Theorem 3.2, to all $s\in\mathbb{R}$. In fact, by applying (3.5) to $(I-\Delta)^{s/2}u$ we see that

$$\sum_{|\alpha|\le 1}\|\partial^\alpha u(t,\,\cdot\,)\|_{H^s}\le C\Big(\sum_{|\alpha|\le 1}\|\partial^\alpha u(0,\,\cdot\,)\|_{H^s}$$
$$+\int_0^t\big(\|Lu(\tau,\,\cdot\,)\|_{H^s}+\|[L,(I-\Delta)^{s/2}]u(\tau,\,\cdot\,)\|_{L^s}\big)\,d\tau\Big).$$

Using the calculus of pseudo-differential operators one can show that

$$\|[L,(I-\Delta)^{s/2}]u(\tau,\,\cdot\,)\|_{L^2}\le C\sum_{|\alpha|\le 1}\|\partial^\alpha u(\tau,\,\cdot\,)\|_{H^s},$$

leading to (3.1) for arbitrary $s\in\mathbb{R}$ after an application of Gronwall's inequality.

§4. Local existence for quasilinear equations

The goal of this section is to prove a local existence theorem for equations of the form

$$\text{(4.1)}\qquad \begin{cases}\sum g^{jk}(u,u')\partial_j\partial_k u=F(u,u')\\ u(0,\,\cdot\,)=f,\quad \partial_0u(0,\,\cdot\,)=g.\end{cases}$$

We shall assume that g^{jk} and F are C^∞ with all derivatives $O(1)$. We also suppose that $F(0,0)=0$, and that

$$\sum|g^{jk}-g_0^{jk}|<1/2,$$

if, as before, g_0^{jk} are the coefficients of the d'Alembertian.

Theorem 4.1. *Let $s \geq n+2$. If $(f,g) \in H^{s+1} \times H^s$ then there is a $T > 0$, depending on the norm of the data, so that the Cauchy problem (4.1) has a unique solution satisfying*

$$\sum_{|\alpha| \leq s+1} \|\partial^\alpha u(t,\,\cdot\,)\|_{L^2} < \infty\,, \quad 0 \leq t \leq T\,. \tag{4.2}$$

Also, if T_* is the supremum over all such times T, then either $T_* = \infty$ or

$$\sum_{|\alpha| \leq (s+3)/2} |\partial^\alpha u(t,x)| \notin L^\infty([0,T_*) \times \mathbb{R}^n)\,. \tag{4.3}$$

Proof. Let us start with existence since uniqueness will follow from the arguments we shall use to establish this half. We shall find a local solution to (4.1) using the method of Picard iteration. To do this we let

$$u_{-1} \equiv 0\,,$$

and then define u_m, $m = 0, 1, \ldots$, inductively by

$$\begin{cases} \sum g^{jk}(u_{m-1}, u'_{m-1})\partial_j\partial_k u_m = F(u_{m-1}, u'_{m-1}) \\ u_m(0,\,\cdot\,) = f\,, \quad \partial_0 u_m(0,\,\cdot\,) = g\,. \end{cases} \tag{4.1$'$}$$

We shall assume for the sake of convenience that the data are in $\mathcal{S}$, since at the end one can remove this assumption and replace it by the one in the theorem by an approximation argument which is similar to the one used in the proof of Theorem 3.2. If we make this assumption we can apply the local existence theorem for linear equations and an induction argument to see that for every m there is a C^∞ solution of (4.1$'$).

We claim that if T is small enough there is a constant A so that we have the uniform bounds

$$A_m(t) = \sum_{|\alpha| \leq s} \Big(\|\partial^\alpha u_m(t,\,\cdot\,)\|_{L^2} + \|\partial^\alpha u'_m(t,\,\cdot\,)\|_{L^2} \Big) \leq A < \infty\,, \quad 0 \leq t \leq T\,. \tag{4.4}$$

Recall that Sobolev's theorem (see Appendix) implies that

$$|\partial^\alpha u_m(t,x)| \leq C \sum_{|\beta| \leq |\alpha| + [(n+2)/2]} \|\partial^\beta u_m(t,\,\cdot\,)\|_{L^2}\,.$$

Therefore, if (4.4) held we would have

$$|\partial^\alpha u_m(t,x)| \leq C A_m(t)\,, \quad 0 \leq |\alpha| \leq s+1-[(n+2)/2]\,. \tag{4.5}$$

By Proposition 2.1, or more precisely, (3.5), (4.4) holds for any fixed T when $m = 0$. So to proceed, let us suppose that the estimate holds for a given T and constant A, to be determined later, when m is replaced by $m-1$. We shall then prove L^2 estimates for $|\partial^\alpha u_m| + |\partial^\alpha u_m'|$ if $|\alpha| \le s$, using (3.6) and

$$(4.6) \quad \sum g^{jk}(u_{m-1}, u_{m-1}')\partial_j\partial_k\partial^\alpha u_m$$
$$= \partial^\alpha F(u_{m-1}, u_{m-1}') - \sum [\partial^\alpha, g^{jk}(u_{m-1}, u_{m-1}')]\partial_j\partial_k u_m\,,$$

if $[\cdot, \cdot]$ denotes the commutator.

To apply our energy estimate (2.5), we need to estimate the L^2 norm of the right side. We claim that each term on the right is a linear combination of a bounded factor times factors involving derivatives of u_{m-1}, where at most one such factor involves a derivative of order $> [(s+3)/2]$. If this were the case, since

$$[(s+3)/2] \le s + 1 - [(n+2)/2] \iff s \ge n+2\,,$$

we can use (4.5) to pointwise estimate all but one of the factors, and leave the term with the highest number of derivatives inside the L^2 norm to be estimated by A_{m-1}.

Let us start by seeing that our claim is valid for the first term in the right side of (4.6). By Leibnitz's rule, it is a finite linear combination of terms of the form

$$F^{(\gamma)}\, \partial^{\alpha_1} u_{m-1} \cdots \partial^{\alpha_r} u_{m-1} \partial^{\beta_1}\partial_{j_1} u_{m-1} \cdots \partial^{\beta_s}\partial_{j_s} u_{m-1}\,,$$

where

$$\sum |\alpha_l| + \sum |\beta_l| \le |\alpha| \le s\,.$$

Hence, at most one of the α_l or β_l can have order larger than $s/2$, which means that there is at most one factor where u_{m-1} is differentiated more than $1 + s/2 = (s+2)/2$ times. Therefore, as pointed out before, we can use (4.5) and the induction hypothesis to pointwise estimate all but one of the factors by CA. Since $F^\gamma = O(1)$ and $F(0,0) = 0$, we therefore get that

$$\| \partial^\alpha (F(u_{m-1}, u_{m-1}')(t, \cdot)) \|_{L^2} \le C_A\,, \quad 0 \le t \le T\,,$$

where C_A can be taken to be a fixed constant times $(1+A)^s$.

We can similarly estimate the term involving the commutator. We first notice that it is a finite linear combination of terms of the form

$$\partial^{\alpha_1}(g^{jk}(u_{m-1}, u_{m-1}')) \cdot \partial^{\alpha_2}\partial_k u_m\,,$$

where

$$|\alpha_1| > 0, \quad |\alpha_2| \le s, \quad \text{and } |\alpha_1| + |\alpha_2| = |\alpha| + 1 \le s+1.$$

This, in turn, can be written as a finite combination of terms of the form

$$(\partial^\sigma g^{jk})(u_{m-1}, u'_{m-1})\partial^{\alpha_1} u_{m-1} \cdots \partial^{\alpha_r} u_{m-1}\, \partial^{\gamma_1} u'_{m-1} \cdots \partial^{\gamma_s} u'_{m-1} \partial^\beta u'_m ,$$

where at most one of the integers $|\alpha_j|$, $1 + |\gamma_j|$, or $1 + |\beta|$ can be larger than $1 + [(s+1)/2] = [(s+3)/2]$. This means that in the above expression there is at most one factor where u_{m-1} or u_m is differentiated more than $[(s+3)/2]$ times. Thus, by (4.5) and the induction hypothesis, it follows that the L^2 norm of the last term in (4.6) is controlled by

$$C_A(A_m(t) + 1),$$

where C_A is as above.

If we use the estimates we have just obtained for the L^2 norm of the right side of (4.6), we can apply the energy inequality, (2.5), and the argument used to establish (3.5) to obtain for $|\alpha| \le s$

$$\begin{aligned} &\|\partial^\alpha u_m(t, \cdot)\|_{L^2} + \|\partial^\alpha u'_m(t, \cdot)\|_{L^2} \\ &\quad \le C\Big(\|\partial^\alpha u_m(0, \cdot)\|_{L^2} + \|\partial^\alpha u'_m(0, \cdot)\|_{L^2} + C_A \int_0^t (A_m(\tau) + 1)\, d\tau \Big) \\ &\qquad \times \exp\Big(\int_0^t 2 \sum \|\partial_i (g^{jk}(u_{m-1}, u'_{m-1}))(\tau, \cdot)\|_{L^\infty}\, d\tau \Big), \end{aligned}$$

where C is independent of m and remains bounded if t is in a compact interval. Since (4.5) and the induction hypothesis imply that the last factor is $\le \exp(CAt)$, for some C, if we sum over $|\alpha| \le s$, we deduce that

$$A_m(t) \le Ce^{CAt}\left(A_m(0) + C_A \int_0^t (A_m(\tau) + 1)\, d\tau \right). \tag{4.7}$$

If we use Gronwall's inequality, we see that this in turn gives bounds

$$A_m(t) \le Ce^{CAt}\left(A_m(0) + C_A t \right) \exp(tCC_A e^{CAt}). \tag{4.8}$$

Using the equation defining the u_m, one sees that $A_m(0)$ can be controlled by a constant A_0 independent of m, and hence, if $A > CA_0$, one gets (4.4) for sufficiently small T.

To finish the existence proof we need to see that the u_m converge to a solution of the Cauchy problem. To do this it suffices to see that for $t \in [0,T]$

$$(4.9)\quad C_m(t) = \|u_m(t,\,\cdot\,) - u_{m-1}(t,\,\cdot\,)\|_{L^2} + \|u'_m(t,\,\cdot\,) - u'_{m-1}(t,\,\cdot\,)\|_{L^2} = O(2^{-m})\,.$$

For then (u_m, u'_m) converges in $C([0,T]; H^1)\times C^1([0,T]; L^2)$ to some (u, u') which must solve the Cauchy problem by the remarks at the end of §1.

To prove (4.9) we first notice that we can write

$$\sum g^{jk}(u_{m-1}, u'_{m-1})\partial_j\partial_k(u_m - u_{m-1}) = \sum (g^{jk}(u_{m-2}, u'_{m-2}) - g^{jk}(u_{m-1}, u'_{m-1}))\partial_j\partial_k u_{m-1} + F(u_{m-1}, u'_{m-1}) - F(u_{m-2}, u'_{m-2})\,.$$

Notice that our hypotheses on the nonlinear terms imply that the right side of the equation is

$$O\big(\,(\,|u_{m-1} - u_{m-2}| + |u'_{m-1} - u'_{m-2}|\,)\cdot(\,1 + |u''_{m-1}|\,)\,\big)\,.$$

Also, recall that u_m and u_{m-1} have the same Cauchy data at $t=0$. Therefore, by (2.5) and (4.5), if T as above is fixed then there must be a uniform constant C so that

$$C_m(t) \le C\int_0^t C_{m-1}(\tau)\,d\tau\,, \quad 0\le t<T\,.$$

Thus, if $0\le t\le T$

$$C_m(t) \le C^m \int\cdots\int_{0\le\tau_1\le\tau_2\le\cdots\le\tau_m\le t} C_0(\tau_1)\,d\tau_1\cdots d\tau_m \le \frac{(Ct)^m}{m!}\sup_{0\le t\le T} C_0(t)\,,$$

and since this implies (4.9), we have finished the existence part of the theorem. Note that the proof shows that T can be bounded from below by a fixed positive constant if one assumes that the $H^{s+1}\times H^s$ norm of the data is smaller than a fixed constant.

The uniqueness part of the theorem follows from the proof of (4.9). In fact, suppose u and $\tilde{u}$ are both in $L^\infty([0,T]; H^{s+1})\cap C^{0,1}([0,T]; H^s)$ and solve (4.1) with the same data. Then, arguing as above shows that, for fixed T, there must be an absolute constant C, depending on their norms, so that

$$\sum_{|\alpha|\le 1}\|\,\partial^\alpha(u-\tilde{u})(t,\,\cdot\,)\,\|_{L^2} \le C\int_0^t \sum_{|\alpha|\le 1}\|\,\partial^\alpha(u-\tilde{u})(\tau,\,\cdot\,)\,\|_{L^2}\,d\tau\,.$$

From this one deduces via Gronwall's inequality that, for $t \in [0,T]$, $u(t,\cdot\,)-\widetilde{u}(t,\cdot\,)$ must have zero H^1 norm, which of course implies that $u=\widetilde{u}$ on $[0,T]\times\mathbb{R}^n$.

We still have to prove the last part of the theorem. To do so, it suffices to show that if $T_*<\infty$ and if (4.2) holds for all $0\le T<T_*$ then

$$\sup_{0\le t<T_*}\sum_{|\alpha|\le s+1}\|\partial^\alpha u(t,\cdot\,)\|_{L^2}<\infty\,, \tag{4.10}$$

if

$$\sup_{(t,x)\in[0,T_*)\times\mathbb{R}^n}\sum_{|\alpha|\le (s+3)/2}|\partial^\alpha u(t,x)|\le A<\infty\,. \tag{4.11}$$

For if (4.10) held then u would extend to a function in

$$L^\infty([0,T_*];H^{s+1})\cap C^{0,1}([0,T_*];H^s)\,.$$

Hence we could use the first part of the theorem to see that u extends to a solution verifying (4.2) for some $T>T_*$.

The proof of the existence part of the theorem shows that (4.11) implies (4.10). In fact, if

$$A(t)=\sum_{|\alpha|\le s+1}\|\partial^\alpha u(t,\cdot\,)\|_{L^2}\,,$$

then arguing as in the first part of the proof shows that if (4.11) held then

$$A(t)\le C_{T_*,A}\left(A(0)+C_A\int_0^t(A(\tau)+1)\,d\tau\right),\quad 0\le t<T_*\,.$$

An application of Gronwall's inequality then gives (4.10) and finishes the proof. □

The hypotheses on s are far from optimal. For instance, using interpolation inequalities one can modify the proof of Theorem 4.1 to see that one only needs to assume that $s>(n+2)/2$. Similarly, one can replace $[(s+3)/2]$ in (4.3) by 2. See, e.g., Hörmander [5]. Later, we shall see that, in certain cases, the regularity assumptions on the data even further, using estimates related to the restriction theorem for the Fourier transform.

These limitations are not serious, though, since we shall mainly be concerned with Cauchy problems with C_0^∞ data, and, in this case, Theorem 4.1 easily implies the following

Theorem 4.2. *If $f, g \in C_0^\infty$ then there is a $T > 0$ so that (4.1) has a solution $u \in C^\infty([0,T] \times \mathbb{R}^n)$. If T_* denotes the supremum of such times T, then either $T_* = \infty$ or*

$$\sum_{|\alpha| \le (n+6)/2} |\partial^\alpha u(t,x)| \notin L^\infty([0,T_*) \times \mathbb{R}^n). \tag{4.12}$$

Proof. For the first part, by Sobolev's lemma, it suffices to see that there is a $T > 0$ so that

$$\sum_{|\alpha| \le s+1} \|\partial^\alpha u(t,x)\|_{L^2} \le C_s, \quad 0 \le t \le T, \tag{4.13}$$

for every s. By Theorem 4.1, there is such a T if $s = n + 3$.

We next claim that if, for this value of T, (4.13) holds for a given s, then it must also hold if s is replaced by $s + 1$, assuming, for the induction argument, that $s \ge n + 3$. But this also follows from Theorem 4.1.

To verify this, notice that, by Sobolev's lemma, if (4.13) holds for a given s, then we would have

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \sum_{|\alpha| \le s+1-[(n+2)/2]} |\partial^\alpha u(t,x)| < \infty. \tag{4.13'}$$

And since

$$[(s+4)/2] \le s + 1 - [(n+2)/2] \iff s \ge n + 3,$$

we conclude that derivatives of order $\le ((s+1)+3)/2$ are bounded in $[0,T] \times \mathbb{R}^n$. Hence, by the last part of Theorem 4.1, (4.13) must hold with s replaced by $s + 1$. This completes the induction argument, and, as we pointed out before, u must be C^∞ in this strip since (4.13) holds for all s.

This argument also of course yields the other half of the theorem. In fact, by Theorem 4.1, the T_* in our theorem is exactly the supremum over all T such that (4.13) holds with $s = n + 3$. Hence, by the above induction argument, it also has to be the supremum over all T such that there is a solution in $C^\infty([0,T] \times \mathbb{R}^n)$. □

§5. Local existence for semilinear equations in (1 + 3)-dimensions

In (1+3)-dimensions there is a very simple local existence theorem for semilinear wave equations that is based on the fact that the forward fundamental solution for the d'Alembertian in $\mathbb{R}^{1+3}$ is a nonnegative measure. This and the related C^k regularity properties of the solution to the linear wave equation lead to a very simple result that we shall use later in our study of global existence for semilinear wave equations with small data, and in proving global existence in the large for the "critical" semilinear wave equation in $(1+3)$-dimensions.

Let us be more specific. If $F \in C^k$ and $F(0) = 0$, we shall study the equation

$$\begin{cases} \Box u(t,x) = F(u(t,x))\,, \ t > 0 \\ u(0,x) = f(x)\,, \ \partial_t u(0,x) = g(x)\,. \end{cases} \tag{5.1}$$

As we saw in §1, if the equation were linear, we would have that $u \in C^k(\mathbb{R}^{1+3}_+)$ if $f \in C^{k+1}(\mathbb{R}^3)$ and $g \in C^k(\mathbb{R}^3)$, and, thus, to make use of the regularity of the nonlinearity, it is natural to make this assumption on the data. For simplicity, we shall assume that the data is compactly supported, although, using Huygen's principle, this hypothesis can be removed if the conclusions are suitably modified.

Theorem 5.1. *Assume, as above, that $F \in C^k$, $F(0) = 0$, and that $f \in C_0^{k+1}(\mathbb{R}^3)$, $g \in C_0^k(\mathbb{R}^3)$, with $k = 1, 2, \ldots$. Then there is a $T > 0$ so that (5.1) has a unique solution $u \in C^k([0,T] \times \mathbb{R}^3)$. If the supremum, T_*, of such times T is finite then $\sup_x |u(t,x)| \to \infty$ as $t \to T_*$.*

Proof. As in the proof of Theorem 4.1, we shall obtain u by successive approximations. Therefore, let us set $u_{-1} \equiv 0$ and define u_m for $m = 0, 1, 2, \ldots$ by

$$\begin{cases} \Box u_m = F(u_{m-1})\,, \ t > 0 \\ u_m(0,x) = f(x)\,, \quad \partial_t u_m(0,x) = g(x)\,. \end{cases}$$

Notice that u_0 is a solution of the linear Cauchy problem with data (f,g). Also, notice that, because of our assumption on F and the data, Theorem 1.1 and Duhamel's principle imply that each u_m is in $C^k(\mathbb{R}^{1+3}_+)$. However, showing that u_m converges to a C^k solution on some strip $[0,T] \times \mathbb{R}^3$ will take some work and will involve two main steps. First, we shall show that, if $T > 0$ is small enough, u_m converges to a $C^{k-1,1}([0,T] \times \mathbb{R}^3)$ solution, where we recall $C^{j,1}$ denotes C^j functions whose derivatives of order j are Lipschitz continuous. After this, using the fact that F is C^k and the fact that the free solution, u_0, is C^k, we shall uniformly control the moduli of continuity of the $\partial^\alpha u_m$ when $|\alpha| = k$ and deduce that $u \in C^k$.

For the first step, notice that since $(f,g) \in C^{k+1} \times C^k$ have compact support, (1.5) implies that there must be a uniform constant C_0 so that the solution to the linear Cauchy problem with this data satisfies

$$\sum_{|\alpha| \le k} |\partial^\alpha u_0(t,x)| \le C_0\,.$$

If $C = 2C_0$, we claim that if $T > 0$ is small enough, then if $m = 1, 2, \ldots$

$$(5.3)\quad C_m(T) = \sum_{|\alpha|<k} \sup_{\{(t,x):\, 0\le t\le T\}} |\partial^\alpha u_m(t,x) - \partial^\alpha u_{m-1}(t,x)| \le C2^{-m},$$

and

$$(5.4)\qquad B_m(t) = \sum_{|\alpha|=k} \sup_x |\partial^\alpha u_m(t,x)| \le C, \quad 0 \le t \le T.$$

If this were the case, then u_m would converge uniformly to a $C^{k-1,1}([0,T]\times \mathbb{R}^3)$ solution of (5.1).

Let us start with (5.3). We shall assume that the estimate holds when m is replaced by a smaller number n. We shall then see that (5.3) holds if $T > 0$ is small enough. As a consequence of our inductive hypothesis, we have that

$$(5.5)\qquad \sum_{|\alpha|<k} |\partial^\alpha u_n(t,x)| \le 2C, \quad \text{if } 0 \le t \le T, \text{and } n < m.$$

To use this, note that we can write $u_m = w_m + u_0$, where w_m solves the inhomogeneous wave equation

$$\begin{cases} \Box w_m(t,x) = F(u_{m-1}(t,x)), & t > 0 \\ w_m(0,x) = \partial_t w_m(0,x) = 0. \end{cases}$$

Thus, recalling (1.10), we see that for $m = 1, 2, \ldots$

$$(5.6)\quad |\partial^\alpha u_m(t,x) - \partial^\alpha u_{m-1}(t,x)| \le \int_{|y|<t} \big| \partial^\alpha F(u_{m-1}(t-|y|, x-y)) - \partial^\alpha F(u_{m-2}(t-|y|, x-y)) \big| \frac{dy}{|y|}.$$

If $|\alpha| < k$, then the term inside the first absolute value in the right side is a linear combination of terms of the form

$$F^{(j)}(u_{m-1})\,\partial^{\alpha_1} u_{m-1} \cdots \partial^{\alpha_l} u_{m-1} - F^{(j)}(u_{m-2})\,\partial^{\alpha_1} u_{m-2} \cdots \partial^{\alpha_l} u_{m-2}, \quad j, |\alpha_i| < k,$$

where $l < k-1$. Thus, since $F^{(j)} \in C^1$ and $\partial^{\alpha_i} u \in C^1$, it follows from (5.5) and (5.6) that there is a uniform constant C_1 so that

$$\begin{aligned} &C_m(T) \\ &\le C_1 T^2 \Big[1 + \sum_{|\alpha|<k} \sup_{\{(t,x):0\le t\le T\}} \big(|\partial^\alpha u_{m-1}(t,x)| + |\partial^\alpha u_{m-2}(t,x)|\big)\Big]^k C_{m-1}(T) \\ &\le C_1 (1+4C)^k T^2 C_{m-1}(T), \end{aligned}$$

giving us (5.3) if $T \le (2C_1(1+4C)^k)^{-1/2}$.

To prove (5.4), let us assume that the estimate holds if m is replaced by $m-1$. Then, since $u_m = u_0 + w_m$, using our assumption about u_0, we have that if $|\alpha| = k$,

$$\begin{aligned} |\partial^\alpha u_m(t,x)| &\le B_0(t) + \int_{|y|<t} \big|\partial^\alpha F(u_{m-1}(t-|y|, x-y))\big| dy/|y| \\ &\le C/2 + \int_{|y|<t} \big|\partial^\alpha F(u_{m-1}(t-|y|, x-y))\big| dy/|y| . \end{aligned}$$

To estimate the last term, we note that the term inside the absolute value is a combination of terms of the form

$$F'(u_{m-1})\,\partial^\alpha u_{m-1}$$

plus terms of the form

$$F^{(k)}(u_{m-1})\,\partial_{j_1} u_{m-1} \cdots \partial_{j_k} u_{m-1} ,$$

and other terms involving factors where neither F nor u_{m-1} is differentiated more than $k-1$ times.

The first term is bounded by a uniform constant times C, by the induction hypothesis and the fact that $F'(u_{m-1}) = O(1)$, since we have just shown that (5.5) holds for all m. Since (5.5) also implies that all the other terms are dominated by a fixed constant, we conclude that there must be a fixed constant C_0 so that

$$\begin{aligned} B_m(t) &\le C/2 + C_0(C+1) \int_{|y|<T} dy/|y| \\ &\le C/2 + C_0'\,(C+1)\,T^2 , \end{aligned}$$

which implies (5.4) if $T^2 \le [C_0'(C+1)]^{-1} \cdot C/2$.

We have just shown that $u_m \to u$ in $C^{k-1}([0,T]\times\mathbb{R}^3)$ and also that derivatives of order k of the u_m are uniformly bounded. Since the support of u_m intersected with $[0,T]\times\mathbb{R}^3$ is contained in a fixed compact set, if we could show that the $\partial^\alpha u_m$ are equicontinuous on $[0,T]\times\mathbb{R}^3$ when $|\alpha| = k$, we would conclude, via the Arzela-Ascoli theorem, that $u \in C^k([0,T]\times\mathbb{R}^3)$.

To this end, fix $|\alpha| = k$, and set, for $h = (h_0, h_1, h_2, h_3)$ with $h_0 \ge 0$,

$$\omega_m(h) = \sup_{\{(t,x):\, t+h_0 \le T\}} |\partial^\alpha u_m((t,x)+h) - \partial^\alpha u_m(t,x)| .$$

We then must show that if $T > 0$ is small enough and fixed, then for all $\varepsilon > 0$, there is a $\delta > 0$ so that

$$\omega_m(h) < \varepsilon, \quad \text{if } |h| < \delta . \tag{5.7}$$

To see this, let $h' = (h_1, h_2, h_3)$. Then

$$|\partial^\alpha u_m((t,x)+h) - \partial^\alpha u_m(t,x)|$$

$$\leq |\partial^\alpha u_0((t,x)+h) - \partial^\alpha u_0(t,x)|$$

$$+ \int_{|y|<t} \big|\partial^\alpha F(u_{m-1}(t+h_0-|y|, x+h'-y)) - \partial^\alpha F(u_{m-1}(t-|y|, x-y))\big| \frac{dy}{|y|}$$

$$+ \int_{t<|y|<t+h_0} \big|\partial^\alpha F(u_{m-1}(t+h_0-|y|, x+h'-y))\big| \frac{dy}{|y|}$$

$$\leq \omega_0(h) + CTh_0$$

$$+ \int_{|y|<t} \big|\partial^\alpha F(u_{m-1}(t+h_0-|y|, x+h'-y)) - \partial^\alpha F(u_{m-1}(t-|y|, x-y))\big| \frac{dy}{|y|}$$

$$= \omega_0(h) + CTh_0 + R.$$

To estimate the last term, R, let $(s_1, y_1) = (t+h_0-|y|, x+h'-y)$ and $(s_2, y_2) = (t-|y|, x-y)$. Then the first term inside the absolute value of the integral defining R can be expressed as a linear combination of terms of the form

$$\begin{aligned} I = {} & F^{(j)}(u_{m-1}(s_1,y_1)) \cdot \big(\partial^{\alpha_1} u_{m-1} \cdots \partial^{\alpha_l} u_{m-1}\big)(s_1,y_1) \\ & - F^{(j)}(u_{m-1}(s_2,y_2)) \cdot \big(\partial^{\alpha_1} u_{m-1} \cdots \partial^{\alpha_l} u_{m-1}\big)(s_2,y_2), \quad j, |\alpha_i| < k, \end{aligned}$$

plus

$$II = F'(u_{m-1}(s_1,y_1))\, \partial^\alpha u_{m-1}(s_1,y_1) - F'(u_{m-1}(s_2,y_2))\, \partial^\alpha u_{m-1}(s_2,y_2),$$

as well as

$$\begin{aligned} III = {} & F^{(k)}(u_{m-1}(s_1,y_1))\, \big(\partial_{j_1} u_{m-1} \cdots \partial_{j_k} u_{m-1}\big)(s_1,y_1) \\ & - F^{(k)}(u_{m-1}(s_2,y_2))\, \big(\partial_{j_1} u_{m-1} \cdots \partial_{j_k} u_{m-1}\big)(s_2,y_2), \end{aligned}$$

if $k > 1$.

Since $F \in C^k$, (5.4) and (5.5) imply that

$$|I| \leq C|h|.$$

On the other hand, there is a uniform constant C_0 so that

$$|II| + |III| \leq C_0 \Big(\sum_{j=1}^{k} \sup_{\substack{|v| \leq C_0|h| \\ |u| \leq C_0}} |F^{(j)}(u+v) - F^{(j)}(u)| + \omega_{m-1}(h) + |h| \Big).$$

Substituting these bounds into the integral defining R, we conclude that

$$\omega_m(h) \le \omega_0(h) + CT|h| \\ + CT^2\Big(\sum_{j=1}^{k} \sup_{\substack{|v|\le C_0|h| \\ |u|\le C_0}} |F^{(j)}(u+v) - F^{(j)}(u)| + \omega_{m-1}(h) + |h|\Big).$$

Assuming that T is small enough so that $CT^2 < 1/2$, we deduce that

$$\omega_m(h) \le 2\omega_0(h) + (1+2CT)|h| + \sum_{j=1}^{k} \sup_{\substack{|v|\le C_0|h| \\ |u|\le C_0}} |F^{(j)}(u+v) - F^{(j)}(u)|,$$

which gives implies that (5.7) must hold since u_0 and F are both C^k.

This completes the proof of the existence part of the theorem. As, in the proof the uniqueness part of Theorem 4.1, the uniqueness part here just follows from modifying the arguments we have just used to establish local existence for the equation. We leave this as an exercise for the reader.

To finish the proof we must show that if u is a C^k solution of (5.1) in $[0,T) \times \mathbb{R}^3$, then u extends to a C^k function in the closed strip $[0,T] \times \mathbb{R}^3$ if

$$\sup_{\{(t,x):\, 0\le t<T\}} |u(t,x)| \le A.$$

We claim that this implies that all derivatives of order $\le k$ are uniformly bounded in the strip. To see this, we note for instance that, since $\Box u' = F'(u)u'$, (1.10) implies that

$$\|u'(t,\,\cdot\,)\|_{L^\infty} \le \|u_0'(t,\,\cdot\,)\|_{L^\infty} + \int_0^t (t-s)\|F'(u(s,\,\cdot\,))\,u'(s,\,\cdot\,)\|_{L^\infty}\,ds \\ \le C + CT\int_0^t \|u'(s,\,\cdot\,)\|_{L^\infty}\,ds,$$

which gives us the bounds for u' after applying Gronwall's inequality. The bounds for the other derivatives follow from a similar argument.

This shows that u extends to a $C^{k-1,1}$ function in $[0,T] \times \mathbb{R}^3$. To see that u is C^k, for $|\alpha| = k$ and h as above, let us now set,

$$\omega(t,h) = \begin{cases} \sup_x |\partial^\alpha u((t,x)+h) - \partial^\alpha u(t,x)|, & \text{if } t+h_0 \le T \\ 0, & \text{otherwise}. \end{cases}$$

Then, if we argue as in the proof of (5.8), we find that

$$\omega(t,h) \\ \le \omega_0(h) + C_T\Big(|h| + \sum_{j=1}^{k} \sup_{\substack{|v|\le C|h| \\ |u|\le C}} |F^{(j)}(u+v) - F^{(j)}(u)| + \int_0^t \omega(s,h)\,ds\Big),$$

for some fixed constant C, if $\omega_0(h)$ is as above. This implies, by Gronwall's inequality again, that

$$\omega(t,h) \le C'_T \left(\omega_0(h) + \sum_{j=1}^{k} \sup_{\substack{|v|\le C|h| \\ |u|\le C}} |F^{(j)}(u+v) - F^{(j)}(u)| \right).$$

The right side goes to zero as $h \to 0$ since u_0 and F are both C^k. Hence the modulus of continuity of derivatives of order k of u is under control, which implies that u can be extended to a C^k function in the closed strip and finishes the proof. □

Notes

The material here is standard. We have followed in part Hörmander [2], [5] and John [1], [6], [8]. In various forms the local existence theorems are classical. See Friedrichs [1] and John [2]. For the quasilinear case, however, we have followed an idea of Klainerman [4], allowing us to avoid the use of the interpolation inequalities of Gagliardo [1] and Nirenberg [1] which can be used to obtain better results. See Hörmander [5]. The local existence theorem for semilinear equations in $\mathbb{R}^{1+3}$ is taken from Lindblad [1], although related results go back at least to Jörgens [1].

CHAPTER II

QUASILINEAR EQUATIONS WITH SMALL DATA

The main goal of this chapter is to extend the local existence theorem for quasilinear equations, showing, in particular, that in certain cases the local solutions obtained before extend to global ones if the initial data is compactly supported and sufficiently small. Recall that Sobolev's theorem was used in a key way in the proof of the local existence theorem, when, for instance, we dominated $u(t,x)$ by the sum of the L^2 norms of $\partial_x^\alpha u(t,\,\cdot\,)$ with $|\alpha| \le (n+1)/2$. To obtain better theorems we shall need a better estimate which reflects the decay of the linear solution, as recorded in Theorem 1.1 from the last chapter.

The improvement of the usual Sobolev estimates which are adapted to the wave equation is due to Klainerman [3], [4]. The idea involves using, in the L^2 norms, a larger collection of vector fields that preserve the equation $\Box u = 0$ in order to obtain pointwise bounds for $u(t,x)$ that decay as $t \to \infty$ in the natural way. Armed with these estimates, it will not be hard to adapt the proof of the local existence theorem to obtain global existence theorems in certain cases. In the most straightforward cases where the dimension is large, this will just involve substituting the Klainerman-Sobolev estimates for the Sobolev estimates in the relevant places of the proof. In the harder case where $n = 3$ we shall also require a related estimate for the inhomogeneous wave equation, which is due to Klainerman [3] and Hörmander [4], as well as an adaptation of the energy method involving the expanded collection of vector fields.

After proving the generalized Sobolev estimates in §1, we shall apply them in §2 to prove a global existence theorem for quasilinear equations in $\mathbb{R}^{1+n}_+$ with $n \ge 4$. We shall also see here that there is "almost global existence" when $n = 3$. In general, though, one does not have global existence in the case of three spatial dimensions. One needs the "null condition" which will be introduced in §3, where the corresponding global existence theorem will be treated. Finally, in §4, we shall see that, if one considers a class of equations verifying the null condition, one can prove estimates related to the restriction theorem for the Fourier transform which give a considerable improvement over the earlier regularity assumptions for local existence.

§1. Klainerman-Sobolev inequalities

Before we can state the Klainerman-Sobolev estimates we need to introduce the vector fields involved. They are just the generators of the Poincaré group, the group of linear transformations preserving equations of the form $\Box u = 0$. We shall thus call them the *invariant vector fields*. Recall that the generators are just the derivatives of the group parameters taken at the identity. Consequently, a basis consists of

$$\partial_i\,,\ 0 \le i \le n\,, \quad L_0\,, \quad \text{and } \Omega_{ij}\,,\ 0 \le i < j \le n\,, \tag{1.1}$$

where $\partial_i = \partial/\partial x_i$ are just the usual translation invariant vector fields in $\mathbb{R}^{1+n}$, L_0 is the radial vector field there,

$$L_0 = t\partial_t + \sum_{i=1}^{n} x_i \partial_i\,, \tag{1.2}$$

and the Ω_{ij} are the angular derivatives in the Minkowski metric $\sum g_0^{jk} dx_j dx_k$, that is

$$\Omega_{ij} = g_0^{ii} x_i \partial_j - g_0^{jj} x_j \partial_i\,. \tag{1.3}$$

As before, $g_0^{jk} = \text{diag}\,(1, -1, \dots, -1)$ are the coefficients of the d'Alembertian Thus,

$$\Omega_{ij} = x_j \partial_i - x_i \partial_j\,, \quad 0 < i < j \le n \tag{1.4}$$

are the spatial angular momentum operators, and

$$\Omega_{0j} = t\partial_j + x_j \partial_t \quad 0 < j \le n\,, \tag{1.5}$$

are the remaining ones for the full set of angular momentum operators for the Lorentz group. This, we recall, is just the group of linear transformations preserving $\Box$, and hence $\{\,\partial_i\,, \Omega_{jk}\,\}$ are generators of this group. The radial vector field, L_0, must be added to obtain generators of the larger Poincaré group.

The Ω_{ij} of course commute with $\Box$:

$$[\Box, \Omega_{ij}] = 0\,, \quad 0 \le i < j \le n\,. \tag{1.6}$$

However, the radial vector field does not since it is not a generator for the Lorentz group. Instead,

$$[\Box, L_0] = 2\Box\,. \tag{1.7}$$

In what follows, we shall let

$$\partial_0, \dots, \partial_n, L_0, \Omega_{01}, \dots, \Omega_{n-1\,n}$$

be denoted, respectively, by Γ_i, $i = 0, \dots, (n+4)(n+1)/2$. At times we shall suppress the subscript. Also, if $(\alpha_0, \dots, \alpha_m)$, $m = (n+4)(n+1)/2$ is a multi-index, we shall write

$$\Gamma^\alpha = \Gamma_1^{\alpha_1} \cdots \Gamma_m^{\alpha_m} .$$

We shall call the vector fields in (1.2) and (1.3) the homogeneous vector fields since their coefficients are homogeneous of degree one. Notice that for these we have the commutativity relations

$$[\Gamma_i, \Gamma_j] = \sum c_{ijk} \Gamma_k , \tag{1.8}$$

for certain fixed constants, where the sum just involves homogeneous vector fields. Thus, the commutator of two homogeneous vector fields is a linear combination of homogeneous vector fields. If we form the commutator of ∂_j with a homogeneous vector field we get a translation invariant vector field:

$$[\Gamma_k, \partial_j] = \sum_{i=0}^{n} a_{ijk} \partial_i , \tag{1.9}$$

since

$$[\partial_k, L_0] = \partial_k , \ [\partial_k, \Omega_{0j}] = \delta_{0k}\partial_j + \delta_{jk}\partial_0 , \quad \text{and} \ [\partial_k, \Omega_{ij}] = \delta_{jk}\partial_i - \delta_{ik}\partial_j , 0 < i < j \le n . \tag{1.10}$$

Another fact that will be useful for us concerns the span of the homogeneous vector fields at a given point $(t,x) \in \mathbb{R}^{1+n}\backslash 0$. Specifically, notice that if $t^2 \neq |x|^2$, that is, if (t,x) is not on the light cone, then these vector fields span the full tangent space above (t,x). On the other hand if $t^2 = |x|^2$, they only span the tangent space to the light cone, while the missing normal component vanishes only to first order in view of (1.10).

Since their coefficients are homogeneous of degree one, the following proposition quantifies this.

Proposition 1.1. *Let $r = |x|$ and $\partial_r = r^{-1}\sum_{i=1}^n x_i \partial_i$. Then in $\mathbb{R}^{1+n}_+\backslash 0$ we can write*

$$(t-r)\partial_r = a_0(t,x) L_0 + \sum_{i=0}^{n} a_i(t,x) \Omega_{0i} , \tag{1.11}$$

where the coefficients are smooth, homogeneous of degree zero and satisfy bounds of the form

$$|\partial^\alpha a_j(t,x)| \le C_\alpha (t+|x|)^{-|\alpha|}$$

for all α if $|x| > \delta t$, with $\delta > 0$ fixed. Also,

$$(1.12) \qquad (t-r)^2 \sum_{i=0}^{n} |\partial_i u(t,x)|^2 \le |L_0 u(t,x)|^2 + \sum_{0\le j<k\le n} |(\Omega_{jk}u)(t,x)|^2 .$$

Proof. The first part just follows from the identity

$$(1.13) \qquad (t-r)\partial_r = \frac{1}{r+t}\Big(t\sum_{i=1}^{n} \frac{x_i}{|x|}\Omega_{0i} - rL_0 \Big) .$$

To prove (1.12) we shall use this as well as

$$(1.14) \qquad (t-r)\partial_t = \frac{1}{r+t}\Big(tL_0 - \sum_{i=1}^{n} x_i \Omega_{0i} \Big) .$$

Since (1.12) is invariant under rotations, in proving it we may assume that $x_1 > 0$, and $x_2 = \cdots = x_n = 0$. Then

$$(1.15) \qquad (t^2+|x|^2)\sum_{j=2}^{n} |\partial_j u|^2 = \sum_{0<j<k} |\Omega_{jk}u|^2 + \sum_{k=2}^{n} |\Omega_{0k}u|^2 .$$

Also, by (1.13) and (1.14),

$$(t^2-|x|^2)\partial_t u = tL_0 u - x_1\Omega_{01}u , \quad (t^2-|x|^2)\partial_1 u = -x_1 L_0 + t\Omega_{01}u ,$$

and so

$$\begin{aligned}(t^2-|x|^2)^2\big(|\partial_t u|^2 + |\partial_1 u|^2 \big) &\le 2(t+x_1)^2\big(|L_0 u|^2 + |\Omega_{01}u|^2 \big) \\ &\le (t^2+x_1^2)\big(|L_0 u|^2 + |\Omega_{01}u|^2 \big) .\end{aligned}$$

Combining this with (1.15) yields (1.12) and finishes the proof. □

The proof of the estimates we shall require will use this proposition as well as the following variant of Sobolev's theorem for $\mathbb{R}^n$.

Lemma 1.2. *Fix $\delta > 0$. Then if $f \in C^\infty\mathbb{R}^n)$*

$$|f(x)|^2 \le C_{n,\delta} \sum_{|\alpha| \le \frac{n+2}{2}} \int_{|y|<\delta} |(\partial_x^\alpha f)(x+y)|^2\, dy\,. \tag{1.16}$$

If $u \in C^\infty(S^{n-1})$, then

$$|u(\omega)|^2 \le C_n \sum_{|\alpha| \le \frac{n+1}{2}} \int_{S^{n-1}} \big|(\partial_\nu^\alpha u)(\nu)\big|^2\, d\sigma(\nu)\,, \tag{1.17}$$

if $\partial_\nu^\alpha = \Omega_{12}^{\alpha_1} \cdots \Omega_{n-1\,n}^{\alpha_\mu}$, $\mu = n(n-1)/2$, with Ω_{ij}, $1 \le i < j \le n$ denoting the restriction to S^{n-1} of the vector fields in (1.4). Also, if $v(r,\omega)$ is in $C^\infty(\mathbb{R}_+ \times S^{n-1})$,

$$|v(r,\omega)|^2 \le C_{n,\delta} \sum_{j+|\alpha| \le \frac{n+2}{2}} \iint_{|q|<\delta} \big|(\partial_q^j \partial_\nu^\alpha v)(r+q,\nu)\big|^2\, dq d\sigma(\nu)\,. \tag{1.18}$$

Inequality (1.16) is just a local form of the Euclidean Sobolev estimates and follows from the latter by a simple argument using cutoff functions. The other inequalities in the lemma follow from (1.16), using local coordinates on S^{n-1}, since the $\{\Omega_{ij}\}_{1\le i<j\le n}$ span the tangent space at any point of S^{n-1}.

We are now ready for the Klainerman-Sobolev estimates:

Theorem 1.3. *Let $u \in C^\infty(\mathbb{R}^{1+n})$ vanish when $|x|$ is large. Then, if $t > 0$*

$$\big(1+t+|x|\big)^{\frac{n-1}{2}} \big(1+\big|t-|x|\big|\big)^{\frac{1}{2}} |u(t,x)| \le C \sum_{|\alpha| \le \frac{n+2}{2}} \|\Gamma^\alpha u(t,\,\cdot\,)\|_{L^2}\,. \tag{1.19}$$

Proof. The case where $t + |x| \le 1$ follows from (1.16). So in what follows we shall assume that $t + |x| > 1$.

Case 1: $|x| \notin [t/2, 3t/2]$.

Since we are assuming that (t,x) is away from the light cone, here we shall want to use (1.12). In fact, since $\big|t - |x+y|\big| \ge c(t+|x|)$, with $c > 0$ if $|y| < (t+|x|)/8$, (1.9) and (1.12) yield

$$(t+|x|)^n \sum_{|\alpha| \le \frac{n+2}{2}} \int_{|y|<1/8} \big|\partial_y^\alpha\big(u(t,\,x+(t+|x|)y)\big)\big|^2\, dy$$

$$= \sum_{|\alpha| \le \frac{n+2}{2}} \int_{|y|<(t+|x|)/8} \big|\big((t+|x|)\partial_y\big)^\alpha u(t,x+y)\big|^2\, dy$$

$$\le C \sum_{|\alpha| \le \frac{n+2}{2}} \|\Gamma^\alpha u(t,\,\cdot\,)\|_{L^2}^2\,,$$

where the last sum only involves homogeneous vector fields. Since (1.16) implies that $(t+|x|)^n|u(t,x)|^2$ is dominated by the left side, we get (1.19) for this case.

Case 2: $|x| \in [t/2, 3t/2]$.

Here we need to exploit the other half of Proposition 1.1. To use (1.11), let us set

$$q = |x| - t = r - t,$$

and write $x = r\omega = (t+q)\omega$, with $\omega \in S^{n-1}$. In these modified polar coordinates we have

$$dx = (t+q)^{n-1} dq d\sigma(\omega), \quad \partial_r = \partial_q, \quad q\partial_q = (r-t)\partial_r.$$

Hence, if we abuse notation a bit and write $u(t,q,\omega) = u(t,(t+q)\omega)$, we get

(1.20)

$$\begin{aligned} t^{n-1} &\sum_{j+k+|\alpha|\le \frac{n+2}{2}} \iint_{-\frac{3t}{4}<q<t} |(q\partial_q)^j \partial_q^k \partial_\nu^\alpha u(t,q,\nu)|^2 \, dq d\sigma(\nu) \\ &\le 4^{n-1} \sum_{j+k+|\alpha|\le \frac{n+2}{2}} \iint_{r\in[t/4,2t]} |((t-r)\partial_r)^j \partial_r^k \partial_\nu^\alpha u(t,r\nu)|^2 \, r^{n-1} dr d\sigma(\nu) \\ &\le C \sum_{|\alpha|\le\frac{n+2}{2}} \|\Gamma^\alpha u(t,\cdot)\|_{L^2}^2 . \end{aligned}$$

In the first inequality, we exploited the fact that, since $|x| \in [t/2, 3t/2]$, to control $u(t,x)$, we need only to integrate over, say, $t/4 < r < 2t$, allowing us to absorb t^{n-1} in the volume element. In the last inequality, we of course used (1.11).

Inequality (1.20) gives us the bounds (1.19) when $|\, t - |x|\, | \le 1$. In fact if $|x| \in [t/2, 3t/2]$ (and hence $|q| < t/2$), (1.18) implies that $t^{n-1}|u(t,x)|^2$ is dominated by the modification of the left side of (1.20) where the sum is only taken over $k + |\alpha| \le (n+2)/2$.

These two steps imply that $(t+|x|)^{\frac{n-1}{2}}|u(t,x)|$ is dominated by the right side of (1.19). This is all that will be needed in the applications; however, let us finish the proof by showing how (1.11) leads to the extra decay involving $t - |x|$, if $|x| \in [t/2, 3t/2]$.

Here we shall want to use the part of the sum in (1.20) involving $j = 0$, but $k \ge 0$. To exploit the extra factor of q in each derivative, we need to use a variation on the argument used for the first part of the proof. More specifically, if we fix $x = x_0$, here we shall want to introduce cutoff functions so that the integrals are only taken over a region where $q \approx q_0 = t - |x_0|$. Since we have just obtained the desired bounds when $|t - |x_0|| \le 1$, let us assume that $1 \le q_0 \le t/2$. We then pick a $\chi \in C_0^\infty((-1/2, 1/2))$ which equals one near the origin and set

$$v(t,q,\nu) = \chi((q-q_0)/q_0)\, u(t,q,\nu),$$

in which case

$$v(t,q,\omega) = u(t,q,\omega) = u(t,x_0), \quad \text{if } q = q_0 .$$

Notice that, since q must be comparable to q_0 on the support of v, and since

$$|(q_0\partial_q)^j \chi((q-q_0)/q_0)| \le C_j ,$$

where C_j depends only on χ, one must have the uniform bounds

$$\sum_{j+|\alpha|\le \frac{n+2}{2}} |(q_0\partial_q)^j \partial_\nu^\alpha v(t,q,\nu)| \le C \sum_{j+|\alpha|\le \frac{n+2}{2}} |(q\partial_q)^j \partial_\nu^\alpha u(t,q,\nu)| .$$

Notice also that the cutoff function vanishes unless

$$|q - (t - |x_0|)| \le \tfrac{1}{2}|t - |x_0||,$$

and since $|t - |x_0|| \le t/2$, this means that

$$v(t,q,\nu) = 0 \quad \text{if } |q| \ge 3t/4 .$$

Thus,

$$\begin{aligned}
&\sum_{j+|\alpha|\le \frac{n+2}{2}} \iint |\partial_q^j \partial_\nu^\alpha (v(t, q_0 + q_0 q, \nu))|^2\, dq d\sigma(\nu) \\
&\quad = |q_0|^{-1} \sum_{j+|\alpha|\le \frac{n+2}{2}} \iint |((q_0\partial_q)^j \partial_\nu^\alpha v)(t, q_0 + q, \nu)|^2\, dq d\sigma(\nu) \\
&\quad \le |q_0|^{-1} \sum_{j+|\alpha|\le \frac{n+2}{2}} \iint_{|q|\le \frac{3t}{4}} |((q\partial_q)^j \partial_\nu^\alpha u)(t,q,\nu)|^2\, dq d\sigma(\nu) .
\end{aligned}$$

Since (1.18) implies that $|v(t,q_0,\omega)|^2$ is dominated by the left side, we conclude, using (1.20) that $|q_0|\, t^{n-1}\, |u(t,x_0)|^2$ is also dominated by square of the right side of (1.19).

This finishes the proof. □

Remark 1.4. Recall that we saw in Chapter 1 that the solution of the Cauchy problem

$$\begin{cases} \Box u(t,x) = 0\,, & t > 0 \\ u(0,x) = f(x)\,, & \partial_t u(0,x) = g(x)\,. \end{cases}$$

is $O\big(\,(1+t)^{-\frac{n-1}{2}}\,\big)$, if the data belongs to C_0^∞. Let us see that the Klainerman-Sobolev inequality can be used to give an alternative proof of this if the dimension is *odd.* To do this, we first observe that, for any $n \geq 2$, Theorem 1.3 implies that

$$\text{(1.21)}\qquad \begin{aligned} &|u'(t,x)| \\ &\leq C\,(1+t)^{-\frac{n-1}{2}}\,(1+|\,t-|x|\,|)^{-\frac{1}{2}} \sum_{|\alpha|\leq\frac{n+2}{2}} \|\Gamma^\alpha u'(t,\,\cdot\,)\|_{L^2} \\ &\leq C'\,(1+t)^{-\frac{n-1}{2}}\,(1+|\,t-|x|\,|)^{-\frac{1}{2}} \sum_{|\alpha|\leq\frac{n+2}{2}} \|\,(\Gamma^\alpha u)'(t,\,\cdot\,)\,\|_{L^2}\,, \end{aligned}$$

using (1.9) in the last step. Notice that $\Box\Gamma^\alpha u = 0$, since $[\Box,\Gamma^\alpha]u = c\Box u = 0$. Therefore, combining the last inequality with the energy inequality for $\Box$ gives

$$\text{(1.21}'\text{)}\qquad |u'(t,x)| \leq C(1+t)^{-\frac{n-1}{2}}\,(1+|\,t-|x|\,|)^{-\frac{1}{2}} \sum_{|\alpha|\leq\frac{n+2}{2}} \|(\Gamma^\alpha u)'(0,\,\cdot\,)\|_{L^2}\,.$$

Using the equation $\Box u = 0$, one sees that, like the Cauchy data for u, the Cauchy data of $\Gamma^\alpha u$ belongs to C_0^∞. In fact, the Cauchy data of $\Gamma^\alpha u$ just involves a finite combination of differential operators applied to f and g. The coefficients of the operators need not be constant, but that is irrelevant since f and g are compactly supported. Based on this, we conclude that

$$|u'(t,x)| \leq C(1+t)^{-\frac{n-1}{2}}\,(1+|\,t-|x|\,|)^{-\frac{1}{2}}\,,$$

where the constant C depends on the data and comes from the right side of (1.21′).

If the spatial dimension n is odd, then $u(t,x) = 0$ if $|\,t-|x|\,| > R$ and f and g are supported in the ball of radius R centered at the origin. Thus, using the fundamental theorem of calculus, we conclude that $|u(t,x)| \leq C(1+t)^{-\frac{n-1}{2}}$ for odd n. On the other hand, if n is even, then we can only say that $u(t,x) = 0$ if $|x| > t+R$, and so here (1.21′) only leads to bounds of the form

$$\text{(1.21}''\text{)}\qquad |u(t,x)| \leq C\,(1+t)^{-\frac{n-1}{2}}\,(1+|\,t-|x|\,|)^{\frac{1}{2}}$$

for even n, which of course are much worse than the ones in Theorem 1.1 from Chapter 1.

We shall be able to get bounds like (1.21″) for solutions of certain nonlinear equations using this argument along with the uniqueness theorem of John for nonlinear equations that was proved in §2 of the last chapter.

Let us conclude this section by proving an estimate of Hörmander [4] (see also Klainerman [3]) for the inhomogeneous wave equation in $(1+3)$-dimensions:

$$\begin{cases} \Box w(t,x) = F(t,x)\,, \ (t,x) \in \mathbb{R}^{1+3}_+ \\ w(0,x) = \partial_t w(0,x) = 0\,. \end{cases}$$

We shall not need this estimate until §3; however, it is convenient to deal with it now since its proof is somewhat similar to that of Theorem 1.3.

Theorem 1.5. *Let $F \in C^2(\mathbb{R}^{1+3}_+)$. Then, if $t + |x| > 1$,*

$$(1.22) \qquad (t+|x|)|w(t,x)| \le C \iint_{[0,t]\times\mathbb{R}^3} \sum_{|\alpha|\le 2} |\Gamma^\alpha F(s,y)| \frac{ds dy}{1+s+|y|}\,.$$

Proof. Let us assume first that F vanishes in a neighborhood of the origin. In the proof we shall need to split F up into various pieces using partitions of unity which involve functions ψ which are homogeneous of degree zero and belong to $C^\infty(\mathbb{R}^{1+n}\backslash 0)$. Since the coefficients of our vector fields have homogeneity of degree one or zero, it follows that for such ψ

$$\sum_{|\alpha|\le 2} |\Gamma^\alpha(\psi F)| \le C \sum_{|\alpha|\le 2} |\Gamma^\alpha F|.$$

On account of this, in what follows, for the sake of notation, we shall suppress the elements of the partition of unity.

To prove (1.22) we shall need to make use of the following formula from Chapter 1:

$$(1.23) \qquad w(t,x) = \frac{1}{4\pi} \int_{|y|<t} F(t-|y|, x-y)\, \frac{dy}{|y|}\,.$$

We shall split the proof of (1.22) into two main cases: $|x| < t/2$, and $|x| > t/2$.

Case 1: $|x| < t/2$.

Let us first note that, since w has vanishing Cauchy data, we can write

$$w(t,x) = \int_0^1 \frac{d}{d\theta} w(\theta t, \theta x)\, d\theta = \int_0^1 (L_0 w)(\theta t, \theta x)\, d\theta\,,$$

if L_0 is the radial vector field in (1.2). Since $\Box L_0 w = L_0 \Box w + [\Box, L_0]w = L_0 F + 2F$, we can use (1.23) to see that

$$|w(t,x)| \le \int_0^1 \int_{|y|\le \theta t} \big(|L_0 F| + |F| \big)(\theta t - |y|, \theta x - y) \frac{dy}{|y|} d\theta .$$

If we make the change of variables $(s,z) = (\theta t - |y|, \theta x - y)$, then $dy d\theta / |y| = ds dz / |t|y| - x\cdot y|$. But $|\,|y| - x\cdot y/t\,| \ge |y|/2$, as $|x| < t/2$. Also, $|z - sx/t| = |\,|y|x/t - y| \le 3|y|/2$, and hence

$$\text{(1.24)} \qquad \begin{aligned} t|w(t,x)| &\le 2 \iint \big(|L_0 F| + |F| \big)(s,z) \frac{dsdz}{|y|} \\ &\le 3 \iint \big(|L_0 F| + |F| \big)(s,z) \frac{dsdz}{|z - sx/t|} . \end{aligned}$$

If $|z| > 2s/3$ on the support of F, then (1.22) follows trivially from this, since $|z - sx/t| > 3|z|/2$.

If, say, $|z| < 3s/4$ on supp F then we need to use a different argument, ultimately exploiting (1.12). To do this we require the following

Lemma 1.6. *If $\varphi(r)$ is C^1 and vanishes for large r*

$$\int_0^\infty |\varphi(r)|\, r dr \le \frac{1}{2} \int_0^\infty |\varphi'(r)|\, r^2 dr .$$

Assuming this for now, we can finish the proof for Case 1. In fact, if we take polar coordinates around $z = sx/t$, we deduce that

$$\iint \big(|L_0 F| + |F| \big)(s,z) \frac{dzds}{|z - sx/t|} \le \iint \big(|\partial_z L_0 F| + |\partial_z F| \big)(s,z)\, dzds .$$

Since we are assuming that F vanishes when $|z| > 3s/4$, if we combine this with (1.24) and use (1.12), we conclude that

$$\text{(1.25)} \qquad t\,|w(t,x)| \le C \sum_{1\le|\alpha|\le 2} \iint |\Gamma^\alpha F(s,z)|\, dsdz/(s + |z|) ,$$

where the sum only involves homogeneous vector fields.

Case 2: $|x| > t/2$.

Let us assume first that $|y| < s/3$ on supp F. Then Huygen's principle implies that $w(t,x) = 0$ if $|x| \ge t$. On account of this, we get the desired bounds here from (1.25), since, like in the last step, we are assuming that F is supported in a conic set contained in the interior of the light cone.

To finish the case where $|x| > t/2$, we now assume that $|y| > s/4$ on supp F. We then define the radial majorant of F:

$$F^*(t,r) = \sup_{\omega} |F(t, r\omega)| .$$

If we then let $w^*(t,x)$ solve $\Box w^*(t,x) = F^*(t,|x|)$ with zero data, then the comparison theorem mentioned in §1 of Chapter 1 implies that

$$|w| \le w^* .$$

Also, since w^* is radial, formula (1.11) from Chapter 1 gives

$$rw^*(t,r) = \frac{1}{2}\int_0^t \int_{|r-(t-s)|}^{r+(t-s)} F^*(s,\rho)\, \rho d\rho ds .$$

But if we use Sobolev's lemma on S^2, we get the bounds

$$F^*(s,\rho) \le C \sum_{|\alpha|\le 2} \int_{S^2} \big| (\Gamma^\alpha F)(s,\rho\omega) \big| \, d\sigma(\omega) ,$$

where the sum only involves operators coming from the Euclidean rotations, that is, $\Gamma = \Omega_{ij}$, $1 \le i < j \le n$. Since $|w| \le w^*$, we can plug this into the last formula to deduce that

$$\begin{aligned} |x|\,|w(t,x)| &\le C \sum_{|\alpha|\le 2} \iint |(\Gamma^\alpha F)(s,\rho\omega)|\, \rho d\rho d\sigma(\omega) ds \\ &= C \sum_{|\alpha|\le 2} \iint |(\Gamma^\alpha F)(s,y)|\, dsdy/|y| . \end{aligned} \tag{1.26}$$

From this we obtain (1.22) again, since we are now assuming that $|x| > t/2$, and that $|y| > s/4$ on supp F.

This completes the proof of (1.22) if F vanishes near the origin. To finish the proof of the theorem we now must show that the inequality holds if F vanishes when, say, $s + |y| > 1/100$. By Huygen's principle this implies that $w(t,x) = 0$ if $|\,t - |x|\,| > 1/10$. Since we are assuming in (1.22) that $t+|x| > 1$, this means that we only have to consider the case where $|x| > t/2$. If we then use (1.26) again we conclude that

$$\begin{aligned} &(t + |x|)|w(t,x)| \\ &\quad \le C \iint |F(s,y)|\, dsdy/|y| + C \sum_{1\le|\alpha|\le 2} \iint |(\Gamma^\alpha F)(s,y)|\, dsdy/|y| . \end{aligned}$$

If we use Lemma 1.6, we see that the first term on the right is dominated the integral of $|\partial_y F|$. The term involving the sum is also dominated by the right side of (1.22) since $|\Gamma^\alpha F(s,y)| \le C|y| \sum_{1\le|\beta|\le 2} |(\partial_y^\beta F)(s,y)|$, since each Γ is one of the Euclidean angular momentum operators Ω_{ij}, and

$$|\Omega_{ij}F(s,y)| \le |y| \sum_{k=1}^{n} |\partial_k F(s,y)|\,.$$

This completes the proof. $\square$

Proof of Lemma 1.6. We first observe that

$$\begin{aligned}\int_0^R |(R-r)\varphi(r)|rdr &= \int_0^R \Big| \int_r^R \frac{\partial}{\partial\rho}\big((R-\rho)\varphi(\rho) \big)\, d\rho \,\Big| \,dr \\ &\le \iint\limits_{0<r<\rho<R} r\big|\,(R-\rho)\varphi'(\rho) - \varphi(\rho)\,\big|\, d\rho dr \\ &\le \tfrac{1}{2}\int_0^R \big(R|\varphi'(\rho)| + |\varphi(\rho)| \big)\, \rho^2\, d\rho\,.\end{aligned}$$

From this we deduce that

$$\begin{aligned}2R\int_0^R |\varphi(r)|\, rdr &\le 2\int_0^R |(R-r)\varphi(r)| r\, dr + 2\int_0^R |\varphi(r)|\, r^2\, dr \\ &\le \int_0^R \big(R|\varphi'(\rho)| + 3|\varphi(\rho)| \big)\, \rho^2\, d\rho\,.\end{aligned}$$

This of course leads to the desired inequality after dividing by R and then letting $R \to +\infty$, since we are assuming that φ is C^1 and vanishes for large r. $\square$

§2. Global existence in higher dimensions

We shall now consider long-time existence questions for equations of the form

$$\begin{cases} \sum_{j,k=0}^{n} g^{jk}(u')\partial_j\partial_k u = F(u') \\ u(0,x) = \varepsilon f(x), \quad \partial_t u(0,x) = \varepsilon g(x)\,. \end{cases} \tag{2.1}$$

We shall assume that F and the coefficients g^{jk} are C^∞ and that the linearization of (2.1) is the linear homogeneous wave equation with the above data. The latter means that

$$\sum_{j,k=0}^{n} g^{jk}(0)\partial_j\partial_k = \square\,, \tag{2.2}$$

and also that

$$F(0) = 0, \quad \text{and } dF(0) = 0, \tag{2.3}$$

so that $F(u') = O(|u'|^2)$. As before, $u' = (\partial_0 u, \ldots, \partial_n u)$ denotes the full gradient of u. We shall also assume that (f, g) are fixed, smooth, and compactly supported.

Our first result says that if the dimension is large enough then there is always global existence for (2.1) if $\varepsilon > 0$ is sufficiently small:

Theorem 2.1. *Let $n \geq 4$ and fix $f, g \in C_0^\infty(\mathbb{R}^n)$. Then if $\varepsilon > 0$ is sufficiently small the Cauchy problem (2.1) always has a global C^∞ solution.*

The assumption about compact support is needed. Without it one could construct C^∞ data for which equations of the form (2.1) blow-up in finite time using classical blow-up results from the case where the spatial dimension n is one. For instance in $\mathbb{R} \times \mathbb{R}^1_+$, equations of the form

$$\partial_t^2 u - c^2(\partial_x u)\partial_x^2 u = 0$$

with nontrivial data always blow-up in finite time if c is a positive function with non-vanishing derivative. (See e.g. John [8].) From this one can see that the theorem cannot hold without the assumption of compact support simply by considering data depending only on the first spatial variable. The smallness assumption is also necessary.

Proof. As we pointed out before, the strategy will be to adapt the proof of the local existence theorem from the last chapter, mainly by substituting the Klainerman-Sobolev estimates in places where we used the usual Sobolev estimates in the earlier argument.

We shall use the so-called continuity method. We know that for a given $\varepsilon > 0$ there is always a $T > 0$ so that (2.1) has a smooth solution on the strip $[0, T] \times \mathbb{R}^n$. We know further, by Theorem 4.2 of the last chapter, that, if ε is fixed, the set of such T is open. Therefore, it suffices to show that this set is closed if $\varepsilon > 0$ is small enough.

Because of this reduction, we may assume that there is a $T_* = T_*(\varepsilon) > 0$ so that (2.1) has a smooth solution on $[0, T_*) \times \mathbb{R}^n$. We then must show that u can actually be extended to a C^∞ solution in the closed strip $[0, T_*] \times \mathbb{R}^n$ if $0 < \varepsilon \leq \varepsilon_0$, with ε_0 depending only on f and g.

By the local existence theorem, we know that, for a given $\varepsilon > 0$, if u cannot be extended to a solution in the closed strip, then $\sum_{|\alpha| \leq \frac{n+6}{2}} |\partial^\alpha u| \notin L^\infty([0, T_*) \times \mathbb{R}^n)$. Thus, our task is equivalent to obtaining pointwise bounds on $\partial^\alpha u$ in the half-open strip when $|\alpha| \leq (n+6)/2$. In other words, it suffices to show that, if $T_* = T_*(\varepsilon)$ is as above,

$$\sup_{(t,x) \in [0, T_*) \times \mathbb{R}^n} \sum_{|\alpha| \leq (n+6)/2} |\partial^\alpha u(t, x)| < \infty, \tag{2.4}$$

if $0 < \varepsilon < \varepsilon_0$, with ε_0 depending only on the data.

Recall that, by Theorem 2.2 from Chapter 1, $u(t,x) = 0$ when $|x| > R + |t|$, if the Cauchy data vanish when $|x| > R$. Keeping this in mind, we claim that, if

$$A(t) = \sum_{|\alpha| \le s} \| (\Gamma^\alpha u)'(t, \cdot)\|_{L^2(\mathbb{R}^n)} ,$$

then it suffices to see that, for small $\varepsilon > 0$,

$$\sup_{0 \le t < T_*} A(t) < \infty , \quad \text{if } s \ge n+4 . \tag{2.4'}$$

To see this, notice that by the Klainerman-Sobolev estimates

$$\begin{aligned}(1+t)^{\frac{n-1}{2}} (1 + |\, t - |x|\,|)^{\frac{1}{2}}\, |(\Gamma^\beta u)'(t,x)| &\\ \le C \sum_{|\alpha| \le (n+2)/2} \|\Gamma^\alpha (\Gamma^\beta u)'(t, \cdot)\|_{L^2} &\\ \le C' \sum_{|\alpha| \le |\beta| + (n+2)/2} \|(\Gamma^\alpha u)'(t, \cdot)\|_{L^2} ,&\end{aligned}$$

with the last step using (1.10), which says that $[\partial_j, \Gamma]$ is either zero or $\pm\partial_i$ for some i. Thus,

$$\begin{aligned}&|(\Gamma^\beta u)'(t,x)| \\ &\quad \le C\,(1+t)^{-\frac{n-1}{2}} (1 + |\, t - |x|\,|)^{-\frac{1}{2}} A(t) , \quad |\beta| \le s - [(n+2)/2] .\end{aligned} \tag{2.5}$$

From this, and the support properties of u, we can argue as in Remark 1.4 to get that

$$|\Gamma^\beta u(t,x)| \le C\,(1+t)^{-\frac{n-1}{2}} (1 + |\, t - |x|\,|)^{\frac{1}{2}} A(t) , \quad \text{if } |\beta| \le s - [(n+2)/2] .$$

Since $t - |x| = O(t)$ in the support, we get (2.4) from this, since

$$s - [(n+2)/2] \ge [(n+6)/2] , \quad \text{if } s \ge n+4 .$$

To prove (2.4′), let us fix $s \ge n+4$ and assume that A is large enough so that

$$A(0) \le \frac{A\varepsilon}{16} . \tag{2.6}$$

Since u satisfies (2.1), $A(0)$ depends only on the initial data (f, g). Therefore, A can be chosen to be independent of ε. We then claim that, if ε is small enough and $0 < T < T_*$, then

$$A(t) \le \frac{A\varepsilon}{2} , \quad \text{if } 0 \le t \le T . \tag{2.7}$$

Since $A(t) \in C([0,T_*))$ this is true for small T. Thus, the set of such T is nonempty and closed in $[0,T_*)$. To show that it is also open, and hence the entire interval, it therefore suffices to show that the weaker bound

$$A(t) \le A\varepsilon\,, \quad t \in [0,T] \subset [0,T_*) \tag{2.8}$$

implies (2.7) if ε is sufficiently small.

To do this notice that, by using (1.10) again, we can always express $\Gamma^\alpha \partial_j$ as a linear combination of terms of the form $\partial_k \Gamma^\beta$ with $|\beta| \le |\alpha|$, due to the fact that $[\partial, \Gamma]$ is always 0 or $\pm\partial_i$ for some i. Thus, by (2.5)

$$\begin{aligned} &|\Gamma^\alpha u'(t,x)| \\ &\quad \le CA\varepsilon/(1+t)^{\frac{n-1}{2}}\,, \quad 0 \le t \le T\,,\ 0 \le |\alpha| \le s - [(n+2)/2]\,. \end{aligned} \tag{2.9}$$

Thus, if g_0^{jk} denote the coefficients of the d'Alembertian, and if we let $r^{jk} = g_0^{jk} - g^{jk}(u')$, we have

$$\sum |r^{jk}| \le C'A\varepsilon/(1+t)^{\frac{n-1}{2}}\,, \tag{2.10}$$

and

$$\sum \|\partial_i g^{jk}(t,\,\cdot\,)\|_{L^\infty} \le C'A\varepsilon/(1+t)^{\frac{n-1}{2}}\,, \quad 0 \le t \le T\,.$$

Since the last factor is integrable when $n \ge 4$, we see that

$$\exp\Big(\int_0^T 2\sum \|\partial_i g^{jk}(t,\,\cdot\,)\|_{L^\infty}\,dt\Big) \le \exp\Big(\int_0^\infty C'A\varepsilon(1+t)^{-\frac{n-1}{2}}\,dt\Big)\,. \tag{2.11}$$

In what follows, we shall assume that $\varepsilon_0 > 0$ is chosen so that, whenever $0 < \varepsilon < \varepsilon_0$, the right sides of (2.10) and (2.11) are smaller than 1/2 and 2, respectively. Under this assumption, we shall be able to estimate the L^2 norm of $(\Gamma^\alpha u)'$, $|\alpha| \le s$ using the energy inequality from §2 of Chapter 1 and the equation

$$\begin{aligned} &\sum g^{jk}(u')\partial_j\partial_k\Gamma^\alpha u \\ &\quad = \Gamma^\alpha \sum g^{jk}(u')\partial_j\partial_k u + [\Box, \Gamma^\alpha]u + [\sum r^{jk}(u')\partial_j\partial_k, \Gamma^\alpha]u \\ &\quad = \Gamma^\alpha F(u') + [\Box,\Gamma^\alpha]u + \sum [r^{jk}(u'),\Gamma^\alpha]\partial_j\partial_k u + \sum r^{jk}(u')[\partial_j\partial_k,\Gamma^\alpha]u\,. \end{aligned} \tag{2.12}$$

We claim that each term on the right is a linear combination of terms of the form

$$a(u')\,\Gamma^{\alpha_1}u' \cdots \Gamma^{\alpha_N}u'\,, \quad |\alpha_j| \le |\alpha|\,, \tag{2.13}$$

where at most one $|\alpha_j|$ is larger than $(s+1)/2$ and

$$a(u') = O\big(\min\{|u'|^{2-N}\,,\,1+|u'|\}\big)\,. \tag{2.14}$$

Note that

$$(s+1)/2 < s - [(n+2)/2]\,, \quad \text{if } s \geq n+4\,.$$

Therefore, if we fix $0 \leq t \leq T$, and if $N \geq 2$, we can estimate the L^2 of (2.13), by using (2.9) to pointwise estimate all of the $\Gamma^{\alpha_j}u$ except the term with the highest number of derivatives, which we leave in the L^2 norm. By doing this, we would conclude that (2.13) has L^2 norm

$$\leq C_A\varepsilon(1+t)^{-\frac{n-1}{2}}A(t)\,,$$

where C_A just depends on A, if, as above s is fixed. Using (2.14), we reach the same conclusion if $N < 2$.

Thus, if the claim held, the right side of (2.12) would have L^2 norms with this bound, and, hence, Proposition 2.1 from the last chapter would yield

$$A(t) \leq 4A(0) + 4C_A\,\varepsilon\int_0^t (1+\tau)^{-\frac{n-1}{2}}\,A(\tau)\,d\tau\,,$$

if $0 < \varepsilon \leq \varepsilon_0$ and $0 \leq t \leq T$. Using Gronwall's inequality, we conclude from this that

$$A(t) \leq 4A(0)\exp\Big(\int_0^t 4C_A\,\varepsilon\,(1+\tau)^{-\frac{n-1}{2}}\,d\tau\Big)\,, \quad 0 \leq t \leq T\,. \tag{2.15}$$

Since $\int_0^\infty (1+\tau)^{-\frac{n-1}{2}}\,d\tau < \infty$ if $n \geq 4$, after possibly choosing $\varepsilon_0 > 0$ smaller, we can assume that the last factor in (2.15) is ≤ 2 if $0 < \varepsilon < \varepsilon_0$. This would then imply the desired improvement (2.7), because of (2.6).

Verifying the claim of course just involves arguments very similar to the ones used in the proof of the local existence theorem.

In fact, to start, notice that the first term in the right of (2.12) can be expressed as a linear combination of terms of the form

$$F^{(k)}(u')\,\Gamma^{\alpha_1}u'\cdots\Gamma^{\alpha_k}u'\,,$$

where

$$\sum|\alpha_j| \leq |\alpha| \leq s\,.$$

Hence, at most one of the α_j can have order larger than $s/2$. This, together with (2.3), shows that the first term on the right side of (2.12) is as in (2.13).

These arguments also show that the third term in the right of (2.12) has this form. In fact it can be expressed as a linear combination of terms of the form

$$(\partial^\gamma r^{jk})(u')\,\Gamma^{\alpha_1}u'\cdots\Gamma^{\alpha_l}u'\,\Gamma^\beta\partial_k u\,,$$

where $|\alpha_1|>0$ and

$$|\alpha_1|+\cdots+|\alpha_l|+|\beta|\le|\alpha|+1\le s+1\,.$$

Consequently at most one of the integers $|\beta|$, $|\alpha_1|,\dots,|\alpha_l|$ can be larger than $(s+1)/2$, leading to the claim for the third term.

It is easier to verify the claim for the other two terms. In fact, since $[\partial_j\partial_k,\Gamma^\alpha]$ can always be expressed as a linear combination of terms of the form $\Gamma^\beta\partial_i$, with $|\beta|\le|\alpha|\le s$, and since $r^{jk}(0)=0$, we conclude that the last term in the right of (2.12) is a combination of terms of the form (2.13) with $N=1$ there. To verify the claim for the remaining term, recall that, by (1.6) and (1.7), $[\Box,\Gamma^\alpha]$ can be expressed as a linear combination of terms of the form $\Gamma^\beta\Box$ with $|\beta|<|\alpha|$. But (2.1) implies that

$$\Gamma^\beta\Box u=\Gamma^\beta F(u')-\Gamma^\beta\sum r^{jk}(u')\partial_j\partial_k u\,,$$

and since we have just shown that the terms occurring in the right are as in (2.13), this completes the proof of the claim.

Since we showed earlier that the claim implies (2.7), this completes the proof of the theorem. $\Box$

In lower dimensions one does not have global existence; however, the following result holds.

Theorem 2.2. *There is a constant c, depending only on $(f,g)\in C_0^\infty(\mathbb{R}^n)$, such that for small $\varepsilon>0$ (2.1) has a smooth solution for $0\le t\le T_\varepsilon$, with*

$$T_\varepsilon=\begin{cases} e^{c/\varepsilon}\,, & n=3\\ (c/\varepsilon)^2\,, & n=2\\ c/\varepsilon\,, & n=1\,.\end{cases}\tag{2.16}$$

Since the life-span becomes exponentially large as the data gets smaller and smaller, the result for $n=3$ is called an almost global existence theorem, and it is due to John and Klainerman [1].

Proof. We just apply the proof of the last result. In fact, by applying the local existence theorem like before, we see that it suffices to show that, if $\varepsilon>0$ is small, then

$$\sum_{|\alpha|\le(n+6)/2}|\partial^\alpha u(t,x)|\in L^\infty([0,T_*)\times\mathbb{R}^n)\,,$$

if u has a C^∞ solution of (2.1) in $[0,T_*)\times\mathbb{R}^n$, with $T_*=T_*(\varepsilon)\le T_\varepsilon$. But this would be a consequence of (2.4′), which, in turn, would follow from showing that, for small $\varepsilon>0$, (2.7) holds if one assumes (2.8), with A as in (2.6).

Since everything is like in the earlier proof, we conclude that (2.15) must also hold here, and hence, if $0\le t\le T<T_*\le T_\varepsilon$,

$$A(t)\le 4A(0)\exp\Big(\int_0^t 4C_A\varepsilon(1+\tau)^{-\frac{n-1}{2}}\,d\tau\Big) \le \frac{A\varepsilon}{4}\exp\Big(\int_0^t 4C_A\varepsilon(1+\tau)^{-\frac{n-1}{2}}\,d\tau\Big). \tag{2.17}$$

Note that when $1\le n\le 3$, T_ε has been chosen so that

$$\int_0^{T_\varepsilon}\varepsilon(1+\tau)^{-\frac{n-1}{2}}\,d\tau\le C_nc,\quad 0<\varepsilon<1,$$

where c is as in (2.16). Hence, if c and $\varepsilon>0$ are small enough, we conclude that the last factor in (2.17) is ≤ 2. Since this means that (2.7) holds, the proof is complete. □

The bounds for the life-span of u in the last theorem are sharp in general. See, e.g., John [4], [8].

On the other hand, if $n=2$ or $n=3$, one can still obtain global existence for small data if (2.1) agrees to higher order with the linear Cauchy problem. In fact, if that (2.2) and (2.3) are replaced by

$$\partial^m r^{jk}(0)=0,\ \forall j,k,\ \text{if } 0\le m\le\kappa-1, \tag{2.2′}$$

and

$$\partial^m F(0)=0,\ \text{if } 0\le m\le\kappa, \tag{2.3′}$$

respectively, then one has the following result:

Theorem 2.3. *Let $n=2$ or $n=3$. Then, if $f,g\in C_0^\infty(\mathbb{R}^n)$ are fixed, (2.1) has a global smooth solution for small enough ε, provided that (2.2′) and (2.3′) hold with $\kappa=2$ when $n=3$, and $\kappa=3$ when $n=2$. Thus, there is global existence for (2.1) with small data if the equation is a third or fourth order perturbation of the linear Cauchy problem with $n=3$ or $n=2$, respectively.*

The main difference between the proof of this result and that of Theorem 2.1 is that, if we assume (2.2′) and (2.3′), then the right side of (2.12) is a linear combination of terms of the form (2.13), where the first factor is better behaved:

$$a(u') = O\big(\min\{|u'|^{\kappa-N}, 1+|u'|\}\big).$$

On account of this, for fixed t, the right side of (2.12) has L^2 norm

$$\le C_A\varepsilon(1+t)^{-\frac{n-1}{2}\kappa} A(t).$$

So, (2.15) can be replaced by

$$A(t) \le 4A(0)\exp\Big(\int_0^t 4C_A\varepsilon\,(1+\tau)^{-\frac{n-1}{2}\kappa}\,d\tau\Big),$$

leading to (2.7), for small $\varepsilon > 0$, as, by construction,

$$\int_0^\infty (1+\tau)^{-\frac{n-1}{2}\kappa}\,d\tau < \infty.$$

To close this section, we should point out that if one allows the nonlinearities to depend on u as well the situation can change dramatically. For instance, if the right side of (2.1) is replaced by $F(u,u')$ and if we assume that $F(0,0)$ and $dF(0,0)$ are both zero, so that, as in Theorem 2.1 the nonlinearity is still quadratic, then the existence results can be much different. If, for instance $F = u^2$, then the life-span of solutions is no longer $O(\exp(c/\varepsilon^2))$, but only $O(\varepsilon^{-2})$. This is part of a blow-up theorem of John [3] which will be treated in the next chapter. In the other direction, Lindblad [2] showed that when $n = 3$ there is global existence for equations of the form $\Box u = G(u,u',u'')$ if $G(0,0,u'') = 0$ and if G is quadratic in u' and vanishes to third order in u, thus removing the assumption that the nonlinearity is linear in the second derivatives. Also, if $F = u^2$, then there is no longer global existence for small data when $n = 4$. See Sideris [1] and Zhou [2]. However, if $n \ge 5$, then there is global existence for $\Box u = u^2$. See Klainerman [1].

§3. The null condition and global existence when $n = 3$

In the last section we saw that there is global existence for equations of the form (2.1) when $n \ge 4$. When $n = 3$, though, the proof of this result breaks down, in effect, because the decay factor $(1+t)^{-1}$ corresponding to $n = 3$ is no longer integrable.

Not only does the proof of Theorem 2.1 break down when $n = 3$, but the conclusion is no longer valid there as well. In fact, if $n = 3$, then every non-trivial C^3 solution of

$$\Box u = (\partial_t u)^2 \tag{3.1}$$

with compactly supported Cauchy data blows up in finite time. See John [4].

On the other hand, the seemingly similar equation

$$\Box u = (\partial_t u)^2 - \sum_{j=1}^{3} (\partial_j u)^2 \tag{3.2}$$

always has a global C^∞ solution if the data belongs to C_0^∞ and is small. This example is due to Nirenberg. To verify the assertion, one makes the substitution $v = 1 - e^{-u}$. Then the Cauchy problem for (3.2) is transformed to the linear Cauchy problem

$$\Box v = 0\,, \;\; v(0,x) = 1 - e^{-u(0,x)}\,, \;\; \partial_t v(0,x) = \partial_t u(0,x) e^{-u(0,x)}\,.$$

Since $u = -\log(1 - v)$, we conclude that (3.2) has a global solution if $v(t,x) < 1$. And since formula (1.5) from Chapter 1 implies that this is the case if, say, $u(0,x) = \varepsilon f(x)$ and $\partial_t u(0,x) = \varepsilon g(x)$, with $f, g \in C_0^\infty(\mathbb{R}^3)$ fixed and $\varepsilon > 0$ small, we get the claim.

From these two examples, we conclude that, if we wish to have a version of Theorem 2.1 which is valid when $n = 3$, we must only allow certain types of nonlinearities. Like in (2.1), they may involve the coefficients of $\partial_j \partial_k u$; however, for the sake of exposition, let us now concentrate on equations of the form $\Box u = F(u, u')$, postponing the more general case for the moment. As we shall see, the special case we are about to consider includes equations arising in certain geometric problems involving the so-called wave maps.

To handle this class of equations (and also because Nirenberg's example essentially describes the scalar case), we need to consider nonlinear hyperbolic systems. Specifically, for now, we shall consider equations of the form

$$\begin{cases} \Box u^I(t,x) = F^I(u(t,x), u'(t,x))\,, \; (t,x) \in \mathbb{R}_+^{1+3}\,, \quad I = 1, \dots, N \\ \\ u(0,x) = \varepsilon f(x)\,, \quad \partial_t u(0,x) = \varepsilon g(x)\,. \end{cases} \tag{3.3}$$

Here, of course, u is an $\mathbb{R}^N$-valued function

$$u = (u^1, \dots, u^N)\,,$$

and if we let $F = (F^1, \dots, F^N) \in C^\infty$, we can rewrite the first part of the equation as

$$\Box u = F(u, u')\,.$$

As in the last section, we shall assume that the linearization of (3.3) is the linear Cauchy problem $\Box u = 0$ with the above data. Hence, we assume that

$$F(0,0) = 0\,, \text{ and } dF(0,0) = 0\,. \tag{3.4}$$

In view of the above discussion, we then need to make certain assumptions on the quadratic part of the nonlinearity if we wish to have global existence results. These assumptions are contained in the following

Definition 3.1. Assume that F satisfies (3.4), and let F_0 denote the quadratic part of F so that

$$F = F_0 + O(\,|u|^3 + |u'|^3\,)\,.$$

We then say that F satisfies the *null condition* if F_0 is independent of u, and, moreover, for every $I = 1, \dots, N$,

$$F_0^I = F_0^I(u') = \sum_{L,M=1}^{N} \sum_{j,k=0}^{3} f_{jk}^{ILM} \partial_j u^L \partial_k u^M\,, \tag{3.5}$$

where f_{jk}^{ILM} are constants satisfying

$$\sum_{j,k=0}^{3} f_{jk}^{ILM} \xi_j \xi_k = 0\,, \quad \text{whenever } \xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2\,.$$

The null condition was introduced by Klainerman [2]. Under this assumption, we have the following theorem of Christodoulou [1] and Klainerman [6].

Theorem 3.2. *Let $n = 3$ and assume that F as above satisfies the null condition. Then, if $f, g \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^N)$ are fixed, (3.3) always has a global solution if $\varepsilon > 0$ is sufficiently small.*

As we pointed out before, a more general version holds, involving equations of the form $\Box u = F(u, u', u'')$ satisfying an appropriate null condition. However, we are only considering equations of the form (3.3) for now, in order to try to not get too bogged down with technicalities. The proof of the more general theorem of Christodoulou and Klainerman only involves straightforward, but somewhat technical, modifications of the proof of Theorem 3.2 and it will be sketched at the end of this section.

Before turning to the proof of Theorem 3.2, we need to collect several facts relating to the special form of the nonlinearity involved.

We first notice that if we assume the null condition, then the main part of the nonlinearity in (3.3) must have a special form. In fact, if we let

$$Q_0(v,w) = \partial_t v \partial_t w - \sum_{j=1}^{3} \partial_j v \partial_j w, \tag{3.6}$$

and

$$Q_{ab}(v,w) = \partial_a v \partial_b w - \partial_a w \partial_b v, \quad 0 \le a < b \le 3, \tag{3.7}$$

then we claim that $F_0(u')$ must just involve linear combinations of these "null forms." Specifically, there must be constants c_0^{IJK} and c_{ab}^{IJK} so that

$$F_0^I = \sum_{J,K=1}^{N} c_0^{IJK} Q_0(u^J, u^K) + \sum_{J,K=1}^{N} \sum_{0\le a<b\le 3} c_{ab}^{IJK} Q_{ab}(u^J, u^K). \tag{3.5'}$$

This claim of course just follows from the fact that if $B(\xi, \eta)$ is a bilinear form on $\mathbb{R}^{1+3} \times \mathbb{R}^{1+3}$ satisfying $B(\xi,\xi) = 0$ when $\xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$, then B must be a linear combination of $Q_0(\xi,\eta) = \xi_0\eta_0 - \sum_{j=1}^3 \xi_j\eta_j$ and the $Q_{ab}(\xi,\eta) = \xi_a\eta_b - \xi_b\eta_a$, $0 \le a < b \le 3$.

In the proof of Theorem 3.2, we shall see that $\Gamma^\alpha u = O((1+|(t,x)|)^{-1})$, using Theorem 1.5 and the decay for the solution of the linear equation. Assuming this, the following key lemma will allow us to see that, because of their special form, the quadratic terms in (3.3) behave like the cubic error terms in terms of their time decay.

Lemma 3.3. *Let Q be one of the null forms in* (3.6) – (3.7). *Then if* $t > 0$

$$|Q(v,w)(t,x)| \le \frac{C}{1+t+|x|} \sum_{|\alpha|=1} |\Gamma^\alpha v(t,x)| \sum_{|\alpha|=1} |\Gamma^\alpha w(t,x)|.$$

Proof. We need only consider the case where $t + |x| > 1$, since the other case is trivial. If $1 \le i < j \le 3$, a calculation gives

$$Q_{ij}(v,w) = t^{-1}\big[\partial_t v \Omega_{ij} w + \Omega_{0i} v \partial_j w - \Omega_{0j} v \partial_i w\big],$$

leading to the desired bound for this null form if $t > |x|$. For the other case we need to use

$$\partial_i = \sum_{j=1}^{3} \frac{x_i x_j}{|x|^2}\partial_j + \sum_{j=1}^{3} \frac{x_j \Omega_{ij}}{|x|^2}, \quad 1 \le i \le 3.$$

From this we deduce that

$$Q_{ij}(v,w) = |x|^{-1}\left[\partial_r v\Omega_{ij}w + (A_i v\partial_j w - A_j v\partial_i w)\right],$$

if $A_i = \sum_{j=1}^3 x_i\Omega_{ij}/|x|$. Hence, the bound holds for Q_{ij} if $1 \le i < j < 3$.

For $i = 0$, $j \ge 1$, we write

$$Q_{0j}(v,w) = t^{-1}\left(\partial_t v\Omega_{0j}w - \partial_t w\Omega_{0j}v\right)$$

to get the right bounds here when $t > |x|$. For the other case we use

$$Q_{0j}(v,w) = |x|^{-1}\left[\tfrac{x_j}{|x|}(\Omega_r v\partial_r w - \partial_r v\Omega_r w) + (\partial_t v A_j w - A_j v\partial_t w)\right],$$

where A_j is above and $\Omega_r = |x|^{-1}\sum_{j=1}^3 x_j\Omega_{0j} = t\partial_r + r\partial_t$.

To handle the remaining null form we just write

$$\begin{aligned} Q_0(v,w) &= t^{-1}\left[\partial_t v L_0 w - \sum_{i=1}^3 \Omega_{0i}v\partial_i w\right] \\ &= |x|^{-1}\left[(\Omega_r v\partial_t w - \partial_r v L_0 w) - \sum_{i=1}^3 \partial_i v A_i w\right], \end{aligned}$$

which of course leads to the desired bounds here as well. □

We shall also need to know the action of the invariant vector fields on $F_0(u')$. In view of (3.5′), this amounts to knowing the form of $\Gamma Q(v,w)$ if Γ is one of the invariant vector fields, and Q is a null form. To this end, let us define the "commutator" of this action by

$$[\Gamma, Q](v,w) = \Gamma Q(v,w) - Q(\Gamma v, w) - Q(v, \Gamma w).$$

Clearly, if $\Gamma = \partial_j$, the commutator is zero. On the other hand, using the commutation relations for the invariant vector fields one easily verifies that

$$\begin{aligned} [\Omega_{jk}, Q_0] &= [L_0, Q_{jk}] = 0, \quad 0 \le j \le k \le 3 \\ [L_0, Q_0] &= -2Q \\ [\Omega_{jk}, Q_{ab}] &= -g_0^{jj}\left(g_0^{aj}Q_{bk} - g_0^{ak}Q_{bj} + g_0^{bj}Q_{ak} - g_0^{bj}Q_{aj}\right). \end{aligned} \tag{3.8}$$

Using (3.8) and Lemma 3.3 we obtain the following important result.

Proposition 3.4. *Let Q be one of the null forms. Then, if $t > 0$,*

$$(1+t+|x|)\sum_{|\alpha|\le M}|\Gamma^\alpha Q(v,w)| \le C_M \sum_{1\le|\alpha|\le M+1}|\Gamma^\alpha v|\cdot\sum_{1\le|\alpha|\le\frac{M+2}{2}}|\Gamma^\alpha w| + C_M \sum_{1\le|\alpha|\le M+1}|\Gamma^\alpha w|\cdot\sum_{1\le|\alpha|\le\frac{M+2}{2}}|\Gamma^\alpha v|.$$

Consequently, if $F(u,u')$ satisfies the null condition and if F_0, as in (3.5) denotes its quadratic part, then

$$(3.9)\quad \sum_{|\alpha|\le M}|\Gamma^\alpha F_0(u')| \le \frac{C_M}{(1+t+|x|)}\sum_{1\le|\alpha|\le M+1}|\Gamma^\alpha u|\cdot\sum_{1\le|\alpha|\le\frac{M+2}{2}}|\Gamma^\alpha u|.$$

The proof of the theorem requires one last ingredient. This involves a modification of the energy method from the last chapter involving more general vector fields than ∂_t. So we seek other first order differential operators $X(\partial)$, this time involving the invariant vector fields, so that $X(\partial)v\Box v$ can be written as an exact divergence. As before, we shall also want the associated energy (the integral of the zero-component of the divergence) to always be nonnegative, in order to get a non-trivial energy inequality. It turns out that, up to Lorentz transformations, the only such operators involve combinations of ∂_t and

$$X_0(\partial) = (t^2+|x|^2)\partial_t + 2t(\sum_{i=1}^3 x_i\partial_i + 1).$$

See, e.g., Hörmander [5] for a proof of this fact. The operator X_0 was introduced by Morawetz [1] in connection with decay estimates for the wave equation outside of an obstacle.

For our energy arguments we shall require the most simple combination of these two operators, namely,

$$\begin{aligned} X(\partial) &= \partial_t + X_0(\partial) \\ &= (1+t^2+|x|^2)\partial_t + 2t\sum_{i=1}^3 x_i\partial_i + 2t \qquad (3.10)\\ &= \vec{X}\cdot\nabla_{t,x} + 2t, \end{aligned}$$

where the coefficients of the principal part are given by

$$\vec{X} = (1+t^2+|x|^2, 2tx_1, 2tx_2, 2tx_3).$$

If, as in §2 of Chapter 1 we let $\mathbf{1} = (1,0,0,0)$, then we have the following important formula

$$X(\partial)v\,\Box v = \operatorname{div}\left(X(\partial)v g_0 v' - \tfrac{1}{2}g_0(v',v')\overrightarrow{X} - v^2\mathbf{1}\right). \tag{3.11}$$

To see this, we note that div $\overrightarrow{X} = 8t$ and

$$\sum_{j,k=0}^{3} g_0^{jj}\partial_j X_k\,\partial_j v\,\partial_k v = 2tg_0(v',v'),$$

if X_j, $j = 0,\ldots,3$, denote the coordinates of $\overrightarrow{X}$. Consequently,

$$\begin{aligned}
&\operatorname{div}\left(X(\partial)vg_0v' - \tfrac{1}{2}g_0(v',v')\overrightarrow{X} - v^2\mathbf{1}\right)\\
&= \operatorname{div}\left((\overrightarrow{X}\cdot v')g_0v' - \tfrac{1}{2}g_0(v',v')\overrightarrow{X}\right) + \operatorname{div}\left(2tvg_0v' - v^2\mathbf{1}\right)\\
&= (\overrightarrow{X}\cdot v')\Box v + \sum g_0^{jj}\partial_j v\,\partial_j X_k\,\partial_k v - \tfrac{1}{2}g_0(v',v')\operatorname{div}\overrightarrow{X} + 2t\sum g_0^{jj}\partial_j(v\partial_j v)\\
&= (\overrightarrow{X}\cdot v' + 2tv)\Box v + 2tg_0(v',v') - 4tg_0(v',v') + 2tg_0(v',v')\\
&= X(\partial)v\,\Box v,
\end{aligned}$$

as claimed.

Let us now compute the energy density associated with (3.11). Notice that

$$\tfrac{1}{2}|L_0v|^2 + \tfrac{1}{2}\sum_{0\le j<k\le 3}|\Omega_{jk}v|^2 = \tfrac{1}{2}(t^2+|x|^2)|v'|^2 + 2t\partial_t v\,x\cdot\partial_x v.$$

Using this we see that the zero-component of the divergence in (3.11) is

$$\begin{aligned}
e_0(v) &= X(\partial)v\,\partial_t v - \tfrac{1}{2}(1+t^2+|x|^2)g_0(v',v') - v^2\\
&= \tfrac{1}{2}(1+t^2+|x|^2)|v'|^2 + 2t\partial_t v\partial_x v\cdot x + 2tv\partial_t v - v^2\\
&= \tfrac{1}{2}\Big[\,|v'|^2 + |L_0v|^2 + \sum_{0\le j<k\le 3}|\Omega_{jk}u|^2\Big] + 2tv\partial_t v - v^2.
\end{aligned}\tag{3.12}$$

Let $E_0(t;v)$ be the associated energy:

$$\begin{aligned}
E_0(t;v) &= \int_{\mathbb{R}^3} e_0(v)(t,x)\,dx\\
&= \frac{1}{2}\int_{\mathbb{R}^3}\Big(|v'|^2 + |L_0v|^2 + \sum_{0\le j<k\le 3}|\Omega_{jk}v|^2\Big)\,dx + \int_{\mathbb{R}^3}\left(2tv\partial_t v - v^2\right)dx
\end{aligned}$$

Note that $2vt\partial_t v = 2vL_0 v - \sum_{j=1}^3 x_j \partial_j v^2$, and so

$$\int 2tv\partial_t v\, dx = \int 2vL_0 v\, dx + 3\int v^2\, dx, \tag{3.13}$$

or, equivalently,

$$\int (2tv\partial_t v - v^2)\, dx = \int (2vL_0 v + 2v^2)\, dx.$$

From this, we deduce that

$$\tfrac{1}{2}\|L_0 v + 2v\|_{L^2}^2 = \int \left(\tfrac{1}{2}|L_0 v|^2 + 2tv\partial_t v - v^2 \right) dx,$$

and so

$$E_0(t; v) = \frac{1}{2}\Big(\|v'\|_{L^2}^2 + \sum_{0\le j<k\le 3} \|\Omega_{jk} v\|_{L^2}^2 + \|L_0 v + 2v\|_{L^2}^2 \Big). \tag{3.14}$$

To apply the estimates from §1, we shall need to compare the energy with L^2 norms involving the invariant vector fields. Specifically, we require the following

Lemma 3.5. *There is a constant C so that*

$$C^{-1} \sum_{|\alpha|\le 1} \|\Gamma^\alpha v(t,\,\cdot\,)\|_{L^2}^2 \le E_0(t; v) \le C \sum_{|\alpha|\le 1} \|\Gamma^\alpha v(t,\,\cdot\,)\|_{L^2}^2. \tag{3.15}$$

Proof. The upper bound clearly exists by (3.14) and Minkowski's inequality. To prove the lower bound it suffices to show that

$$\|v(t,\,\cdot\,)\|_{L^2}^2 + \|L_0 v(t,\,\cdot\,)\|_{L^2}^2 \le C E_0(t; v)$$

for some constant C.

For this, as in the proof of Lemma 3.3, let $\Omega_r = |x|^{-1}\sum_{j=1}^3 x_j \Omega_{0j} = t\partial_r + r\partial_t$. Notice, by using polar coordinates and integrating by parts, that

$$\begin{aligned} \int 2tv\partial_t v\, dx &= \int 2tr^{-1} v\Omega_r v\, dx - \int 2t^2 r^{-1} v\partial_r v\, dx \\ &= \int 2tr^{-1} v\Omega_r v\, dx + \int t^2 r^{-2} v^2\, dx. \end{aligned} \tag{3.16}$$

Thus, if we take $3/4$ times the right side of (3.13) plus $1/4$ of the right side of (3.16), we conclude that

$$\int (2tv\partial_t v - v^2)\, dx = \int \left(\tfrac{3}{2} vL_0 v + \tfrac{5}{4} v^2 + \tfrac{1}{2} tr^{-1} v\Omega_r v + \tfrac{1}{4} t^2 r^{-2} v^2 \right) dx.$$

Hence,

$$E_0(t;v) = \frac{1}{2}\int \Big(|v'|^2 + \sum_{0\le j<k\le 3} |\Omega_{jk}v|^2 - |\Omega_r v|^2 \Big)\, dx$$
$$+ \frac{1}{2}\int \Big((L_0 v)^2 + 3vL_0 v + \tfrac{5}{2}v^2 \Big)\, dx$$
$$+ \frac{1}{2}\int \Big((\Omega_r v)^2 + tr^{-1} v\Omega_r v + \tfrac{1}{2}t^2 r^{-2} v^2 \Big)\, dx .$$

Using the definition of Ω_r we see that the first integrand is nonnegative. Elementary calculations show that the same is true for the last two integrals. Moreover, since the second integrand is bounded from below by a positive constant times $(L_0 v)^2 + v^2$, the proof is complete. □

Using the last result and the energy integral method from §2 of the last chapter, we get the following important result.

Proposition 3.6. *Let $v \in C^2(\mathbb{R}^{1+3}_+)$ vanish for large x. Then, if $t > 0$,*

$$\frac{d}{dt}E_0(t;v) \le 2\big\| \,(1+t+|x|)\Box v(t,\,\cdot\,)\, \big\|_{L^2} E_0(t;v)^{1/2} . \tag{3.17}$$

Moreover, for fixed $M = 0, 1, \dots,$

$$\sum_{|\alpha|\le M+1} \|\Gamma^\alpha v(t,\,\cdot\,)\|_{L^2} \tag{3.18}$$
$$\le C \sum_{|\alpha|\le M+1} \|\Gamma^\alpha v(0,\,\cdot\,)\|_{L^2} + C \sum_{|\alpha|\le M} \int_0^t \|\,(1+\tau+|x|)\Gamma^\alpha \Box v(\tau,\,\cdot\,)\,\|_{L^2}\, d\tau .$$

Proof. Since $e_0(v)$ is the zero-component of the divergence in (3.11)

$$\frac{d}{dt}E_0(t;v) = \int \frac{d}{dt}e_0(v)(t,x)\, dx$$
$$= \int X(\partial)v\, \Box v\, dx$$
$$\le \big\| \,(1+|(t,x)|)^{-1} X(\partial)v\, \big\|_{L^2} \,\|\,(1+|(t,x)|)\,\Box v\,\|_{L^2} .$$

But $|X(\partial)v| = |\vec{X}\cdot v' + 2tv|$. and

$$\vec{X}\cdot v' = \partial_t v + (t,x)\cdot (L_0 v, \Omega_{01}v, \Omega_{02}v, \Omega_{03}v) .$$

Hence,

$$|X(\partial)v|^2 \le 2(\partial_t v)^2 + 2|(t,x)|^2 \left[|L_0 v + 2v|^2 + \sum |\Omega_{0j} v|^2 \right]$$
$$\le 4(1+|(t,x)|)^2 \left[\tfrac{1}{2}|\partial_t v|^2 + \tfrac{1}{2}|L_0 v + 2v|^2 + \tfrac{1}{2}\sum |\Omega_{0j} v|^2 \right].$$

Combining the two inequalities leads to (3.17), in view of (3.14).

Clearly (3.17) implies

$$(3.17') \qquad E_0(t;v)^{1/2} \le E_0(0;v)^{1/2} + \int_0^t \| (1+\tau+|x|)\Box v(\tau,\,\cdot\,) \|_{L^2}\, d\tau\,.$$

Since $[\Box, \Gamma]$ is either 0 or $2\Box$, if Γ is one of the invariant vector fields, this, along with Lemma 3.5, implies (3.18). $\Box$

We are finally ready to prove Theorem 3.2. The key ingredients are the two propositions from this section, along with the Klainerman-Sobolev estimates and Hörmander's estimate for the linear inhomogeneous wave equation from §1.

Proof of Theorem 3.2. As in the proof of Theorem 2.1, we shall use the continuity method. Since the local existence theorems from the last chapter are also valid for systems of the form (3.3), it suffices to show that if u is a C^∞ solution of (3.3) in $[0,T_*) \times \mathbb{R}^3$, then $\sum_{|\alpha|\le 4} |\partial^\alpha u| \in L^\infty([0,T_*)\times\mathbb{R}^3)$, if $0<\varepsilon<\varepsilon_0$, where ε_0 depends only on f and g in (3.3).

To do this, let u_0 be the solution of the linear wave equation $\Box u_0 = 0$ with the same Cauchy data: $u_0(0,\,\cdot\,) = \varepsilon f$, $\partial_t u_0(0,\,\cdot\,) = \varepsilon g$. Then since $\Box\Gamma u_0 = 0$, if Γ is one of the invariant vector fields, if k is fixed

$$(3.19) \qquad \sum_{|\alpha|\le k} |\Gamma^\alpha u_0(t,x)| \le \frac{A\varepsilon}{2(1+t+|x|)}\,,$$

for some absolute constant A. We then claim that there is an $\varepsilon_0 > 0$ so that, if $0<\varepsilon<\varepsilon_0$, then

$$(3.20) \qquad \sum_{|\alpha|\le k} |\Gamma^\alpha u(t,x)| \le \frac{A\varepsilon}{(1+t+|x|)}\,, \quad 0\le t<T_*\,.$$

To prove the theorem, we need bounds like this for some $k\ge 4$, so let us fix such a k in what follows.

Using (3.3) and (3.4), we see that $\Gamma^\alpha u(0,\,\cdot\,) - \Gamma^\alpha u_0(0,\,\cdot\,) = O(\varepsilon^2)$. Hence, if $\varepsilon>0$ is small enough and fixed, we conclude that (3.19) implies that the bound in (3.20) holds if $0\le t\le T$, with T sufficiently small. So let us assume that $0<T<T_*$ and that

$$(3.20') \qquad \sum_{|\alpha|\le k} |\Gamma^\alpha u(t,x)| \le \frac{2A\varepsilon}{(1+t+|x|)}\,, \quad t\in[0,T]\subset[0,T_*)\,.$$

We then claim that, if ε is sufficiently small, then this implies that there must be absolute constants A_0 and C_0 so that

$$\sum_{|\alpha|\le k+3} \|\Gamma^\alpha u(t,\,\cdot\,)\|_{L^2} \le A_0(1+t)^{C_0\varepsilon} \sum_{|\alpha|\le k+3} \|\Gamma^\alpha u(0,\,\cdot\,)\|_{L^2}\,, \text{ if } 0\le t\le T. \tag{3.21}$$

Assume that we could additionally show that this inequality and (3.20′) imply the following improvement over (3.20′) if $\varepsilon > 0$ is small:

$$\sum_{|\alpha|\le k} |\Gamma^\alpha u(t,x)| \le \frac{A\varepsilon}{1+t+|x|}, \quad 0\le t\le T. \tag{3.22}$$

We then would conclude that, if $\varepsilon > 0$ is small enough, the set of $T\in[0,T_*)$ so that (3.22) holds is both open and relatively closed in $[0,T_*)$. Since we have just observed that it is also non-empty, we conclude that this "continuous induction" argument would imply that (3.20) must hold, giving us the theorem.

So, to summarize the proof will consist of two steps. First, we must show that (3.20′) implies (3.21). Then, we must show that (3.20′) and (3.21) imply (3.22).

Step 1: (3.20′) $\implies$ (3.21).

Here we shall want to apply Proposition 3.6. So we notice that

$$\Gamma^\alpha\Box u = \Gamma^\alpha F(u,u') = \Gamma^\alpha F_0(u') + \Gamma^\alpha R(u,u')\,,$$

where, if $F_0(u')$ as in Definition 3.1 is the quadratic part of F, $R(u,u') = F(u,u') - F_0(u')$. To apply the proposition we need to estimate the L^2 norm of $(1+t+|x|)$ times each of these two terms if $|\alpha|\le (k+3)-1 = k+2$. For the main term, we just use Proposition 3.4 to see that, if $0\le t\le T$,

$$\begin{aligned}
\|(1+t+|x|)(\Gamma^\alpha F_0(u'))(t,\,\cdot\,)\|_{L^2}
&\le C \sum_{|\beta|\le k+3} \|\Gamma^\beta u(t,\,\cdot\,)\|_{L^2} \sum_{|\beta|\le \frac{k+4}{2}} \|\Gamma^\beta u(t,\,\cdot\,)\|_{L^\infty} \\
&\le \frac{2CA\varepsilon}{1+t} \sum_{|\beta|\le k+3} \|\Gamma^\beta u(t,\,\cdot\,)\|_{L^2}\,,
\end{aligned}$$

with the last step using (3.20′) and the fact that $(k+4)/2\le k$ if $k\ge 4$. The remainder term is also easy to estimate. Since it vanishes to third order at

$(u, u') = 0$, by Taylor's theorem and Leibnitz's rule, $\Gamma^\alpha R$ must be a linear combination of terms of the form

$$a(u,u')\Gamma^{\alpha_1}u^{(j_1)}\Gamma^{\alpha_2}u^{(j_2)}\Gamma^{\alpha_3}u^{(j_3)}\,, \tag{3.23}$$

where the $|\alpha_1| + |\alpha_2| + |\alpha_3| \le |\alpha| \le k+2$, $j_i = 0$, or 1, and $a = O(1)$. Hence, here too there is at most one factor involving more than $(k+4)/2$ derivatives, and so we can apply (3.20′) to see that for fixed t

$$\|(1+t+|x|)\Gamma^\alpha R(u,u')(t,\,\cdot\,)\|_{L^2} \le C\varepsilon^2(1+t)^{-1} \sum_{|\beta|\le k+3} \|\Gamma^\beta u(t,\,\cdot\,)\|_{L^2}\,.$$

Hence, if, say, $0 < \varepsilon < 1$, and if (3.20′) holds, we conclude that there must a uniform constant C_1 so that

$$\|(1+t+|x|)\Gamma^\alpha \Box u(t,\,\cdot\,)\|_{L^2} \le C_1\varepsilon(1+t)^{-1} \sum_{|\beta|\le k+3} \|\Gamma^\beta u(t,\,\cdot\,)\|_{L^2}\,.$$

Using (3.18), we see that this yields

$$\sum_{|\alpha|\le k+3} \|\Gamma^\alpha u(t,\,\cdot\,)\|_{L^2}$$

$$\le C \sum_{|\alpha|\le k+3} \|\Gamma^\alpha u(0,\,\cdot\,)\|_{L^2} + \int_0^t CC_1\varepsilon(1+\tau)^{-1} \sum_{|\alpha|\le k+3} \|\Gamma^\alpha u(\tau,\,\cdot\,)\|_{L^2}\,d\tau\,.$$

Since, by Gronwall's inequality this yields (3.21), with C_0 fixed multiple of CC_1, this completes the first step.

Step 2: (3.21) **and** (3.20′) $\Longrightarrow$ (3.22).

Recall that A has been chosen so that the solution to the linear wave equation, $\Box u_0 = 0$, with data $(\varepsilon f, \varepsilon g)$ satisfies (3.19). Also, as we pointed out before, the equation implies that the Cauchy data of $\Gamma^\alpha u - \Gamma^\alpha u_0$ is $O(\varepsilon^2)$, and, therefore, by Theorem 1.1 from Chapter 1, the solution of the linear wave equation with this data must be $O(\varepsilon^2(1+t+|x|)^{-1})$. Consequently, if we fix $|\alpha| \le k$, we see that Theorem 1.5 implies that (3.22) would follow if we could show that, for small $\varepsilon > 0$,

$$\int_0^T \int_{\mathbb{R}^3} |\,\Gamma^\beta \Box \Gamma^\alpha u(t,x)\,| \frac{dx dt}{1+t+|x|} \le C\varepsilon^2\,, \quad |\beta| \le 2\,,\ |\alpha| \le k\,. \tag{3.22′}$$

In fact, if this inequality held, then in view of the above discussion, we would have that $\Gamma^\alpha u - \Gamma^\alpha u_0 = O(\varepsilon^2(1+t+|x|)^{-1})$, by Theorem 1.5, which of course yields (3.22) for small $\varepsilon > 0$ because of (3.19).

Since the commutator of $\Box$ with one of the invariant vector fields is 0 or $2\Box$, (3.22′) would of course be a consequence of the estimate

$$\int_0^T \int_{\mathbb{R}^3} | \Gamma^\alpha \Box u(t,x) | \frac{dx dt}{1+t+|x|} \le C\varepsilon^2 , \quad |\alpha| \le k+2 . \tag{3.24}$$

To show that $\Gamma^\alpha F_0(u')$ satisfies this estimate, we note that Proposition 3.4 implies that it is pointwise dominated by $(1+t+|x|)^{-1} \sum_{|\alpha| \le k+3} |\Gamma^\alpha u|^2$. Hence, by (3.21),

$$\int \big| \Gamma^\alpha F_0(u')(t,x) \big| \, dx \le C\varepsilon^2 (1+t)^{-1+2C_0\varepsilon} .$$

To estimate the remainder, we note that it is a combination of terms of the form (3.23). If we use (3.20′) to pointwise estimate the factor involving the fewest number of derivatives and apply Schwarz's inequality to estimate the integral of the remaining factors, we conclude that

$$\int \big| \Gamma^\alpha R(u,u')(t,x) \big| \, dx \le C\varepsilon^3 (1+t)^{-1+2C_0\varepsilon} .$$

Since $\Box u = F_0 + R$, we conclude that (3.24) must hold if $2C_0\varepsilon < 1$.

This completes the proof. $\Box$

Example (wave maps). Let $S \subset \mathbb{R}^{N+1}$ be a smooth hypersurface. Then we say that a map u from $\mathbb{R}^{1+n}$ to S is harmonic with respect to the Lorentz metric if it is stationary with respect to the Lagrangian

$$L(u;R) = \frac{1}{2} \int_R \big(|\partial_x u|^2 - |\partial_t u|^2 \big) \, dt dx , \quad R \subset \mathbb{R}^{1+n} ,$$

with the integral being interpreted formally if it is divergent. When we say that u is stationary with respect to L, we mean that if $\psi \in C_0^\infty(R; T_u S)$, then

$$0 = \frac{d}{d\varepsilon} L(u + \varepsilon\psi; R) \big|_{\varepsilon=0} = \int \Box u \cdot \psi \, dt dx .$$

In other words, u must satisfy

$$\Box u \perp T_u S .$$

Written in local coordinates $(u^1, \dots, u^N)$ on S, this means that u must satisfy

$$\Box u^I = \sum_{J,K=1}^N \Gamma^I_{JK}(u) Q_0(u^J, u^K) ,$$

where Γ^I_{JK} are the Christoffel symbols for S, and, as before, $Q_0(v,w) = \partial_t v \partial_t w - \partial_x v \cdot \partial_x w$. Maps with this property are called wave maps, and the full wave map equation of course involves initial conditions specifying the values of u and $\partial_t u$ when $t = 0$.

Further Remarks. Theorem 3.2 can be extended to include certain types of nonlinear equations where the coefficients of $\partial_j \partial_k u$ are allowed to depend on u'. Specifically, the theorem of Christodoulou and Klainerman includes certain quasi-diagonal systems of the form

$$(3.3') \quad \begin{cases} \Box u^I(t,x) = \sum_{j,k=0}^{3} g^{jk}(u,u')\partial_j\partial_k u^I + F^I(u,u'), \quad I = 1,\dots,N \\ u(0,x) = \varepsilon f(x), \quad \partial_t u(0,x) = \varepsilon g(x), \end{cases}$$

where $g^{jk}(u,u')$ satisfies a null condition contained in the following

Definition 3.7. We say that $\{g^{jk}(u,u')\}$ satisfies the null condition if

$$g^{jk}(u,u') = \sum_{I=1}^{N}\sum_{l=0}^{3} g_I^{jkl}\partial_l u^I + O(|u|^2 + |u'|^2),$$

where, for every $I = 1,\dots,N$, the constants g_I^{jkl} satisfy

$$\sum_{j,k,l=0}^{3} g_I^{jkl}\xi_j\xi_k\xi_l = 0, \quad \text{whenever } \xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2.$$

Assuming that the full null condition is satisfied, we have the following

Theorem 3.2'. *Fix $f, g \in C^\infty(\mathbb{R}^3; \mathbb{R}^N)$ and assume that $F(u,u')$ and $\{g^{jk}(u,u')\}$ satisfy the null condition. Then, if $\varepsilon > 0$ is sufficiently small, (3.3') has a global C^∞ solution.*

As we said before, the proof just involves rather technical modifications of the proof of Theorem 3.2. To describe them, let us first observe that if

$$G(u',u'') = \sum_{I=1}^{N}\sum_{j,k,l=0}^{3} g_I^{jkl}\partial_l u^I \partial_j\partial_k u$$

is the main term in the right side of the equation, then $G(u',u'')$ must be a linear combination of terms of the form $Q(u^I, \partial u)$, where Q is one of the null forms. Hence, using Proposition 3.4, we find that

$$(3.9') \quad \sum_{|\alpha|\le M} |\Gamma^\alpha G(u',u'')| \le C(1+t+|x|)^{-1} \sum_{|\alpha|\le M+2} |\Gamma^\alpha u| \sum_{|\alpha|\le \frac{M+4}{2}} |\Gamma^\alpha u|.$$

This is one of the two main additional ingredients in the proof of this result versus the earlier one. The other involves a modification of the energy arguments used before to take into account the variable coefficients of $\partial_j \partial_k u$ in (3.3′). This is in the spirit of the technical differences between the proofs of the energy inequality (2.2) in Chapter 1 for the d'Alembertian and its generalization there, (2.5), to variable coefficients.

In the present context, let us abuse notation a bit and let

$$g^{jk} = g_0^{jk} - g^{jk}(u, u'),$$

where $g^{jk}(u, u')$ is as in (3.3′). Then one can write

$$(3.11') \quad X(\partial)v \sum_{j,k=0}^{3} g^{jk}\partial_j\partial_k v = \text{div}\,\left(X(\partial)vgv' - \tfrac{1}{2}g(v', v')\overrightarrow{X} - v^2\mathbf{1} \right) + R,$$

where the remainder, R, can of course be computed explicitly. The important thing, though, is that, if we assume (3.20′), as in the proof of Theorem 3.2, then (3.9′) can be used to see that if $v = u$, then there is a fixed constant C_1 so that,

$$\int |R|\, dx \le C_1\varepsilon(1+t)^{-1} \sum_{|\alpha|\le 1} \|\Gamma^\alpha u(t, \cdot)\|_{L^2}^2, \quad 0 \le t \le T,$$

assuming that $\varepsilon > 0$ in (3.19′) is sufficiently small. In a similar vein, if $u = v$ and if, for fixed t, $E(t; u)$ denotes the zero-component of the divergence in (3.11′), then, for small enough $\varepsilon > 0$ in (3.19′) we have

$$(3.15') \quad (2C)^{-1} \sum_{|\alpha|\le 1} \|\Gamma^\alpha u(t, \cdot)\|_{L^2}^2 \le E(t; u) \le 2C \sum_{|\alpha|\le 1} \|\Gamma^\alpha u(t, \cdot)\|_{L^2}^2,$$

with C being the constant in (3.15).

Using (3.15′) and the estimate for the remainder in (3.11′) we can modify the proof of Proposition 3.6 to see that

$$\begin{aligned}\frac{d}{dt}E(t; u) \le \|\,(1+t+|x|)\sum g^{jk}\partial_j\partial_k u(t, \cdot)\,\|_{L^2} \cdot E(t; u)^{1/2} \\ + C_1'\varepsilon(1+t)^{-1}E(t; u).\end{aligned}$$

After integrating and applying Gronwall's inequality this in turn yields an estimate of the form

$$\begin{aligned}&E(t; u)^{1/2} \\ &\le C(1+t)^{C\varepsilon}\Big(E(0; u)^{1/2} + \int_0^t (1+\tau)^{C\varepsilon}\|(1+\tau+|x|)\sum g^{jk}\partial_j\partial_k u(\tau, \cdot)\|_{L^2} d\tau\Big)\end{aligned}$$

If one uses this estimate along with (3.15′) and (3.9′), it is not hard to modify the proof of Theorem 3.2 to obtain the more general result.

§4. The restriction theorem and local existence revisited

In this section we shall show that, for the type of equations arising from wave maps, one can significantly improve the regularity assumptions in the local existence theorem from Chapter 1. The type of equations we shall consider here are of the form

$$\text{(4.1)}\qquad \begin{cases} \Box u^I = \sum_{J,K=1}^{N} \Gamma^I_{J,K}(u) Q^I_{J,K}(u^J, u^K), \quad I = 1, \dots, N \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g. \end{cases}$$

Here $\Gamma^I_{J,K} \in C^\infty$ for all I, J, K, and $Q^I_{J,K}$ is a linear combination of the null forms Q_0 and Q_{ab} introduced in the last section, that is,

$$Q^I_{J,K}(v,w) = c^I_{JK} Q_0(v,w) + \sum_{0 \le a < b \le n} c^{Iab}_{JK} Q_{ab}(v,w),$$

for certain constants c^I_{JK} and c^{Iab}_{JK}. For wave maps one just has $Q^I_{JK} = Q_0$, and Γ^I_{JK} are Christoffel symbols.

Let us first state a result of Klainerman and Machedon [1] which says that when $n = 3$ one needs much less regularity than was assumed before.

Theorem 4.1. *Let $n = 3$ and fix $(f, g) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. Then there is a $T > 0$ so that (4.1) has a unique solution verifying*

$$\text{(4.2)}\qquad u \in C([0,T]; H^2) \cap C^1([0,T]; H^1),$$

and

$$\text{(4.3)}\qquad Q(u^J, u^K) \in H^1([0,T] \times \mathbb{R}^3), \quad \forall J, K,$$

whenever Q is a null form.

We said before that the techniques used in the proof of Theorem 4.1 from the last chapter could be adapted to show that there is local existence for quasilinear equations with data in $H^\gamma(\mathbb{R}^n) \times H^{\gamma-1}(\mathbb{R}^n)$ if $\gamma > (n+2)/2$. Thus, Theorem 4.1 says that for the special case of equations like (4.1) here, there is an improvement of $1/2$ of a derivative in the regularity assumptions. On the other hand, the example of Nirenberg given in the last section shows that one might expect that the theorem should hold if $\gamma > 3/2$ if $n = 3$. On the other hand, Lindblad [3],[4] showed that for the equation $\Box u = u_t^2$ one cannot have $\gamma \le 2$ if $n = 3$, thus showing that Theorem 4.1 is in some sense sharp.

Returning to Theorem 4.1, notice that since (4.3) implies that the right side of (4.1) belongs to $L^1([0,T]; H^1)$ if $T < \infty$, we see that the existence theorem for linear equations essentially implies that (4.2) holds if (4.3) is satisfied. So the main step in proving the theorem will be to obtain an estimate which will ensure that (4.3) holds. This is contained in the following result which is also due to Klainerman and Machedon.

Theorem 4.2. *For $j = 1,2$, let $u_j(t,x)$ solve*

$$\begin{cases} \Box u_j(t,x) = F_j(t,x)\,, & (t,x) \in \mathbb{R}^{1+3}_+ \\ u_j(0,\cdot) = f_j\,, & \partial_t u_j(0,\cdot) = g_j\,. \end{cases} \tag{4.4}$$

Then, if Q is a null form

$$\sum_{|\alpha|\le 1} \| \partial^\alpha Q(u_1,u_2) \|_{L^2(\mathbb{R}^{1+3}_+)} \le C \prod_{j=1,2} \big(\|f_j\|_{H^2(\mathbb{R}^3)} + \|g_j\|_{H^1(\mathbb{R}^3)} + \int_0^\infty \|F_j(t,\cdot)\|_{H^1(\mathbb{R}^3)}\, dt \big)\,. \tag{4.5}$$

Notice that (4.5) is a bilinear estimate, which is important in view of the quadratic nature of the nonlinearity in (4.1). It is related to an earlier linear estimate of Strichartz which says that

$$\|u\|_{L^4(\mathbb{R}^{1+3}_+)} \le C \big(\|f\|_{\dot H^{1/2}(\mathbb{R}^3)} + \|g\|_{\dot H^{-1/2}(\mathbb{R}^3)} + \int_0^\infty \|F(t,\cdot)\|_{L^2(\mathbb{R}^3)}\, dt \big)\,, \tag{4.6}$$

if u solves the linear equation $\Box u = F$ with data (f,g). Here, $\dot H^\alpha(\mathbb{R}^n)$, denotes the homogeneous Sobolev space with norm

$$\|f\|_{\dot H^\alpha(\mathbb{R}^n)} = \|(-\Delta)^{\alpha/2} f\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\Big(\int_{\mathbb{R}^n} |\,\hat f(\xi)\,|\xi|^\alpha\,|^2\, d\xi\Big)^{1/2}.$$

As we shall see in the next chapter, (4.6) is equivalent to the following estimate concerning the restriction of the Fourier transform to the light cone in $\mathbb{R}^{1+3}_+$:

$$\Big(\int_{\mathbb{R}^3} |\hat F(|\xi|,\xi)|^2\, d\xi/|\xi|\Big)^{1/2} \le C\|F\|_{L^{4/3}(\mathbb{R}^{1+3})}\,, \quad F \in \mathcal{S}(\mathbb{R}^{1+3})\,. \tag{4.6'}$$

Here $\hat F$ denotes the space-time Fourier transform of F.

We postpone the proof of (4.6) until the next chapter where it will be used. The proof of (4.5), however, more closely resembles that of the following sharp restriction theorem of Carleson and Sjölin [1] and Zygmund [1] which we shall present now to motivate estimates like (4.6) as well as Theorem 4.2.

Theorem 4.3. (Restriction theorem for S^1) *If $1 \le p < 4/3$ there is a constant C_p so that*

$$\|\hat f\|_{L^{p/3(p-1)}(S^1)} \le C_p\|f\|_{L^p(\mathbb{R}^2)}\,, \quad f \in \mathcal{S}(\mathbb{R}^2)\,. \tag{4.7}$$

Proof of Theorem 4.3. The inequality for $p = 1$ is a consequence of the Riemann-Lebesgue lemma, so in what follows let us assume that $1 < p < 4/3$. To handle this case, let S denote the dual of the restriction operator:

$$(Sg)(x) = (2\pi)^{-2} \int_0^{2\pi} e^{ix\cdot(\cos\theta,\sin\theta)} g(\theta)\, d\theta\,.$$

Then if $g \in C^\infty$

$$\begin{aligned} \Big| \int_0^{2\pi} \hat{f}(\cos\theta, \sin\theta)\, \overline{g(\theta)}\, d\theta \Big| &= \Big| \int_{\mathbb{R}^2} f(x)\, \overline{Sg(x)}\, dx \Big| \\ &\le \|f\|_{L^p(\mathbb{R}^2)} \, \|Sg\|_{L^{p/(p-1)}(\mathbb{R}^2)}\,. \end{aligned}$$

If we take the supremum over g with $\|g\|_{L^{p/(3-2p)}} = 1$ of the left side of this inequality, then we get the left side of (4.7). Since $q = p/(p-1) \in (4,\infty)$ if $1 < p < 4/3$, we conclude that (4.7) for $1 < p < 4/3$ is equivalent to the inequality

$$(4.7') \qquad \|Sg\|_{L^q(\mathbb{R}^2)} \le C_q \, \|g\|_{L^{q/(q-3)}([0,2\pi])}\,, \quad 4 < q < \infty\,,$$

since $q/(q-3) = p/(3-2p)$ if $q = p/(p-1)$.

To prove this version, we may assume without loss of generality that $g(\theta) = 0$ if $\theta \notin [0, \pi/2]$. Then, to take advantage of the fact that $q > 4$, note that

$$\begin{aligned} &\big((Sg)(x) \big)^2 \\ &\quad = (2\pi)^{-4} \cdot \int_0^{\pi/2} \int_0^{\pi/2} e^{ix\cdot(\cos\theta_1+\cos\theta_2, \sin\theta_1+\sin\theta_2)}\, g(\theta_1) g(\theta_2)\, d\theta_1\, d\theta_2\,. \end{aligned}$$

If we make the change of variables

$$\xi = (\cos\theta_1 + \cos\theta_2, \sin\theta_1 + \sin\theta_2)\,,$$

then the Jacobian satisfies

$$\Big| \frac{d\xi}{d(\theta_1, \theta_2)} \Big| = |\cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1| \ge c|\theta_1 - \theta_2|\,,$$

for some $c > 0$, as $\theta_j \in [0, \pi/2]$ and hence we avoid antipodal points.

Notice that in these coordinates the last integral is just a Fourier transform. So if we now use the Hausdorff-Young inequality, we see that, since $q/2 > 2$ and $2/q + (q-2)/q = 1$,

$$
\begin{aligned}
&\|(Sg)^2\|_{L^{q/2}(\mathbb{R}^2)} \\
&\le \Big(\iint \Big| g(\theta_1)g(\theta_2) \Big| \frac{d\xi}{d(\theta_1,\theta_2)} \Big|^{-1} \Big|^{\frac{q}{q-2}} d\xi \Big)^{\frac{q-2}{q}} \\
&= \Big(\iint \Big| |g(\theta_1)|^{\frac{q}{q-2}} |g(\theta_2)|^{\frac{q}{q-2}} \Big| \frac{d\xi}{d(\theta_1,\theta_2)} \Big|^{-\frac{q}{q-2}+1} d\theta_1 d\theta_2 \Big)^{\frac{q-2}{q}} \\
&\le C\Big(\iint \Big| |g(\theta_1)|^{\frac{q}{q-2}} |g(\theta_2)|^{\frac{q}{q-2}} |\theta_1-\theta_2|^{-\frac{2}{q-2}} d\theta_1 d\theta_2 \Big)^{\frac{q-2}{q}} .
\end{aligned}
\tag{4.8}
$$

To finish, let us recall the Hardy-Littlewood inequality for fractional integrals (see Appendix) which says that if $1 < r,p < \infty$:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(s)g(t)}{|t-s|^{\alpha}}\, ds dt \\
&\qquad \le C_{p,r}\|f\|_{L^p}\|g\|_{L^r}\,, \quad \alpha = 2-\frac{1}{p}-\frac{1}{r}\,, \quad \frac{1}{p}+\frac{1}{r} > 1\,.
\end{aligned}
\tag{4.9}
$$

In (4.8) we want to take $\alpha = 2/(q-2)$, and $p = r = (q-2)/(q-3)$. The hypotheses of the Hardy-Littlewood inequality are then fulfilled since $2 - 2(q-3)/(q-2) = \alpha$, and $1/r + 1/p = 2(q-3)/(q-2) > 1$ as $q > 4$. Hence, (4.8) and (4.9) yield

$$\|Sg\|^2_{L^q(\mathbb{R}^2)} = \|(Sg)^2\|_{L^{q/2}(\mathbb{R}^2)} \le C_q\|g\|^2_{L^{q/(q-3)}}\,,$$

which completes the proof. □

Let us now see how Theorem 4.2 follows from related arguments. The presence of the null forms in (4.5), rather than arbitrary bilinear forms acting on the gradients, will create cancellation which will allow us to avoid using the Hardy-Littlewood inequality in the proof. Let us turn to the details.

Proof of Theorem 4.2. For simplicity, let us assume that $g_j = 0$ and $F_j = 0$ in (4.4). The proof for the more general case follows from the arguments used to establish this special case.

Let us first see that (4.5) holds when $|\alpha| = 1$. Hence, under the above assumptions we want to see that

$$\|\partial Q(u_1,u_2)\|_{L^2(\mathbb{R}^{1+3})} \le C\|f_1\|_{H^2(\mathbb{R}^3)}\,\|f_2\|_{H^2(\mathbb{R}^3)}\,,$$

if Q is one of the null forms. Since $\partial_k Q(u_1,u_2) = Q(\partial_k u_1,u_2)+Q(u_1,\partial_k u_2)$, we conclude, by symmetry, that this estimate would follow from

$$\|Q(\partial_k u_1,u_2)\|_{L^2(\mathbb{R}^{1+3})} \le C\|f_1\|_{H^2}\|f_2\|_{H^2}\,, \quad k = 0,1,2,3\,.$$

For the sake of notation, let us just show that this inequality holds when $k = 0$, since the other cases follow from the same argument.

To proceed, notice that we can write $u_j(t,x) = (u_j^+(t,x) + u_j^-(t,x))/2$, where

$$u_j^{\pm}(t,x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it|\xi|} \hat{f}_j(\xi)\, d\xi\,.$$

Hence $Q(\partial_t u_1, u_2)$ is the sum of four terms $Q(\partial_t u_1^{\pm}, u_2^{\pm})/4$. Therefore, it suffices to see that

$$(4.5') \quad \|Q(\partial_t u_1^+, u_2^+)\|_{L^2(\mathbb{R}^{1+3})} + \|Q(\partial_t u_1^+, u_2^-)\|_{L^2(\mathbb{R}^{1+3})} \le C\|f_1\|_{H^2}\|f_2\|_{H^2}\,.$$

Let us first see that the first term in the left satisfies the desired bounds. To do this, we first note that

$$Q_0(\partial_t u_1^+, u_2^+) \\ = (2\pi)^{-6} \iint e^{ix\cdot(\xi+\eta)+it(|\xi|+|\eta|)}\, i|\xi|\, q_0(\xi,\eta)\, \hat{f}_1(\xi)\, \hat{f}_2(\eta) d\xi d\eta\,,$$

where

$$q_0(\xi,\eta) = \xi\cdot\eta - |\xi|\,|\eta|\,.$$

Also,

$$Q_{ij}(\partial_t u_1^+, u_2^+) \\ = (2\pi)^{-6} \iint e^{ix\cdot(\xi+\eta)+it(|\xi|+|\eta|)}\, i|\xi|\, q_{ij}(\xi,\eta)\, \hat{f}_1(\xi)\, \hat{f}_2(\eta)\, d\xi d\eta\,,$$

where

$$q_{ij}(\xi,\eta) = \begin{cases} -(\xi_i\eta_j - \xi_j\eta_i)\,, & 1 \le i < j \le 3 \\ -(|\xi|\eta_j - |\eta|\xi_j)\,, & 0 = i < j \le 3\,. \end{cases}$$

The important thing is that if q is q_0 or one of the q_{ij}, then

$$|q(\xi,\eta)| \le C|\xi|\,|\eta| \cdot \left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right|,$$

for some constant C. This will allow us to make a change of variables like in the proof of the restriction theorem and absorb the Jacobian factor that arises.

Let us be more precise. If we use polar coordinates and write $\eta = \rho\omega$, $\rho > 0$, $\omega \in S^2$, then we find that

$$(4.10) \quad \|Q(\partial_t u_1^+, u_2^+)\|_{L^2(\mathbb{R}^{1+3})} \le \int_{S^2} \|T_\omega^+(f_1, f_2)(t,x)\|_{L^2(\mathbb{R}^{1+3})}\, d\sigma(\omega)\,,$$

where

$$T_\omega^+(f_1,f_2)(t,x) = \iint e^{ix\cdot(\xi+\rho\omega)+it(|\xi|+\rho)}\,|\xi|^2\hat f_1(\xi)\rho\hat f_2(\rho\omega)\,q(\xi/|\xi|,\omega)\,d\xi\,\rho^2 d\rho\,.$$

But if, for fixed ω, we make the change of variables

$$(\tau,\zeta)=(|\xi|+\rho,\xi+\rho\omega)\,,$$

then the Jacobian satisfies

$$|d(\tau,\zeta)/d(\rho,\xi)| = \left|1-\omega\cdot\frac{\xi}{|\xi|}\right| \ge c\left|\omega-\frac{\xi}{|\xi|}\right|^2,$$

for some $c>0$. Thus, by Plancherel's theorem

$$\begin{aligned}\|T_\omega^+(f_1,f_2)\|_{L^2}^2 &\le C\iint \Big|\,|\xi|^2\hat f(\xi)\,\rho^3\hat f(\rho\omega)\,q(\tfrac{\xi}{|\xi|},\omega)\,\Big|\tfrac{d(\tau,\zeta)}{d(\rho,\xi)}\Big|^{-1}\Big|^2 d\tau d\zeta\\ &= \iint \Big|\,|\xi|^2\hat f(\xi)\,\rho^3\hat f(\rho\omega)\,q(\tfrac{\xi}{|\xi|},\omega)\,\Big|\tfrac{d(\tau,\zeta)}{d(\rho,\xi)}\Big|^{-\frac12}\Big|^2 d\rho d\xi\\ &\le C\iint \Big|\,\rho^2\hat f(\rho\omega)\,|\xi|^2\hat f(\xi)\,\Big|^2\rho^2 d\rho d\xi\,,\end{aligned}$$

with the last step coming from the fact that $|q(\xi/|\xi|,\omega)|\,\big|\frac{d(\tau,\zeta)}{d(\rho,\xi)}\big|^{-\frac12}\le C$. Using this along with (4.10), we conclude that

$$\begin{aligned}\|Q(\partial_t u_1^+,u_2^+)\|_{L^2}^2 &\le C\iint \Big|\,\rho^2\hat f(\rho\omega)\,|\xi|^2\hat f(\xi)\,\Big|^2\rho^2 d\rho d\sigma(\omega) d\xi\\ &= C(2\pi)^{-6}\|f_1\|_{\dot H^2}^2\|f_2\|_{\dot H^2}^2 \le C(2\pi)^{-6}\|f_1\|_{H^2}^2\|f_2\|_{H^2}^2\,,\end{aligned}$$

as desired.

The argument for $Q(\partial_t u_1^+,u_2^-)$ is similar. Here one has a modified version of (4.10) involving T_ω^- in the right with

$$T_\omega^-(f_1,f_2) = \iint e^{ix\cdot(\xi+\rho\omega)+it(|\xi|-\rho)}\,|\xi|^2\hat f_1(\xi)\rho\hat f_2(\rho\omega)\,\widetilde q(\xi/|\xi|,\omega)\,d\xi\,\rho^2 d\rho\,,$$

where, now,

$$|\widetilde q(\xi,\eta)| \le C|\xi|\,|\eta|\,\Big|\tfrac{\xi}{|\xi|}+\tfrac{\eta}{|\eta|}\Big|\,.$$

Thus, if for fixed ω, we make the change of variables $(\tau,\zeta)=(|\xi|-\rho,\xi+\rho\omega)$, then the above arguments show that T_ω^- satisfies the same estimate as T_ω^+, since $|d(\tau,\zeta)/d(\rho,\xi)|^{-1/2}\tilde{q}(\xi/|\xi|,\omega)| \le C$. This completes the proof of (4.5) for the terms in the left involving $|\alpha| = 1$.

To finish we still need to show that the L^2 norm of $Q(u_1, u_2)$ can be controlled by the right side of (4.5). But, if we assume that u_j are as above, then the above arguments give

$$\|Q(u_1,u_2)\|_{L^2(\mathbb{R}^{1+3})} \le C\|f_1\|_{\dot H^1}\|f_2\|_{\dot H^2} \le C\|f_1\|_{H^2}\|f_2\|_{H^2},$$

and thus one can estimate the L^2 norm of Q by the arguments used to handle its gradient. □

Proof of Theorem 4.1. We shall only prove the existence part since uniqueness follows from similar arguments. To handle existence, let us write (4.1) as $\Box u = F(u, u')$, where of course F^I is a combination of terms of the form $\Gamma(u)Q(u^J, u^K)$, with Q being a null form and $\Gamma \in C^\infty$. We then, as in the proofs of the earlier local existence results, let $u_{-1} \equiv 0$, and define u_m, $m = 1, 2, \dots$ recursively by

$$\begin{cases} \Box u_m = F(u_{m-1}, u'_{m-1}) \\ u_m(0,x) = f(x), \quad \partial_t u_m(0,x) = g(x). \end{cases}$$

We then claim that if $T > 0$ is sufficiently small and if C_0 is sufficiently large then

$$\sum_{|\alpha|\le 1} \| \partial^\alpha(F(u_m, u'_m) - F(u_{m-1}, u'_{m-1})) \|_{L^2([0,T]\times\mathbb{R}^3)} \le C_0 2^{-m}, \quad m = 0, 1, 2, \dots. \tag{4.11}$$

Notice that, since $\Box u_0 = 0$, this inequality must hold if $m = 0$ for some constant C_0, by Theorem 4.2 and our assumption on the data. We then claim that for the same constant (4.11) holds for $m = 1, 2, \dots$, if T is small.

If this were the case then, by the energy inequality for the d'Alembertian we would have

$$\sum_{|\alpha|\le 2} \|\partial^\alpha(u_{m+1}(t,\cdot) - u_m(t,\cdot))\|_{L^2(\mathbb{R}^3)} \le C_0' 2^{-m}, \quad m = 0, 1 \dots, \tag{4.12}$$

assuming, say, that $T < 1$. Since $\Box u_0 = 0$, our assumptions on the data imply that $\|\partial^\alpha u_0(t,\cdot)\|_{L^2} \le C$ if $|\alpha| \le 2$ and $0 < t < 1$. Hence, there must be an absolute constant C_1 so that if (4.12) holds for all u_n with $n \le m$, then

$$\sum_{|\alpha|\le 2} \|\partial^\alpha u_{m+1}(t,\cdot)\|_{L^2} \le C_1, \quad 0 \le t \le T, \tag{4.12'}$$

By Sobolev's theorem, this in turn gives

$$|u_{m+1}(t,x)| \le C_1', \quad (t,x) \in [0,T] \times \mathbb{R}^3 . \tag{4.13}$$

By (4.11) and (4.12) u_m converges in $L^\infty([0,T];H^2) \cap C^{0,1}([0,T];H^1)$ to a solution u of (4.1). Since (4.11) implies that $F(u,u') \in L^1([0,T];H^1)$, the linear existence theorem implies that (4.2) must hold. To see that (4.3) is also valid, we also need that

$$\sum_{|\alpha|\le 1} \|\partial^\alpha (Q(u_m^J, u_m^K) - Q(u_{m-1}^J, u_{m-1}^K))\|_{L^2([0,T]\times\mathbb{R}^3)} \le C2^{-m}, \tag{4.11$'$}$$

if Q is a null form.

Let us assume that (4.11) holds if whenever a given $m \ge 1$ is replaced by $n < m$. We then claim that this implies that (4.11) holds if T is small enough. To verify this we note that the I-th component of the term inside the L^2 norm involves derivatives of terms which are linear combinations of expressions of the form

$$\begin{aligned}\Gamma(u_{m-1})\big(Q(u_m^J, u_m^K) - Q(u_{m-1}^J, u_{m-1}^K)\big)& \\ + \big(\Gamma(u_m) - \Gamma(u_{m-1})\big) Q(u_m^J, u_m^K) &= I + II .\end{aligned} \tag{4.14}$$

To estimate the first term we note that the second factor in its definition can be written as $Q(u_m^J - u_{m-1}^J, u_m^K) + Q(u_{m-1}^J, u_m^K - u_{m-1}^K)$. Let us estimate the first term here. To do so we note that, since u_{m-1} is bounded, if the following norms are taken over $[0,T]\times\mathbb{R}^3$, then

$$\begin{aligned}&\sum_{|\alpha|\le 1} \|\partial^\alpha \big(\Gamma(u_{m-1}) Q(u_m^J - u_{m-1}^J, u_m^K)\big)\|_{L^2} \\ &\le C \sum_{|\alpha|\le 1} \|\partial^\alpha Q(u_m^J - u_{m-1}^J, u_m^K)\|_{L^2} + \|\partial(\Gamma(u_{m-1}))\cdot Q(u_m^J - u_{m-1}^J, u_m^K)\|_{L^2} .\end{aligned} \tag{4.15}$$

If we use Theorem 4.2, we find that the first term in the right is dominated by

$$\begin{aligned}&\int_0^T \sum_{|\alpha|\le 1} \|\partial^\alpha \big(F(u_{m-1}, u_{m-1}') - F(u_{m-2}, u_{m-2}')\big)\|_{L^2(\mathbb{R}^3)}\, dt \\ &\qquad \times \Big(\|f\|_{H^2} + \|g\|_{H^1} + \int_0^T \sum_{|\alpha|\le 1} \|\partial^\alpha F(u_{m-1}, u_{m-1}')\|_{L^2(\mathbb{R}^3)}\, dt\Big) \\ &\le C T^{1/2} C_0 2^{-(m-1)},\end{aligned}$$

using in the last step Schwartz's inequality and the fact that we are assuming that (4.11) holds if m is replaced by any $n < m$. The other term in the right side of (4.15) can be estimated in a similar manner. One first uses Hölder's inequality and the fact that $|\partial\Gamma(u_{m-1})| \le C|\partial u_{m-1}|$, to see that it is dominated by

$$\sup_{0\le t\le T} \|\partial u_{m-1}(t,\,\cdot\,)\|_{L^4(\mathbb{R}^3)} \cdot \Big(\int_0^T \|Q(u_m^J - u_{m-1}^J, u_m^K)\|^2_{L^4(\mathbb{R}^3)}\, dt \Big)^{1/2}$$
$$\le C \sup_{0\le t\le T} \sum_{|\alpha|\le 2} \|\partial^\alpha u_{m-1}(t,\,\cdot\,)\|_{L^2(\mathbb{R}^3)}$$
$$\times \sum_{|\alpha|\le 1} \|\partial^\alpha Q(u_m^J - u_{m-1}^J, u_m^K)\|_{L^2([0,T]\times\mathbb{R}^3)}\,,$$

using Sobolev's theorem in the last step. But the first factor on the right is $O(1)$ by (4.12′), and the other one is $O(T^{1/2}2^{-(m-1)})$ by the last step.

Since similar arguments apply to $\Gamma(u_{m-1})Q(u_{m-1}^J, u_m^K - u_{m-1}^K)$ we conclude that if I is as in (4.14), then

$$\sum_{|\alpha|\le 1} \|\partial^\alpha I\|_{L^2([0,T]\times\mathbb{R}^3)} \le CT^{1/2}C_0 2^{-(m-1)}\,,$$

where C_0 is as in (4.11).

These arguments can also be used to estimate the other term in (4.14). In fact, $\sum_{|\alpha|\le 1} \|\partial^\alpha II\|_{L^2}$ is dominated by

$$\text{(4.16)}\quad \|u_m - u_{m-1}\|_{L^\infty} \sum_{|\alpha|\le 1} \|\partial^\alpha Q(u_m^J, u_m^K)\|_{L^2}$$
$$+ \sup_{0\le t\le T} \sum_{|\alpha|\le 1} \|\partial^\alpha(u_m(t,\,\cdot\,) - u_{m-1}(t,\,\cdot\,))\|_{L^4(\mathbb{R}^3)} \Big(\int_0^T \|Q(u_m^J, u_m^K)\|^2_{L^4}\, dt \Big)^{1/2},$$

where the norms are taken over the strip $[0,T]\times\mathbb{R}^3$. Notice that, by Sobolev's theorem and the energy inequality,

$$\|u_m - u_{m-1}\|_{L^\infty}$$
$$\le C \sup_{0\le t\le T} \sum_{|\alpha|\le 2} \|\partial^\alpha(u_m(t,\,\cdot\,) - u_{m-1}(t,\,\cdot\,))\|_{L^2}$$
$$\le C \sum_{|\alpha|\le 1} \int_0^T \big\| \partial^\alpha(F(u_{m-1}, u'_{m-1}) - F(u_{m-2}, u'_{m-2})) \big\|_{L^2}\, dt$$
$$\le CT^{1/2}C_0 2^{-(m-1)}\,.$$

Since, by the last step, $\sum_{|\alpha|\le 1}\|\partial^\alpha Q(u_m^J, u_m^K)\|_{L^2} = O(1)$, we conclude that the first term in (4.16) is $\le C'T^{1/2}C_0 2^{-(m-1)}$. The other term in (4.16) can be estimated in the same way, after noting that, by Sobolev's theorem, it is

$$\le C \sup_{0\le t\le T} \sum_{|\alpha|\le 1} \|\partial^\alpha (u_m(t,\cdot) - u_{m-1}(t,\cdot))\|_{L^2} \sum_{|\alpha|\le 1} \|\partial^\alpha Q(u_m^J, u_m^K)\|_{L^2}.$$

Thus, II satisfies the same bounds as I. This, in turn, implies that

$$\sum_{|\alpha|\le 1} \| \partial^\alpha (F(u_m, u_m') - F(u_{m-1}, u_{m-1}')) \|_{L^2([0,T]\times\mathbb{R}^3)} \le CT^{1/2}C_0 2^{-(m-1)},$$

which of course yields (4.11) if $CT^{1/2} < 1/2$.

If we take $\Gamma \equiv 1$ in the above argument we see that we also have (4.11′), and so the proof is complete. □

Notes

When $n = 1$ solutions of quasilinear equations with data $(\varepsilon f, \varepsilon g) \in C_0^\infty$ in general only have lifespan $T_\varepsilon = O(\varepsilon^{-1})$. Using energy estimates alone one can derive the same lower bounds for the lifespan in higher dimensions. This was improved considerably, though, in John [2], where it was shown that $T_\varepsilon^{-1} = O(\varepsilon^2)$ if $n \ge 4$ and $O((\varepsilon \log 1/\varepsilon)^{-4})$ if $n = 3$. After this breakthrough, Klainerman [1] showed that for small $\varepsilon > 0$ there is global existence when $n \ge 6$. Later John [5] greatly improved his earlier result showing that, when $n = 3$, $T_\varepsilon^{-1} = O(\varepsilon^N)$ for any N. In this paper John also made the important observation that, if u solves the linear equation in $\mathbb{R}_+^{1+3}$ with C_0^∞ data, then $\Omega u(t,x) = O((1+t)^{-2})$, if $\Omega = t^{-1}L_0$ or $t^{-1}\Omega_{ij}$, $0 \le i < j \le 3$. Klainerman [3], [4] then obtained his generalization of the usual Sobolev estimates reflecting this decay and also formulated his null condition [2]. A related result can be found in Choquet-Bruhat and Christodoulou [1]. In John and Klainerman [1] the almost global existence theorem for $n = 3$ was proved using Klainerman's [3] version of the estimates for the inhomogeneous wave equation. Then in Klainerman [4] a simplified proof was given, which we have followed along with Hörmander [5]. In this paper it was also shown that there is global existence for small $\varepsilon > 0$ if $n \ge 4$. The estimates for the inhomogeneous wave equation are due to Hörmander [4]. The proof we have presented is taken from Hörmander [5], [6] and Lindblad [2].

The low regularity local existence theorem from §4 is due to Klainerman and Machedon [1]. The proof we have given involves a modification of their arguments and is due to Beals and Bezard [1]. In Sogge [2] it was shown that this result holds for variable coefficient operators as well.

This was shown independently by Georgiev and Shirmer [1], where it was also observed that the special case corresponding to $\mathbb{R}_+ \times S^3$ with the standard metric can be used in conjunction with the conformal method of Christodoulou [1] to give global existence for the equations in §4 with small compactly supported data in $H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. Recently, Klainerman and Machedon [2], using an idea of Bourgain [1], have shown that there is local existence for systems only involving Q_0 and data in $H^{n/2+\varepsilon}(\mathbb{R}^n) \times H^{n/2-1+\varepsilon}(\mathbb{R}^n)$, $\varepsilon > 0$, if $n = 3$, respectively. The version for $n = 2$ has also been announced by Grillakis [3]. Both are essentially sharp in view of the example of Nirenberg. However, when $n = 2$ the important open problem of deciding when there is local existence for wave maps with data of finite energy remains open. See Grillakis [3] and Shatah [1] for a discussion of this problem.

SEMILINEAR EQUATIONS WITH SMALL DATA

We saw in the last chapter that equations in $\mathbb{R}^{1+3}_+$ of the form $\Box u = cu^3$ have global solutions if the Cauchy data is in C_0^∞ and sufficiently small. We also remarked that, in general, solutions of $\Box u = u^2$ must blowup in finite time. The goal of this chapter is to determine when equations of the form $\Box u = |u|^\kappa$ can have global solutions for arbitrary small data. The surprising answer, which is due to John, is that for small C_0^∞ data global solutions always exist when $\kappa > 1+\sqrt{2}$ but not when $\kappa < 1+\sqrt{2}$. After proving this, we shall be interested in determining how much regularity is needed for the data. Answering this will allow us to in some sense quantify the smallness assumption. We shall find that in the non-radial case a different type of regularity assumption is needed depending on whether κ is larger or smaller than the "conformal power" $\kappa = 3$. In the case of spherical symmetry, though, the behavior changes at John's power $\kappa = 1+\sqrt{2}$. Both of these low regularity results will be obtained via certain generalized "Strichartz inequalities." Certain of these inequalities will be important for us in the next chapter when we study the so-called "critical wave equation" $\Box u + u^5 = 0$ in $\mathbb{R}^{1+3}_+$. This terminology, though, is in some sense a misnomer, because, in view of the above discussion, there are really three critical powers for semilinear wave equations in $(1+3)$-dimensions: $\kappa = 1+\sqrt{2}$, $\kappa = 3$ and $\kappa = 5$.

§1. John's Existence Theorem for $\mathbb{R}^{1+3}$

In this chapter we shall study solutions of semilinear equations of the form

$$\begin{cases} \Box u(t,x) = F_\kappa(u) \\ u(0,x) = \varepsilon f(x), \quad \partial_t u(0,x) = \varepsilon g(x), \end{cases} \tag{1.1}$$

where we assume that f and g are fixed compactly supported functions. If $\kappa \geq 2$ we shall assume that $F_\kappa \in C^2$, $F_\kappa(0) = F'_\kappa(0) = 0$, and

$$|F''_\kappa(u)| \leq C|u|^{\kappa-2}, \quad \text{if } |u| \leq 1. \tag{1.2}$$

If $1 < \kappa < 2$, we weaken these assumptions to $F_\kappa \in C^1$, $F_\kappa(0) = 0$, and

$$|F'_\kappa(u)| \leq C|u|^{\kappa-1}, \quad \text{if } |u| \leq 1. \tag{1.3}$$

The main result of this chapter then is the following

Theorem 1.1. *Suppose that $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$ have compact support. Then if $\kappa > 1+\sqrt{2}$, (1.1) has a unique global solution $u \in C^2(\mathbb{R}^{1+3}_+)$ if $\varepsilon > 0$ is small enough. On the other hand, if we set*

$$T_\varepsilon = \begin{cases} \exp(c\varepsilon^{-\kappa(\kappa-1)}), & \kappa = 1+\sqrt{2} \\ c\varepsilon^{\frac{\kappa(\kappa-1)}{\kappa^2-2\kappa-1}}, & 1<\kappa<1+\sqrt{2}, \end{cases} \tag{1.4}$$

then, if $c, \varepsilon > 0$ are sufficiently small, there is a unique solution $u \in C^2([0,T_\varepsilon) \times \mathbb{R}^3)$ if $2 \le \kappa < 1+\sqrt{2}$. Also, if $1 < \kappa < 2$ and $f \in C^2(\mathbb{R}^3)$ and $g \in C^1(\mathbb{R}^3)$ are compactly supported then there is a unique (weak) solution $u \in C^1([0,T_\varepsilon) \times \mathbb{R}^3)$ if T_ε is as above. Furthermore, if $F_\kappa(u) = |u|^\kappa$, then in general (1.1) has no global solution even for small $\varepsilon > 0$ and the lifespan estimates are of the best possible nature when $2 \le \kappa < 1+\sqrt{2}$.

The positive results for $\kappa > 1+\sqrt{2}$ and $\kappa = 2$ are due to John [3]. He also showed that if $\kappa < 1+\sqrt{2}$ and $F_\kappa(u) = |u|^\kappa$, then the lifespan estimates given above are sharp. Lindblad [1] extended the positive results to include $1 < \kappa < 1+\sqrt{2}$, and then Zhou [1] handled the case where $\kappa = 1+\sqrt{2}$. In the latter works it was also shown that the above lifespan estimates are sharp for $1 < \kappa < 2$ and $\kappa = 1+\sqrt{2}$, respectively; however, we shall just give John's original argument which leads to the blow-up results stated in the theorem.

John's proof of the existence results for $\kappa > 1+\sqrt{2}$ involved an iteration argument in the space with norm

$$\sup_{t>0}(1+t)\,\Big\|\,(1+|t-r|)^{\kappa-2}\sup_{\omega\in S^2}|u(t,r\omega)|\,\Big\|_{L^\infty_r}. \tag{1.5}$$

The extensions of his results of Lindblad and Zhou used a different method; however, all use arguments exploiting the positivity of the forward fundamental solution for $\Box$ in $(1+3)$-dimensions. The proof we shall give is based on joint work with Lindblad [2]. It also uses the comparison theorem for the inhomogeneous wave equation in $\mathbb{R}^{1+3}_+$. However, the estimates we shall use differ somewhat from the ones used by John, and they are based on applying the Hardy-Littlewood maximal theorem to the formula for the solution of the inhomogeneous wave equation under the assumption of spherical symmetry.

Before turning to the proofs, let us explain what happens in other dimensions. In $(1+n)$-dimensions it is conjectured that there should be global existence for semilinear equations with small compactly supported data if $\kappa > \kappa_n$, where κ_n is the positive root of

$$(n-1)\kappa^2 - (n+1)\kappa - 2 = 0, \tag{1.6}$$

that is

$$\kappa_n = \frac{n+1+\sqrt{(n+1)^2+8(n-1)}}{2(n-1)}.$$

When $n=2$ such global existence results were established by Glassey [1], using arguments similar to those of John for the $(1+3)$-dimensional case. When $n=4$, in which case $\kappa_4=2$, Zhou [2] verified this conjecture, using arguments based on the Klainerman-Sobolev estimates which are similar in spirit to the proof of global existence for quasilinear equations in higher dimensions. In higher dimensions, though, only partial results are known. Lindblad and the author [2] showed that, under the assumption of spherical symmetry, the conjecture holds for any dimension, using inequalities similar to the ones given at the end of the next section. In this work it was also shown that without the assumption of spherical symmetry the conjecture holds if $n \le 6$. This result is based on a partial substitute for the lack of a comparison theorem in higher dimensions; however, the fact that fundamental solutions for $\Box$ become increasingly singular as n increases seems to make the conjecture difficult for very high dimensions. The above conjecture was made by Strauss [3]. The strange numerics coming from (1.6) appeared first in his work [2] on the different context of scattering for semilinear Schrödinger equations.

The proof of Theorem 1.1 will occupy much of this chapter. In the next section we shall prove the radial estimates that will be used for the existence part of the theorem. Then, in §3, we shall show that if $\kappa < 1+\sqrt{2}$ then u can blow-up in time $\approx T_\varepsilon$ if $F_\kappa = |u|^\kappa$. The rest of the chapter will concern generalizations of Theorem 1.1, including certain existence results for weak solutions of (1.1) under minimal regularity assumptions on the data.

The radial estimates we shall require involve mixed-norms. To set the notation that will be used, suppose that $G(x,y)$ is a function of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$. We then define its $L^p_x L^q_y(\mathbb{R}^n \times \mathbb{R}^m) = L^p_x(\mathbb{R}^n; L^q(\mathbb{R}^m))$ norm by

$$\|G\|_{L^p_x L^q_y} = \big\| \, \|G(x,\,\cdot\,)\|_{L^q(\mathbb{R}^m)} \, \big\|_{L^p(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} \|G(x,\,\cdot\,)\|^p_{L^q(\mathbb{R}^m)} \, dy \Big)^{1/p},$$

assuming in the last equality that $p<\infty$. For $p=\infty$, of course, $\|G\|_{L^\infty_x L^q_y} = \sup_{x\in\mathbb{R}^n} \|G(x,\,\cdot\,)\|_{L^q(\mathbb{R}^m)}$.

Mixed-norm estimates will play an important role in almost all of the long-time existence results for semilinear equations that we shall give in this course. The ones that are needed for our proof of John's theorem are contained in the following

Theorem 1.2. *Suppose that $F(t,x) \in C(\mathbb{R}^{1+3}_+)$ is spherically symmetric in the spatial variables, and let w solve the inhomogeneous wave* *equation* $\Box w = F$ *in* $\mathbb{R}^{1+3}_+$ *with zero Cauchy data at* $t = 0$. *Then if* $S_T = \{ (t,x) : 0 \le t \le T \}$,

(1.7)
$$\| \, |x|^{\frac{q-2}{q}} w(T, \cdot) \, \|_{L^q(\mathbb{R}^3)} \le C_q \|F\|_{L^q_t L^1_x(S_T)}, \quad 1 \le q < \infty,$$

(1.8)
$$\| \, w(T, \cdot) \, \|_{L^q(|x|<T)} \le C_q T^{\frac{3-q}{q}} \|F\|_{L^\infty_t L^1_x(S_T)}, \; 1 \le q < 3.$$

Since we are not assuming spherical symmetry in (1.1), we shall need estimates for the radial majorant of u:

$$u^*(t, |x|) = \sup_{\omega \in S^2} |u(t, |x|\omega)| \, .$$

Using Theorem 1.2 and the comparison theorem for $\Box$ we have the following useful estimates for u^*:

Corollary 1.3. *Suppose that* $F \in C(\mathbb{R}^{1+3}_+)$ *satisfies*

$$F(t,x) = 0 \, , \quad \text{if } |x| \ge t + 1 \, .$$

Then, if w *is the solution of* $\Box w = F$ *with zero Cauchy data at* $t = 0$ *and if* $T > 10$,

(1.9)
$$\|w^*(T, |x|)\|_{L^q_x(\mathbb{R}^3)} \le C(1+T)^{\frac{2-q}{q}} \|F^*\|_{L^q_t L^1_x(S_T)} + C_q T^{\frac{3-q}{q}} \|F^*\|_{L^\infty_t L^1_x([T/4,T]\times\mathbb{R}^3)} \, ,$$

provided that $1 \le q < 3$. *Also, for such* q

(1.10)
$$\|w^*(T, |x|)\|_{L^q_x(\mathbb{R}^3)} \le C_q T^{\frac{3-q}{q}} \|F^*\|_{L^\infty_t L^1_x(S_T)} \, .$$

For simplicity we have stated the above inequalities under the assumption that $F(t,x) = 0$ if $t + |x| > 1$. However, for the applications, since we are not assuming that the data in (1.1) is supported in the unit ball, we shall want to replace this condition by $F(t,x) = 0$ when $|x| > t + R$, if $f(x) = g(x) = 0$ for $|x| > R$. But, by a scaling argument, (1.9) and (1.10) extend to this case, with constants depending on R and q, since $\Box\big(w(Rt, Rx) \big) = R^2 F(Rt, Rx)$, so applying (1.9) and (1.10) to this equation yields estimates for $w(t,x)$.

Proof of Corollary 1.3. If $\widetilde{w}$ is the solution of $\Box \widetilde{w} = F^*$ with zero data then $|w| \le \widetilde{w}$, by the positivity of the forward fundamental solution for $\Box$. Since $\widetilde{w}$ is spherically symmetric, it follows that $w^*(T, |x|) \le \widetilde{w}(T, |x|)$ as well. Using (1.7) and this majorization, we conclude that

$$\|w^*(T, |x|)\|_{L^q(|x| > \frac{T+1}{4})} \le C(1+T)^{\frac{2-q}{q}} \|F^*\|_{L^q_t L^1_x(S_T)} \, .$$

To show that the estimate also holds if $|x| < (T+1)/4$, we recall that

$$|x|\widetilde{w}(T,x) = \frac{1}{2}\int_0^T \int_{|T-r-s|}^{T+r-s} F^*(s,\rho)\,\rho d\rho ds\,, \quad r = |x|\,.$$

If $r = |x| < (T+1)/4$ and $s < T/4$ the integral is zero if $T > 10$, for then $|T-r-s| \geq s+1$ and $F^*(s,\rho) = 0$ for $\rho \geq s+1$, by the support properties of F. Therefore, if we use (1.8) we see that, if $T > 10$,

$$\begin{aligned}&\|w^*(T,|x|)\|_{L^q(|x|\leq \frac{T+1}{4})}\\ &\qquad \leq \|\widetilde{w}(T,|x|)\|_{L^q(|x|\leq\frac{T+1}{4})} \leq CT^{\frac{3-q}{q}}\|F^*\|_{L_t^\infty L_x^1([T/4,T]\times\mathbb{R}^3)}\,.\end{aligned}$$

This, along with the last inequality implies (1.9).

To prove (1.10), we first notice that (1.8) yields

$$\|w^*(T,|x|)\|_{L^q(|x|<T)} \leq \|\widetilde{w}(T,|x|)\|_{L^q(|x|<T)} \leq CT^{\frac{3-q}{q}}\|F^*\|_{L_t^\infty L_x^1(S_T)}\,.$$

Since Huygen's principle and the support properties of F imply that $w(T,x)$ vanishes if $|x| > T+1$, to finish, we only need to show that this estimate also holds if the norm is taken over $T < |x| < T+1$. For this, we use (1.7) to get

$$\begin{aligned}\|w^*(T,|x|)\|_{L^q(T<|x|<T+1)} &\leq T^{\frac{2-q}{q}}\|\,|x|^{\frac{q-2}{q}}w^*(T,|x|)\,\|_{L^q(T<|x|<T+1)}\\ &\leq CT^{\frac{2-q}{q}}\|F^*\|_{L_t^q L_x^1(S_T)}\\ &\leq CT^{\frac{3-q}{q}}\|F^*\|_{L_t^\infty L_x^1(S_T)}\,,\end{aligned}$$

which completes the proof. □

Proof of existence part of Theorem 1.1. Let us recall that the local existence theorem for semilinear equations in $\mathbb{R}^{1+3}$ says that, given $\varepsilon > 0$, there is a $T_* > 0$ so that (1.1) has a solution $u \in C^2([0,T_*)\times\mathbb{R}^3)$ if $\kappa \geq 2$, and $u \in C^1([0,T_*)\times\mathbb{R}^3)$ if $1 < \kappa < 2$, since we are assuming that $(f,g) \in C^3 \times C^2$ when $\kappa \geq 2$ and $(f,g) \in C^2 \times C^1$ when $1 < \kappa < 2$. If $T_\varepsilon = +\infty$ for $\kappa > 1+\sqrt{2}$, and if, for $1 < \kappa \leq 1+\sqrt{2}$, T_ε is given by (1.4), we wish to show that we can take $T_* = T_\varepsilon$ if $\varepsilon > 0$ is small enough. If we invoke the local existence theorem again, we see that in order to do this it suffices to show that if $\varepsilon > 0$ is small then

$$(1.11)\quad T_* < T_\varepsilon \text{ and } u \in C^{[\kappa]}([0,T_*)\times\mathbb{R}^3) \implies u \in L^\infty([0,T_*)\times\mathbb{R}^3)\,.$$

Here $[\kappa]$ denotes the greatest integer $\leq \kappa$, and we are assuming, as we may by (1.2), that $1 < \kappa < 3$.

Slightly different arguments are needed for the two cases where $2 \leq \kappa < 3$ and $1 < \kappa < 2$. Let us start out with the former.

Case 1: $2 \le \kappa < 3$.

Consider the linear version of (1.1):

$$\begin{cases} \Box u_0 = 0 \\ u_0(0,x) = \varepsilon f(x), \quad \partial_t u_0(0,x) = \varepsilon g(x) \end{cases} \tag{1.12}$$

We are assuming that $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$ are compactly supported. If they vanish when $|x| > R$ then, by the strong Huygen's principle in $(1+3)$-dimensions

$$u_0(t,x) = 0, \quad \text{if } |x| \notin [t-R, t+R]. \tag{1.13}$$

Also, by decay estimates for the linear equation from Chapter 1, $D^\alpha u_0(t,x) = O(\varepsilon/(1+t))$ if $|\alpha| \le 2$. Therefore, there must be a constant A, depending on f and g, so that

$$(1+t)^{\frac{\kappa-2}{\kappa}} \sum_{|\alpha|\le 2} \| (D^\alpha u_0)^*(t,|x|) \|_{L^\kappa(\mathbb{R}^3)} \le \frac{A\varepsilon}{4}, \tag{1.14}$$

if, as before, $G^*(t,|x|)$ denotes the radial majorant of a function $G(t,x)$, and if $D = \partial/\partial x$.

Based on this, we claim that if $\varepsilon > 0$ is small enough and T_* is as in (1.11), then whenever $T < T_*$

$$(1+t)^{\frac{\kappa-2}{\kappa}} \sum_{|\alpha|\le 2} \| (D^\alpha u)^*(t,|x|) \|_{L^\kappa(\mathbb{R}^3)} \le \frac{A\varepsilon}{2}, \text{ if } 0 \le t \le T. \tag{1.11'}$$

By Sobolev's theorem, this would yield (1.11) for $2 \le \kappa < 3$.

As in the last chapter, we shall prove this using the continuity method. Since (1.11′) must hold for small enough times, by (1.14), we conclude that it suffices to show that bounds of the form

$$(1+t)^{\frac{\kappa-2}{\kappa}} \sum_{|\alpha|\le 2} \| (D^\alpha u)^*(t,|x|) \|_{L^\kappa(\mathbb{R}^3)} \le A\varepsilon, \ 0 \le t \le T, \tag{1.15}$$

actually imply that the stronger estimate (1.11′) holds. Notice again that, by Sobolev's theorem, (1.15) implies that $|u(t,x)| \le CA\varepsilon$, $0 \le t \le T$, for some fixed constant C. We shall initially assume that $\varepsilon < 1/CA$ so that we can use (1.2) to deduce that on S_T

$$|F_\kappa^{(j)}(u)| \le C|u|^{\kappa - j}, \ |j| \le 2.$$

Next, notice that we can write $u = u_0 + w$, where u_0 is as in (1.12) and where w has zero Cauchy data at $t = 0$ and satisfies $\Box w = F_\kappa(u)$.

Therefore, if we call the left side of (1.15) $A(t)$, we can apply Corollary 1.3 to obtain

$$(1.16)\quad A(t) \le \tfrac{A\varepsilon}{4} + C \sum_{|\alpha|\le 2} \| (D^\alpha F_\kappa(u))^* \|_{L^\kappa_s L^1_x(S_t)} + C(1+t)^{\frac{\kappa-2}{\kappa}} t^{\frac{3-\kappa}{\kappa}} \sum_{|\alpha|\le 2} \| (D^\alpha F_\kappa(u))^* \|_{L^\infty_s L^1_x([t/4,t]\times\mathbb{R}^3)} \cdot$$

For simplicity, here we have assumed that $t > 10$ to be able to apply (1.9); however, if $0 < t < 10$ one can use (1.10) and similar arguments.

To estimate the right side of (1.16), notice that (1.2) and Leibnitz's rule give

$$\sum_{|\alpha|\le 2} |D^\alpha F_\kappa(u(t,x))| \le C \sum_{|\alpha|\le 2} |D^\alpha u(t,x)|^\kappa .$$

Since the same inequality must hold for the radial majorants, (1.16) gives

$$(1.17)\quad \begin{aligned} &A(t) \\ &\le \tfrac{A\varepsilon}{4} + C \sum_{|\alpha|\le 2} \Big(\|(D^\alpha u)^*\|^\kappa_{L^{\kappa^2}_s L^\kappa_x(S_t)} + t^{\frac{1}{\kappa}} \|(D^\alpha u)^*\|^\kappa_{L^\infty_s L^\kappa_x([t/4,t]\times\mathbb{R}^3)} \Big) \\ &\le \tfrac{A\varepsilon}{4} + C \sup_{s\le T} A^\kappa(s) \cdot \Big[\Big(\int_0^T (1+s)^{\kappa^2 \frac{2-\kappa}{\kappa}} \, ds \Big)^{\frac{1}{\kappa}} + (1+t)^{\frac{1}{\kappa} - \kappa \frac{\kappa-2}{\kappa}} \Big] \\ &= \tfrac{A\varepsilon}{4} + C \sup_{s\le T} A^\kappa(s) \cdot \Big[\Big(\int_0^T (1+s)^{\kappa(2-\kappa)} \, ds \Big)^{\frac{1}{\kappa}} + (1+t)^{-\frac{\kappa^2-2\kappa-1}{\kappa}} \Big]. \end{aligned}$$

To proceed, notice first that

$$\kappa(2-\kappa) < -1 \text{ and } \kappa^2 - 2\kappa - 1 > 0 \iff \kappa > 1+\sqrt{2}\,.$$

Hence, if $\kappa > 1+\sqrt{2}$, the last term in the brackets in (1.17) can be bounded independently of T. Consequently, by (1.15), we obtain, for some fixed constant C,

$$A(t) \le \tfrac{A\varepsilon}{4} + C(A\varepsilon)^\kappa , \quad \text{if } 0 \le t \le T \text{ and } \kappa > 1+\sqrt{2}\,.$$

But this implies (1.11′) in this case if ε is small enough so that $C(A\varepsilon)^\kappa \le A\varepsilon/4$.

To handle the case where $2 \le \kappa \le 1+\sqrt{2}$, we notice that the last factor in (1.17) is

$$\le C \cdot \begin{cases} (\log(1+T_\varepsilon))^{1/\kappa} , & \kappa = 1+\sqrt{2}\,, \\ (1+T_\varepsilon)^{-\frac{\kappa^2-2\kappa-1}{\kappa}} , & 2 \le \kappa < 1+\sqrt{2}\,. \end{cases}$$

Thus, by (1.17),

$$A(t) \le \tfrac{A\varepsilon}{4} + C(A\varepsilon)^\kappa \cdot \begin{cases} (\log(1+T_\varepsilon))^{1/\kappa}, & \kappa = 1+\sqrt{2}, \\ (1+T_\varepsilon)^{-\frac{\kappa^2-2\kappa-1}{\kappa}}, & 2 \le \kappa < 1+\sqrt{2}. \end{cases}$$

From this, we deduce that (1.11′) also holds if

$$C(A\varepsilon)^{\kappa-1}(\log(1+T_\varepsilon))^{1/\kappa} \le 1/4, \quad \kappa = 1+\sqrt{2},$$

or

$$C(A\varepsilon)^{\kappa-1}(1+T_\varepsilon)^{-\frac{\kappa^2-2\kappa-1}{\kappa}} \le 1/4, \quad 2 \le \kappa < 1+\sqrt{2}.$$

Since both are satisfied if T_ε is as in (1.4), with $c > 0$ small enough, this finishes the proof of the existence results for $2 \le \kappa < 3$.

Case 2: $1 < \kappa < 2$.

Since F_κ is only C^1 for this range of κ, we cannot hope to be able to control the L^∞ norm of u directly as in the previous case. Here, in order to obtain (1.11), we shall need to apply the energy inequality to get around this fact. For simplicity, let us assume for the moment that (1.3) also holds when $|u| \ge 1$. At the end we shall be able to remove this assumption since we shall show that $u = O(\varepsilon^\gamma)$ on $[0, T_\varepsilon) \times \mathbb{R}^3$, for some $\gamma = \gamma(\kappa) > 0$.

Assuming this, if $0 < T_* < T_\varepsilon$ is as in (1.11), since (1.14) still holds, we can repeat the proof of (1.11′) to see here that, if $\varepsilon > 0$ and $c > 0$ are small, then

$$(1.18) \qquad (1+t)^{\frac{\kappa-2}{\kappa}} \sum_{|\alpha|\le 1} \|(D^\alpha u)^*(t,|x|)\|_{L^\kappa(\mathbb{R}^3)} \le \frac{A\varepsilon}{2}, \quad 0 \le t < T_*,$$

if A is as in the bounds (1.14) for the linear solution.

If we use these bounds for u_0 as well as (1.10), we can argue as in (1.16) and (1.17) to see that, if $\delta > 0$ and $0 < t < T_*$,

$$\begin{aligned}\sum_{|\alpha|\le 1} \|(D^\alpha u)^*(t,\,\cdot\,)\|_{L^{3-\delta}(\mathbb{R}^3)} &\le C\varepsilon + Ct^{\frac{\delta}{3-\delta}} \sum_{|\alpha|\le 1} \|(D^\alpha u)^*\|^\kappa_{L^\infty_s L^\kappa_x(S_t)} \\ &\le C\varepsilon + C' T_\varepsilon^{\frac{\delta}{3-\delta}}\, \varepsilon^\kappa\, T_\varepsilon^{2-\kappa},\end{aligned}$$

with the last inequality coming from (1.18). But, if we use the definition of T_ε, we see that

$$\varepsilon^\kappa T_\varepsilon^{2-\kappa} \le C\varepsilon\, \varepsilon^{\kappa-1}\, \varepsilon^{(\kappa-1)\frac{\kappa(2-\kappa)}{\kappa^2-2\kappa-1}}.$$

Consequently, since $\frac{\kappa(2-\kappa)}{\kappa^2-2\kappa-1} > -1$ we conclude that if δ is close to zero and fixed, and if ε is sufficiently small

$$(1.19) \qquad \sum_{|\alpha|\le 1} \|D^\alpha u(t,\,\cdot\,)\|_{L^{3-\delta}(\mathbb{R}^3)} \le C_\delta\, \varepsilon, \quad 0 \le t < T_*.$$

From this we deduce, using Sobolev's theorem, that if $3 < q < \infty$ is fixed, then

$$\|u(t,\,\cdot)\|_{L^q(\mathbb{R}^3)} \le C_q\,\varepsilon\,, \quad 0 < t < T_*\,, \tag{1.20}$$

provided that ε is smaller than some ε_q depending on q. We shall use this inequality when $q = 6$.

Also, if we use Hölder's inequality, we see that (1.18) and (1.19) yield

$$\sum_{|\alpha|\le 1} \|\,(D^\alpha u)(t,\,\cdot\,)\,\|_{L^p(\mathbb{R}^3)} \le C_p\,\varepsilon\,,\ \kappa \le p < 3\,,\ 0 \le t < T_*\,. \tag{1.21}$$

In particular $u(t,\,\cdot)$ and $\nabla_x u(t,\,\cdot)$ belong uniformly to $L^2(\mathbb{R}^3)$. To complete the proof, it suffices to show that, for some $\gamma > 0$, we also have the following bounds on second derivatives

$$\|(D^\alpha u)(t,\,\cdot)\|_{L^2(\mathbb{R}^3)} \le C\varepsilon^\gamma\,, 0 \le t < T_*\,,\ |\alpha| = 2\,, \tag{1.22}$$

for, assuming $0 < \gamma < 1$, this inequality along with the last one and Sobolev's theorem imply that $u = O(\varepsilon^\gamma)$ in $[0, T_*) \times \mathbb{R}^3$.

To obtain (1.22), we first use the energy inequality to deduce that, when $|\alpha| = 2$,

$$\|D^\alpha u(t,\,\cdot)\|_{L^2} \le C\varepsilon + C\int_0^t \|\nabla_x u(s,\,\cdot)\cdot u^{\kappa-1}(s,\,\cdot)\|_{L^2}\,ds\,.$$

But if we use Hölder's inequality we see that

$$\|\nabla_x u(s,\,\cdot)\cdot u^{\kappa-1}(s,\,\cdot)\|_{L^2} \le \|\nabla_x u(s,\,\cdot)\|_{L^q}\|u(s,\,\cdot)\|_{L^6}^{\kappa-1}\,, \quad q = 6/(4-\kappa)\,.$$

Since $2 < 6/(4-\kappa) < 3$ if $1 < \kappa < 2$, (1.20), (1.21) and Hölder's inequality imply that the right side is $O(\varepsilon^\kappa)$. Therefore, we conclude that

$$\|D^\alpha u(t,\,\cdot)\|_{L^2} \le C\varepsilon + C\varepsilon^\kappa T_\varepsilon\,,\ |\alpha| = 2\,.$$

But $\varepsilon^\kappa T_\varepsilon = O(\varepsilon^\gamma)$, for some $\gamma = \gamma(\kappa) > 0$ if $\kappa < 2$, which gives us (1.22) and completes the proof of the existence part of Theorem 1.1. □

§2. Inequalities of Hardy and Littlewood and radial estimates

Let us start out by proving Theorem 1.2. Recall that we are assuming that $F(t,x) = F(t,|x|)$ is spherically symmetric and we wish to estimate the solution of

$$\begin{cases} \Box w(t,x) = F(t,x)\,, & (t,x) \in \mathbb{R}^{1+3}_+ \\ w(0,x) = \partial_t w(0,x) = 0\,. \end{cases}$$

The two estimates we must prove under these assumptions are

$$(2.1)\qquad \| \, |x|^{\frac{q-2}{q}} w(T,\,\cdot\,) \, \|_{L^q(\mathbb{R}^3)} \le C_q \, \|F\|_{L^q_t L^1_x(S_T)} \,, \quad 1 \le q < \infty$$

and

$$(2.2)\qquad \|w(T,\,\cdot\,)\|_{L^q(|x|<T)} \le C_q \, T^{\frac{3-q}{q}} \, \|F\|_{L^\infty_t L^1_x(S_T)} \,, \quad 1 \le q < 3\,.$$

The proof will involve two tools: the formula for the solution of the inhomogeneous wave equation for radial functions, as well as the maximal inequality of Hardy and Littlewood. Recall that the later says that, if

$$\big(\mathcal{M}f\big)(t) = \sup_{\rho>0} \frac{1}{2\rho} \int_{t-\rho}^{t+\rho} |f(s)| \, ds\,,$$

then

$$\|\mathcal{M}f\|_{L^p(\mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R})}\,, \quad 1 < p \le \infty\,.$$

For a proof of this inequality, see the Appendix.

Proof of (2.1). Since F is spherically symmetric, we have, setting $r = |x|$,

$$rw(T,r) = \frac{1}{2} \int_0^T \int_{|r+s-T|}^{r+T-s} \rho F(s,\rho)\, d\rho ds\,.$$

Hence, if we set

$$(2.3)\qquad G(y) = \int_0^T \int_{|y-s|\le\rho} \rho^2 F(s,\rho)\, \frac{d\rho}{\rho} ds\,,$$

then

$$(2.4)\qquad rw(T,r) \le G(T-r)\,.$$

Notice that since w is spherically symmetric

$$\| \, |x|^{\frac{q-2}{q}} w(T,\,\cdot\,) \, \|^q_{L^q(\mathbb{R}^3)} = 4\pi \int_0^\infty |rw(T,r)|^q \, dr\,.$$

Consequently, (2.1) would follow from

$$(2.1')\qquad \|G\|_{L^q(\mathbb{R})} \le C_q \|F\|_{L^q_t L^1_x(S_T)}\,, \quad 1 \le q < \infty\,.$$

To prove this, we note that the left side of (2.1′) equals the supremum over $f \in L^{\frac{q}{q-1}}(\mathbb{R})$ with norm 1 of

$$\Big|\int f(r)G(r)\,dr\Big| = \Big|\iint \Big(\frac{1}{\rho}\int_{s-\rho}^{s+\rho} f(r)\,dr\Big)\, F(s,\rho)\,\rho^2 d\rho ds\Big|$$
$$\leq 2\iint \mathcal{M}(f)(s)\,|F(s,\rho)|\,\rho^2 d\rho ds$$
$$\leq 8\pi\,\|F\|_{L^q_t L^1_x}\cdot \|\mathcal{M}(f)\|_{L^{\frac{q}{q-1}}(\mathbb{R})}.$$

Since $1 < q/(q-1) \leq \infty$ if $1 \leq q < \infty$, we conclude that (2.1′) must hold, by the Hardy-Littlewood maximal inequality. □

Proof of (2.2). We first notice that, by a scaling argument, we can take $T = 1$. In fact, if for a given $T > 0$ we let $w_T(t,x) = w(Tt,Tx)$ then $\Box w_T = T^2F(Tt,Tx)$. Hence, if (2.2) held for the special case $T = 1$ we would obtain the general case after noting that we would then have

$$\|w(T,\,\cdot\,)\|_{L^q(|x|<T)} = T^{3/q}\,\|w_T(1,\,\cdot\,)\|_{L^q(|x|<1)}$$
$$\leq C_q\, T^{3/q}\,\|T^2F(Tt,Tx)\|_{L^\infty_t L^1_x(S_1)}$$
$$= C_q T^{-1+3/q}\,\|F\|_{L^\infty_t L^1(S_T)},$$

as claimed.

Assuming then for simplicity that $T = 1$, if we argue as above we find that the left side of (2.2) is dominated by the supremum over $f \in L^{\frac{q}{q-1}}([0,1])$ with norm 1 of

$$\iint \mathcal{M}\big(r^{\frac{2-q}{q}} f\big)(s)\, F(s,\rho)\,\rho^2 d\rho ds.$$

We need only consider f supported in $[0,1]$ since the norm is the left side of (2.2) is only taken over $r = |x| < 1$. If we choose $\sigma > 1$ close to 1 the last expression is dominated by

$$\|\,\mathcal{M}(r^{\frac{2-q}{q}} f)\,\|_{L^\sigma}\cdot \|F\|_{L_t^{\frac{\sigma}{\sigma-1}} L^1_x(S_1)} \leq C_\sigma\,\|r^{\frac{2-q}{q}} f\|_{L^\sigma}\|F\|_{L^\infty_t L^1_x},$$

using the Hardy-Littlewood maximal theorem in the last step. On account of this, we would have (2.2) if we could show that for $2 \leq q < 3$ we can choose $\sigma > 1$ so that the product of the first two factors on the right is $O(1)$. But, since $\frac{1}{\sigma} - \frac{q-1}{q} = \frac{q-\sigma(q-1)}{\sigma q}$, by Hölder's inequality and the support property of f, the product must be

$$\leq \|f\|_{L^{\frac{q}{q-1}}}\Big(\int_0^1 r^{\frac{2-q}{q}\frac{\sigma q}{q-\sigma(q-1)}}\,dr\Big)^{\frac{1}{\sigma}-\frac{q-1}{q}} = C_{\sigma,q},$$

with the last inequality holding if we can choose $\sigma > 1$ so that

$$\tfrac{(2-q)\sigma}{q-\sigma(q-1)} > -1 .$$

Since this can be done if and only if $q < 3$, the proof of (2.2) is complete. □

If we use the Hardy-Littlewood inequality for fractional integrals we can prove the following variant of these inequalities which will be useful when we turn to sharp regularity theorems under the assumption of radial symmetry.

Theorem 2.1. *As above, let $\square w = F$ and assume that w has vanishing Cauchy data at $t = 0$ and that $F \in C(\mathbb{R}^{1+3}_+)$ is spherically symmetric. Then if $T > 0$,*

$$\|w\|_{L_t^{\frac{q(q-1)}{3-q}} L_x^q(S_T)} \le C_q \|F\|_{L_t^{\frac{q-1}{3-q}} L_x^1(S_T)}, \quad 1+\sqrt{2} < q < 3 . \tag{2.5}$$

Also,

$$\|w\|_{L_t^\sigma L_x^q(S_T)} \le T^{\frac{1}{\sigma}-\frac{q-2}{q}} \|F\|_{L_t^q L_x^1(S_T)} , \quad 2 \le q < 3 , \sigma < \tfrac{q}{q-2} . \tag{2.6}$$

Proof. If we recall (2.4), we see that

$$\|w\|^q_{L_t^{\frac{q(q-1)}{3-q}} L_x^q} \le \left(\int_0^\infty \left(\int_0^\infty |G(t-r)|^q \, r^{-q+2} \, dr \right)^{\frac{q-1}{3-q}} dt \right)^{\frac{3-q}{q-1}} .$$

Recall also the one-dimensional Hardy-Littlewood inequality for fractional integrals (see Appendix):

$$\Big\| \int H(t-r) \, |r|^{-1+1/p_1-1/p_2} \, dr \Big\|_{L^{p_2}} \le C_{p_1,p_2} \|H\|_{L^{p_1}} , \quad 1 < p_1 < p_2 < \infty .$$

If we take $p_1 = \frac{q-1}{q(3-q)}$ and $p_2 = \frac{q-1}{3-q}$, then of course $p_1 < p_2$. Also, $p_2 < \infty$, since we are assuming that $q < 3$. The other condition on the exponents in the Hardy-Littlewood inequality is also satisfied, for $p_1 > 1$ here is equivalent to our other assumption in (2.5) that $q > 1 + \sqrt{2}$. Finally, since $q - 2 = 1 + 1/p_1 - 1/p_2$, if we take $H(t) = |G(t)|^q$ in the fractional integral inequality, we obtain

$$\|w\|^q_{L_t^{\frac{q(q-1)}{3-q}} L_x^q} \le C \|H\|_{L^{\frac{q-1}{q(3-q)}}} = C \|G\|^q_{L^{\frac{q-1}{3-q}}} .$$

If we now apply (2.1′), we see that we can dominate the right side by $\|F\|^q_{L_t^{\frac{q-1}{3-q}} L_x^1}$, finishing the proof of (2.5).

The proof of (2.6) is similar. By scaling considerations, we see that we may take $T = 1$. Also, after possibly applying Hölder's inequality, we see that we may also assume that $q \le \sigma < q/(q-2)$. Under these assumptions, the left side of (2.6) is controlled by

$$\Big(\int_0^1 \Big(\int_0^\infty |G(t-r)|^q\, r^{-(q-2)}\, dr\Big)^{\sigma/q} dt\Big)^{1/\sigma} \le \Big(\int \Big(\int K(t,s)\, |G(s)|^q\, ds\Big)^{\sigma/q} dt\Big)^{1/\sigma},$$

where $K(t,s) = |t-s|^{-(q-2)}$ if $0 \le t \le 1$ and 0 otherwise. If we now use Minkowski's inequality, we obtain

$$\begin{aligned}\Big(\int \Big(\int K(t,s)\, H(s)\, ds\Big)^{\sigma/q} dt\Big)^{q/\sigma} &\le \sup_s \Big(\int |K(t,s)|^{\sigma/q}\, dt\Big)^{q/\sigma} \cdot \int |H(s)|\, ds \\ &= \sup_s \Big(\int_0^1 |s-t|^{-\sigma\frac{q-2}{q}}\, dt\Big)^{q/\sigma} \cdot \int |H(s)|\, ds \\ &= C_\sigma \int |H(s)|\, ds.\end{aligned}$$

Taking $H(s) = |G(s)|^q$, yields (2.6). □

§3. Blow-up for small powers

Fix a spherically symmetric and compactly supported $0 \le g \in C^2(\mathbb{R}^3)$ satisfying $g(x) \ge 1$ when $|x| \le 1$. We shall then show that for $1 < \kappa < 1+\sqrt{2}$ there is no global solution of

$$\begin{cases} \Box u = |u|^\kappa \\ u(0,x) = 0, \quad \partial_t u(0,x) = \varepsilon g(x), \end{cases} \tag{3.1}$$

even if $\varepsilon > 0$ is very small. We are assuming that g is spherically symmetric only for simplicity; however, this assumption can easily be removed using ideas in the proof of the existence part of the theorem, this time controlling u from below by its radial minorant. Also, one will see from the construction that the Cauchy data need not have this special form. What is only needed is that the solution to the linear Cauchy problem with the given data is nonnegative.

To proceed, let us recall that if u solves (3.1), then we can write $u = u_0 + w$, where u_0 is the solution of the Cauchy problem $\Box u_0 = 0$ with data $(0, \varepsilon g)$, and w is the solution of the inhomogeneous wave equation $\Box w = |u|^\kappa$ with zero Cauchy data at $t = 0$. Notice that $u_0 \geq 0$, by formula (1.5) from Chapter 1. Hence, if we write $r = |x|$ and abuse notation by writing $u(t,r) = u(t,x)$, then

$$u(t,r) \geq w(t,r) = \frac{1}{2r} \int_0^t \int_{|r-(t-s)|}^{r+t-s} |u(s,\rho)|^\kappa \, \rho d\rho ds \,. \tag{3.2}$$

Since $u(t,r) \geq u_0(t,r) \geq \varepsilon t$ if $0 < t, |x| < 1/2$, (3.2) implies that

$$u(s,\rho) \geq c_0 \varepsilon / \rho \,, \quad \text{if } s + \rho > 1 \,, \; 0 \leq s - \rho \leq 1/2 \,, \tag{3.3}$$

where $c_0 > 0$.

Now let

$$\Sigma = \{ (t,r) : \, 0 \leq r < t - 1 \} \,.$$

Then we claim that (3.2) and (3.3) yield

$$u(t,r) \geq \frac{(c_0\varepsilon)^\kappa}{2r} \iint\limits_{\substack{t-r<\rho+s<t+r \\ 0<s-\rho<1/2}} \rho^{-\kappa} \rho \, d\rho ds \,, \quad (t,r) \in \Sigma \,. \tag{3.4}$$

To see this, we first notice that on the support of the integral the two conditions in (3.3) are valid. So, to finish the claim, we need to check that the domain of integration in (3.4) is contained in that of (3.2). To this end, let us first check that $|t - r - s| < \rho < t + r - s$ in the domain of integration in (3.4). The upper bound is clear, as is the lower bound if $t - r - s > 0$. If $t - r - s < 0$, then we have $\rho > r + s - t$, since in (3.4) $s - \rho < 1/2 < t - r$ as $t - r > 1$ if $(t,r) \in \Sigma$. Similarly, we have $0 \leq s \leq t$ in (3.4) since $s < \rho + 1/2 < t + r - s + 1/2 < 2t - s - 1/2$, with the last inequality holding as $(t,r) \in \Sigma$.

Now let us establish a lower bound for the left side of (3.4). Changing variables

$$\alpha = \rho + s \quad \beta = s - \rho \,,$$

we see that, if $(t,r) \in \Sigma$ and $\kappa > 1$,

$$\begin{aligned} u(t,r) &\geq \frac{(c_0\varepsilon)^\kappa}{2r} \int_0^{1/2} \int_{t-r}^{t+r} \big((\alpha - \beta)/2 \big)^{1-\kappa} \, d\alpha d\beta \\ &\geq (c_0\varepsilon)^\kappa 2^{\kappa-2} \frac{1}{2r} \int_{t-r}^{t+r} \alpha^{1-\kappa} \, d\alpha \\ &\geq (c_0\varepsilon)^\kappa 2^{\kappa-2} (t+r)^{1-\kappa} \,. \end{aligned}$$

The Sobolev exponent γ will always be smaller than $3/2$, and hence the data, and consequently u, need not be bounded. On account of this we must strengthen our earlier assumptions about the nonlinearity somewhat. We now assume that for a given $\kappa > 2$, F_κ is a C^1 function satisfying

$$|F_\kappa(u)| \le C|u|^\kappa , \quad \text{and } C^{-1}|F_\kappa(u)| \le |uF'_\kappa(u)| \le C|F_\kappa(u)| , \tag{4.2}$$

where C is an absolute constant and the inequality is assumed to hold for all $u \in \mathbb{R}$. Thus, $|F_\kappa^{(j)}(u)| \le C|u|^{\kappa-j}$ for $j = 0, 1$.

Under these hypotheses, we wish to find the minimal γ, depending on κ, such that the conditions on the data in (4.1) are strong enough to ensure that for some $0 < T \le \infty$ there is a (weak) solution $(u, \partial_t u) \in C([0,T]; \dot{H}^\gamma \times \dot{H}^{\gamma-1})$ of (4.1). Specifically, our main result is the following

Theorem 4.1. *For a given $\kappa > 2$ set*

$$\gamma = \gamma(\kappa) = \begin{cases} \frac{3}{2} - \frac{2}{\kappa-1} , & \kappa \ge 3 \\ 1 - \frac{1}{\kappa-1} , & 2 < \kappa \le 3 . \end{cases} \tag{4.3}$$

Then there is a $T > 0$ and a unique (weak) solution of (4.1) verifying

$$(u, \partial_t u) \in C([0,T]; \dot{H}^\gamma \times \dot{H}^{\gamma-1}) \text{ and } u \in L_t^s L_x^{2(\kappa-1)}([0,T] \times \mathbb{R}^3) , \quad s = \max\{ 2(\kappa-1), 2(\kappa-1)/(\kappa-2) \} . \tag{4.4}$$

Moreover, if $\kappa \ge 3$ there is an $\varepsilon(\kappa) > 0$, so that if

$$\|f\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)} < \varepsilon(\kappa) , \tag{4.5}$$

then one can take $T = \infty$, in which case (4.1) has a unique global (weak) solution satisfying (4.4). Finally, for any $\kappa > 2$ if T_* denotes the supremum of all $T > 0$ such that there is a solution of (4.1) satisfying (4.4) then either $T_* = \infty$ or $u \notin L^{2(\kappa-1)}([0,T_*) \times \mathbb{R}^3)$.

Notice that (4.3) can be rewritten more succinctly as

$$\gamma = \gamma(\kappa) = \max\{ 3/2 - 2/(\kappa-1) , 1 - 1/(\kappa-1) \} . \tag{4.3'}$$

A simple scaling argument shows that one must always have $\gamma \ge 3/2 - 2/(\kappa-1)$ for well-posedness of (4.1). To see this, recall that by results from the last section we can always choose $g \in C_0^\infty(\mathbb{R}^3)$ so that the solution to $\Box u = |u|^\kappa$ with data $(0, g)$ has lifespan $0 < T_* < \infty$. But then $u_\varepsilon(t,x) = \varepsilon^{-2/(\kappa-1)}u(t/\varepsilon, x/\varepsilon)$ solves the same equation, but with data $(0, g_\varepsilon)$, with $g_\varepsilon = \varepsilon^{-2/(\kappa-1)-1}g(x/\varepsilon)$. The lifespan of u_ε is clearly $T_\varepsilon = \varepsilon T_*$. Also, using the definition of the homogeneous Sobolev norm, we find that

$\|g_\varepsilon\|_{\dot H^{\gamma-1}}/\|g\|_{\dot H^{\gamma-1}} = \varepsilon^{(3/2-2/(\kappa-1)-\gamma)}$. Thus, if $\gamma < 3/2 - 2/(\kappa-1)$ both the lifespan and the norm of the data go to zero with ε. Since g is compactly supported, one could thus add up suitable translates of normalized dilates of the data and obtain new data for which there is no local existence in a strip.

The fact that one also needs $\gamma \geq 1 - 1/(\kappa-1)$ is harder to see. The argument is based on the fact that

$$u_{\alpha\beta}(t,x) = \frac{c_\alpha(1-\beta^2)^{\alpha/2}}{(\varepsilon-(t-\beta x_1))^\alpha}, \quad c_\alpha = (\alpha(\alpha+1))^{\alpha/2}, \ \alpha = \frac{2}{\kappa-1}$$

satisfies $\Box u_{\alpha\beta} = |u_{\alpha\beta}|^\kappa$ and blows up when $t - \beta x_1 = \varepsilon$. The Cauchy data of course does not have compact support since $u_{\alpha\beta}$ depends only on t and x_1. Nonetheless, it turns out that one can cutoff the data outside $|x| > \varepsilon$ so that the resulting solution will still blow up when $t = \varepsilon$ and $x = 0$. The data that arises from this construction will have $\dot H^\gamma \times \dot H^{\gamma-1}$ norm which is $O((1-\beta^2)^{1-1/(\kappa-1)-\gamma}\varepsilon^{3/2-2/\kappa-\gamma})$, and so if $\gamma < 1 - 1/(\kappa-1)$ we can choose sequences $\beta \nearrow 1$ and $\varepsilon \searrow 0$ so that both the $\dot H^\gamma \times \dot H^{\gamma-1}$ norm of the data and the lifespan go to zero. Hence, by the above reasoning, we conclude that there cannot be well-posedness for (4.1) if $\gamma < 1 - 1/(\kappa-1)$. For more details see Lindblad and Sogge [1].

In the preceding construction the Cauchy data involved functions which were concentrated on highly eccentric sets. In the next section, we shall see that, if one assumes spherical symmetry, the conclusions of Theorem 4.1 corresponding to the assumption $\kappa \geq 3$ remain valid for the larger range $\kappa > 1+\sqrt{2}$. The power 3 turns out to be the unique power for which equations of the form $\Box u = c|u|^{\kappa-1}u$ are preserved under Lorentz transformations of $\mathbb{R}^{1+3}$. Thus, in the non-radial case, Theorem 4.1 says that for well-posedness in $\dot H^\gamma$ spaces, one has different types of results corresponding to whether one is in the superconformal range $\kappa \geq 3$ or the subconformal range $\kappa \leq 3$. On the other hand, as we just pointed out, under the assumption of spherical symmetry John's power $\kappa = 1+\sqrt{2}$ plays the role of the conformal power for such problems.

The proof of Theorem 4.1 will require different arguments for the cases where $\kappa > 5$ and $\kappa \leq 5$. The latter is somewhat more straightforward and will be treated first. For "critical" and "subcritical" exponents $\kappa \leq 5$ the results will be a consequence of the following estimates.

Theorem 4.2. *Suppose that u is a weak solution of the linear equation*

$$\begin{cases} \Box u(t,x) = F(t,x), & (t,x) \in \mathbb{R}^{1+3}_+ \\ u(0,\cdot) = f, & \partial_t u(0,\cdot) = g. \end{cases} \tag{4.6}$$

Then if $1/2 \le \gamma \le 1$ there is a constant C, depending only on q and γ so that for every $T > 0$

$$(4.7)\quad \|u\|_{L_t^{\frac{2q}{(3-2\gamma)q-6}} L_x^q(S_T)} + \|u(T,\,\cdot\,)\|_{\dot H^\gamma(\mathbb{R}^3)} + \|\partial_t u(T,\,\cdot\,)\|_{\dot H^{\gamma-1}(\mathbb{R}^3)}$$
$$\le C\big(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} + \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}(S_T)}\big),$$

provided that

$$6/(2-\gamma) \le q \le 2/(1-\gamma) \text{ if } 1/2 \le \gamma < 1, \text{ and } 6 \le q < \infty \text{ if } \gamma = 1.$$

Also, if $0 < \gamma < 1$,

$$(4.8)\quad \|u\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}(S_T)} + \|u(T,\,\cdot\,)\|_{\dot H^\gamma(\mathbb{R}^3)} + \|\partial_t u(T,\,\cdot\,)\|_{\dot H^{\gamma-1}(\mathbb{R}^3)}$$
$$\le C_\gamma\big(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} + \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}(S_T)}\big).$$

Remarks. The special case of (4.7) or (4.8) where $\gamma = 1/2$ is just

$$(4.9)\quad \|u\|_{L^4(S_T)} + \|u(T,\,\cdot\,)\|_{\dot H^{1/2}(\mathbb{R}^3)} + \|\partial_t u(T,\,\cdot\,)\|_{\dot H^{-1/2}(\mathbb{R}^3)}$$
$$\le C\big(\|f\|_{\dot H^{1/2}(\mathbb{R}^3)} + \|g\|_{\dot H^{-1/2}(\mathbb{R}^3)} + \|F\|_{L^{4/3}(S_T)}\big).$$

This important inequality is due to Strichartz [1], [2]. As we pointed out at the end of the last chapter, it is equivalent to a restriction theorem for the Fourier transform. To see this, recall that if $g = 0$ and $F = 0$ then

$$u(t,x) = (2\pi)^{-3}\int e^{ix\cdot\xi}\cos(t|\xi|)\,\hat f(\xi)\,d\xi.$$

Note that

$$\|f\|^2_{\dot H^{1/2}} = (2\pi)^{-3}\int \big|\,\hat f(\xi)\,|\xi|^{1/2}\,\big|^2\,d\xi.$$

Therefore, if we replace f by the function whose inverse Fourier transform is $\hat f(\xi)/|\xi|^{1/2}$, we conclude that the special case of (4.9) corresponding to $g = 0$ and $F = 0$ implies that

$$T:\, L^2(\mathbb{R}^3) \to L^4(\mathbb{R}^{1+3}),$$

if

$$Tf(t,x) = (2\pi)^{-3}\int_{\mathbb{R}^3} e^{ix\cdot\xi + it|\xi|}\,\hat f(\xi)\,d\xi/|\xi|^{1/2}.$$

By duality, the last statement of course is equivalent to

$$T^* : L^{4/3}(\mathbb{R}^{1+3}) \to L^2(\mathbb{R}^3)\,,$$

with

$$\begin{aligned}(T^*G)(x) &= (2\pi)^{-3} \iint_{\mathbb{R}^{1+3}} e^{ix\cdot\xi - it|\xi|}\widetilde{G}(t,\xi)\,dt\,d\xi/|\xi|^{1/2} \\ &= (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\cdot\xi}\hat{G}(|\xi|,\xi)\,d\xi/|\xi|^{1/2}\,.\end{aligned}$$

Here $\widetilde{G}$ and $\hat{G}$ denote the spatial and space-time Fourier transforms of $G(t,x)$, respectively. Next, if we use Plancherel's theorem we see that

$$\|T^*G\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^{-3} \int_{\mathbb{R}^3} |\hat{G}(|\xi|,\xi)|^2\,d\xi/|\xi|\,,$$

and hence our argument shows that Strichartz's estimate (4.9) implies the following restriction theorem for the light cone in $\mathbb{R}^{1+3}_+$:

$$\int_{\mathbb{R}^3} |\hat{G}(|\xi|,\xi)|^2\,d\xi/|\xi| \le C\|G\|^2_{L^{4/3}(\mathbb{R}^{1+3})}\,,\quad G\in\mathcal{S}\,. \tag{4.9'}$$

The reader can check, using arguments to follow, that the above argument can be reversed to show that this inequality implies (4.9) and hence the two estimates are equivalent. The above restriction theorem and (4.9) extend to other dimensions $n \ge 2$. In this case, for instance, $G \in L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^{1+n})$ has a Fourier transform which restricts to L^2 of the light cone with respect to the Lorentz invariant measure $d\xi/|\xi|$ as above.

The other extreme case of (4.7) where $\gamma = 1$ is due to Pecher [1]. Here the inequality reads

$$\begin{aligned}&\|u'(T,\,\cdot\,)\|_{L^2(\mathbb{R}^3)} + \|u\|_{L_t^{\frac{2q}{q-6}}L_x^q(S_T)} \\ &\qquad \le C_q\|u'(0,\,\cdot\,)\|_{L^2(\mathbb{R}^3)} + C_q\int_0^T \|F(t,\,\cdot\,)\|_{L^2(\mathbb{R}^3)}\,dt\,,\ 6\le q<\infty\,.\end{aligned} \tag{4.10}$$

Notice that, except for the mixed-norm term in the left, this inequality agrees with the energy inequality for the d'Alembertian. In the next chapter, (4.10) will be a key ingredient in our proof of global existence for the "critical wave equation" $\Box u = -u^5$ with arbitrary (not necessarily small) smooth data. A final comment about (4.10) is that even though it breaks down when $q = \infty$, the inequality does hold for this exponent under the assumption of spherical symmetry. In this case, the inequality follows from the energy inequality and the dual version of (1.7) with $q = 2$ there.

Returning to Theorem 4.1, the proof for $\kappa \le 5$ will actually only require the following special cases of Theorem 4.2.

Corollary 4.3. *Suppose that u is a weak solution of* (4.6). *Then there is a constant C depending only on γ so that if $T > 0$*

$$\text{(4.11)}\quad \|u\|_{L_t^{\frac{2\kappa}{1+\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T)} + \|u(T,\,\cdot)\|_{\dot H^\gamma(\mathbb{R}^3)} + \|\partial_t u(T,\,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} \le C\left(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} + \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}(S_T)} \right),$$

provided that

$$\text{(4.12)}\qquad 3 \le \kappa \le 5\,, \text{ and } \gamma = \tfrac{3}{2} - \tfrac{2}{\kappa-1}\,.$$

Also,

$$\text{(4.13)}\quad \|u\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T)} + \|u(T,\,\cdot)\|_{\dot H^\gamma(\mathbb{R}^3)} + \|\partial_t u(T,\,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} \le C_\gamma\left(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} + \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}(S_T)} \right),$$

if

$$\text{(4.14)}\qquad 2 < \kappa \le 3\,, \text{ and } \gamma = 1 - \tfrac{1}{\kappa-1}\,.$$

Note that (4.12) can be rewritten as

$$\text{(4.12}'\text{)}\qquad 1/2 \le \gamma \le 1\,, \text{ and } \kappa = \kappa_\gamma = \frac{7-2\gamma}{3-2\gamma}\,.$$

Since, for this value of κ and this range of γ, $\frac{6}{2-\gamma} \le \frac{2\kappa}{2-\gamma} \le \frac{2}{1-\gamma}$, we see that (4.11) is a special case of (4.7). Inequality (4.13) likewise follows from (4.8) since (4.14) is equivalent to

$$\text{(4.14}'\text{)}\qquad 0 < \gamma \le 1/2\,, \text{ and } \kappa = \kappa_\gamma = \frac{2-\gamma}{1-\gamma}\,.$$

The proof of the part of Theorem 4.1 corresponding to the "supercritical" range $\kappa > 5$ requires a different argument based on the following

Theorem 4.4. *Suppose that u is a weak solution of* (4.6). *Then, if $4 \le q < \infty$ and $\gamma = \gamma(q) = 3/2 - 4/q$,*

(4.15)

$$\|u\|_{L^q(S_T)} + \|\,(\sqrt{-\Delta_x})^{\gamma-1/2} u\,\|_{L^4(S_T)} + \|u(T,\,\cdot)\|_{\dot H^\gamma(\mathbb{R}^3)} + \|\partial_t u(T,\,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} \le C_q\left(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} + \|\,(\sqrt{-\Delta_x})^{\gamma-1/2} F\,\|_{L^{4/3}(S_T)} \right).$$

Remark. If $\Box u_0 = 0$ with data (f, g), the last inequality is essentially equivalent to

$$(4.15') \quad \|u_0\|_{L^q(\mathbb{R}^{1+3}_+)} \le C\big(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} \big), \quad 4 \le q < \infty, \quad \gamma = \tfrac{3}{2} - \tfrac{4}{q}.$$

Unfortunately, no such inequality holds for any $q < 4$. For related reasons, one needs more than $3/2 - 2/(\kappa - 1)$ derivatives for existence in (4.1) if $\kappa < 3$. However, if one assumes spherical symmetry, the results improve considerably. First of all, as we shall see, (4.15′) holds for $3 < q < \infty$, and using this we shall see that for $\kappa > 1 + \sqrt{2}$ one only needs the above regularity for existence in the radial case.

We shall postpone the proof of Theorems 4.2 and 4.4 until the next section. For the remainder of this section, though, let us show how the above inequalities imply Theorem 4.1. We shall first handle existence. Let us start with the "critical" and "subcritical" case first since the argument for $\kappa > 5$ is a little less elementary.

Proof of existence for $\kappa \le 5$. Slightly different arguments are needed to handle the subconformal and superconformal cases. Let us start with the former.

Case 1: $2 < \kappa \le 3$.

As usual, let $u_{-1} \equiv 0$ and let u_m, $m = 0, 1, 2, \dots$ be defined inductively by

$$\begin{cases} \Box u_m = F_\kappa(u_{m-1}) \\ u_m(0, \cdot\,) = f, \quad \partial_t u_m(0, \cdot\,) = g, \end{cases}$$

where $(f, g) \in \dot H^\gamma \times \dot H^{\gamma-1}$ are as in (4.1). The main step then involves the following

Lemma 4.5. *For a given $2 < \kappa \le 3$ let $\gamma = 1 - 1/(\kappa - 1)$. If we then set*

$$(4.16) \quad A_m(T) = \|u_m\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T)}, \quad \text{and } B_m(T) = \|u_m - u_{m-1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T)}$$

there is an $\varepsilon_0 = \varepsilon_0(\kappa) > 0$ so that if $m = 0, 1, 2, \dots$

$$(4.17) \quad A_m(T) \le 2A_0(T), \; B_{m+1}(T) \le \frac{1}{2} B_m(T), \; \text{if } 2A_0(T) T^{\frac{1}{\kappa-1} - \frac{1}{2}} \le \varepsilon_0.$$

Proof. We wish to apply (4.13) and induction. Let us first notice, though, that $2\kappa/(2 - \gamma) = 2(\kappa - 1)$ if γ is as above, since then $2 - \gamma = \kappa/(\kappa - 1)$. Hence,

$$\tfrac{2-\gamma}{2} = \tfrac{2-\gamma}{2\kappa} + \tfrac{1}{2}.$$

Because of this, if we write

$$\Box(u_{m+1}-u_{j+1}) = V_\kappa(u_m,u_j)(u_m-u_j)\,,\qquad \text{with } V_\kappa(u,v)=\frac{F_\kappa(u)-F_\kappa(v)}{u-v}\,,$$

then (4.13) and Hölder's inequality give

$$\begin{aligned}(4.18)\qquad \|u_{m+1}-u_{j+1}\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2\kappa}{2-\gamma}}(S_T)} &\le C_\gamma\|V_\kappa(u_m,u_j)\cdot(u_m-u_j)\|_{L_t^{\frac{2}{1+\gamma}}L_x^{\frac{2}{2-\gamma}}(S_T)}\\ &\le \frac{1}{2}\|u_m-u_j\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2\kappa}{2-\gamma}}(S_T)}\,,\end{aligned}$$

if

$$\|V_\kappa(u_m,u_j)\|_{L^2(S_T)} \le 1/2C_\gamma\,.$$

Since $|V_\kappa(u,v)|\le C(|u|+|v|)^{\kappa-1}$, there must be an $\varepsilon_0>0$ such that this is true if

$$(4.19)\qquad \|u_k\|_{L^{2(\kappa-1)}(S_T)}\le\varepsilon_0\,,\quad k=m,j\,.$$

By Hölder's inequality, this holds if

$$(4.20)\qquad A_k(T)T^{\frac{1}{2(\kappa-1)}-\frac{\gamma}{2}} = A_k(T)T^{\frac{1}{\kappa-1}-\frac{1}{2}}\le\varepsilon_0\,.$$

Next, we want to use induction to prove that $A_m(T)\le 2A_0(T)$, since by the last part of (4.17) this would imply (4.20). Therefore, let us assume that this estimate holds and show that it yields the bounds for A_{m+1}. To do this, we choose $j=-1$ in (4.18) to obtain

$$A_{m+1}\le A_0+\frac{1}{2}A_m\,,\quad \text{if } A_m(T)T^{\frac{1}{\kappa-1}-\frac{1}{2}}\le\varepsilon_0\,.$$

Since the last condition holds by the induction hypothesis and our assumption in (4.17), we obtain the first part of (4.17) as desired. The other part just involves taking $j=m-1$ in (4.18). □

Using the lemma we easily get existence for $2<\kappa\le 3$. We first notice that by (4.13)

$$A_0(T)\le C_\gamma\left(\|f\|_{\dot H^\gamma}+\|g\|_{\dot H^{\gamma-1}}\right),$$

and so we can always choose T as satisfying the last part of (4.17). Since $B_0(T)=A_0(T)$, it follows that u_m converges to a limit in $L_t^{\frac{2}{\gamma}}L_x^{\frac{2\kappa}{2-\gamma}}(S_T)$ and hence in the sense of distributions. To see that $F_\kappa(u_m)$ converges weakly to $F_\kappa(u)$, it suffices to see that

$$\|F_\kappa(u_{m+1}) - F_\kappa(u_m)\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{1-\gamma}}(S_T)} \le C2^{-m},$$

for then $F_\kappa(u_m) \to F_\kappa(u)$ in $L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{1-\gamma}}(S_T)$. However, this follows from the above since

$$\begin{aligned}\|F_\kappa(u_{m+1}) - F_\kappa(u_m)\|&_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{1-\gamma}}(S_T)} \\ &\le \|V_\kappa(u_m, u_{m-1})\|_{L^2(S_T)} \|u_{m+1} - u_m\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2\kappa}{1-\gamma}}(S_T)}.\end{aligned}$$

Thus, we have shown that u must be a weak solution of (4.1) which satisfies the second part of (4.4). The first part follows immediately from (4.13) and (4.17) if we assume, say, that the data belong to C_0^∞, for then $(u_m, \partial_t u_m)$ must be a Cauchy sequence in $C([0,T]; \dot H^\gamma \times \dot H^{\gamma-1})$ converging to $(u, \partial_t u)$. As in the linear case, this assumption about the data can be removed by a standard approximation argument using (4.13). This completes the existence part of Theorem 4.1 for $2 < \kappa \le 3$. □

Case 2: $3 \le \kappa \le 5$.
Here the main step is to obtain the following

Lemma 4.6. *For a given $3 \le \kappa \le 5$, let $\gamma = 3/2 - 2/(\kappa - 1)$. Then if we set*

$$\text{(4.21)} \quad A_m(T) = \|u_m\|_{L_t^{\frac{2\kappa}{1+\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T)}, \quad \text{and } B_m(T) = \|u_m - u_{m-1}\|_{L_t^{\frac{2\kappa}{1+\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T)},$$

there is an $\varepsilon_0 > 0$ so that if $m = 0, 1, 2, \ldots$

$$\text{(4.22)} \qquad A_m(T) \le 2A_0(T), \quad B_{m+1}(T) \le \frac{1}{2} B_m(T), \quad \text{if } 2A_0(T) \le \varepsilon_0.$$

Proof. The argument is like the one used to establish Lemma 4.5, but slightly simpler as the norms in (4.11) behave well with respect to iterations. We first notice that this inequality yields

$$\begin{aligned}\|u_{m+1} - u_{j+1}\|&_{L_t^{\frac{2\kappa}{1+\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T)} \\ &\le C\|V_\kappa(u_m, u_j)\,(u_m - u_j)\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}(S_T)} \\ &\le C' \big(\|u_m\|^{\kappa-1}_{L_t^{\frac{2\kappa}{1+\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T)} + \|u_j\|^{\kappa-1}_{L_t^{\frac{2\kappa}{1+\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T)} \big) \|u_m - u_j\|_{L_t^{\frac{2\kappa}{1+\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T)},\end{aligned}$$

using Hölder's inequality and the fact that $V_\kappa = O(|u_m|^{\kappa-1} + |u_j|^{\kappa-1})$. Thus, if $\varepsilon_0^{\kappa-1}C' < 1/4$ and if we assume that $A_m(T) \le 2A_0(T)$ then by taking $j = -1$ above we get

$$A_{m+1}(T) \le A_0(T) + \frac{1}{2}A_m(T),$$

yielding the first part of (4.22) by induction. Taking $j = m-1$ gives the other part. □

To use this lemma to obtain existence for (4.1) for this range of κ, we first notice that (4.11) implies that

$$\|u_0\|_{L_t^{\frac{2\kappa}{1+\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(\mathbb{R}_+^{1+3})} \le C(\|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}}).$$

Hence we must have $2A_0(T) \le \varepsilon_0$ for all T if the data has small norm, or, if not, this inequality will be satisfied for some $T > 0$ by the dominated convergence theorem. Therefore, if we let $T = \infty$ in the first case and T be this finite time in the second, we can argue as before to conclude that there must be a weak solution of (4.1) verifying the first part of (4.4) as well as

$$u \in L_t^{\frac{2\kappa}{1+\gamma}} L_x^{\frac{2\kappa}{2-\gamma}}(S_T) = L_t^{\frac{4\kappa(\kappa-1)}{5\kappa-9}} L_x^{\frac{4\kappa(\kappa-1)}{\kappa+3}}(S_T).$$

By Sobolev's theorem we have for $0 \le t \le T$

$$\|u(t,\cdot)\|_{L_x^{\frac{6}{3-2\gamma}}(\mathbb{R}^3)} \le C\|u(t,\cdot)\|_{\dot H^\gamma(\mathbb{R}^3)},$$

and so

$$u \in L_t^\infty L_x^{\frac{6}{3-2\gamma}}(S_T) = L_t^\infty L_x^{\frac{3(\kappa-1)}{2}}(S_T).$$

Hence we obtain the other part of (4.4) as, by Hölder's inequality,

$$\begin{aligned} &\|u\|_{L_t^{2(\kappa-1)} L_x^{2(\kappa-1)}(S_T)} \\ &\quad \le \|u\|^\theta_{L_t^{\frac{4\kappa(\kappa-1)}{5\kappa-9}} L_x^{\frac{4\kappa(\kappa-1)}{\kappa+3}}(S_T)} \|u\|^{1-\theta}_{L_t^\infty L_x^{\frac{3(\kappa-1)}{2}}(S_T)}, \quad \theta = 2\kappa/(5\kappa-9), \end{aligned}$$

which completes the proof of the existence part of Theorem 4.1 when $3 \le \kappa \le 5$. □

End of existence proof: $\kappa > 5$. The key ingredient to prove the existence results for supercritical powers is the following consequence of Theorem 4.4

Lemma 4.7. *Given $\kappa > 5$ let $\gamma = 3/2 - 2/(\kappa-1)$ and $q = 2(\kappa-1)$. Also, let*

$$A_m(T) = \|\,(\sqrt{-\Delta_x})^{1-2/(\kappa-1)}u_m\,\|_{L^4(S_T)} + \|u_m\|_{L^q(S_T)}\,, \tag{4.23}$$

$$B_m(T) = \|u_m - u_{m-1}\|_{L^4(S_T\cap\Lambda_{R,0})}\,, \tag{4.24}$$

where $\Lambda_{R,0} = \{\,(t,x) \in \mathbb{R}^{1+3}_+ : |x| < R-t\,,\, t \ge 0\}$ and $R < \infty$. Then there is an $\varepsilon_0 > 0$ so that, if $m = 0, 1, 2, \dots,$

$$A_m(T) \le 2A_0(T)\,, \quad B_{m+1}(T) \le \frac{1}{2}B_m(T)\,, \text{ if } A_0(T) \le \varepsilon_0\,. \tag{4.25}$$

The proof is a bit less elementary than that of the previous two lemmas since it requires the so-called Liebnitz rule for fractional derivatives. This result says that if $F(u) \in C^1$ satisfies $C_0^{-1} \le |uF'(u)|\,/\,|F(u)| \le C_0$ for some constant C_0 then, if $1 < q < p, r < \infty$ and $0 < \sigma \le 1$,

$$\begin{aligned} &\|\,(\sqrt{-\Delta_x})^{\sigma}F(u)\,\|_{L^q} \\ &\qquad \le C\|F'(u)\|_{L^p}\,\|\,(\sqrt{-\Delta_x})^{\sigma}u\,\|_{L^r}\,,\ 1/q = 1/p + 1/r\,, \end{aligned} \tag{4.26}$$

where C depends on C_0, σ, p, q, and r. (See Christ [1].)

Proof of Lemma 4.7. We shall want to apply (4.26) with $q = 4/3$, $p = 2$ and $r = 4$. Specifically, this inequality along with (4.15) applied to the equation $\Box(u_{m+1} - u_0) = F_\kappa(u_m)$ gives

$$\begin{aligned} A_{m+1} &\le C_q\|F_\kappa'(u_m)\|_{L^2(S_T)}A_m + A_0 \\ &\le C_q'\|u_m\|^{\kappa-1}_{L^q(S_T)}A_m + A_0 \\ &\le C_q'\,A_m^\kappa + A_0\,. \end{aligned}$$

So we want to choose ε_0 in (4.25) small enough so that $C2^\kappa\varepsilon_0^{\kappa-1} < 1$, for then $A_{m+1} \le 2A_0$ by induction. The estimate for B_m now follows from the proof of Lemma 4.5 since we have shown that (4.19) holds. In fact, (4.9) and domain of dependence considerations give that

$$B_{m+1}(T) \le C\|F_\kappa(u_m) - F_\kappa(u_{m-1})\|_{L^{4/3}(S_T\cap\Lambda_{R,0})} \le C\,\varepsilon_0^{\kappa-1}\,B_m(T)\,,$$

leading to the desired bound if $C\varepsilon_0^{\kappa-1} < 1/2$. $\Box$

Let us now see how this lemma implies the existence results for (4.1) when $\kappa > 5$. Arguing as before we see that we can always choose $T > 0$

so that (4.25) holds. Also, if the data has small enough norm we can take $T = \infty$.

Next, note that since $2(\kappa - 1) > 4$, Hölder's inequality implies that $B_0(T) \le C_R A_0(T)$. Thus, by (4.25), u_m must tend to a limit in $L^4_{\mathrm{loc}}(S_T)$ and hence in $\mathcal{D}'$ and almost everywhere. Similarly, using Hölder's inequality, one sees that $F_\kappa(u_m)$ converges to $F_\kappa(u)$ in L^1_{loc} and hence u is a weak solution of (4.1).

To verify that it satisfies (4.4), we first note that, by Fatou's lemma,

$$(4.27)\quad \|u\|_{L^q(S_T)} \le \liminf_{m\to\infty} \|u_m\|_{L^q(S_T)} \le 2A_0(T) < \infty\,, \quad q = 2(\kappa-1)\,,$$

which gives us the second half of (4.4). Also, if $\phi \in C_0^\infty(S_T)$, $\langle u_m, \phi\rangle \to \langle u, \phi\rangle$ as $m \to \infty$. Therefore, since

$$|\langle u_m, \phi\rangle| \le 2A_0\, \|\, (\sqrt{-\Delta_x})^{-(1-2/(\kappa-1))}\phi\|_{L^{4/3}}\,,$$

we conclude that

$$|\langle u, \phi\rangle| \le 2A_0\|\, (\sqrt{-\Delta_x})^{-(1-2/(\kappa-1))}\phi\|_{L^{4/3}}\,,$$

and hence $(\sqrt{-\Delta_x})^{1-2/(\kappa-1)}u \in L^4(S_T)$. This together with (4.26) and (4.27) implies that $(\sqrt{-\Delta_x})^{1-2/(\kappa-1)}F_\kappa(u) \in L^{4/3}(S_T)$. Hence, we can apply Theorem 4.4 to see that the first part of (4.4) also holds. □

Uniqueness.

To finish the proof of Theorem 4.1 we shall require the following

Theorem 4.8. *Assume that $V \in L^2([0,T] \times \mathbb{R}^3)$ and that $(f, g) \in \dot H^\gamma(\mathbb{R}^3) \times \dot H^{\gamma-1}(\mathbb{R}^3)$ with $0 < \gamma < 1$. Then the equation*

$$(4.28)\qquad \Box u = Vu\,,\ u(0,\,\cdot\,) = f\,,\ \partial_t u(0,\,\cdot\,) = g$$

has a unique solution satisfying

$$(u, \partial_t u) \in C([0,T]; \dot H^\gamma(\mathbb{R}^3) \times \dot H^{\gamma-1}(\mathbb{R}^3)) \text{ and } u \in L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}(S_T)\,.$$

Moreover, there is a universal constant C_γ so that for $0 \le t \le T$

$$(4.29)\quad \|u(t,\,\cdot\,)\|_{\dot H^\gamma} + \|\partial_t u(t,\,\cdot\,)\|_{\dot H^{\gamma-1}} \le 2\exp\big(\, C_\gamma \int_{S_T} |V(t,x)|^2\, dtdx\,\big)\,(\,\|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}}\,)\,.$$

Also, if $1/2 \le \gamma < 3/2$ suppose that $u \in L^{\frac{8}{3-2\gamma}}(S_T)$, and $(\sqrt{-\Delta_x})^{\gamma-1/2}u \in L^4(S_T)$. Then, if $\Box u = F(u)$ with data (f,g),

$$(4.30)\quad \|u(t,\,\cdot\,)\|_{\dot H^\gamma} + \|\partial_t u(t,\,\cdot\,)\|_{\dot H^{\gamma-1}} \le 2\exp\big(\, C_\gamma \int_{S_T} |F'(u(t,x))|^2\, dtdx\,\big)\,(\,\|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}}\,)\,.$$

Proof. Let us first prove (4.29). We want to use (4.8) applied to the equation $\Box(u-u_0)=Vu$, where u_0 is the solution to $\Box u_0=0$ with the same data as u when $t=0$. Let $T_1\le T$ be the largest number so that

$$\|V\|_{L^2(S_{T_1})}\le \varepsilon_\gamma\,,$$

where ε_γ is to be specified later. In particular if $\varepsilon_\gamma\le C_\gamma/2$, where C_γ is the constant in (4.8), then

$$\|u-u_0\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}(S_T)}\le\frac{1}{2}\|u\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}(S_T)}\,,$$

which implies that

$$\|u\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}(S_T)}\le 2\|u_0\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}(S_T)}\,,\quad 0\le T\le T_1\,, \tag{4.31}$$

provided that the left side is bounded. This immediately gives uniqueness for (4.28), because if u_1 and u_2 are two solutions with the same data, then $u=u_1-u_2$ solves the same equation with the corresponding $u_0=0$. As in the existence proofs above we can construct a solution of (4.28) for which the left side of (4.31) is bounded by putting $u=\lim_{m\to\infty}u_m$, where $u_{-1}=0$ and for $m=0,1,\ldots$, $\Box u_m=Vu_{m-1}$ with data as in (4.28).

Next, if we apply (4.8) and argue as in the proof of Lemma 4.5, we find that

$$\begin{aligned}\|(u-u_0)(T,\,\cdot\,)\|_{\dot H^\gamma}&+\|\partial_t(u-u_0)(T,\,\cdot\,)\|_{\dot H^{\gamma-1}}\\ &\le C_\gamma\|V\|_{L^2(S_T)}\|u\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}(S_T)}\\ &\le 2C_\gamma\|V\|_{L^2(S_T)}\|u_0\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}(S_T)}\,.\end{aligned}$$

But if we apply (4.8) one more time we can estimate the last factor:

$$\|u_0\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}(S_T)}\le C_\gamma(\|f\|_{\dot H^\gamma}+\|g\|_{\dot H^{\gamma-1}})\,.$$

This gives us

$$\|u(T,\,\cdot\,)\|_{\dot H^\gamma}+\|\partial_t u(T,\,\cdot\,)\|_{\dot H^{\gamma-1}}\le 2(\|f\|_{\dot H^\gamma}+\|g\|_{\dot H^{\gamma-1}})\,,\quad T\le T_1\,, \tag{4.32}$$

if $2C_\gamma^2\varepsilon_\gamma<1$, since by the energy inequality

$$\|u_0(T,\,\cdot\,)\|_{\dot H^\gamma}+\|\partial_t u_0(T,\,\cdot\,)\|_{\dot H^{\gamma-1}}=\|f\|_{\dot H^\gamma}+\|g\|_{\dot H^{\gamma-1}}.$$

This proves (4.29) if (4.31) holds. If T is too large so that (4.31) does not hold with $T_1 = T$ we can repeat this argument. For a given T we choose $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = T$ so that

$$\|V\|_{L^2([T_k,T_{k+1}]\times\mathbb{R}^3)} = \varepsilon_\gamma\,,\ k = 0,\ldots,N-1, \text{ and } \|V\|_{L^2([T_N,T_{N+1}]\times\mathbb{R}^3)} \le \varepsilon_\gamma.$$

Then

$$\int_{S_T} |V(t,x)|^2\,dtdx = \sum_{k=0}^{N} \|V\|^2_{L^2([T_k,T_{k+1}]\times\mathbb{R}^3)} \ge N\varepsilon_\gamma^2\,. \tag{4.33}$$

Repeating the argument that lead to (4.32) N times yields

$$\|u(T,\,\cdot\,)\|_{\dot H^\gamma} + \|\partial_t u(T,\,\cdot\,)\|_{\dot H^{\gamma-1}} \le 2^N(\,\|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}}\,)\,,\ \ 0 \le T \le T_N\,.$$

If we now use (4.33) we get (4.29) with $C_\gamma = \varepsilon_\gamma^{-2}\log 2$.

The proof of (4.30) is similar. Here one uses (4.15) instead of (4.8). □

End of proof of Theorem 4.1. Note that if $2 \le \kappa \le 3$ and $\gamma = 1-1/(\kappa-1)$, then $2/(1-\gamma) = 2(\kappa-1)$ and $2/\gamma = 2(\kappa-1)/(\kappa-2)$. Hence (4.4) and (4.29) give the uniqueness part of Theorem 4.1 for the subconformal range $\kappa \le 3$. If one takes $v = F_\kappa(u)/u$, then it also clear that Theorem 4.8 implies that if for any $\kappa > 2$ (4.4) holds for all $0 \le T < T_* < \infty$ then either $u \notin L^{2(\kappa-1)}(S_{T_*})$ or else u can be extended to a solution in a larger strip.

Thus, the only thing remaining is the uniqueness part of the Theorem 4.1 for $\kappa > 3$. Strictly speaking this does not follow from the previous theorem; however, it does follow easily from Strichartz's inequality (4.9). In fact, if u_1 and u_2 are two solutions of (4.1) satisfying (4.4) then the difference $w = u_1 - u_2$ satisfies the equation $\Box w = Vw$ with zero Cauchy data and $V = (F_\kappa(u_1) - F_\kappa(u_2))/(u_1 - u_2) \in L^2(S_T)$. So, if $\Lambda_{R,0}$ is as in Lemma 4.7, and if we apply (4.9) we obtain

$$\begin{aligned}\|u_1 - u_2\|_{L^4(S_T\cap\Lambda_{R,0})} &\le C\|V\cdot(u_1-u_2)\|_{L^{4/3}(S_T\cap\Lambda_{R,0})}\\ &\le C\|V\|_{L^2(S_T)}\,\|u_1-u_2\|_{L^4(S_T\cap\Lambda_{R,0})}\end{aligned}$$

Since $2(\kappa-1) > 4$, $u_1 - u_2 \in L^4(S_T\cap\Lambda_{R,0})$, and hence we conclude that if T is small enough $\|u_1-u_2\|_{L^4(S_T\cap\Lambda_{R,0})} = 0$ and hence $u_1 = u_2$ in $S_T\cap\Lambda_{R,0}$. Repeating this argument a finite number of times will show that the same is true for any fixed $T > 0$ giving us the remaining uniqueness statement.

§5. Proof of the inequalities

Let us first prove Theorem 4.2. The strategy will be to make several reductions until we are left with proving inequalities which follow from arguments similar to the ones in the proof of the restriction theorem for S^1 given in the last chapter.

The first reduction is based on the observation that to prove Theorem 4.2 it suffices to establish (4.8) and the special case where $\gamma = 1$. The other inequalities in (4.7) just follow from the energy inequality and interpolation with the ones we have singled out. In fact, if $1/2 \le \gamma < 1$, then if $q = 2/(1-\gamma)$ in (4.7) this is just (4.8) for this range of γ. The version of (4.7) corresponding to $q = 6/(2-\gamma)$ comes from interpolating between (4.8) for $\gamma = 1/2$ (i.e., (4.9)) and

$$\|u\|_{L_t^\infty L_x^6} + \|u'(t,\,\cdot\,)\|_{L^2} \le C(\,\|f\|_{\dot H^1} + \|g\|_{L^2} + \|F\|_{L_t^1 L_x^2}\,)\,.$$

But this inequality follows from the energy inequality since Sobolev's theorem gives

$$\|u(t,\,\cdot\,)\|_{L_x^6} \le C\|u(t,\,\cdot\,)\|_{\dot H^1}\,.$$

We have thus argued that, for $1/2 \le \gamma < 1$, the versions of (4.7) for the two endpoints where $q = 6/(2-\gamma)$ or $q = 2/(1-\gamma)$ follow from (4.8). Since the versions for $6/(2-\gamma) < q < 2/(1-\gamma)$ follow from these special cases via interpolation, we have established our claim that we need only prove (4.7) when $\gamma = 1$. This inequality, as we pointed out before, is (4.10), and it will be needed in the next chapter as well.

Next let us split our task into proving estimates for the inhomogeneous wave equation and the Cauchy problem. To this, we recall that, if u solves (4.6), then we can write $u = v + w$, where v solves the Cauchy problem with the given data

$$\begin{cases} \Box v = 0 \\ v(0,\,\cdot\,) = f\,, \quad \partial_t v(0,\,\cdot\,) = g\,, \end{cases}$$

and w solves the inhomogeneous wave equation associated with (4.6),

$$\begin{cases} \Box w = F \\ w(0,\,\cdot\,) = \partial_t w(0,\,\cdot\,) = 0\,. \end{cases}$$

On account of this splitting and the above reduction, to prove Theorem 4.2, it suffices to show that

(5.1)
$$\|v\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}} + \|v'(T,\,\cdot\,)\|_{\dot H^{\gamma-1}} \le C_\gamma(\|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}})\,,\ 0 < \gamma < 1$$

(5.2)
$$\|v\|_{L_t^{\frac{2q}{q-6}} L_x^q} + \|v'(T,\,\cdot\,)\|_{L^2} \le C_q\,(\|f\|_{\dot H^1} + \|g\|_{L^2})\,,\ 6 \le q < \infty\,.$$

as well as

(5.3)
$$\|w\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}} + \|w'(T,\cdot)\|_{\dot H^{\gamma-1}} \le C_\gamma \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}}, \, 0<\gamma<1$$

(5.4)
$$\|w\|_{L_t^{\frac{2q}{q-6}} L_x^q} + \|w'(T,\cdot)\|_{L^2} \le C_q \|F\|_{L_t^1 L_x^2}, \, 6 \le q < \infty .$$

As usual u' denotes the space-time gradient of u, and therefore

$$\|u'(T,\cdot)\|_{\dot H^{\gamma-1}} \approx \|u(T,\cdot)\|_{\dot H^\gamma} + \|\partial_t u(T,\cdot)\|_{\dot H^{\gamma-1}} .$$

Also, in (5.1)-(5.4) all of the norms are taken over $\mathbb{R}_+^{1+3}$. Such inequalities of course imply ones over strips $S_T = [0,T]\times\mathbb{R}^3$ by domain of dependence considerations, that is, Duhamel's principle.

Let us start with the inhomogeneous estimates. The ones for v are slightly more straightforward and will follow from similar arguments.

Recall that, if we use the spatial Fourier transform, then

$$w(t,x) = (2\pi)^{-3} \int_0^t \int_{\mathbb{R}^3} e^{ix\cdot\xi} \frac{\sin(t-s)|\xi|}{|\xi|} \hat F(s,\xi)\, d\xi ds .$$

Therefore, if we let

$$\big(W^\alpha F\big)(t,x) = \int_0^t \int_{\mathbb{R}^3} e^{ix\cdot\xi + i(t-s)|\xi|} \hat F(s,\xi)\, |\xi|^{-\alpha}\, d\xi ds ,$$

then (5.3) is a consequence of

(5.5)
$$\|W^\alpha F\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}} \le C_\gamma \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}}, \; 0<\gamma<1, \; \alpha = 1,$$

and

(5.6)
$$\|W^\alpha F\|_{L_t^\infty L_x^2} \le C_\gamma \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}}, \; 0<\gamma<1, \; \alpha = 1-\gamma .$$

In fact, since the imaginary part of W^α, $\alpha = 1$ is the operator sending F to w, (5.5) implies that the first term in the left of (5.3) satisfies the desired bounds, while (5.6) and Plancherel's theorem implies that the same is true for the other term. Similarly, (5.4) follows from

(5.7)
$$\|W^\alpha F\|_{L_t^{\frac{2q}{q-6}} L_x^q} \le C_q \|F\|_{L_t^1 L_x^2}, \; 6 \le q < \infty, \; \alpha = 1,$$

since, by the energy inequality, the second term in the left side of (5.4) is $\le \|F\|_{L_t^1 L_x^2}$.

The proof of these inequalities will require a couple of results which are simple consequences of basic facts from harmonic analysis.

Lemma 5.1. *Let* $K(t,x;s,y) \in C(\mathbb{R}^{1+n} \times \mathbb{R}^{1+n})$ *and set*

$$(TG)(t,x) = \int_{\mathbb{R}^{1+n}} K(t,x;s,y)G(s,y)\,dsdy\,.$$

For fixed s *and* t *define the frozen operator*

$$\big(T_{s,t}g\big)(x) = \int_{\mathbb{R}^n} K(t,x;s,y)\,g(y)\,dy$$

and suppose that

$$\|T_{s,t}g\|_{L^q(\mathbb{R}^n)} \le C_0\,|t-s|^{-1+(1/r_1-1/r_2)}\,\|g\|_{L^p(\mathbb{R}^n)}\,.$$

Then if $1 < r_1 < r_2 < \infty$ *and if* C_{r_1,r_2} *is the constant in the* $L^{r_1}(\mathbb{R}) \to L^{r_2}(\mathbb{R})$ *inequality for fractional integrals,*

$$\|TG\|_{L_t^{r_2}L_x^q(\mathbb{R}^{1+n})} \le C_0\,C_{r_1,r_2}\,\|G\|_{L_t^{r_1}L_x^p(\mathbb{R}^{1+n})}\,.$$

Lemma 5.2. *Let* $\beta \in C_0^\infty(\mathbb{R}_+)$ *satisfy* $\sum_{-\infty}^{\infty}\beta(s/2^j) = 1$, $s > 0$, *and define Littlewood-Paley operators*

$$G_j(t,x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\beta(|\xi|/2^j)\,\hat{G}(t,\xi)\,d\xi\,,$$

using the spatial Fourier transform. Then

$$\|G\|^2_{L_t^sL_x^q} \le C\sum_{j=-\infty}^{\infty}\|G_j\|^2_{L_t^sL_x^q}\,,\ \textit{if}\ 2\le q<\infty\,,\ \textit{and}\ 2\le s\le\infty\,,$$

and

$$\sum_{j=-\infty}^{\infty}\|G_j\|^2_{L_t^rL_x^p} \le C\|G\|^2_{L_t^rL_x^p}\,,\ \textit{if}\ 1<p\le 2\,,\ \textit{and}\ 1\le r\le 2\,.$$

Proof of Lemma 5.1. If we apply Minkowski's integral inequality we get

$$\|TG\|_{L_t^{r_2}L_x^q} \le \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^n}\big|\,(T_{t,s}G(s,\,\cdot\,))(x)\,\big|^q\,dx\Big)^{\frac{1}{q}}\,ds\Big)^{r_2}\,dt\Big)^{\frac{1}{r_2}}\,.$$

If we use our assumption about the frozen operator, we conclude that the last term is

$$\le C_0\Big(\int\Big(\int\|G(s,\,\cdot\,)\|_{L^p(\mathbb{R}^n)}\,|t-s|^{-1+(\frac{1}{r_1}-\frac{1}{r_2})}\,ds\Big)^{r_2}\,dt\Big)^{\frac{1}{r_2}}\,.$$

Finally, if we apply the Hardy-Littlewood inequality, we conclude that the last expression is

$$\leq C_0 C_{r_1,r_2} \Big(\int \|G(t,\cdot)\|_{L^p(\mathbb{R}^n)}^{r_1} \, dt \Big)^{1/r_1} = C_0 C_{r_1,r_2} \|G\|_{L_t^{r_1} L_x^p(\mathbb{R}^{1+n})} .$$

Proof of Lemma 5.2. The proof relies on Littlewood-Paley theory. Specifically, we shall use the fact that if $1 < p < \infty$ there is a constant $0 < C = C_p < \infty$ so that

$$C^{-1} \leq \| \big(\sum |G_j(t,\cdot)|^2 \big)^{1/2} \|_{L_x^p}^2 / \|G(t,\cdot)\|_{L_x^p}^2 \leq C .$$

For a proof of this see, e.g., Stein [1]. The constants in the inequalities in the lemma are just this constant C.

To see this, let us focus on the first inequality. If we use the first Littlewood-Paley inequality, we get that

$$\|G(t,\cdot)\|_{L_x^q}^2 \leq C \Big(\int_{\mathbb{R}^n} \big(\sum |G_j(t,x)|^2 \big)^{q/2} dx \Big)^{2/q} .$$

However, since $q/2 \geq 1$, Minkowski's inequality implies that the last expression is

$$\leq C \sum \|G_j(t,\cdot)\|_{L_x^q}^2 .$$

If we now use this and the fact that $s/2 \geq 1$, we similarly get

$$\begin{aligned} \|G\|_{L_t^s L_x^q}^2 &= \Big(\int \|G(t,\cdot)\|_{L_x^q}^{2\cdot\frac{s}{2}} dt \Big)^{2/s} \\ &\leq C \Big(\int \Big(\sum \|G_j(t,\cdot)\|_{L_x^q}^2 \Big)^{\frac{s}{2}} dt \Big)^{2/s} \\ &= C \sum \|G_j\|_{L_t^s L_x^q}^2 . \end{aligned}$$

The proof of the other inequality uses the second Littlewood-Paley inequality and is similar. □

Let us return to inequalities (5.5)-(5.7). If $\beta \in C_0^\infty(\mathbb{R}_+)$ is as in Lemma 5.2, and if we define dyadic operators

$$W_j^\alpha F(t,x) = \int_0^t \int_{\mathbb{R}^3} e^{ix\cdot\xi + i(t-s)|\xi|} \beta(|\xi|/2^j) \, \hat{F}(s,\xi) \, |\xi|^{-\alpha} \, d\xi ds ,$$

then we first claim that it suffices to prove dyadic versions of (5.5)-(5.7), where W^α is replaced by W_j^α and the constant involved is independent of

$j \in \mathbb{Z}$. This follows from Lemma 5.2 since all the exponents in the left sides of the inequalities are ≥ 2, while the ones in the right are ≤ 2. Keeping this in mind, let us just verify the claim for (5.5), since the argument for the other two inequalities is the same.

To show that this inequality follows from uniform dyadic estimates, we first observe that there must be a uniform constant C_0 so that

$$W_j^\alpha F = \sum_{\{k:\, |j-k| \leq C_0\}} W_j^\alpha F_k \,,$$

since

$$(W_j^\alpha F_k)(t,x) = \int_0^t \int_{\mathbb{R}^3} e^{ix\cdot\xi + i(t-s)|\xi|}\, \beta(|\xi|/2^j)\beta(|\xi|/2^k)\, \hat{F}(s,\xi)\, |\xi|^{-\alpha}\, d\xi ds \,,$$

and $\beta(|\xi|/2^j)\beta(|\xi|/2^k) \equiv 0$ if $|j-k|$ is large as $\beta \in C_0^\infty(\mathbb{R}_+)$. Therefore, uniform dyadic estimates and Lemma 5.2 would yield

$$\begin{aligned}
\|W^\alpha F\|^2_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}} &\leq C \sum_{-\infty}^{\infty} \|W_j^\alpha F\|^2_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}} \\
&\leq C' \sum_{j=-\infty}^{\infty} \sum_{|j-k| \leq C_0} \|F_k\|^2_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}} \\
&\leq C'' \|F\|^2_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}} \,,
\end{aligned}$$

as desired.

Having seen that the inequalities follow from uniform dyadic estimates, our next reduction will involve seeing that scaling considerations imply that these in turn must follow from the special case where $j = 0$:

(5.5′)
$$\|W_0^\alpha F\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}} \leq C_\gamma \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}} \,, \ 0 < \gamma < 1 \,, \ \alpha = 1$$

(5.6′)
$$\|W_0^\alpha F\|_{L_t^\infty L_x^2} \leq C_\gamma \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}} \,, \ 0 < \gamma < 1 \,, \ \alpha = 1 - \gamma$$

(5.7′)
$$\|W_0^\alpha F\|_{L_t^{\frac{2q}{q-6}} L_x^q} \leq C_q \|F\|_{L_t^1 L_x^2} \,, \ 6 \leq q < \infty \,, \ \alpha = 1 \,.$$

To see this, we need to use the identity

$$\big(W_j^\alpha F\big)(t,x) = \lambda^{-1-\alpha} \big(W_0^\alpha F_\lambda\big)(\lambda t, \lambda x) \,,$$
$$\text{if } F_\lambda(t,x) = \lambda^{-3} F(t/\lambda, x/\lambda) \,, \ \lambda = 2^j \,.$$

Because of this, if (5.5′) held, then by changing variables we would get for $\alpha = 1$

$$\begin{aligned}\|W_j^\alpha F\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}} &= \lambda^{-2}\lambda^{-\frac{\gamma}{2}}\lambda^{-3\cdot\frac{1-\gamma}{2}}\|W_0^\alpha F_\lambda\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}} \\ &\le C_\gamma \lambda^{-2}\lambda^{-\frac{\gamma}{2}}\lambda^{-3\cdot\frac{1-\gamma}{2}}\|F_\lambda\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}} \\ &= C_\gamma \lambda^{-2}\lambda^{-\frac{\gamma}{2}}\lambda^{-3\cdot\frac{1-\gamma}{2}}\lambda^{\frac{1+\gamma}{2}}\lambda^{3\cdot\frac{2-\gamma}{2}}\|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}} \\ &= C_\gamma \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}}\,.\end{aligned}$$

Hence (5.5′) implies the same bounds for the operators W_j^α, $\alpha = 1$. Since this argument applies to the other two inequalities, we are left now with proving (5.5′) – (5.7′).

If we let $a(t,s;\xi) = \beta(|\xi|)/|\xi|^\alpha$ if $0 \le s \le t$ and 0 otherwise, then we can rewrite $W^\alpha F$ more succinctly as

$$(TF)(t,x) = \iint e^{ix\cdot\xi + i(t-s)|\xi|}\, a(t,s;\xi)\,\hat{F}(s,\xi)\,d\xi ds\,.$$

We then claim that the remaining estimates are easy consequences of the following

Proposition 5.3. *Suppose that, as $t,s \in \mathbb{R}$, $\xi \to a(t,s;\xi)$ are spherically symmetric functions belonging to a bounded subset of C^∞ and that $a(t,s;\xi) = 0$ if $|\xi| \notin [C^{-1}, C]$, where $1 < C < \infty$ is a fixed constant. Then, if $1 < p \le 2$*

$$\|TF\|_{L_t^{\frac{2p}{2-p}} L_x^{\frac{p}{p-1}}(\mathbb{R}^{1+3})} \le C_p \|F\|_{L_t^{\frac{2p}{3p-2}} L_x^p(\mathbb{R}^{1+3})} \tag{5.8}$$

$$\|TF\|_{L_t^\infty L_x^2(\mathbb{R}^{1+3})} \le C_p \|F\|_{L_t^{\frac{2p}{3p-2}} L_x^p(\mathbb{R}^{1+3})} \tag{5.9}$$

Also, if $2 \le q < \infty$, then

$$\|TF\|_{L_t^s L_x^q(\mathbb{R}^{1+3})} \le C_q \|F\|_{L_t^1 L_x^2(\mathbb{R}^{1+3})}\,, \quad \tfrac{2q}{q-2} \le s \le \infty\,. \tag{5.10}$$

Clearly (5.9) and (5.10) imply (5.6′) and (5.7′), respectively. To get (5.5′), we first notice that interpolating between (5.8) and (5.9) yields

$$\|TF\|_{L_t^{\frac{2q}{q-2}} L_x^q} \le C_{p,q}\, \|F\|_{L_t^{\frac{2p}{3p-2}} L_x^p}\,, \tag{5.11}$$

provided that

$$1 < p \le 2 \quad \text{and} \quad 2 \le q \le p/(p-1)\,.$$

The special case where $p = 2/(2-\gamma)$ and $q = 2/(1-\gamma)$ yields (5.5′) for $0 \le \gamma \le 1/2$, since here $q = 2/(1-\gamma) \le p/(p-1) = 2/\gamma$. To prove the inequality for $1/2 \le \gamma < 1$, we notice that interpolating between (5.8) and (5.10) implies that (5.11) must also hold when

$$2 \le q < \infty \quad \text{and} \quad q/(q-1) \le p \le 2\,.$$

This gives (5.5′) with $1/2 \le \gamma < 1$, for if we take $q = 2/(1-\gamma)$ and $p = 2/(2-\gamma)$, then $q/(q-1) = 2/(1+\gamma) \le 2/(2-\gamma)$.

Thus, we have finally reduced the inhomogeneous estimates (5.3) and (5.4) to the above proposition whose proof we now give.

Proof of Proposition 5.3. First observe that, when $s = 2q/(q-2)$, inequality (5.10) follows from (5.9) by duality since T^* is the same type of operator as T and the dual space of $L_t^{\frac{2q}{q-2}} L_x^q$ is $L_t^{\frac{2p}{3p-2}} L_x^p$ with $p = q/(q-1)$. The version of (5.10) with $s = \infty$, on the other hand, follows easily from an application of Sobolev's theorem and Plancherel's inequality:

$$\begin{aligned}\big\|(TF)(t,\,\cdot\,)\big\|_{L^q(\mathbb{R}^3)} &\le \int \Big\| \int e^{ix\cdot\xi + i(t-s)|\xi|}\, a(t,s;\xi)\hat{F}(s,\xi)\, d\xi \,\Big\|_{L^q(dx)}\, ds \\ &\le C \int \|F(s,\,\cdot\,)\|_{L^2(\mathbb{R}^3)}\, ds\,.\end{aligned}$$

Since the other inequalities in (5.10) follow from interpolating between the special cases where $s = 2q/(q-2)$ or $s = \infty$, we conclude that it suffices to prove (5.8) and (5.9). But our next claim is that (5.9) actually follows from (5.8). To see this, we note that, if t_0 is fixed, and if we let $(SF)(x) = (TF)(t_0, x)$, then

$$\begin{aligned}\|TF(t_0,\,\cdot\,)\|_{L^2}^2 &= \int SF(x)\, \overline{SF(x)}\, dx \\ &= \iint (S^*SF)(t,x)\, \overline{F(t,x)}\, dtdx \\ &\le \|S^*SF\|_{L_t^{\frac{2p}{2-p}} L_x^{\frac{p}{p-1}}} \cdot \|F\|_{L_t^{\frac{2p}{3p-2}} L_x^p}\,,\end{aligned} \tag{5.12}$$

using Hölder's inequality in the last step. But

$$(S^*SF)(t,x) = \iint e^{ix\cdot\xi + i(t-s)|\xi|}\, \tilde{a}(t,s;\xi)\, \hat{F}(s,\xi)\, d\xi ds\,,$$

where $\tilde{a}(t,s;\xi) = (2\pi)^3 a(t,t_0;\xi)\, \overline{a(t_0,s;\xi)}$. Thus S^*S is the same type of operator as T. Hence, since the proof of (5.8) will show that the constants involved depend only on the size of finitely many derivatives of a, we conclude that this inequality implies that the right side of (5.12) is

$$\le C_p^2 \|F\|^2_{L_t^{\frac{2p}{3p-2}} L_x^p},$$

yielding (5.9).

Thus, we have reduced matters to proving (5.8). For this we shall need to use Lemma 5.1, so let us define the frozen operators

$$\big(T_{t,s} f\big)(x) = \int e^{ix\cdot\xi + i(t-s)|\xi|}\, a(t,s;\xi)\, \hat{f}(\xi)\, d\xi\,.$$

We then claim that

$$\|T_{t,s} f\|_{L^{\frac{p}{p-1}}(\mathbb{R}^3)} \le C|t-s|^{-2(1/p-1/2)}\, \|f\|_{L^p(\mathbb{R}^3)}\,. \tag{5.8$'$}$$

This along with Lemma 5.1 would yield (5.8) since $(3p-2)/2p-(2-p)/2p = 2-2/p$.

To prove (5.8′), it suffices to prove the special cases where $p=1$ and $p=2$, since the others follow from these by interpolation. The inequality with $p=2$ is a trivial consequence of Plancherel's theorem with $C = \sup|a|$. To prove the $L^1 \to L^\infty$ bounds, let $x \to K(t,s;x)$ denote the convolution kernel of $T_{t,s}$:

$$K(t,s;x) = \int e^{ix\cdot\xi + i(t-s)|\xi|}\, a(t,s;\xi)\, d\xi\,.$$

Then it suffices to show that

$$|K(t,s;x)| \le C_N \big(1+|t-s|\big)^{-1} \big(1+\big|\,|t-s|-|x|\,\big|\big)^{-N}, \ \forall N\,. \tag{5.8$''$}$$

For the $L^1 \to L^\infty$ estimate we only need that $K = O(|t-s|^{-1})$; however, this more precise estimate is no harder to obtain and will be used in the proof of Theorem 4.4.

To prove (5.8″) we need the following identity for the Fourier transform of Lebesgue measure on S^2:

$$\widehat{d\sigma}(\xi) = \int_{S^2} e^{i\omega\cdot\xi}\, d\sigma(\omega) \;=\; 4\pi\, \frac{\sin|\xi|}{|\xi|}\,. \tag{5.13}$$

Let us postpone its proof for the moment and see how it can be used to prove (5.8″). The first step is to notice that it gives

$$\begin{aligned} K(t,s;x) &= \int_0^\infty \int_{S^2} e^{ix\cdot\rho\omega}\, e^{i(t-s)\rho}\, a(t,s;\rho)\, d\sigma(\omega)\, \rho^2 d\rho \\ &= \frac{4\pi}{|x|} \int_0^\infty \sin(|x|\rho)\, e^{i(t-s)\rho}\, a(t,s;\rho)\, \rho d\rho\,. \end{aligned}$$

If we integrate by parts with respect to ρ, we see that (5.8″) holds when $|x| < 1$.

To see that the bounds also hold for $|x| > 1$, let $b(t,s;\rho) = \rho\, a(t,s;\rho)$, when $\rho > 0$ and 0 otherwise. Then the Fourier transform with respect to the last variable satisfies $|\hat{b}(t,s;\tau)| \le C_N(1+|\tau|)^{-N}$ for any N, where the constants are independent of s and t, because of our assumptions regarding a. To use this we note that we can write $K = K_+ + K_-$, where

$$K_\pm(t,s;x) = \frac{2\pi}{i|x|}\int e^{i((t-s)\pm|x|)\rho} b(t,s;\rho)\,d\rho$$
$$= \frac{2\pi}{i|x|}\,\hat{b}(t,s;\mp|x| - (t-s))\,.$$

Since our observation about the Fourier partial transform of b implies that the last expression satisfies the bounds in (5.8″) if $|x| > 1$, we only have to prove (5.13).

The proof of this formula is easy. By symmetry we may assume that $\xi = (0,0,\xi_3)$. If we then argue as in the proof of formula (1.11) from Chapter 1, we find that

$$\widehat{d\sigma}(0,0,\xi_3) = 2\pi\int_0^\pi e^{i\xi_3\cos\theta}\sin\theta\,d\theta = 2\pi\int_{-1}^1 e^{i\xi_3\tau}\,d\tau = 4\pi\,\sin\xi_3/\xi_3\,,$$

as desired. □

End of Proof of Theorem 4.2. We still must obtain the homogeneous estimates (5.1) and (5.2). Since all but the mixed norm terms in the left sides of the inequality can be estimated by the energy inequality, it suffices to see that if $\Box v = 0$ with data (f,g), then

$$\|v\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}(\mathbb{R}^{1+3})} \le C_\gamma\left(\|f\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)}\right), \quad 0<\gamma<1$$

$$\|v\|_{L_t^{\frac{2q}{q-6}} L_x^q(\mathbb{R}^{1+3})} \le C_q\left(\|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)}\right), \quad 6\le q<\infty\,.$$

However, if we recall the Fourier transform representation of v from §1 of Chapter 1, we see that, if

$$(R^\alpha f)(t,x) = \int e^{ix\cdot\xi+it|\xi|}\,\hat{f}(\xi)\,d\xi/|\xi|^\alpha\,,$$

then the above inequalities are equivalent to

$$\text{(5.1}'\text{)}\qquad \|R^\alpha f\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}(\mathbb{R}^{1+3})} \le C_\gamma\|f\|_{L^2(\mathbb{R}^3)}\,, \quad \alpha=\gamma$$

$$\text{(5.2}'\text{)}\qquad \|R^\alpha f\|_{L_t^{\frac{2q}{q-6}} L_x^q(\mathbb{R}^{1+3})} \le C_q\|f\|_{L^2(\mathbb{R}^3)}\,, \quad \alpha=1\,.$$

To prove these two inequalities, let $\beta \in C_0^\infty(\mathbb{R}_+)$ be as above and define dyadic operators

$$(R_j^\alpha f)(t,x) = \int e^{ix\cdot\xi + it|\xi|}\, \beta(|\xi|/2^j)\, \hat{f}(\xi)\, d\xi/|\xi|^\alpha\,.$$

Then, as before, Littlewood-Paley theory and scaling arguments show that the above inequalities follow from the unit dyadic estimates

(5.1″)
$$\|R_0^\alpha f\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}(\mathbb{R}^{1+3})} \le C_\gamma \|f\|_{L^2(\mathbb{R}^3)}, \quad \alpha = \gamma$$

(5.2″)
$$\|R_0^\alpha f\|_{L_t^{\frac{2q}{q-6}} L_x^q(\mathbb{R}^{1+3})} \le C_q \|f\|_{L^2(\mathbb{R}^3)}, \quad \alpha = 1,\ 6 \le q < \infty.$$

Let $(R_0^\alpha)^*$ denote the adjoint operator:

$$\big((R_0^\alpha)^* F\big)(x) = \iint e^{ix\cdot\xi - is|\xi|}\, \beta(|\xi|)\, \hat{F}(s,\xi)\, |\xi|^{-\alpha}\, d\xi ds\,.$$

Then, by duality, (5.1″) of course is equivalent to the statement that

$$\|\,(R_0^\alpha)^* F\,\|_{L^2(\mathbb{R}^3)} \le C_\gamma\, \|F\|_{L_t^{\frac{2}{2-\gamma}} L_x^{\frac{2}{1+\gamma}}(\mathbb{R}^{1+3})}\,, \quad 0 < \gamma < 1\,,\ \alpha = \gamma\,.$$

Note that if T is as in Proposition 5,3, then

$$\big((R_0^\alpha)^* F\big)(x) = \big(TF\big)(0,x)\,,$$

provided that the symbol involved is given by $a(t,s;\xi) = \beta(|\xi|)/|\xi|^\alpha$. On account of this, the last inequality, and hence (5.1), follows from (5.9). Here we have used the fact that if $p = 2/(1+\gamma)$, then $2p/(3p-2) = 2/(2-\gamma)$.

The remaining inequality essentially follows from this argument. Note that the dual version of (5.2″) is

(5.2‴)
$$\|\,(R_0^\alpha)^* F\,\|_{L^2(\mathbb{R}^3)} \le C_q\, \|F\|_{L_t^{\frac{2q}{q+6}} L_t^{\frac{q}{q-1}}(\mathbb{R}^{1+3})}\,, \quad 6 \le q < \infty\,.$$

Here, if $p = q/(q-1)$, then $2p/(3p-2) = 2q/(q-2)$, and so we only have $1 \le 2q/(q+6) < 2p/(3p-2)$. Consequently, we cannot apply (5.9) directly to get (5.2‴). To get around this, we notice that Sobolev estimates give that $T: L_t^1 L_x^p \to L_x^2$, and so we conclude, by interpolation, that (5.9) generalizes to

$$\|TF\|_{L_t^\infty L_x^2(\mathbb{R}^{1+3})} \le C_p\, \|F\|_{L_t^r L_x^p(\mathbb{R}^{1+3})}\,,\ 1 < p \le 2\,,\ 1 \le r \le \tfrac{2p}{3p-2}\,.$$

Using this we get (5.2‴), which completes the proof Theorem 4.1. □

Proof of Theorem 4.4. We first observe that (4.9) yields

$$\begin{aligned}\|(\sqrt{-\Delta_x})^{\gamma-1/2}u\|_{L^4(S_T)} + \|u(T,\cdot)\|_{\dot H^\gamma(\mathbb{R}^3)} + \|\partial_t u(T,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} \\ \le C_q\big(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} + \|(\sqrt{-\Delta_x})^{\gamma-1/2}F\|_{L^{4/3}(S_T)}\big),\end{aligned}$$

since $\Box(\sqrt{-\Delta_x})^{\gamma-1/2}u = (\sqrt{-\Delta_x})^{\gamma-1/2}F$ and $(\sqrt{-\Delta_x})^{\gamma-1/2}u$ has Cauchy data $((\sqrt{-\Delta_x})^{\gamma-1/2}f, (\sqrt{-\Delta_x})^{\gamma-1/2}g)$. Since (4.9) is a special case of inequality (4.7) in Theorem 4.1, we conclude that, to prove (4.15), it suffices to show that

$$\|u\|_{L^q(S_T)} \le C_q\big(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} + \|(\sqrt{-\Delta_x})^{\gamma-1/2}F\|_{L^{4/3}(S_T)}\big),$$
$$\text{if } 4 \le q < \infty,\ \gamma = 3/2 - 4/q.$$

Splitting u into its homogeneous and inhomogeneous parts as before, shows that our task amounts to showing that if γ and q are as above then

$$\|v\|_{L^q(\mathbb{R}^{1+3}_+)} \le C_q\big(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)}\big) \tag{5.14}$$

$$\|w\|_{L^q(\mathbb{R}^{1+3}_+)} \le C_q\|(\sqrt{-\Delta_x})^{\gamma-1/2}F\|_{L^{4/3}(S_T)}. \tag{5.15}$$

If R_0^α is as above, then the first inequality would follow as before from showing that, for a given α,

$$\|R_0^\alpha f\|_{L^q(\mathbb{R}^{1+3}_+)} \le C_q\|f\|_{L^2(\mathbb{R}^3)},\quad 4 \le q < \infty. \tag{5.14′}$$

The estimate for $q = 4$ is just (5.1″) with $\gamma = 1/2$. Also, by Sobolev's lemma and Plancherel's theorem $R_0^\alpha : L^2(\mathbb{R}^3) \to L^\infty(\mathbb{R}^{1+3}_+)$. Interpolating between the last two results gives (5.14′).

To prove (5.15), we note that, if W_j^α is as in the proof of the inhomogeneous estimates in Theorem 4.1, then (5.15) would follow from the unit dyadic estimates

$$\|W_0^\alpha F\|_{L^q(\mathbb{R}^{1+3}_+)} \le C\|F\|_{L^{4/3}(\mathbb{R}^{1+3}_+)},\quad 4 \le q \le \infty. \tag{5.15′}$$

The inequality with $q = 4$ was proved before. Since, for a given α, the kernel of W_0^α satisfies the bounds in (5.8″), we conclude that $K(t,x,\cdot,\cdot) \in L^4(\mathbb{R}^{1+3}_+)$ uniformly, and hence (5.15′) also holds when $q = \infty$. Since interpolation between the special cases where $q = 4$ or $q = \infty$ yields the remaining inequalities, the proof is complete. □

§6. Improved results under spherical symmetry

The purpose of this section is to show that, since one has better Strichartz estimates under the assumption of spherical symmetry, one in turn has better existence results for equations of the form $\Box u = F_\kappa(u)$ with radial data. In particular, in this case, John's power, $\kappa = 1+\sqrt{2}$, in some sense plays the role of the conformal power $\kappa = 3$, at least in terms of existence theorems under minimal regularity assumptions. Specifically, using Theorem 2.1 and some related estimates for the linear Cauchy problem we have the following

Theorem 6.1. *Let $1+\sqrt{2} < \kappa < 3$ and fix F_κ satisfying (4.2). Assume also that f and g are spherically symmetric functions. There is a $\varepsilon(\kappa) > 0$ so that, if*

$$\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} < \varepsilon(\kappa)\,, \quad \gamma = \tfrac{3}{2} - \tfrac{2}{\kappa-1}\,, \tag{6.1}$$

then there is a unique global (weak) solution $u \in L_t^{\frac{\kappa(\kappa-1)}{3-\kappa}} L_x^\kappa(\mathbb{R}^{1+3}_+)$ of

$$\Box u = F_\kappa(u)\,,\ u(0,\,\cdot\,) = f\,,\ \partial_t u(0,\,\cdot\,) = g\,. \tag{6.2}$$

Also, if $2 \le \kappa < 1+\sqrt{2}$ there is $\varepsilon(\kappa) > 0$ so that if $0 < \varepsilon < \varepsilon(\kappa)$ and

$$\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)} < \varepsilon\,, \quad \gamma = \tfrac{1}{2} - \tfrac{1}{\kappa}\,, \tag{6.3}$$

then there is a unique solution $u \in L_t^{\kappa^2} L_x^\kappa([0,T_\varepsilon]\times\mathbb{R}^3)$, with $T_\varepsilon = \varepsilon^{\frac{\kappa(\kappa-1)}{\kappa^2-2\kappa-1}}$.

We pointed out in the last section that one always needs $\gamma \ge 3/2 - 2/(\kappa-1)$ for global existence theorems. One also needs $\gamma \ge 1/2 - 1/\kappa$ even for local existence results. To see this, we note that if $g(x) = \chi(x)/|x|^{2+1/\kappa}$, with $\chi \in C_0^\infty(\mathbb{R}^3)$, then $g \in \dot H^{\gamma-1}(\mathbb{R}^3)$ for any $\gamma < 1/2 - 1/\kappa$. On the other hand, if $\chi \ge 0$ and $\chi(0) = 1$, the solution to the Cauchy problem $\Box u_0 = 0$ with data $(0,g)$ is a nonnegative function which is not in $L^\kappa(\mathcal{N})$ if $\mathcal{N}$ is any neighborhood of the origin in $\mathbb{R}^{1+3}_+$, as $u_0 \approx t^{-1}\,|t-|x||^{-1/\kappa}$. On account of this and the comparison theorem for $\mathbb{R}^{1+3}_+$, (6.2) cannot have a local weak solution near the origin. In fact, if there were such a solution, then we would have that $u \ge u_0$, and hence $|u|^\kappa \notin L^1$ near the origin in $\mathbb{R}^{1+3}_+$. Hence, $|u|^\kappa$ cannot be a distribution and so (6.2) makes no sense, giving us our contradiction.

Thus, even for compactly supported data, the condition that

$$\gamma \ge \max\{3/2 - 2/(\kappa-1),\ 1/2 - 1/\kappa\}$$

in the theorem is necessary. It is interesting to note that, for $\kappa > 0$,

$$3/2 - 2/(\kappa-1) = 1/2 - 1/\kappa \iff \kappa = 1+\sqrt{2}\,.$$

For the proof, we need to recall that Theorem 2.1 says that if w solves the inhomogeneous equation $\Box w = F$ in $\mathbb{R}_+^{1+3}$ with zero Cauchy data, and if F is spherically symmetric, then

$$\text{(6.4)} \qquad \|w\|_{L_t^{\frac{\kappa(\kappa-1)}{3-\kappa}} L_x^{\kappa}(\mathbf{R}_+^{1+3})} \le C_\kappa \|F\|_{L_t^{\frac{\kappa-1}{3-\kappa}} L_x^1(\mathbf{R}_+^{1+3})}, \quad 1+\sqrt{2} < \kappa < 3,$$

and

$$\text{(6.5)} \quad \|w\|_{L_t^{\kappa^2} L_x^{\kappa}([0,T]\times\mathbf{R}^3)} \le CT^{\frac{1+2\kappa-\kappa^2}{\kappa^2}} \|F\|_{L_t^{\kappa} L_x^1([0,T]\times\mathbf{R}^3)}, \quad 2 \le \kappa < 1+\sqrt{2},$$

with the last inequality following from (2.6) and the fact that $\kappa^2 < \kappa/(\kappa-2)$ if $\kappa < 1+\sqrt{2}$.

In addition to these estimates we shall need the following estimates for the linear Cauchy problem which, as we said before, are based on improved Strichartz estimates under the assumption of spherical symmetry.

Theorem 6.2. *Let u_0 be the solution of the linear Cauchy problem in $\mathbb{R}_+^{1+3}$:*

$$\Box u_0 = 0, \; u_0(0,\cdot) = f, \; \partial_t u_0(0,\cdot) = g.$$

Then, if f and g are spherically symmetric,

$$\text{(6.6)} \quad \|u_0\|_{L_t^{\frac{\kappa(\kappa-1)}{3-\kappa}} L_x^{\kappa}(\mathbf{R}_+^{1+3})} \le C_\kappa \left(\|f\|_{\dot H^\gamma(\mathbf{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbf{R}^3)} \right), \; \gamma = \tfrac{3}{2} - \tfrac{2}{\kappa-1}, \; 1+\sqrt{2} < \kappa < 3,$$

and

$$\text{(6.7)} \quad \|u_0\|_{L_t^{\kappa^2} L_x^{\kappa}([0,T]\times\mathbf{R}^3)} \le C_\kappa T^{\frac{1+2\kappa-\kappa^2}{\kappa^2}} \left(\|f\|_{\dot H^\gamma(\mathbf{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbf{R}^3)} \right), \; \gamma = \tfrac{1}{2} - \tfrac{1}{\kappa}, \; 2 \le \kappa < 1+\sqrt{2}.$$

Let us postpone the proof for a moment and see now how it yields the improved existence results for spherically symmetric solutions of (6.2).

Proof of Theorem 6.1. We shall only prove the existence part since the uniqueness assertion follows from similar arguments.

Let us first suppose that $1+\sqrt{2} < \kappa < 3$. As usual, let $u_{-1} \equiv 0$, and then for $m = 0, 1, 2, \dots$, let u_m be defined recursively by $\Box u_m = F_\kappa(u_{m-1})$ with the same Cauchy data (f, g) as in (6.2). We then claim that, if $\varepsilon(\kappa) > 0$ is small enough,

$$\text{(6.8)} \qquad A_m = \|u_m - u_{m-1}\|_{L^{\frac{\kappa(\kappa-1)}{3-\kappa}} L_x^{\kappa}(\mathbf{R}_+^{1+3})} \le 2^{-m} C_\kappa \varepsilon, \; 0 < \varepsilon < \varepsilon(\kappa),$$

provided that the data satisfy (6.1) and C_κ is as in (6.6). If this is the case then u_m must converge in $L_t^{\frac{\kappa(\kappa-1)}{3-\kappa}} L_x^\kappa(\mathbb{R}_+^{1+3})$ to a weak solution of (6.2).

By (6.6), (6.8) holds if $m = 0$. So let us assume that the estimate holds whenever $m \geq 1$ is replaced by $n < m$ and show that then (6.8) must hold if $\varepsilon(\kappa) > 0$ is small. Notice that, under this induction hypothesis, we must have

$$(6.8') \qquad \|u_n\|_{L_t^{\frac{\kappa(\kappa-1)}{3-\kappa}} L_x^\kappa(\mathbb{R}_+^{1+3})} \leq 2C_\kappa\varepsilon\,. \quad n < m\,.$$

To use this, we first apply (6.4) to see that there are constants C_j, $j = 1, 2$, depending on κ, so that

$$\begin{aligned} A_m &\leq C_1 \| F_\kappa(u_{m-1}) - F_\kappa(u_{m-2}) \|_{L_t^{\frac{\kappa-1}{3-\kappa}} L_x^1(\mathbb{R}_+^{1+3})} \\ (6.9) \qquad &\leq C_2 \left(\|u_{m-1}\|^{\kappa-1}_{L_t^{\frac{\kappa(\kappa-1)}{3-\kappa}} L_x^\kappa(\mathbb{R}_+^{1+3})} + \|u_{m-2}\|^{\kappa-1}_{L_t^{\frac{\kappa(\kappa-1)}{3-\kappa}} L_x^\kappa(\mathbb{R}_+^{1+3})} \right) \cdot A_{m-1} \\ &\leq 2C_2(2\varepsilon)^{\kappa-1} A_{m-1}\,, \end{aligned}$$

using Hölder's inequality and the fact that

$$F_\kappa(u_{m-1}) - F_\kappa(u_{m-2}) = O\big(\left(|u_{m-1}|^{\kappa-1} + |u_{m-2}|^{\kappa-1}\right) \cdot |u_{m-1} - u_{m-2}| \big)\,.$$

By (6.9) and the induction hypothesis, we see that (6.8) must hold if $2C_2(2\varepsilon(\kappa))^{\kappa-1} < 1/2$, which finishes the proof of Theorem 6.1 when $1 + \sqrt{2} < \kappa < 3$.

To prove the other half, let $2 \leq \kappa < 1 + \sqrt{2}$ and set

$$B_m = \|u_m - u_{m-1}\|_{L_t^{\kappa^2} L_x^\kappa([0,T_\varepsilon]\times\mathbb{R}^3)}\,.$$

We then claim that, if $\varepsilon(\kappa) > 0$ is small enough and if C_κ is as (6.7), then

$$(6.10) \qquad B_m \leq C_\kappa\, \varepsilon\, T_\varepsilon^{\frac{1+2\kappa-\kappa^2}{\kappa^2}}\, 2^{-m}\,, \quad 0 < \varepsilon < \varepsilon(\kappa)\,.$$

As before, this implies that the u_m converge to a weak solution of the equation.

By (6.7), we conclude that (6.10) must hold when $m = 0$. Therefore, let us assume that the estimate holds when $m \geq 1$ is replaced by any $n < m$ and show that this yields (6.10) if ε is small enough. However, if we use (6.5) and argue as above, we see that there must be an absolute constant B so that

$$B_m \leq B\left(2C_\kappa \varepsilon T_\varepsilon^{\frac{1+2\kappa-\kappa^2}{\kappa^2}}\right)^{\kappa-1} B_{m-1} = B\,(2C_\kappa)^{\kappa-1}\,\varepsilon^{\frac{\kappa-1}{\kappa}}\, B_{m-1}\,.$$

Since $\kappa > 1$, as claimed, the induction hypothesis yields (6.10) if $\varepsilon > 0$ is small enough. □

To conclude this section we must establish Theorem 6.2. The proof will be based on the following improvement of the Strichartz estimates from the last section which is valid under the assumption of spherical symmetry.

Proposition 6.3. *Let* $\Box u_0 = 0$, $u_0(0,x) = f(x)$, $\partial_t u_0(0,x) = g(x)$. *If* f *and* g *are spherically symmetric, then*

$$(6.11)\quad \|u_0\|_{L^p_{t,x}(\mathbb{R}^{1+3}_+)} \le C_p\left(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)}\right),\ \gamma = \tfrac{3}{2} - \tfrac{4}{p},\quad 3 < p < \infty,$$

and

$$(6.12)\quad \|u_0\|_{L^p_{t,x}(\{(t,x)\in\mathbb{R}^{1+3}_+:\ \min\{t,|x|\}<1\})} \le C_p\left(\|f\|_{\dot H^\gamma(\mathbb{R}^3)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^3)}\right),\ \gamma = \tfrac{1}{2} - \tfrac{1}{p},\ 2 \le p < 3.$$

Assuming this, we easily obtain Theorem 6.2:

Proof of Theorem 6.2. To prove (6.6), we just interpolate between (6.11) and the trivial estimate

$$(6.13)\qquad \|u\|_{L^\infty_t L^2_x} \le C\|f\|_{L^2} + C\|g\|_{\dot H^{-1}}.$$

If $1+\sqrt{2} < q < 3$, we can find a $3 < p < \infty$ and a corresponding $0 < \theta < 1$ so that

$$\begin{aligned} \tfrac{1}{q} &= \theta\tfrac{1}{p} + (1-\theta)\tfrac{1}{2} \\ \tfrac{3-q}{q(q-1)} &= \theta\tfrac{1}{p} + (1-\theta)\tfrac{1}{\infty}. \end{aligned}$$

If these relations hold, then, by interpolation, (6.11) and (6.13) yield (6.6) with

$$\gamma = \theta\left(\tfrac{3}{2} - \tfrac{4}{p}\right).$$

But

$$\tfrac{\theta}{p} = \tfrac{3-q}{q(q-1)},\quad \text{and}\quad \theta = 1 - \tfrac{4q-8}{q(q-1)}.$$

So

$$\gamma = \tfrac{3}{2} - \tfrac{6q-12}{q(q-1)} - \tfrac{12-4q}{q(q-1)} = \tfrac{3}{2} - \tfrac{2}{q-1},$$

as desired.

To prove (6.7), notice that, since the inequality is scale invariant, we may take $T = 1$. In this case the inequality just comes from interpolating between (6.12) and (6.13). $\Box$

Proof of Proposition 6.3. We shall take $g = 0$ since the argument for non-zero g is similar. If $f \in \mathcal{S}(\mathbb{R}^3)$ is spherically symmetric and

$$(Tf)(t,|x|) = \int_{\mathbb{R}^3} e^{ix\cdot\xi}\cos(t|\xi|)\,\hat f(\xi)\,d\xi,$$

then since $\hat f(\xi) = \hat f(|\xi|)$ is also spherically symmetric, it suffices to show that

$$(6.11') \quad \|Tf\|_{L^p_{t,x}(\mathbb{R}^{1+3}_+)} \le C_p \Big(\int_{-\infty}^{\infty} \big| \hat f(\rho)\, \rho^{5/2-4/p} \big|^2 \, d\rho \Big)^{1/2}, \quad 3 < p < \infty,$$

and

$$(6.12') \quad \|Tf(t,|x|)\|_{L^p_{t,x}(\{(t,x)\in\mathbb{R}^{1+3}_+ :\, |x|<1\})} + \sup_{t>0} \|Tf(t,|x|)\|_{L^p(|x|>1)} \le C_p \Big(\int_{-\infty}^{\infty} \big| \hat f(\rho) \rho^{3/2-1/p} \big|^2 \, d\rho \Big)^{1/2}, \ 2 \le p < 3.$$

Let us start with (6.11′). As in (5.13), let $\widehat{d\sigma}$ denote the Fourier transform of Lebesgue measure on S^2. Then, using polar coordinates, $\xi = \rho\omega$, we can write

$$\begin{aligned}(Tf)(t,|x|) &= \int_0^\infty \int_{S^2} e^{ix\cdot\rho\omega} \cos(t\rho)\, \hat f(\rho)\, d\sigma(\omega)\, \rho^2 d\rho \\ &= \int \cos(t\rho)\, \widehat{d\sigma}(\rho|x|)\, \hat f(\rho)\, \rho^2\, d\rho .\end{aligned}$$

Viewing this as an inverse Fourier transform in t, for fixed $|x|$, we can apply Sobolev's theorem and the Plancherel inequality to get

$$\Big(\int_0^\infty |(Tf)(t,|x|)|^p \, dt \Big)^{2/p} \le C \int_0^\infty \big| \hat f(\rho)\, \widehat{d\sigma}(\rho|x|) \rho^2 \, \rho^{1/2-1/p} \big|^2 \, d\rho .$$

Thus, since $p > 2$, the above and Minkowski's inequality give

$$\begin{aligned}\|(Tf)(t,|x|)\|_{L^p_{t,x}} &\le C \|\hat f(\rho)\, \rho^{5/2-1/p}\, \widehat{d\sigma}(\rho|x|)\|_{L^p_x L^2_\rho} \\ &\le C \|\hat f(\rho)\, \rho^{5/2-1/p}\, \widehat{d\sigma}(\rho|x|)\|_{L^2_\rho L^p_x} .\end{aligned}$$

Since $\|\widehat{d\sigma}(\rho|x|)\|_{L^p_x(\mathbb{R}^3)} \le C\rho^{-3/p}$, this yields (6.11′).

This argument also shows that the first term in the left side of (6.12′) satisfies the desired bounds since (5.13) gives $\| \widehat{d\sigma}(\rho|x|) \|_{L^p_x(|x|<1)} \le C/\rho$. To handle the other term, notice that, if we fix t, then for $2 \le p < \infty$

$$\begin{aligned}\|Tf(t,\cdot)\|_{L^p(|x|>1)} &\le C \Big\| \int_0^\infty \cos(t\rho) \hat f(\rho) \sin(\rho r)\, \rho\, d\rho \Big\|_{L^p((1,\infty);\, r^{2-p} dr)} \\ &\le C \Big\| \int_0^\infty \cos(t\rho) \hat f(\rho) \sin(\rho r)\, \rho\, d\rho \Big\|_{L^p((0,\infty); dr)} \\ &\le C_p \, \|\hat f(\rho)\, \rho\, \rho^{1/2-1/p}\|_{L^2(d\rho)} ,\end{aligned}$$

using again Sobolev's theorem and Plancherel's inequality in the last step. □

Notes

We have already commented on historical background related to John's theorem concerning the power $\kappa = 1 + \sqrt{2}$. The radial inequalities we used to prove the positive results in his theorem are due to Lindblad and Sogge [2], but these are related to earlier ones due to several authors. As we pointed out before, the special case of $q = 2$ in (1.7) is a limiting case of estimates of Pecher [1] which requires spherical symmetry. A variant of this case of (1.7) was also obtained by Lindblad [3]. The inequality for $q = 2$ was proved by Klainerman and Machedon [1] using the Hardy-Littlewood maximal theorem. Somewhat related arguments can also be found in Müller and Seeger [1]. We saw in the last chapter the expediency of using estimates involving L^1 norms in the right when studying quasilinear equations. In the case of semilinear equations, this important idea goes back to Lindblad [1] and this is perhaps the first departure from John's [3] approach of using L^∞ norms. The Strichartz estimates as stated in Theorem 4.2 are due to Lindblad and Sogge [1]; however, these to are related to many earlier ones. All are related to the original estimate (4.9) of Strichartz [1], [2] and use ideas from his proof. An earlier version of his estimate though is due to Segal [1], and his estimates for the homogeneous inequality was inspired by the L^2 restriction theorem of Fefferman and Stein (see Fefferman [1], [2]) and the sharp version of Stein and Tomas (see Tomas [1]). Mixed-norm estimates for the wave equation were introduced by Marshall [1], and Pecher [1] later obtained (4.10). Besov space versions of estimates in Theorem 4.2 are due to several authors, including Brenner [1], Ginibre and Velo [1]–[3] and Kapitansky [1], [3]. We refer the reader to the excellent surveys of Ginibre and Velo [4] and Strauss [4] for further comments.

CHAPTER IV

GLOBAL EXISTENCE FOR SEMILINEAR EQUATIONS WITH LARGE DATA

Here we shall study equations in $\mathbb{R}_+^{1+3}$ which are modeled after $\Box u + |u|^{\kappa-1}u = 0$. The conserved energy here is

$$\int \left(\tfrac{1}{2}|u'|^2 + \tfrac{1}{\kappa+1}u^{\kappa+1} \right) dx\,.$$

Therefore, the above equations split naturally into two classes depending on whether or not the kinetic energy dominates the potential term: $\kappa \leq 5$ and $\kappa > 5$. We shall study the first range which involves subcritical exponents $\kappa < 5$ and the critical power $\kappa = 5$. Not much is known about the supercritical case where $\kappa > 5$. Using special cases of the mixed-norm estimate (4.10) from the last chapter, we shall show that one has global classical solutions to the above equations for both the subcritical and critical range if the Cauchy data is sufficiently smooth. These mixed-norm estimates along with the energy identity are all that will be needed for the subcritical case since here scaling works to our advantage. For the critical case, though, we shall also need local energy arguments based on a Morawetz-Pohožaev identity.

§1. Main results

In this chapter we shall prove global existence results in $\mathbb{R}^{1+3}$ for a natural class of semilinear equations with repulsive nonlinearities. Specifically, we shall study equations of the form

$$(1.1)\qquad \begin{cases} \Box u = -\varphi_\kappa(u) \\ u(0,x) = f(x) \in C^3(\mathbb{R}^3)\,, \quad \partial_t u(0,x) = g(x) \in C^2(\mathbb{R}^3)\,. \end{cases}$$

For simplicity, we shall only consider real solutions, so we shall always assume that the data are real-valued.

To take advantage of our regularity assumptions on the data, as in §5 of Chapter 1, we shall assume that

$$\varphi_\kappa \in C^2(\mathbb{R})\,.$$

Then, if the data have compact support, (1.1) has a local C^2 solution.

In view of John's blow-up results from the last chapter, we need to impose some additional hypotheses on the nonlinearity if we wish to ensure that (1.1) always has a global solution. The first condition will be that we shall assume that φ_κ and its first derivatives have power growth at infinity:

$$|\varphi_\kappa(u)| + |u\varphi'_\kappa(u)| \le C(1+|u|)^\kappa \quad \text{some } \kappa > 1 . \tag{1.2}$$

As we saw before, this condition is not sufficient, since there can be blow-up for (1.1) if, say, $-\varphi_\kappa(u) = |u|^\kappa$. To rule out this situation we need to make an assumption on the primitive of φ_κ. Specifically, if

$$\Phi_\kappa(u) = \int_0^u \varphi_\kappa(\tau)\, d\tau ,$$

then we shall assume that

$$\Phi_\kappa(u) \ge 0 , \tag{1.3}$$

and, moreover,

$$|u|^{\kappa+1} \le C_0\,(1+\Phi_\kappa(u)) . \tag{1.4}$$

For the critical case we shall require an additional assumption, namely, that, when $|u|$ is larger than a fixed constant,

$$u\varphi_\kappa(u) - 4\Phi_\kappa(u) \ge 0 , \quad \text{if } \kappa = 5 . \tag{1.5}$$

This condition is clearly satisfied in the model case where $\varphi_\kappa = u^5$. Although we shall not use it in the subcritical case, notice that, if $\varphi_\kappa = |u|^{\kappa-1}u$, then (1.5) holds if and only if κ is in the superconformal range $\kappa \ge 3$.

Note that (1.2) implies that the reverse inequality to (1.4) essentially holds, that is,

$$\Phi_\kappa(u) \le C(1+|u|)^{\kappa+1} . \tag{1.6}$$

Later, we shall see that, if, say, the data is compactly supported and if u is a C^2 solution in $[0,T] \times \mathbb{R}^3$, then the energy associated with (1.1),

$$E(u;t) = \int_{\mathbb{R}^3} \left(\tfrac{1}{2}|u'(t,x)|^2 + \Phi_\kappa(u) \right) dx , \tag{1.7}$$

must be constant for $0 \le t \le T$. Here, as usual, $u' = (\partial_t u, \partial_x u)$ denotes the space-time gradient of u. Notice that the energy involves two terms: a kinetic term and a term involving the nonlinear potential $\Phi_\kappa(u)$. To make sure that the nonlinear contribution is controlled by the kinetic energy, in view of (1.6) and (1.7), we need to assume that $\kappa \le 5$. This is because the Sobolev embedding for $\mathbb{R}^3$ says that in general a function whose gradient is in L^2 is in L^q_{loc} only when $q \le 6$. Thus, since $\Phi_\kappa(u)$ behaves like $|u|^{\kappa+1}$, we want $\kappa + 1 \le 6$, or, as stated, $\kappa \le 5$. The case where $\kappa = 5$ is called the critical case since $q = 6$ is the critical exponent for the above Sobolev embedding, while the range $1 < \kappa < 5$ is called the subcritical case.

With all of this in mind, our main result then is the following

Theorem 1.1. *Let $1 < \kappa \le 5$ and assume that φ_κ satisfies (1.2) – (1.4). Assume also that (1.5) is satisfied when $|u|$ is large if $\kappa = 5$. Then (1.1) always has a global solution $u \in C^2(\mathbb{R}^{1+3}_+)$. Furthermore, if one assumes additionally that φ_κ and the Cauchy data are C^∞, then $u \in C^\infty(\mathbb{R}^{1+3}_+)$.*

The subcritical part of the theorem was proved by Jörgens [1] in 1961. The critical case where $\kappa = 5$ was settled some 30 years later by Grillakis [1]. His work followed the earlier results of Rauch [1] for data with small energy and the work of Struwe [1] handling the spherically symmetric case.

Remark. The model case involves the equation $\Box u = -u^5$. We saw in the last chapter that if one instead considers the related *attractive* equation, $\Box u = u^5$, then in general there is blow-up even for smooth compactly supported data. If one allows, as above, data without compact support, then it is much easier to see that there can be blow up. In fact,

$$u(t,x) = (3/4)^{1/4}\,(1-t)^{-1/2}\,,$$

solves $\Box u = u^5$ in $[0,1) \times \mathbb{R}^3$ and of course blows up as $t \nearrow 1$.

The statement about C^∞ solutions at the end of Theorem 1.1 follows from the seemingly weaker conclusion about C^2 solutions by the local existence theorem from Chapter 1. Therefore we only need to prove the first part of the theorem.

The next thing to notice is that, in proving the global existence theorem, we need only consider compactly supported data. This reduction will simplify the energy arguments that are to follow. To see this we just use a simple approximation argument.

Let us be more specific. Fix $\chi \in C_0^\infty(\mathbb{R}^3)$ satisfying $\chi = 1$ when $|x| \le 1$, and set $f_R = \chi(x/R)f(x)$ and $g_R = \chi(x/R)g(x)$. If we assume that Theorem 1.1 is valid for compactly supported data, and if we let $u_R(t,x)$ be the solution of (1.1) with data (f_R, g_R), we claim that, as $R \to \infty$, u_R converges in $C^2(\mathbb{R}^{1+3}_+)$ to a solution of (1.1) with data (f,g). To see this, if $t_0 \in \mathbb{R}_+$, let

$$\Lambda_{t_0,0} = \{\,(t,x) : \; 0 \le t \le t_0\,, \quad |x| \le t_0 - t\,\}$$

denote the backward light cone through $(t_0, 0)$. Then, by the uniqueness theorem from §2 of Chapter 1, we must have that $u_{R_1} = u_{R_2}$ in $\Lambda_{t_0,0}$ if $R_1, R_2 > t_0$, since u_{R_1} and u_{R_2} both have Cauchy data (f, g) in $\Lambda_{t_0,0} \cap \{ (0,x) : x \in \mathbb{R}^3 \}$. Thus, since $\mathbb{R}^{1+3}_+ = \cap_{t_0>0}\Lambda_{t_0,0}$, u_R must converge to a solution (1.2).

This reduction will also allow us to apply Pecher's generalization of the energy inequality for the d'Alembertian in a straightforward way. We shall only need special cases of this inequality, that is, inequality (4.10) from the last chapter. The ones we require are especially natural and important when one studies equations with critical nonlinearities.

Let us be more specific. If $v \in C^1$ is a (weak) solution of

$$\begin{cases} \Box v(t,x) = F(t,x)\,, & (t,x) \in \mathbb{R}^{1+3}_+ \\ v(0,x) = f(x)\,, & \partial_t v(0,x) = g(x)\,, \end{cases}$$

then the inequalities we require say that, if $T > 0$ and $S_T = [0,T] \times \mathbb{R}^3$, then

$$\|v\|_{L^4_t L^{12}_x(S_T)} \le C\left(\|v'(0,\,\cdot\,)\|_{L^2(\mathbb{R}^3)} + \|F\|_{L^1_t L^2_x(S_T)} \right), \tag{1.8}$$

as well as

$$\sup_{0\le t\le T} \|v(t,\,\cdot\,)\|_{L^6(\mathbb{R}^3)} \le C\left(\|v'(0,\,\cdot\,)\|_{L^2(\mathbb{R}^3)} + \|F\|_{L^1_t L^2_x(S_T)} \right). \tag{1.9}$$

As we pointed out before, the second inequality follows from the energy inequality for the d'Alembertian since Sobolev's theorem implies that the left side is dominated by $\sup_{0\le t\le T} \|v'(t,\,\cdot\,)\|_{L^2}$. Inequality (1.8), though, required a different argument.

Recall that, by the local existence theorem, if u is a C^2 solution of (1.1) in a half-open strip $[0, T_*) \times \mathbb{R}^3$ with compactly supported data, then either u extends to a C^2 solution in a larger strip or else u blows up pointwise, that is, $u \notin L^\infty([0,T_*) \times \mathbb{R}^3)$. Our next step is to see that we can replace L^∞ by the mixed-norm in the left side of (1.8) even if (1.4) does not hold.

Proposition 1.2. *Let $1 < \kappa \le 5$ and suppose that $\varphi_\kappa \in C^2$ satisfies (1.2). Then, if $f \in C^3$ and $g \in C^2$ are fixed compactly supported functions there is a $T > 0$ so that (1.1) has a solution $u \in C^2([0,T] \times \mathbb{R}^3)$. Moreover, if T_* is the supremum of all such times, then either $T_* = +\infty$, or*

$$u \notin L^4_t L^{12}_x([0,T_*) \times \mathbb{R}^3)\,.$$

Proof. The first part of the theorem follows from the local existence theorem in the first chapter. To prove the second half, suppose that $0 <$

$T_* < \infty$ and that u is a C^2 solution of (1.1) in the half-open strip $[0, T_*) \times \mathbb{R}^3$ satisfying

$$u \in L_t^4 L_x^{12}([0, T_*) \times \mathbb{R}^3). \tag{1.10}$$

We then must show that u extends to a C^2 solution in the closed strip $[0, T_*] \times \mathbb{R}^3$. By the local existence theorem again, our task is equivalent to showing that (1.10) implies

$$u \in L^\infty([0, T_*) \times \mathbb{R}^3), \tag{1.11}$$

assuming, of course, as above that u solves (1.1) with compactly supported data and $u \in C^2([0, T_*) \times \mathbb{R}^3)$.

Let $0 < R < \infty$ be so large that the data vanishes when $|x| > R$. We then recall, by the uniqueness theorem from §2 of the first chapter, that $u(t, x) = 0$ when $|x| > R + t$. Therefore, our assumption (1.2) implies that, if $0 \le t_0 < s < T_*$, then

$$\begin{aligned} &\| \partial_x^\alpha (\varphi_\kappa(u)) \|_{L_t^1 L_x^2([t_0, s] \times \mathbb{R}^3)} \\ &\qquad \le C + C \| \, |u|^{\kappa-1} \partial_x^\alpha u \|_{L_t^1 L_x^2([t_0, s] \times \mathbb{R}^3)}, \quad |\alpha| = 0, 1, \end{aligned} \tag{1.12}$$

where the constant can be taken to depend on R, T_* and the constant in (1.2), but not on t_0 or s.

If we use (1.9), with t replaced by $t - t_0$, then this observation gives

(1.13)

$$\begin{aligned} &\sup_{t_0 \le t \le s} \sum_{|\alpha| \le 1} \|\partial_x^\alpha u(t, \cdot)\|_{L^6(\mathbb{R}^3)} \\ &\le C\Big(1 + \sum_{|\alpha| \le 1} \big(\| (\partial_x^\alpha u)'(t_0, \cdot) \|_{L^2(\mathbb{R}^3)} + \| \, |u|^{\kappa-1} \partial_x^\alpha u \|_{L_t^1 L_x^2([t_0, s] \times \mathbb{R}^3)} \big)\Big) \\ &\le C(t_0) + C \sum_{|\alpha| \le 1} \| \, |u|^{\kappa-1} \partial_x^\alpha u \|_{L_t^1 L_x^2([t_0, s] \times \mathbb{R}^3)}. \end{aligned}$$

Here, of course, $C(t_0)$ is independent of s and finite since $u(t_0, \cdot)$ is C^2 and vanishes for large $|x|$.

To handle the last term, let us first consider the case where $\kappa = 5$. We then can apply Hölder's inequality to see that the last term in (1.13) is

$$\begin{aligned} &\le C \sup_{t_0 \le t \le s} \sum_{|\alpha| \le 1} \|\partial_x^\alpha u(t, \cdot)\|_{L^6(\mathbb{R}^3)} \cdot \| \, |u|^4 \|_{L_t^1 L_x^3([t_0, s] \times \mathbb{R}^3)} \\ &\le C \sup_{t_0 \le t \le s} \sum_{|\alpha| \le 1} \|\partial_x^\alpha u(t, \cdot)\|_{L^6(\mathbb{R}^3)} \cdot \|u\|^4_{L_t^4 L_x^{12}([t_0, s] \times \mathbb{R}^3)}. \end{aligned}$$

This step of course motivated our choice of mixed-norm in the proposition. Since we are assuming (1.10), the last factor must go to zero as $t_0 \nearrow T_*$. Therefore, for t_0 sufficiently close to T_*, we conclude that the last term in (1.13) is smaller than half of the left side and hence

$$\sup_{t_0 \le t \le s} \sum_{|\alpha| \le 1} \|\partial_x^\alpha u(t,\,\cdot)\|_{L^6(\mathbf{R}^3)} \le 2C(t_0)\,. \tag{1.14}$$

If we let $s \nearrow T_*$ we conclude that

$$\sup_{0 \le t \le T_*} \sum_{|\alpha| \le 1} \|\partial_x^\alpha u(t,\,\cdot)\|_{L^6(\mathbf{R}^3)} < \infty\,. \tag{1.11$'$}$$

This clearly implies (1.11), by Sobolev's theorem, and so we have handled the critical case, $\kappa = 5$.

The subcritical case is handled similarly. If we use Hölder's inequality as above, we see that, for $1 < \kappa < 5$, the last term in (1.13) must be

$$\le C \sup_{t_0 \le t \le s} \sum_{|\alpha| \le 1} \|\partial_x^\alpha u(t,\,\cdot)\|_{L^6(\mathbf{R}^3)} \cdot \|u\|^{\kappa-1}_{L_t^{\kappa-1} L_x^{3(\kappa-1)}([t_0,s]\times\mathbf{R}^3)}\,.$$

Note that $\kappa - 1 < 4$ and $3(\kappa - 1) < 12$. Therefore, if we recall the support properties of u, we can apply Hölder's inequality again to see that there must be a constant C_{R,T_*} so that the last factor is

$$\le C_{R,T_*} \|u\|^{\kappa-1}_{L_t^4 L_x^{12}([t_0,s]\times\mathbf{R}^3)}\,.$$

Since $\kappa > 1$, this term also goes to zero as $t_0 \nearrow T_*$, and hence we can repeat the arguments used in the critical case to see that (1.14) must hold if t_0 is sufficiently close to T_*. Since this in turn implies (1.11$'$) and hence (1.11) for this range of κ, the proof is complete. □

On account of this proposition, to prove Theorem 1.1, we may assume that u is as above, and then it suffices to show that (1.10) holds. We shall of course want to make use of the $L_t^4 L_x^{12}$-estimate (1.8). We shall also need to see that energy defined in (1.7) is conserved. This will be handled in the next section. There we shall also see that these two tools are all that is needed to complete the proof for the subcritical case. The critical case, however, is more delicate and it will be handled in §3. Here we shall also need local energy estimates which are based on an identity of Morawetz.

§2. Energy estimates and the subcritical case

The next step in the proof of the global existence theorem will be to see that the energy defined by (1.7) is conserved. More specifically, we shall require the following

Proposition 2.1. *Suppose that φ_κ is as in Theorem 1.1. Suppose also that $0 < T_* < \infty$ and that $u \in C^2([0,T_*) \times \mathbb{R}^3)$ solves (1.1) and that the Cauchy data vanish when $|x| > R$. Then*

$$E(u;t) = \int_{\mathbb{R}^3} \big(\tfrac12 |u'(t,x)|^2 + \Phi_\kappa(u(t,x)) \big)\, dx = E(u;0)\,, \ 0 < t < T_*\,. \tag{2.1}$$

Moreover, for fixed data as above, there is a constant C_{R,T_*} so that

$$\int_{\mathbb{R}^3} \big(|u'(t,x)|^2 + |u(t,x)|^{\kappa+1} \big)\, dx \le C_{R,T_*}\,, \quad 0 < t < T_*\,. \tag{2.1$'$}$$

Proof. Recall that $u(t,x) = 0$ if $|x| > t + R$. Therefore, (2.1) implies (2.1$'$) since (1.4) holds. In fact, the constant in (2.1$'$) can be taken to be a multiple of $E(u;0) + (R + T_*)$. Note that

$$E(u;0) = \int_{\mathbb{R}^3} \big(\tfrac12 |\partial_x f(x)|^2 + \tfrac12 |g(x)|^2 + \Phi_\kappa(f(x)) \big)\, dx \ < \infty\,,$$

in view of our assumptions on the data.

The proof of (2.1) just involves a simple variation on the energy arguments from §2 of the first chapter. In fact, if we multiply the equation $\Box u + \varphi_\kappa(u)$ by $\partial_t u$, then our earlier formula (2.1) from that section changes to

$$0 = \partial_t u \,\big(\Box u + \varphi_\kappa(u) \big) \ = \ \operatorname{div}_{t,x} e(u)\,, \tag{2.2}$$

where, in the present context,

$$e(u) = \big(\tfrac12 |u'|^2 + \Phi_\kappa(u)\,,\ -\partial_t u \nabla_x u \big)\,. \tag{2.3}$$

If we fix $0 < t < T_*$, then u is C^2 and has compact support in $[0,t] \times \mathbb{R}^3$. Therefore, integrating (2.2) leads to

$$\begin{aligned} 0 &= \int_0^t \int_{\mathbb{R}^3} \operatorname{div}_{\tau,x} e(u)\, dx d\tau \\ &= \int_{\mathbb{R}^3} \int_0^t \frac{\partial}{\partial \tau} \big(\tfrac12 |u'|^2 + \Phi_\kappa(u) \big)\, d\tau dx \\ &= \int_{\mathbb{R}^3} \big(\tfrac12 |u'(t,x)|^2 + \Phi_\kappa(u(t,x))\big)\, dx \\ &\qquad\qquad - \int_{\mathbb{R}^3} \big(\tfrac12 |u'(0,x)|^2 + \Phi_\kappa(u(0,x))\big)\, dx, \end{aligned}$$

giving us (2.1). $\square$

We almost ready to finish the proof of global existence for the subcritical case. First, though, we must prove the following simple lemma which will also be used to handle the critical case.

Lemma 2.2. *Let $0 < C_0 < \infty$ and suppose that $0 \le y(s) \in C([a,b))$ satisfies $y(a) = 0$ and*

$$y(s) \le C_0 + \varepsilon y(s)^\sigma ,$$

for some $\sigma > 0$. Then, if $\varepsilon < 2^{-\sigma} C_0^{1-\sigma}$, it follows that

$$y(s) \le 2C_0, \quad s \in [a,b) .$$

Proof. Since $C_0 + \varepsilon x_1^\sigma - x_1 < 0$ if $\varepsilon < 2^{-\sigma} C_0^{1-\sigma}$ and $x_1 = 2C_0$, it follows that if

$$0 \le C_0 + \varepsilon x^\sigma - x , \quad \forall x \in [0, x_0) ,$$

then $x_0 < x_1 = 2C_0$. Since $y(s)$ must be smaller than the supremum of such x_0, the lemma follows. □

End of proof for subcritical case. Recall that we may assume that the Cauchy data in (1.1) vanishes for $|x| > R$. We then must show that if $0 < T_* < \infty$ and if $u \in C^2([0,T_*) \times \mathbb{R}^3)$, then

$$u \in L_t^4 L_x^{12}([0,T_*) \times \mathbb{R}^3) . \tag{2.4}$$

Let $0 \le t_0 < s < T_*$. If we then use (1.8) and (1.12), we conclude that there must be a constant C, depending only on T_* and R, so that

$$\begin{aligned} &\|u\|_{L_t^4 L_x^{12}([t_0,s]\times\mathbb{R}^3)} \\ &\le C\left(1 + \|u'(t_0,\,\cdot\,)\|_{L^2(\mathbb{R}^3)} + \|\,|u|^\kappa\,\|_{L_t^1 L_x^2([t_0,s]\times\mathbb{R}^3)}\right) \\ &\le C\big(1 + (2E(u;0))^{1/2}\big) + C\|\,|u|^\kappa\,\|_{L_t^1 L_x^2([t_0,s]\times\mathbb{R}^3)} , \end{aligned} \tag{2.5}$$

using the conservation of energy in the last step.

To handle the last term, note that

$$1 = \tfrac{5-\kappa}{4} + \tfrac{\kappa-1}{4} , \text{ and } \tfrac{1}{2} = \tfrac{7-\kappa}{12} + \tfrac{\kappa-1}{12} .$$

Hence, if we write $|u|^\kappa = |u|^{\kappa-1} \cdot |u|$, we can apply Hölder's inequality to see that

$$\begin{aligned} &\|\,|u|^\kappa\,\|_{L_t^1 L_x^2([t_0,s]\times\mathbb{R}^3)} \\ &\le \|u\|_{L_t^{\frac{4}{5-\kappa}} L^{\frac{12}{7-\kappa}}([t_0,s]\times\mathbb{R}^3)} \cdot \left\|\,|u|^{\kappa-1}\,\right\|_{L_t^{\frac{4}{\kappa-1}} L_x^{\frac{12}{\kappa-1}}([t_0,s]\times\mathbb{R}^3)} \\ &= \|u\|_{L_t^{\frac{4}{5-\kappa}} L^{\frac{12}{7-\kappa}}([t_0,s]\times\mathbb{R}^3)} \cdot \|u\|^{\kappa-1}_{L_t^4 L_x^{12}([t_0,s]\times\mathbb{R}^3)} . \end{aligned} \tag{2.6}$$

Notice that

$$\tfrac{12}{7-\kappa} < \kappa + 1 , \quad 1 < \kappa < 5 .$$

Therefore, since $u(t,x) = 0$ when $|x| \geq t + R$, we can apply Hölder's inequality again to see that

$$\begin{aligned} &\|u\|_{L_t^{\frac{4}{5-\kappa}} L_x^{\frac{12}{7-\kappa}}([t_0,s]\times\mathbb{R}^3)} \\ &\qquad \leq (T_* - t_0)^{\frac{5-\kappa}{4}} \sup_{t_0 \leq t \leq s} \|u(t,\cdot)\|_{L^{\frac{12}{7-\kappa}}(\mathbb{R}^3)} \\ &\qquad \leq C(T_* - t_0)^{\frac{5-\kappa}{4}} (T_* + R)^{3(\frac{7-\kappa}{12} - \frac{1}{\kappa+1})} \sup_{t_0 \leq t \leq s} \|u\|_{L^{\kappa+1}(\mathbb{R}^3)} \\ &\qquad \leq C_{R,T_*} (T_* - t_0)^{\frac{5-\kappa}{4}}, \end{aligned} \tag{2.7}$$

using (2.1′) in the last step.

Let

$$\varepsilon(t_0) = CC_{R,T_*} (T_* - t_0)^{\frac{5-\kappa}{4}},$$

where C is as in (2.5). Then (2.5)-(2.7) give

$$\|u\|_{L_t^4 L_x^{12}([t_0,s]\times\mathbb{R}^3)} \leq C\left(1+(2E(u;0))^{1/2}\right)+\varepsilon(t_0)\,\|u\|^{\kappa-1}_{L_t^4 L_x^{12}([t_0,s]\times\mathbb{R}^3)}. \tag{2.8}$$

Note that $\varepsilon(t_0) \to 0$ as $t_0 \nearrow T_*$, since we are assuming that $\kappa < 5$. Therefore, Lemma 2.2 implies that, if t_0 is sufficiently close to T_*, then

$$\|u\|_{L_t^4 L_x^{12}([t_0,s]\times\mathbb{R}^3)} \leq 2C\left(1 + (2E(u;0))^{1/2}\right), \quad t_0 \leq s < T_*. \tag{2.4′}$$

This clearly gives (2.4), since, in $[0, t_0] \times \mathbb{R}^3$, u is bounded and compactly supported. □

Remark. Notice how the proof breaks down in the critical case where $\kappa = 5$. It will work, though, if $E(u;0)$ is sufficiently small and (1.4) holds without the 1 in the right, for instance when $\varphi_\kappa(u) = u^5$, which allows us to recover the result of Rauch [1]. For, in this case, the constant in (2.5) can be taken to be independent of T_* and R, and so we can apply Lemma 2.2 if the $L_t^\infty L_x^6$ norm of u is small enough, or, equivalently, because of our assumption about (1.4) here, if it has small enough energy. On the other hand, if one does not make this small data assumption, and, if one tried to use this proof, one would be stuck because $\varepsilon(t_0)$, as defined above, would just be a possibly large constant. Nonetheless, if we just try to prove that, say, u belongs to $L_t^4 L_x^{12}$ with the norm just taken over a backward light cone in $[0, T_*) \times \mathbb{R}^3$, then, in the critical case, we could modify the definition of $\varepsilon(t)$ so that it involves an L^6 norm taken over a ball of radius $T_* - t$. If we could show that these go to zero as $t \to T_*$, then the above proof will show that u must be in $L_t^4 L_x^{12}$ of the backward light cone. This, will be the main step in proving (2.4) for the critical case. Showing that these L^6 norms over smaller and smaller balls go to zero, or, equivalently, that the integral of $\Phi_\kappa(u)$ over these sets goes to zero, will be the main new step, which is due to Struwe [1] based on local energy arguments.

§3. A decay lemma and the critical case

Let u be as in Proposition 2.1. To prove local existence in the critical case we require a local version of the energy identity. Fix $x_0 \in \mathbb{R}^3$. Then this will say that if, for $0 \leq t_0 < s < T_*$ and $\delta \geq 0$ we let

$$\Lambda(\delta; t_0, s) = \{ (t,x) : t_0 \leq t \leq s, |x - x_0| \leq \delta + T_* - t \} \tag{3.1}$$

be a portion of the backward light cone through $(T_* + \delta, x_0)$, then the energy in the larger bottom bounding ball,

$$D_{t_0} = \{ (t,x) \in \Lambda(\delta; t_0, s) : t = t_0 \},$$

equals the energy in the smaller top bounding ball,

$$D_s = \{ (t,x) \in \Lambda(\delta; t_0, s) : t = s \},$$

plus the energy flux across the rest of the boundary,

$$M_{t_0}^s = \{ (t,x) \in \Lambda(\delta; t_0, s) : t_0 \leq t \leq s \,, |x - x_0| = \delta + T_* - t \}.$$

To be more specific, let

$$E(u; D_t) = \int_{D_t} \left(\tfrac{1}{2} |u'(t,x)|^2 + \Phi_\kappa(u(t,x)) \right) dx \,, \quad 0 \leq t < T_* \,, \tag{3.2}$$

and let

$$\text{Flux}\,(u; M_{t_0}^s) = \int_{M_{t_0}^s} \langle e(u), \vec{\nu} \rangle \, d\sigma \,, \quad 0 \leq t_0 < s \leq T_* \,, \tag{3.3}$$

where $e(u)$ is as in (2.3), $\vec{\nu}$ is the outward normal through a given point on M_t^s, and $d\sigma$ denotes Lebesgue measure on this set. We then claim that, if, as above, $0 \leq t_0 < s < T_*$, then

$$E(u; D_{t_0}) = E(u; D_s) + \text{Flux}\,(u; M_{t_0}^s) \,. \tag{3.4}$$

The proof of this formula is an easy variant of the proof of (2.1). One simply integrates $\text{div}_{t,x}\, e(u)$ over $\Lambda(\delta; t_0, s)$, leading to (3.4) after an application of the divergence theorem.

The next thing we need to see is that our assumption $\Phi_\kappa(u) \geq 0$ implies that the energy flux is nonnegative. To verify this, we note that $M_{t_0}^s$ consists of points of the form $(\delta + T_* - |x|, x_0 - x)$, with $\delta + T_* - |x| \in [t_0, s]$. At such a point, the outward normal is $(1, -x/|x|)/\sqrt{2}$, and thus

$$\begin{aligned} \sqrt{2}\,\langle e(u), \vec{\nu} \rangle &= \tfrac{1}{2}|u'|^2 + \Phi_\kappa(u) - \partial_t u \, \frac{x}{|x|} \cdot \nabla_x u \\ &= \tfrac{1}{2} \Big| \frac{x}{|x|} \partial_t u - \nabla_x u \Big|^2 + \Phi_\kappa(u) \geq 0 \,, \end{aligned} \tag{3.5}$$

establishing our claim.

Since Flux ≥ 0, we conclude from (3.4) that $t \to E(u; D_t)$ is a non-increasing function on $[0, T_*)$. It is also bounded, since $E(u; D_t) \leq E(u;t) = E(u;0) < \infty$, on account of our assumptions on the data. Hence, the first two terms in (3.4) must approach a common limit. This in turn gives the important fact that

$$\text{Flux } (u; M_t^{T_*}) \to 0 \quad \text{as } t \to T_* .$$

Using the monotonicity of $E(u, D_t)$, we shall be able to adapt the proof of Proposition 2.1 to obtain the following result.

Proposition 3.1. *Let $\kappa = 5$ and suppose that $u \in C^2([0, T_*) \times \mathbb{R}^3)$ solves (1.1) with Cauchy data vanishing when $|x| > R$. Fix $x_0 \in \mathbb{R}^3$ and assume that*

$$\int_{|x-x_0|\leq T_*-t_0} \left(\tfrac{1}{2}|u'(t_0,x)|^2 + \Phi_\kappa(u(t_0,x)) \right) dx < \varepsilon . \tag{3.6}$$

Then there is an $\varepsilon_0 > 0$, depending only on T_*, R and $E(u;0)$, so that, if $0 < \varepsilon < \varepsilon_0$ and $0 \leq t_0 < T_*$

$$u \in L_t^4 L_x^{12}(\Lambda(\delta; t_0, T_*)) , \tag{3.7}$$

provided that $\delta > 0$ and $T_* - t_0$ are sufficiently small.

We have only required that t_0 is close to T_* for the proof. Once we show that (3.7) holds, we can of course replace t_0 by 0, with the same δ, since u is in $C^2([0, T_*) \times \mathbb{R}^3)$ and vanishes when $|x|$ is large.

Proof. Let C_0 be the constant in (1.4). The first step is to see that (3.6) and our assumption (1.4) about lower bounds for Φ_κ, $\kappa = 5$, imply that, if (3.6) holds for a given $\varepsilon > 0$, then

$$\sup_{t_0 \leq t < T_*} \int_{|x-x_0|\leq \delta + T_* - t} |u(t,x)|^6 \, dx < 2C_0\varepsilon , \tag{3.6$'$}$$

provided that $\delta > 0$ and $T_* - t_0$ are sufficiently small. To prove this, we first notice that, if we replace the integral in (3.6) by one over the larger region where $|x - x_0| < \delta + T_* - t_0$, then the resulting quantity must be smaller than, say, $3\varepsilon/2$, if δ is small enough. Since $E(u; D_t)$ is a non-increasing function of t, we must also have that

$$\sup_{t_0 \leq t < T_*} \int_{|x-x_0|\leq \delta + T_* - t} \left(\tfrac{1}{2}|u'(t,x)|^2 + \Phi_\kappa(u(t,x)) \right) dx < 3\varepsilon/2 .$$

Finally, if C_0 is the constant in (1.4), and $t_0 \le t < T_*$, then

$$\begin{aligned}\int_{|x-x_0|\le\delta+T_*-t} &|u(t,x)|^6\,dx \\ &\le \frac{4\pi}{3}C_0(\delta+T_*-t_0)^3 + C_0\int_{|x-x_0|\le\delta+T_*-t}\Phi_\kappa(u(t,x))\,dx \\ &\le \frac{4\pi}{3}C_0(\delta+T_*-t_0)^3 + \frac{3C_0\varepsilon}{2},\end{aligned}$$

and so (3.6′) must hold if δ and t_0 are chosen so that $4\pi C_0(\delta+T_*-t_0)^3/3 < \varepsilon/2$.

Using (3.6′) we can easily adapt the proof of Proposition 1.2 to see that (3.7) must hold if ε is small enough. We shall want to apply (1.8) with $v = u$ and $F = -\varphi_\kappa(u)$. If $t_0 < s < T_*$ and if the norm in the left is only taken over $\Lambda(\delta; t_0, s)$, then the norm involving F need only be taken over the same set, by Huygen's principle. Thus,

(3.8)
$$\begin{aligned}\|u\|_{L^4_tL^{12}_x(\Lambda(\delta;t_0,s))} &\le C\|u'(t_0,\,\cdot\,)\|_{L^2(\mathbf{R}^3)} + C\|\varphi_\kappa(u)\|_{L^1_tL^2_x(\Lambda(\delta;t_0,s))} \\ &\le C(2E(u;0))^{1/2} + C\|\varphi_\kappa(u)\|_{L^1_tL^2_x(\Lambda(\delta;t_0,s))},\end{aligned}$$

using the conservation of the total energy in the last step.

If we use a variant of (1.12) and Hölder's inequality, we see that there must be a constant C_1, depending only on T_* and R, so that

$$\begin{aligned}\|\varphi_\kappa(u)\|_{L^1_tL^2_x(\Lambda(\delta;t_0,s))} &\le C_1 + C_1\big\|\,|u|^4u\,\big\|_{L^1_tL^2_x(\Lambda(\delta;t_0,s))} \\ &\le C_1 + C_1\|u\|_{L^\infty_tL^6_x(\Lambda(\delta;t_0,s))}\cdot\big\|\,|u|^4\,\big\|_{L^1_tL^3_x(\Lambda(\delta;t_0,s))} \\ &\le C_1 + C_1\|u\|_{L^\infty_tL^6_x(\Lambda(\delta;t_0,s))}\cdot\|u\|^4_{L^4_tL^{12}_x(\Lambda(\delta;t_0,s))}.\end{aligned}$$

By (3.6′), the $L^\infty_tL^6_x$ norm is $\le (2C_0\varepsilon)^{1/6}$. Therefore, if we let $C_2 = C_1C$, then combining the last two inequalities gives

$$\|u\|_{L^4_tL^{12}_x(\Lambda(\delta;t_0,s))} \le \big(C(2E(u;0))^{1/2}+C_2\big) + C_2(2C_0\varepsilon)^{1/6}\,\|u\|^4_{L^4_tL^{12}_x(\Lambda(\delta;t_0,s))}.$$

Finally, since the constants are independent of s, an application of Lemma 2.2 gives that

$$\|u\|_{L^4_tL^{12}_x(\Lambda(\delta;t_0,s))} \le 2\big(C(2E(u;0))^{1/2}+C_2\big),$$

provided that

$$C_2(2C_0\varepsilon)^{1/6} < 2^{-4}\big(C(2E(u;0))^{1/2}+C_2\big)^{-3}.$$

Since, as desired, ε here depends only on T_*, R and $E(u;0)$, the proof is complete. □

Using the last result we can reduce our task to showing that the energy cannot concentrate at any point (T_*, x_0). More precisely, we claim that, to finish the proof of the global existence theorem for the critical case, it suffices to show that, for every $x_0 \in \mathbb{R}^3$,

$$\lim_{t \nearrow T_*} \int_{|x-x_0|<T_*-t} \left(\tfrac{1}{2}|u'(t,x)|^2 + \Phi_\kappa(u(t,x)) \right) dx = 0. \tag{3.9}$$

To verify this, we just notice that (3.9) would of course imply that (3.6) always holds for a given $\varepsilon > 0$ if t_0 is close to T_*. Hence, for every fixed $x_0 \in \mathbb{R}^3$, there must be a $\delta > 0$ so that (3.7) holds. As we pointed out before, since $u(t,x)$ is bounded on $[0,t_0] \times \mathbb{R}^3$, (3.7) can be strengthened to

$$u \in L^4_t L^{12}_x(\Lambda(\delta; 0, T_*)).$$

Since u vanishes outside of a relatively compact subset of $[0,T_*) \times \mathbb{R}^3$, we can cover its support by finitely many of these sets $\Lambda(\delta; 0, T_*)$. Hence, $u \in L^4_t L^{12}_x([0,T_*) \times \mathbb{R}^3)$, which implies that u can be extended to a global C^2 solution, by Proposition 1.2.

Clearly (3.9) must hold if u extends to a C^2 function on $[0,T_*] \times \mathbb{R}^3$. Thus, we have reduced our task of showing that such an extension always exist to proving the seemingly much weaker statement about non-concentration of energy. The next step is to show that we can reduce matters further and see that we only need to see that the part of the energy coming from the nonlinear potential cannot concentrate:

Proposition 3.2. *Let $\kappa = 5$ and let u be as above. Then* (3.9) *must hold if*

$$\lim_{t \nearrow T_*} \int_{|x-x_0|<T_*-t} \Phi_\kappa(u(t,x))\, dx = 0. \tag{3.10}$$

Proof. We must show that the kinetic energy cannot concentrate. If we use the lower bound (1.4) for $\Phi_\kappa(u)$, we see that (3.10) is equivalent to the statement that

$$\lim_{t \nearrow T_*} \int_{|x-x_0|<T_*-t} |u(t,x)|^6\, dx = 0.$$

The proof of Proposition 3.1 shows that this in turn implies that, for the backward light cone through (T_*, x_0), we have

$$u \in L^4_t L^{12}_x(\Lambda(0; 0, T_*)). \tag{3.10'}$$

To use this, let $0 \le t_0 < s < T_*$. Then applying (1.9) to the equation $\Box u' = -\varphi_\kappa'(u)u'$, and arguing as in the proof of Proposition 3.1, gives

$$\sup_{t_0\le t\le s}\Big(\int_{|x-x_0|<T_*-t}|u'(t,x)|^6\,dx\Big)^{1/6} = \|u'\|_{L^\infty_t L^6_x(\Lambda(0;t_0,s))}$$

$$\le C\sum_{|\alpha|=2}\|\partial^\alpha u(t_0,\,\cdot\,)\|_{L^2(\mathbb{R}^3)} + C\|\varphi_\kappa'(u)u'\|_{L^1_t L^2_x(\Lambda(0;t_0,s))}$$

$$\le C\sum_{|\alpha|=2}\|\partial^\alpha u(t_0,\,\cdot\,)\|_{L^2(\mathbb{R}^3)} + C'\big(1+\|\,u^4\,u'\,\|_{L^1_t L^2_x(\Lambda(0;t_0,s))}\big)$$

$$\le C\sum_{|\alpha|=2}\|\partial^\alpha u(t_0,\,\cdot\,)\|_{L^2(\mathbb{R}^3)} + C'\big(1+\|u\|^4_{L^4_t L^{12}_x(\Lambda(0;t_0,s))}\|u'\|_{L^\infty_t L^6_x(\Lambda(0;t_0,s))}\big)$$

$$\le C(t_0) + C'\|u\|^4_{L^4_t L^{12}_x(\Lambda(0;t_0,s))}\|u'\|_{L^\infty_t L^6_x(\Lambda(0;t_0,s))}\,.$$

The constant $C(t_0)$ is finite since $u \in C^2([0,T_*)\times\mathbb{R}^3))$ and since u vanishes when $|x|$ is large. Also, (3.10′) implies that the $L^4_t L^{12}_x$ norm in the inequality goes to zero as $t_0 \to T_*$. We therefore conclude that, if t_0 is close to T_*,

$$\sup_{t_0\le t<T_*}\Big(\int_{|x-x_0|<T_*-t}|u'(t,x)|^6\,dx\Big)^{1/6} \le 2C(t_0)\,.$$

But an application of Hölder's inequality shows that this leads to

$$\Big(\int_{|x-x_0|<T_*-t}|u'(t,x)|^2\,dx\Big)^{1/2} \le 2C(t_0)\cdot\Big(\frac{4\pi}{3}(T_*-t)^3\Big)^{1/3},\quad t_0\le t<T_*\,.$$

Hence,

$$\lim_{t\nearrow T_*}\int_{|x-x_0|<T_*-t}\tfrac12|u'(t,x)|^2\,dx = 0\,,$$

as desired. □

A Morawetz-Pohožaev identity and the end of the proof.

To finish the proof of the global existence theorem we are just left with showing that (3.10) holds. To do this we shall need to use a Morawetz identity which is related to an identity of Pohožaev [1] for elliptic equations. To simplify the notation to follow, it is convenient to shift (T_*,x_0) in (3.10) to the origin. Hence, from now on, we shall assume that

$$u\in C^2([-T_*,0)\times\mathbb{R}^3)$$

solves $\Box u + \varphi_\kappa(u) = 0$ with compactly supported Cauchy data at $t = -T_*$. The last step in the proof of Theorem 1.1 will be to show that we must have

$$\lim_{t \nearrow 0} \int_{\{x:\, |x|<|t|\}} \Phi_\kappa(u)\, dx = 0. \tag{3.11}$$

To derive this Morawetz identity that will be used in the proof of (3.11) we shall use Noether's principle. Thus, we need to use the Lagrangian associated with our equation,

$$L(q,p) = \tfrac{1}{2}|p_0|^2 - \tfrac{1}{2}\sum_{j=1}^{3} |p_j|^2 - \Phi_\kappa(q),$$

where $(q,p) \in \mathbb{R} \times \mathbb{R}^{1+3}$. On account of our current assumption that $\Box u + \varphi_\kappa(u) = 0$ on $[-T_*, 0) \times \mathbb{R}^3$), we must have

$$\begin{aligned}\frac{d}{d\varepsilon} \int_{[-T_*,0)\times\mathbb{R}^3} & L(u+\varepsilon\psi, (u+\varepsilon\psi)')\, dtdx \Big|_{\varepsilon=0} \\ &= -\int_{[-T_*,0)\times\mathbb{R}^3} (\Box u + \varphi_\kappa(u))\, \psi\, dtdx = 0,\end{aligned}$$

whenever $\psi \in C_0^\infty((-T_*, 0) \times \mathbb{R}^3)$. Thus, u must satisfy the Euler-Lagrange equation associated with our equation,

$$\frac{\partial L}{\partial q}(u,u') - \sum_{j=0}^{3} \partial_j \Big[\frac{\partial L}{\partial p_j}(u,u') \Big] = 0,$$

where, as usual, $\partial_0 = \partial_t$.

If u_r were a one-parameter C^1 deformation of u, then

$$\partial_r L(u_r, u_r') = \frac{\partial L}{\partial q}(u_r, u_r')\partial_r u_r + \sum_{j=0}^{3} \frac{\partial L}{\partial p_j}(u_r, u_r')\partial_j \partial_r u_r.$$

If we assume that $u_{r_0} = u$, then we could use the Euler-Lagrange equation to obtain

$$\partial_r L(u_r, u_r')\big|_{r=r_0} = \sum_{j=0}^{3} \partial_j \Big[\frac{\partial L}{\partial p_j}(u,u')\partial_r u \Big]. \tag{3.12}$$

The Morawetz identity we need arises from the deformation

$$u_r(t,x) = ru(rt, rx)$$

with $r_0 = 1$. In this case,

$$\partial_r u\Big|_{r=1} = u + \sum_{j=0}^{3} x_j \partial_j u\,.$$

Note also that

$$L(u_r, u_r') = r^4 \cdot \big(L(u,u')\big)(rt,rx) + r^4 \Phi_\kappa(u(rt,rx)) - \Phi_\kappa(u_r(t,x))\,,$$

and hence, taking $x_0 = t$,

$$\partial_r L(u_r, u_r')\big|_{r=1} = \sum_{j=0}^{3} x_j \partial_j L(u,u') + 4L(u,u') + 4\Phi_\kappa(u) - u\Phi_\kappa'(u)\,.$$

Combining this with (3.12) leads to the desired identity

$$\sum_{j=0}^{3} \partial_j \Big[\frac{\partial L}{\partial p_j}(u,u')\,\big(u + \sum_{k=0}^{3} x_k \partial_k u\big) - x_j L(u,u') \Big] = 4\Phi_\kappa(u) - u\varphi_\kappa(u)\,.$$

Substituting the definition of our Lagrangian, we see that we can rewrite this as

$$\text{(3.13)} \qquad \operatorname{div}_{t,x}\big(tQ + \partial_t u\, u\,,\, -tP\big) = 4\Phi_\kappa(u) - u\varphi_\kappa(u)\,,$$

where

$$Q = \tfrac{1}{2}|u'|^2 + \Phi_\kappa(u) + t^{-1}\partial_t u\, x \cdot \nabla_x u$$

$$P = \big(\tfrac{1}{2}|\partial_t u|^2 - \tfrac{1}{2}|\nabla_x u|^2 - \Phi_\kappa(u)\big)\, x/t \;+\; \big(t^{-1}u + \partial_t u + t^{-1}x \cdot \nabla_x u\big)\, \nabla_x u\,.$$

Remark. This formula also comes from the fact that

$$\text{(3.13}'\text{)} \quad (t\partial_t u + x \cdot \nabla_x u + u)(\Box u + \varphi_\kappa(u)) = \operatorname{div}_{t,x}\big(tQ + \partial_t u\, u\,,\, -tP\big) - 4\Phi_\kappa(u) + u\varphi_\kappa(u)\,.$$

Note that the operator occurring in the left, $t\partial_t + r\partial_r + 1$, is the generator of the deformation $u = u_r$ used above. A related remark is that, if we instead let u_r be defined by time-translations, that is, $u_r(t,x) = u(t+r,x)$, then we could have derived (2.2) differently as

$$\partial_t \Big[\frac{\partial L}{\partial p_0}(u,u') - L(u,u') \Big] + \sum_{j=0}^{3} \partial_j \Big[\frac{\partial L}{\partial p_j}(u,u')\partial_t u \Big] = 0\,.$$

Returning to the proof, let us see how (3.13) can be used to prove (3.11). Since we have changed notation to simplify the identity needed for the proof we have to modify the earlier definitions used in the local energy arguments. So, now, if $T_* < T < S \le 0$, we set

$$D_T = \{ (T,x) : |x| \le -T \}$$
$$\Lambda(T,S) = \{ (t,x) : T \le t \le S \, , \ |x| \le -t \} \, ,$$

and

$$M_T^S = \{ (t,x) \, : T \le t \le S \, , \ |x| = -t \} \, .$$

Thus, $\Lambda(T,S)$ is the part of the backward light cone through the origin which intersects $[T,S] \times \mathbb{R}^3$, and the boundary can be split up into three parts, D_T, D_S, and M_T^S.

If we integrate (3.13) and apply the divergence theorem we obtain

$$\int_{D_S} (\, SQ + u\partial_t u \,) \, dx - \int_{D_T} (\, TQ + u\partial_t u \,) \, dx + \frac{1}{\sqrt{2}} \int_{M_T^S} (\, tQ + u\partial_t u + x \cdot P \,) \, d\sigma$$
$$= \iint_{\Lambda(T,S)} (\, 4\Phi_\kappa(u) - u\varphi_\kappa(u) \,) \, dt dx \, .$$

Using Proposition 2.1 and Hölder's inequality one finds that the first term goes to zero as $S \nearrow 0$. Thus, taking limits, we are lead to the identity

$$I + II = \iint_{\Lambda(T,0)} (\, 4\Phi_\kappa(u) - u\varphi_\kappa(u) \,) \, dt dx \, ,$$

where

$$I = - \int_{D_T} (\, TQ + u\partial_t u \,) \, dx$$

$$II = \frac{1}{\sqrt{2}} \int_{M_T^0} (\, tQ + u\partial_t u + x \cdot P \,) \, d\sigma \, .$$

So far we have not made use of our assumption (1.5), which, we recall, says that $u\varphi_\kappa(u) - 4\Phi_\kappa(u) \ge 0$ when $|u|$ is larger than a fixed constant. Based on this, we conclude that

$$I + II \le CT^4 \, . \tag{3.14}$$

To exploit this inequality, we need to try to a obtain lower bounds for I and II, keeping in mind that I involves the term we are trying to control.

Let us start out by trying to handle II, since, as we shall see, this will suggest the lower bounds needed for I. The first step is to realize that since $t = -|x|$ on M_T^0, we have

$$II = \frac{1}{\sqrt{2}} \int_{M_T^0} \Big[-|x|\,|\partial_t u|^2 + 2\partial_t u\, x \cdot \nabla_x u - \frac{(x \cdot \nabla_x u)^2}{|x|} - u \frac{x}{|x|} \cdot \nabla_x u + u \partial_t u \Big] d\sigma$$
$$= -\frac{1}{\sqrt{2}} \int_{M_T^0} \Big[|x| \Big(\frac{x \cdot \nabla_x u}{|x|} - \partial_t u \Big)^2 + u \Big(\frac{x \cdot \nabla_x u}{|x|} - \partial_t u \Big) \Big] d\sigma .$$

If we parameterize M_T^0 by

$$y \to (-|y|, y), \quad |y| \le |T|,$$

then $d\sigma = \sqrt{2} dy$. Also, if we set $v(y) = u(-|y|, y)$, then

$$y \cdot \frac{\nabla v}{|y|} = \frac{x \cdot \nabla_x u}{|x|} - \partial_t u .$$

Therefore,

$$II = -\int_{|y| \le |T|} \Big(\frac{|y \cdot \nabla v|^2}{|y|} + v \frac{y \cdot \nabla v}{|y|} \Big) dy$$
$$= -\int_{|y| \le |T|} \frac{1}{|y|} |y \cdot \nabla v + v|^2 \, dy + \int_{|y| \le |T|} \Big[\frac{v^2}{|y|} + v \frac{y \cdot \nabla v}{|y|} \Big] dy .$$

To evaluate the last term, note that, if we use polar coordinates, $y = r\omega$, then $vy \cdot \nabla v/|y| = v\partial_r v = \frac{1}{2}\partial_r(v^2)$. Hence, integration by parts gives

$$\int_{|y| \le |T|} v \frac{y \cdot \nabla v}{|y|} dy = \frac{1}{2} \int_{S^2} \int_0^{|T|} \partial_r \big(v^2(r\omega)\big)\, r^2 dr\, d\sigma(\omega)$$
$$= \frac{1}{2} \int_{S^2} v^2(|T|\omega)\, |T|^2 \, d\sigma(\omega) - \int_{S^2} \int_0^{|T|} v^2(r\omega)\, r dr d\sigma(\omega)$$
$$= \frac{1}{2} \int_{\partial D_T} u^2 \, d\sigma - \int_{|y| \le |T|} v^2 \frac{dy}{|y|} .$$

Combining this with the earlier formula and switching back to the original coordinates gives

$$II = \frac{1}{\sqrt{2}} \int_{M_T^0} t \,\Big| \frac{x}{|x|} \cdot \nabla_x u - \partial_t u + \frac{u}{t^2} \Big|^2 d\sigma + \frac{1}{2} \int_{\partial D_T} u^2 \, d\sigma . \tag{3.15}$$

To handle I, we first notice that the integrand in its definition is

$$TQ + u\partial_t u = T\big(\tfrac{1}{2}|u'|^2 + \Phi_\kappa(u)\big) + \partial_t u\,(u + x\cdot\nabla_x u).$$

But

$$|\partial_t u(u + x\cdot\nabla_x u)| \le -T\big[\tfrac{1}{2}(\partial_t u)^2 + \tfrac{1}{2}\big|\nabla_x u + \frac{x}{|x|^2}u\big|^2\big],$$

and so

$$\begin{aligned} I &\ge -T\int_{D_T}\Phi_\kappa(u)\,dx - T\int_{D_T}\big(\tfrac{1}{2}\big|\nabla_x u\big|^2 - \tfrac{1}{2}\big|\nabla_x u + \frac{x}{|x|^2}u\big|^2\big)\,dx \\ &= |T|\int_{D_T}\Phi_\kappa(u)\,dx + T\Big(\int_{D_T} u\frac{x}{|x|^2}\cdot\nabla_x u\,dx + \frac{1}{2}\int_{D_T}\frac{u^2}{|x|^2}\,dx\Big). \end{aligned}$$

If we integrate by parts like before, we find that

$$\int_{D_T} u\frac{x}{|x|^2}\cdot\nabla_x u\,dx + \frac{1}{2}\int_{D_T}\frac{u^2}{|x|^2}\,dx = \frac{1}{2}\int_{\partial D_T}\frac{-u^2}{T}\,d\sigma,$$

and hence

$$I \ge |T|\int_{D_T}\Phi_\kappa(u)\,dx - \frac{1}{2}\int_{\partial D_T} u^2\,d\sigma.$$

If we combine this with (3.14) and (3.15), we see that

(3.16)

$$\begin{aligned} |T|\int_{D_T}\Phi_\kappa(u)\,dx &\le CT^4 + \frac{1}{\sqrt{2}}\int_{M_T^0}|t|\,\big|\,\partial_t u + \frac{x}{|x|}\cdot\nabla_x u + \frac{u}{|x|}\,\big|^2\,d\sigma \\ &\le CT^4 + |T|\int_{M_T^0}\big|\,\partial_t u + \frac{x}{|x|}\cdot\nabla_x u\,\big|^2\,d\sigma + \int_{M_T^0}\frac{u^2}{|t|}\,d\sigma. \end{aligned}$$

By (3.5), the second to last term is $\le |T|\,\text{Flux}\,(u; M_T^0)$. The last term can also be controlled by the energy flux. In fact, if we use (1.4) and Hölder's inequality we see that it is

$$\begin{aligned} &\le \Big(\int_{M_T^0}|t|^{-3/2}\,d\sigma\Big)^{2/3}\Big(\int_{M_T^0}u^6\,d\sigma\Big)^{1/3} \\ &\le |T|\cdot C_0\Big(\int_{M_0^T}(\Phi_\kappa(u)+1)\,d\sigma\Big)^{1/3} \\ &\le C_0|T|\big(\text{Flux}\,(u;M_T^0)\big)^{1/3} + C_0|T|\cdot(4\pi|T|^3/3)^{1/3}. \end{aligned}$$

If we plug in our estimates for the last two terms into (3.16), we conclude that, for small $|T|$,

$$\int_{D_T}\Phi_\kappa(u)\,dx \le C\big(|T| + \text{Flux}\,(u;M_T^0) + (\text{Flux}\,(u;M_T^0))^{1/3}\big).$$

This finally gives us (3.10), since we saw before that $\text{Flux}\,(u;M_T^0)\to 0$ as $T\nearrow 0$.

This completes the proof of Theorem 1.1. □

Notes

As we pointed out before the subcritical part of Theorem 1.1 is due to Jörgens [1], while the critical case is due to Grillakis [1], after earlier work of Rauch [1] and Struwe [1]. The proof we have presented here, though, more closely resembles later arguments of Grillakis [2] and Shatah and Struwe [1]. In these two papers a combination of the Strichartz estimate (4.9) and the estimate (4.10) of Pecher stated in the last chapter was used for the proof, as well as Leibnitz's rule for fractional derivatives in the latter paper. In Smith and Sogge [1] it was observed that, by using a combination of ideas from these two papers, one could present a somewhat simplified argument based only on the $L^4_tL^{12}_x$ estimate (see also Keel [1]), and this is the proof we have followed. Related arguments showing how mixed-norm estimates lead to existence results can be found in many places, such as Brenner and von Wahl [1], Ginibre and Velo [1] and Ginibre, Soffer and Velo [1].

APPENDIX: SOBOLEV ESTIMATES AND HARDY-LITTLEWOOD INEQUALITIES

We shall prove the main inequalities used in the course. Let us start out with Sobolev's theorem:

Theorem 1. (Sobolev Inequalities) *Suppose that for a given* $m \in \mathbb{N}$, *we have*

$$1 \le p < q < \infty \text{ and } (1/p - 1/q) = m/n\,.$$

Then there is a constant $C = C(p, q, m, n)$ *so that*

$$\|f\|_{L^q(\mathbf{R}^n)} \le C \sum_{|\alpha|=m} \|\partial^\alpha f\|_{L^p(\mathbf{R}^n)}\,, \quad f \in C_0^\infty(\mathbb{R}^n)\,. \tag{1}$$

Also,

$$\|f\|_{L^\infty(\mathbf{R}^n)} \le C \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{L^p(\mathbf{R}^n)}\,, \quad \text{if } p > n/m\,, \tag{2}$$

with $C = C(p, m, n)$.

All the estimates except the ones corresponding to $p = 1$ in (1) are due to Sobolev [1]. The endpoints for (1) are due to Gagliardo [1] and Nirenberg [1]. Our proof will follow arguments in Aubin [1] and Hörmander [5].

Let us first prove (1) since it is the most difficult. We shall see that it is a consequence of the following special case:

Proposition 2. *Let* $C_n = n^{-1}\,(n/\omega_{n-1})^{1/n}$. *Then*

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbf{R}^n)} \le C_n \|f'\|_{L^1(\mathbf{R}^n)}\,, \tag{3}$$

whenever f *is Lipschitz and vanishes for large* x.

Proof. When $n = 1$ the result is a simple consequence of the fundamental theorem of calculus, so let us assume that $n > 1$. In this case, since $|f|$ is Lipschitz and $|f'| = |\,|f|'\,|$ almost everywhere, we conclude that we may assume that $f \ge 0$. By a limiting argument, we may also assume that $f \in C_0^\infty$.

Under these assumptions, let

$$\chi_\lambda(x) = \begin{cases} 1, & \text{if } f(x) > \lambda \\ 0, & \text{otherwise}. \end{cases}$$

Then, $f(x) = \int_0^\infty \chi_\lambda(x)\, d\lambda$, and so

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \int_0^\infty \|\chi_\lambda\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}\, d\lambda,$$

by Minkowski's integral inequality. If λ is not a critical value of f, then χ_λ is the characteristic function of a set E_λ with smooth boundary. For such λ, the isoperimetric inequality gives

$$\begin{aligned} \|\chi_\lambda\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} &= |E_\lambda|^{\frac{n-1}{n}} \\ &\le C_n\, |\,\partial E_\lambda\,| = C_n \int_{\{x:\, f(x)=\lambda\}} d\sigma_\lambda(x). \end{aligned}$$

The set of critical values is closed and of measure zero. Therefore, since

$$\int_0^\infty |\,\partial E_\lambda\,|\, d\lambda = \int |f'|\, dx,$$

we get (3) by combining the last two inequalities □

We remark that (3) is sharp; see Aubin [1]. Except for the constant, the inequality of Gagliardo [1] and Nirenberg [1] is stronger since it says that

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \frac{1}{2} \prod_{j=1}^{n} \|\,\partial f/\partial x_j\,\|_{L^1(\mathbb{R}^n)}^{1/n}.$$

Proof of (1). Let us assume first that $m = 1$. Then to prove (1), we may assume $q > n/(n-1)$, since the case where $q = n/(n-1)$ is (3). We then write $q = \sigma n/(n-1)$, where $\sigma = (n-1)q/n > 1$. Applying (3) to $|f|^\sigma$ gives

$$\begin{aligned} \|f\|_{L^q}^\sigma = \|\,|f|^\sigma\,\|_{L^{\frac{n}{n-1}}} &\le C_n \sigma \int_{\mathbb{R}^n} |f'|\,|f|^{\sigma-1}\, dx \\ &\le C_n \sigma \|f'\|_{L^p}\, \|f\|_{L^{(\sigma-1)\frac{p}{p-1}}}^{\sigma-1}. \end{aligned}$$

Since $\sigma - 1 = \frac{(n-1)q-n}{n}$ and $\frac{p}{p-1} = \frac{nq}{(n-1)q-n}$, if p is as in (1) with $m = 1$, we obtain this special case of (1).

The inequalities with $m = 2, 3, \ldots$ follow from the case where $m = 1$ by iteration. □

Proof of (2). Using (1), we see that we need only consider the case where $m = 1$. To handle this fix a function ψ with integral 1 which is in $C_0^\infty(\mathbb{R}^n)$ and supported in the unit ball. Then

$$|f(0)| = \left|f(0)\int \psi(x)\,dx\right| \le \left|\int \psi(x)\,(\,f(x)-f(0)\,)\,dx\,\right| + \left|\int \psi(x)f(x)\,dx\,\right|.$$

By Hölder's inequality the last term is $\le \|\psi\|_{L^{\frac{p}{p-1}}}\|f\|_{L^p} = C\|f\|_{L^p}$, and so it suffices to show that

$$(2') \qquad \left|\int \psi(x)\,(\,f(x) - f(0)\,)\,dx\,\right| \le C_p\|f'\|_{L^p(\mathbb{R}^n)}, \quad p > n.$$

If we use polar coordinates, we see that the left side equals

$$\left|\int_{S^{n-1}}\int \psi(r\omega)\,(f(r\omega) - f(0))\,r^{n-1}dr d\sigma(\omega)\,\right|$$
$$= \left|\int_{S^{n-1}}\int_0^1 \frac{\partial f}{\partial r}(r\omega)\Psi(r\omega)\,dr d\sigma(\omega)\,\right|,$$

if $\Psi(r\omega) = \int_r^\infty \tau^{n-1}\psi(\tau\omega)\,d\tau$, so that Ψ vanishes for $r > 1$ and satisfies $\frac{\partial}{\partial r}\Psi(r\omega) = r^{n-1}\psi(r\omega)$. Since Ψ is bounded, Hölder's inequality implies that the last expression is dominated by

$$\left(\int_{S^{n-1}}\int \left|\frac{\partial f}{\partial r}\right|^p r^{n-1}dr d\sigma\right)^{1/p}\cdot\left(\int_0^1 r^{-\frac{n-1}{p}\cdot\frac{p}{p-1}}\,dr\right)^{(p-1)/p} \le C_p\,\|f'\|_{L^p},$$

with the last step using the fact that $(n-1)/(p-1) < 1$. □

Corollary 3. *If $\delta > 0$ and $p > n/m$ are fixed then there is a constant C so that*

$$|f(x)| \le C\sum_{|\alpha|\le m}\left(\int_{|x-y|<\delta}|\partial^\alpha f(y)|^p\,dx\right)^{1/p}, \quad f \in C^\infty(\mathbb{R}^n).$$

This just follows from applying (2) to $\chi(y)f(y)$ where $\chi \in C_0^\infty$ satisfies $\chi(x) = 1$ and $\chi(y) = 0$ if $|x - y| > \delta$. The special case corresponding to $p = 2$ is the main part of Lemma 1.2 in Chapter 2.

A related inequality concerns fractional integration:

Theorem 4. (Hardy-Littlewood Fractional Integral Inequality) *Fix* $0 < \alpha < 1$ *and exponents* $1 < p < q < \infty$ *satisfying*

$$1 - (1/p - 1/q) = \alpha .$$

Then, if

$$I_\alpha f(t) = \int_{-\infty}^{\infty} f(s)\, |t-s|^{-\alpha}\, ds ,$$

there is a constant $C = C(\alpha, p, q)$ *so that*

$$\|I_\alpha f\|_{L^q(\mathbb{R})} \le C \|f\|_{L^p(\mathbb{R})} . \tag{4}$$

Remark. It turns out that one can recover the estimates in (1) corresponding to $p > 1$. The first step is to notice that using Lemma 5.1 from Chapter 3 one can see that the one-dimensional fractional integral inequality implies Sobolev's extension to higher dimensions. Specifically, if

$$I_\alpha f(x) = \int_{\mathbb{R}^n} f(y)\, |x-y|^{-\alpha}\, dy ,$$

then

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n)} \le C_{p,q,n} \|f\|_{L^p(\mathbb{R}^n)} , \tag{4'}$$

if $1 < p < q < \infty$ and $1 - (1/p - 1/q) = \alpha/n$. To verify this, one just uses the freezing arguments that were employed in Chapter 3 to prove (4′) inductively using (4). (See Sogge [1, pp. 25-26].) Alternatively, one could just use the proof of (4) to follow.

To give Sobolev's proof of (1) for $m = 1$ and $p > 1$, we need to use (4′) with $\alpha = n - 1$. The exponent q in the left is the same then as the one in (1) for the above parameters. To prove this inequality, let us fix $x \in \mathbb{R}^n$ and use polar coordinates around x, that is, we write $y = x + r\omega$, with $r = |x - y|$ and $\omega \in S^{n-1}$. If then ω is fixed and if $f \in C_0^\infty$, we have

$$f(x) = -\int_0^\infty \frac{\partial f}{\partial r}(x + r\omega)\, dr = -\int_0^\infty |x-y|^{1-n} \frac{\partial f}{\partial r}(x + r\omega)\, r^{n-1} dr .$$

Hence,

$$|f(x)| \le \int_0^\infty |x-y|^{1-n} \,|\, (\nabla f)(x + r\omega) \,|\, r^{n-1} dr .$$

Integration over S^{n-1} then yields

$$|f(x)| \le \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} |\nabla f(y)|\, |x-y|^{1-n}\, dy ,$$

giving us (1) with $m = 1$ and $p > 1$, after applying (4′). As before, versions for $m = 2, 3, \ldots$ follow by iteration.

The one-dimensional fractional integral inequality was used repeatedly in the second half of the course. It turns out that it follows from the estimates for the Hardy-Littlewood maximal function that were also used in Chapter 3:

Theorem 5. (Hardy-Littlewood Maximal Inequality) *Let*

$$(\mathcal{M}f)(t) = \sup_{r>0} \frac{1}{2r} \int_{t-r}^{t+r} |f(s)|\, ds\,.$$

Then, if $1 < p \le \infty$,

$$\|\mathcal{M}f\|_{L^p(\mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R})}\,. \tag{5}$$

Proof of Theorem 4. We shall use an argument of Hedberg [1]. See also Stein [3, p. 354].

We may assume that $f \ge 0$. We then write

$$(I_\alpha f)(t) = \int_{|s|<R} f(t-s)|s|^{-\alpha}\, ds + \int_{|s|>R} f(t-s)|s|^{-\alpha}\, ds\,. \tag{6}$$

Note that

$$\begin{aligned}\int_{R2^{-k}<|s|<R2^{-k+1}} f(t-s)\,|s|^{-\alpha}\, ds &\le (R2^{-k})^{-\alpha}\,(2\cdot R2^{-k+1})\,\mathcal{M}f(t)\\ &\le 4\cdot 2^{-k(1-\alpha)}\, R^{1-\alpha}\mathcal{M}f(t)\end{aligned}$$

Hence, if we sum over $k = 1, 2, 3, \ldots$, we conclude that the first term in the right of (6) is

$$\le C_\alpha R^{1-\alpha}\mathcal{M}f(t) = C_\alpha R^{\frac{1}{p}-\frac{1}{q}}\mathcal{M}f(t)\,.$$

If we apply Hölder's inequality, we see that the second term in the right of (6) is

$$\le 2\|f\|_{L^p}\cdot \Big(\int_R^\infty s^{-\alpha\cdot\frac{p}{p-1}}\, ds\Big)^{\frac{p-1}{p}}\,.$$

But $\alpha\frac{p}{p-1} = \frac{\alpha q}{\alpha q-1} > 1$, and so the last factor equals

$$C_\alpha R^{-\alpha+\frac{p-1}{p}} = C_\alpha R^{-\frac{1}{q}}\,.$$

Thus,

$$(I_\alpha f)(t) \le C_\alpha\big(R^{\frac{1}{p}-\frac{1}{q}}\,(\mathcal{M}f)(t) + R^{-\frac{1}{q}}\,\|f\|_{L^p}\big)\,.$$

If we choose R so that the last two terms agree, that is,

$$R = (\|f\|_{L^p}/(\mathcal{M}f)(t))^p,$$

we conclude that

$$(I_\alpha f)(t) \le 2C_\alpha \|f\|_{L^p}^{1-\frac{p}{q}}\,(\mathcal{M}f(t))^{p/q}\,.$$

From this we see that (4) follows from (5) as claimed. □

To prove the Hardy-Littlewood maximal theorem we shall use the following covering lemma, found in Garsia [1] and Garnett [1], which exploits the ordering property of $\mathbb{R}$.

Lemma 6. *Let* $\{I_1, \dots I_N\}$ *be a finite collection of open intervals* $I_j = (a_j, b_j)$. *Then there is a pairwise disjoint subfamily* $\{J_1, \dots, J_M\}$ *satisfying*

$$|\cup I_j| \le 2\sum_{k=1}^{M} |J_k|\,.$$

Proof. We can replace the original family by one whose union is the same and for which no interval is contained in a union of other ones. Assuming the intervals have this property, label the I_j so that their left endpoints satisfy $a_1 < a_2 < \cdots < a_N$. Then $b_{j+1} > b_j$ since otherwise $I_{j+1} \subset I_j$, and $a_{j+1} > b_{j-1}$ since otherwise $I_j \subset I_{j-1} \cup I_{j+1}$. Thus $I_j \cap I_k = \emptyset$ if both j and k are either odd or even. Since $|\cup I_j| \le \sum_{j \text{ even}} |I_j| + \sum_{j \text{ odd}} |I_j|$, we just let $\{J_k\}$ be the even or odd numbered intervals, depending on which sum is the largest. □

Proof of (5). The case where $p = \infty$ is trivial, so we shall assume that $1 < p < \infty$.

The main step then is to show that one has the weak-type (1,1) bounds

$$(5') \qquad |\{t \in \mathbb{R} : \mathcal{M}f(t) > \lambda\}| \le 2\lambda^{-1}\|f\|_{L^1(\mathbb{R})}\,, \quad f \in L^1(\mathbb{R})\,.$$

Note that if E_λ denotes the set where $\mathcal{M}f > \lambda$, then E_λ is open. Also, given $t \in E_\lambda$, there is an open interval centered at t so that

$$(6) \qquad |I| \le \lambda^{-1}\int_I |f|\, ds\,.$$

Let $K \subset E_\lambda$ be compact. Then we can cover K be finitely many intervals $\{I_j\}$ satisfying (6). If $\{J_k\}$ is a pairwise disjoint family satisfying the conclusion of the lemma, then

$$\begin{aligned} |K| \le |\cup I_j| &\le 2\sum_k |J_k| \\ &\le 2\sum_k \lambda^{-1}\int_{J_k} |f|\, ds = 2\lambda^{-1}\int_{-\infty}^{\infty} |f|\, ds\,. \end{aligned}$$

Since $|E_\lambda|$ is the supremum of the measures of its compact subsets, this yields (5′).

We can now complete the proof of the Hardy-Littlewood maximal theorem using an argument of Marcinkiewicz [1]. We first recall that, if $m(\lambda)$ denotes the distribution function for $\mathcal{M}f$, that is, the left side of (5′), then

$$(7) \qquad \int |\mathcal{M}f|^p\, dt = -\int_0^\infty \lambda^p\, dm(\lambda) = p\int_0^\infty \lambda^{p-1} m(\lambda)\, d\lambda\,.$$

Next, given $\lambda > 0$, let $f_\lambda(x)$ be the truncation of f at level $\lambda/2$, that is, $f_\lambda(x) = f(x)$ if $|f(x)| \geq \lambda/2$ and zero otherwise. Then clearly $\mathcal{M}f(t) \leq \mathcal{M}f_\lambda(t) + \lambda/2$, and so

$$\begin{aligned} m(\lambda) &\leq \big|\{t : \mathcal{M}f_\lambda(t) > \lambda/2\}\big| \\ &\leq 2(\lambda/2)^{-1}\int |f_\lambda|\,ds = 4\lambda^{-1}\int_{|f|>\lambda/2} |f|\,dx\,, \end{aligned}$$

using (5′) in the last step. Plugging this into (7) yields

$$\begin{aligned} \int (\mathcal{M}f(t))^p\,dt &\leq p\int_0^\infty \lambda^{p-1}\Big(4\lambda^{-1}\int_{|f|>\lambda/2} |f|\,ds\Big)\,d\lambda \\ &= 4p\int_{-\infty}^{\infty}\Big(\int_0^{2|f(s)|}\lambda^{p-2}\,d\lambda\Big)\,|f(s)|\,ds \\ &= \frac{p\,2^{p+1}}{p-1}\int_{-\infty}^{\infty} |f(s)|^p\,ds\,. \end{aligned}$$

This completes the proof. □

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