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Blowup and Blowup at Infinity for Quasilinear Wave Equations

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We consider in this paper 3D quasilinear wave equations satisfying the null condition, for which we construct classical solutions blowing up at infinity. This is done using conformal inversion and local techniques for constructing blowup solutions. It turns out that we are dealing with non genuinely nonlinear points; hence, we are forced to consider weakly nonlinear points, the definition of which involves a new set of invariant quantities. These new invariants are of interest in themselves, and we extend their definition to first-order general systems as well.

1 Introduction

In this paper, we investigate smooth solutions of quasilinear wave equations in $\mathbf{R}^3_x \times [0, +\infty[$

$$\Box u + Q_2(u) + Q_3(u) + Q_4(u) = 0,$$

where

$$Q_2(u) = g_2^{\alpha\beta\gamma}(\partial_\gamma u)(\partial^2_{\alpha\beta} u),$$

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$$Q_3(u) = g_3^{\alpha\beta\gamma\delta}(\partial_\gamma u)(\partial_\delta u)(\partial^2_{\alpha\beta} u),$$

$$Q_4(u) = g_4^{\alpha\beta\gamma\delta\epsilon}(\partial_\gamma u)(\partial_\delta u)(\partial_\epsilon u)(\partial^2_{\alpha\beta} u).$$

Here and in the whole paper, the coefficients g are assumed to be real constants. We assume moreover that Q_2 satisfies the null condition. Our aim is to understand the behavior at infinity and the stability of global C^∞ solutions, not necessarily small.

We know from the results of Christodoulou [9] and Klainerman [11] (see also [10]) that small solutions are global and stable, with a smooth free representation (see [7] for basic definitions)

$$u(x,t) = \frac{1}{r} F(r-t, \omega, 1/r), \quad r = |x|, \quad x = r\omega.$$

In a recent work [8], we give sufficient conditions for global C^∞ *large* solutions to be stable, or to have a smooth free representation. Results for systems are given in [6]. Here, we concentrate instead on the construction of global smooth solutions which blowup at infinity.

It turns out (as pointed out by Christodoulou) that a convenient tool to study quasilinear wave equations is the conformal inversion

$$X = \frac{x}{t^2 - r^2}, \quad T = -\frac{t}{t^2 - r^2}.$$

This transformation reduces the problem of the behavior of u at infinity to a local problem of smoothness, up to the boundary $\{|X| + T = 0\}$ of a cone $\mathcal{T} = \{|X| < -T, T < 0\}$, for the solution v of a new quasilinear wave equation. The function v is related to u by

$$u(x,t) = (T^2 - R^2) v(X, T).$$

To obtain a solution u blowing up at infinity, the idea is then to use previously developed techniques of construction of local blowup solutions, in the context of the equation for v. More precisely, we use the "geometric blowup" techniques developed in [1, 2]. Hence the paper is divided into two parts: finite time blowup and blowup at infinity.

The first part of this paper gives a self-contained treatment of the geometric blowup construction in two cases: the genuinely nonlinear case and some linearly degenerate case. The proofs that we give here are more unified and transparent than previously published proofs. But this is not the only reason: in view of the application we have in sight, we have to face two new features:

(i) The v-equation is of general form, with coefficients depending on $(X, T, v, \partial v)$, a case not considered in [5], where the goal was simply to illustrate a new phenomenon. Since the computations are rather heavy, we have to get a *geometric understanding* of the required conditions (to construct blowup solutions) in this general setting. This leads us to discover new nonlinear invariants, defining what we call a "weakly nonlinear point.' Such a point is a point which is not genuinely nonlinear, but not uniformly linearly degenerate either. We believe that this concept should turn out to be useful in other context as well (see Appendix for a definition applicable to any first-order system for a simple eigenvalue).

(ii) In many cases (in particular, if $Q_3 = Q_4 = 0$), the v-equation presents at the boundary a nongenuinely nonlinear point, as observed in [8]. This forces us to consider this apparently "non-generic" case.

The statements summarizing the blowup constructions in the genuinely nonlinear case and in the weakly nonlinear case are given in Theorems I.1 and I.2 in Section 4.

In the second part, we apply the results of Part I to the equation on v (that is, after conformal inversion). The issue is to decide whether there can be solutions v with smooth data which blowup at boundary points. We give two theorems summarizing the blowup constructions for the v equation: Theorems 10.1 and 10.3 (along with Corollary 10.4).

The corresponding theorem for u is Theorem 10.8. We can state roughly what we obtain as follow: let m^0_∞ be a "point at infinity" given by

$$\omega = x/r = \omega^0, \quad r - t = \sigma^0, \ \ r \to +\infty.$$

Under various conditions on the coefficients g_i at $(\omega^0, -1)$, we construct a C^∞ function u in a neighborhood of m^0_∞ such that

(i) u and $\partial_\alpha u$ have continuous (up to $z = 1/r = 0$) free representations

$$u(x, t) = \frac{1}{r} F(r - t, \omega, 1/r), \quad (\partial_\alpha u)(x, t) = \frac{1}{r} F_\alpha(r - t, \omega, 1/r),$$

(ii) For some continuous $a \neq 0$ and B,

$$u'' = \frac{1}{j}\frac{1}{r}(a\omega^t\omega + O(1/r)) + B/r,$$

where $j(m) \to 0$ as $m \to m_\infty^0$. The precise behavior of j is described in Theorems 10.1 and 10.3 according to the various cases. Since u exists only locally, the fact that u actually blows up at infinity from smooth data has to be precisely explained: it is reflected in the fact that the point corresponding to m_∞^0 is a hyperbolic blowup point for v, as explained after Theorem 10.1. We admit that it would be more satisfying to get a solution u existing everywhere (corresponding to a solution v existing in a strip $T_1 \leq T \leq T_2$ for instance), but we do not know how to do that for now.

Finally, in Section 10.3, we show, in the spirit of [12] and [14], that blowup at infinity for u is related to the existence of some incomplete null geodesic for the metric corresponding to the linearized u-equation.

Part I: Finite Time Blowup for a Second-Order Equation

As explained in Section 1, the construction of blowup solutions at infinity requires applying the technique of [5] to quasilinear equations obtained by conformal inversion. These equations do not fit into the framework of [5], for two reasons: first, they have variable coefficients, a case not considered in [5] where the goal was simply to illustrate a phenomenon. Secondly, there is a high degeneracy at the points of interest to us; this requires a more intrinsic understanding of the assumptions to be made than that was done in [5]. In this part, we extend the theory of [5], keeping the same notation for convenience.

2 Notation

Let a quasilinear equation with real *analytic* coefficients be

$$P(u) = \Sigma_{1 \leq i, j \leq n+1} p^{ij}(x, u, \partial u) \partial_{ij}^2 u + r(x, u, \partial u) = 0, \quad p^{ij} = p^{ji}, \quad p^{n+1,n+1} \neq 0,$$

where the variables are

$$x_1, y = (x_2, \ldots, x_n), t = x_{n+1}, \quad x = (x_1, y, t), \quad \partial u = (\partial_1 u, \ldots, \partial_{n+1} u),$$

with dual variables

$$\xi_1, \eta = (\xi_2, \ldots, \xi_n), \quad \tau = \xi_{n+1}, \quad \xi = (\xi_1, \eta, \tau).$$

The principal symbol is

$$p(x, u, U, \xi) = \Sigma p^{ij}(x, u, U) \xi_i \xi_j = p^{ij} \xi_i \xi_j, \quad U = (U_1, \ldots, U_{n+1}).$$

The variables ξ and U are indexed below since ξ_i stands for ∂_i and U_i stands for $\partial_i u$. We do not write sums for summation over repeated indices. We assume given a real base point $\underline{m} = (\underline{x}, \underline{u}, \underline{U}, \underline{\xi})$ (with $\underline{x} = 0$ for convenience) where

$$\underline{\xi}_1 = -1, \quad p(\underline{m}) = 0, \quad (\partial_\tau p)(\underline{m}) \neq 0.$$

We denote by $\tau = \lambda(x, u, U, \xi_1, \eta)$ the corresponding simple root of p. The derivatives with respect to U will be denoted by D

$$D^j p = \partial_{U_j} p,$$

and the derivatives with respect to ξ_i will be denoted by ∂_ξ^i.

3 Some Nonlinear Invariants

In this section only, the splitting of variables $x = (x_1, y, t)$ does not play a role, so we use x only, especially in Lemmas 3.2 and 3.5.

3.1

Definition 3.1. We define the function q associated to the symbol $p = p(x, u, U, \xi)$ by

$$q(x, u, U, \xi) = \xi_i D^i p(x, u, U, \xi).$$

The function q corresponds exactly to the function q defined in Appendix for first-order systems, if one transforms the operator into such a system. The function q for systems, in turn, is inspired by the concept of "genuinely nonlinear eigenvalue" introduced by Lax [13] for 1D systems. □

Lemma 3.2. The function q is invariant by the change of coordinates. More precisely, if $y = \phi(x)$ are the new variables with dual variables η and $u = v(\phi)$, the function $\tilde{q}$ corresponding to the new equation $\tilde{P}(v) = 0$ is given by

$$\tilde{q}(y, v, V, \eta) = q(\phi^{-1}(y), v, {}^t\phi'(\phi^{-1}(y))V, {}^t\phi'(\phi^{-1}(y))\eta).$$

Here, $(\phi')_{ij} = \partial_j \phi^i$. □

Lemma 3.3. The function q is invariant by the change of unknown function. More precisely, if $u=\phi(v)$ gives the new equation $\tilde{P}(v)=0$, the corresponding function $\tilde{q}$ is

$$\tilde{q}(x, v, V, \xi) = (\phi'(v))^2 q(x, \phi(v), \phi' V, \xi).$$

□

The straightforward proofs of these lemmas are left to the reader.

To handle linearly degenerate cases, one needs to introduce the following objects, the meaning of which will be explained after the definition at the end of this section.

3.2

Definition 3.4. We define the symmetric matrix C^{ij} associated to the functions p, q, and the function $\underline{C}$ associated to the functions p, q, and to $r=r(x, u, U)$ by

$$\underline{C}(p, q, r) = \underline{C} = \{p, q\} + (\partial_u q)(U\partial_\xi p) - (\partial_u p)(U\partial_\xi q) - r(\xi Dq),$$

$$C^{ij}(p, q) = C^{ij} = \partial_\xi^i p D^j q - \partial_\xi^i q D^j p + \partial_\xi^j p D^i q - \partial_\xi^j q D^i p - 2p^{ij}(\xi Dq)$$

for $1 \leq i, j \leq n+1$. Here, $\{f, g\}$ is the usual Poisson bracket

$$\{f, g\} = (\partial_\xi f)(\partial_x g) - (\partial_\xi g)(\partial_x f).$$

□

Remarkably enough, these coefficients turn out to be, in some sense, indicated in the Lemmas 3.5 and 3.6, invariant with respect to both changes of variables and unknown functions.

Lemma 3.5. Under the change of variables $y=\phi(x)$,

(i) $\tilde{C}^{ij} = (\partial_k \phi^i)(\partial_l \phi^j) C^{kl}$, or, equivalently, $\tilde{C} = (\phi')C({}^t\phi')$,

(ii) $\underline{\tilde{C}} = \underline{C} + \frac{1}{2}(\partial_{kl}^2 \phi^j V_j) C^{kl}$.

□

Proof. Let $y=\phi(x)$. With the notation of Lemma 3.2, in the new equation on v, the lower order terms are

$$\tilde{r} = p^{ij}(\partial_{ij}^2 \phi^k)\partial_k v + r.$$

(a) Let us first prove (i): $\partial_\eta^i \tilde{p} = \partial_\xi^j p \partial_j \phi^i$, $\partial_\eta^i \tilde{q} = \partial_\xi^j q \partial_j \phi^i$,

$$D^i \tilde{p} = D^j p \partial_j \phi^i, \; D^i \tilde{q} = D^j q \partial_j \phi^i.$$

In particular, we note $\eta D\tilde{q} = \eta_i D^j q \partial_j \phi^i = \xi_j D^j q = \xi Dq$. Hence

$$\tilde{C}^{ij} = \partial_\xi^k p \partial_k \phi^i D^l q \partial_l \phi^j - \partial_\xi^k q \partial_k \phi^i D^l p \partial_l \phi^j + \partial_\xi^k p \partial_k \phi^j D^l q \partial_l \phi^i$$
$$- \partial_\xi^k q \partial_k \phi^j D^l p \partial_l \phi^i - 2p^{kl} \partial_k \phi^i \partial_l \phi^j (\eta D\tilde{q}) = (\partial_k \phi^i)(\partial_l \phi^j) C^{kl}.$$

(b) To prove (ii), we note

$$\partial_i \tilde{p} = \partial_j p \partial_i ((\phi^{-1})^j) + [\partial_i (\partial_k \phi^j (\phi^{-1}))](D^k p V_j + (\partial_\xi^k p)\eta_j)$$

and similarly for $\tilde{q}$. Since $\partial_i((\phi^{-1})^l) = (\phi^{-1})'_{li} = ((\phi')^{-1})_{li} = (({}^t\phi')^{-1})_{il}$,

$$\partial_i \tilde{p} = \partial_j p (({}^t\phi')^{-1})_{ij} + \partial_{kl}^2 \phi^j (({}^t\phi')^{-1})_{il} (D^k p V_j + (\partial_\xi^k p)\eta_j).$$

Hence

$$\partial_\eta^i \tilde{p} \partial_i \tilde{q} = \partial_\xi^j p \partial_j \phi^i \partial_i \tilde{q} = \partial_\xi^j p \partial_k q (\partial_j \phi^i)(({}^t\phi')^{-1})_{ik}$$
$$+ \partial_\xi^q p (\partial_q \phi^i) \partial_{kl}^2 \phi^j (({}^t\phi')^{-1})_{il} (D^k q V_j + \partial_\xi^k q \eta_j)$$
$$= \partial_\xi^i p \partial_i q + (\partial_{kl}^2 \phi^j \eta_j)(\partial_\xi^l p)(\partial_\xi^k q) + (\partial_{kl}^2 \phi^j V_j)(\partial_\xi^l p)(D^k q).$$

Finally,

$$\{\tilde{p}, \tilde{q}\} = \{p, q\} + (\partial_{kl}^2 \phi^j V_j)(\partial_\xi^l p D^k q - \partial_\xi^l q D^k p).$$

Note that

$$V \partial_\eta \tilde{p} = V_i \partial_\xi^j p \partial_j \phi^i = U_j \partial_\xi^j p = U \partial_\xi p.$$

Thus it remains to compute $\{\tilde{p}, \tilde{q}\} - \tilde{r}(\eta D\tilde{q})$. Taking into account the expression for $\tilde{r}$, we get

$$\{\tilde{p}, \tilde{q}\} - \tilde{r}(\eta D\tilde{q}) = \{p, q\} - r(\xi Dq) + E,$$

$$E = (\partial^2_{kl}\phi^j V_j)(\partial^l_\xi p D^k q + \partial^l_\xi q D^k p - p^{kl}),$$

$$E = \tfrac{1}{2}(\partial^2_{kl}\phi^j V_j) C^{kl}. \qquad \blacksquare$$

Lemma 3.6. Under the change of unknown function $u = \phi(v)$,

(i) $\tilde{C}^{ij} = (\phi')^4 C^{ij}$,

(ii) At a point where $p = q = 0$, $\underline{\tilde{C}} = (\phi')^3 \underline{C} + \frac{1}{2}(\phi')(\phi'') U_i U_j C^{ij}$. □

Proof.

(a) According to the proof of Lemma 3.3,

$$D^i \tilde{p} = \phi'^2 D^i p, \tilde{q} = \phi'^2 q.$$

Hence $\xi D\tilde{q} = \phi'^3 \xi Dq$, $\tilde{C}^{ij} = \phi'^4 C^{ij}$.

(b) The change in $\underline{C}$ is more subtle. First,

$$V\partial_\xi \tilde{p} = \phi' V_i \partial^i_\xi p = U\partial_\xi p, \; V\partial_\xi \tilde{q} = \phi' U\partial_\xi q.$$

Next, $\partial_v \tilde{p} = \phi'' p + \phi'^2 \partial_u p + \phi'\phi'' V_i D^i p$, and similarly for $\tilde{q}$. Hence, at a point where $p = q = 0$, taking into account the additional term in $\tilde{r}$,

$$\begin{aligned}\underline{\tilde{C}} &= \phi'^3\{p, q\} + \phi'^3[\partial_u q(U\partial_\xi p) - \partial_u p(U\partial_\xi q)] - \phi'^3 r(\xi Dq)\\ &\quad + \phi'\phi''[(UDq)(U\partial_\xi p) - (UDp)(U\partial_\xi q) - (\xi Dq) p^{ij} U_i U_j]\\ &= \phi'^3 \underline{C} + \tfrac{1}{2}\phi'\phi'' U_i U_j C^{ij}. \qquad \blacksquare\end{aligned}$$

Finally, we give here some algebraic relations between the coefficients C^{ij}.

Lemma 3.7. At a point where $p = q = 0$, for all j, $\xi_i C^{ij} = 0$. □

Proof. We have

$$\xi_i C^{ij} = D^j q(\xi\partial_\xi p) - D^j p(\xi\partial_\xi q) + (\partial^j_\xi p)(\xi Dq) - (\partial^j_\xi q)(\xi Dp) - 2(\xi Dq)(\xi_i p^{ij}).$$

Using the homogeneity of p and q in ξ, this reduces to $\xi_i C^{ij} = (\xi Dq)(\partial_\xi^j p - 2\xi_i p^{ij})$. But $p = p^{ij}\xi_i\xi_j$, hence $\partial_\xi^j p = 2p^{ij}\xi_i$. ■

The invariants C and $\underline{C}$ are new to us: we did not find them in the literature. They probably have a geometric interpretation in terms of contact structures. The above invariance lemmas allow us to define two concepts.

Definition 3.8. A genuinely nonlinear point $\underline{m} = (\underline{x}, \underline{u}, \underline{U}, \underline{\xi})$ is a point where $p(\underline{m}) = 0$, $q(\underline{m}) \neq 0$. □

This concept is in fact, in a slightly less general form, due to Lax [13].

Definition 3.9. A weakly nonlinear point $\underline{m}$ is a point where $p(\underline{m}) = q(\underline{m}) = 0$, and the coefficients $\underline{C}$, C^{ij} are not all zero. □

Thus, the coefficients C, $\underline{C}$ measure the nonuniformity in the linear degeneracy of the point. They were found by brute force computations, when trying to construct blowup solutions for such points. Recall that these concepts make sense for a quasilinear equation $P(u) = 0$, and are invariant under a change of coordinates and a change of unknown function u.

3.3 Example and comments

In the simplest 1D case

$$\partial_t^2 u - c^2(\partial_x u, \partial_t u)\partial_x^2 u = 0,$$

we have $q = -2c\xi^2\tilde{q}$, with $\tilde{q} = \xi\partial_1 c + \tau\partial_2 c$. At the characteristic point, $\tau = \epsilon c\xi$ $(\epsilon = \pm 1)$, $q = 0$ is equivalent to $\partial_1 c + \epsilon c\partial_2 c = 0$. We have then

$$C^{11}(p, \tilde{q}) = c\xi^2(-c\partial_1^2 c + c^3\partial_2^2 c + 2(\partial_1 c)^2),$$

$$C^{12}(p, \tilde{q}) = -2\epsilon\xi^2(-c\partial_1^2 c + c^3\partial_2^2 c + 2(\partial_1 c)^2),$$

$$C^{22} = \frac{\xi^2}{c}(-c\partial_1^2 c + c^3\partial_2^2 c + 2(\partial_1 c)^2).$$

Hence, since $\underline{C} = 0$, the point is weakly nonlinear if

$$-c\partial_1^2 c + c^3\partial_2^2 c + 2(\partial_1 c)^2 \neq 0.$$

In general, note first that the condition that all C^{ij} vanish is invariant by a change of coordinate or unknown function. The coefficients C^{ij} can be viewed as $Z^{ij}q$, where the fields Z are defined by

$$Z^{ij} = \partial_\xi^i pD^j - D^j p\partial_\xi^j + \partial_\xi^j pD^i - D^i p\partial_\xi^i - 2p^{ij}\xi D.$$

The fields Z^{ij} are "vertical" fields (in the sense that they have no (x,t) component) tangent to $\{p=0\}$ at a point where $p=q=0$. The field $H_p + (U\partial_\xi p)\partial_u - (\partial_u p)U\partial_\xi - r\xi D$, which defines $\underline{C}$ in contrast, is always nonvertical, since $\partial_\tau p \neq 0$.

4 Main Results About Blowup

We come back now to the framework of Section 2.

Theorem I.1 (genuinely nonlinear case). Assume that $q(\underline{m}) \neq 0$. Let $\mu < \underline{\tau}$, and define

$$\mathcal{D} = \{(x_1, y, t), t \leq 0\} \cup \{(x_1, y, t), t \geq 0, x_1 \leq y\underline{\eta} + \mu t\}.$$

There exists, locally near the origin in $\mathcal{D}$, a solution u of $P(u) = 0$ such that

(i) $u \in C^1(\mathcal{D})$,

(ii) u is C^∞ in $\mathcal{D}$ except at the origin,

(iii) $u'' = S/j + R$, where S is symmetric of rank one, and R and S are C^∞ in $\mathcal{D}$. The function j vanishes only at the origin, $j \geq C|t|$ for $t \leq 0$, $j \geq Ct^{2/3}$ for $t \geq 0$ and $j \sim C|t|$ for some curve reaching the origin. □

According to [4], the blowup of u is a "geometric blowup of cusp type," the precise description of which is given at the end of the proof of Theorem I.1 in Section 6. Note that, in comparison with previous works, the existence domain of the solution has been extended to a region in $\{t \geq 0\}$. In the case of Burgers equation, for instance, we can extend the solution from the left all the way to the right branch of the blowup cusp tangent to the characteristic (and similarly, from the right all the way to the other branch of the cusp to the left of the characteristic). Here, however, we do not quite reach the characteristic, since the position of the "cusp" with respect to the characteristic plane is not clear.

Theorem I.2 (weakly nonlinear case). Assume that $\underline{m}$ is a weakly nonlinear point for the equation $P(u)=0$. Then there exists, close to the origin, a solution u of $P(u)=0$ such that

(i) $u \in C^1$ and $u \in C^\infty$ outside the origin,

(ii) $u'' = S/j + R$, where S is symmetric of rank one, and R and S are C^∞. For some $c>0$, $j \geq c|x|^2$ with $j \sim c|x|^2$ on some curve reaching the origin. □

As for Theorem I.1, the precise description of j is given at the end of the proof in Section 7.

Theorem I.1 has been essentially proved in [3, 4]; however, in these versions of the theorem, the function u is only constructed for $t \leq 0$. The extension here is crucial for applications in Part II. Theorem I.2 is proved in [5] in the special case where the coefficients p^{ij} and r depend only on ∂u, under non geometric assumptions. The goal there was to illustrate the existence of unstable higher order blowup, while the general case we consider here is required for the applications in Part II. However, we give in Sections 5–7 a simplified and more transparent proof of both theorems, in a self-contained way, for the convenience of the reader.

Remark: Two challenging examples

The typical cases to which the given Theorems I.1 and I.2 do not apply are the totally linearly degenerate cases where $q \equiv 0$ on $p=0$. For example

$$(1+\partial_1 u)\partial_t^2 u - \partial_1^2 u - (\partial_t u)\partial_{t1}^2 u = 0$$

in one space variable, $q \equiv 0$. Setting as usual $v_0 = \partial_t u,\ v_1 = \partial_1 u$, this equation can be reduced to a first-order quasilinear 2×2 system. The identical vanishing of q reflects then in the fact that both eigenvalues of this system are linearly degenerate, hence no blowup can occur, as easily seen by elementary computations. Now, if we add some variable, we get

$$(1+\partial_1 u)\partial_t^2 u - \partial_1^2 u - \partial_2^2 u - (\partial_t u)\partial_{t1}^2 u = 0,$$

an equation for which we do not know how to construct blowup solutions, if at all.

Another similar 2D example has been already mentioned in [5]:

$$\Box u + (\partial_1 u)\partial_2^2 u - (\partial_2 u)\partial_{12}^2 u = 0.$$

Here again, $q \equiv 0$, and we do not know how to construct blowup solutions for which the linearized equation is hyperbolic.

5 Generalities about the Proof of the Theorems: Geometric Blowup and the Blowup System

Associated to a real function ϕ, we introduce the change of variables Φ

$$\Phi : (s, y, t) \mapsto (x_1 = \phi(s, y, t), y, t), \quad \phi(0) = 0,$$

and set

$$w = u(\Phi), \quad v = (\partial_1 u)(\Phi).$$

We have the obvious "auxiliary equation"

$$\mathcal{A} \equiv \partial_s w - v \partial_s \phi = 0.$$

Writing

$$\bar{\partial} = (0, \partial_y, \partial_t), \quad \hat{\phi} = (-1, \partial_y \phi, \partial_t \phi),$$

we get

$$(\partial u)(\Phi) = \bar{\partial} w - v \hat{\phi},$$

$$(\partial^2_{ij} u)(\Phi) = \bar{\partial}^2_{ij} w - v \bar{\partial}^2_{ij} \phi - (\hat{\phi}_i \bar{\partial}_j v + \hat{\phi}_j \bar{\partial}_i v) + \hat{\phi}_i \hat{\phi}_j \left(\frac{\partial_s v}{\partial_s \phi} \right).$$

Hence the equation $P(u) = 0$ is satisfied as soon as the functions (v, w, ϕ) satisfy the blowup system

$$\mathcal{A} = 0, \quad \mathcal{E} \equiv \Sigma p^{ij}(\phi, y, t, w, v, \partial_y w - v \partial_y \phi, \partial_t w - v \partial_t \phi) \hat{\phi}_i \hat{\phi}_j = 0,$$

$$\mathcal{R} \equiv \Sigma p^{ij}(\phi, y, t, w, v, \partial_y w - v \partial_y \phi, \partial_t w - v \partial_t \phi)[\bar{\partial}^2_{ij} w - v \bar{\partial}^2_{ij} \phi - (\hat{\phi}_i \bar{\partial}_j v + \hat{\phi}_j \bar{\partial}_i v)]$$
$$+ r(\phi, y, t, w, \partial_y w - v \partial_y \phi, \partial_t w - v \partial_t \phi) = 0.$$

To satisfy the eikonal equation $\mathcal{E} = 0$, we choose the root $\tau = \lambda$, that is, we ask

$$\partial_t \phi = \lambda(\phi, y, t, w, v, \partial_y w - v \partial_y \phi, \partial_t w - v \partial_t \phi, -1, \partial_y \phi).$$

Since ∂_t and ∂_1 are characteristic directions for the blowup system, we perform the change of variables

$$T = s + t, \quad S = t - s, \quad y = y, \quad \partial_t = \partial_T + \partial_S, \quad \partial_s = \partial_T - \partial_S.$$

The system $\mathcal{A} = 0$, $\mathcal{E} = 0$ is then equivalent to

$$\partial_T w - \partial_S w = v(\partial_T \phi - \partial_S \phi), \quad \partial_T \phi + \partial_S \phi = \lambda\left(\phi, y, \frac{T+S}{2}, w, v, B_y, 2B_S, -1, \partial_y \phi\right),$$

or, equivalently,

$$\partial_T w = \partial_S w + (\lambda - 2\partial_S \phi) v, \quad \partial_T \phi = -\partial_S \phi + \lambda.$$

Here and in the sequence, we note $B_S = \partial_S w - v\partial_S \phi$, $B_y = \partial_y w - v\partial_y \phi$. The residual equation $\mathcal{R} = 0$ becomes

$$\begin{aligned}
\mathcal{R} = {} & p^{n+1,n+1}[(\partial_T + \partial_S)^2 w - v(\partial_T + \partial_S)^2 \phi - 2(\partial_T \phi + \partial_S \phi)(\partial_T v + \partial_S v)] \\
& + 2\Sigma_{2 \le i \le n} p^{i,n+1}[\partial_i(\partial_T + \partial_S) w - v\partial_i(\partial_T + \partial_S)\phi \\
& - (\partial_i \phi)(\partial_T + \partial_S) v + (\partial_i v)(\partial_T + \partial_S)\phi] + 2p^{1,n+1}(\partial_T + \partial_S) v \\
& + \Sigma_{i,j \le n} p^{ij}[\bar{\partial}^2_{ij} w - v\bar{\partial}^2_{ij}\phi - (\hat{\phi}_i \bar{\partial}_j v + \hat{\phi}_j \bar{\partial}_i v)] = 0.
\end{aligned}$$

Using the above equations giving $\partial_T w$ and $\partial_T \phi$, we can eliminate all T-derivatives of w and ϕ from this equation $\mathcal{R} = 0$. It is always understood that the coefficients p^{ij} and λ are taken at the point

$$\left(\phi, y, \frac{T+S}{2}, w, v, B_y, 2B_S, -1, \partial_y \phi\right).$$

In the sequence, to avoid any confusion, we will sometimes denote by $[f]$ a function taken at this point. If a function f depends also on τ and is taken for $\tau = [\lambda]$, we note simply $\bar{f}$. The following lemma gives the expressions for the second-order derivatives of ϕ and w (compare with [5, Lemma 2.1]).

Lemma 5.1. From the equations of the blowup system, we obtain

$$\partial_S[\lambda] = \partial_1 \lambda \partial_S \phi + \tfrac{1}{2}\partial_t \lambda + \partial_u \lambda \partial_S w + \Lambda \partial_S v + D\lambda B_{Sy} + 2D^{n+1}\lambda B_{SS} + \partial_\eta \lambda \partial^2_{Sy}\phi,$$

$$\partial_{y_j}[\lambda] = \partial_1 \lambda \partial_{y_j}\phi + \partial_{y_j}\lambda + \partial_u \lambda \partial_{y_j} w + \Lambda \partial_{y_j} v + D\lambda B_{y_j y} + 2D^{n+1}\lambda B_{y_j S} + \partial_\eta \lambda \partial^2_{y_j y}\phi,$$

$$\partial_T[\lambda] = \partial_1\lambda\partial_T\phi + \tfrac{1}{2}\partial_t\lambda + \partial_u\lambda\partial_T w + \Lambda\partial_T v + D\lambda B_{SY} + 2D^{n+1}\lambda B_{SS}$$
$$+ (\lambda - 2\partial_S\phi)(D\lambda\partial_Y v + 2D^{n+1}\lambda\partial_S v) + \partial_\eta\lambda(-\partial^2_{SY}\phi + \partial_Y[\lambda]),$$
$$\partial_T[\lambda] = \partial_1\lambda(\partial_T\phi + \partial_\eta\lambda\partial_Y\phi) + \partial_Y\lambda\partial_\eta\lambda + \tfrac{1}{2}\partial_t\lambda + \partial_u\lambda(\partial_T w + \partial_\eta\lambda\partial_Y w)$$
$$+ \Lambda(\partial_T v + \partial_\eta\lambda\partial_Y v) + (\lambda - 2\partial_S\phi)(D\lambda\partial_Y v + 2D^{n+1}\lambda\partial_S v) + 2D^{n+1}\lambda B_{SS}$$
$$+ (D\lambda + 2D^{n+1}\lambda\partial_\eta\lambda)B_{SY} + D\lambda\partial_\eta\lambda B_{YY} - \partial^2_{SY}\phi\partial_\eta\lambda + \partial_\eta\lambda\partial_\eta\lambda\partial^2_{YY}\phi,$$
$$\partial^2_{TS}\phi = -\partial^2_S\phi + \partial_S[\lambda],\ \ \partial^2_{TY}\phi = -\partial^2_{SY}\phi + \partial_Y[\lambda],$$
$$\partial^2_{TS}w = \partial^2_S w + (\lambda - 2\partial_S\phi)\partial_S v + v(\partial_S[\lambda] - 2\partial^2_S\phi),$$
$$\partial^2_{TY}w = \partial^2_{SY}w + (\lambda - 2\partial_S\phi)\partial_Y v + v(\partial_Y[\lambda] - 2\partial^2_{SY}\phi).$$

Here,

$$B_{SS} = \partial^2_S w - v\partial^2_S\phi,\ B_{SY} = \partial^2_{SY}w - v\partial^2_{SY}\phi,\ B_{YY} = \partial^2_{YY}w - v\partial^2_{YY}\phi,$$
$$\Lambda = [D^1\lambda] - \partial_Y\phi[D\lambda] - 2\partial_S\phi[D^{n+1}\lambda].$$

The following lemma clarifies the equation $\mathcal{R} = 0$. □

Lemma 5.2. The equation $\mathcal{R} = 0$ can be written as

$$-(\partial_\tau^{\bar{}} p)(\partial_T v + \partial_S v) - \Sigma_{2\le i\le n}(\partial_\eta^{\bar{i}} p)\partial_{Y_i} v + p^{n+1,n+1}[4B_{SS} + (\lambda - 2\partial_S\phi)(3\partial_S v + \partial_T v)]$$
$$+ 2\Sigma_{2\le i\le n}p^{i,n+1}[2B_{SY_i} + (\lambda - 2\partial_S\phi)\partial_{Y_i} v] + \Sigma_{2\le i,j\le n}p^{ij}B_{Y_iY_j} + r = 0.$$ □

Proof. We have first

$$\partial^2_T w = \partial^2_{ST}w + (\lambda - 2\partial_S\phi)\partial_T v + v(\partial_T[\lambda] - 2\partial^2_{ST}\phi),$$
$$(\partial_T + \partial_S)^2 w = 4B_{SS} + (\lambda - 2\partial_S\phi)(3\partial_S v + \partial_T v) + v(\partial_T + \partial_S)[\lambda],$$
$$(\partial_T + \partial_S)^2 w - v(\partial_T + \partial_S)^2\phi = 4B_{SS} + (\lambda - 2\partial_S\phi)(3\partial_S v + \partial_T v),$$

For $2 \le i \le n$,

$$\partial_i(\partial_T + \partial_S)w - v\partial_i(\partial_T + \partial_S)\phi = 2B_{SY_i} + (\lambda - 2\partial_S\phi)\partial_{Y_i} v.$$

Hence the residual equation reads

$$\begin{aligned}
&p^{n+1,n+1}[4B_{SS} + (\lambda - 2\partial_S\phi)(3\partial_S v + \partial_T v) - 2\lambda(\partial_S v + \partial_T v)]\\
&\quad + 2\Sigma_{2\le i\le n} p^{i,n+1}[2B_{SY_i} + (\lambda - 2\partial_S\phi)\partial_{Y_i} v - (\partial_i\phi)(\partial_T v + \partial_S v) - \lambda\partial_{Y_i} v]\\
&\quad + 2p^{1,n+1}(\partial_S v + \partial_T v) + \Sigma_{2\le i,j\le n} p^{ij}[\partial^2_{ij} w - v\partial^2_{ij}\phi - 2\partial_i\phi\partial_j v]\\
&\quad + 2\Sigma_{2\le i\le n} p^{1,i}\partial_i v + r = 0.
\end{aligned}$$

Now, since

$$p = p^{n+1,n+1}\tau^2 + 2\Sigma_{2\le i\le n} p^{i,n+1}\tau\eta_i + 2p^{1,n+1}\tau\xi_1 + \Sigma_{2\le i,j\le n} p^{ij}\eta_i\eta_j + 2\Sigma_{2\le i\le n} p^{1,i}\xi_1\eta_i + p^{11}\xi_1^2,$$

we have

$$\begin{aligned}
\partial_\tau p &= 2\tau p^{n+1,n+1} + 2\Sigma_{2\le i\le n} p^{i,n+1}\eta_i + 2p^{1,n+1}\xi_1,\\
\partial^i_\eta p &= 2p^{i,n+1}\tau + 2\Sigma_{2\le j\le n} p^{ij}\eta_j + 2p^{1,i}\xi_1.
\end{aligned}$$

The residual equation is then

$$\begin{aligned}
&-(\partial_{\bar\tau} p)(\partial_T v + \partial_S v) - \Sigma_{2\le i\le n}(\partial^{\bar i}_\eta p)\partial_{Y_i} v + p^{n+1,n+1}[4B_{SS} + (\lambda - 2\partial_S\phi)(3\partial_S v + \partial_T v)]\\
&\quad + 2\Sigma_{2\le i\le n} p^{i,n+1}[2B_{SY_i} + (\lambda - 2\partial_S\phi)\partial_{Y_i} v] + \Sigma_{2\le i,j\le n} p^{ij} B_{Y_i Y_j} + r = 0.
\end{aligned}$$

■

As a consequence of these lemma, we introduce $\partial_S\phi, \partial_Y\phi, \partial_S w, \partial_Y w$ as new unknowns, and thus obtain a fully nonlinear first-order system in the unknowns $v, \phi, w, \partial_S\phi, \partial_Y\phi, \partial_S w, \partial_Y w$, resolved with respect to the T-derivative, and for which $T = 0$ is non-characteristic (at a point where $\phi_s = 0$). The coefficients being analytic by assumption, we use the Cauchy–Kovalevsky theorem, prescribing the unknowns on $\{T = 0\}$, to solve the system locally near the origin. At the origin, we choose of course

$$\phi = \underline{x}_1 = 0, \quad w = \underline{u}, \quad v = \underline{U}_1, \quad B_Y = \underline{U}, \quad 2B_S = \underline{U}_{n+1}, \quad \partial_Y\phi = \underline{\eta}.$$

To obtain a blowup solution, we also choose $\partial_s\phi(0) = 0$, which corresponds to $\lambda = 2\partial_S\phi$. Finally, we observe that the values of v on the one hand, the values of ϕ, w and the values of their first derivatives on the other hand, are fixed at the given point. The following simple lemma relates the functions Λ and q.

Lemma 5.3. The function $\Lambda = [D^1\lambda] - \partial_Y\phi[D\lambda] - 2\partial_S\phi[D^{n+1}\lambda]$ defined in Lemma 5.2 is related to q by the formula

$$(\overline{\partial_\tau p})\Lambda + ([\lambda] - 2\partial_S\phi)\overline{D^{n+1} p} = \bar{q}. \qquad \square$$

Proof. Since

$$p(x, u, U, \xi) = (\tau - \lambda(x, u, U, \xi_1, \eta))h(x, u, U, \xi),$$

we obtain, taking derivatives and setting $\tau = \lambda$,

$$\overline{D^j p} = -\bar{h}D^j\lambda, \quad \overline{\partial_\tau p} = \bar{h}.$$

This gives

$$(\overline{\partial_\tau p})\Lambda = [-D^1 p + \partial_Y\phi \overline{Dp} + 2\partial_S\phi D^{n+1} p]$$
$$= [\xi_1 D^1 p + \eta \overline{Dp} + \tau D^{n+1} p] - (\lambda - 2\partial_S\phi)\overline{D^{n+1} p}. \qquad \blacksquare$$

We set $F = \partial_T\phi - \partial_S\phi = [\lambda] - 2\partial_S\phi$.

Lemma 5.4. The first-order derivatives of F are

$$\partial_S F = \partial_S[\lambda] - 2\partial_S^2\phi,\ \partial_Y F = \partial_Y[\lambda] - 2\partial_{SY}^2\phi,$$
$$\partial_T F = \Lambda(\partial_T v - \partial_S v) + F(\partial_1\lambda + v\partial_u\lambda + D\lambda\partial_Y v + 2D^{n+1}\lambda\partial_S v) - \partial_S F + \partial_\eta\lambda\partial_Y F. \qquad \square$$

Proof. We have

$$\partial_T F = \partial_T^2\phi - \partial_{ST}^2\phi = \partial_T[\lambda] - 2\partial_{ST}^2\phi = 2\partial_S^2\phi + \partial_T[\lambda] - 2\partial_S[\lambda].$$

We express $\partial_T[\lambda]$ using the first formula in Lemma 5.1, noting that

$$D\lambda B_{SY} + 2D^{n+1}B_{SS} = \partial_S[\lambda] - \Lambda\partial_S v - \partial_\eta\lambda\partial_{SY}^2\phi - \partial_1\lambda\partial_S\phi - \tfrac{1}{2}\partial_t\lambda - \partial_u\lambda\partial_S w.$$

We obtain

$$\partial_T F = \Lambda(\partial_T v - \partial_S v) - \partial_S F + \partial_\eta \lambda \partial_y F + (\lambda - 2\partial_S \phi)(D\lambda \partial_y v + 2D^{n+1}\lambda \partial_S v)$$
$$+ \partial_1 \lambda(\partial_T \phi - \partial_S \phi) + \partial_u \lambda(\partial_T w - \partial_S w),$$

which gives the formula of the lemma, using the equations on w and ϕ. ■

6 Proof of Theorem I.1

Lemma 6.1. In the genuinely nonlinear case, we can choose the initial values of v, w, ϕ in such a way that, at the origin,

$$\partial_s \phi = 0, \quad \partial^2_{st}\phi < 0, \quad \partial^2_{yy}\phi = 0, \quad \partial^2_{ss}\phi = \partial^2_{sy}\phi = 0, \quad \mathrm{Hess}_{s,y}(\partial_s \phi) \gg 0.$$ □

Proof. (a) We want first to ensure $\partial_s F = \partial_y F = 0$. In the new coordinates, this means $\partial_T F = \partial_S F$ and $\partial_y F = 0$. According to Lemma 5.4, this is equivalent to $\partial_y F = 0$ and $2\partial_S F = (\partial_T v - \partial_S v)\Lambda$. Taking Lemma 5.1 into account, we can choose the blocks B_{SS}, B_{Sy} and B_{yy} arbitrarily, impose $\partial^2_y \phi = 0$, and then choose $\partial^2_S \phi$ and $\partial^2_{Sy}\phi$ appropriately.

We have then $\partial_t F = \partial_T F + \partial_S F = (\partial_T v - \partial_S v)\Lambda$. In the genuinely nonlinear case, we have $\Lambda \neq 0$. Using Lemma 5.2, we can choose $\partial_y v$ arbitrarily and then choose $\partial_S v$ to obtain $\partial_t F < 0$.

(b) We now compute the hessian of F with respect to (s, y) in the new variables, using the equation on F that we write here for simplicity $\partial_T F = g + aF - \partial_S F + b\partial_y F$:

$$\partial^2_{ST} F = \partial_S g + a\partial_S F - \partial^2_S F + b\partial^2_{Sy} F, \; \partial^2_{Ty} F = \partial_y g - \partial^2_{Sy} F + b\partial^2_y F,$$
$$\partial^2_T F = \partial_T g + a\partial_T F - \partial^2_{ST} F + b\partial^2_{Ty} F = \partial_T g - \partial_S g + b\partial_y g + \partial^2_S F - 2b\partial^2_{Sy} F + b^2 \partial^2_{yy} F.$$

From this we deduce

$$\partial^2_{ss} F = E + 4\partial^2_S F - 4b\partial^2_{Sy} F + b^2 \partial^2_{yy} F,$$
$$E = \partial_T g - 3\partial_S g + b\partial_y g - ag, \; \partial_{sy} F = \partial_y g - 2\partial^2_{Sy} F + b\partial^2_{yy} F.$$

Hence, with arbitrary A, B, the partial hessian of F is

$$H = A^2\partial_s^2 F + 2AB\partial_{sY}^2 F + B^2\partial_{YY}^2 F = A^2 E + 2AB\partial_Y g + H_0,$$
$$H_0 = 4A^2\partial_S^2 F - 4A\partial_{SY}^2 F(B+bA) + \partial_{YY}^2 F(B+bA)^2.$$

Setting $C = B + bA$, we obtain

$$H = A^2(E - 2b\partial_Y g) + 2AC\,\partial_Y g + \mathrm{Hess}_{S,Y}F(-2A, C).$$

It remains to compute $\mathrm{Hess}_{S,Y}F$:

$$\partial_S^2[\lambda] = \Lambda\partial_S^2 v + D\lambda B_{SSY} + 2D^{n+1}\lambda B_{SSS} + \partial_\eta\lambda\partial_{SSY}^3\phi + \cdots\ ,$$
$$\partial_{SY}^2[\lambda] = \Lambda\partial_{SY}^2 v + D\lambda B_{SYY} + 2D^{n+1}\lambda B_{SSY} + \partial_\eta\lambda\partial_{SYY}^3\phi + \cdots\ ,$$
$$\partial_{YY}^2[\lambda] = \Lambda\partial_{YY}^2 v + D\lambda B_{YYY} + 2D^{n+1}\lambda B_{SYY} + \partial_\eta\lambda\partial_Y^3\phi + \cdots\ .$$

and, according to Lemma 5.4

$$\partial_S^2 F = \partial_S^2[\lambda] - 2\partial_S^3\phi, \quad \partial_{SY}^2 F = \partial_{SY}^2[\lambda] - 2\partial_{SSY}^3\phi, \quad \partial_{YY}^2 F = \partial_{YY}^2[\lambda] - 2\partial_{SYY}^3\phi.$$

Here, the dots denote terms involving at most derivatives of order 1 of v and derivatives of order 2 of w, ϕ; we also have set (subscripts do not denote derivatives!)

$$B_{SSS} = \partial_S^3 w - v\partial_S^3\phi, \quad B_{SSY} = \partial_{SSY}^3 w - v\partial_{SSY}^3\phi, \text{ etc.}$$

We discuss now the terms in H involving first-order derivatives of $g = \Lambda(\partial_T v - \partial_S v)$: the residual equation written in Lemma 5.2 shows that second-order derivatives of v depend only on the third-order blocks B_{SSS}, B_{SSY}, etc. and on terms involving at most first-order derivatives of v and second-order derivatives of w, ϕ. We proceed as follows: we choose arbitrarily the third-order blocks and the tangential second-order derivatives $\partial_S^2 v$, $\partial_{SY}^2 v$, $\partial_{YY}^2 v$ of v. At this point, the quadratic form $A^2(E - 2b\partial_Y g) + 2AC\,\partial_Y g$ in H is fixed. Now we choose the third-order derivatives $\partial_S^3\phi$, $\partial_{SSY}^3\phi$ and $\partial_{SYY}^3\phi$ to get $\mathrm{Hess}_{S,Y}F \geq \alpha$. If α is chosen big enough positive, H will be positive definite in the original variables A, B. ■

To summarize the preceding results, one can describe the behavior of $j=(\partial_s\phi)(\Phi^{-1})$ by recalling the fundamental properties of ϕ:

$$\Phi(s,y,t)=(\phi(s,y,t),y,t),$$

$$\phi(0)=0,\quad (\partial_s\phi)(0)=0,\quad (\partial_t\phi)(0)=\underline{\tau},\quad (\partial_y\phi)(0)=\underline{\eta},$$

$$(\partial_s^2\phi)(0)=0,\quad (\partial_{sy}^2\phi)(0)=0,\quad (\partial_{st}^2\phi)(0)<0,\quad (\partial_{yy}^2\phi(0))=0,$$

$$\mathrm{Hess}_{s,y}(\partial_s\phi)(0)\gg 0.$$

According to Lemma 6.1, the surface $S=\{(s,y,t),\ \partial_s\phi(s,y,t)=0\}$ is given, for $t\geq 0$, by an equation $t=\psi(s,y)$, with, at the origin,

$$\partial_s\psi=\partial_y\psi=0,\quad \mathrm{Hess}\,\psi\gg 0.$$

Now, at the origin, since we chose $\partial_y^2\phi=0$,

$$\phi(s,y,t)=y(\partial_y\phi)+t(\partial_t\phi)+O(|s|\,|t|+|y|\,|t|+t^2+|s|^3+|y|^3).$$

For (y_0,t_0) given close to $(0,0)$, either the line $\{y=y_0,t=t_0\}$ does not meet S, or it meets S at exactly two points $s_-(y_0,t_0)\leq s_+(y_0,t_0)$ (equality occurs if and only if (y_0,t_0) belongs to the apparent contour of S). In this later case, $|s_\pm|+|y_0|=O(t_0^{1/2})$. Hence, the value $x=\phi(s_-,y_0,t_0)$ satisfies $x=y_0(\partial_y\phi)+t_0(\partial_t\phi)+O(t_0^{3/2})$. This shows that the image $\Phi(\{\partial_s\phi=0\})$ is contained in the region $x_1>y\underline{\eta}+\mu t$, except for the origin. To obtain the more precise estimate of j in Theorem I.1, we write

$$\phi(s,y,t)=y\underline{\eta}+t(\underline{\tau}+o(1))+s(h(s,y)+O(|s|^3+|y|^3))+O(y^3),$$

where h is a positive definite quadratic form. Hence, the condition $x_1\leq y\underline{\eta}+\mu t$ implies

$$s(s^2+|y|^2)\leq -\alpha t+\beta|y|^3$$

for some constants $\alpha>0$ and β. If $2\beta|y|^3\leq\alpha t$, then s is negative and

$$\alpha t\leq 2|s|(s^2+y^2)\leq 2(s^2+y^2)^{3/2},$$

which implies $\partial_s\phi \geq Ct^{2/3}$. If $2\beta|y|^3 \geq \alpha t$, then $|y|^2 \geq Ct^{2/3}$ and also $\partial_s\phi \geq Ct^{2/3}$.

7 Proof of Theorem I.2

First, we start with a simple lemma.

Lemma 7.1. Let F be a C^2 function satisfying the differential equation

$$ZF = g + cF.$$

We assume here $Z = \partial_T + \partial_S + b\partial_Y$, and, at the origin,

$$g = 0, \quad F = 0, \quad \partial_S F = \partial_Y F = 0, \quad \text{Hess}_{S,Y} F \geq \alpha id > 0.$$

Then a necessary condition for Hess F to be positive definite is $Zg > 0$. This condition is sufficient if α is big enough with respect to $Zg(0)$, $(\partial_S g)(0)$, $(\partial_Y g)(0)$. □

Proof. The necessity is obvious. At the origin, the complete hessian of F, evaluated on (ξ, η, τ) is

$$H = \tau^2\partial_T^2 F + 2\tau\xi\partial_{TS}^2 F + 2\tau\eta\partial_{TY}^2 F + H_0(\xi, \eta),$$

where H_0 is the partial hessian $\text{Hess}_{SY} F$. This gives

$$\begin{aligned} H &= \tau^2(\partial_T g - \partial_S g + b\partial_Y g) + 2\tau_x i\partial_S g + 2\tau\eta\partial_Y g + H_0(\xi - \tau, \eta + b\tau) \\ &= \tau^2 Zg + 2\partial_S g\tau(\xi - \tau) + 2\partial_Y g\tau(\eta + b\tau) + H_0(\xi - \tau, \eta + b\tau). \end{aligned}$$

This implies the desired conclusion. ■

Secondly, we apply this lemma to the equation on F given in Lemma 5.4, with $g = \Lambda(\partial_T v - \partial_S v)$, since in the weakly nonlinear case, we have $\Lambda = 0$. It remains to check $(\partial_{\tau}^{-} p)Z\Lambda = Z[q]$.

Lemma 7.2. With $Z = \partial_T + \partial_S - (\partial_\eta\lambda)\partial_Y$,

$$Z[q] = \underline{C} + 2B_{SS}C^{n+1,n+1} + 2B_{SY_i}C^{i,n+1} + \tfrac{1}{2}B_{Y_iY_j}C^{ij}.$$ □

Proof. The exact quantity we have to derive is

$$[q] = q(\phi, y, (S+T)/2, w, v, B_Y, 2B_S, -1, \partial_Y\phi, [\lambda]), \quad B_S = \partial_S w - v\partial_S\phi, \quad B_Y = \partial_Y w - v\partial_Y\phi.$$

Hence

$$Z[q] = \partial_1 q Z\phi + (\partial_y q)Zy + \partial_t q + \partial_u q Zw + D^1 qZv$$
$$+ D^i qZB_{Y_i} + 2D^{n+1}qZB_S + (\partial_\eta q)Z(\partial_Y\phi) + \partial_\tau qZ[\lambda].$$

Since λ is homogeneous in ξ of degree 1,

$$Z\phi = \partial_T\phi + \partial_S\phi - \partial_\eta\lambda\partial_Y\phi = \lambda - (\partial_Y\phi)\partial_\eta\lambda = -\partial_\xi^1\lambda.$$

Using the equation on w

$$Zw = 2\partial_S w - (\partial_\eta\lambda)\partial_Y w,$$
$$Z(\partial_Y w) = 2\partial^2_{SY} w - (\partial_\eta\lambda)\partial^2_{YY} w,$$
$$Z(\partial_S w) = 2\partial^2_S w - (\partial_\eta\lambda)\partial^2_{SY} w.$$

Similarly,

$$Z(\partial_S\phi) = 2\partial^2_S\phi - (\partial_\eta\lambda)\partial^2_{SY}\phi,$$
$$Z(\partial_Y\phi) = 2\partial^2_{SY}\phi - (\partial_\eta\lambda)\partial^2_{YY}\phi.$$

This gives us

$$ZB_S = 2B_{SS} - (\partial_\eta\lambda)B_{SY} - (\partial_S\phi)Zv,$$
$$ZB_Y = 2B_{SY} - (\partial_\eta\lambda)B_{YY} - (\partial_Y\phi)Zv.$$

We can alternatively write

$$Z(\partial_Y\phi) = \partial_Y[\lambda] - \partial_\eta\lambda\partial^2_{YY}\phi = \partial_1\lambda\partial_Y\phi + \partial_Y\lambda + \partial_u\lambda\partial_Y w + D^i\lambda B_{YY_i} + 2D^{n+1}\lambda B_{SY}.$$

Finally, we need

$$\begin{aligned}Z[\lambda] &= \partial_T[\lambda] + \partial_S[\lambda] - (\partial_\eta\lambda)\partial_Y[\lambda]\\ &= \partial_1\lambda\partial_S\phi + \tfrac{1}{2}\partial_t\lambda + (\partial_u\lambda)\partial_S w + D^i\lambda B_{SY_i} + 2D^{n+1}\lambda B_{SS} + \partial_\eta\lambda\partial^2_{SY}\phi\\ &\quad + \partial_1\lambda\partial_T\phi + \tfrac{1}{2}\partial_t\lambda + \partial_u\lambda\partial_T w + D^i\lambda B_{SY_i} + 2D^{n+1}\lambda B_{SS} - (\partial_\eta\lambda)\partial^2_{SY}\phi\\ &= \lambda\partial_1\lambda + \partial_t\lambda + 2\partial_u\lambda\partial_S w + 2D^i\lambda B_{SY_i} + 4D^{n+1}\lambda B_{SS}.\end{aligned}$$

Using all these expressions in $Z[q]$, and using the alternative version of $Z(\partial_Y\phi)$ for the coefficient of $\partial_\eta q$, we get

$$\begin{aligned}E &= -\partial^1_\xi\lambda\partial_1 q + (\partial_1\lambda)(\partial_Y\phi)\partial_\eta q + (\partial_Y\lambda)\partial_\eta q - \partial_Y q\partial_\eta\lambda + \partial_t q + (\partial_u\lambda)(\partial_\eta q)\partial_Y w\\ &\quad + (2\partial_S w - \partial_\eta\lambda\partial_Y w)\partial_u q + \partial_\tau q(\lambda\partial_1\lambda + \partial_t\lambda + 2\partial_S w\partial_u\lambda),\\ Z[q] &= E + (Zv)(D^1 q - \partial_{Y_i}\phi D^i q - 2\partial_S\phi D^{n+1}q) + 4B_{SS}(D^{n+1}q + D^{n+1}\lambda\partial_\tau q)\\ &\quad + 2B_{SY_i}(D^i q + D^i\lambda\partial_\tau q + D^{n+1}\lambda\partial^i_\eta q - D^{n+1}q\partial^i_\eta\lambda) + B_{Y_iY_j}(D^i\lambda\partial^j_\eta q - D^i q\partial^j_\eta\lambda).\end{aligned}$$

Recall now that the values of v, B_Y and B_S have been chosen in such a way that

$$v = U_1, \quad B_{Y_i} = U_i, \quad 2B_S = U_{n+1}, \quad \partial_Y\phi = \eta, \quad \partial_S\phi = \lambda/2.$$

This gives us

$$\partial_{Y_i} w = U_i + U_1\eta_i, \quad 2\partial_S w = U_{n+1} + \lambda U_1.$$

On the other hand, since q is homogeneous in ξ of degree 3,

$$\lambda\partial_\tau q + \partial_Y\phi\partial_\eta q - \partial^1_\xi q = 3q = 0.$$

Hence we can write

$$\begin{aligned}E &= -\partial^1_\xi\lambda\partial_1 q + \partial_1\lambda\partial^1_\xi q + \partial_Y\lambda\partial_\eta q - \partial_Y q\partial_\eta\lambda + \partial_t q + \partial_t\lambda\partial_\tau q + G,\\ G &= (\partial_u\lambda)(U\partial_\eta q + U_{n+1}\partial_\tau q + U_1\partial^1_\xi q) + (\partial_u q)(U_{n+1} + \lambda U_1 - \partial^i_\eta\lambda(U_i + U_1\eta_i)).\end{aligned}$$

Since $\lambda - \eta_i \partial^i_\eta \lambda = -\partial^1_\xi \lambda$, we get finally

$$G = (\partial_u \lambda)(U \partial_\xi q) + (\partial_u q)(U_{n+1} - U_i \partial^i_\eta \lambda - U_1 \partial^1_\xi \lambda).$$

To compute Zv, we use the equation on v, which gives

$$(\partial_{\bar{\tau}} p) Z v = 4 B_{SS} p^{n+1,n+1} + 4 B_{SY_i} p^{i,n+1} + p^{ij} B_{Y_i Y_j} + r.$$

It remains now to multiply $Z[q]$ by $(\partial_{\bar{\tau}} p)$, and use that, for any derivative ∂,

$$(\partial_\tau p)\partial\lambda = -\partial p.$$

We obtain

$$(\partial_\tau p) G = -(\partial_u p)(U \partial_\xi q) + (\partial_u q)(U_{n+1} \partial_\tau p + U_i \partial^i_\eta p + U_1 \partial^1_\xi p) = (\partial_u q)(U \partial_\xi p) - (\partial_u p)(U \partial_\xi q).$$

The remaining terms in E become, after multiplication by $\partial_\tau p$

$$\partial_1 q \partial^1_\xi p - \partial_1 p \partial^1_\xi q + (\partial_t q)(\partial_\tau p) - (\partial_t p)(\partial_\tau q) + \partial_y q \partial_\eta p - \partial_y p \partial_\eta q = \{p, q\}.$$

Distributing the terms of Zv, we get the desired formula. ∎

Finally, according to Lemma 3.7, since $\xi_1 = -1$,

$$C^{1i} = \sum_{j \geq 2} \xi_j C^{ij},$$

hence

$$C^{11} = \sum_{j \geq 2} \xi_j C^{1j} = \sum_{2 \leq i, j} \xi_i \xi_j C^{ij}.$$

This shows that if the coefficients $\underline{C}, C^{ij}$ are not all zero, then the coefficients $\underline{C}, C^{ij}$ for $2 \leq i, j$ cannot be all zero either. Hence the formula of Lemma 7.2 allows us to choose B_{SS}, B_{Sy}, and B_{yy} such that $Z[q] \neq 0$. Hence, $Z\Lambda \neq 0$ and it is possible to adjust the sign of $\partial_T v - \partial_S v$ to obtain $Zg > 0$. Since we know from the proof of Lemma 6.1 that it is possible to choose the third-order derivatives of ϕ to get $\mathrm{Hess}_{Sy} F$ as big as we want, we finally obtain $\mathrm{Hess}\, F \gg 0$.

To summarize the preceding results, one can describe the behavior of $j=(\partial_s\phi)(\Phi^{-1})$ by recalling the fundamental properties of ϕ:

$$\Phi(s,y,t)=(\phi(s,y,t),y,t),$$

$$\phi(0)=0,\quad (\partial_s\phi)(0)=0,\quad (\partial_t\phi)(0)=\underline{\tau},\quad (\partial_y\phi)(0)=\underline{\eta},$$

$$\partial(\partial_s\phi)(0)=0,\quad \mathrm{Hess}(\partial_s\phi)(0)\gg 0.$$

To prove $j\geq C|x|^2$ is to prove

$$\partial_s\phi(s,y,t)\geq C(\phi^2(s,y,t)+y^2+t^2).$$

But $\phi(s,y,t)=O(|s|+|y|+|t|)$, so this is obvious.

Part II: Blowup at Infinity for Second-Order Equations

We use here the results of Part I to discuss blowup at infinity for an equation in $\mathbf{R}^3_x\times[0,+\infty[$ (with $x_0=t$)

$$\Box u+Q(u)=0,\quad Q(u)=Q_2(u)+Q_3(u)+Q_4(u),$$

where

$$Q_2(u)=g_2^{\alpha\beta\gamma}(\partial_\gamma u)(\partial^2_{\alpha\beta}u),$$

$$Q_3(u)=g_3^{\alpha\beta\gamma\delta}(\partial_\gamma u)(\partial_\delta u)(\partial^2_{\alpha\beta}u),$$

$$Q_4(u)=g_4^{\alpha\beta\gamma\delta\epsilon}(\partial_\gamma u)(\partial_\delta u)(\partial_\epsilon u)(\partial^2_{\alpha\beta}u).$$

Here, the coefficients g are real constants with the obvious symmetries in the indices, and

$$g_2^{00\gamma}=g_3^{00\gamma\delta}=g_4^{00\gamma\delta\epsilon}=0.$$

This last condition means that we normalized the coefficient of ∂_t^2 to be one. We could have included higher order $Q_i(u)$ as well ($i\geq 5$), but these play no role in the discussion of blowup at infinity. We assume that g_2 (and only g_2) satisfies the null condition

$$\xi_0^2 = \sum \xi_i^2 \Rightarrow g_2(\xi) \equiv g_2^{\alpha\beta\gamma} \xi_\alpha \xi_\beta \xi_\gamma = 0.$$

For $X = (X_0 = T, X_1, X_2, X_3)$ we define the functions

$$g_3(X) = g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma X_\delta, \quad g_4(X) = g_4^{\alpha\beta\gamma\delta\epsilon} X_\alpha X_\beta X_\gamma X_\delta X_\epsilon.$$

8 Some Results and Notations from [8]

Let u_0 and u_1 be two C_0^∞ functions of x vanishing for $|x| \geq M$. We take $t_0 > 2M$ and consider the Cauchy problem

$$\Box u + Q(u) = 0, \quad u(x, t_0) = u_0(x), \quad (\partial_t u)(x, t_0) = u_1(x).$$

The solution vanishes for $r \geq M + t - t_0$, hence its support is contained in

$$\Gamma = \{(x, t) \in \mathbf{R}^4,\ t^2 > |x|^2\}.$$

We use in Γ the conformal inversion I defined by

$$X = \frac{x}{t^2 - r^2}, \quad T = -\frac{t}{t^2 - r^2}.$$

Note that $I^2 = id$ and $I(\Gamma) = \Gamma$. We set

$$K_0 = 2TS - \Delta\partial_T, \quad K_i = 2X_i S + \Delta\partial_i, \quad \Delta = T^2 - R^2, \quad X_0 = T, \quad S = T\partial_T + \sum X_i \partial_i.$$

The fields K_0, K_i are the image under I of ∂_t, ∂_i. To a function $u = u(x, t)$ we associate the function $v = v(X, T)$ defined by

$$\tilde{u}(X, T) \equiv u(I)(X, T) = (T^2 - R^2) v(X, T).$$

We have then the relations

$$(\partial_\alpha u)(I) = (T^2 - R^2)\bar{K}_\alpha v,\, \bar{K}_\alpha = K_\alpha + 2X_\alpha,$$
$$\Box v = (T^2 - R^2)^{-3}(\Box u)(I).$$

The function u being the solution of the above Cauchy problem, the function v is the C^∞ solution in $\mathcal{T} = \{(X, T), T_0 \leq T < 0, |X| < |T|\}$ of the Cauchy problem

$$\Box v + \bar{Q}(v) = 0, \quad v(X, T_0) = v_0(X), \quad (\partial_T v)(X, T_0) = v_1(X), \quad T_0 = -1/t_0.$$

Here, $\bar{Q}$ is a nonlinear expression that we shall compute, and v_0, v_1 are C^∞ functions compactly supported in $|X| < |T_0|$.

The fundamental question of [8] is this: is v smooth up to the boundary of $\mathcal{T}$? If this is the case, we say that no blowup at infinity occurs for u, and that u has a free representation. In this case, u is also stable. If v is singular at some point $m_0 = (X_0, T_0)$, $|X_0| = -T_0$ of $\partial\mathcal{T}$, we say that u blows up at infinity; in this case, it seems that u is not stable, though this is only a conjecture.

Now arises the question: is it possible for v to have such a singularity on $\partial\mathcal{T}$? It could be, in fact, that, because of the special structure of the new equation on v that we shall examine soon, the smoothness of the data implies automatically $v \in C^\infty(\bar{\mathcal{T}})$. This is the case, for instance, for the very special example

$$\Box u + (\partial_t u)^2 - \sum (\partial_i u)^2 = 0,$$

as discussed in [8, Section 5.1].

To answer this question, we will use the results of Part I to construct explicitly solutions v which blow up at the boundary of $\mathcal{T}$.

9 The Equation on v

Lemma 9.1. The new equation on v is

$$\Box v + \bar{Q}(v) = 0, \quad \bar{Q}(v) = \bar{Q}_2(v) + \bar{Q}_3(v) + \bar{Q}_4(v),$$

where the nonlinear expressions $\bar{Q}_i$ are

$$\begin{aligned}
\bar{Q}_2(v) &= 4[2A(X)(Sv+v) + Y_1 v]S^2 v + 4(Sv+v)Y_2 Sv \\
&\quad + 2\Delta(Sv+v)(g_2^{\alpha\beta\gamma}\epsilon_\alpha\epsilon_\beta X_\gamma \partial^2_{\alpha\beta} v) + 4\Delta Y_3 Sv + O(\Delta^2) + r_2(X, v, \partial v), \\
\bar{Q}_3(v) &= 16 g_3(X)(Sv+v)^2(S^2 v + 3Sv + 2v) \\
&\quad + 16\Delta(Sv+v)(S^2 v + 3Sv + 2v)[g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \partial_\delta v] \\
&\quad + 8\Delta(Sv+v)^2[g_3^{\alpha\beta\gamma\delta} X_\gamma X_\delta (Y_{\alpha\beta} Sv + Y_{\alpha\beta} v + \epsilon_\alpha \delta_{\alpha\beta}(Sv+v))] + O(\Delta^2), \\
\bar{Q}_4(v) &= 32\Delta g_4(X)(Sv+v)^3(S^2 v + 3Sv + 2v) + O(\Delta^2).
\end{aligned}$$

Here, we have used the notation from [8]: ϵ_α is -1 for $\alpha = 0$ and 1 otherwise, and

$$A(X) = 2g_2^{0i0} X_i, \quad Y_1 = g_2^{\alpha\beta\gamma} X_\alpha X_\beta \epsilon_\gamma \partial_\gamma,$$
$$Y_2 = 2g_2^{\alpha\beta\gamma} \epsilon_\alpha X_\beta X_\gamma \partial_\alpha, \quad Y_3 = g_2^{\alpha\beta\gamma} \epsilon_\alpha \epsilon_\gamma X_\beta (\partial_\gamma v)\partial_\alpha.$$

It is proved in [8] that the fields Y_1 and Y_2 are tangent to the boundary $\{|X| = -T\}$. □

Proof.

(a) The formula for $\bar{Q}_2$ is taken from [8].

(b) Since $\partial_\alpha u$ is transformed into $\Delta \bar{K}_\alpha v$, $\partial^2_{\alpha\beta} u$ is transformed into $\Delta \bar{K}_\alpha \bar{K}_\beta v$. Hence $Q_3(u)$ is transformed into

$$\Delta^3 g_3^{\alpha\beta\gamma\delta} (\bar{K}_\gamma v)(\bar{K}_\delta v)\bar{K}_\alpha \bar{K}_\beta v.$$

Since

$$\begin{aligned}
\bar{K}_\alpha \bar{K}_\beta v &= 4X_\alpha X_\beta (S^2 v + 3Sv + 2v) + 2\Delta(Y_{\alpha\beta} Sv + Y_{\alpha\beta} v + \epsilon_\alpha \delta_{\alpha\beta}(Sv+v)) \\
&\quad + O(\Delta^2), \; Y_{\alpha\beta} = \epsilon_\alpha X_\beta \partial_\alpha + \epsilon_\beta X_\alpha \partial_\beta,
\end{aligned}$$

we obtain the desired formula for $\bar{Q}_3$ and $\bar{Q}_4$. ■

Let p_2, p_3, p_4 be the principal symbols corresponding to the expressions $\bar{Q}_2, \bar{Q}_3, \bar{Q}_4$, so that the principal symbol p of the equation on v is $p = \tau^2 - |\xi|^2 + p_2 +$

$p_3 + p_4$. The first-order terms corresponding to the expressions $\bar{Q}_2$, $\bar{Q}_3$, $\bar{Q}_4$ are denoted by r_2, r_3, r_4, so that the first-order terms r in the equation for v are $r = r_2 + r_3 + r_4$. According to the general theory of Part I, we note $q_i = \xi D p_i$, so that $q = q_2 + q_3 + q_4$. In the following lemma, which gives the expressions of all these symbols, we keep ∂v instead of V for clarity.

Lemma 9.2. With $\tau = \xi_0$ as usual and $s = \sum X_\alpha \xi_\alpha$, y_2 and y_3 the symbols, respectively, of S, Y_2 and Y_3,

$$p_2 = 4[2A(X)(Sv + v) + Y_1 v]s^2 + 4(Sv + v)y_2 s$$
$$+ 2\Delta(Sv + v)(g_2^{\alpha\beta\gamma}\epsilon_\alpha\epsilon_\beta X_\gamma \xi_\alpha\xi_\beta) + 4\Delta y_3 s + O(\Delta^2),$$
$$p_3 = 16g_3(X)(Sv + v)^2 s^2 + 16\Delta(Sv + v)s^2[g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \partial_\delta v]$$
$$+ 16\Delta(Sv + v)^2 s[g_3^{\alpha\beta\gamma\delta}\epsilon_\alpha\xi_\alpha X_\beta X_\gamma X_\delta] + O(\Delta^2),$$
$$p_4 = 32\Delta g_4(X)(Sv + v)^3 s^2 + O(\Delta^2),$$
$$q_2 = \Delta\tilde{q}_2,\ \tilde{q}_2 = 2A(X)s(\tau^2 - \sum \xi_i^2) + O(\Delta),$$
$$q_3 = 32g_3(X)(Sv + v)s^3 + \Delta\tilde{q}_3,\ \tilde{q}_3 = 16s^3(g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \partial_\delta v)$$
$$+ 16s^2(Sv + v)(g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \xi_\delta) + 32s^2(Sv + v)(g_3^{\alpha\beta\gamma\delta}\epsilon_\alpha\xi_\alpha X_\beta X_\gamma X_\delta) + O(\Delta),$$
$$q_4 = \Delta\tilde{q}_4,\ \tilde{q}_4 = 96g_4(X)(Sv + v)^2 s^3 + O(\Delta).$$

Note in particular that the boundary $\{|X| = -T\}$ is characteristic for the linearized operator on v. □

Proof. The formulas for p_i follow immediately from the expressions of $\bar{Q}_i$ given in Lemma 3.2.

(a) From the expression for p_2, we get

$$q_2 = 8A(X)s^3 + 4s^2(y_1 + y_2) + 2\Delta s(g_2^{\alpha\beta\gamma}\epsilon_\alpha\epsilon_\beta X_\gamma \xi_\alpha\xi_\beta) + 4\Delta s(g_2^{\alpha\beta\gamma}\epsilon_\alpha\xi_\alpha X_\beta \epsilon_\gamma \xi_\gamma) + O(\Delta^2).$$

The computation of $y_1 + y_2$ has been done in [8] on $R = -T$. We do it here in general:

$$y_1 + y_2 = g_2^{\alpha\beta\gamma} X_\alpha X_\beta \epsilon_\gamma \xi_\gamma + 2 g_2^{\gamma\alpha\beta} X_\alpha X_\beta \epsilon_\gamma \xi_\gamma$$
$$= -\tau (g_2^{\alpha\beta 0} X_\alpha X_\beta + 2 g_2^{0\alpha\beta} X_\alpha X_\beta) + \xi_i (g_2^{\alpha\beta i} X_\alpha X_\beta + 2 g_2^{i\alpha\beta} X_\alpha X_\beta).$$

The coefficient of $-\tau$ is, using [8, Lemma 1.2],

$$4 g_2^{0i0} T X_i + (g_2^{ij0} + 2 g_2^{0ij}) X_i X_j = 2T A(X).$$

The second sum is

$$A(\xi)(T^2 - R^2) - 2A(X)(X \cdot \xi) + 2T[g_2^{ij0} \xi_i X_j + (g_2^{0ij} + g_2^{0ji}) \xi_i X_j]$$
$$= A(\xi)(T^2 - R^2) - 2A(X)(X \cdot \xi).$$

Hence

$$y_1 + y_2 = -2A(X)s + \Delta A(\xi).$$

Now

$$g_2^{\alpha\beta\gamma} \epsilon_\alpha \xi_\alpha (\epsilon_\beta \xi_\beta X_\gamma + 2 X_\beta \epsilon_\gamma \xi_\gamma) = -2T\tau A(\xi) + \tau^2 A(X) - 2\tau [g_2^{ij0} \xi_i X_j + (g_2^{0ij} + g_2^{0ji}) \xi_i X_j]$$
$$+ T(g_2^{ij0} \xi_i \xi_j + 2 g_2^{0ij} \xi_i \xi_j) + g_2^{ijk} \xi_i \xi_j X_k + 2 g_2^{ijk} \xi_i \xi_k X_j$$
$$= -2A(\xi)s + A(X)\left(\tau^2 - \sum \xi_i^2\right).$$

This gives the formula for $\tilde{q}_2$.

(b) The formula for $\tilde{q}_3$ and $\tilde{q}_4$ follow easily from the expressions of p_3 and p_4. ■

10 Main Results

We first give the blowup results for the equation on v, as explained in Section 9. We then comment how these translate to give information about the equation on u.

10.1 Blowup for the v-equation

In all following theorems and corollaries, we fix a point $m^0 = (X^0, T^0)$ of $\partial\mathcal{T}$ with

$$R^0 + T^0 = 0, \quad T^0 < 0, \quad X_1^0 \neq 0.$$

The constraint $X_1^0 \neq 0$ is purely technical, and correspond to the fact that we choose to have X_1 play a special role in the blowup constructions.

Theorem 10.1 (genuinely nonlinear case). Assume that there exists a point $\underline{m} = (m^0, \underline{v}, \underline{V}, \underline{\xi}, \underline{\tau})$ such that $p(\underline{m}) = 0$ and $g_3(m^0)(Sv + v)s \neq 0$ at $\underline{m}$. Then one can construct a solution v of the equation $\Box v + \bar{Q}(v) = 0$ in a neighborhood V of m^0 in $\{|X| \leq -T\}$, so that

(i) $v \in C^1(V)$, and $v \in C^\infty$ away from m^0,
(ii) the point m^0 is a hyperbolic blowup point for v,
(iii) $v'' = S/j + R$, where S is symmetric of rank one, and S and R are C^∞. The function j vanishes only at m^0, and $j \geq C|T^0 - T|$ for $T \leq T^0$ while $j \geq C|T^0 - T|^{2/3}$ for $T \geq T^0$. □

Remark 10.2. The expression "m^0 is a hyperbolic blowup point for v" means that

(i) the linearized operator on v is strictly hyperbolic with respect to T,
(ii) m^0 belongs to a domain of determination of a compact portion of some hypersurface $\{T = T^1\}$ $(T^1 < T^0)$ on which the Cauchy data of v are C^∞.

This theorem follows from an application of Theorem I.1 to the equation $\Box v + \bar{Q}(v) = 0$ at the point m^0. The precise description of the way j vanishes at m^0 follows from the proof of Theorem I.1. □

Theorem 10.3 (weakly nonlinear case). Assume $g_3(m^0) = 0$. Set

$$c = 8A(X)s^2(\tau^2 - |\xi|^2) + 128s^4(Sv + v)^2 Y_2(g_3) + 64s^4(g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \partial_\delta v)$$
$$+ 64s^3(Sv + v)(g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \xi_\delta) + 384s^4 g_4(Sv + v)^2$$

and assume that there exists a point $\underline{m} = (m^0, \underline{v}, \underline{V}, \underline{\xi}, \underline{\tau})$ such that $p(\underline{m}) = 0$ and $c \neq 0$ at $\underline{m}$.

Then one can construct a solution v of $\Box v + \bar{Q}(v) = 0$ in a neighborhood V of m^0 in $\{|X| \leq -T\}$, so that

(i) $v \in C^1(V)$, and $v \in C^\infty$ away from m^0,
(ii) the point m^0 is a hyperbolic blowup point for v,
(iii) $v'' = S/j + R$, where S is symmetric of rank one, and S and R are C^∞. For some $C > 0$, d being the distance to m^0, $j \geq C d^2$ and $j \sim C d^2$ on some curve reaching the origin. □

Corollary 10.4. Assume only quadratic terms are present ($Q_3 = Q_4 = 0$) in the u-equations. Assume that there exists a point $\underline{m} = (m^0, \underline{v}, \underline{V}, \underline{\xi}, \underline{\tau})$ such that $p(\underline{m}) = 0$ and $A(m^0) s(\tau^2 - |\xi|^2) \neq 0$ at $\underline{m}$. Then the conclusion of Theorem 10.3 holds. □

Corollary 10.5. Assume $g_3(m^0) \neq 0$. Then there exists a point $\underline{m} = (m^0, \underline{v}, \underline{V}, \underline{\xi}, \underline{\tau})$ satisfying the assumptions of Theorem 10.1. □

Corollary 10.6. Assume $g_3(m^0) = 0$. Assume that, at m^0, the fields Y_1 and $Y_4 \equiv g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \partial_\delta$ are independent. Then there exists a point $\underline{m} = (m^0, \underline{v}, \underline{V}, \underline{\xi}, \underline{\tau})$ satisfying the assumptions of Theorem 10.3. □

Corollary 10.7. Assume only quadratic terms are present in the u-equations. Assume $A(m^0) \neq 0$. Then there exists a point $\underline{m} = (m^0, \underline{v}, \underline{V}, \underline{\xi}, \underline{\tau})$ satisfying the assumptions of Corollary 10.4. □

10.2 Blowup at infinity for the u-equation

It is tedious to completely translate the results of Section 10.1 about v into results about u. However, we will translate the assumptions of Corollary 10.5–10.7 and describe roughly the corresponding behavior of u at infinity.

First, for given ω^0, σ^0, let us define the "point at infinity" m^0_∞ in x-polar coordinates as

$$\omega = \omega^0, \quad t^0 - r^0 = \sigma^0, \quad t = +\infty.$$

We say that $m \to m^0_\infty$ if

$$\omega \to \omega^0, \quad t - r \to \sigma^0, \quad t \to +\infty.$$

A neighborhood of m^0_∞ is defined by

$$|\omega - \omega^0| < C_1, \quad |t - r - \sigma^0| < C_2, \quad t > C_3.$$

Theorem 10.8. Let m^0_∞ be a given point at infinity, and note $\omega^0_0 = -1$. Assume that one of the following three assumptions is true:

(i) $g_3(\omega^0, -1) \neq 0$,

(ii) $g_3(\omega^0, -1) = 0$, $A(\omega^0, -1) \neq 0$ and the fields

$$Y_1 = g_2^{\alpha\beta\gamma} \omega^0_\alpha \omega^0_\beta \epsilon_\gamma \partial_\gamma, \quad Y_4 = g_3^{\alpha\beta\gamma\delta} \omega^0_\alpha \omega^0_\beta \omega^0_\gamma \epsilon_\delta \partial_\delta$$

are independent,

(iii) $Q_3 = Q_4 \equiv 0$ and $A(\omega^0, -1) \neq 0$.

Then there exists a C^∞ function u defined in an appropriate neighborhood of m^0_∞, such that

(a) u and ∂u have a free representation

$$u(x, t) = \frac{1}{r} F(r - t, \omega, 1/r), \quad (\partial_\alpha u)(x, t) = \frac{1}{r} F_\alpha(r - t, \omega, 1/r),$$

where $F \in C^1$, $F_\alpha \in C^0$ up to $z = 1/r = 0$,

(b) For some continuous $a \neq 0$ and B,

$$u''(x, t) = \frac{1}{j} \frac{1}{r} (a\omega^t \omega + O(1/r)) + B/r,$$

where $j(m) > 0$, $j(m) \to 0$ as $m \to m^0_\infty$,

(c) u is in the domain of determination of a hyperboloid for which the Cauchy data of u have a free representation.

The behavior of j at infinity depends on which of the assumptions (i), (ii), or (iii) is made; these assumptions correspond to the assumptions of Corollary 10.5–10.7, respectively. □

Remark 10.9.

(i) The property (c) in Theorem 10.8 is a rough way of translating the fact that v has a hyperbolic blowup at m^0 with smooth data, as pointed out in the proofs of Theorems 10.1 and 10.3.

(ii) The function j in Theorem 10.8 is, with the notation of Section 10.1, $j(I) = (\partial_s \phi)(\Phi^{-1})$. □

10.3 Null incomplete geodesics

We believe it interesting to look at the metric corresponding to the linearized u-equation, to see if blowup at infinity for u is related to the existence of some incomplete null geodesic. It turns out that the u-blowup corresponding to the strong v-blowup associated with a weakly nonlinear point (as in Theorem I.2) is connected with such a geodesic.

Proposition 10.10. Let the solution u of $\Box u + Q(u) = 0$ correspond to a solution v of $\Box v + \bar{Q}(v) = 0$. Assume that v blows up at the point m^0 as in Theorem 10.3. Then there exists an incomplete null geodesic for the metric associated to the linearized equation of $\Box u + Q(u) = 0$. □

Proof. (a) We denote here the v-equation by $p^{ij}\partial^2_{ij} v + r = 0$, and use the notation of Section 5. First, we observe that the bicharacteristics for the linearized v-equation are not well defined in a standard way, since v is not C^2. However, once v is constructed by the procedure explained in Section 5, the linearized operator on v corresponds, by the change of variables Φ, to the linear operator with principal symbol $(\partial_s\phi)^{-1} q$, with $q = \partial_s \phi p^{ij} \bar{\xi}_i \bar{\xi}_j - 2 p^{ij} \hat{\phi}_i \bar{\xi}_j \zeta$, where ζ is the dual variable of s.

We investigate the null bicharacteristic of q starting from a point $(0, \zeta^0, \bar{\xi}^0)$ with $\zeta^0 \neq 0$, $p^{ij} \hat{\phi}_i \bar{\xi}^0_j = 0$. Denoting by a prime the derivation with respect to the parameter, we have

$$(\partial_s \phi)' = (\partial^2_s \phi) s' + (\partial^2_{sy} \phi) y' + (\partial^2_{st} \phi) t' = 0,$$

$$(\partial_s \phi)'' = (\partial^2_s \phi)' s' + (\partial^2_{sy} \phi)' y' + (\partial^2_{st} \phi)' t',$$

$$(\partial_s \phi)'' = \text{Hess}(\partial_s \phi)(s', y', t').$$

Since $s' = 0$ but $t' = -2\zeta p^{n+1,i}\hat{\phi}_i \neq 0$, we have $(\partial_s \phi)'' > 0$. If we denote by $X(\sigma_1)$ the bicharacteristic curve for the symbol q, and by $Y(\sigma_2)$ the bicharacteristic curve for $(\partial_s \phi)^{-1} q$, we have $X(\sigma_1) = Y(\psi(\sigma_1))$, with $\frac{d\psi}{d\sigma_1} = (\partial_s \phi)(X(\sigma_1))$. This gives $\psi(\sigma_1) = c\sigma_1^3 + O(\sigma_1^4)$. Since $X(\sigma_1) = X^0 \sigma_1 + O(\sigma_1^2)$, we finally get $Y(\sigma_2) = Y^0 \sigma_2^{1/3} + O(\sigma_2^{2/3})$.

(b) The image by Φ of the curve $Y(\sigma_2)$ is a curve $W(\sigma_2) = W^0 \sigma_2^{1/3} + O(\sigma_2^{2/3})$, with

$$W^0 = \Phi' X^0 = -2\zeta (\Sigma_{j\geq 2} p^{ij}\hat{\phi}_i \hat{\phi}_j, \, p^{ij}\hat{\phi}_j).$$

Note that $W^0 \neq 0$.

c. We return now to the u-equation. The transform $\tilde{\ell}$ by conformal inversion of the symbol ℓ of the linearized equation on u is related to the symbol p of the linearized equation on v by the formula $\tilde{\ell} = \Delta^2 p$, $\Delta = R^2 - T^2$. The null geodesic $W(\sigma_2)$ for the linearized v-equation and the null geodesic $Z(\sigma_3)$ for $\tilde{\ell}$ are related by $Z(h(\sigma_2)) = W(\sigma_2)$ with $\frac{dh}{d\sigma_2} = \Delta^{-2}(W(\sigma_2))$. Since $\Delta(W(\sigma_2)) = \Delta^0 \sigma_2^{1/3} + O(\sigma_2^{2/3})$, $\frac{dh}{d\sigma_2}$ is integrable, hence the geodesic is incomplete. ∎

11 The Genuinely Nonlinear Case

Theorem 10.1 is an immediate application of Theorem I.1, taking into account the expressions of q_2, q_3 and q_4 given in Lemma 9.2. To prove Corollary 10.5, assume that at the given point $m^0 = (X^0, T^0)$, $R^0 = -T^0$, $g_3 \neq 0$ and $X_1^0 \neq 0$. We choose $\underline{\eta_i} = -X_i^0 / X_1^0$. We have to select the root $\underline{\tau}$ in such a way that $s \neq 0$. Since the values of v and ∂v are at our disposal, we can take them small enough (with $Sv + v \neq 0$) to make sure that $\Box v + \bar{Q}(v)$ is hyperbolic with respect to T and that τ is close enough to $\pm|\xi|$. We have $s = -R^0 \tau - (R^0)^2 / X_1^0$, and $|\xi| = R^0 / |X_1^0|$, hence $s = -R^0[\tau + (sgn X_1^0)|\xi|]$. So, we choose τ of the same sign as X_1^0. The constructed blowup solution exists for $T \leq T^0$ and, for $T \geq T^0$, according to Theorem I.1, in the region

$$X_1 - X_1^0 + (1/X_1^0) \sum_{2\leq i} (X_i - X_i^0) X_i^0 \leq \mu (T - T^0).$$

This region contains a neighborhood of m^0 in $\{|X| \leq -T\}$. Since the boundary $\partial \mathcal{T}$ is characteristic for the linearized operator on v, we do have a hyperbolic blowup at m^0. The behavior of j follows from Theorem I.1.

12 The Weakly Nonlinear Case

We assume now $g_3(m^0) = 0$, $s \neq 0$. Hence $q = 0$. We need to examine the coefficients $\underline{C}$ and C^{ij} at the point under consideration.

Lemma 12.1. At m^0, all coefficients C^{ij} vanish. The coefficient $\underline{C}$ is

$$\underline{C} = 8A(X)s^2(\tau^2 - |\xi|^2) + 128s^4(Sv + v)^2 Y_2(g_3) + 64s^4(g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \partial_\delta v)$$
$$+ 64s^3(Sv + v)(g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \xi_\delta) + 384s^4 g_4(Sv + v)^2. \qquad \square$$

Proof. The function q is the sum of three functions; q_2 and q_4 contain the factor Δ, and q_3 is a linear combination of g_3 and Δ. Since $g_3 = \Delta = 0$ at m^0, all derivatives of q with respect to either U or ξ vanish, hence all coefficients C^{ij} vanish. Similarly, $\underline{C} = \{p, q\}$. Hence

$$\underline{C} = \{p, \Delta\}(\tilde{q}_2 + \tilde{q}_3 + \tilde{q}_4) + 32s^3(Sv + v)\{p, g_3\}.$$

Now

$$\{\tau^2 - |\xi|^2, \Delta\} = 4\tau T + 4\sum \xi_i X_i = 4s,$$

and, if Z is a vector field tangent to $R + T = 0$ with symbol z, $\{z, \Delta\} = Z(\Delta) = 0$. Hence

$$\{p_2, \Delta\} = 4[2A(X)(Sv + v) + Y_1 v]\{s^2, \Delta\} + 4(Sv + v)\{y_2 s, \Delta\} = 0,$$
$$\{p_3, \Delta\} = \{p_4, \Delta\} = 0.$$

Also

$$\{\tau^2 - |\xi|^2, g_3\} = 2(\tau \partial_T g_3 - \sum \xi_i \partial_i g_3),$$
$$\{p_2, g_3\} = 4[2A(X)(Sv + v) + Y_1 v]\{s^2, g_3\} + 4(Sv + v)\{y_2 s, g_3\},$$
$$\{p_3, g_3\} = \{p_4, g_3\} = 0.$$

Finally, if f is a homogeneous function of degree ν, $\{s, f\} = \nu f$, hence

$$\{s^2, g_3\} = 2s\{s, g_3\} = 8g_3 = 0,$$

$$\{y_2 s, g_3\} = 4y_2 g_3 + s\{y_2, g_3\} = sY_2(g_3),$$

$$\{p_2, g_3\} = 4(Sv + v)sY_2(g_3).$$

Since

$$\partial_\alpha g_3 = 2[g_3^{\alpha\beta\gamma\delta} + g_3^{\beta\gamma\delta\alpha}]X_\beta X_\gamma X_\delta,$$

we see that

$$\{\tau^2 - |\xi|^2, g_3\} = -4[g_3^{\alpha\beta\gamma\delta} + g_3^{\beta\gamma\delta\alpha}]\epsilon_\alpha \xi_\alpha X_\beta X_\gamma X_\delta = -4(I + II),$$

$$\tilde{q}_3 = 16s^3(g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \partial_\delta v) + 16s^2(Sv + v)(2I + II) + O(\Delta).$$

Hence

$$4s\tilde{q}_3 + 32s^3(Sv + v)\{p, g_3\} = 128s^4(Sv + v)^2 Y_2(g_3) + 64s^4(g_3^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma \epsilon_\delta \partial_\delta v)$$
$$+ 64s^3(Sv + v)II.$$

This gives the desired formula. ∎

Theorem 10.3 follows immediately from Lemma 12.1 and Theorem I.2. We now prove Corollary 10.6: according to the assumptions, we can choose $\underline{V}$ such that, at $\underline{m}$, $Y_1 v \neq 0$ and $Y_4 v = 0$. We choose then $\underline{v}$ such that $Sv + v = 0$ at $\underline{m}$. Thus

$$p = \tau^2 - |\xi|^2 + 4(Y_1 v)s^2, \quad \underline{C} = 8As^2(\tau^2 - |\xi|^2) + 64s^4(Y_4 v).$$

As in the proof of Corollary 10.5, taking V small enough, we can choose $(\underline{\eta}, \underline{\tau})$ such that $p = 0$ and $s \neq 0$. Then

$$\underline{C} = 8s^4[8(Y_4 v) - A(Y_1 v)] = -8s^4 A(Y_1 v) \neq 0.$$

The case of Corollary 10.7 is not included in the assumptions of Corollary 10.6, since $g_3 \equiv 0$. If Y_1 is not zero at m^0, we take $\underline{V}$ with $Y_1 v \neq 0$ and then $Sv + v = 0$. Thus

$$p = \tau^2 - |\xi|^2 + 4(Y_1 v)s^2, \quad \underline{C} = 8As^2(\tau^2 - |\xi|^2).$$

As before, we can choose $\underline{m}$ such that $p = 0, s \neq 0$ and then, automatically, $\underline{C} \neq 0$. If Y_1 vanishes at m^0, we get

$$p = \tau^2 - |\xi|^2 + 4s(Sv + v)[As + y_2], \quad \underline{C} = 8As^2(\tau^2 - |\xi|^2).$$

We know from [8, Section 5], that $y_1 + y_2 = -2As$. Hence $As + y_2 = -As$. We take $(\underline{v}, \underline{V})$ such that $Sv + v \neq 0$ but small enough, so that we can solve $p = 0$ with a root $(\underline{\eta}, \underline{\tau})$ for which $s \neq 0$. Then $\underline{C} \neq 0$ and this finishes the proof.

13 Proof of Theorem 10.8

Set $\omega_0 = -1, \omega_i = x_i/r$ as usual. If $m^0 = (X^0, T^0)$ satisfies $R^0 = -T^0$, then $X^0_\alpha = -T^0\omega_\alpha$, so that the values at m^0 of all coefficients and vector fields which are homogeneous in X can be deduced from the values at ω^0. On the other hand, since

$$\sigma \equiv r - t = (T - R)^{-1}, \quad \omega = \omega, \quad z \equiv 1/r = (T + R)\frac{T - R}{R},$$

the functions (σ, ω, z) are local coordinates close to m^0, for which $z = 0$ describes $\partial\mathcal{T}$. Hence, the fact that the constructed function v is C^1 reflects in the fact that u and $\partial_\alpha u$ have continuous (up to $z = 0$) free representations

$$u(x, t) = \frac{1}{r}F(\sigma, \omega, z), \quad (\partial_\alpha u)(x, t) = \frac{1}{r}F_\alpha(\sigma, \omega, z).$$

From the theory of Section 5, we see that

$$(\partial^2_{\alpha\beta} v) = \frac{\partial_s v}{\partial_s \phi} b^t b + C,$$

where ∂_v, b and c are smooth, $\partial_s v \neq 0$, $b = \hat{\phi}$. Now

$$(\partial^2_{\alpha\beta} u)(I) = 4\Delta X_\alpha X_\beta S^2 v + O(\Delta^2).$$

We have

$$S^2 v = \frac{1}{\partial_s \phi} \sum X_\alpha X_\beta \hat{\phi}_\alpha \hat{\phi}_\beta + C^0 = \frac{1}{\partial_s \phi} (s(\hat{\phi}))^2 + C^0.$$

By construction, we arranged $s(\hat{\phi}) \neq 0$ at $\underline{m}$. Hence

$$u'' = \frac{a}{rj}(\omega^t \omega + O(1/r)) + B/r,$$

with $a \neq 0$, where by an abuse of notation, $j = j(I)$.

Appendix: The Case of First-Order Systems

In this appendix, we present the concept of weakly nonlinear point in the framework of a first-order general system. We also state and prove the analog of Theorem I.2. We postponed this part at the end of the paper in order not to interrupt the natural continuation between Parts I and II.

Notation

We consider in $\mathbf{R}_x^{n+1}$ a first-order quasilinear system

$$L(u) \equiv A^{n+1}(x, u)\partial_t u + \sum_{1 \leq i \leq n} A^i(x, u)\partial_i u + B(x, u) = 0.$$

Here, the coordinates notations are the same as in Section 2, $u = u(x) = (u_1, \ldots, u_N) \in \mathbf{R}^N$, the matrices A^i are smooth real functions of (x, u), B is a smooth real vector in $\mathbf{R}^N$. Note that it is natural to index the u-coordinates below, since the reduction in a scalar wave equation in some unknown v to a system is done by setting $u_i = \partial_i v$. According to this, we vectors in $\mathbf{R}_u^N$ will be indexed above, etc. We set

$$A(x, u, \xi) = \sum_{1 \leq i \leq n+1} A^i(x, u)\xi_i.$$

Let $\underline{m} = (\underline{x}, \underline{u}, \underline{\xi})$, $\underline{\xi}_1 = -1$ such that $A(\underline{m})$ has a simple eigenvalue $\underline{\lambda} = 0$, with associated left and right eigenvectors $\underline{r}, \underline{\ell}$. For m close to $\underline{m}$, let $\lambda(m)$ the small eigenvalue of $A(m)$ with $\lambda(\underline{m}) = 0$, and $r(m), \ell(m)$ be the associated right and left eigenvectors, normalized to satisfy ${}^t\ell r \equiv 1$. We assume ${}^t\ell A^{n+1} r \neq 0$ at $\underline{m}$.

We complete $\underline{r}$ into a basis of $\mathbf{R}^N$ by vectors r^k, $2 \leq k \leq N$; similarly, we assume that we can choose ℓ^k, $2 \leq k \leq N$ such that ${}^t\ell^k A(\underline{m})$ complete ${}^t\ell A^{n+1}$ into a basis of $\mathbf{R}^N$.

Note that the system we consider is not necessarily symmetric (since the symmetry is destroyed by the change of unknown functions, Lemmas A.3 and A.6 would not make sense). For a symmetric hyperbolic system, that is, with $A^{n+1} \gg 0$, we can take $\ell = r$, $\ell^k = r^k$, and ${}^t\ell A^{n+1} r \neq 0$ and the other assumptions are automatically verified. We denote

$$\partial_u^i = \partial_{u_i}, \quad \partial_\xi^i = \partial_{\xi_i}.$$

Some nonlinear invariants

Definition A.1. The function $q(x, u, \xi)$ associated to the symbol A is defined by

$$q(x, u, \xi) = (r \cdot \nabla_u \lambda)(x, u, \xi). \qquad \square$$

This definition is obviously inspired by Lax [13].

Lemma A.2. The function q is invariant under the change of coordinates. More precisely, if $y = \phi(x)$ are the new variables with dual variables η and $u = v(\phi)$, the function $\tilde{q}$ corresponding to the new system $\tilde{L}(v) = 0$ is given by

$$\tilde{q}(y, v, \eta) = q(\phi^{-1}(y), v, {}^t\phi'(\phi^{-1}(y))\eta). \qquad \square$$

Lemma A.3. The function q is invariant by the change of unknown functions $u(x) = \phi(x, v(x))$. More precisely,

$$\tilde{q}(x, v, \xi) = q(x, \phi(x, v), \xi). \qquad \square$$

Definition A.4. Associated to the system, we define the vectors

$$C^i = (C^{i1}, C^{i2}, \ldots, C^{iN}), \quad 1 \leq i \leq n+1$$

and the function $\underline{C}$ by

$$C^{i1} = (\partial_\xi^i \lambda)(r\partial_u q) - (\partial_\xi^i q)(r\partial_u \lambda) - ({}^t\ell A^i r)(r\partial_u q),$$

$$C^{ik} = (\partial_\xi^i \lambda)(r^k\partial_u q) - (\partial_\xi^i q)(r^k\partial_u \lambda) - ({}^t\ell A^i r^k)(r\partial_u q),\ k \geq 2,$$

$$\underline{C} = \{\lambda, q\} - {}^t\ell B(r\partial_u q).$$

Here, $\{f, g\}$ is the usual Poisson bracket

$$\{f, g\} = (\partial_\xi f)(\partial_x g) - (\partial_\xi g)(\partial_x f). \qquad \square$$

Lemma A.5. Under the change of variables $y = \phi(x)$

$$\tilde{C}^{ij} = \sum \partial_k \phi^i C^{kj}, \quad \underline{\tilde{C}} = \underline{C}. \qquad \square$$

Proof. Since

$$\partial_\eta^i \tilde{\lambda} = \partial_j \phi^i \partial_\xi^j \lambda, \quad \tilde{A}^i = \sum \partial_j \phi^i A^j,$$

the formula for the $\tilde{C}^{ij}$ is immediate. The invariance of $\underline{C}$ results from the invariance of the Poisson bracket. ■

Lemma A.6. Under the change of unknown function $u(x) = \phi(x, v(x))$, if we define $\tilde{r}^k = (\phi'_v)^{-1}(\underline{x}, \underline{v}) r^k$,

$$\tilde{C}^{ij} = C^{ij},$$

$$\underline{\tilde{C}} = \underline{C} + \sum \alpha_{ij} C^{ij}$$

for some coefficients α_{ij}. $\square$

Proof. Through this change of function, the ξ-derivatives do not change, and the fields $r^k \partial_u$ are invariants. Since ${}^t\tilde{\ell} = {}^t\ell\phi'_v$, the scalars ${}^t\ell A^i r^k$ do not change either. This gives the result for $\tilde{C}^{ij}$. Now

$$\partial_i \tilde{\lambda} = \partial_i \lambda + \partial_i \phi_j \partial_u^j \lambda,$$

and similarly for q. Hence

$$\{\tilde{\lambda}, \tilde{q}\} = \{\lambda, q\} + I,$$

$$I = (\partial_\xi^i \lambda) \partial_i \phi_j \partial_u^j q - (\partial_\xi^i q) \partial_i \phi_j \partial_u^j \lambda.$$

If we express the j basis vector e^j in $\mathbf{R}_u^N$ as $e^j = \beta_k^j r^k$, we get

$$I = \sum (\partial_i \phi_j) \beta_k^j C^{ik} + \left(\sum \partial_i \phi_j {}^t\ell A^i e^j \right) (r \partial_u q).$$

Now, the additional term in $\tilde{B}$ gives

$$^t\tilde{\ell}\tilde{B} = {}^t\ell B + \sum {}^t\ell A^i(\partial_i\phi_j e^j). \qquad \blacksquare$$

Lemma A.7. We have, at a point $\underline{m}$ where $p = q = 0$,

(i) For all i, $1 \le i \le n+1$, $C^{i1} = 0$,
(ii) For all j, $1 \le j \le N$, $\xi_i C^{ij} = 0$. $\square$

Proof. Since $(A - \lambda)r = 0$,

$$(A - \lambda)(\partial_\xi^i r) + A^i r - (\partial_\xi^i \lambda)r = 0.$$

Multiplying to the left by ${}^t\ell$, we get ${}^t\ell A^i r = \partial_\xi^i \lambda$, which proves (i).

We have

$$\xi_i C^{ij} = (r^j\partial_u q)(\xi_i\partial_\xi^i\lambda) - (r^j\partial_u\lambda)(\xi_i\partial_\xi^i q) - (r\partial_u q)\left({}^t\ell \sum \xi_i A^i\right) r^j.$$

Taking into account the homogeneity in ξ of λ and q,

$$\xi_i\partial_\xi^i\lambda = \lambda = 0, \quad \xi_i\partial_\xi^i q = 0,$$

and also, by definition, ${}^t\ell A = 0$. $\blacksquare$

These invariance lemmas lead us to introduce the following definitions.

Definition A.8. A point $\underline{m} = (\underline{x}, \underline{u}, \underline{\xi})$ is a genuinely nonlinear point if

$$p(\underline{m}) = 0, \quad q(\underline{m}) = (r\partial_u\lambda)(\underline{m}) \neq 0. \qquad \square$$

This definition has been introduced by Lax for the case $n = 1$.

Definition A.9. A point $\underline{m} = (\underline{x}, \underline{u}, \underline{\xi})$ is a weakly nonlinear point if

$$p(\underline{m}) = 0, \quad q(\underline{m}) = 0$$

and the coefficients C^{ij}, $\underline{C}$ are not all zero. $\square$

Note that this definition does not depend on the choice of the vectors r^k, $k \geq 2$. This concept seems to be new in the literature. Recall that it is invariant under the change of coordinates and unknown function.

Examples and comments

- The simplest example is the 1D scalar ($N = 1$) Burgers-type equation

$$\partial_t u + c(x, t, u)\partial_x u = 0.$$

Then $q = \xi \partial_u c$, and

$$\underline{C} = \xi(\partial_t + c\partial_x)(\partial_u c).$$

A weakly nonlinear point is thus such that

$$\partial_u c = 0, (\partial_t + c\partial_x)(\partial_u c) \neq 0.$$

- Consider now a 2×2 system in 1D in diagonal form

$$\partial_t u_1 + \lambda_1(u)\partial_x u_1 = 0,$$

$$\partial_t u_2 + \lambda_2(u)\partial_x u_2 = 0.$$

Assume for instance that $\underline{\tau} + \lambda_1(\underline{u})\underline{\xi} = 0$. Then $\lambda = \tau + \lambda_1(u)\xi$, and $q = \xi\partial_1\lambda_1$. We have then

$$C^{12} = \xi\lambda_1\partial_{12}^2\lambda_1, \quad C^{22} = \xi\partial_{12}^2\lambda_1.$$

Hence a weakly nonlinear point, characteristic for the first mode, is such that

$$\partial_1\lambda_1 = 0, \quad \partial_{12}^2\lambda_1 \neq 0.$$

As for the case of a scalar second order equation, we have $C^{ij} = Z^{ij}q$, where the fields

$$Z^{ij} = \partial_\xi^i\lambda r^k\partial_u - (r^k\partial_u\lambda)\partial_\xi^i - ({}^t\ell A^i r^k)r\partial_u$$

are vertical fields tangent to $\{\lambda = 0\}$ at a point where $\lambda = q = 0$.

Blowup for quasilinear systems

Theorem A.10. Assume that $\underline{m}$ (with $\underline{x}=0$) is a weakly nonlinear point for the system under consideration. Then there exists, close to the origin, a solution of $L(u)=0$ such that

(i) $u \in C^0$ and $u \in C^\infty$ outside the origin,

(ii) $u' = \frac{S}{j} + R$, where S has rank one, R and S are C^∞. For some $c>0$, $j \geq c|x|^2$ with $j \sim c|x|^2$ on some curve reaching the origin. □

Proof. The proof of this theorem is of course very similar to the one of Theorem 10.3. However, we give it here completely, for the convenience of the reader.

(a) Let us go back to the general theory, analogous to but simpler than the corresponding theory sketched in Section 5. We follow here, with some modifications, the notations and the approach of this section. First, note that, close to $\underline{m}$, $\lambda=0$ is equivalent to $\tau = \mu(x, u, \xi_1, \eta)$, since

$$\partial_\tau \lambda(\underline{m}) = {}^t\ell A^{n+1} r \neq 0$$

by assumption.

With $\phi(s, y, t)$ and Φ as before, we set

$$u(\phi(s, y, t), y, t) = v(s, y, t).$$

Hence, with the same notations for $\bar{\partial}$ and $\hat{\phi}$,

$$(\partial u)(\Phi) = \bar{\partial} v - \hat{\phi} \frac{\partial_s v}{\partial_s \phi}.$$

The new system is

$$-(\partial_s \phi)^{-1} [\hat{\phi}_i A^i(\phi, y, t, v)] \partial_s v + A^i(\phi, y, t, v) \bar{\partial}_i v + B(\phi, y, t, v) = 0.$$

We impose now the eikonal equation $\lambda(\phi, y, t, v, -1, \partial_y \phi, \partial_t \phi) = 0$, which is equivalent to

$$\partial_t \phi = \mu(\phi, y, t, v, -1, \partial_y \phi).$$

Multiplying to the left successively by ${}^t\ell(\phi, y, t, v, \hat{\phi})$ and ${}^t\ell^k$, we obtain the equivalent system

$$
\begin{gathered}
{}^t\ell[A^i\bar{\partial}_i v + B] = 0, \\
{}^t\ell^k A(\phi, y, t, v, \hat{\phi})\partial_s v - (\partial_s\phi)\,{}^t\ell^k[A^i\bar{\partial}_i v + B] = 0, \quad k \geq 2.
\end{gathered}
$$

Using now the new coordinates $T = s + t$, $S = t - s$, we obtain the $N+1 \times N+1$ blowup system

$$
\begin{gathered}
\partial_T\phi + \partial_S\phi = \mu(\phi, y, (S+T)/2, -1, \partial_y\phi), \\
{}^t\ell A^{n+1}(\partial_T v + \partial_S v) + {}^t\ell[A^i\partial_{y_i} v + B] = 0, \\
{}^t\ell^k A(\partial_T v - \partial_S v) - (\partial_T\phi - \partial_S\phi)\,{}^t\ell^k[A^i\partial_{y_i} v + A^{n+1}(\partial_T v + \partial_S v) + B] = 0.
\end{gathered}
$$

Defining

$$
f = {}^t\ell[A^i\partial_{y_i} v + 2A^{n+1}\partial_S v + B],
$$

we rewrite the last N equations as

$$
M(\partial_T v - \partial_S v) = {}^t(-f, g),\, g^k = (\partial_T\phi - \partial_S\phi)\,{}^t\ell^k[A^i\partial_{y_i} v + A^{n+1}(\partial_T v + \partial_S v) + B], \quad k \geq 2,
$$

where M is the $N \times N$ matrix with lines $({}^t\ell A^{n+1}, {}^t\ell^k A)$. Note that, by construction, this matrix is invertible at $\underline{m}$, and that

$$
Mr = ({}^t\ell A^{n+1} r, 0, \ldots, 0).
$$

One can solve the blowup system using the Cauchy–Kovalesky theorem, with arbitrary analytic data on $\{T = 0\}$, satisfying

$$
v(0) = \underline{v},\, \phi(0) = 0, \quad \partial_y\phi(0) = \underline{\eta}.
$$

We carry out the analysis just as in Section 5, setting $F(S, y, T) = \partial_T\phi - \partial_S\phi$. Straightforward computations on the eikonal equation give then

$$
ZF \equiv \partial_T F + \partial_S F - \partial_\eta\mu\,\partial_y F = F\partial_1\mu + (\partial_u\mu)(\partial_T v - \partial_S v).
$$

We choose $\partial_S \phi(0) = 2\mu = 2\underline{\tau}$ to get $F(0) = 0$.

(b) In accordance with Lemma 7.1, we now compute $Z[q]$ at $\underline{m}$, where

$$[q] = q(\phi, y, (S+T)/2, v, -1, \partial_y \phi, [\mu]).$$

We have first

$$Z[q] = \partial_1 q Z\phi + \partial_y q Z y + \partial_t q + \partial_u q Z v + \partial_\xi^i q Z(\partial_{y_i}\phi) + \partial_\tau q Z[\mu].$$

Next,

$$\begin{aligned} Z\phi &= -\partial_\xi^1 \mu, \\ Z(\partial_{y_i}\phi) &= \partial_{y_i}[\mu] - \partial_\xi^j \mu \partial^2_{y_i y_j}\phi \\ &= \partial_1 \mu \partial_{y_i}\phi + \partial_{y_i}\mu + \partial_u \mu \partial_{y_i} v, \\ Z[\mu] &= -\partial_\xi^1 \mu \partial_1 \mu - (\partial_y \mu)(\partial_\eta \mu) + \partial_t \mu + \partial_u \mu Z v + (\partial_\eta \mu)(\partial_1 \mu \partial_y \phi + \partial_y \mu + \partial_u \mu \partial_y v) \\ &= \mu \partial_1 \mu + \partial_t \mu + \partial_u \mu Z v + (\partial_\eta \mu)(\partial_u \mu)\partial_y v. \end{aligned}$$

Hence, since $q = 0$ at $\underline{m}$, we get at $\underline{m}$

$$\begin{aligned} Z[q] = {} & -\partial_\xi^1 \mu \partial_1 q + \partial_1 \mu \partial_\xi^1 q + \partial_\eta q \partial_y \mu - \partial_\eta \mu \partial_y q + \partial_t q + \partial_\tau q \partial_t \mu \\ & + \partial_u q Z v + \partial_u \mu[(\partial_\eta q + \partial_\tau q \partial_\eta \mu)\partial_y v + \partial_\tau q Z v). \end{aligned}$$

Now, inverting M, we obtain at $\underline{m}$

$$\partial_T v - \partial_S v = -\frac{f}{{}^t\ell A^{n+1} r} r,$$

hence

$$Z v = -\frac{f}{{}^t\ell A^{n+1} r} r + 2\partial_S v - \partial_\eta \mu \partial_y v.$$

(c) Since $\lambda(x, u, -1, \eta, \mu) \equiv 0$, we get, for any derivative ∂,

$$(\partial_\tau \lambda)\partial \mu = -\partial \lambda.$$

Hence, remembering that $\partial_\tau \lambda = {}^t\ell A^{n+1} r$,

$$(\partial_\tau \lambda) Z[q] = \{\lambda, q\} + E(\partial_u q \partial_\tau \lambda - \partial_u \lambda \partial_\tau q) + (\partial_\eta \lambda \partial_u q - \partial_\eta q \partial_u \lambda)\partial_Y v,$$

where

$$E = 2\partial_S v - \frac{f}{\partial_\tau \lambda} r.$$

Now, using the explicit expression of f,

$$E = -\frac{{}^t\ell A^i \partial_{Y_i} v + {}^t\ell B}{\partial_\tau \lambda} r + 2\left(\partial_S v - r\frac{{}^t\ell A^{n+1}\partial_S v}{\partial_\tau \lambda}\right).$$

Decomposing $\partial_S v$ and $\partial_{Y_i} v$ on the basis r, r^k

$$\partial_S v = \alpha r + \alpha_k r^k,\ \partial_{Y_i} v = \beta_i r + \beta_{ik} r^k,$$

we have

$$\partial_S v - r\frac{{}^t\ell A^{n+1}\partial_S v}{\partial_\tau \lambda} = (\partial_\tau \lambda)^{-1}[\alpha_k (r^k \partial_\tau \lambda - r {}^t\ell A^{n+1} r^k)],$$

$$(\partial_\eta \lambda \partial_u q - \partial_\eta q \partial_u \lambda)\partial_Y v - {}^t\ell A^i \partial_{Y_i} v (r\partial_u q) = (r\partial_u q)\beta_i(\partial_\eta^i \lambda - {}^t\ell A^i r) + \beta_{ik}[\partial_\eta^i \lambda r^k \partial_u q$$
$$- \partial_\eta^i q r^k \partial_u \lambda - (r\partial_u q)({}^t\ell A^i r^k)].$$

To summarize, using the notations of Section A.1, and noting that $\partial_\eta^i \lambda = {}^t\ell A^i r$

$$(\partial_\tau \lambda) Z[q] = \underline{C} + \sum_{2\le i\le n} \beta_{ik} C^{ik} + \frac{2}{\partial_\tau \lambda}\alpha_k C^{n+1,k}.$$

(d) We proceed now as follows: according to Lemma A.7, the assumption of the theorem implies that the coefficients $\underline{C}$ and C^{ij} (i from 2 to n, k from 2 to N) are not all zero. Hence we can choose $\partial_S v$, $\partial_Y v$, more precisely the coefficients α_k and β_i, β_{ik}, to get $Z[q] \neq 0$. After that, we choose $\partial_Y^2 \phi$ arbitrarily, and then $\partial_{SY}^2 \phi$ and $\partial_S^2 \phi$ such that

$$\partial_S F = \partial_Y F = 0.$$

Since $(\partial_T v - \partial_S v)\partial_u \lambda$ vanishes at $\underline{m}$, this implies also $\partial_T F = 0$. Inverting M, we obtain with an irrelevant coefficient $*$

$$\partial_T v - \partial_S v = -\frac{f}{{}^t\ell A^{n+1} r} r + *F,$$

hence, since F has a critical point,

$$\begin{aligned} Z[(\partial_T v - \partial_S v)\partial_u \mu] &= Z\left[-\frac{f}{\partial_\tau \lambda} r \partial_u \mu\right] \\ &= \frac{f}{(\partial_\tau \lambda)^2} Z(r\partial_u \lambda). \end{aligned}$$

It remains to observe that

$$f = \text{known quantity} + 2({}^t\ell A^{n+1} r)\alpha,$$

thus we can choose α to get $f \neq 0$ with the right sign, so that finally $Z[(\partial_T v - \partial_S v)\partial_u \mu] > 0$.

(e) The last step is analogous to that in Section 7: we can choose the third-order derivatives of ϕ such that the partial hessian of F with respect to (S, y) is positive definite big enough. ■

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