Stability of Large Solutions to Quasilinear Wave Equations

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ABSTRACT. We investigate the stability of (large) global C^∞ solutions to quasilinear wave equations satisfying the null condition in $\mathbb{R}^3_x \times [0, +\infty[$.

We give sufficient conditions for such a solution to be stable and have a free representation, and discuss the connection between stability and blowup at infinity. This latter concept is defined using a conformal inversion.

INTRODUCTION

In this paper, we study the behavior of solutions to the Cauchy problem

$$\Box u + Q(u) = 0, \quad Q(u) = g^{\alpha\beta\gamma}(\partial_\gamma u)(\partial^2_{\alpha\beta} u),$$
$$u(x,0) = u_0(x), \quad (\partial_t u)(x,0) = u_1(x).$$

We consider the simplest case where $u_i \in C_0^\infty(\mathbf{R}^3_x)$, and assume that $Q(u)$ satisfies the null condition

$$g^{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma = 0$$

whenever $\xi_0^2 = \sum \xi_i^2$.

Let us review first the classical results obtained in the case of "small solutions" by Christodoulou [11] and Klainerman [13].The approach of [13] uses the Lorentz fields

$$Z = \partial_\alpha,\ R = x \wedge \partial,\ H_i = t\partial_i + x_i\partial_t,\ S = t\partial_t + r\partial_r.$$

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Defining a higher order energy (for an appropriate N)

$$E_N(t) = \sum_{k \le N} \|Z^k \partial u(\cdot, t)\|_{L^2}^2,$$

it is shown that if $E_N(0) = \varepsilon^2$ is small enough, the solution u exists globally and $E_N(t)$ remains of the order of ε^2 for all times. If we try an asymptotic analysis in the spirit of Hörmander [12] or Lindblad-Rodnianski [15], introducing the slow time $\tau = \varepsilon \log t$ and looking for u in the representation

$$u(x,t) = \frac{1}{r} F(r-t,\, \omega,\, \tau), \quad r = |x|, \quad x = r\omega,$$

we find that $Q(u) = O(t^{-3})$ and $\partial_\tau F \equiv 0$. Thus, in a first approximation, u behaves like a free solution of the wave equation $(1/r)F(r-t, \omega)$.

The approach by Christodoulou uses the embedding of Minkowski space into the Einstein cylinder (see also [12]): taking advantage of the null condition, it transforms the original problem into a quasilinear hyperbolic Cauchy problem

$$\Box v + \bar{Q}(v) = 0$$

for a certain function v in an open domain $\mathcal{D}$. The global existence of v in $\mathcal{D}$ yields the global existence of u, the smoothness of v in $\bar{\mathcal{D}}$ giving the asymptotic behavior of u at infinity. It is not clear how to compare the smallness assumptions in both approaches.

In the present paper, we consider large C^∞ solutions, supposed to exist globally, and investigate both their behavior at infinity and their stability. The possibility of using a conformal compactification (namely, in this paper, conformal inversion) allows us to give a precise meaning to the expression "blowup at infinity" : this means that the function $v \in C^\infty(\mathcal{D})$ has some singularity on the boundary of $\mathcal{D}$. It is not clear, however, if this can actually happen : may be all solutions v, smooth in the open domain $\mathcal{D}$, are automatically smooth on the closure. Though we do not believe this, we have no proof that smooth solutions blowing up at infinity do exist.

In this context, we distinguish between

(i) Existence of a representation of u,
(ii) Stability of u.

The first property means simply $v \in C^\infty(\bar{\mathcal{D}})$, since by conformal inversion this corresponds to an actual representation of u identical to that of a free solution (see [12])

$$u(x,t) = \frac{1}{r} F\left(r-t,\, \omega,\, \frac{1}{r}\right).$$

The proof of (i) is analogous to the proof of a finite blowup criteria for a quasilinear wave equation or system (see for instance [16]). The statement follows the same scheme : if the solution is smooth in the open domain and if a certain a priori condition is satisfied, then the solution is smooth in the closed domain. For technical reasons (similar to that of [14]), we restrict ourselves in this paper to discussing representation or stability at null infinity, that is, in a domain $t \leq \underline{t}+r$, $t \to +\infty$ but $|r-t| \leq C$. The a priori conditions that we impose on the solution are of two types :

(i) Smallness conditions,
(ii) Decay conditions.

These last conditions are formulated in terms of pointwise decay of a certain number of $Z^k \partial u$; they involve no small constants. The first ones are the delicate point, since we do not want to fall back in the framework of the previous papers. The only smallness assumptions that we make on u are about energy properties for the linearized operator

$$\mathcal{L} = \Box + g^{\alpha\beta\gamma}(\partial_\gamma u)\partial^2_{\alpha\beta}.$$

Roughly, we assume that the field

$$K_0 = (r^2+t^2)\partial_t + 2rt\partial_r$$

is timelike, and that a certain family of hyperboloids is spacelike. These conditions are of course inspired by the necessity of having good energy estimates for the equation on v, the field K_0 and the hyperboloids corresponding to ∂_T and $\{T = C\}$ by conformal inversion. The precise statements are given in the representation theorem and the first stability theorem of Section 3.

The representation property clearly implies stability of u. However, the other implication is not clear, though we conjecture that it is true. In fact, if it were not true, this would mean that there are solutions $v \in C^\infty(\mathcal{D})$ of $\Box v + \bar{Q}(v) = 0$ with some singularities on $\partial\mathcal{D}$, such that *all* small perturbations of the data produce again a solution in $C^\infty(\mathcal{D})$.

If, with the aim of handling more general situations in the future, we are willing to ignore conformal inversion, we are left with this : a representation of u with a smooth profile F

$$u(x,t) = \frac{1}{r}F\left(r-t,\,\omega,\,\frac{1}{r}\right)$$

implies that u remains $O(t^{-1})$ under the action of any number of operators $\bar{K}_\alpha = K_\alpha + 2x_\alpha$, where K_0 and

$$K_i = 2x_iS + (t^2-r^2)\partial_i$$

are the image of ∂_T and ∂_{X_i} by conformal inversion. In fact,

$$\bar{K}_\alpha u = \frac{1}{r}\bar{F}_\alpha\left(r-t,\,\omega,\,\frac{1}{r}\right)$$

for a new profile $\bar{F}_\alpha$ explicitly deduced from F. Note that this behavior is far from being obvious, since it would seem from the expression of $\bar{K}_\alpha$ that only $\bar{K}_\alpha u = O(1)$; in fact, several cancellations of "principal terms" take place in the computation of $\bar{K}_\alpha u$. On the other hand, the second stability theorem shows a stability condition, using Lorentz fields, in the spirit of [13]. From what has been said above, it is hard to believe that this stability property does not necessarily correspond to a representation with a $v \in C^\infty(\bar{\mathcal{D}})$. However, we are not able to prove this directly, since this would require to prove directly the $O(t^{-1})$ behavior of $\bar{K}_\alpha u$. The difficulty here is to express $\bar{K}_\alpha Q(u)$ in an appropriate way analogous to what can be done for $ZQ(u)$ for instance.

Finally, we would like to emphasize the obvious fact that conformal inversion allows us to connect blowup at infinity with finite time blowup : if we can describe the way a certain solution $v \in C^\infty(\mathcal{D})$ of $\Box v + \bar{Q}(v) = 0$ blows up at a point $m_0 \in \partial\mathcal{D}$, this will describe the behavior at infinity of the corresponding u. Taking into account previous work on finite time blowup for quasilinear wave equations [2], [4], [5], we distinguish fundamentally genuinely nonlinear points from linearly degenerate points. In this spirit, the linear degeneracy theorem of Section 5 shows that all points of $\partial\mathcal{D}$ are linearly degenerate, a highly non-obvious result.

The plan of the paper is as follows : in Section 1, we recall some basic facts about the null condition, while we introduce the conformal inversion in Section 2. The main results are stated in Section 3, and proved in Section 4. Finally, Section 5 is devoted to a somewhat heuristic discussion of blowup at infinity and finite time blowup.

1. NOTATION AND BASIC FACTS ABOUT THE NULL CONDITION

In this paper, we deal with the Cauchy problem for the quasilinear wave equation in $\mathbf{R}^3_x \times [0, +\infty[$

$$\Box u + Q(u) = 0, \quad Q(u) \equiv g^{\alpha\beta\gamma}(\partial_\gamma u)(\partial^2_{\alpha\beta} u).$$

Here, $x = (x_1, x_2, x_3)$, $x_0 = t$: greek indices run from 0 to 3, while latin indices run from 1 to 3. For simplicity, we take $g^{\alpha\beta\gamma}$ to be real given constants with

$$g^{00\gamma} = 0, \quad g^{\alpha\beta\gamma} = g^{\beta\alpha\gamma},$$

and the sum sign on repeated indices in the expression of Q is omitted. The data

$$u_0(x) = u(x, 0), \quad u_1(x) = (\partial_t u)(x, 0)$$

are supposed to be C^∞ functions on $\mathbf{R}^3$ supported for $|x| \le M$. Note that if u is a global solution of $\Box u + Q(u) = 0$, then the function u_λ defined by

$$u_\lambda(x, t) = \lambda^{-1} u(\lambda x, \lambda t)$$

is also a solution, with data supported for $|x| \leq M/\lambda$. Whenever we consider a global smooth solution u of $\square u + Q(u) = 0$, we assume that u is small enough to make the linearized operator

$$\square + g^{\alpha\beta\gamma}(\partial_\gamma u)\partial^2_{\alpha\beta}$$

strictly hyperbolic with respect to t.

1.1. The null condition. We assume that the constants g satisfy the null condition, that is

$$g^{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma = 0$$

whenever $\xi_0^2 = \sum \xi_i^2$. Let us recall the fundamental result due Christodoulou [11] and Klainerman [13] : if the data are sufficiently small, there exists a global smooth solution u to the Cauchy problem for $\square u + Q(u) = 0$. If the null condition is not satisfied, finite time blowup of small enough smooth solutions has been proved for (almost) all initial data [8].

Recall that the Lorentz fields

$$\partial_\alpha, \quad R = x \wedge \partial, \quad H_i = t\partial_i + x_i\partial_t, \quad S = t\partial_t + \sum x_i\partial_i,$$

that we denote generically by Z, commute with $\square$, with the exception of $[\square, S] = 2\square$. For an integer k, Z^k denotes a product of k Lorentz fields. The quadratic form Q enjoyes two important properties in connection with the fields Z (see [12] for proofs).

Lemma 1.1 (Estimation Lemma 1). *For any smooth functions u, v,*

$$|g^{\alpha\beta\gamma}(\partial_\gamma u)(\partial^2_{\alpha\beta}v)| \leq C(1+t)^{-1}\Big(|Zu|\ |\partial^2 v| + |\partial u|\ |Z\partial v|\Big),$$

where the sum over all Lorentz fields Z is omitted.

Note the well-known decomposition

$$\partial_i = \omega_i\partial_r - \left[\omega \wedge \left(\frac{R}{r}\right)\right]_i.$$

Here and in the whole paper, (r, ω) are the polar coordinates

$$r = |x|, \quad x = r\omega, \quad \partial_r = \sum \omega_i\partial_i.$$

We will often use $\tilde{\partial}_i = \partial_i - \omega_i\partial_r$, which is tangent to the spheres, with the decomposition

$$|\nabla_x u|^2 = \sum(\partial_i u)^2 = (\partial_r u)^2 + \sum(\tilde{\partial}_i u)^2 = (\partial_r u)^2 + \left|\frac{R}{r}u\right|^2.$$

Note that in an exterior region $r \geq ct$ ($c > 0$), if Q contains no t-derivative, the above estimate of Q can be obtained using only the rotations R and not all Lorentz fields, since we also have the nice formula

$$\partial^2_{ij}u = \omega_i\omega_j\partial^2_r u + \tilde{\partial}_i\partial_j u + \partial_i\tilde{\partial}_j u - \tilde{\partial}_j\tilde{\partial}_i u - \frac{1}{r}(\omega_i\partial_j - \omega_j\partial_i)u.$$

Lemma 1.2 (Commutation Lemma). *For any Lorentz field Z and any smooth functions u, v,*

$$\begin{aligned} &Z[g^{\alpha\beta\gamma}(\partial_\gamma u)(\partial^2_{\alpha\beta}v)] \\ &\quad = g^{\alpha\beta\gamma}(\partial_\gamma Zu)(\partial^2_{\alpha\beta}u) + g^{\alpha\beta\gamma}(\partial_\gamma u)(\partial^2_{\alpha\beta}Zv) + \tilde{g}^{\alpha\beta\gamma}(\partial_\gamma u)(\partial^2_{\alpha\beta}v),\end{aligned}$$

where the new sum with constant coefficients $\tilde{g}$ satisfies again the null condition.

1.2. Algebraic identities for the null condition. It is sometimes convenient to split the coordinates in (x, t)

$$\begin{aligned} &Q(u) = 2g^{0i\gamma}(\partial_\gamma u)(\partial^2_{ti}u) + g^{ij\gamma}(\partial_\gamma u)(\partial^2_{ij}u) \\ &= 2g^{0i0}(\partial_t u)(\partial^2_{ti}u) + 2g^{0ij}(\partial_j u)(\partial^2_{ti}u) + g^{ij0}(\partial_t u)(\partial^2_{ij}u) + g^{ijk}(\partial_k u)(\partial^2_{ij}u).\end{aligned}$$

Lemma 1.3. *[Algebraic identities] The null condition implies the identities*

(i) $\forall \xi \in \mathbf{R}^3$, $g^{ij0}\xi_i\xi_j + 2g^{0ij}\xi_i\xi_j = 0$,

(ii) $\forall \xi \in \mathbf{R}^3$, $g^{ijk}\xi_i\xi_j\xi_k + 2(g^{0i0}\xi_i)(\sum \xi_i^2) = 0$,

(iii) $\forall \xi \in \mathbf{R}^4$, $g(\xi) \equiv g^{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma = A(\xi)(\xi_0^2 - \sum \xi_i^2)$, $A(\xi) = 2g^{0i0}\xi_i$,

(iv) $\forall \xi, \eta \in \mathbf{R}^3$, $g^{ijk}\xi_i\xi_j\eta_k + 2g^{ijk}\xi_j\xi_k\eta_i = -2A(\xi)(\xi\cdot\eta) - A(\eta)(\sum \xi_i^2)$, *where* $\xi\cdot\eta = \sum \xi_i\eta_i$.

Proof. Let us fix ξ with $\xi_0^2 = \sum \xi_i^2$. The null condition reads

$$2g^{0i0}\xi_0^2\xi_i + g^{ijk}\xi_i\xi_j\xi_k + (\xi_0)[2g^{0ij}\xi_i\xi_j + g^{ij0}\xi_i\xi_j] = 0.$$

Replace ξ_0 by $-\xi_0$ in the above equation : by comparison, we obtain (i) and (ii). These two identities in turn imply (iii).

To obtain (iv), set

$$f(\xi) \equiv g^{ijk}\xi_i\xi_j\xi_k = -A(\xi)|\xi|^2.$$

For fixed ξ and η, consider the function

$$\phi(\lambda) = f(\xi + \lambda\eta) = -A(\xi)|\xi|^2 - 2\lambda A(\xi)(\xi\cdot\eta) - \lambda A(\eta)|\xi|^2 + O(\lambda^2).$$

Then

$$\phi'(0) = \eta \cdot \nabla f(\xi) = g^{ijk}\xi_i\xi_j\eta_k + 2g^{ijk}\xi_j\xi_k\eta_i$$

gives the claim. $\square$

1.3. Pointwise behavior of the solution. For a smooth solution u of $\Box u + Q(u) = 0$, relatively weak decay assumptions on a certain number of functions $Z^k u$ imply a much better decay for u. The following lemma is an illustration, among many possible variations, of this statement.

Lemma 1.4. *Assume that* u *is a global smooth solution of* $\Box u + Q(u) = 0$ *satisfying for some* C *and* $\nu > \frac{1}{2}$

$$\sum_{k\le 4} |Z^k u| \le C\langle r-t\rangle^{1/2}(1+t)^{-\nu}, \quad \nu > \frac{1}{2}.$$

Then $|u| \le C\langle r-t\rangle^{-1}(1+t)^{-1}$.

Proof. Using the estimation lemma, and the inequality $\langle r-t\rangle|\partial v| \le C\sum|Zv|$, we get from

$$\sum_{k\le 2} |Z^k u| \le C(1+t)^{-\nu}\langle r-t\rangle^{1/2}$$

the estimate

$$|Q(u)| \le C(1+t)^{-(2\nu+1)}\langle r-t\rangle^{-1}.$$

Using the commutation lemma, we have the same estimate also for ZQ and Z^2Q. Proposition 3.1 of [7] shows then

$$\sum_{k\le 2} |Z^k u| \le C(1+t)^{-1}.$$

Using once more the estimation lemma, we get $|Q(u)| \le \langle r-t\rangle^{-2}(1+t)^{-3}$, hence by the same proposition, $|u| \le C\langle r-t\rangle^{-1}(1+t)^{-1}$. $\square$

The following lemma is a weaker version of the preceding result in a region where $|r-t| \le C$.

Lemma 1.4′. *Assume that* u *is a global smooth solution in a region*

$$t \le \underline{t} + r,$$

satisfying there

$$\sum_{k\le 1} |Z^k \partial u| \le C(1+t)^{-\nu}, \quad \nu > \frac{1}{2}.$$

Then $|u| \le C(1+t)^{-1}$.

Proof. The assumptions imply $|Q(u)| \le C(1+t)^{-1-2\nu}$. Hence Proposition 3.1 of [7] gives the result. $\square$

2. CONFORMAL INVERSION

2.1. Basic formula. We recall here the definition and some basic formula. Let

$$\Gamma = \{(x,t) \in \mathbf{R}^4,\ t^2 > |x|^2\}.$$

The conformal inversion I is the application from Γ to itself defined by

$$X = \frac{x}{t^2 - r^2}, \quad T = \frac{-t}{t^2 - r^2}.$$

Note that $I^2 = \text{id}$ on Γ. We will also note $X_0 = T$, $R = |X|$, $X/R = \omega = x/r$, and $\tilde{u}(X,T)$ for the transform of a function $u(x,t)$

$$\tilde{u}(X,T) = u(x,t), \quad \tilde{u}(I) = u.$$

We write the Lorentz fields on the (X,T) side with the same letters, when no ambiguity arises : for instance, $S = T\partial_T + R\partial_R$, etc.

Lemma 2.1. *The fields ∂_t, ∂_i, are transformed by I into*

$$K_0 = (R^2 + T^2)\partial_T + 2RT\partial_R = 2TS - (T^2 - R^2)\partial_T,$$
$$K_i = 2X_iS + (T^2 - R^2)\partial_i.$$

The fields

$$L = \partial_t + \partial_r, \quad \underline{L} = \partial_t - \partial_r$$

are transformed into

$$(R+T)^2(\partial_T + \partial_R), \quad (R-T)^2(\partial_T - \partial_R).$$

The rotation fields R_i are preserved, while the hyperbolic rotations H_i and the scaling field S are transformed into their opposite.

2.2. The wave equation. From now on, we use the following notation : $u(x,t)$ is given, $\tilde{u}(X,T)$ is its transformed function, and we set

$$v(X,T) = \frac{\tilde{u}(X,T)}{T^2 - R^2}.$$

The following lemma summarizes well-known facts (see [1] for instance).

Lemma 2.2. *Set $\bar{K}_\alpha = K_\alpha + 2x_\alpha$. The following formula hold :*

(i) $\Box v = (T^2 - R^2)^{-3}(\Box u)(I)$,

(ii) $(\partial_\alpha u)(I) = (T^2 - R^2)\bar{K}_\alpha v$,

(iii) $R\tilde{u} = (T^2 - R^2)Rv$, $H\tilde{u} = (T^2 - R^2)Hv$, $S\tilde{u} = (T^2 - R^2)(Sv + 2v)$,

(iv) $\Box\bar{K}_\alpha v = (K_\alpha + 6X_\alpha)\Box v$.

2.3. The operators L_1 and L_2. As shown by formula (iv) in Lemma 2.2 above, $\bar{K}_\alpha w = O(t^{-1})$ if w is a solution of the wave equation with C_0^∞ data. To understand which cancellations take place, it is convenient to introduce the following operators.

Definition.

The operator L_1 is defined by

$$L_1 w = (r+t)Lw + 2w.$$

The operator L_2 is defined by

$$L_2 w = L_1^2 w + 2L_1 w = (r+t)^2 L^2 w + 8(r+t)Lw + 8w.$$

To understand the origin of these definitions, let us compute $L_1 w$ and $L_2 w$ on a free solution of $\Box w = 0$, represented as $w(x,t) = r^{-1}F(r-t,\ \omega,\ r^{-1})$ (see [12]). Here, F is smooth, and we note $\sigma = r - t$, $z = r^{-1}$. We have first

$$Lw = -\frac{1}{r^2}F - \frac{1}{r^3}\partial_z F,$$

$$L^2 w = \frac{2}{r^3}F + \frac{4}{r^4}\partial_z F + O(r^{-5}),$$

Then

$$L_1 w = \frac{r-t}{r^2}F - \frac{r+t}{r^3}\partial_z F,$$

$$L_2 w = \frac{2F}{r^3}[(r+t)^2 - 4r(r+t) + 4r^2] + \frac{4(r+t)\partial_z F}{r^4}(r+t-2r) + O(r^{-3})$$

$$= \frac{2(r-t)^2 F}{r^3} + \frac{4(t^2-r^2)\partial_z F}{r^4} + O(r^{-3}).$$

The point is that, for a function w satisfying $\sum_{k\le 1} |Z^k w| = O(t^{-1})$, one has $Lw = O(t^{-2})$ and one would expect $L_1 w = O(t^{-1})$; in the present case however, we see that $L_1 w = O(t^{-2})$ as long as $|r-t| \le C$. Similarly, and even more strikingly, one would expect $L_2 w = O(t^{-1})$, since $L^2 w = O(t^{-3})$; in contrast, we obtain here $L_2 w = O(t^{-3})$ as long as $|r-t| \le C$: note the double cancellation occuring for the F and $\partial_z F$ coefficients in the above computation.

The following lemma displays the relations between $\bar{K}_\alpha$, L_1 and L_2.

Lemma 2.3. *The following identities hold :*

(i) $\bar{K}_0 w = tL_1 w + (r-t)\sum \omega_i H_i w,$

(ii) $\bar{K}_i w = x_i L_1 w + (t-r)H_i w + r(t-r)\tilde{\partial}_i w,\ \tilde{\partial}_i = \partial_i - \omega_i \partial_r,$

(iii) $$\bar{K}_0^2 w = \frac{(r+t)^2}{4} L_2 w + \frac{t+r}{2r}(t^2 - r^2) L_1 w - 2\frac{(t-r)^3}{r} w$$
$$+ \frac{(t-r)^4}{4} \underline{L}^2 w + \frac{(r-t)^3(t-3r)}{2r} \underline{L} w + \frac{(t^2-r^2)^2}{2}\left(L\underline{L}w - \frac{2}{r}\partial_r w\right).$$

Proof. Since $H_0 \equiv \sum \omega_i H_i = t\partial_r + r\partial_t$, we have

$$\begin{aligned} &tL_1 w + (r-t)(t\partial_r w + r\partial_t w) \\ &\quad = (\partial_t w)(t(r+t) + r(r-t)) + (\partial_r w)(t(r+t) + t(r-t)) + 2tw \\ &\quad = \bar{K}_0 w. \end{aligned}$$

Similarly,

$$\begin{aligned} &x_i L_1 w + (t-r)(t\partial_i w + x_i \partial_t w) + r(t-r)(\partial_i w - \omega_i \partial_r w) \\ &\quad = 2x_i w + x_i(\partial_t w)(r+t+t-r) + x_i(\partial_r w)(r+t-(t-r)) \\ &\qquad + (\partial_i w)(t(t-r) + r(t-r)) \\ &\quad = 2x_i S w + (t^2 - r^2)\partial_i w + 2x_i w = \bar{K}_i w. \end{aligned}$$

Formula (iii) has the interest of introducing L_2. The proof is straightforward : first, we check that

$$4K_0^2 w = (r+t)^4 L^2 w + (r-t)^4 \underline{L}^2 w + 2(t^2-r^2)^2 L\underline{L} w + 4(r+t)^3 L w + 4(t-r)^3 \underline{L} w.$$

Then

$$\bar{K}_0^2 w = (K_0 + 2t)(K_0 w + 2tw) = K_0^2 w + 4tK_0 w + 2(r^2 + 3t^2) w.$$

The strategy is then to express $L^2 w$ through $L_2 w$, and Lw through $L_1 w$, and this leads immediately to the formula. ◻

As a consequence, in a region $|r - t| \le C$, if $\sum_{k\le 1} |Z^k w| = O(t^{-1})$ and $|L_1 w| = O(t^{-2})$, we have $\bar{K}_\alpha w = O(t^{-1})$ instead of the expected $O(1)$. More precisely, for a solution admitting the smooth representation

$$u(x,t) = \frac{1}{r} F\left(r - t,\, \omega,\, \frac{1}{r}\right),$$

one easily obtains

$$\bar{K}_\alpha u(x,t) = \frac{1}{r}(\bar{F}_\alpha)\left(r - t,\, \omega,\, \frac{1}{r}\right)$$

for some new profile $\bar{F}_\alpha$. For instance,

$$\bar{F}_0 = -\sigma^2(\partial_\sigma F) - 2(1 - \sigma z)(\partial_z F),$$

and similar formula for $\bar{F}_i$.

There are of course formula for the products $\bar{K}_0\bar{K}_i$ and $\bar{K}_i\bar{K}_j$, similar to the one given here for $\bar{K}_0^2$. These formula imply that in a region $|r-t| \le C$ where we assume $\sum_{k\le 2}|Z^k w| = O(t^{-1})$, $L_1 w = O(t^{-2})$ and $L_2 w = O(t^{-3})$, we also have $\bar{K}_\alpha\bar{K}_\beta w = O(t^{-1})$. For a solution w of $\Box w = 0$, these estimates can be obtained just using the commutation formula of Lemma 2.2. In the present case of a smooth solution u of $\Box u + Q(u) = 0$, it is not algebraically clear how Q behaves under the action of $K_\alpha + 6x_\alpha$. This is the reason why we use conformal inversion in Section 4.

The following lemma shows how one can control $L_1 w$ and $L_2 w$.

Lemma 2.4. *Recalling the notation $\Delta_\omega = \sum R_i^2$ for the Laplace operator on the spheres, we have the formula*

(i) $$\underline{L}L_1 w = (r+t)\Box w + \frac{r+t}{r^2}\Delta_\omega w + r^{-1}\Big(Sw + \sum \omega_i H_i w\Big) + \frac{r-t}{r}\underline{L}w,$$

(ii) *With* $\underline{\mathcal{L}} = (r+t)L + 8 - (r+t)/r$,

$$\begin{aligned}\underline{L}L_2 w = (r+t)\underline{\mathcal{L}}\Box w + r^{-1}L_2 w + \frac{r+t}{r^2}L_1\Delta_\omega w - 2\frac{r+t}{r^2}L_1 w \\ + 3\frac{r^2-t^2}{r^3}\Delta_\omega w + 2\frac{(r-t)^2}{r^2}\underline{w} + 4\frac{t-r}{r^2}w.\end{aligned}$$

Proof. We have first

$$\begin{aligned}\underline{L}L_1 w &= (r+t)\underline{L}Lw + 2\underline{L}w = (r+t)\left(\Box w + \frac{2}{r}\partial_r w + \frac{1}{r^2}\Delta_\omega w\right) + 2\underline{L}w \\ &= (r+t)\Box w + \frac{r+t}{r^2}\Delta_\omega w + \frac{r+t}{r}Lw + \frac{r-t}{r}\underline{L}w.\end{aligned}$$

Next, since $L_2 w = (r+t)^2 L^2 w + 8(r+t)Lw + 8w$,

$$\underline{L}L_2 w = (r+t)^2\underline{L}L^2 w + 8(r+t)\underline{L}Lw + 8\underline{L}w.$$

Replacing systematically $\underline{L}L$ by $\Box + \frac{1}{r}(L - \underline{L}) + \frac{1}{r^2}\Delta_\omega$, we obtain

$$\underline{L}L^2 w = \left(L - \frac{1}{r}\right)\Box w + \frac{1}{r}L^2 w + \frac{1}{r^2}L\Delta_\omega w - \frac{3}{r^3}\Delta_\omega w - \frac{4}{r^2}\partial_r w.$$

Expressing repeatedly $L^2 w$ and Lw using $L_2 w$ and $L_1 w$, we get the desired expression. ◻

With the help of this lemma, we can obtain in a strip $|r-t| \leq C$, the desired estimates $L_1 w = O(t^{-2})$ and $L_2 w = O(t^{-3})$, provided we have a good control of $\Box w$ and of $\sum_{k \leq N} |Z^k w|$ for an appropriate N. For the present paper however, we need only estimate $L_1 u$, and this will be done in the next section 2.4.

Remark. To estimate $L_2 u$ in the framework of the present paper, Lemma 2.4 shows that it would be necessary to prove $\underline{L}Q(u) = O(t^{-4})$. This can actually be done in the following way: the estimation lemma from Section 1.1 shows that $Q(u) = O(t^{-3})$, an estimate also valid for $ZQ(u)$, etc. However, a more precise computation shows some cancellation in the expression of $rLQ(u)$, namely $rLQ(u) + 3Q(u) = O(t^{-4})$. Since we will not use this in the sequence, we omit the details.

2.4. Translation of the assumptions on u. In the whole paper, we will fix $t_0 > 2M$, and consider for $t \geq t_0$ the Cauchy problem with data on $\{t = t_0\}$

$$\Box u + Q(u) = 0,\ u(x, t_0) = u_0(x),\ (\partial_t u)(x, t_0) = u_1(x).$$

Since the Cauchy data vanish for $|x| \geq M$, the solution u (if it exists) vanishes for $r \geq M + t - t_0$. We will always use the conformal inversion I in these coordinates (x, t), and set $T_0 = -1/t_0$. Recall that we associate to the function u the function v defined by

$$u(I)(X, T) = (T^2 - R^2) v(X, T).$$

The image by I of the hyperboloid H_{t_0, t_1} defined in Section 3 is just $\{T = T_1 = -1/t_1\}$. The assumptions on u in the theorems of Section 3 have to be translated into assumptions on v. In the new coordinates (x, t), the assumptions of these theorems are that u is a C^∞ solution of $\Box u + Q(u) = 0$ in the region

$$\left\{ (x, t),\ t_0 \leq t \leq -\frac{1}{2T_1} + \left(r^2 + \frac{1}{4T_1^2} \right)^{1/2} \right\}$$

for some T_1, $-1/t_0 < T_1 < 0$.

Lemma 2.5. *The function v is a C^∞ function in the region*

$$\mathcal{T} = \left\{ (X, T),\ T_0 \leq T \leq T_1 < 0,\ R < |T| \right\}.$$

The functions $v(X, T_0)$, $(\partial_T v)(X, T_0)$ are compactly supported in $\{|X| < |T_0|\}$. Denoting by θ any smooth vector field in $\mathcal{T}$ tangent to the boundary $\{R = |T|\}$, the decay assumptions on u imply, for some constant C,

$$|v| + |\partial v| + |\theta \partial v| + (T^2 - R^2)|\partial^2 v| \leq C.$$

Proof. The plane $\{T = T_0\}$ corresponds under I to the upper branch of the hyperboloid $H_{T_0} = \{t^2 - r^2 + t/T_0 = 0\}$. Since $t_0 > 2M$, u vanishes identically near H_{T_0} but on a compact set, hence the properties of the traces of v on $\{T = T_0\}$ are established.

On $\bar{\mathcal{T}}$, $0 < -T_1 \le R - T \le -2T_0$. Hence, for any function w,

$$|w(X,T)| \le C \Leftrightarrow \left|[(T^2 - R^2)w](I)(x,t)\right| \le \frac{C}{t}.$$

In fact, $T^2 - R^2 = (R - T)|R + T|$, and $|R + T| = 1/(r + t)$. On the other hand, the fields S, R and

$$(R + T)(\partial_T + \partial_R) = S + \sum \omega_i H_i$$

generate all tangent fields θ. Hence the boundedness properties of v are implied by

$$|v| + |\partial v| + |\partial R v| + |\partial S v| + |\partial H v| \le C.$$

According to Lemmas 2.1, 2.2, these conditions on v are equivalent to

$$|u| + |\bar{K}_\alpha u| + |Ru| + |Su| + |Hu| + |\bar{K}_\alpha R u| + |\bar{K}_\alpha S u| + |\bar{K}_\alpha H u| \le \frac{C}{t}.$$

Lemmas 2.3, 2.4 show that $|\bar{K}_\alpha u| = O(t^{-1})$ is implied by $|Zu| = O(t^{-1})$, $|\Delta_\omega u| = O(t^{-1})$ and $|\Box u| = O(t^{-3})$: this last requirement is implied by the estimation lemma (Lemma 1.1), since $|\Box u| = |Q(u)|$. To prove the further bounds on $\bar{K}_\alpha Z u$, we proceed in exactly the same way, using the commutation lemma, which shows that also $|ZQ(u)| = O(t^{-3})$. □

2.5. The equation on v. The computation in this section is due to Christodoulou [11], who even performed it in a more general context (embedding of the Minkowski space into the Einstein cylinder, see also [12]). Since we need very precise information on the equation satisfied by v, beyond the fact that no singular terms appear, we do it again.

Lemma 2.6. *Recall that v is defined by $u(I)(X,T) = (T^2 - R^2)v(X,T)$.*

(i) *If the function u satisfies the equation $\Box u + Q(u) = 0$, the function v satisfies the equation $\Box v + \bar{Q}(v) = 0$, where*

$$\begin{aligned}\bar{Q}(v) = {}& 4[2A(Sv + v) + Y_1 v](S^2 v + 3Sv + 2v)\\ &+ 4(Sv + v)[Y_2 S v + Y_2 v + (g^{\alpha\beta\gamma}\varepsilon_\alpha \delta_{\alpha\beta} X_\gamma)(Sv + v)]\\ &+ 2(T^2 - R^2)(Sv + v)(g^{\alpha\beta\gamma}\varepsilon_\alpha\varepsilon_\beta X_\gamma \partial^2_{\alpha\beta} v + g^{\alpha\beta\gamma}\varepsilon_\alpha\varepsilon_\gamma\delta_{\alpha\beta}\partial_\gamma v)\\ &+ 4(T^2 - R^2)(Y_3 S v + Y_3 v) + (T^2 - R^2)^2 g^{\alpha\beta\gamma}\varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma(\partial_\gamma v)(\partial^2_{\alpha\beta} v).\end{aligned}$$

We have used here the following notation : ε_α *is* -1 *for* $\alpha = 0$ *and* 1 *otherwise; the vector field* $Y_{\alpha\beta}$ *is*

$$Y_{\alpha\beta} = \varepsilon_\alpha X_\beta \partial_\alpha + \varepsilon_\beta X_\alpha \partial_\beta.$$

The vector fields Y_1, Y_2 *and* Y_3 *are defined by*

$$\begin{aligned} Y_1 &= g^{\alpha\beta\gamma} X_\alpha X_\beta \varepsilon_\gamma \partial_\gamma, \\ Y_2 &= g^{\alpha\beta\gamma} X_\gamma Y_{\alpha\beta} = 2g^{\alpha\beta\gamma} \varepsilon_\alpha X_\beta X_\gamma \partial_\alpha, \\ Y_3 &= \tfrac{1}{2} g^{\alpha\beta\gamma} \varepsilon_\gamma (\partial_\gamma v) Y_{\alpha\beta} = g^{\alpha\beta\gamma} \varepsilon_\alpha \varepsilon_\gamma X_\beta (\partial_\gamma v) \partial_\alpha. \end{aligned}$$

(ii) *The vector fields* Y_1 *and* Y_2 *are tangent to the boundary* $\{R + T = 0\}$,

(iii) *The boundary* $\{R + T = 0\}$ *is characteristic for the operator* $g^{\alpha\beta\gamma} \varepsilon_\alpha \varepsilon_\beta X_\gamma \partial^2_{\alpha\beta}$.

Proof. By conformal inversion, the transform of $\partial_\alpha u$ is $\Delta \bar{K}_\alpha v$, according to lemma 2.2 (here, $\Delta = T^2 - R^2$). Hence the transform of $\partial^2_{\alpha\beta} u$ is $\Delta \bar{K}_\alpha \bar{K}_\beta v$. Now, since $K_\alpha = 2X_\alpha S + \varepsilon_\alpha \Delta \partial_\alpha$,

$$\begin{aligned} \bar{K}_\alpha \bar{K}_\beta v &= 2X_\alpha S(\bar{K}_\beta v) + \varepsilon_\alpha \Delta \partial_\alpha (\bar{K}_\beta v) + 2X_\alpha \bar{K}_\beta v \\ &= 2X_\alpha S(2X_\beta(Sv + v) + \varepsilon_\beta \Delta \partial_\beta v) + \varepsilon_\alpha \Delta \partial_\alpha (2X_\beta(Sv + v) + \varepsilon_\beta \Delta \partial_\beta v) \\ &\qquad + 2X_\alpha (2X_\beta (Sv + v) + \varepsilon_\beta \Delta \partial_\beta v) \\ &= 4X_\alpha X_\beta [S^2 v + 3Sv + 2v] + 2\Delta[Y_{\alpha\beta} Sv + Y_{\alpha\beta} v + \varepsilon_\alpha \delta_{\alpha\beta}(Sv + v)] \\ &\qquad + \Delta^2 \varepsilon_\alpha \varepsilon_\beta \partial^2_{\alpha\beta} v. \end{aligned}$$

In the sum $g^{\alpha\beta\gamma}(\bar{K}_\gamma v)(\bar{K}_\alpha \bar{K}_\beta v)$, the only terms which do not contain a factor Δ are

$$8(Sv + v)(S^2 v + 3Sv + 2v)(g^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma).$$

By the algebraic properties displayed in Lemma 1.3, $g^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma = A(X)\Delta$; hence the transform of $Q(u)$ is $\Delta^3 \bar{Q}(v)$, with

$$\begin{aligned} \bar{Q}(v) = {} & 4[S^2 v + 3Sv + 2v][2A(Sv + v) + g^{\alpha\beta\gamma} X_\alpha X_\beta \varepsilon_\gamma \partial_\gamma v] \\ & + 4(Sv + v) g^{\alpha\beta\gamma} X_\gamma (Y_{\alpha\beta} Sv + Y_{\alpha\beta} v + \varepsilon_\alpha \delta_{\alpha\beta}(Sv + v)) \\ & + 2\Delta[(Sv + v)(g^{\alpha\beta\gamma} \varepsilon_\alpha \varepsilon_\beta X_\gamma \partial^2_{\alpha\beta} v + g^{\alpha\beta\gamma} \varepsilon_\alpha \varepsilon_\gamma \delta_{\alpha\beta} \partial_\gamma v) \\ & \quad + g^{\alpha\beta\gamma} \varepsilon_\gamma (\partial_\gamma v)(Y_{\alpha\beta} Sv + Y_{\alpha\beta} v)] + \Delta^2 g^{\alpha\beta\gamma} \varepsilon_\alpha \varepsilon_\beta \varepsilon_\gamma (\partial_\gamma v)(\partial^2_{\alpha\beta} v), \end{aligned}$$

which is the result stated in the lemma.

Since $\partial_\alpha(R+T) = \varepsilon_\alpha(X_\alpha)/R$ on $R+T=0$, we have there

$$RY_1(R+T) = g^{\alpha\beta\gamma}X_\alpha X_\beta X_\gamma = 0,$$

showing that Y_1 is tangent to the boundary. Similarly, we have on the boundary

$$RY_2(R+T) = 2g^{\alpha\beta\gamma}\varepsilon_\alpha X_\beta X_\gamma \varepsilon_\alpha X_\alpha = 0,$$

hence (ii) is proved. Finally, on $R+T=0$,

$$R^2 g^{\alpha\beta\gamma}\varepsilon_\alpha\varepsilon_\beta X_\gamma[\partial_\alpha(R+T)][\partial_\beta(R+T)] = g^{\alpha\beta\gamma}X_\alpha X_\beta X_\gamma = 0. \qquad \square$$

2.6. The energy condition. Let us denote by $\mathcal{L}$ the linearized operator on v corresponding to the equation $\square v + \bar{Q}(v) = 0$, and by $\mathcal{L}_2$ its principal part. According to Lemma 2.6,

$$\mathcal{L}_2 = \square + 4aS^2 + 4(Sv+v)Y_2S + \Delta c^{\alpha\beta}\partial^2_{\alpha\beta} + 4\Delta Y_3 S \equiv \square + \ell^{\alpha\beta}\partial^2_{\alpha\beta},$$

where

$$\begin{aligned}
\Delta &= T^2 - R^2,\\
Y_3 &= g^{\alpha\beta\gamma}\varepsilon_\alpha\varepsilon_\gamma X_\beta(\partial_\gamma v)\partial_\alpha,\\
a &= 2A(Sv+v) + Y_1 v,\\
c^{\alpha\beta} &= [g^{\alpha\beta\gamma}\varepsilon_\alpha\varepsilon_\beta][2X_\gamma(Sv+v) + \Delta\varepsilon_\gamma\partial_\gamma v].
\end{aligned}$$

We define now a smallness assumption on v, the "energy condition."

Definition (Energy Condition.). *Let $T_0 < T_1 < 0$. We say that u satisfies the energy condition in the region*

$$\left\{(x,t),\ t_0 \le t \le -\frac{1}{2T_1} + \left(r^2 + \frac{1}{4T_1^2}\right)^{1/2},\ t > r\right\}$$

if the function v satisfies in the region

$$\{(X,T),\ R < |T|,\ T_0 \le T \le T_1\}$$

the following two conditions : For some constant $\alpha_0 < 1$,

(i) $|\ell^{00}| \le \alpha_0$,

(ii) $|\sum \ell^{ij}\xi_i\xi_j| \le \alpha_0|\xi|^2$.

Condition (i) ensures that the surfaces $\{T = C\}$ are (uniformly) non-characteristic, the coefficient of ∂_T^2 in $\mathcal{L}_2$ being $1 + \ell^{00}$. Condition (ii) ensures that the energy corresponding to the operator $\mathcal{L}_2$, the multiplier ∂_T and the surfaces $\{T = C\}$ is at least $(1 - \alpha_0)$ times the standard energy. Note in particular that these conditions imply that ∂_T is everywhere timelike (in the sense of the metric corresponding to the operator).

We formulated the energy condition as above to emphasize its geometric character. It is clear, however, that it is a smallness assumption on u, for which we give a rough sufficient condition in the following lemma.

Lemma 2.7. *There exists a constant ε_0 depending only on g, such that if*

$$|u| + |Ru| + |Su| + |H_0 u| \leq \frac{\varepsilon_0}{t(t-r)|T_0|^3}, \quad H_0 = \sum \omega_i H_i = r\partial_t + t\partial_r,$$

the energy condition on u is satisfied.

Proof. We claim first that there exists an ε_1, depending only on g, such that the conditions

$$|v| + |Rv| + |Sv| + |H_0 v| \leq \varepsilon_1 |T_0|^{-3}$$

imply the energy condition. In fact, Y_1 and Y_2 can be expressed as a linear combination of R, S and $(T+R)(\partial_T + \partial_R)$ with coefficients homogeneous of order 1. Similarly, ΔY_3^0 is a combination of Sv and $H_0 v$ with coefficients homogeneous of degree 2. Hence the first energy condition is ensured if ε_1 is small enough.

For the second condition, we inspect the coefficients ℓ^{ij}: they all are linear combinations of Rv, Sv and $H_0 v$ with coefficients homogeneous of degree 3 ; hence the second energy condition is also satisfied if ε_1 is small enough. Finally, the given condition on u is just the translation of the conditions on v. $\square$

3. MAIN RESULTS

Let $M > 0$, and assume given two functions $u_0, u_1 \in C_0^\infty(\mathbf{R}_x^3)$, vanishing for $|x| \geq M$. Choose $t_0 > 2M$, $t_1 > 2(t_0 - M)$. Define the (half) hyperboloid H_{t_0,t_1} by

$$H_{t_0,t_1} = \left\{ (x,t),\ t = -t_0 + \frac{t_1}{2} + \left(r^2 + \frac{t_1^2}{4} \right)^{1/2} \right\}.$$

Assume that there exists, in the region between $\{t = 0\}$ and H_{t_0,t_1}, a C^∞ solution u of the Cauchy problem

$$\Box u + g^{\alpha\beta\gamma}(\partial_\gamma u)(\partial^2_{\alpha\beta} u) = 0,$$
$$u(x,0) = u_0(x),\ (\partial_t u)(x,0) = u_1(x).$$

Recall that we have assumed that g satisfies the null condition, with $g^{00\gamma} = 0$. We make on u two series of assumptions :

Smallness assumptions. *We assume the following:*

(i) *The linearized operator* $\Box + g^{\alpha\beta\gamma}(\partial_\gamma u)\partial^2_{\alpha\beta}$ *is strictly hyperbolic with respect to* t,

(ii) *For some* $\underline{t} > 0$, *the function* u *satisfies the energy condition of Section 2.6 for* $t \geq \underline{t}$.

Decay assumptions. *For some constant* C, *the function* u *satisfies*

$$\sum_{k\leq 2} |Z^k \partial u| \leq \frac{C}{1+t}, \quad \sum_{k\leq 1} |Z^k \Delta_\omega u| \leq \frac{C}{1+t}, \quad \Delta_\omega = \sum R_i^2.$$

Note that these assumptions imply in particular that the region $t \geq \underline{t}$ is a domain of determination. Our first result gives a representation of u analogous to the representation of the free solutions of $\Box u = 0$ (see [12]).

Theorem (Representation Theorem). *Under the assumptions above, there exists a* C^∞ *function*

$$F : \left[t_0 - \frac{t_1}{2} - 1,\ M\right] \times S^2 \times [0,1] \to \mathbf{R}$$

such that, for $r \geq 1$,

$$u(x,t) = \frac{1}{r} F\left(r - t,\ \omega,\ \frac{1}{r}\right).$$

The second result expresses the stability of u.

Stability Theorem 1. *Let the assumptions above on* u *be satisfied. Let* $\underline{u}_0$, $\underline{u}_1$ *be* C_0^∞ *functions on* $\mathbf{R}^3_x$, *vanishing for* $|x| \geq M$. *There exists* $\varepsilon_0 > 0$ *such that, for* $\varepsilon \leq \varepsilon_0$, *the solution of the Cauchy problem*

$$\Box w + g^{\alpha\beta\gamma}(\partial_\gamma w)(\partial^2_{\alpha\beta} w) = 0,$$
$$w(x,0) = u_0(x) + \varepsilon \underline{u}_0(x), \quad (\partial_t w)(x,0) = u_1(x) + \varepsilon \underline{u}_1(x)$$

exists globally in the same region as u *and has a representation*

$$w(x,t) = \frac{1}{r} G\left(r - t,\ \omega,\ \frac{1}{r}\right),$$

where G *has the same smoothness properties as* F.

In the following corollary, in order to make scaling invariant assumptions, we define M as the smallest M such that the supports of the Cauchy data u_0 and u_1 are contained in the ball $|x| \leq M$.

Corollary. *Assume* $M = 1$. *Then there exists* $\varepsilon_1 > 0$ *such that if, for some* $\nu > \frac{1}{2}$, *the solution* u *satisfies the assumptions:*

Smallness Assumptions: *For some* $\underline{t} > 0$ *and* $t \geq \underline{t}$, $\sum_{k\leq 2} |Z^k \partial u| \leq \varepsilon_1 (1+t)^{-\nu}$.
Decay Assumptions: *For some constant* C, $\sum_{k\leq 4} |Z^k \partial u| \leq C(1+t)^{-\nu}$.

then the conclusions of both theorems are true.

Finally, as explained in the introduction, the following theorem indicates a case where stability is proved directly, without knowing about any representation of the solution u. In contrast with the two preceding theorems, the solution u is assumed here to exist in the whole of $\mathbf{R}^3_x \times [0, +\infty[$.

Stability Theorem 2. *Let* $u \in C^\infty(\mathbf{R}^3 \times [0, \infty[)$. *Assume that* u *is a solution of* $\Box u + Q(u) = 0$ *and satisfies the following properties :*
Smallness Property: *For some* $\alpha_0 < 1$, $|g^{ij\gamma}(\partial_\gamma u)\xi_i \xi_j| \leq \alpha_0 |\xi|^2$.
Decay property: *For some* C_0, $\sum_{k\leq 7} |Z^k \partial u| \leq C_0 (1+t)^{-1} \langle r-t \rangle^{-1/2}$.

Then the solution u *is stable, in the same sense as in the first stability theorem.*

Comments.

1. The possibility of a representation formula for the solution and its stability are of course properties of the solution for large t : that explains the formulation in the theorems.
2. We formulate the "energy condition", which is the only smallness condition on the solution u, in terms of the function v, which depends on the choice of t_0 (see Sections 2.4– 2.6). It is of course possible to translate this explicit condition on v into a condition on u, either directly by replacing v by u, or indirectly by discussing the energy density produced on the hyperboloids by integration of $(\Box u + Q(u))(\bar{K}_0 u)$. However, this last approach did not seem very transparent to us either. The point of this explicit smallness condition is its geometric character, which makes it, we believe, easily acceptable. It seems to us reasonable to think that the weakest requirement of the energy condition is obtained by taking for M its minimum and choosing t_0 as close to $2M$ as possible.
3. As computed in Section 2.3, if the solution admits the smooth representation
$$u(x,t) = \frac{1}{r} F\left(r-t,\ \omega,\ \frac{1}{r}\right),$$
then $\bar{K}_\alpha u$ admits a similar representation, with a new profile $\bar{F}_\alpha$. Thus a necessary condition for u to admit such a representation is $\bar{K}^l u = O(t^{-1})$ for all products of l operators $\bar{K}_\alpha$. It turns out, thanks to the conformal inversion, that this condition is also sufficient. However, we were not able to produce a direct proof of this (that is, working only on the "u side").
4. If $u(x,t) = (1/r)F(r-t,\ \omega,\ 1/r)$ were a solution of $\Box u = 0$, F would satisfy the equation
$$2\partial^2_{\sigma z} F + z\partial^2_z F - \Delta_\omega F - 2z\partial_z F = 0, \quad \sigma = r - t, \quad z = \frac{1}{r}.$$

Hence F would be determined, at least as a formal power series in $1/r$, from its value at infinity $F(\sigma, \omega, 0)$. The same occurs here for the profile F of a solution of $\Box u + Q(u) = 0$.

5. The formulation of the stability in the stability theorems can evidently be improved by introducing an appropriate norm.
6. For a discussion of the link between instability and blowup at infinity, see Section 5.
7. The corollary after the theorems is just one example of the following possibility : we can weaken the decay rate to any $t^{-\nu}$ $(\nu > 0)$ if we ask

$$\sum_{k \le N} |Z^k \partial u| \le C(1+t)^{-\nu}$$

for N big enough.

8. We discussed only the representation and stability of solutions in an exterior region roughly $t \le \underline{t} + r$. The reason for this is technical : to obtain the smoothness of v up to the origin from the equation on v, we have to require boundedness (or at least reasonable behavior) of a certain number of derivatives of v. For instance, $|\partial_T v| \le C$ is equivalent to $|\bar{K}_0 u| \le C(1+t)^{-1}\langle r - t\rangle^{-1}$. We can of course take these decay properties of u as our assumptions, and obtain then representation and stability for u. But we do not know how to obtain these decay properties of u from decay properties of, say, $Z^k u$ for $k \le N$, using the equation on u.
9. In Stability Theorem 2 we do not know whether u has a smooth representation in the sense of the representation theorem, but we believe so.
10. It is possible of course to weaken the decay assumptions on u in the second stability theorem by using more Z fields, as indicated in Lemma 1.4. We kept the assumptions on u in this form since they are expressed in the same terms as that on the perturbation v, which come from Klainerman's inequality.
11. The assumption that the Cauchy data are compactly supported does not seem important to us. It can certainly be replaced by a strong enough decay at infinity, using the embedding of a Minkowski space into the Einstein cylinder as in [11]. We chose to use conformal inversion since it gives simpler computations.

4. Proof of the Theorems

Recall here from Section 2.4 that we introduce a new (translated) t-coordinate by considering u as the solution for $t \ge t_0$ of the Cauchy problem

$$\Box u + Q(u) = 0, \quad u(x, t_0) = u_0(x), \quad (\partial_t u)(x, t_0) = u_1(x).$$

We use then conformal inversion in these coordinates.

4.1. Determination domains. Recall from Lemma 2.5 the notation

$$\mathcal{T} = \Big\{(X,T),\ T_0 \le T \le T_1 < 0,\ R < |T|\Big\}.$$

Let us choose an integer $k > T_1 - T_0$, and define the subdomain $\mathcal{T}_\varepsilon \subset \mathcal{T}$ by

$$\mathcal{T}_\varepsilon = \Big\{(X,T),\ T_0 \le T \le T_1,\ R \le -T - \varepsilon(k + T - T_1)\Big\}.$$

This domain is a compact truncated cone of revolution contained in $\mathcal{T}$, with normal "vector" to the lateral boundary $N = (\omega, 1+\varepsilon)$.

Lemma 4.1. *Let $\mathcal{L}$ be the linearized operator on v of the equation $\Box v + \bar{Q}(v) = 0$. Integrating as usual the product $(\mathcal{L}w)(\partial_T w)$ in $\mathcal{T}_\varepsilon \cap \{T_0 \le T \le T'\}$, we obtain energy terms in w on the plane $\{T = T'\}$ and on the lateral boundary Λ_ε of the domain :*

(i) *There exists $\alpha_0 < 1$ such that, on $\{T = T'\}$, the energy is greater than*

$$(1-\alpha_0)\frac{1}{2}\int \Big[(\partial_T w)^2 + \sum(\partial_i w)^2\Big]\,\mathrm{d}X.$$

(ii) *On Λ_ε, the energy is non-negative.*

Proof. The first point is exactly the energy condition of Section 2.6. For the second point, we note that the energy condition implies that ∂_T is timelike. Thus it is enough to check that the normal to Λ_ε (in the sense of the metric) is also timelike, that is, $p(N) \ge 0$, p denoting the principal symbol of $\mathcal{L}$. Since $N = (N_i, N_0) = (\omega, 1+\varepsilon)$, we have

$$\begin{aligned} p(N) = N_0^2 - \sum N_i^2 &+ 4[2A(Sv+v) + Y_1 v]s^2 + 4(Sv+v)y_2 s \\ &+ 2\Delta(Sv+v)(g^{\alpha\beta\gamma}\varepsilon_\alpha\varepsilon_\beta X_\gamma N_\alpha N_\beta) + 4\Delta y_3 s \\ &+ \Delta^2(g^{\alpha\beta\gamma}\varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma(\partial_\gamma v)N_\alpha N_\beta), \end{aligned}$$

s, y_2, and y_3 being the symbols of S, Y_2 and Y_3 taken on N. Now, on Λ_ε,

$$R + T = O(\varepsilon),\ \Delta = O(\varepsilon),\ X_\alpha = R\omega_\alpha,\ \omega_\alpha = \varepsilon_\alpha N_\alpha + O(\varepsilon).$$

Hence

$$s = TN_0 + \sum X_i N_i = R + T + \varepsilon T = O(\varepsilon), \quad y_2 = g^{\alpha\beta\gamma}\varepsilon_\alpha N_\alpha X_\beta X_\gamma = O(\varepsilon),$$
$$g^{\alpha\beta\gamma}\varepsilon_\alpha\varepsilon_\beta X_\gamma N_\alpha N_\beta = O(\varepsilon).$$

These estimates correspond of course to points (ii) and (iii) of Lemma 2.6. We thus obtain

$$p(N) = 2\varepsilon + \varepsilon^2 + O(|\partial v|\varepsilon^2).$$

Since, by the assumptions of the theorems, we already know $|\partial v| \le C$ everywhere independently of ε, the claim is proved for small enough ε. ◻

Lemma 4.2. *For the linearized operator $\mathcal{L}$, we have the standard energy inequality in $\mathcal{T}_\varepsilon$,*

$$|\partial w(T')|_{L^2} \leq C|\partial w(T_0)|_{L^2} + C\int_{T_0}^{T'} |\mathcal{L}w(\cdot,s)|_{L^2}\, \mathrm{d}s.$$

Here, the constants do not depend on ε.

Proof. Since, according to the preceding lemma, we control the standard energy on the surfaces $\{T' = C\}$, it is enough, to prove the lemma,

(i) to control $|\partial a|_{L^\infty}$, if a is a coefficient of the principal part,
(ii) to control $|b|_{L^\infty}$, if b is a coefficient of the lower order terms.

Coefficients such as a involve only at most one tangential derivative θv of v ; coefficients such as b involve at most two derivatives of v, one of them being tangential. In both cases, the assumptions on u, translated into properties of v (see Lemma 2.5) garantee a uniform control in L^∞ norm. ◻

4.2. End of the proof of the representation theorem.

a. To prove that $v \in C^\infty(\bar{\mathcal{T}})$, we proceed as usual, working in $\mathcal{T}_\varepsilon$. We commute ∂_X^α with the equation on v (after normalizing the coefficient of $\partial_T^2 v$ to one), and use interpolation in X (for fixed T). One has to be careful, however, about the structure of the terms, since we do not control the L^∞ norm of $\partial^2 v$, but only of $\partial\theta v$. Symbolically, we can write the equation on v as

$$\Box v + \bar{Q}(v) = \Box v + (v+\theta v)\partial\theta v + v^2 + v\theta v + (\theta v)^2 + (\theta v)\partial v = 0.$$

The first order terms do not cause any problem. For a typical second order term $(\theta v)(\partial\theta v)$, we write

$$\partial_X^k[(\theta v)(\partial\theta v)] = (\theta v)\partial\theta\partial_X^k v + (\partial\theta v)(\theta\partial_X^k v) + r,$$

where the first two terms belong to the linearized operator acting on $w = \partial_X^k v$, and

$$|r| \leq C \sum_{1\leq l\leq k-1} [\partial_X^{\ell-1}\partial\theta v|\,|\partial_X^{k-\ell}\partial\theta v|.$$

Using now the inequality (see for instance [10])

$$\|(\partial_X^p f)(\partial_X^q g)\|_{L^2} \leq C\Big(\|f\|_{L^\infty}\,\|g\|_{H^{p+q}} + \|g\|_{L^\infty}\,\|f\|_{H^{p+q}}\Big)$$

with the functions $f = \partial\theta v$, $g = \partial\theta v$ and $p+q = k-1$, we get

$$\|r\|_{L^2} \leq C\|\partial\theta v\|_{L^\infty}\,\|\partial v\|_{H^k}.$$

In this way, we get a control of the L^2 norm in $\mathcal{T}_\varepsilon$ of all terms $\partial_X^k v$, for all k ; this control is, of course, uniform in ε. Using the equation, we extend this control to all derivatives of v. Since ε is arbitrary, we get that $v \in C^\infty(\bar{\mathcal{T}})$.

b. We have given the above argument in the full domain $\mathcal{T}_\varepsilon$ in order to simplify the notation. If we assume the energy condition only for $t \geq \underline{t}$ as in the theorems, we will in fact work for v in the domain

$$\mathcal{T}_\varepsilon \cap \{R \geq \left(T^2 + \frac{T}{\underline{t}}\right)^{1/2}\}.$$

Since v is C^∞ in a neighborhood of the boundary

$$R = \left(T^2 + \frac{T}{\underline{t}}\right)^{1/2}$$

of this domain, the argument runs in exactly the same way.

c. We now have

$$u(x,t) = \frac{1}{t^2 - r^2} v\left(\frac{x}{t^2 - r^2}, -\frac{t}{t^2 - r^2}\right).$$

Since $t/(t+r)$ and $r/(t+r)$ are smooth functions of $(t-r)/r$, we obtain the representation of u in the coordinates (x,t), which yields the representation with the original coordinates r, ω and $t - r$.

d. The stability of u is a consequence of the stability of v. □

4.3. Proof of Stability Theorem 2

4.3.1. Set

$$P_z = \Box + g^{\alpha\beta\gamma}(\partial_\gamma z)\partial^2_{\alpha\beta},$$

where z is some smooth given function (it will be $u + v$ in application). We first establish an energy inequality for P_z.

Lemma 4.3 (Energy inequality).
Assume that z satisfies the following properties:
Smallness Property: *For some $\alpha_1 < 1$, $|g^{ij\gamma}(\partial_\gamma z)\xi_i\xi_j| \leq \alpha_1|\xi|^2$,*
Decay Property: *For some C, $\sum_{k\leq 1}|Z^k\partial z| \leq \underline{C}(1+t)^{-1}\langle r - t\rangle^{-1/2}$.*
Then, with $\eta = 0.1$, $T_i = \partial_i + \omega_i\partial_t$ and

$$E_w(t) = \frac{1}{2}\int\left[(\partial_t w)^2 + \sum(\partial_i w)^2\right](x,t)\,dx,$$

the following energy inequality holds:

$$E_w(t) + \int_{0\leq s\leq t}\langle r - s\rangle^{-1-\eta}\sum|T_i w|^2\,dx\,ds$$
$$\leq \left\{CE_w(0) + C\int_{0\leq s\leq t}|P_z w|\,|\partial_t w|\,dx\,ds\right\}\exp C(\underline{C} + \underline{C}^2).$$

Proof. We take essentially the proof from [3], but repeat it for convenience.

a. Note the pointwise inequality

$$|T_i w| \le C(1+t)^{-1} \sum |Zw|.$$

b. With $a = a(r-t)$, we have first

$$(\square w)(\partial_t w)e^a = \frac{1}{2}\partial_t\Big[e^a(\partial_t w)^2 + e^a \sum(\partial_i w)^2\Big] + \sum \partial_i(\cdots) + \frac{1}{2}e^a a' \sum (T_i w)^2.$$

From now on, we choose

$$a(s) = \int_{-\infty}^{s} \langle\sigma\rangle^{-1-\eta} d\sigma,$$

hence $0 \le a \le C$. We also have (omitting indices in q_1 and q_2)

$$\begin{aligned} 2e^a(\partial_\gamma z)(\partial^2_{\alpha\beta} w)(\partial_t w) &= \div(\cdots) + e^a(a' q_1 + q_2), \\ q_1 &= -(\partial_\gamma z)[\omega_\alpha(\partial_\beta w)(\partial_t w) + \omega_\beta(\partial_\alpha w)(\partial_t w) + (\partial_\alpha w)(\partial_\beta w)], \\ q_2 &= -(\partial_\alpha w)(\partial_t w)(\partial^2_{\beta\gamma} z) - (\partial_\beta w)(\partial_t w)(\partial^2_{\alpha\gamma} z) + (\partial_\alpha w)(\partial_\beta w)(\partial^2_{t\gamma} z). \end{aligned}$$

c. We use now systematically the identity $\partial_\alpha = T_\alpha - \omega_\alpha \partial_t$, with $T_0 = 0$. The coefficient of $-(\partial_\gamma z)$ in q_1 is

$$\begin{aligned} &\omega_\alpha(T_\beta w - \omega_\beta \partial_t w)(\partial_t w) + \omega_\beta(T_\alpha w - \omega_\alpha \partial_t w)(\partial_t w) \\ &\qquad + (T_\alpha w)(\partial_\beta w) - \omega_\alpha(\partial_t w)(T_\beta w - \omega_\beta \partial_t w) \\ &\qquad = -\omega_\alpha\omega_\beta(\partial_t w)^2 + (T_\alpha w)(\partial_\beta w) + \omega_\beta(T_\alpha w)(\partial_t w). \end{aligned}$$

When summing with the coefficients $g^{\alpha\beta\gamma}$, using the null condition, we get symbolically

$$g^{\alpha\beta\gamma} q_1 = (\partial z)(\partial w)(Tw) + (Tz)(\partial w)^2.$$

Proceeding similarly with q_2, we obtain symbolically

$$\begin{aligned} q_2 &= (\partial w)(Tw)(\partial^2 z) + b(\partial_t w)^2, \\ b &= \omega_\alpha \partial^2_{\beta\gamma} z + \omega_\beta \partial^2_{\alpha\gamma} z + \omega_\alpha\omega_\beta \partial^2_{t\gamma} z. \end{aligned}$$

Again, we can write symbolically

$$b = \omega_\alpha\omega_\beta\omega_\gamma(\partial_t^2 z) + T\partial z,$$

which gives upon summing

$$g^{\alpha\beta\gamma} q_2 = (\partial w)(Tw)(\partial^2 z) + (\partial w)^2(T\partial z).$$

d. The terms containing $(\partial w)^2$ in $a' g^{\alpha\beta\gamma} q_1$ and $g^{\alpha\beta\gamma} q_2$ have a coefficient bounded by

$$C(1+t)^{-1}\Big\{\langle r-t\rangle^{-1-\eta}\sum|Zz|+\sum|Z\partial z|\Big\}\le C\underline{C}(1+t)^{-2}.$$

The terms in $(Tw)(\partial w)$ in $a' g^{\alpha\beta\gamma} q_1$ and $g^{\alpha\beta\gamma} q_2$ have a coefficient bounded by

$$|\partial^2 z| + a'|\partial z| \le C\underline{C}(1+t)^{-1}\langle r-t\rangle^{-3/2}.$$

e. The energy term for the operator P is

$$\frac{1}{2}\int\Big[e^a(\partial_t w)^2 + e^a\sum(\partial_i w)^2 + e^a g^{ij\gamma}(\partial_\gamma z)(\partial_i w)(\partial_j w)\Big](x,t)\,dx.$$

Hence the smallness assumption on z implies that it is bounded below by

$$\frac{1}{2}\int\Big\{e^a(1-\alpha_1)\Big[(\partial_t w)^2+\sum(\partial_i w)^2\Big]\Big\}(x,t)\,dx.$$

This implies that the terms $\int C\underline{C}(1+t)^{-2}|\partial w|^2$ can be handled using Gronwall's lemma. To handle the terms in $(Tw)(\partial w)$, we write, with $\eta' > 0$ as small as desired,

$$\begin{aligned}&C\underline{C}\int_{0\le s\le t}\langle r-s\rangle^{-3/2}(1+s)^{-1}|Tw|\,|\partial w|\,dx\,ds\\ &\le \eta'\int_{0\le s\le t}\langle r-s\rangle^{-1-\eta}\sum(T_i w)^2\,dx\,ds + CC_{\eta'}\underline{C}^2\int_{0\le s\le t}(1+s)^{-2}|\partial w|^2\,dx\,ds.\end{aligned}$$

Choosing η' small enough, and using Gronwall's lemma, we obtain the result. $\square$

4.3.2. Let $u+v$ be the solution corresponding to the perturbed data,

$$v(x,0)=\varepsilon\underline{u}_0(x),\ (\partial_t v)(x,0)=\varepsilon\underline{u}_1(x).$$

We proceed by induction on time, and make on v the induction hypothesis

$$\sum_{k\le 6}\|\partial Z^k v\|_{L^2}\le C_1\varepsilon.$$

Using the Klainerman inequality, this implies

$$\sum_{k\le 4}|Z^k\partial v|(x,t)\le CC_1\varepsilon(1+t)^{-1}\langle r-t\rangle^{-1/2}.$$

The equation satisfied by v is, with $z=u+v$,

$$P_z v + g^{\alpha\beta\gamma}(\partial^2_{\alpha\beta}u)(\partial_\gamma v)=0.$$

Taking into account the smallness assumption on u, we see that the smallness assumption required on z in Lemma 4.3 is satisfied if $C_1\varepsilon$ is small enough, and the decay assumption is satisfied with $\underline{C} = C(C_0 + C_1\varepsilon)$. We write the additional term as

$$(\partial^2 u)(Tv) - c(\partial_t v),\ c = g^{\alpha\beta\gamma}\omega_\gamma\partial^2_{\alpha\beta}u.$$

Just as before, we have $|c| \le C(1+t)^{-1}|Z\partial u| \le CC_0(1+t)^{-2}$. Hence

$$E_v(t) + \int_{0\le s\le t}\langle r-s\rangle^{-1-\eta}\sum(T_i v)^2\,\mathrm{d}x\,\mathrm{d}s \le CE_v(0)\exp C(\underline{C} + \underline{C}^2).$$

4.3.3. We now take $k \le 6$ and commute a product Z^k with the equation on v. In this process, we use repeatedly the commutation lemma of Section 1.1. We obtain

$$P_z(Z^k v) + g^{\alpha\beta\gamma}(\partial^2_{\alpha\beta}(u+v))\partial_\gamma Z^k v + R = 0,$$

$$R = \sum_{p\le k,\ q\le k-1,\ p+q\le k} h^{\alpha\beta\gamma}_{pq}(\partial_\gamma Z^p u)(\partial^2_{\alpha\beta}Z^q v)$$
$$+ \sum_{p\le k-1,\ q\le k-1,\ p+q\le k} \ell^{\alpha\beta\gamma}_{pq}(\partial_\gamma Z^p v)(\partial^2_{\alpha\beta}Z^q v) + \sum_{p\le k,\ q\le k-1,\ p+q\le k} m^{\alpha\beta\gamma}_{pq}(\partial^2_{\alpha\beta}Z^p u)(\partial_\gamma Z^q v).$$

Here, for each couple (p,q), the constants $h^{\alpha\beta\gamma}_{pq}$, $\ell^{\alpha\beta\gamma}_{pq}$ and $m^{\alpha\beta\gamma}_{pq}$ satisfy the null condition. We will use the energy inequality for P_z ; we have to control the additional terms

$$g^{\alpha\beta\gamma}(\partial^2_{\alpha\beta}z)(\partial_\gamma w)(\partial_t w),\ R(\partial_t w)$$

with $w = Z^k v$. The first term is handled just as for the case $k = 0$. To control the other terms, we have to modify slightly the estimation lemma of Section 1.1 (Lemma 1.1).

Lemma 4.4. *If g satisfies the null condition, we have the two inequalities (which we write symbolically for simplicity)*

$$G \equiv |g^{\alpha\beta\gamma}(\partial_\gamma w_1)(\partial^2_{\alpha\beta}w_2)| \le C(1+t)^{-1}|Zw_1|\,|\partial^2 w_2| + C|\partial w_1|\,|T\partial w_2|,$$
$$G \le C|Tw_1|\,|\partial^2 w_2| + C(1+t)^{-1}|\partial w_1|\,|Z\partial w_2|.$$

Using the first inequality of the lemma, we bound an h-term of $R(\partial_t w)$ as

$$|h^{\alpha\beta\gamma}_{pq}(\partial_\gamma Z^p u)(\partial^2_{\alpha\beta}Z^q v)(\partial_t w)|$$
$$\le C(1+t)^{-1}|Z^{p+1}u|\,|\partial^2 Z^q v|\,|\partial_t w| + C|\partial Z^p u|\,|T\partial Z^q v|\,|\partial_t w|.$$

The first term is bounded by

$$CC_0(1+t)^{-3/2}[|\partial^2 Z^q v|^2 + |\partial_t w|^2].$$

The second term is bounded for small $\eta' > 0$ by

$$\eta' \langle r-t\rangle^{-1-\eta}|T\partial Z^q v|^2 + CC_{\eta'}C_0^2(1+t)^{-3/2}|\partial_t w|^2.$$

The m-terms are handled similarly, using the second line of the lemma, with the same type of bounds.

To handle an ℓ-term, we distinguish the cases $p \geq q$ or $p \leq q$. If $p \geq q$, then $q \leq 3$; using the second line of the lemma, we bound the term by

$$C|TZ^p v|\,|\partial^2 Z^q v|\,|\partial_t w| + C(1+t)^{-1}|\partial Z^p v|\,|Z\partial Z^q v|\,|\partial_t w|.$$

Since $q \leq 3$,

$$|\partial^2 Z^q v| \leq CC_1\varepsilon(1+t)^{-1}\langle r-t\rangle^{-1/2},$$

and the first term is less than

$$\eta' \langle r-t\rangle^{-1-\eta}|TZ^q v|^2 + CC_{\eta'}(C_1\varepsilon)^2(1+t)^{-3/2}|\partial_t w|^2.$$

Similarly, the second term is less than

$$C(C_1\varepsilon)(1+t)^{-2}[|\partial Z^p v|^2 + |\partial_t w|^2].$$

Assume now $p \leq q$, hence $p \leq 3$. Using the first line of the lemma, we bound the ℓ-term by

$$C(1+t)^{-1}|Z^{p+1} v|\,|\partial^2 Z^q v|\,|\partial_t w| + C|\partial Z^p v|\,|T\partial Z^q v|\,|\partial_t w|.$$

The first term is less than

$$C(C_1\varepsilon)(1+t)^{-3/2}[|\partial Z^{q+1} v|^2 + |\partial_t w|^2].$$

The second term is less than

$$\eta' \langle r-t\rangle^{-1-\eta}|T\partial Z^q v|^2 + CC_{\eta'}(C_1\varepsilon)^2(1+t)^{-3/2}|\partial_t w|^2.$$

4.3.4. Summing all inequalities, and using Gronwall's lemma, we obtain with

$$\phi(t) = \sum_{k\leq 6} \Big|\, |\partial Z^k v(\cdot,t)|\, \Big|_{L^2}$$

the inequality

$$\phi(t) \leq C_2\phi(0)\exp C_2(C_0 + C_1\varepsilon + C_0^2 + C_1^2\varepsilon^2),$$

where C_2 is a numerical constant (independent of C_0 and C_1). Since

$$\phi(0) \le C_3\varepsilon,$$

we choose

$$C_1 = 4C_2C_3 \exp C_2(C_0 + C_0^2).$$

Then we choose $\varepsilon_0 > 0$ small enough to satisfy $C_1\varepsilon_0$ small and

$$\exp C_2(C_1\varepsilon_0 + C_1^2\varepsilon_0^2) \le 2.$$

For $\varepsilon \le \varepsilon_0$, this gives

$$\phi(t) \le \frac{1}{2}C_1\varepsilon$$

and finishes the proof. ◻

5. Remarks on the blowup at infinity

Recall that for some $t_0 > 2M$, we consider u as a solution of the Cauchy problem

$$\square u + Q(u) = 0,\ u(x,t_0) = u_0(x),\ (\partial_t u)(x,t_0) = u_1(x).$$

Using the conformal inversion I, we set $T_0 = -1/t_0$ and associate to u the function v defined by

$$u(I)(X,T) = (T^2 - R^2)v(X,T).$$

The concept of "blowing up at infinity" is not clear to us in general situations, and we think that it would be interesting to clarify the concept in general situations for quasilinear wave equations. In the present case however, the use of conformal inversion allows us to give precise definitions.

Definition. *We say that a given global C^∞ solution u of $\square u + Q(u) = 0$ does not blow up at infinity if the corresponding function v is C^∞ on the compact region*

$$\{(X,T),\ T_0 \le T \le 0,\ T^2 - R^2 \ge 0\}.$$

For a C^∞ solution u only defined below an hyperboloid

$$\left\{(x,t),\ t_0 \le t \le -\frac{1}{2T_1} + \left(r^2 + \frac{1}{4T_1^2}\right)^{1/2}\right\},\quad T_0 < T_1 < 0,$$

the same definition makes sense, the function v being here C^∞ on the compact region

$$\{(X,T),\ T_0 \le T \le T_1,\ T^2 - R^2 \ge 0\}.$$

With such definitions, it is clear that solutions which do not blowup at infinity are stable. Global solutions blowing up at infinity (if they exist) correspond to functions v having some singularity only on the boundary of the domain : this appears as an unstable limiting case, since it would seem that some perturbations of v produce functions having a singularity inside the domain, corresponding to finite time blowup for u. It is not clear however if the limiting case of global smooth solutions corresponds to global solutions or to solutions blowing up in finite type, as shown by the three very simple examples below.

5.1. Three simple examples.

a. Let us consider the ODE

$$y' = -y + y^2, \quad y(0) = y_0 > 0.$$

The explicit solution is

$$y(t) = (1 + Ae^t)^{-1}, \quad A = \frac{1}{y_0} - 1.$$

If $y_0 < 1$, then $A > 0$ and the solution y is global and stable, with behavior $y(t) \to 0$ at infinity. For $y_0 > 1$, then $A < 0$ and finite time blowup occurs. For the limiting case $y_0 = 1$, $A = 0$ and the solution is simply $y \equiv 1$, which is global with blowup at infinity.

b. Let us consider the ODE

$$y' = -2(t-1)y^2, \quad y(0) = y_0 > 0.$$

The explicit solution is

$$y(t) = \left[(t-1)^2 + \frac{1}{y_0} - 1 \right]^{-1}.$$

If $y_0 < 1$, the solution is global and goes to zero at infinity. If $y_0 > 1$, the solution blows up in finite time. For the limiting case $y_0 = 1$, the solution is simply $y(t) = (t-1)^{-2}$, which blows up in finite time.

c. The explanation of the difference between these two very simple examples is this: in a, the finite time singularities are stable (with minimal blowup rate), hence they cannot occur for the limiting case. In contrast, the finite time singularity $(t-1)^{-2}$ is unstable (with higher blowup rate), and it does occur as a limiting case singularity. In the first case, the change

$$s = e^{-t}, \quad w(s) = s^{-1} y(-\log s)$$

reduces the problem to

$$w' = -w^2.$$

Thus we have a regular compactification of the problem, analogous to the one obtained using conformal inversion for our quasilinear wave equation. In the second case, it does not seem possible to obtain such a compactification.

d. Consider the Cauchy problem with data on $\{t = t_0\}$ for the equation

$$\Box u + (\partial_t u)^2 - \sum (\partial_i u)^2 = 0.$$

As is well known, setting $\underline{u} = e^u - 1$, this equation reads $\Box \underline{u} = 0$. Define as before v and $\underline{v}$ by

$$u(I)(X,T) = (T^2 - R^2) v(X,T), \quad \underline{u}(I)(X,T) = (T^2 - R^2)\underline{v}(X,T).$$

We already know that $\Box \underline{v} = 0$. The transformed equation for v is easily seen to be

$$\Box v + 4v(Sv) + (T^2 - R^2)\Big[(\partial_T v)^2 - \sum (\partial_i v)^2\Big] + 4v^2 = 0.$$

We also have

$$\tilde{u}(X,T) = \log(1 + \tilde{\underline{u}}) = \log(1 + (T^2 - R^2)\underline{v}) = (T^2 - R^2)v.$$

Suppose now that u is a global C^∞ solution of our equation : this implies $\underline{u} = (T^2 - R^2)\underline{v} > -1$ everywhere. For s close to zero, $\log(1 + s) = sf(s)$ for some $f \in C^\infty$. Since $\underline{v}$ is C^∞ everywhere, close to $\{T^2 = R^2\}$, we get

$$v(X,T) = \underline{v} f((T^2 - R^2)\underline{v}).$$

Hence v is automatically in $C^\infty(\bar{\mathcal{T}})$, and u does not blowup at infinity.

This example is of course very special : the equation on v has the property that all solutions which exist and are C^∞ in the interior of $\mathcal{T}$ are automatically $C^\infty(\bar{\mathcal{T}})$. But the question is : what tells us that this is not the case for the general equation $\Box v + \bar{Q}(v) = 0$ obtained in Lemma 2.6 ?

Let us mention an apparent "paradox" connected with this example : it is easy to construct C^∞ data

$$v_0(X) = v(X, T_0),\ v_1(X) = (\partial_T v)(X, T_0)$$

for the equation on v, compactly supported in $\{|X| < |T_0|\}$, such that the corresponding solution v blows up inside $\mathcal{T}$. If we consider now the solution v_ε of the v-equation with data $(\varepsilon v_0, \varepsilon v_1)$ on $\{T = T_0\}$, it certainly belongs to $C^\infty(\bar{\mathcal{T}})$ for ε small enough. Define ε_0 by

$$\varepsilon_0 = \sup\{\varepsilon > 0,\ \forall\, 0 \le \varepsilon' \le \varepsilon,\ v_{\varepsilon'} \in C^\infty(\bar{\mathcal{T}})\}.$$

One would be tempted to believe that v_{ε_0} is C^∞ inside $\mathcal{T}$, but this is not the case in view of the above considerations. This example is similar to example 2 above : the singularity of v_{ε_0} inside $\mathcal{T}$ is unstable.

5.2. Heuristics. To summarize what has been said before :

(i) If a solution does not blowup at infinity, it is stable,
(ii) We do not know if blow up at infinity implies instability of the solution, but we believe so,
(iii) We do not know if there are actually solutions which blow up at infinity, but we believe there are (otherwise this paper would be meaningless).

5.3. Local blowup for quasilinear wave equations. From what has been said before, it seems important to us to investigate, on the cone $\{R + T = 0\}$, blowup for the solutions of the equation $\Box v + \bar{Q}(v) = 0$. Let us recall in a sketchy way some basic facts from [4] about blowup.

Consider to simplify an (hyperbolic) equation of the form

$$\sum p^{\alpha\beta}(\partial v)\partial^2_{\alpha\beta}v = 0.$$

One could allow $p^{\alpha\beta}$ to also depend on (x_α, v), but only the dependence on ∂v is significant. The principal symbol $p = p^{\alpha\beta}(V)\xi_\alpha\xi_\beta$ of the linearized operator on v is a function of ξ and $V = (\partial_\alpha v)$. Denote by D_α the derivative with respect to V_α. A characteristic point (V_0, ξ_0) is said to be genuinely nonlinear (following Lax) if

$$p(V_0, \xi_0) = 0, \quad \xi_0 \cdot Dp(V_0, \xi_0) \neq 0.$$

In this case, one can construct singular solutions for which $\partial^2 v$ blows up at some point m of a spacelike surface $\{d = 0\}$ with the minimal rate d^{-1}. This type of singularity is believed to be stable, that is, slightly modified data yield a modified solution blowing up close to m with the same minimal rate (for a precise statement, see [6]). A linearly degenerate point (V_0, ξ_0) is defined by

$$p(V_0, \xi_0) = 0, \quad \xi_0 \cdot Dp(V_0, \xi_0) = 0.$$

In this case, one can construct singular solutions for which $\partial^2 v$ blows up at some point m of a spacelike surface $\{d = 0\}$ with the higher rate d^{-2} ; moreover, such solutions are unstable, meaning that some slight modifications of the data yield a minimal rate blowup solution, while some other modifications yield a solution which does not blow up at all (see [5]). The condition of being a linearly degenerate point is invariantly defined with respect to change of variables or nonlinear change of unknown function. Note that the null condition does not necessarily implies that all points are linearly degenerate ; however, there are equations satisfying the null condition for which all points are linearly degenerate, for instance

$$\Box u + (\partial_1 u)(\partial_2^2 u) - (\partial_2 u)(\partial_{12}^2 u) = 0.$$

For such an equation on u, the equation on v, away from the boundary $\{R+T\} = 0$, will have only linearly degenerate points, since the equation on v is the result of a change of variables (the conformal inversion) coupled with a change of unknown function. The remarkable fact however, is that for *all* equations satisfying the null condition, the boundary points are linearly degenerate.

Theorem (Linear Degeneracy Theorem). *On the boundary $\{R + T = 0\}$, all points are linearly degenerate points for the equation $\Box v + \bar{Q}(v) = 0$.*

Proof. The principal symbol p of the linearized equation $\mathcal{L}$ is actually a function of $(X, T, v, \partial v)$ in our case, but we disregard the dependence on (X, T, v) as irrelevant. On $R + T = 0$, the principal symbol is (with (ξ, τ) dual to (X, T))

$$p = \tau^2 - |\xi|^2 + 4[2A(Sv + v) + Y_1 v]s^2 + 4(Sv + v)y_2 s,$$

with $s = T(\tau - \omega\xi)$,

$$\begin{aligned} T^{-2}y_1 &= \tau(2g^{0i0}\omega_i - g^{ij0}\omega_i\omega_j) + g^{ijk}\omega_i\omega_j\xi_k - 2g^{0ij}\omega_i\xi_j, \\ T^{-2}y_2 &= \tau(2g^{0i0}\omega_i - 2g^{0ij}\omega_i\omega_j) - 2g^{ij0}\omega_i\xi_j + 2g^{ijk}\omega_k\omega_i\xi_j \\ &\qquad + 2g^{0i0}\xi_i - 2g^{0ij}\omega_j\xi_i. \end{aligned}$$

Since

$$D_0 p = 4(2AT + Y_1^0)s^2 + 4Ty_2 s, \; D_i p = 4(-2AT\omega_i + Y_1^i)s^2 - 4T\omega_i y_2 s,$$

the expression $E = \frac{1}{4}[\tau D_0 p + \sum \xi_i D_i p]$ is

$$\begin{aligned} E &= (2AT(\tau - \omega\xi) + y_1)s^2 + sy_2 T(\tau - \omega\xi) \\ &= T^2(\tau - \omega\xi)^2[2AT(\tau - \omega\xi) + y_1 + y_2]. \end{aligned}$$

Now

$$\begin{aligned} T^{-2}(y_1 + y_2) = \tau[2A(\omega) &- (2g^{0ij}\omega_i\omega_j + g^{ij0}\omega_i\omega_j)] \\ &+ A(\xi) - 2[g^{ij0}\omega_i\xi_j + g^{0ij}(\omega_i\xi_j + \omega_j\xi_i)] \\ &+ g^{ijk}\omega_i\omega_j\xi_k + 2g^{ijk}\omega_i\omega_k\xi j. \end{aligned}$$

Taking into account the algebraic properties displayed in Section 1.1, with $X = \omega$ and $Y = \xi$ for the last line, we obtain

$$\begin{aligned} T^{-2}(y_1 + y_2) &= 2A(\omega)\tau + A(\xi) - 2A(\omega)(\omega\xi) - A(\xi)|\omega|^2 \\ &= 2A(\omega)(\tau - \omega\xi). \end{aligned}$$

Hence finally

$$2A(X)T(\tau - \omega\xi) + y_1 + y_2 = -2A(\omega)T^2(\tau - \omega\xi) + 2A(\omega)T^2(\tau - \omega\xi) = 0. \qquad \square$$

This computation shows that, despite appearances, the equation $\square v + \bar{Q}(v) = 0$ still contains many hidden cancellations ! We still dont know how to exploit this result, but we feel that it is important to understand blowup at infinity for the solutions of $\square u + Q(u) = 0$.

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