THE NULL CONDITION AND GLOBAL EXISTENCE OF SOLUTIONS TO SYSTEMS OF WAVE EQUATIONS WITH DIFFERENT SPEEDS

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In this paper, we consider the initial value problems to systems of quasilinear wave equations with different speeds in two space dimensions. Applying John-Shatah observations to our problem, we introduce the null condition for the system with different speeds. Moreover, we prove a global existence theorem for a class satisfying the null condition.

1 Introduction.

We shall start this paper with the description of John-Shatah observations on the null condition. We consider the scalar quasilinear wave equations with quadratic nonlinearity in three space dimensions. Introducing the space-time gradient of unknown, one can find that components of the gradient satisfy some hyperbolic system of first order. The plane wave solutions of this system satisfy hyperbolic systems of first order in *one* space dimension. Making use of the results of F. John,[3] F. John and J. Shatah have proved in F. John's book [5] the following remarkable fact: The requirement that no plane wave solution of this system is genuinely nonlinear leads to a class of equations which satisfy the null condition (S. Klainerman [9]).

We apply John-Shatah observations to a system of quasilinear wave equations with different speeds in two space dimensions. We consider the system with unknown vector $u(t,x) = {}^t(u^1(t,x),\dots,u^m(t,x))$ in the form

$$\partial_t^2 u^i - c_i^2 \Delta u^i = \sum_{j=1}^{m} \sum_{\alpha,\beta=0}^{2} C_{ij}^{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u^j \quad (i = 1,\dots,m), \tag{1.1}$$

where ∂u stands for space-time gradient of u, i.e.

$$\begin{aligned} &\partial u = {}^t(\partial u^1,\dots,\partial u^m),\\ &\partial u^i = {}^t(\partial_0 u^i, \partial_1 u^i, \partial_2 u^i),\\ &\partial_0 = \partial_t = \frac{\partial}{\partial t},\ \partial_1 = \frac{\partial}{\partial x_1},\ \partial_2 = \frac{\partial}{\partial x_2} \end{aligned} \tag{1.2}$$

and the c_i $(i = 1,\dots,m)$ are positive constants different from each other. We assume that $C_{ij}^{\alpha\beta}$ are C^∞-functions of their arguments which vanish at $\partial u = 0$

of second order and $C_{ij}^{\alpha\beta} = C_{ij}^{\beta\alpha}$. Set $v = \partial u$. Then one can find from (1.1) and (1.2) that the vector v satisfies the system of first order which is hyperbolic near $v = 0$:

$$\sum_{\alpha=0}^{2} a^{\alpha}(v)\partial_{\alpha} v = 0. \tag{1.3}$$

For the concrete expression of $a^{\alpha}(v)$ see section 2. We next consider the plane wave solutions w of the system (1.3):

$$v(t,x) = w(t,s),\ s = \sum_{i=1}^{2} \zeta_i x_i \tag{1.4}$$

where $\zeta = (\zeta_1, \zeta_2) \in \mathbf{R}^2$ and $\zeta \neq 0$. Then one can find from (1.3) and (1.4) that the vector w satisfies the system in one space dimension:

$$a^0(w)\partial_t w + \sum_{i=1}^{2} \zeta_i a^i(w)\partial_s w = 0. \tag{1.5}$$

We take the initial values for the solutions w of (1.5) in the form

$$w(0,s) = \varepsilon\varphi(s), \tag{1.6}$$

where φ has compact support and ε is small positive constant.

Since the system (1.1) has the cubic nonlinearity, the system (1.5) is not genuinely nonlinear. Thus it is natural to require that the lifespan T_ε of solutions to (1.5) and (1.6) is at least of order ε^{-3} for any ζ. Making use of the results of Li Ta-tsien, Kong De-xing and Zhou Yi,[12] we shall prove in section 2 that the requirement above is equivalent to the following fact: it holds that

$$\sum_{\alpha,\beta,\gamma,\delta=0}^{2} \left.\frac{\partial^2 C_{ii}^{\alpha\beta}}{\partial(\partial_\gamma u^i)\partial(\partial_\delta u^i)}\right|_{\partial u=0} X_\alpha^i X_\beta^i X_\gamma^i X_\delta^i = 0 \quad (i = 1, \ldots, m) \tag{1.7}$$

for any real vector $X^i = (X_0^i, X_1^i, X_2^i)$ satisfying

$$(X_0^i)^2 - c_i^2 \sum_{j=1}^{2} (X_j^i)^2 = 0. \tag{1.8}$$

Thus we can interpret this as the null condition for the system (1.1) with different speeds. If

$$\left.\frac{\partial^2 C_{ii}^{\alpha\beta}}{\partial(\partial_\gamma u^i)\partial(\partial_\delta u^i)}\right|_{\partial u=0} = 0 \qquad \begin{pmatrix} i = 1, \ldots, m \\ \alpha,\beta,\gamma,\delta = 0,1,2 \end{pmatrix} \tag{1.9}$$

then the null condition (1.7) is automatically satisfied. The main aim of this paper is to prove the global existence of solution to (1.1) with small data under the assumption (1.9). It is still open whether the null condition (1.7) guarantees the global existence of solution to (1.1).

The null condition is also considered with respect to nonlinear elastic wave equations. For this problem, see T. C. Sideris' work.[13]

In section 3, we introduce some notations and in section 4 we state the main result. In section 5, using the representation of solution to (1.1) of Kovalyov,[10] we estimate the first order derivatives of solution. Finally we prove the main results in section 7 using the estimates and energy inequalities in section 6.

2 The Null Condition.

In this section we introduce the null condition for the system (1.1) with different speeds stated in Introduction.

We consider the system in the form

$$\sum_{j=1}^{m} \sum_{\alpha,\beta=0}^{2} a_{ij}^{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u^j = 0 \quad (i = 1, \ldots, m), \tag{2.1}$$

where ∂u stands for space-time gradient of u

$$\begin{aligned} &\partial u = {}^t(\partial u^1, \ldots, \partial u^m), \\ &\partial u^i = {}^t(\partial_0 u^i, \partial_1 u^i, \partial_2 u^i), \\ &\partial_0 = \partial_t = \frac{\partial}{\partial t}, \, \partial_1 = \frac{\partial}{\partial x_1}, \, \partial_2 = \frac{\partial}{\partial x_2}. \end{aligned}$$

We assume that

$$\begin{aligned} &a_{ij}^{\alpha\beta} = a_{ij}^{\beta\alpha}, \, a_{ii}^{00}(0) = 1, \\ &a_{ij}^{\alpha\beta}(0) = -c_i^2\delta_{ij}\delta_{\alpha\beta} \quad \text{for} \quad (\alpha, \beta) \neq (0,0), \\ &c_i > 0, \, c_i \neq c_j \quad \text{for} \quad i \neq j \end{aligned} \tag{2.2}$$

and $a_{ii}^{00}(\partial u) - 1$, $a_{ij}^{\alpha\beta}(\partial u) + c_i^2\delta_{ij}\delta_{\alpha\beta}$ vanish at $\partial u = 0$ at least of second order, that is,

$$\begin{aligned} a_{ii}^{00}(\partial u) &= 1 + O(|\partial u|^2) \\ a_{ij}^{\alpha\beta}(\partial u) &= -c_i^2\delta_{ij}\delta_{\alpha\beta} + O(|\partial u|^2) \end{aligned} \tag{2.3}$$

near $\partial u = 0$.

Set

$$\begin{aligned} v &= {}^t(v^1, \ldots, v^m) \\ v^i &= {}^t(v_0^i, v_1^i, v_2^i), \, v_\alpha^i = \partial_\alpha u^i. \end{aligned} \tag{2.4}$$

Then we find from (2.1) and (2.4) that the vector v satisfies a system of first order which is hyperbolic near $v = 0$:

$$\sum_{\alpha=0}^{2} a^{\alpha}(v)\partial_{\alpha}v = 0 \tag{2.5}$$

Here the $3m \times 3m$ matrices a^{α} are defined by

$$a^{\alpha} = \left(A^{\alpha}_{ij} \,:\, i,j = 1,\ldots,m\right), \tag{2.6}$$

where

$$\begin{aligned} A^{0}_{ij} &= \begin{pmatrix} a^{00}_{ij} & 0 & 0 \\ 0 & \delta_{ij} & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix}, \; A^{1}_{ij} = \begin{pmatrix} 2a^{10}_{ij} & a^{11}_{ij} & a^{12}_{ij} \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ A^{2}_{ij} &= \begin{pmatrix} 2a^{20}_{ij} & a^{21}_{ij} & a^{22}_{ij} \\ 0 & 0 & 0 \\ -\delta_{ij} & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.7}$$

We next consider the plane wave solution $v(t,x)$ of the equation (2.5):

$$v(t,x) = w(t,s), \; s = \sum_{i=1}^{2} \zeta_i x_i, \tag{2.8}$$

where $\zeta = (\zeta_1, \zeta_2) \in \mathbf{R}^2$ and $\zeta \neq 0$. Then we find from (2.5) and (2.8) that the vector w satisfies a system in one space dimension.

$$\partial_t w + a(w)\partial_s w = 0 \tag{2.9}$$

where

$$a(w) = a^{0}(w)^{-1} \sum_{i=1}^{2} \zeta_i a^{i}(w). \tag{2.10}$$

We take the initial values for the solution w of (2.9) in the form

$$w(0,s) = \varepsilon\varphi(s), \tag{2.11}$$

where φ has compact support and $\varepsilon > 0$.

We shall seek the eigenvalues $\lambda = \lambda(w)$ of the matrix $a(w)$ and the corresponding eigenvector $\xi = \xi(w)$. By the definition, λ satisfies an equation

$$\det\left(\lambda a^{0}(w) - \sum_{i=1}^{2} \zeta_i a^{i}(w)\right) = 0. \tag{2.12}$$

We can verify by induction that

$$\det\left(\lambda a^0(w) - \sum_{i=1}^{2} \zeta_i a^i(w)\right) = \lambda^m \det(p_{ij} \,:\, i,j = 1,\ldots,m) \tag{2.13}$$

where

$$p_{ij} = a_{ij}^{00}\lambda^2 - 2\lambda \sum_{k=1}^{2} a_{ij}^{k0}\zeta_k + \sum_{k,l=1}^{2} a_{ij}^{kl}\zeta_k\zeta_l. \tag{2.14}$$

Therefore we find from (2.2), (2.3), (2.12), (2.13) and (2.14) that the eigenvalues $\lambda_i^{\pm}(0)(i = 1,\ldots,m)$ of the matrix $a(0)$, aside from the trivial multiple eigenvalue $\lambda = 0$, become

$$\lambda_i^{\pm}(0) = \pm c_i|\zeta|,\ |\zeta| = (\zeta_1^2 + \zeta_2^2)^{1/2}. \tag{2.15}$$

According to (2.4) we arrange the components of a vector $\xi \in \mathbf{R}^{3m}$ as follows:

$$\begin{aligned} \xi &= {}^t(\xi^1,\ldots,\xi^m), \\ \xi^i &= {}^t(\xi_0^i,\xi_1^i,\xi_2^i). \end{aligned} \tag{2.16}$$

Then we find from (2.2), (2.3), (2.7) and (2.12) that the eigenvector $\xi_i^{\pm}(0)$ corresponding to $\lambda_i^{\pm}(0)$ becomes

$$\begin{aligned} \left(\xi_i^{\pm}(0)\right)^j &= {}^t(0,0,0) \quad \text{for} \quad j \neq i \\ \left(\xi_i^{\pm}(0)\right)^i &= {}^t(\mp 1, \zeta_1/c_i|\zeta|, \zeta_2/c_i|\zeta|) \end{aligned} \tag{2.17}$$

and the eigenvectors ξ_i $(i = 1,\ldots,m)$ corresponding to the trivial eigenvalue 0 become

$$\begin{aligned} (\xi_i)^j &= {}^t(0,0,0) \quad \text{for} \quad j \neq i, \\ (\xi_i)^i &= {}^t(0,\zeta_2,-\zeta_1). \end{aligned}$$

Since $\xi_i^{\pm}(0), \xi_i$ $(i = 1,\ldots,m)$ are linearly independent, we see that the system (2.9) is hyperbolic near $w = 0$.

We now require that a solution $w(t,s)$ to the initial value problem (2.9), (2.11) has a lifespan T_ε which is at least of order ε^{-3} for any $\zeta \in \mathbf{R}^2$. This requirement is equivalent to the following facts [12]:

$$\sum_{j=1}^{m}\sum_{\alpha=0}^{2} \left.\frac{\partial \lambda_i^{\pm}}{\partial w_\alpha^j}\right|_{w=0} \left(\xi_i^{\pm}(0)\right)_\alpha^j = 0 \tag{2.18}$$

and

$$\sum_{j,k=1}^{m} \sum_{\alpha,\beta=0}^{2} \left. \frac{\partial^2 \lambda_i^{\pm}}{\partial w_\alpha^j \partial w_\beta^k} \right|_{w=0} \left(\xi_i^{\pm}(0)\right)_\alpha^j \left(\xi_i^{\pm}(0)\right)_\beta^k = 0 \tag{2.19}$$

for $i = 1, \ldots, m$ and $\zeta \in \mathbf{R}^2$.

Set

$$P(\lambda) = \det\left(p_{ij}(\lambda) \,:\, i, j = 1, \ldots, m\right).$$

Differentiating the equations

$$P\left(\lambda_i^{\pm}(w)\right) = 0 \quad (i = 1, \ldots, m) \tag{2.20}$$

in a variable w_α^j and evaluating the results at $w = 0$, we get

$$2c_i|\zeta|^{2m-1} \prod_{l \neq i} (c_i^2 - c_l^2) \left. \frac{\partial \lambda_i^{\pm}}{\partial w_\alpha^j} \right|_{w=0} = 0$$

which implies

$$\left. \frac{\partial \lambda_i^{\pm}}{\partial w_\alpha^j} \right|_{w=0} = 0 \tag{2.21}$$

for all i, j, α. Therefore it follows from (2.2), (2.3), (2.14) and (2.21) that the condition (2.18) holds trivially and

$$\left. \frac{\partial \left(p_{ij}(\lambda_k^{\pm})\right)}{\partial w_\alpha^l} \right|_{w=0} = 0 \tag{2.22}$$

for all i, j, k, l, α. Next differentiating twice the equations (2.20) in variables w_α^j and w_β^k and evaluating the results at $w = 0$, we get

$$|\zeta|^{2m-2} \prod_{l \neq i} (c_i^2 - c_l^2) \left. \frac{\partial^2 \left(p_{ii}(\lambda_i^{\pm})\right)}{\partial w_\alpha^j \partial w_\beta^k} \right|_{w=0} = 0 \tag{2.23}$$

for all i, j, k, α, β. By the definition (2.14) of $p_{ii}(\lambda)$, we have

$$\begin{aligned} \left. \frac{\partial^2 \left(p_{ii}(\lambda_i^{\pm})\right)}{\partial w_\alpha^j \partial w_\beta^k} \right|_{w=0} &= 2\lambda_i^{\pm}(0) \left. \frac{\partial^2 \lambda_i^{\pm}}{\partial w_\alpha^j \partial w_\beta^k} \right|_{w=0} + \lambda_i^{\pm}(0)^2 \left. \frac{\partial^2 a_{ii}^{00}}{\partial w_\alpha^j \partial w_\beta^k} \right|_{w=0} \\ &\quad -2\lambda_i^{\pm}(0) \sum_{l=1}^{2} \left. \frac{\partial^2 a_{ii}^{l0}}{\partial w_\alpha^j \partial w_\beta^k} \right|_{w=0} \zeta_l \\ &\quad + \sum_{h,l=1}^{2} \left. \frac{\partial a_{ii}^{hl}}{\partial w_\alpha^j \partial w_\beta^k} \right|_{w=0} \zeta_h \zeta_l. \end{aligned} \tag{2.24}$$

Then it follows from (2.15),(2.17), (2.23) and (2.24) that

$$\left.\frac{\partial^2 \lambda_i^\pm}{\partial w_\alpha^j \partial w_\beta^k}\right|_{w=0} = \frac{\mp c_i|\zeta|}{2} \sum_{\gamma,\delta=0}^{2} \left.\frac{\partial^2 a_{ii}^{\gamma\delta}}{\partial w_\alpha^j \partial w_\beta^k}\right|_{w=0} \left(\xi_i^\pm(0)\right)_\gamma^i \left(\xi_i^\pm(0)\right)_\delta^i \qquad (2.25)$$

Therefore we find from (2.17) and (2.25) that the condition (2.19) is equivalent to

$$\sum_{\alpha,\beta,\gamma,\delta=0}^{2} \left.\frac{\partial^2 a_{ii}^{\gamma\delta}}{\partial w_\alpha^i \partial w_\beta^i}\right|_{w=0} \left(\xi_i^\pm(0)\right)_\alpha^i \left(\xi_i^\pm(0)\right)_\beta^i \left(\xi_i^\pm(0)\right)_\gamma^i \left(\xi_i^\pm(0)\right)_\delta^i = 0 \qquad (2.26)$$

for $i = 1, \ldots, m$ and $\zeta \in \mathbf{R}^2$ By the definition (2.17) of $(\xi_i^\pm(0))^i$, we have

$$\left\{\left(\xi_i^\pm(0)\right)_0^i\right\}^2 - c_i^2 \sum_{j=1}^{2} \left\{\left(\xi_i^\pm(0)\right)_j^i\right\}^2 = 0 \qquad (2.27)$$

for all $\zeta \in \mathbf{R}^2$

Consequently we have proved the following

Proposition 2. 1 *The lifespan T_ε of a unique solution $w(t,s)$ of the initial value problem (2.9), (2.11) is at least of order ε^{-3} for any $\zeta \in \mathbf{R}^2$ if and only if it holds that*

$$\sum_{\alpha,\beta,\gamma,\delta=0}^{2} \left.\frac{\partial^2 a_{ii}^{\gamma\delta}}{\partial w_\alpha^i \partial w_\beta^i}\right|_{w=0} X_\alpha^i X_\beta^i X_\gamma^i X_\delta^i = 0 \quad (i = 1, \ldots, m) \qquad (2.28)$$

for all real vector $X^i = (X_0^i, X_1^i, X_2^i)$ satisfying

$$(X_0^i)^2 - c_i^2 \sum_{j=1}^{2} (X_j^i)^2 = 0. \qquad (2.29)$$

Setting

$$\begin{aligned} C_{ii}^{00}(\partial u) &= 1 - a_{ii}^{00}(\partial u), \\ C_{ij}^{\alpha\beta}(\partial u) &= -c_i^2 \delta_{ij}\delta_{\alpha\beta} - a_{ij}^{\alpha\beta}(\partial u) \quad (\alpha,\beta) \neq (0,0), \end{aligned}$$

we see that the null condition (1.7), (1.8) follows from Proposition 2. 1 .

3 Notations.

To begin with, we introduce some notations that are used throughout the paper.
Partial derivatives are denoted by

$$\partial_0 = \partial_t = \frac{\partial}{\partial t}, \partial_1 = \frac{\partial}{\partial x_1}, \partial_2 = \frac{\partial}{\partial x_2}$$

We also use the angular derivative:

$$\Omega = x_1\partial_2 - x_2\partial_1.$$

We set

$$\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3) = (\partial_1, \partial_2, \Omega)$$

and define

$$\mathcal{D}^A = \mathcal{D}_1^{A_1}\mathcal{D}_2^{A_2}\mathcal{D}_3^{A_3}, |A| = A_1 + A_2 + A_3,$$

where $A = (A_1, A_3, A_3)$ is a multi-index.

Let $u = {}^t(u^1, \ldots, u^m)$ be an unknown vector and set

$$w_i(t,r) = (r+1)^{1/2-\gamma}(t+r+1)^\gamma(|r-c_it|+1)^{1/2} \tag{3.1}$$

for $0 < \gamma < 1/2$. Then we define, for a non-negative integer k,

$$\begin{aligned} |\partial u(t,x)| &= \sum_{i=1}^m \sum_{\alpha=0}^2 |\partial_\alpha u^i(t,x)| \\ |\partial u(t)|_k &= \sum_{|A|\le k} \sum_{i=1}^m \sum_{\alpha=0}^2 \sup_{x\in\mathbf{R}^2} \left|\mathcal{D}^A\partial_\alpha u^i(t,x)\right| \\ \|\partial u(t)\|_k &= \sum_{|A|\le k} \sum_{i=1}^m \sum_{\alpha=0}^2 \left\|\mathcal{D}^A\partial_\alpha u^i(t,\cdot)\right\|_{L^2(\mathbf{R}^2)} \\ [\partial u(t)]_k &= \sum_{|A|\le k} \sum_{i=1}^m \sum_{\alpha=0}^2 \sup_{x\in\mathbf{R}^2} \left|w_i(t,|x|)\mathcal{D}^A\partial_\alpha u^i(t,x)\right|. \end{aligned} \tag{3.2}$$

Moreover, we define

$$\begin{aligned} |\partial u|_k(t) &= \sup_{0<s<t} |\partial u(s)|_k, \\ [\partial u]_k(t) &= \sup_{0<s<t} [\partial u(s)]_k. \end{aligned} \tag{3.3}$$

In what follows, M denotes various constant depending on F_i, f^i, g^i and c_i.

4 Statement of the Main Result.

The initial value problem to be considered is

$$\begin{cases} \partial_t^2 u^i - c_i^2 \Delta u^i = F_i(\partial u, \partial^2 u) & \text{in} \quad [0,\infty) \times \mathbf{R}^2 \\ u^i(0,\cdot) = \varepsilon f^i,\ \partial_t u^i(0,\cdot) = \varepsilon g^i & \text{in} \quad \mathbf{R}^2 \end{cases} \qquad (i = 1, \dots, m) \tag{4.1}$$

where c_i are positive constants and $\varepsilon > 0$ is small parameter. Moreover, f^i and g^i are C^∞-functions with compact support. We describe some assumptions on the initial value problem (4.1) and state the main theorem.

First, we assume that F_i are of first degree with respect to the second derivatives of u:

$$F_i(\partial u, \partial^2 u) = \sum_{j=1}^{m} \sum_{\alpha,\beta=0}^{2} C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta u^j + E_i(\partial u). \tag{4.2}$$

Here, $C_{ij}^{\alpha\beta}$ and E_i are C^∞-functions of ∂u in $\{|\partial u| < 1\}$ that satisfy

$$C_{ij}^{\alpha\beta} = C_{ij}^{\beta\alpha} = C_{ji}^{\alpha\beta}, \tag{4.3}$$

$$\left| C_{ij}^{\alpha\beta}(\partial u) \right| \le M |\partial u|^2, \tag{4.4}$$

$$|E_i(\partial u)| \le M |\partial u|^3 \tag{4.5}$$

Assuming (4.2)-(4.5), M. Kovalyov [10] proved the almost global existence of the solution to (4.1).

Second, we assume the null condition (1.9) for different speeds introduced in Introduction:

$$c_i \neq c_j \quad \text{for} \quad i \neq j \tag{4.6}$$

$$\left. \frac{\partial^2 C_{ii}^{\alpha\beta}}{\partial(\partial_\gamma u^i) \partial(\partial_\delta u^i)} \right|_{\partial u = 0} = 0 \qquad \begin{pmatrix} i = 1, \dots, m \\ \alpha, \beta, \gamma, \delta = 0, 1, 2 \end{pmatrix}. \tag{4.7}$$

The conditions for E_i are

$$\left. \frac{\partial^3 E_i}{\partial(\partial_\alpha u^i) \partial(\partial_\beta u^i) \partial(\partial_\gamma u^i)} \right|_{\partial u = 0} = 0 \qquad \begin{pmatrix} i = 1, \dots, m \\ \alpha, \beta, \gamma = 0, 1, 2 \end{pmatrix} \tag{4.8}$$

in accordance with (4.7).

Theorem *Let us assume (4.2)-(4.8). Then there exists a positive constant ε_0 depending on given functions such that the initial value problem (4.1)* *has a unique* C^∞*- solution in* $[0,\infty)\times\mathbf{R}^2$ *for all* ε *with* $0<\varepsilon<\varepsilon_0$.

M. Kovalyov showed in his paper[11] that the theorem holds when $C_{ij}^{\alpha\beta}=0$ and E_i $(i=1,\dots,m)$ satisfy the condition

$$\left.\frac{\partial^3 E_i}{\partial(\partial_\alpha u^j)\partial(\partial_\beta u^j)\partial(\partial_\gamma u^j)}\right|_{\partial u=0}=0 \qquad \begin{pmatrix} i,j=1,\dots,m \\ \alpha,\beta,\gamma=0,1,2\end{pmatrix}$$

instead of (4.8).

5 Estimate of the First Derivatives of the Solutions to Initial Value Problems.

The aim of this section is to estimate the first derivatives of the solution to the initial value problem:

$$\begin{cases} \partial_t^2 u-\Delta u=F(t,x) & \text{in} \quad [0,T)\times\mathbf{R}^2 \\ u(0,\cdot)=\partial_t u(0,\cdot)=0 & \text{in} \quad \mathbf{R}^2 \end{cases} \tag{5.1}$$

Here, F is a C^∞ function in $[0,T)\times\mathbf{R}^2$ For this purpose, we use the representation formula of the solution to (5.1) which has proved by M. Kovalyov [10]:

Proposition 5. 1 *Let* $u\in C^\infty([0,T)\times\mathbf{R}^2)$ *be the solution of the initial value problem (5.1). Then,* u *has the following representation:*

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi}\iint_{D'} rdrds\int_{-\varphi}^{\varphi} K_1F(s,re^{\sqrt{-1}(\theta+\psi)})d\psi \\ &\quad +\frac{1}{2\pi}\chi(t-a)\iint_{D''} rdrds\int_{-\pi}^{\pi} K_1F(s,re^{\sqrt{-1}(\theta+\psi)})d\psi \end{aligned}$$

where

$$\begin{aligned} x &= (a\cos\theta,a\sin\theta)=ae^{\sqrt{-1}\theta} \\ \varphi &= \arccos\frac{a^2+r^2-(t-s)^2}{2ar} \quad \text{for} \quad (s,r)\in D' \\ K_1 &= \frac{1}{\{(t-s)^2-a^2-r^2+2ar\cos\psi\}^{1/2}} \\ \chi(s) &= \begin{cases} 1 & (s>0) \\ 0 & (s\le 0)\end{cases} \end{aligned}$$

Moreover, the domains D' and D'' are defined as follows.

$$D' = \{(s,r) \,|\, 0 < s < t,\ r_1 < r < r_2\}$$

$$D'' = \begin{cases} \{(s,r) \,|\, 0 < s < t-a,\ 0 < r < r_1\} & \text{for} \quad t > a \\ \emptyset & \text{for} \quad t \le a \end{cases}$$

where

$$r_1 = |a - t + s|,\ r_2 = a + t - s. \tag{5.2}$$

Next, we derive representation formulae for the first derivatives of the solution of the initial value problem (5.1) from Proposition 5. 1 . In order to present the formulae, we set

$$\begin{aligned} \delta &= \min\{1/2, a\} \\ \tilde{\delta} &= \min\{1/2, (t-a)/2\} \end{aligned} \qquad (a = |x|) \tag{5.3}$$

and split the domains D' and D'' as follows:

$$D' = \mathit{blue} \cup \mathit{white}$$

$$\mathit{blue} = \{(s,r) \in D' \,|\, r_1 < r \le r_1 + \delta \ \text{ or } \ r_2 - \delta \le r < r_2\}$$

$$\mathit{white} = \begin{cases} D' \setminus \mathit{blue} & \text{for} \quad \delta = 1/2 \\ \emptyset & \text{for} \quad \delta = a \end{cases}$$

$$D'' = \mathit{black} \cup \mathit{red}$$

$$\mathit{black} = \{(s,r) \in D'' \,|\, r_1 - \tilde{\delta} \le r < r_1 \ \text{ or } \ 0 < r \le \tilde{\delta}\}$$

$$\mathit{red} = \begin{cases} D'' \setminus \mathit{black} & \text{for} \quad \tilde{\delta} = 1/2 \\ \emptyset & \text{for} \quad \tilde{\delta} = (t-a)/2 \end{cases}$$

We set

$$I_{blue}(F)(t,x) = \iint_{blue} r dr ds \int_{-\varphi}^{\varphi} K_1 F(s, re^{\sqrt{-1}(\theta+\psi)}) d\psi$$

$$
\begin{aligned}
I_{white}(F)(t,x) &= \iint_{white} rdrds \int_{-\varphi}^{\varphi} K_1 F(s, re^{\sqrt{-1}(\theta+\psi)}) d\psi \\
I_{black}(F)(t,x) &= \iint_{black} rdrds \int_{-\pi}^{\pi} K_1 F(s, re^{\sqrt{-1}(\theta+\psi)}) d\psi \\
I_{red}(F)(t,x) &= \iint_{red} rdrds \int_{-\pi}^{\pi} K_1 F(s, re^{\sqrt{-1}(\theta+\psi)}) d\psi
\end{aligned}
$$

Then, by Proposition 5. 1 , $\partial_\mu u$ ($\mu = 0, 1, 2$) is represented as

$$
\begin{aligned}
\partial_\mu u \;=\; & \frac{1}{2\pi}\{I_{blue}(\partial_\mu F) + \chi(a-1/2)I_{white}(\partial_\mu F) \\
& + \chi(t-a)I_{black}(\partial_\mu F) + \chi(t-a-1)I_{red}(\partial_\mu F)\}.
\end{aligned}
\tag{5.4}
$$

Following M. Kovalyov,[10] we change the variable of integration from ψ to τ by the map $\psi = \Psi$, where

$$
\begin{aligned}
\Psi &= \arccos[1 + P\tau - \tau], \\
P &= \frac{a^2 + r^2 - (t-s)^2}{2ar}.
\end{aligned}
$$

Then we have the following

Proposition 5. 2

$$
\begin{aligned}
& I_{white}(\partial_\mu F)(t,x) \\
&= \sum_{j,\alpha=0}^{1} \left\{ \iint_{\partial(white)} d\sigma \int_0^1 rK_2 a_\mu^\alpha(\theta+\Psi_j) n_\alpha F(s, re^{\sqrt{-1}(\theta+\Psi_j)}) d\tau \right. \\
&\quad - \iint_{white} drds \int_0^1 \nabla_\alpha \{rK_2 a_\mu^\alpha(\theta+\Psi_j)\} F(s, re^{\sqrt{-1}(\theta+\Psi_j)}) d\tau \\
&\quad \left. - \iint_{white} rdrds \int_0^1 K_2 a_\mu^\alpha(\theta+\Psi_j)(\Omega F)(s, re^{\sqrt{-1}(\theta+\Psi_j)}) \nabla_\alpha \Psi_j d\tau \right\} \\
&\quad + \iint_{white} drds \int_{-\varphi}^{\varphi} K_1 a_\mu^2(\theta+\psi)(\Omega F)(s, re^{\sqrt{-1}(\theta+\psi)}) d\psi
\end{aligned}
$$

$$
\begin{aligned}
& I_{red}(\partial_\mu F)(t,x) \\
&= \sum_{\alpha=0}^{1} \iint_{\partial(red)} rd\sigma \int_{-\pi}^{\pi} K_1 a_\mu^\alpha(\theta+\psi) n_\alpha F(s, re^{\sqrt{-1}(\theta+\psi)}) d\psi \\
&\quad - \sum_{\alpha=0}^{1} \iint_{red} drds \int_{-\pi}^{\pi} \nabla_\alpha \{rK_1\} a_\mu^\alpha(\theta+\psi) F(s, re^{\sqrt{-1}(\theta+\psi)}) d\psi \\
&\quad + \iint_{red} drds \int_{-\pi}^{\pi} K_1 a_\mu^2(\theta+\psi)(\Omega F)(s, re^{\sqrt{-1}(\theta+\psi)}) d\psi
\end{aligned}
$$

where

$$\begin{aligned} K_2 &= \frac{1}{\{2ar\tau(1-\tau)(2+P\tau-\tau)\}^{1/2}}, \\ \Psi_j &= (-1)^j \Psi, \end{aligned}$$

$$a^0_\mu(\theta) = \delta^0_\mu, \quad a^1_\mu(\theta) = \begin{cases} 0 & (\mu = 0) \\ \cos\theta & (\mu = 1) \\ \sin\theta & (\mu = 2) \end{cases}, \quad a^2_\mu(\theta) = \begin{cases} 0 & (\mu = 0) \\ -\sin\theta & (\mu = 1) \\ \cos\theta & (\mu = 2) \end{cases},$$

$$\nabla_\alpha = \begin{cases} \partial_s & (\alpha = 0) \\ \partial_r & (\alpha = 1) \end{cases},$$

$\vec{n} = {}^t(n_0, n_1)$ is the unit outer normal vector field on $\partial(white) \cup \partial(red)$, and $d\sigma$ is the line element on $\partial(white) \cup \partial(red)$.

M. Kovalyov used these formulae in his work,[10] but he has omitted the terms containing the first derivatives of Ψ_j in the above formulae. So we show the proof for completeness.

Proof. We denote $F(s, re^{\sqrt{-1}\theta}) = G(s, r, \theta)$. Then,

$$(\partial_\mu F)(s, re^{\sqrt{-1}\theta}) = \sum_{\alpha=0}^{1} a^\alpha_\mu(\theta)(\nabla_\alpha G)(s, r, \theta) + \frac{a^2_\mu(\theta)}{r}(\partial_\theta G)(s, r, \theta).$$

Therefore,

$$\begin{aligned} I_{white}(\partial_\mu F) &= \sum_{\alpha=0}^{1} \iint_{white} rdrds \int_{-\varphi}^{\varphi} K_1 a^\alpha_\mu(\theta+\psi)(\nabla_\alpha G)(s, r, \theta+\psi)d\psi \\ &\quad + \iint_{white} drds \int_{-\varphi}^{\varphi} K_1 a^2_\mu(\theta+\psi)(\partial_\theta G)(s, r, \theta+\psi)d\psi. \end{aligned}$$

Changing variable from ψ to τ by the map $\psi = \Psi$, we have

$$\begin{aligned} &\iint_{white} rdrds \int_{-\varphi}^{\varphi} K_1 a^\alpha_\mu(\theta+\psi)(\nabla_\alpha G)(s, r, \theta+\psi)d\psi \\ &= \sum_{j=0}^{1} \iint_{white} rdrds \int_0^1 K_2 a^\alpha_\mu(\theta+\Psi_j)(\nabla_\alpha G)(s, r, \theta+\Psi_j)d\tau. \end{aligned} \tag{5.5}$$

Notice that

$$(\nabla_\alpha G)(s,r,\theta+\Psi_j) = \nabla_\alpha\{G(s,r,\theta+\Psi_j)\} - (\partial_\theta G)(s,r,\theta+\Psi_j)\nabla_\alpha\Psi_j. \quad (5.6)$$

Substituting (5.6) into (5.5) and integrating by parts give

$$\begin{aligned}
&I_{white}(\partial_\mu F)\\
&= \sum_{j,\alpha=0}^{1}\left\{\iint_{\partial(white)} d\sigma \int_0^1 rK_2 a_\mu^\alpha(\theta+\Psi_j)n_\alpha G(s,r,\theta+\Psi_j)d\tau\right.\\
&\quad - \iint_{white} drds \int_0^1 \nabla_\alpha\{rK_2 a_\mu^\alpha(\theta+\Psi_j)\}G(s,r,\theta+\Psi_j)d\tau\\
&\quad \left.- \iint_{white} rdrds \int_0^1 K_2 a_\mu^\alpha(\theta+\Psi_j)(\partial_\theta G)(s,r,\theta+\Psi_j)\nabla_\alpha\Psi_j d\tau\right\}\\
&\quad + \iint_{white} drds \int_{-\varphi}^{\varphi} K_1 a_\mu^2(\theta+\psi)(\partial_\theta G)(s,r,\theta+\psi)d\psi.
\end{aligned} \quad (5.7)$$

Similarly,

$$\begin{aligned}
I_{red}(\partial_\mu F) &= \sum_{\alpha=0}^{1}\iint_{red} rdrds \int_{-\pi}^{\pi} K_1 a_\mu^\alpha(\theta+\psi)(\nabla_\alpha G)(s,r,\theta+\psi)d\psi\\
&\quad + \iint_{red} drds \int_{-\pi}^{\pi} K_1 a_\mu^2(\theta+\psi)(\partial_\theta G)(s,r,\theta+\psi)d\psi\\
&= \sum_{\alpha=0}^{1}\iint_{\partial(red)} rd\sigma \int_{-\pi}^{\pi} K_1 a_\mu^\alpha(\theta+\psi)n_\alpha G(s,r,\theta+\psi)d\psi \quad (5.8)\\
&\quad - \sum_{\alpha=0}^{1}\iint_{red} drds \int_{-\pi}^{\pi} \nabla_\alpha\{rK_1\}a_\mu^\alpha(\theta+\psi)G(s,r,\theta+\psi)d\psi\\
&\quad + \iint_{red} drds \int_{-\pi}^{\pi} K_1 a_\mu^2(\theta+\psi)(\partial_\theta G)(s,r,\theta+\psi)d\psi.
\end{aligned}$$

Thus we get the representation formula from (5.7) and (5.8). ∎

The following proposition is used to estimate the terms appearing in Proposition 5. 2 . This was shown in M. Kovalyov,[10] except the estimates containing the derivatives of Ψ. For the sake of completeness, we give the proof of all.

Proposition 5. 3

I. *Let* $(s,r) \in D'$. *Then the following estimates hold:*

$$\text{(i)} \quad \int_{-\varphi}^{\varphi} K_1 d\psi = 2\int_0^1 K_2 d\tau$$

$$\leq \frac{M}{(ar)^{1/2}} \log\left[2 + \frac{ar}{(r-r_1)(r+r_2)}\chi(t-s-a)\right]$$

(ii) $$\int_0^1 \{|\partial_s K_2| + |\partial_r K_2|\}\, d\tau \leq \frac{M}{(ar)^{1/2}(r+s+a-t)}$$

(iii) $$\int_0^1 K_2 \{|\partial_s \Psi| + |\partial_r \Psi|\}\, d\tau \leq \frac{M(a+r)}{\{ar(r^2-r_1^2)(r_2^2-r^2)\}^{1/2}}$$

II. *Let* $(s,r) \in D''$. *Then the following estimates hold:*

(i) $$\int_{-\pi}^{\pi} K_1 d\psi \leq \frac{M}{\{(r+r_1)(r_2-r)\}^{1/2}} \log\left[2 + \frac{ar}{(r_1-r)(r_2+r)}\right]$$

(ii) $$\int_{-\pi}^{\pi} \{|\partial_s K_1| + |\partial_r K_1|\}\, d\psi \leq \frac{M}{(r_1-r)\{(r+r_1)(r_2-r)\}^{1/2}}$$

Proof. The following identity can be easily verified by simple computation.

$$1+P = \frac{(r+r_2)(r+a-t+s)}{2ar},\ 1-P = \frac{(r_2-r)(t-s-a+r)}{2ar} \tag{5.9}$$

I-(i). Changing variable by the map $\psi = \Psi$, we have

$$\begin{aligned} \int_{-\varphi}^{\varphi} K_1 d\psi &= 2\int_0^1 K_2 d\tau \\ &= \frac{2^{1/2}}{(ar)^{1/2}} \int_0^1 \{\tau(1-\tau)(2+P\tau-\tau)\}^{-1/2} d\tau. \end{aligned} \tag{5.10}$$

First, we notice that in the domain D',

$$\begin{aligned} &|P| < 1 \\ &2 + P\tau - \tau = (P+1)\tau + 2(1-\tau) \geq 2(1-\tau) \quad \text{for} \quad \tau > 0. \end{aligned}$$

Thus, splitting the interval of integration into two pieces, we have

$$\int_0^{1/2} \{\tau(1-\tau)(2+P\tau-\tau)\}^{-1/2} d\tau \leq 2^{1/2} \int_0^{1/2} \tau^{-1/2} d\tau = 2, \tag{5.11}$$

$$\begin{aligned} &\int_{1/2}^1 \{\tau(1-\tau)(2+P\tau-\tau)\}^{-1/2} d\tau \\ &\quad \leq 2^{1/2} \int_{1/2}^1 \{(1-\tau)(2+P\tau-\tau)\}^{-1/2} d\tau \end{aligned}$$

$$
\begin{aligned}
&\leq 2^{1/2}\int_0^1 \partial_\tau\{-2(1-\tau)^{1/2}\}(2+P\tau-\tau)^{-1/2}d\tau \\
&= 2+2^{1/2}(1-P)\int_0^1 (1-\tau)^{1/2}(2+P\tau-\tau)^{-3/2}d\tau \\
&\leq 2+(1-P)\int_0^1 (2+P\tau-\tau)^{-1}d\tau \\
&\leq M\log\left[2+\frac{1}{1+P}\right]. \qquad (5.12)
\end{aligned}
$$

Since $P+1>1/2$ for $t-s\leq a$, the estimate **I**-(i) follows from (5.10)-(5.12) and (5.9).

I-(ii). Since

$$
\begin{aligned}
\partial_s K_2 &= \frac{-(t-s)\tau^{1/2}}{(2ar)^{3/2}(1-\tau)^{1/2}(2+P\tau-\tau)^{3/2}} \\
\partial_r K_2 &= -\frac{1}{2r}K_2 - \frac{\tau^{1/2}}{2^{3/2}(ar)^{1/2}(1-\tau)^{1/2}(2+P\tau-\tau)^{3/2}}\left(\frac{1}{a}-\frac{P}{r}\right),
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\int_0^1 \{|\partial_s K_2| + |\partial_r K_2|\}d\tau \\
&\quad\leq \frac{1}{2r}\int_0^1 K_2 d\tau + \frac{t-s+a+r}{(2ar)^{3/2}}\int_0^1 \frac{\tau^{1/2}}{(1-\tau)^{1/2}(2+P\tau-\tau)^{3/2}}d\tau. \qquad (5.13)
\end{aligned}
$$

By **I**-(i), we have

$$
\begin{aligned}
\frac{1}{2r}\int_0^1 K_2 d\tau &\leq Ma(ar)^{-3/2}\log[2+(1+P)^{-1}] \\
&\leq Ma(ar)^{-3/2}(1+P)^{-1}. \qquad (5.14)
\end{aligned}
$$

On the other hand, since

$$
\int_0^1 \frac{\tau^{1/2}}{(1-\tau)^{1/2}(2+P\tau-\tau)^{3/2}}d\tau \leq \int_0^1 (1-\tau)^{-1/2}(2+P\tau-\tau)^{-3/2}d\tau,
$$

we have by the method from which (5.12) was derived,

$$
\int_0^1 \frac{\tau^{1/2}}{(1-\tau)^{1/2}(2+P\tau-\tau)^{3/2}}d\tau \leq \frac{3}{2^{1/2}}\frac{1}{1+P}. \qquad (5.15)
$$

Therefore it follows from (5.13), (5.14) and (5.15) that

$$\begin{aligned}\int_0^1 \{|\partial_s K_2| + |\partial_r K_2|\} d\tau &\leq \frac{M(t-s+a+r)}{(ar)^{3/2}} \frac{1}{1+P} \\ &\leq \frac{M}{(ar)^{1/2}(r+s+a-t)}.\end{aligned}$$

I-(iii). We can easily verify that

$$\begin{aligned}\partial_r \Psi &= \left(\frac{P}{r} - \frac{1}{a}\right) \frac{\tau^{1/2}}{\{(1-P)(2+P\tau-\tau)\}^{1/2}} \\ \partial_s \Psi &= -\frac{t-s}{ar} \frac{\tau^{1/2}}{\{(1-P)(2+P\tau-\tau)\}^{1/2}}.\end{aligned}$$

We use the same method as we used in **I**-(i) and obtain

$$\begin{aligned}&\int_0^1 K_2\{|\partial_r \Psi| + |\partial_s \Psi|\} d\tau \\ &= \frac{1}{(2ar)^{1/2}} \left\{ \left|\frac{1}{a} - \frac{P}{r}\right| + \frac{t-s}{ar} \right\} \frac{1}{(1-P)^{1/2}} \int_0^1 \frac{d\tau}{(1-\tau)^{1/2}(2+P\tau-\tau)} \\ &\leq \frac{M}{(ar)^{1/2}} \frac{a+r}{ar} \frac{1}{\{(1-P)(1+P)\}^{1/2}}.\end{aligned}$$

Thus we get the estimate **I**-(iii).

II-(i). In the domain D'', $P < -1$ and $t-s > a+r$. Therefore,

$$\begin{aligned}\int_{-\pi}^{\pi} K_1 d\psi &= \frac{2}{\{(t-s)^2 - a^2 - r^2\}^{1/2}} \int_0^{\pi} \frac{d\psi}{(1-P^{-1}\cos\psi)^{1/2}} \\ &\leq \frac{2}{\{(t-s)^2 - a^2 - r^2\}^{1/2}} \left\{ \int_0^{3\pi/4} \frac{d\psi}{(1-2^{-1/2})^{1/2}} \right. \\ &\qquad \left. + \int_{3\pi/4}^{\pi} \frac{d\psi}{(1-P^{-1}\cos\psi)^{1/2}} \right\} \qquad (5.16)\end{aligned}$$

Further,

$$\begin{aligned}\int_{3\pi/4}^{\pi} \frac{d\psi}{(1-P^{-1}\cos\psi)^{1/2}} &= \int_0^{\pi/4} \frac{d\psi}{(1+P^{-1}\cos\psi)^{1/2}} \\ &= \int_0^{\pi/4} \frac{d\psi}{\{1-\cos\psi + (1+P^{-1})\cos\psi\}^{1/2}}\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{\pi/4} \frac{d\psi}{\{2\pi^{-2}\psi^2 + (1+P^{-1})2^{-1/2}\}^{1/2}} \\
&\leq M\{1 - \log(1+P^{-1})\} \\
&= M\left\{1 + \log\left(1 - \frac{1}{1+P}\right)\right\} \qquad (5.17)
\end{aligned}$$

From (5.17), (5.9) and

$$(t-s)^2 - a^2 - r^2 = -2arP \geq ar(1-P) = \frac{1}{2}(r+r_1)(r_2-r),$$

we get **II**-(i).

II-(ii). We can easily see that

$$\begin{aligned}
\partial_s K_1 &= \frac{t-s}{((t-s)^2 - a^2 - r^2 + 2ar\cos\psi)^{3/2}} \\
\partial_r K_1 &= \frac{r - a\cos\psi}{((t-s)^2 - a^2 - r^2 + 2ar\cos\psi)^{3/2}}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{-\pi}^{\pi} \{|\partial_s K_1| + |\partial_r K_1|\} d\psi \\
&\leq 2(t-s)\int_{-\pi}^{\pi} \frac{d\psi}{((t-s)^2 - a^2 - r^2 + 2ar\cos\psi)^{3/2}} \\
&= \frac{4(t-s)}{(-2arP)^{3/2}} \int_0^{\pi} \frac{d\psi}{(1 - P^{-1}\cos\psi)^{3/2}}. \qquad (5.18)
\end{aligned}$$

We get by the same way as the proof of **II**-(i) that

$$\begin{aligned}
&\int_0^{\pi} \frac{d\psi}{(1 - P^{-1}\cos\psi)^{3/2}} \\
&\leq \int_0^{4\pi/3} \frac{d\psi}{(1 - 2^{-1/2})^{3/2}} + \int_0^{\pi/4} \frac{d\psi}{\{2\pi^{-2}\psi^2 + (1+P^{-1})2^{-1/2}\}^{3/2}} \\
&\leq M\left(1 + \frac{P}{1+P}\right) \\
&\leq M\frac{P}{1+P}. \qquad (5.19)
\end{aligned}$$

Therefore it follows from (5.18) and (5.19) that

$$\int_{-\pi}^{\pi} \{|\partial_s K_1| + |\partial_r K_1|\} d\psi \leq \frac{M(t-s)}{(-2arP)^{3/2}} \frac{P}{1+P}$$

$$
\begin{aligned}
&\leq \frac{M(t-s)}{(ar)^{3/2}(1-P)^{1/2}} \frac{-1}{1+P} \\
&\leq \frac{M}{(r_1 - r)(ar)^{1/2}(1-P)^{1/2}}
\end{aligned}
$$

and we get the estimate **II**-(ii). ∎

Now we can show the estimates for the first derivatives of the solution to the initial value problem (5.1).

Proposition 5. 4 *Let $u \in C^\infty([0,T) \times \mathbf{R}^2)$ be the solution of the initial value problem (5.1). And let w(s,r) be a positive function that satisfies*

$$
\text{(i)} \quad k-1 \leq r \leq k+1 \Rightarrow \frac{1}{M} w(s,k) \leq w(s,r) \leq M w(s,k) \tag{5.20}
$$
$$
(k = 1, 2, \cdots; M \text{ is independent of } s, k, r)
$$

$$
\begin{aligned}
\text{(ii)} \quad \frac{1}{w(s,r)} \leq M \Bigg\{ &\sum_{i=1}^{L} \frac{1}{(r+s+1)(|r - c_i s| + 1)} \\
&+ \frac{1}{(r+s+1)^{1+2\gamma}(r+1)^{1-2\gamma}} \\
&+ \frac{1}{(r+s+1)^{1+\epsilon}(|r-s|+1)^{1-\epsilon}} \Bigg\}
\end{aligned} \tag{5.21}
$$
$$
(c_i \neq 1 \ (i = 1, 2, \cdots, L), \ 0 < \gamma < 1/2, \ 0 < \epsilon < 1)
$$

Then the following estimate holds:

$$
\begin{aligned}
|\partial u(t,x)| \leq \; & \frac{M}{(|x|+1)^{1/2-\gamma}(|x|+t+1)^{\gamma}(||x|-t|+1)^{1/2}} \cdot \\
& \cdot \Bigg\{ \sum_{|A| \leq 3} \sup_{0<s<t} \left\| w(s, |\cdot|) \mathcal{D}^A F(s, \cdot) \right\|_0 \\
& + \sum_{|A| \leq 2} \sup_{0<s<t} \left\| w(s, |\cdot|) \mathcal{D}^A \partial_t F(s, \cdot) \right\|_0 \Bigg\}.
\end{aligned} \tag{5.22}
$$

Proof. By (5.4) and Proposition 5. 2 , we have

$$
\begin{aligned}
|\partial u(t,x)| \leq & \left\{ \sup_{0<s<t} \sup_{x\in\mathbf{R}^2} |x|^{1/2} w(s,|x|) |F(s,x)| \right. \\
& + \sup_{0<s<t} \sup_{x\in\mathbf{R}^2} |x|^{1/2} w(s,|x|) |\partial F(s,x)| \\
& \left. + \sup_{0<s<t} \sup_{x\in\mathbf{R}^2} |x|^{1/2} w(s,|x|) |\Omega F(s,x)| \right\} \cdot \\
& \cdot \{ I_1' + \cdots + I_5' + I_1'' + \cdots + I_4'' \}
\end{aligned} \tag{5.23}
$$

where I_i' $(i=1,\ldots,5)$ and I_i'' $(i=1,\ldots,4)$ are defined as follows.

$$
\begin{aligned}
I_1' &= \iint_{blue} \frac{r^{1/2}}{w(s,r)} dr ds \int_{-\varphi}^{\varphi} K_1 d\psi \\
I_2' &= \iint_{\partial(white)} \frac{r^{1/2}}{w(s,r)} d\sigma \int_0^1 K_2 d\tau \\
I_3' &= \iint_{white} \frac{1}{r^{1/2} w(s,r)} dr ds \int_0^1 K_2 d\tau \\
I_4' &= \iint_{white} \frac{r^{1/2}}{w(s,r)} dr ds \int_0^1 \{ |\partial_s K_2| + |\partial_r K_2| \} d\tau \\
I_5' &= \iint_{white} \frac{r^{1/2}}{w(s,r)} dr ds \int_0^1 K_2 \{ |\partial_s \Psi| + |\partial_r \Psi| \} d\tau \\
I_1'' &= \iint_{black} \frac{r^{1/2}}{w(s,r)} dr ds \int_{-\pi}^{\pi} K_1 d\psi \\
I_2'' &= \iint_{\partial(red)} \frac{r^{1/2}}{w(s,r)} d\sigma \int_{-\pi}^{\pi} K_1 d\psi \\
I_3'' &= \iint_{red} \frac{1}{r^{1/2} w(s,r)} dr ds \int_{-\pi}^{\pi} K_1 d\psi \\
I_4'' &= \iint_{red} \frac{r^{1/2}}{w(s,r)} dr ds \int_{-\pi}^{\pi} \{ |\partial_s K_1| + |\partial_r K_1| \} d\psi
\end{aligned}
$$

Here, I_i' $(i=1,\ldots,5)$ are integrals that are related to the domain D', and I_i'' $(i=1,\ldots,4)$ to the domain D''. We show in the following that

$$
I_i' \leq \frac{M}{(a+1)^{1/2-\gamma}(a+t+1)^{\gamma}(|a-t|+1)^{1/2}} \quad (i=1,\ldots,5), \tag{5.24}
$$

$$
I_i'' \leq \frac{M}{(a+1)^{1/2-\gamma}(a+t+1)^{\gamma}(|a-t|+1)^{1/2}} \quad (i=1,\ldots,4), \tag{5.25}
$$

where $a = |x|$. M. Kovalylov showed in his paper[10] that

$$|x||f(x)|^2 w(s,|x|)^2 \le M \sum_{|A|\le 2} \|w(s,|\cdot|)\mathcal{D}^A f\|_0^2 \tag{5.26}$$

for $f \in C_0^\infty(\mathbf{R}^2)$. Then we get the estimate (5.22) from (5.23), (5.24), (5.25) and (5.26).

First, we prove (5.24). To prove this, we introduce some notations. Set

$$\begin{aligned}
\xi(s,r) &= \xi_1(s,r) + \xi_2(s,r),\\
\xi_1(s,r) &= \sum_{i=1}^{L} \frac{1}{(r+s+1)(|r-c_i s|+1)} + \frac{1}{(r+s+1)^{1+2\gamma}(r+1)^{1-2\gamma}},\\
\xi_2(s,r) &= \frac{1}{(r+s+1)^{1+\epsilon}(|r-s|+1)^{1-\epsilon}}.
\end{aligned}$$

Then by the assumption (5.21) on $w(s,r)$,

$$\frac{1}{w(s,r)} \le M\xi(s,r). \tag{5.27}$$

Moreover, set

$$\begin{aligned}
\eta(s,r) &= \eta_1(s,r) + \eta_2(s,r),\\
\eta_1(s,r) &= \sum_{i=1}^{L} \frac{1}{(r+s+1)^{\tilde\lambda}(|r-c_i s|+1)} + \frac{1}{(r+s+1)^{\tilde\lambda+2\gamma}(r+1)^{1-2\gamma}},\\
\eta_2(s,r) &= \frac{1}{(r+s+1)^{\tilde\lambda+\epsilon}(|r-s|+1)^{1-\epsilon}},
\end{aligned}$$

where

$$0 < \lambda < \min\{\gamma, 1/2-\gamma\},\ \tilde\lambda = 1/2-\gamma-\lambda. \tag{5.28}$$

Since $r+s \ge |a-t|$ for $(s,r) \in D'$, we have

$$\xi_i(s,r) \le \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}\eta_i(s,r) \quad (i=1,2) \tag{5.29}$$

$$\frac{1}{w(s,r)} \le \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}\eta(s,r) \tag{5.30}$$

for $(s,r) \in D'$. But in the estimate of I_1' and I_5', $\xi_2(s,r)$ is treated in another way.

(i) *Estimate of* I_1'

By Proposition 5. 3 .**I**.(i),

$$I_1' \leq \frac{M}{a^{1/2}} \iint_{blue} \frac{1}{w(s,r)} \log\left[2 + \frac{ar}{(r-r_1)(r+r_2)}\chi(t-s-a)\right] drds. \quad (5.31)$$

Therefore it follows from (5.31) and (5.20) that

$$\begin{aligned} I_1' \quad \leq \quad & \frac{M}{a^{1/2}}\left\{\int_0^t \frac{ds}{w(s,r_1)}\int_{r_1}^{r_1+\delta} \log\left[2+\frac{ar}{(r-r_1)(r+r_2)}\chi(t-s-a)\right]dr\right. \\ & \left. + \int_0^t \frac{ds}{w(s,r_2)}\int_{r_2-\delta}^{r_2} \log\left[2+\frac{ar}{(r-r_1)(r+r_2)}\chi(t-s-a)\right]dr\right\}. \end{aligned} \quad (5.32)$$

Let us consider the integrals of $\log[2 + ar/(r-r_1)(r+r_2)\cdot\chi(t-s-a)]$. For $0 < s < t-a$ and $r_2 - \delta < r < r_2$, it follows from (5.2) and (5.3) that

$$r - r_1 > r_2 - \delta - r_1 = 2a - \delta > a.$$

Then we have

$$\log\left[2+\frac{ar}{(r-r_1)(r+r_2)}\right] \leq \log\left[2+\frac{a}{r-r_1}\right] \leq \log 3. \quad (5.33)$$

For $0 < s < t-a$ and $r_1 < r < r_1 + \delta$, we have

$$\begin{aligned} & \int_{r_1}^{r_1+\delta} \log\left[2+\frac{ar}{(r-r_1)(r+r_2)}\right]dr \\ & \leq \int_{r_1}^{r_1+\delta} \log\left[2+\frac{r}{r-r_1}\right]dr \\ & = \delta\left[\{\log(3\delta+r_1) - \log\delta\} + \frac{r_1}{3\delta}\log(1+3\delta/r_1)\right] \\ & \leq \delta\log(3/2+t-a) + \delta^{1/2}2e^{-1} + \delta \\ & \leq M\delta^{1/2}\log[2+|a-t|]. \end{aligned} \quad (5.34)$$

Therefore it follows from (5.32), (5.33) and (5.34) that

$$I_1' \leq \frac{M\delta^{1/2}}{a^{1/2}}\log[2+|a-t|]\left\{\int_0^t \frac{ds}{w(s,r_1)} + \int_0^t \frac{ds}{w(s,r_2)}\right\}. \quad (5.35)$$

We next show

$$\int_0^t \frac{ds}{w(s,r_i)} \leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \quad (i=1,2). \quad (5.36)$$

We use (5.30) for $1/w(s,r_2)$ and obtain

$$\int_0^t \frac{ds}{w(s,r_2)} \le \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \int_0^t \eta(s,r_2)ds. \tag{5.37}$$

Moreover,

$$\begin{aligned}
\int_0^t \eta(s,r_2)ds &\le M\int_0^t \left\{ \sum_{i=1}^{L} \frac{1}{(|r_2-c_i s|+1)^{1+\bar\lambda}} \right. \\
&\qquad \left. + \frac{1}{(|r_2-s|+1)^{1+\bar\lambda}} + \frac{1}{(r_2+1)^{1+\bar\lambda}} \right\} ds \\
&\le M\int_{-\infty}^{\infty} \left\{ \sum_{i=1}^{L} \frac{1}{(|r_2-c_i s|+1)^{1+\bar\lambda}} \right. \\
&\qquad \left. + \frac{1}{(|r_2-s|+1)^{1+\bar\lambda}} + \frac{1}{(|r_2|+1)^{1+\bar\lambda}} \right\} ds \\
&\le M.
\end{aligned} \tag{5.38}$$

Therefore, from (5.37) and (5.38) we have (5.36) for $i=2$. The treatment for $i=1$ is slightly different. We remark that $|r_1-s|=|a-t|$ for $(t-a)_+ < s < t$ by the definition (5.2), where $x_+ = \max\{0,x\}$. Then we see from (5.28) that

$$\xi_2(s,r_1) \le \frac{1}{(|a-t|+1)^{1/2+\gamma+\lambda}(r_1+s+1)^{1+\min\{\epsilon,\bar\lambda\}}} \tag{5.39}$$

for $(t-a)_+ < s < t$. Therefore it follows (5.27), (5.29) and (5.39) that

$$\begin{aligned}
\int_0^t \frac{ds}{w(s,r_1)} &\le \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \left\{ \int_0^t \eta_1(s,r_1)ds + \int_0^{(t-a)_+} \eta_2(s,r_1)ds \right. \\
&\qquad \left. + \int_{(t-a)_+}^t \frac{ds}{(r_1+s+1)^{1+\min\{\epsilon,\bar\lambda\}}} \right\} \\
&\le \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}.
\end{aligned}$$

Combining (5.35) and (5.36), we have

$$I_1' \le \frac{M\delta^{1/2}}{a^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \log[2+|a-t|]$$

$$\leq \frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma}}$$
$$\leq \frac{M}{(a+1)^{1/2-\gamma}(a+t+1)^{\gamma}(|a-t|+1)^{1/2}}. \tag{5.40}$$

Here we use the fact that

$$\frac{a+t+1}{(a+1)(|a-t|+1)} \leq 4 \quad \text{for} \quad a, t \geq 0. \tag{5.41}$$

(ii) *Estimate of* I_2'

By Proposition 5. 3 .**I**.(i),

$$I_2' \leq \frac{M}{a^{1/2}} \iint_{\partial(white)} \frac{1}{w(s,r)} \log\left[2 + \frac{ar}{(r-r_1)(r+r_2)}\chi(t-s-a)\right] d\sigma. \tag{5.42}$$

Let $0 \leq s \leq t-a$ and $(s,r) \in \overline{white}$. Then

$$\frac{ar}{(r-r_1)(r+r_2)} \leq \frac{r}{r-r_1} \leq \frac{r_1+1/2}{1/2} \leq 2(t-a+1/2).$$

So we have

$$\frac{ar}{(r-r_1)(r_2+r)}\chi(t-s-a) \leq 2(2+|a-t|) \tag{5.43}$$

for $(s,r) \in \overline{white}$. Hence from (5.42) and (5.43) we get

$$I_2' \leq \frac{M}{a^{1/2}} \log[2+|a-t|] \iint_{\partial(white)} \frac{d\sigma}{w(s,r)}. \tag{5.44}$$

We have already computed the integral of $1/w(s,r)$ in the estimate of I_1', except the one on $\{0\} \times (|a-t|+\delta, a+t-\delta)$. Applying (5.30) for $s=0$, we have

$$\int_{|a-t|+\delta}^{a+t-\delta} \frac{dr}{w(0,r)} \leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}. \tag{5.45}$$

Therefore it follows from (5.44), (5.36) and (5.45) that

$$I_2' \leq \frac{M}{a^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \log[2+|a-t|]. \tag{5.46}$$

Since $a \geq \delta = 1/2$ when the domain *white* is not empty, we have (5.24) for $i=2$ by the way from which (5.40) was derived.

(iii) *Estimate of* I_3'

By Proposition 5. 3 .**I**.(i),

$$I_3' \le \frac{M}{a^{1/2}} \iint_{white} \frac{1}{rw(s,r)} \log\left[2 + \frac{ar}{(r-r_1)(r+r_2)}\chi(t-s-a)\right] drds. \tag{5.47}$$

Further, by (5.30) and (5.43),

$$I_3' \le \frac{M}{a^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \log[2+|a-t|] \iint_{white} \frac{\eta(s,r)}{r} drds. \tag{5.48}$$

Since $r \ge \delta = 1/2$ in the domain *white*, we have

$$\begin{aligned} \frac{\eta(s,r)}{r} \le \; & M\left\{\sum_{i=1}^{L} \frac{1}{(r+1)^{1+\bar\lambda/2}(|r-c_i s|+1)^{1+\bar\lambda/2}}\right. \\ & \left. + \frac{1}{(r+1)^{2+\bar\lambda}} + \frac{1}{(r+1)^{1+\bar\lambda/2}(|r-s|+1)^{1+\bar\lambda/2}}\right\} \end{aligned} \tag{5.49}$$

for $(s,r) \in white$. Concerning the right-hand side of (5.49), the integral of the first and the third term are shown to be bounded by a constant M in the same way as (5.38). As for the second term, we see that

$$\begin{aligned} \iint_{white} \frac{drds}{(r+1)^{2+\bar\lambda}} &\le \int_0^t ds \int_{r_1}^{r_2} \frac{dr}{(r+1)^{2+\bar\lambda}} \\ &\le \frac{1}{1+\bar\lambda}\int_0^t \frac{ds}{(r_1+1)^{1+\bar\lambda}} \le M. \end{aligned} \tag{5.50}$$

Therefore it follows from (5.49) and (5.50) that

$$\iint_{white} \frac{\eta(s,r)}{r} drds \le M. \tag{5.51}$$

Hence from (5.48) and (5.51) we have (5.24) for $i = 3$.

(iv) *Estimate of* I_4'

By Proposition 5. 3 .**I**.(ii),

$$I_4' \le \frac{M}{a^{1/2}} \iint_{white} \frac{1}{w(s,r)(r+s+a-t)} drds. \tag{5.52}$$

Applying (5.30) to the right-hand side of (5.52), we obtain

$$I_4' \le \frac{M}{a^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \iint_{white} \frac{\eta(s,r)}{r+s+a-t} drds. \tag{5.53}$$

Since $r+s+a-t \geq \delta = 1/2$ in the domain *white*, we have

$$\begin{aligned}&\frac{\eta(s,r)}{r+s+a-t}\\&\leq \frac{M}{r+s+a-t+1}\left\{\sum_{i=1}^{L}\frac{1}{(r+s+1)^{\bar{\lambda}/2}(|r-c_i s|+1)^{1+\bar{\lambda}/2}}\right.\\&\quad\left.+\frac{1}{(r+s+1)^{\bar{\lambda}/2}(r+1)^{1+\bar{\lambda}/2}}+\frac{1}{(r+s+1)^{\bar{\lambda}/2}(|r-s|+1)^{1+\bar{\lambda}/2}}\right\}\end{aligned}$$

for $(s,r) \in$ *white*. Hence we find by the change of variables $(\alpha,\beta) = (s+r, s-r)$ that

$$\iint_{white}\frac{\eta(s,r)}{r+s+a-t}drds \leq M. \tag{5.54}$$

Therefore from (5.53), (5.54) and (5.41) we obtain (5.24) for $i=4$.

(v) *Estimate of* I_5'

By Proposition 5. 3 .**I**.(iii),

$$I_5' \leq \frac{M}{a^{1/2}}\iint_{white}\frac{1}{w(s,r)}\frac{a+r}{\{(r^2-r_1^2)(r_2^2-r^2)\}^{1/2}}drds. \tag{5.55}$$

We notice that for $(s,r) \in$ *white*,

$$\begin{aligned}&r_2+r \geq a,\ r_2+r \geq r;\\&r+r_1 \geq a,\ r+r_1 \geq r \quad \text{for} \quad r \geq (r_2-r_1)/2;\\&r_2-r \geq a,\ r_2-r \geq r \quad \text{for} \quad r \leq (r_2-r_1)/2.\end{aligned}$$

Hence we have

$$\frac{a+r}{\{(r^2-r_1^2)(r_2^2-r^2)\}^{1/2}} \leq 2\left\{\frac{1}{(r^2-r_1^2)^{1/2}}+\frac{1}{\{(r-r_1)(r_2-r)\}^{1/2}}\right\} \tag{5.56}$$

for $(s,r) \in$ *white*. Therefore it follows from (5.55), (5.27) and (5.56) that

$$I_5' \leq \frac{M}{a^{1/2}}\iint_{white}\xi(s,r)\left\{\frac{1}{(r^2-r_1^2)^{1/2}}+\frac{1}{\{(r-r_1)(r_2-r)\}^{1/2}}\right\}drds. \tag{5.57}$$

We show in the following that

$$\iint_{white}\xi(s,r)\frac{1}{(r^2-r_1^2)^{1/2}}drds \leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}. \tag{5.58}$$

We use (5.29) for $\xi_1(s,r)$ and obtain

$$\begin{aligned} &\iint_{white} \xi_1(s,r)\frac{1}{(r^2-r_1^2)^{1/2}}drds \\ &\le \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}\iint_{white}\eta_1(s,r)\frac{1}{(r^2-r_1^2)^{1/2}}drds. \end{aligned} \tag{5.59}$$

Since $r \pm (s+a-t) \ge \delta = 1/2$ for $(s,r) \in white$, we have

$$\begin{aligned} \frac{\eta_1(s,r)}{(r^2-r_1^2)^{1/2}} &\le \frac{M}{(r+s+a-t+1)^{1/2}(r-s-a+t+1)^{1/2}}\eta_1(s,r) \\ &\le \frac{M}{r+s+a-t+1}\cdot \\ &\quad \cdot\left\{\sum_{i=1}^{L}\frac{1}{(r+s+1)^{\bar\lambda/2}(|r-c_is|+1)^{1+\bar\lambda/2}}\right. \\ &\quad \left. + \frac{1}{(r+s+1)^{\bar\lambda/2}(r+1)^{1+\bar\lambda/2}}\right\} \\ &\quad + \frac{M}{r-s-a+t+1} \\ &\quad \cdot\left\{\sum_{i=1}^{L}\frac{1}{(|r-s|+1)^{\bar\lambda/2}(|r-c_is|+1)^{1+\bar\lambda/2}}\right. \\ &\quad \left. + \frac{1}{(|r-s|+1)^{\bar\lambda/2}(r+1)^{1+\bar\lambda/2}}\right\} \end{aligned}$$

for $(s,r) \in white$. Hence by the change of variables $(\alpha,\beta) = (s+r, s-r)$ we have

$$\iint_{white}\frac{\eta_1(s,r)}{(r^2-r_1^2)^{1/2}}drds \le M. \tag{5.60}$$

Therefore it follows from (5.59) and (5.60) that

$$\iint_{white}\frac{\xi_1(s,r)}{(r^2-r_1^2)^{1/2}}drds \le \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}. \tag{5.61}$$

On the other hand, we see that, for $(s,r) \in white$,

$$\begin{aligned} &\frac{\xi_2(s,r)}{(r^2-r_1^2)^{1/2}} \\ &\le \frac{M}{(r+s+a-t+1)^{1/2}(r-s-a+t+1)^{1/2}}\cdot \end{aligned}$$

$$
\begin{aligned}
&\cdot\Big[\{1-\chi(|r-s|-|a-t|/2)\}\xi_2(s,r)+\chi(|r-s|-|a-t|/2)\xi_2(s,r)\Big]\\
&\le \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}\frac{1}{(r+s+a-t+1)^{1/2}}\cdot\\
&\quad\cdot\left\{\frac{1}{(r+s+1)^{1/2+\bar{\lambda}+\epsilon}(|r-s|+1)^{1-\epsilon}}\right.\\
&\qquad\left.+\frac{1}{(r-s-a+t+1)^{1/2}(r+s+1)^{1+\min\{\epsilon,\bar{\lambda}\}}}\right\}\\
&\le \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}\frac{1}{(r+s+a-t+1)^{1/2}}\cdot\\
&\quad\cdot\left\{\frac{1}{(r+s+1)^{1/2+\bar{\lambda}/2}(|r-s|+1)^{1+\bar{\lambda}/2}}\right.\\
&\qquad+\frac{1}{(r-s-a+t+1)^{1/2}(|r-s|+1)^{1/2+\min\{\epsilon,\bar{\lambda}\}/2}}\cdot\\
&\qquad\left.\cdot\frac{1}{(r+s+1)^{1/2+\min\{\epsilon,\bar{\lambda}\}/2}}\right\}.
\end{aligned}
$$

Here, we have used that $t-a>0$ on the support of $1-\chi(|r-s|-|a-t|/2)$. Therefore it follows that

$$
\iint_{white}\frac{\xi_2(s,r)}{(r^2-r_1^2)^{1/2}}drds\le\frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}. \tag{5.62}
$$

Combining (5.61) and (5.62), we obtain (5.58). To estimate the second term in the right hand side of (5.57), we note that

$$
\begin{aligned}
&\frac{1}{\{(r-r_1)(r_2-r)\}^{1/2}}\\
&\quad\le\frac{1}{\{(a+t-s-r+1)(r+s+a-t+1)\}^{1/2}}\\
&\quad+\frac{1}{\{(a+t-s-r+1)(r-s-a+t+1)\}^{1/2}}.
\end{aligned} \tag{5.63}
$$

Using (5.29) for the first term of (5.63) and the method above for the second term of (5.63), we get

$$
\iint_{white}\frac{\xi(s,r)}{\{(r-r_1)(r_2-r)\}^{1/2}}drds\le\frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}. \tag{5.64}
$$

Therefore from (5.57), (5.58) and (5.64) we obtain (5.24) for $i = 5$. Consequently, we have proved (5.24).

Next, we prove (5.25). $\xi(s,r)$ and $\eta(s,r)$ are used again, but we do not consider $\xi_1(s,r)$ and $\xi_2(s,r)$ separately. Since $t > a$ when D'' is not empty, $r_1 = t - a - s$.

(vi) *Estimate of I_1''*

By Proposition 5. 3 .**II**.(i),

$$I_1'' \le M \iint_{black} \frac{r^{1/2}}{w(s,r)\{(r+r_1)(r_2-r)\}^{1/2}} \log\left[2 + \frac{ar}{(r_1-r)(r+r_2)}\right] drds. \tag{5.65}$$

In the domain D'', we use the following facts:

$$\frac{1}{\{(r+r_1)(r_2-r)\}^{1/2}} \le \frac{M}{\{(t-a)(a+t)\}^{1/2}} \quad \text{for} \quad r+s \le (t-a)/2, \tag{5.66}$$

$$\frac{1}{w(s,r)} \le \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}\eta(s,r) \quad \text{for} \quad r+s \ge (t-a)/2, \tag{5.67}$$

$$r_2 - r \ge 2a + \tilde{\delta} \quad \text{for} \quad r \le r_1 - \tilde{\delta}, \tag{5.68}$$

$$\begin{aligned} &\frac{ar}{(r_1-r)(r+r_2)} \\ &\quad \le \frac{r}{r_1-r} \le M\{(t-a)\chi(2r-r_1)+1\} \quad \text{for} \quad r \le r_1 - \tilde{\delta}. \end{aligned} \tag{5.69}$$

It follows from (5.66)-(5.69) and (5.3) that

$$\begin{aligned} I_1'' \le\ & \frac{M}{\{(t-a)(a+t)\}^{1/2}} \iint_{\substack{black \\ r+s\le(t-a)/2}} \frac{r^{1/2}}{w(s,r)} drds \\ &+ \frac{M}{(2a+\tilde{\delta})^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \cdot \\ &\cdot \iint_{\substack{black \\ (t-a)/2\le r+s\le t-a-\tilde{\delta}}} \eta(s,r)\frac{\dot{r}^{1/2}}{(r+r_1)^{1/2}} drds \\ &+ \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \cdot \\ &\cdot \iint_{\substack{black \\ t-a-\tilde{\delta}\le r+s}} \eta(s,r)\frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}} \cdot \\ &\cdot \log\left[2 + \frac{ar}{(r_1-r)(r+r_2)}\right] drds. \end{aligned} \tag{5.70}$$

Moreover,

$$\iint_{\substack{black \\ r+s\le(t-a)/2}} \frac{r^{1/2}}{w(s,r)} drds \le M\int_0^{t-a}\xi(s,0)ds\int_0^{\bar\delta} r^{1/2}dr \le M\bar\delta^{3/2}, \quad (5.71)$$

$$\begin{aligned}&\iint_{\substack{black \\ (t-a)/2\le r+s\le t-a-1/2}} \eta(s,r)\frac{r^{1/2}}{(r+r_1)^{1/2}}drds \\ &\le M\int_0^{t-a}\frac{\eta(s,0)}{(1+r_1)^{1/2}}ds\int_0^{1/2}dr \\ &\le M\int_0^{t-a}\frac{1}{(1+t-a-s)^{1/2}}\left\{\sum_{i=1}^{L}\frac{1}{(s+1)^{\tilde\lambda}(c_is+1)}\right. \\ &\quad \left. +\frac{1}{(s+1)^{\tilde\lambda+2\gamma}}+\frac{1}{(s+1)^{1+\tilde\lambda}}\right\}ds \\ &\le M \qquad (5.72)\end{aligned}$$

because $\tilde\lambda + 2\gamma > 1/2$ by (5.28).

It remains to estimate the third term of the right-hand side of (5.70). We show that

$$\begin{aligned}&\int_{(r_1-\bar\delta)_+}^{r_1}\frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}}\log\left[2+\frac{ar}{(r_1-r)(r+r_2)}\right]dr \\ &\le \frac{M}{(a+1)^{1/2}}\log[2+t-a],\end{aligned} \quad (5.73)$$

from which we obtain

$$\begin{aligned}&\iint_{\substack{black \\ t-a-\bar\delta\le r+s}} \eta(s,r)\frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}}\log\left[2+\frac{ar}{(r_1-r)(r+r_2)}\right]drds \\ &\le M\int_0^{t-a}\eta(s,r_1)ds\,\cdot \\ &\quad\cdot\int_{(r_1-\bar\delta)_+}^{r_1}\frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}}\log\left[2+\frac{ar}{(r_1-r)(r+r_2)}\right]dr \\ &\le \frac{M}{(a+1)^{1/2}}\log[2+t-a]. \qquad (5.74)\end{aligned}$$

To prove (5.73), we consider the following two cases separately: (a) $1\le a$ and (b) $0<a<1$.

(a) $1 \le a$
Since $r_2 - r \ge r_2 - r_1 \ge a+1$, we have

$$\int_{(r_1-\tilde{\delta})_+}^{r_1} \frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}} \log\left[2+\frac{ar}{(r_1-r)(r+r_2)}\right] dr \le \frac{1}{(a+1)^{1/2}} \int_{r_1-\tilde{\delta}}^{r_1} \log\left[2+\frac{r}{r_1-r}\right] dr.$$

Hence by the way from which we derive (5.34), we have (5.73) for $a \ge 1$.
(b) $0 < a < 1$
Since $\log[2+ar/(r_1-r)(r+r_2)] \le 1 + a^{1/2}/(r_1-r)^{1/2}$ and $r_2 - r \ge 2a$, we have

$$\begin{aligned}\int_{(r_1-\tilde{\delta})_+}^{r_1} & \frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}} \log\left[2+\frac{ar}{(r_1-r)(r+r_2)}\right] dr \\ &\le \int_{r_1-\tilde{\delta}}^{r_1} \left\{\frac{1}{(r_2-r)^{1/2}} + \frac{1}{(r_1-r)^{1/2}}\right\} dr \\ &\le 4\tilde{\delta}^{1/2} \le 2\sqrt{2}.\end{aligned}$$

Thus we obtain (5.73) for $0 < a < 1$.

Therefore it follows from (5.70)-(5.72) and (5.74) that

$$\begin{aligned} I_1'' &\le \frac{M\tilde{\delta}}{\{(t-a)(a+t)\}^{1/2}} + \frac{M}{(2a+1/2)^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \\ &\quad + \frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \log[2+t-a] \\ &\le \frac{M}{(t-a+1)^{1/2}(t+a+1)^{1/2}} + \frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma}} \\ &\le \frac{M}{(a+1)^{1/2-\gamma}(a+t+1)^{\gamma}(|a-t|+1)^{1/2}}. \end{aligned} \tag{5.75}$$

(vii) *Estimate of I_2''*
By Proposition 5. 3 .**II**.(i),

$$I_2'' \le M \iint_{\partial(red)} \frac{r^{1/2}}{w(s,r)\{(r+r_1)(r_2-r)\}^{1/2}} \log\left[2+\frac{ar}{(r_1-r)(r+r_2)}\right] d\sigma.$$

Moreover, by (5.66)-(5.69), we have

$$I_2'' \le \frac{M}{\{(a+t+1)(t-a+1)\}^{1/2}} \iint_{\substack{\partial(red) \\ s+r\le(t-a)/2}} \frac{r^{1/2}}{w(s,r)} d\sigma$$

$$+\frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}}\log[2+t-a]\cdot$$
$$\cdot\iint_{\substack{\theta(red)\\ s+r\geq(t-a)/2}}\eta(s,r)\frac{r^{1/2}}{(r+r_1)^{1/2}}d\sigma. \tag{5.76}$$

Here we notice that $(t-a)/2\geq\tilde{\delta}=1/2$ when the domain *red* is not empty. We further see that

$$\iint_{\substack{\theta(red)\\ s+r\leq(t-a)/2}}\frac{r^{1/2}}{w(s,r)}d\sigma\leq M, \tag{5.77}$$
$$\iint_{\substack{\theta(red)\\ s+r\geq(t-a)/2}}\eta(s,r)\frac{r^{1/2}}{(r+r_1)^{1/2}}d\sigma\leq M. \tag{5.78}$$

Hence from (5.76), (5.77) and (5.78) we obtain (5.25) for $i=2$.

(viii) *Estimate of* I_3''

By Proposition 5. 3 .**II**.(i),

$$I_3''\leq M\iint_{red}\frac{1}{r^{1/2}w(s,r)\{(r+r_1)(r_2-r)\}^{1/2}}\log\left[2+\frac{ar}{(r_1-r)(r+r_2)}\right]drds.$$

Further, by (5.66)-(5.69), we have

$$\begin{aligned}
I_3'' \quad\leq\quad & \frac{M}{\{(t-a+1)(a+t+1)\}^{1/2}}\iint_{\substack{red\\ s+r\leq(t-a)/2}}\frac{1}{r^{1/2}w(s,r)}drds\\
&+\frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}}\log[2+t-a]\cdot\\
&\iint_{\substack{red\\ s+r\geq(t-a)/2}}\frac{\eta(s,r)}{r^{1/2}(r+r_1)^{1/2}}drds. \qquad (5.79)
\end{aligned}$$

Since $r\geq\tilde{\delta}=1/2$ in the domain *red*, we have

$$\begin{aligned}
\frac{1}{r^{1/2}w(s,r)} \quad\leq\quad & \frac{M\xi(s,r)}{(r+1)^{1/2}}\\
\leq\quad & M\left\{\sum_{i=1}^{L}\frac{1}{(r+1)^{5/4}(|r-c_is|+1)^{5/4}}\right.\\
&+\frac{1}{(r+s+1)^{1+\gamma}(r+1)^{1+1/2-\gamma}}\\
&\left.+\frac{1}{(r+1)^{5/4}(|r-s|+1)^{5/4}}\right\}
\end{aligned}$$

for $(s,r) \in red$. Therefore it follows that

$$\iint_{\substack{red \\ s+r\le(t-a)/2}} \frac{1}{r^{1/2}w(s,r)}drds \le M. \tag{5.80}$$

Moreover,

$$\begin{aligned}
&\frac{\eta(s,r)}{r^{1/2}(r+r_1)^{1/2}} \\
&\le \frac{\eta(s,r)}{(r+1)^{1/2}(r+r_1+1)^{1/2}} \\
&\le M\left\{\sum_{i=1}^{L} \frac{1}{(r+1)^{1+\bar\lambda/2}(|r-c_is|+1)^{1+\bar\lambda/2}}\right. \\
&\quad + \frac{1}{(r-s+t-a+1)^{1/2}(|r-s|+1)^{1/2+\min\{\gamma-\lambda,\bar\lambda/2\}}(r+1)^{1+\bar\lambda/2}} \\
&\quad \left. + \frac{1}{(r+1)^{1+\bar\lambda/2}(|r-s|+1)^{1+\bar\lambda/2}}\right\}
\end{aligned}$$

for $(s,r) \in red$. Therefore it follows that

$$\iint_{\substack{red \\ s+r\ge(t-a)/2}} \frac{\eta(s,r)}{r^{1/2}(r+r_1)^{1/2}}drds \le M. \tag{5.81}$$

Hence from (5.79), (5.80) and (5.81) we obtain (5.25) for $i=3$.

(ix) *Estimate of* I_4''

By Proposition 5. 3 .**II**.(ii),

$$I_4'' \le M \iint_{red} \frac{r^{1/2}}{w(s,r)(r_1-r)\{(r+r_1)(r_2-r)\}^{1/2}}drds.$$

Further, by (5.66), (5.67) and (5.68) we have

$$\begin{aligned}
I_4'' \quad \le \quad & \frac{M}{\{(t-a+1)(a+t+1)\}^{1/2}} \iint_{\substack{red \\ s+r\le(t-a)/2}} \frac{r^{1/2}}{w(s,r)(r_1-r)}drds \\
&+ \frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \\
&\cdot \iint_{\substack{red \\ s+r\ge(t-a)/2}} \eta(s,r)\frac{r^{1/2}}{(r_1-r)(r+r_1)^{1/2}}drds.
\end{aligned} \tag{5.82}$$

Both $r^{1/2}/w(s,r)(r_1-r)$ and $\eta(s,r)r^{1/2}/(r_1-r)(r+r_1)^{1/2}$ are bounded by $\eta(s,r)/(r_1-r)$. And since $r_1 - r \geq \bar{\delta} = 1/2$ in the domain *red*, we have

$$\begin{aligned}
&\iint_{red} \frac{\eta(s,r)}{r_1-r} drds \\
&\leq M \iint_{red} \frac{1}{r_1-r+1} \left\{ \sum_{i=1}^{L} \frac{1}{(r+s+1)^{\bar{\lambda}/2}(|r-c_i s|+1)^{1+\bar{\lambda}/2}} \right. \\
&\quad + \frac{1}{(r+s+1)^{\bar{\lambda}/2}(r+1)^{1+\bar{\lambda}/2}} \\
&\quad \left. + \frac{1}{(r+s+1)^{\bar{\lambda}/2}(|r-s|+1)^{1+\bar{\lambda}/2}} \right\} drds \\
&\leq M. \qquad (5.83)
\end{aligned}$$

Therefore from (5.82) and (5.83) we have (5.25) for $i = 4$. Consequently we have proved the estimate. ∎

6 Energy Estimates.

In this section we prove

Proposition 6. 1 *Let $u = (u^1, \ldots, u^m) \in C^\infty([0,T) \times \mathbf{R}^2; \mathbf{R}^m)$ be a solution of the following system of wave equations with $u(0,\cdot) \in C_0^\infty(\mathbf{R}^2; \mathbf{R}^m)$.*

$$\partial_t^2 u^i - c_i^2 \Delta u^i = \sum_{j=1}^{m} \sum_{\alpha,\beta=0}^{2} C_{ij}^{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u^j + E_i(\partial u) \qquad (6.1)$$

Here, $C_{ij}^{\alpha\beta}, E_i$ $(i,j = 1,2,\ldots,m\,;\, \alpha,\beta = 0,1,2)$ are C^∞-functions in $\{\,|\partial u| < 1\,\}$, which satisfy the conditions (4.3)-(4.8).

Moreover we assume that

$$\left| C_{ij}^{\alpha\beta}(\partial u) \right| \leq \frac{1}{4m} \min\{1, c_i^2, c_j^2\} \quad for \quad |\partial u| < \delta_1 \qquad (6.2)$$

and that there exists a positive number T_1 such that

$$[\partial u]_0(T_1) < 1 \quad and \quad |\partial u|_0(T_1) < \delta_1. \qquad (6.3)$$

Then, we have the following energy estimates for $0 < t < T_1$:

$$\| \partial u(t) \|_N^2 \quad \leq \quad M_N \{ \| \partial u(0) \|_N^2$$

$$+ \int_0^t (s+1)^{-1-\min\{1/3,2\gamma\}} [\partial u(s)]_0^2 \parallel \partial u(s) \parallel_{(N+1)(N+5)}^2 ds \bigg\}, \tag{6.4}$$

$$\parallel \partial u(t) \parallel_N^2 \quad \leq \quad M_N \parallel \partial u(0) \parallel_N^2 (t+1)^{M_N([\partial u]_1(t))^2}. \tag{6.5}$$

Proof. Since Ω commutes $\partial_t^2 - c_i^2 \Delta$,

$$\partial_t^2 \mathcal{D}^A u^i - c_i^2 \Delta \mathcal{D}^A u^i = \sum_{j=1}^m \sum_{\alpha,\beta=0}^2 \mathcal{D}^A \left\{ C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta u^j \right\} + \mathcal{D}^A E_i(\partial u). \tag{6.6}$$

We set

$$a_{ij}^{\alpha\beta} = \begin{cases} 1 - C_{ii}^{00} & (\alpha = \beta = 0,\ j = i) \\ -c_i c_j \delta_{\alpha\beta}\delta_{ij} - C_{ij}^{\alpha\beta} & (\text{otherwise}) \end{cases} \tag{6.7}$$

$$w_A^i = \sum_{j=1}^m \sum_{\alpha,\beta=0}^2 \left[\mathcal{D}^A \left\{ C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta u^j \right\} - C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta \mathcal{D}^A u^j \right] + \mathcal{D}^A E_i(\partial u). \tag{6.8}$$

Then from (6.6), (6.7) and (6.8) we have

$$\sum_{j=1}^m \sum_{\alpha,\beta=0}^2 a_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta \mathcal{D}^A u^j = w_A^i \tag{6.9}$$

Multiplying both sides of (6.9) by $\partial_t \mathcal{D}^A u^i$ and using (4.3), we get

$$\begin{aligned} &\sum_{i,j=1}^m \sum_{\alpha,\beta=0}^2 \left[\partial_\alpha \left\{ 2a_{ij}^{\alpha\beta}(\partial u) \partial_t \mathcal{D}^A u^i \cdot \partial_\beta \mathcal{D}^A u^j \right\} \right. \\ &\quad \left. - \partial_t \left\{ a_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \mathcal{D}^A u^i \cdot \partial_\beta \mathcal{D}^A u^j \right\} \right] \\ &= \sum_{i=1}^m 2\partial_t \mathcal{D}^A u^i \cdot w_A^i \\ &\quad + \sum_{i,j=1}^m \sum_{\alpha\beta=0}^2 \left\{ \partial_t C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \mathcal{D}^A u^i \cdot \partial_\beta \mathcal{D}^A u^j \right. \\ &\quad \left. - 2\partial_\alpha C_{ij}^{\alpha\beta}(\partial u) \partial_t \mathcal{D}^A u^i \cdot \partial_\beta \mathcal{D}^A u^j \right\} \end{aligned} \tag{6.10}$$

Integrating (6.10) over $[0,t]\times\mathbf{R}^2$, we have

$$\begin{aligned}
&\left\|\partial\mathcal{D}^A u(t)\right\|_E^2-\left\|\partial\mathcal{D}^A u(0)\right\|_E^2\\
&=\int_0^t ds\iint_{\mathbf{R}^2}\Bigg[\sum_{i=1}^m 2\partial_t\mathcal{D}^A u^i\cdot w_A^i\\
&\quad+\sum_{i,j=1}^m\sum_{\alpha,\beta=0}^2\Big\{\partial_t C_{ij}^{\alpha\beta}(\partial u)\partial_\alpha\mathcal{D}^A u^i\cdot\partial_\beta\mathcal{D}^A u^j\\
&\quad-2\partial_\alpha C_{ij}^{\alpha\beta}(\partial u)\partial_t\mathcal{D}^A u^i\cdot\partial_\beta\mathcal{D}^A u^j\Big\}\Bigg]dx, \qquad (6.11)
\end{aligned}$$

where

$$\begin{aligned}
\left\|\partial\mathcal{D}^A u(t)\right\|_E^2 &= \iint_{\mathbf{R}^2}\sum_{i,j=1}^m\Big\{a_{ij}^{00}(\partial u)\partial_t\mathcal{D}^A u^i\cdot\partial_t\mathcal{D}^A u^j(t,x)\\
&\quad-\sum_{k,l=1}^2 a_{ij}^{kl}(\partial u)\partial_k\mathcal{D}^A u^i\cdot\partial_l\mathcal{D}^A u^j(t,x)\Big\}dx. \qquad (6.12)
\end{aligned}$$

Notice that by (6.2) and (6.7) we have

$$\frac{1}{M}\left\|\partial\mathcal{D}^A u(t)\right\|_0\le\left\|\partial\mathcal{D}^A u(t)\right\|_E\le M\left\|\partial\mathcal{D}^A u(t)\right\|_0$$

Therefore it follows that

$$\left\|\partial\mathcal{D}^A u(t)\right\|_0^2\le M\left(\left\|\partial\mathcal{D}^A u(0)\right\|_0^2+J_A^{(1)}+J_A^{(2)}\right), \qquad (6.13)$$

where

$$\begin{aligned}
J_A^{(1)} &= \sum_{i=1}^m\int_0^t ds\iint_{\mathbf{R}^2}\left|\partial_t\mathcal{D}^A u^i\cdot w_A^i\right|dx,\\
J_A^{(2)} &= \sum_{i,j=1}^2\sum_{\alpha,\beta=0}^2\int_0^t ds\iint_{\mathbf{R}^2}\left|\partial C_{ij}^{\alpha\beta}(\partial u)\right|\left|\partial\mathcal{D}^A u^i\right|\left|\partial\mathcal{D}^A u^j\right|dx.
\end{aligned}$$

We first prove the estimate (6.4). Since $|\partial u(s)|_0<1$, then by the assumption (4.4)-(4.8), we get

$$\left|\mathcal{D}^B C_{ij}^{\alpha\beta}(\partial u)\right| \le M_B\sum_{l=2}^{|B|+3}\sum_{j_1,\dots,j_l=1}^m\sum_{|B_1|,\dots,|B_l|\le|B|}\delta_{i;jj_1\cdots j_l}\prod_{k=1}^l\left|\mathcal{D}^{B_k}\partial u^{j_k}\right|, \qquad (6.14)$$

$$\left|\mathcal{D}^A E_i(\partial u)\right| \leq M_A \sum_{l=3}^{|A|+4} \sum_{j_1,\dots,j_l=1}^{m} \sum_{|A_1|,\dots,|A_l|\leq|A|} \delta_{i;j_1\cdots j_l} \prod_{k=1}^{l} \left|\mathcal{D}^{A_k}\partial u^{j_k}\right|. \tag{6.15}$$

Here we set

$$\delta_{i;j_1\cdots j_l} = \begin{cases} 1-\delta_{ij_1}\delta_{ij_2}\delta_{ij_3} & (l=3) \\ 1 & (l\geq 4). \end{cases}$$

Since $\left|\det\left(a_{ij}^{00}(\partial u)\right)_{i,j=1}^{m}\right| \geq 1/2^m$ from (6.2), we can solve the following simultaneous linear equation with respect to $\partial_t^2 u^i$:

$$\sum_{j=1}^{m} a_{ij}^{00}(\partial u)\partial_t^2 u^j = E_i(\partial u) - \sum_{j=1}^{m} \sum_{(\alpha,\beta)\neq(0,0)} a_{ij}^{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u^j \quad (i=1,2,\dots,m).$$

Moreover, by (4.4), (4.5), (6.2) and Cramer's formula, we see

$$\partial_t^2 u^l = -\left\{\det\left(a_{ij}^{00}(\partial u)\right)_{i,j=1}^{m}\right\}^{-1} c_l^2 \Delta u^l + \text{higher order terms}. \tag{6.16}$$

Therefore it follows from (6.8), (6.14), (6.15) and (6.16) that

$$\begin{aligned} J_A^{(1)} \leq\ & M_A \sum_{i=1}^{m} \sum_{l=3}^{|A|+5} \sum_{j_1,\dots,j_l=1}^{m} \sum_{|A_1|,\dots,|A_l|\leq|A|} \delta_{i;j_1\cdots j_l} \cdot \\ & \cdot \int_0^t ds \iint_{\mathbf{R}^2} \prod_{k=1}^{l} \left|\mathcal{D}^{A_k}\partial u^{j_k}\right| \left|\mathcal{D}^A \partial u^i\right| dx. \end{aligned} \tag{6.17}$$

Next, we consider $J_A^{(2)}$. By (4.4) and (4.7),

$$\left|\partial C_{ij}^{\alpha\beta}(\partial u)\right| \leq M \sum_{k,l=1}^{m} \delta_{i;jkl} \left|\partial u^k\right| \left|\partial^2 u^l\right|.$$

Therefore it follows from (6.16) that

$$\begin{aligned} J_A^{(2)} \leq\ & M \left\{ \sum_{i,j,k,l=1}^{m} \sum_{h=1}^{2} \int_0^t ds \iint_{\mathbf{R}^2} \delta_{i;jkl} \left|\partial u^k\right| \left|\partial_h \partial u^l\right| \left|\partial\mathcal{D}^A u^i\right| \left|\partial\mathcal{D}^A u^j\right| dx \right. \\ & \left. + \int_0^t ds \iint_{\mathbf{R}^2} |\partial u|^3 \left|\partial\mathcal{D}^A u\right|^2 dx \right\}. \end{aligned} \tag{6.18}$$

Hence we find from (6.17) and (6.18) that

$$J_A^{(1)} + J_A^{(2)} \leq M_A \sum_{l=3}^{|A|+5} \sum_{j_0,\dots,j_l=1}^{m} \sum_{|A_0|,\dots,|A_l|\leq|A|+1} \delta_{j_0;j_1\cdots j_l} \cdot \int_0^t ds \iint_{\mathbf{R}^2} \prod_{k=0}^{l} |\mathcal{D}^{A_k}\partial u^{j_k}|\, dx. \tag{6.19}$$

By Hölder's inequality,

$$\begin{aligned}
&\iint_{\mathbf{R}^2} \prod_{k=0}^{l} |\mathcal{D}^{A_k}\partial u^{j_k}(s,x)|\, dx \\
\leq\; & \left\| \prod_{k=0}^{l-1} \mathcal{D}^{A_k}\partial u^{j_k}(s,\cdot) \right\|_{L^2(\mathbf{R}^2)} \|\mathcal{D}^{A_l}\partial u^{j_l}(s,\cdot)\|_{L^2(\mathbf{R}^2)} \\
\leq\; & \left\| \prod_{k=0}^{l-1} w_{j_k}^{-(l-1)/l}(s,|\cdot|) \right\|_{L^\infty(\mathbf{R}^2)} \\
& \cdot \left\| \prod_{k=0}^{l-1} w_{j_k}^{(l-1)/l}(s,|\cdot|)\mathcal{D}^{A_k}\partial u^{j_k}(s,\cdot) \right\|_{L^2(\mathbf{R}^2)} \|\mathcal{D}^{A_l}\partial u^{j_l}(s,\cdot)\|_{L^2(\mathbf{R}^2)} \\
\leq\; & \left\| \prod_{k=0}^{l-1} w_{j_k}^{-(l-1)/l}(s,|\cdot|) \right\|_{L^\infty(\mathbf{R}^2)} \\
& \prod_{k=0}^{l-1} \left\| w_{j_k}^{(l-1)/l}(s,|\cdot|)\mathcal{D}^{A_k}\partial u^{j_k}(s,\cdot) \right\|_{L^{2l}(\mathbf{R}^2)} \|\mathcal{D}^{A_l}\partial u^{j_l}(s,\cdot)\|_{L^2(\mathbf{R}^2)}.
\end{aligned} \tag{6.20}$$

Without loss of generality we may suppose that $j_0 = j_1 = j_2$ does not hold for $l = 3$. Therefore, it follows from (3.1) that

$$\left\| \prod_{k=0}^{l-1} w_{j_k}^{-(l-1)/l}(s,|\cdot|) \right\|_{L^\infty(\mathbf{R}^2)} \leq M(s+1)^{-1-\min\{1/3,2\gamma\}}. \tag{6.21}$$

In order to estimate (6.20), we need Gagliardo-Nirenberg inequality:
Lemma 6. 1 *Let $f \in C_0^\infty(\mathbf{R}^2)$, $|A| = i \leq k$. Then,*

$$\|\mathcal{D}^A f\|_{L^{rk/i}(X)} \leq M_A \|f\|_{L^\infty(X)}^{1-i/k} \left(\sum_{|B|\leq k} \|\mathcal{D}^B f\|_{L^r(X)} \right)^{i/k},$$

where

$$X = \mathbf{R}^2 \text{ or } \{x \mid x \in \mathbf{R}^2,\, n \le |x| \le n+1\} \quad (n = 0, 1, 2, \ldots).$$

F. John and S. Klainerman proved this lemma in their paper [6] when $X = \mathbf{R}^3$ We modify the proof of them and obtain the above lemma.

Since

$$\frac{1}{M} w_j(s,n) \le w_j(s,r) \le M w_j(s,n) \tag{6.22}$$

for $n \le r \le n+1$, we find from Lemma 6. 1 that

$$\begin{aligned}
&\left\| w_{j_k}(s, |\cdot|)^{(l-1)/l} \mathcal{D}^{A_k} \partial u^{j_k}(s,\cdot) \right\|_{L^{2l}(\mathbf{R}^2)}^{2l} \\
&\le M \sum_{n=0}^{\infty} w_{j_k}(s,n)^{2(l-1)} \left\| \mathcal{D}^{A_k} \partial u^{j_k} \right\|_{L^{2l}(\{n \le |x| \le n+1\})}^{2l} \\
&\le M_A \sum_{n=0}^{\infty} w_{j_k}(s,n)^{2(l-1)} \| \partial u^{j_k}(s,\cdot) \|_{L^\infty(\{n \le |x| \le n+1\})}^{2(l-1)} \cdot \\
&\quad \cdot \left(\sum_{|B| \le l|A_k|} \left\| \mathcal{D}^B \partial u^{j_k}(s,\cdot) \right\|_{L^2(\{n \le |x| \le n+1\})} \right)^2 \\
&\le M_A [\partial u(s)]_0^{2(l-1)} \| \partial u(s) \|_{l|A_k|}^2
\end{aligned} \tag{6.23}$$

Therefore, from (6.3), (6.19)-(6.23) and (6.13), we get (6.4).

Next, we prove (6.5). For the proof, we use the following two lemmas:

Lemma 6. 2 *Let $f, g \in C_0^\infty(\mathbf{R}^2)$. Then,*

$$\left\| \mathcal{D}^A(fg) - f\mathcal{D}^A g \right\|_0 \le M(|f|_1 \|g\|_{|A|-1} + |g|_0 \|f\|_{|A|}).$$

Lemma 6. 3 *Let $f = (f_1, \ldots, f_r) \in C_0^\infty(\mathbf{R}^2; \mathbf{R}^r)$ and let $\omega = \omega(f)$ be a C^∞-function that satisfies*

$$|\omega(f)| \le M|f|^q$$

Then

$$\| \mathcal{D}^A(\omega \circ f) \|_{L^p(\mathbf{R}^2)} \le M |f|_{L^\infty(\mathbf{R}^n)}^{q-1} \sum_{|B| \le |A|} \| \mathcal{D}^B f \|_{L^p(\mathbf{R}^2)}. \tag{6.24}$$

See F. John and S. Klainerman's paper [6] for the proof of Lemma 6. 2 , and M. Kovalyov's paper [10] for Lemma 6. 3 .

By these lemmas,

$$\left\|w_A^i\right\|_{L^2(\mathbf{R}^2)} \leq M_A|\partial u(s)|_0|\partial u(s)|_1\|\partial u(s)\|_{|A|}. \tag{6.25}$$

Therefore from (6.18) and (6.25) we get

$$J_A^{(1)} + J_A^{(2)} \leq M_A \int_0^t |\partial u(s)|_0|\partial u(s)|_1\|\partial u(s)\|_{|A|}^2 ds. \tag{6.26}$$

Further, it follows from (6.13) and (6.26) that

$$\|\partial u(t)\|_N^2 \leq M_N \left\{\|\partial u(0)\|_N^2 + \int_0^t |\partial u(s)|_0|\partial u(s)|_1\|\partial u(s)\|_N^2 ds\right\}.$$

Hence by Gronwall's lemma we find

$$\|\partial u(t)\|_N^2 \leq M\|\partial u(0)\|_N^2 \exp\left(M_N \int_0^t |\partial u(s)|_0|\partial u(s)|_1 ds\right) \tag{6.27}$$

Since

$$|\partial u(s)|_0|\partial u(s)|_1 \leq M_N(s+1)^{-1}[\partial u(s)]_1^2,$$

we obtain (6.5) from (6.27). ∎

7 Proof of the Theorem.

Making use of the method by R. Agemi [1] and F. John,[5] we find that a solution $u(t,x)$ to (4.1) is unique and $u(t,\cdot)\,(t \geq 0)$ has compact support. The local existence theorem of a solution to (4.1) has proved by F. John [4] and T. Kato.[7]

Let $u(t,x)$ be a C^∞-solution to (4.1) in $[0,T) \times \mathbf{R}^2$ We write u as

$$u = u_0 + u_1, \tag{7.1}$$

where u_0 is the solution of the initial value problem

$$\begin{cases} \partial_t^2 u_0^i - c_i^2 \Delta u_0^i = 0 \\ u_0^i(0,\cdot) = \varepsilon f^i, \partial_t u_0^i(0,\cdot) = \varepsilon g^i \quad (i = 1,2,\ldots,m), \end{cases} \tag{7.2}$$

and u_1 is the solution of the initial value problem

$$\begin{cases} \partial_t^2 u_1^i - c_i^2 \Delta u_1^i = F_i(\partial u, \partial^2 u) \\ u_1^i(0,\cdot) = \partial_t u_1^i(0,\cdot) = 0 \quad (i = 1,2,\ldots,m). \end{cases} \tag{7.3}$$

R. T. Glassey has proved in his paper [2] by the method of W. von Wahl [14] that

$$|u_0^i(t,x)| \le \frac{M\varepsilon}{\{(|x|+c_it+1)(||x|-c_it|+1)\}^{1/2}}. \tag{7.4}$$

Here M depends on L^1-norm of f^i, ∂f^i and g^i.

We set

$$\begin{aligned}\frac{1}{\tilde{w}_i(s,r)} &= \sum_{j\neq i}\frac{1}{(r+s+1)(|r-c_js|+1)}+\frac{1}{(r+s+1)^{1+2\gamma}(r+1)^{1-2\gamma}}\\ &\quad+\frac{1}{(r+s+1)^{4/3}(|r-c_is|+1)^{2/3}}.\end{aligned}$$

Then $\tilde{w}_i$ satisfies (5.20) and (5.21). By Proposition 5. 4 , we get

$$\begin{aligned}[\partial u_1(t)]_N &\le M_N\left\{\sum_{i=1}^m\sum_{|A|\le N+3}\sup_{0<s<t}\left\|\tilde{w}_i(s,|\cdot|)\mathcal{D}^AF_i(\partial u,\partial^2u)(s,\cdot)\right\|_{L^2(\mathbf{R}^2)}\right.\\ &\quad\left.+\sum_{i=1}^m\sum_{|A|\le N+2}\sup_{0<s<t}\left\|\tilde{w}_i(s,|\cdot|)\partial_t\mathcal{D}^AF_i(\partial u,\partial^2u)(s,\cdot)\right\|_{L^2(\mathbf{R}^2)}\right\}.\end{aligned} \tag{7.5}$$

Since $[\partial u(t)]_0$ is continuous, we can take for $0<\varepsilon<\varepsilon_1$ a positive number T_1 such that the condition (6.3) holds, provided ε_1 is small enough. We suppose $0<\varepsilon<\varepsilon_1$ and set

$$\tilde{T}_1 = \sup\{T_1 \mid (6.3) \quad \text{holds.}\}.$$

Let $0<t<\tilde{T}_1$ in the following. In particular, $|\partial u|_0(t)<1$. Then by (6.16), we have

$$\left|\mathcal{D}^AF_i(\partial u,\partial^2u)\right| \le M_A\sum_{l=3}^{|A|+6}\sum_{j_1,\dots,j_l=1}^m\sum_{\substack{|A_h|\le|A|+1\\(h=1,2,\dots,l)}}\delta_{i;j_1\cdots j_l}\prod_{k=1}^l\left|\mathcal{D}^{A_k}\partial u^{j_k}\right|, \tag{7.6}$$

$$\left|\partial_t\mathcal{D}^AF_i(\partial u,\partial^2u)\right| \le M_A\sum_{l=3}^{|A|+8}\sum_{j_1,\dots,j_l=1}^m\sum_{\substack{|A_h|\le|A|+2\\(h=1,2,\dots,l)}}\delta_{i;j_1\cdots j_l}\prod_{k=1}^l\left|\mathcal{D}^{A_k}\partial u^{j_k}\right|. \tag{7.7}$$

Hence from (7.5), (7.6) and (7.7) we get

$$[\partial u_1(t)]_N \leq M_N \sum_{l=3}^{N+10} \sum_{j_1,\dots,j_l=1}^{m} \sum_{\substack{|A_h|\leq N+4 \\ (h=1,2,\dots,l)}} \delta_{i;j_1\cdots j_l} \cdot \sup_{0<s<t} \left\| \tilde{w}_i(s,|\cdot|) \prod_{k=1}^{l} \mathcal{D}^{A_k} \partial u^{j_k}(s,\cdot) \right\|_{L^2(\mathbf{R}^2)}. \tag{7.8}$$

We notice that

$$\frac{1}{(w_{j_1} w_{j_2} w_{j_3})^{2/3}(s,r)} \leq \frac{M}{\tilde{w}_i(s,r)},$$

provided $j_1 = j_2 = j_3 = i$ does not hold. Thus we get

$$\delta_{i;j_1\cdots j_l} \tilde{w}_i \leq M(w_{j_1} \cdots w_{j_l})^{(l-1)/l}. \tag{7.9}$$

By (7.9) and Hölder's inequality,

$$\begin{aligned} &\delta_{i;j_1\cdots j_l} \left\| \tilde{w}_i(s,|\cdot|) \prod_{k=1}^{l} \mathcal{D}^{A_k} \partial u^{j_k}(s,\cdot) \right\|_{L^2(\mathbf{R}^2)} \\ &\leq M \left\| \prod_{k=1}^{l} w_{j_k}^{(l-1)/l}(s,|\cdot|) \mathcal{D}^{A_k} \partial u^{j_k}(s,\cdot) \right\|_{L^2(\mathbf{R}^2)} \\ &\leq M \prod_{k=1}^{l} \left\| w_{j_k}^{(l-1)/l}(s,|\cdot|) \mathcal{D}^{A_k} \partial u^{j_k}(s,\cdot) \right\|_{L^{2l}(\mathbf{R}^2)} \end{aligned} \tag{7.10}$$

Hence by (7.8), (7.10) and (6.23) we have

$$[\partial u_1(t)]_N \leq M_N \sup_{0<s<t} [\partial u(s)]_0^2 \|\partial u(s)\|_{N_1}, \tag{7.11}$$

where $N_1 = (N+10)(N+4)$. Therefore it follows from (7.1), (7.4) and (7.11) that

$$[\partial u(t)]_N \leq M_N \left\{ \varepsilon + \sup_{0<s<t} [\partial u(s)]_0^2 \|\partial u(s)\|_{N_1} \right\}. \tag{7.12}$$

By Proposition 6. 1 ,

$$\|\partial u(s)\|_{N_1}^2 \leq M_N \left\{ 1 + [\partial u]_0^2(s) \cdot \int_0^s (\tau+1)^{-1-\min\{1/3,2\gamma\}} \|\partial u(\tau)\|_{(N_1+1)(N_1+5)}^2 d\tau \right\}, \tag{7.13}$$

$$\|\partial u(\tau)\|_{(N_1+1)(N_1+5)} \leq M_N (\tau+1)^{M_N [\partial u]_1^2(\tau)}. \tag{7.14}$$

We fix the constant M_N in (7.12), (7.13) and (7.14) so that

$$M_N \geq \max\{8, 2/\min\{1/3, 2\gamma\}, (2/\delta_1)^{1/2}\}.$$

We take ε_0 to be

$$0 < \varepsilon_0 < \min\{1/4M_N^3, \varepsilon_1\}.$$

Moreover, we suppose that ε_0 is small enough to define the following T_0 for $0 < \varepsilon < \varepsilon_0$:

$$T_0 = \sup\left\{t \;\middle|\; [\partial u]_N(t) \leq 4\varepsilon M_N\right\}.$$

Suppose that $0 < t < T_0$. Then,

$$\begin{aligned}[\partial u(t)]_0 &\leq 4\varepsilon M_N \leq 1/M_N^2 \leq 1/2\\ |\partial u(t)|_0 &\leq 1/M_N^2 \leq \delta_1/2.\end{aligned}$$

Therefore $T_0 \leq \tilde{T}_1$. Then for $0 < \varepsilon < \varepsilon_0$ and $0 < t < T_0$,

$$\begin{aligned}&[\partial u]_1^2(t) \leq (4\varepsilon M_N)^2 \leq 1/M_N^2,\\ &-\min\{1/3, 2\gamma\} + M_N[\partial u]_1^2(\tau) \leq -\min\{1/3, 2\gamma\}/2. \qquad (7.15)\end{aligned}$$

Here from (7.13), (7.14) and (7.15) we get

$$\begin{aligned}\|\partial u(s)\|_{N_1}^2 &\leq M_N\left\{1 + \frac{1}{M_N^2}\cdot M_N\int_0^s (\tau+1)^{-1-\min\{1/3,2\gamma\}/2} d\tau\right\}\\ &\leq 2M_N. \qquad (7.16)\end{aligned}$$

Therefore it follows from (7.12) and (7.16) that

$$\begin{aligned}[\partial u(t)]_N &\leq M_N\left\{\varepsilon + \sup_{0<s<t}[\partial u(s)]_0 \cdot 4\varepsilon M_N \cdot (2M_N)^{1/2}\right\}\\ &\leq M_N\varepsilon + \frac{1}{2}\sup_{0<s<t}[\partial u(s)]_0\\ &\leq M_N\varepsilon + \frac{1}{2}\sup_{0<s<T_0}[\partial u(s)]_N,\end{aligned}$$

which implies

$$\sup_{0<t<T_0}[\partial u(t)]_N \leq 2M_N\varepsilon.$$

Therefore T_0 cannot be finite, and we complete the proof of the theorem. ∎

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