Vanishing Viscosity Limit for Incompressible Viscoelasticity in Two Dimensions

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Abstract

This paper studies the inviscid limit of the two-dimensional incompressible viscoelasticity, which is a system coupling a Navier-Stokes equation with a transport equation for the deformation tensor. The existence of global smooth solutions near the equilibrium with a fixed positive viscosity was known since the work of [35]. The inviscid case was solved recently by the second author [28]. While the latter was solely based on the techniques from the studies of hyperbolic equations, and hence the two-dimensional problem is in general more challenging than that in higher dimensions, the former was relied crucially upon a dissipative mechanism. Indeed, after a symmetrization and a linearization around the equilibrium, the system of the incompressible viscoelasticity reduces to an incompressible system of damped wave equations for both the fluid velocity and the deformation tensor. These two approaches are not compatible. In this paper, we prove global existence of solutions, uniformly in both time $t \in [0, +\infty)$ and viscosity $\mu \geq 0$. This allows us to justify in particular the vanishing viscosity limit for all time. In order to overcome difficulties coming from the incompatibility between the purely hyperbolic limiting system and the systems with additional parabolic viscous perturbations, we introduce in this paper a rather robust method that may apply to a wide class of physical systems of similar nature. Roughly speaking, the method works in the two-dimensional case whenever the hyperbolic system satisfies intrinsically a "strong null condition." For dimensions not less than three, the usual null condition is sufficient for this method to work. © 2018 Wiley Periodicals, Inc.

1 Introduction

One of the common manifestations of anomalous phenomena in complex fluids comes from the elastic effects. The different rheological and hydrodynamic

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properties can be attributed to the special coupling between the transportation of the internal variable and the induced elastic stress. In the variational energetic formulation, these properties can be attributed to the competition between the kinetic energy and the internal elastic effects (see, for instance, [35]).

For isotropic, hyperelastic, and homogeneous incompressible materials, the motion can be described by the following (fundamental elastodynamic) system:

$$\begin{cases}\partial_t v + v\cdot\nabla v + \nabla p = \nabla\cdot\big(\frac{\partial W(F)}{\partial F}F^{\top}\big),\\ \nabla\cdot v = 0.\end{cases} \tag{1.1}$$

Here v is the velocity field, p the scalar pressure (which is the Lagrangian multiplier due to the incompressibility constraint), $W(F)$ the internal elastic energy density, and F the deformation tensor.

The deformation tensor F is often presented in a Lagrangian description using a time-dependent family of orientation-preserving diffeomorphisms $x(t,\cdot)$, $0 \le t < T$. Material points y in the reference configuration are deformed to the spatial positions $x(t, y)$ at time t. We shall use $y(t, x)$ to denote the inverse of $x(t,\cdot)$. The flow map $x(t, y)$ is determined as usual by the velocity $v(t, x)$ via the following ODEs:

$$\begin{cases}\frac{dx(t,y)}{dt} = v(t, x(t, y)),\\ x(0) = y.\end{cases}$$

Such a map $x(t, y)$ would be uniquely defined whenever the velocity field $v(t, x)$ is in an appropriate Sobolev space [11]. The deformation tensor is then defined by

$$\widetilde{F}(t, y) = \frac{\partial x(t, y)}{\partial y}.$$

One simply identifies it as $F(t, x(t, y)) = \widetilde{F}(t, y)$ in the Eulerian coordinates (t, x).

It is easy to check that the incompressible condition is equivalent to $\nabla\cdot F^{\top} = 0$ (see, for instance, [35]). In addition, one can also deduce that

$$\begin{cases}\partial_t F + v\cdot\nabla F = \nabla v F,\\ F_{mj}\nabla_m F_{ik} = F_{lk}\nabla_l F_{ij}, \quad i, j, m, k, l \in \{1, 2, \dots, n\}.\end{cases} \tag{1.2}$$

See, for example, [30, 35]. The above (1.2) is essentially the compatibility condition for the velocity field and the flow map. In what follows, we use the following notations:

$$(\nabla v)_{ij} = \nabla_j v_i, \quad (\nabla v F)_{ij} = (\nabla v)_{ik}F_{kj}, \quad (\nabla\cdot F)_i = \nabla_j F_{ij},$$

and the summation convention over repeated indices will always be applied.

The equations for elastodynamics (1.1) may then be written equivalently as

$$\begin{cases}\partial_t v + v\cdot\nabla v + \nabla p = \nabla\cdot\big(\frac{\partial W(F)}{\partial F}F^{\top}\big),\\ \partial_t F + v\cdot\nabla F = \nabla v F,\\ \nabla\cdot v = 0, \ \ \nabla\cdot F^{\top} = 0,\end{cases} \tag{1.3}$$

with the compatible condition $(1.2)_2$.

Taking into account of viscosity, one leads to the Oldroyd system of viscoelasticity:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \mu \Delta v + \nabla \cdot \big(\frac{\partial W(F)}{\partial F} F^{\top} \big), \\ \partial_t F + v \cdot \nabla F = \nabla v F, \\ \nabla \cdot v = 0, \ \ \nabla \cdot F^{\top} = 0. \end{cases} \tag{1.4}$$

Here $\mu \geq 0$ denotes the fluid viscosity. We notice that the nonlinear coupling structure in (1.4) is universal, and it appears in many physical equations including magnetohydrodynamic equations and liquid crystal flows; see [34].

A main goal of this paper is to justify the global-in-time inviscid limit from the viscoelastic system (1.4) to the elastic system (1.3) in two dimensions. More precisely, we will show that smooth solutions to (1.4) in certain weighted Sobolev spaces exist uniformly in time $t \geq 0$ and $\mu \geq 0$. This allows us to justify the vanishing viscosity limit for all time.

The presence of viscosity requires the use of Eulerian coordinates. Following the standard vector fields method of Klainerman and the "ghost weights" method of Alinhac, a number of rather essential difficulties appear due to the incompatibility between these methods needed for the limiting hyperbolic systems and the equations in the limiting process that possess additional parabolic viscous terms. In particular, the viscous terms would result in "bad" commutators. A reformulation of the system in these coordinates seems necessary in order for us to identify a stronger notion of null condition, which is essential in the two-dimensional case. With this strong null condition we will be able to do various modifications on Klainerman's and Alinhac's methods. We shall discuss it in more detail in Section 2.2 below.

1.1 A Review of Related Results

The study of dynamics of isotropic, hyperelastic, and homogeneous materials has a long history. Compressible elastodynamic systems (commonly referred as elastic waves in literature), are quasilinear wave type systems with multiple wave speeds. For three-dimensional elastic waves, John [19] showed the existence of almost global solutions for small displacement (see also [25]). On the other hand, John [18] proved that a genuine nonlinearity condition leads to formations of finite-time singularities for spherically symmetric, arbitrarily small but nontrivial displacements (see [46] for large displacement singularities). When the genuine nonlinearity condition is excluded, the existence of global small solutions may be expected even in nonsymmetric cases. The difficulty in obtaining global solutions lies in the understanding of the interaction between the fast pressure waves and slow shear waves at a nonlinear level. A breakthrough is due to Sideris [40, 41] and also Agemi [1], under a nonresonance condition which is physically consistent with the system. The proof of Sideris is based on the vector field method of Klainerman [23,24] and the weighted Klainerman-Sideris L^2 energy (introduced in their earlier work [25]). The proof of Agemi relies on a direct estimate of the fundamental solution. We note that the nonresonance condition complements John's genuine nonlinearity condition. With an additional repulsive Poisson term, a global existence was established in [15] which allows a general form for the pressure.

For the incompressible elastodynamics, the only waves presented in the isotropic systems are shear waves which are linearly degenerate. The global well-posedness was obtained by Sideris and Thomases in [42, 44] (see [43] for a unified treatment, and [32] for some improvement on the uniform time-independent bounds on the highest order energies). Based on the aforementioned achievements, the theory of global existence of solutions for the three-dimensional elastic waves with small initial data is relatively satisfactory.

In the two-dimensional case, the proof of long time existence for the elastodynamics is more difficult due to the weaker time decay rate. The first large time existence result is the recent work [31], where the authors showed the almost global existence for the two-dimensional incompressible elastodynamics in Eulerian coordinates. By observing an improved null structure for the system in Lagrangian coordinates (see also discussions in Section 2.2), the second author [28] proved the global well-posedness using the energy method of Klainerman and Alinhac's ghost weight approach. Afterwards, Wang [47] gave a new proof of this latter result using spacetime resonance method [12] and a normal form transformation.

When the viscosity is present and strictly positive, the global well-posedness near equilibrium state was first obtained in [35] for the two-dimensional case. In this case, after a symmetrization and linearization around the equilibrium state, the system becomes a nonstandard (incompressible) damped wave systems for both velocity field and the deformation tensor; see also [34, 36]. This method works both in two-dimensional and three-dimensional cases. Lei and Zhou [33] obtained similar results by working directly on the equations for the deformation tensor through an incompressible limit process. For many related discussions we refer to [30], [9], and [13, 14, 16, 26, 27, 29, 38, 39, 48] and the references therein. In all these works, a dissipative structure of the viscoelastic systems (with a strictly positive viscosity) is a key ingredient to study the long time behavior. Thus the size of the initial data depends on the viscosity in order to have global-in-time existence. Consequently, these arguments cannot be applied to study the vanishing viscosity problem. For the latter, one has to deal with a nonlinear coupled system of equations in which both parabolicity and hyperbolicity can't be ignored.

As in the study of vanishing viscosity limits for classical fluid dynamics, one expects that when the fluid viscosity goes to 0, the limit of solutions to the viscoelastic system converges to a solution to the elastodynamic system. In the case of Navier-Stokes equations, a lot has been learned since the work of Kato [21] and Swann [45] (see also a recent article by [37]). These results are not expected to hold globally in time. If one tries to prove global-in-time convergence, the matter is completely different.

The work of Kessenich [22] established the global well-posedness theory for three-dimensional incompressible viscoelastic materials uniformly in the viscosity and in time. Here, though the presence of viscosity prevents natural hyperbolic scaling invariance, nevertheless Kessenich used the scaling operator. His strategy is to apply first this operator directly to the system, then to deal with the commutators between the scaling operator and the viscosity terms. Sufficiently fast decay rates in three dimensions are the key.

Another important ingredient in [22] is a Hardy-type estimate. It is used to compensate for the derivative loss problem caused by commuting with the viscous terms. In the two-dimensional case neither of these two key steps can be accomplished easily. One of the main reasons is that, while the ghost weight of Alinhac seems to be a necessary tool for the highest-order energy estimates in the two-dimensional problems, one cannot directly apply it here because it would create extra nondecaying terms involving commutators with the viscous term.

Let us also discuss some closely related historical works on quasilinear-wave-type equations. For quasilinear wave equations in dimension 3, and for small initial data, one can obtain an almost global existence [20]. When the spatial dimensions are not bigger than three, the global existence would depend on two basic assumptions: the initial data should be sufficiently small, and the nonlinearities should satisfy a type of null condition [41]. For nonlinear wave equations with sufficiently small initial data and the null condition not satisfied, the finite-time blowup was shown by John [17], Alinhac [4] in three dimensions, and by Alinhac [2, 3, 6] in two. Under the null condition, the fundamental work on global solutions for the three-dimensional scalar wave equation were obtained by Klainerman [24] and by Christodoulou [10]. In two dimensions, the global solutions were proven by Alinhac [5] under the null condition and under the assumption that the initial data is compactly supported.

1.2 Difficulties and Key Ideas

To simplify the presentation, we will focus only on the Hookean elasticity that corresponds to $W(F) = \frac{1}{2}|F|^2$. The general case differs only by the cubic and higher-order terms, which won't make much difference in our arguments; see also the comments in [28, 47]. In the zero viscosity limit, the viscoelastic systems tend to a hyperbolic system. One would naturally try to follow the generalized energy method of Klainerman. An attractive feature of this method is of course that it suffices to use the weighted Sobolev inequalities involving the invariance of the system: translations, rotations, scaling, and the Lorentz invariance. It avoids the delicate estimates of fundamental solutions of wave equations [41]. Similarly, the Alinhac's ghost weight method may enable one to apply Klainerman's generalized energy method to the two-dimensional wave equations [5]. Alinhac's method seems to be a most valuable tool currently to get the highest-order energy estimates for two-dimensional problems in order to obtain a critical decay in time. The latter is needed for global-in-time existence; see for examples [5, 28, 31].

As in Alinhac's works, we would introduce "good unknowns" and explore certain damping mechanisms for these good unknowns due to the outgoing energy flux when ghost weights are applied (very much like the excess term in the energy monotonicity formulae). In the standard energy estimates, the viscosity may also give rise to some dissipative effects, which is good news. However, due to the additional viscous terms that violate hyperbolic scaling, it creates various "bad commutators" when either the vector field method or the method of ghost weights is applied (see (1.5)). As we mentioned earlier, the ghost weights are not needed in the three-dimensional case as one has already established the critical decay in time in estimates of the highest-order energies (see [22]) without using the usual null condition assumption. In addition, there is (see [22]) a Hardy-type inequality that is useful for getting around the difficulties caused by viscosity. Hence for the two-dimensional case, we definitely need a new strategy.

Let us observe more closely how the ghost weight method causes new problems with the highest-order energy estimates for the viscoelastic systems: Let $\omega = x/|x|$, $\sigma = r - t$, and $q(\sigma) = \arctan \sigma$. Suppose one tries to estimate the highest-order energy for the viscoelastic systems (1.4) with the ghost weight $e^{q(\sigma)}$. Formally one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|Z^\kappa v|^2 + |Z^\kappa(F-I)|^2)e^q\,dx + \mu\int_{\mathbb{R}^2}|\nabla Z^\kappa v|^2 e^q\,dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\frac{|Z^\kappa v + Z^\kappa(F-I)\omega|^2 + |Z^\kappa(F-I)\omega^\perp|^2}{1+\sigma^2}e^q\,dx \\
&\quad = \frac{1}{2}\mu\int_{\mathbb{R}^2}|Z^\kappa v|^2\Delta e^q\,dx + \cdots .
\end{aligned} \tag{1.5}$$

Here Z represents a generalized vector field (see Section 2 for precise definitions). Note the estimate is for the difference $F - I$ as we consider the problem when the deformation tensor perturbs around the (equilibrium) identity matrix. The two coercive terms on the second line are due to the viscosity and the ghost weight, respectively. It will be important, and become clear later on, that we observe the quantities $v + (F-I)\omega$ and $(F-I)\omega^\perp$ as "good unknowns." Suppose, for the sake of argument, that we can handle the nonlinear terms (this is far from being trivial and requires the notion of the strong null condition), we are still facing the difficulty of obtaining the expected energy estimates. Since the right-hand side involves the viscous term, it is not clear at all how one can treat them. In fact, these terms cannot be absorbed directly by the coercive terms since a *spatial derivative* is missing. Moreover, it is not integrable in time as $|\Delta e^q| \sim 1$ near the light cone $r \sim t$.

Our first idea to solve this difficulty is to take advantage of the viscous terms presented in the energy estimates at lower-order derivative levels. To do so, we apply operators (1.5), with $\nabla Z^{\kappa-1}$ instead of Z^κ; namely, one of the derivatives has to be a spatial regular derivative and we combine it with an energy estimate (without the ghost weight) when operators $Z^{\kappa-1}$ are applied. The good viscous terms in the lower-order energy estimates are then used to absorb these commutators from the former one. In what follows, we will use $\mathcal{E}_\kappa$ to denote energy estimates with $\nabla Z^{\kappa-1}$ and $E_{\kappa-1}$ to denote energy estimates with all the vector fields $Z^{\kappa-1}$. We will call $\mathcal{E}_\kappa$ the modified generalized energy and $E_{\kappa-1}$ the generalized energy.

In carrying out this procedure, there are a few new difficulties coming from the nonlinear terms. Our second key idea is to transform the viscoelastic system to a fully nonlinear one, together with a transformed fully nonlinear constraint. It turns out that in this fully nonlinear system, the good unknowns in the nonlinear terms always possess an extra spatial derivative and thus the transformed system satisfies the strong null condition (see the definition in Section 2.2). In fact, since v and $F^\top$ are divergence free, there exist potential functions V and $H = (H_1, H_2)$ such that

$$v = \nabla^\perp V, \quad (F - I)^\top = \nabla^\perp H.$$

Then we can reformulate the system of Hookean viscoelasticity as follows (see Section 2 for a detailed derivation):

$$\begin{cases} \partial_t V - \mu\Delta V - \nabla\cdot H = \\ \quad \nabla^\perp\cdot\nabla\cdot\Delta^{-1}(-\nabla^\perp V\otimes\nabla^\perp V + \nabla^\perp H\otimes\nabla^\perp H), \\ \partial_t H - \nabla V = \nabla^\perp H\nabla V, \end{cases} \tag{1.6}$$

with the constraint

$$\nabla^\perp\cdot H = \nabla^\perp H_2\cdot\nabla H_1. \tag{1.7}$$

As in [28], the strong null condition would also mean that in these nonlinear terms, there are always good unknowns in each individual term. The resulting nonlinear structure permits one to perform various integrations by parts and to obtain desired decay estimates. For related discussions on this strong null condition in a more general setting for nonlinear wave equations, we refer to a forthcoming paper [8] on a simplified wave model.

Now we state the main result of this paper as follows:

THEOREM 1.1. *Let $M > 0$ and $0 < \gamma < \frac{1}{8}$ be two given constants,* $(\nabla V_0, \nabla H_0) \in H_\Lambda^{\kappa-1}$, *and* $(V_0, H_0) \in H_\Lambda^{\kappa-1}$ *with* $\kappa \geq 12$. *Suppose that H_0 satisfies the constraint* (1.7) *and*

$$\|(\nabla V_0, \nabla H_0)\|_{H_\Lambda^{\kappa-1}} + \|(V_0, H_0)\|_{H_\Lambda^{\kappa-1}} \leq M, \quad \|(V_0, H_0)\|_{H_\Lambda^{\kappa-3}} \leq \epsilon.$$

There exists a positive constant $\epsilon_0 < e^{-M}$ that depends on M, κ, and γ such that, if $\epsilon \leq \epsilon_0$, then the incompressible Hookean viscoelastic systems (1.6) *with initial data*

$$V(x,0) = V_0(x), \quad H(x,0) = H_0(x),$$

has a unique global classical solution such that

$$\begin{aligned}&\mathcal{E}_\kappa(t) + E_{\kappa-1}(t)\\&\quad + \sum_{|\alpha|+|a|\le\kappa-1} \mu \int_0^t\int_{\mathbb{R}^2} \big(|\Delta \widetilde{S}^\alpha \Gamma^a V(\tau)|^2 + |\nabla \widetilde{S}^\alpha \Gamma^a V(\tau)|^2\big) dx\, d\tau\\&\quad \le C_0 M^2 \langle t\rangle^\gamma,\end{aligned}$$

$$E_{\kappa-3}(t) + \sum_{|\alpha|+|a|\le\kappa-3} \mu \int_0^t\int_{\mathbb{R}^2} |\nabla \widetilde{S}^\alpha \Gamma^a V(\tau)|^2\, dx\, d\tau \le C_0 \epsilon^2 e^{C_0 M}$$

for some $C_0 > 1$ *uniformly for* $0 \le t < \infty$ *and uniformly for* $\mu \ge 0$.

Here $E_{\kappa-1}$ and $E_{\kappa-3}$ are generalized energy, and $\mathcal{E}_\kappa$ is new modified generalized energy. $\widetilde{S}$ and Γ are generalized vector fields. A more detailed discussion follows in Section 2.

Remark 1.2. Here we only need to assume that the viscosity is smaller than a given constant, say $\mu \le 1$. When $\mu \ge 1$, one can use the parabolic method of [30, 35] to get the uniform bound. In the following arguments, we will always make this assumption.

Remark 1.3. One can easily adapt our method to the three-dimensional case. In fact, the conclusion in [22] could be improved slightly by stating that the uniform bound (in terms of the viscosity) for the highest-order energy holds; see also [32].

Remark 1.4. When there is no viscosity, namely $\mu = 0$, the system is reduced to the two-dimensional incompressible elastodynamics. In this case, our proof of global existence also works, and it can be substantially simplified as there is no need to use the modified $\mathcal{E}_\kappa$.

Remark 1.5. The uniform global a priori estimates allow one to justify the vanishing viscosity limit by a usual compactness argument; see, for example, [21, 37, 45].

Let us end this introduction by discussing a couple of additional technical difficulties that one has to resolve in proving the above theorem. The first one is the issue of derivative loss due to the presence of viscous terms, whenever one performs the weighted energy estimates. Heuristically, for the system of elastodynamics, under some smallness assumption, one can verify that $X_{\kappa-1} \lesssim E_{\kappa-1}$. Here, $X_{\kappa-1}$ represents the weighted L^2 generalized energy. We need to clarify here that these quantities are not the ones from [28]; rather they resemble what were defined in [44] (see Section 2 for precise definitions). However, when the viscosity is present, one can only show that $X_{\kappa-2} \lesssim E_{\kappa-1}$. Consequently, when one deals with energies outside of the light cone, one has to be extra careful. The transformation of the original system into a fully nonlinear one turns out to be useful here. Its advantage as discussed above is the presence of an extra spatial derivative in nonlinear terms. It provides more flexibility in using the weighted L^2 energies along with the integration by parts. In the three-dimensional case [22], Kessenich obtained one extra spatial derivative using a Hardy-type inequality along with the weighted ($L^\infty - L^2$)-estimate. But in the two-dimensional case, Hardy's inequality has an additional logarithmic factor, and it is no longer useful. To estimate $X_{\kappa-1}$, we introduce a modified weighted energy $Y_{\kappa-1}$. The latter is useful to capture a better decay property of the good unknowns as in Alinhac's works. The estimates for $Y_{\kappa-1}$ are similar to $X_{\kappa-1}$. Thus the derivative loss problem persists for $Y_{\kappa-1}$ in treating the highest-order energy estimates as well. Fortunately, at this stage we can borrow the full ghost weight energies to close the estimates.

The newly formulated (1.6) elastodynamic system becomes a nonlocal fully nonlinear system. Generally speaking, for quasilinear or fully nonlinear systems, one needs certain symmetries to avoid the derivative loss. For (1.6), a careful and lengthy examination of the nonlinearities shows that the system indeed possesses the desired symmetry.

The proposed method also needs decay-in-time estimates for the lower-order energies as usual. For the two-dimensional case, solutions often decay like $\langle t\rangle^{-1/2}$ for wave equations. Since the viscoelastic system satisfies the usual null condition, one obtains a critical decay for energies and hence the implication of an almost global existence result; see [31]. For the global existence of classical solutions, the strong null structure used here for the system may be needed. One of the contributions of this article is to show that the viscoelastic system possess a strong null condition under Eulerian coordinates. Here it is worth pointing out that, for the scalar quasilinear wave equations that satisfy the usual null condition, Alinhac [5] used a Hardy-type inequality for compactly supported solutions to overcome the issue with critical decays. Here, due to the nonlocality of terms in the system, the compact support property of the initial data would not be preserved.

The remaining part of this paper is organized as follows. In the following section, we will formulate the system of incompressible elastodynamics in Eulerian coordinates and present its basic properties. In Section 3, we will give some linear and nonlinear estimates; then the weighted L^2 norm and some L^∞ norm will be given. The last section corresponds to the various higher-order and lower-order energy estimates.

2 Equations and Basic Properties

In this section, we will rigorously introduce the concept of the strong null condition for general fluid systems and will reformulate the system as a fully nonlinear one in which the strong null condition can be verified explicitly. Then we introduce some necessary notations and discuss the vector fields applied to the system.

2.1 The Equations of Motion

Due to the presence of the viscous term, we will consider the problem in Eulerian coordinates. Here, partial derivatives with respect to Eulerian coordinates are written as $\partial_t = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$. Spatial derivatives are abbreviated as $\nabla = (\partial_1, \partial_2)$. For convenience, we also use the following notations:

$$\omega = \frac{x}{r}, \quad r = |x|, \quad \omega^\perp = \left(\omega_1^\perp, \omega_2^\perp\right) = (-\omega_2, \omega_1), \quad \nabla^\perp = (-\partial_2, \partial_1).$$

We shall consider the equations of motion for incompressible Hookean elasticity (general nonlinear elasticity can be treated similarly), which corresponds to the Hookean strain energy functional $W(F) = \frac{1}{2}|F|^2$. When the deformation tensor perturbs around its equilibrium, $F = I + G$, the incompressible viscoelastic system (1.4) can be rewritten as

$$\begin{cases} \partial_t v - \mu\Delta v - \nabla \cdot G = -\nabla p - v \cdot \nabla v + \nabla \cdot (GG^\mathsf{T}), \\ \partial_t G - \nabla v = -v \cdot \nabla G + \nabla v G, \\ \nabla \cdot v = 0, \quad \nabla \cdot G^\mathsf{T} = 0. \end{cases} \tag{2.1}$$

In the two-dimensional case, it's easy to see that $(1.2)_2$ is equivalent to

$$(\nabla^\perp \cdot G)_i = G_{l2}\nabla_l G_{i1} - G_{l1}\nabla_l G_{i2}. \tag{2.2}$$

Before we reformulate the system, let us explicitly introduce the strong null condition.

2.2 Strong Null Condition and Reformulation of the Viscoelastic System (2.1)–(2.2)

The strong null condition is a more restricted notion of the null condition, which was originally introduced and applied in [28] in the proof of the global well-posedness of incompressible elastodynamics.

We start with the following scalar quasilinear wave equation:

$$\partial_t^2 u - \Delta u = Q(\partial u, \partial^2 u). \tag{2.3}$$

Here Q is a bilinear form.

DEFINITION 2.1. (Strong null condition) We say Q satisfies the strong null condition if

$$Q(\partial u, \partial^2 u) = Q_1(\partial u, g(\partial u)) + \mathcal{R},$$

where the reminder $\mathcal{R}$ satisfies

$$|\mathcal{R}| \lesssim \frac{|\partial u||\partial Z u|}{1+t}, \quad r \geq \frac{t+1}{2}.$$

Here the expression $g(\partial u)$ is a good known in the sense of Alinhac [5]:

$$g(u) = \omega \partial_t u + \nabla u.$$

Remark 2.2. One can compare the strong null condition with the null condition. We say that Q satisfies the null condition if

$$Q(\partial u, \partial^2 u) = Q_1(\partial u, g(\partial u)) + Q_2(g, \partial^2 u) + \mathcal{R},$$

where the reminder term $\mathcal{R}$ satisfies

$$|\mathcal{R}| \lesssim \frac{|\Gamma u||\partial^2 u| + |\partial u||\partial \Gamma u|}{1+t}, \quad r \geq \frac{t+1}{2}.$$

The main point is that by "null condition" we mean that Q contains at least one good unknown of $g(u)$ or $g(\partial u)$. By the *strong null condition*, it requires that Q must contain the good unknown $g(\partial u)$. In general, a quasilinear wave equation (2.3) with null condition may not satisfy the strong null condition.

In [28], it was observed that writing (2.1)–(2.2) in Lagrangian coordinates,

$$\begin{cases} D_t^2 x - \Delta_y x + F^{-T}\nabla_y p = 0, \\ \det(\nabla_y x) = 1, \end{cases} \tag{2.4}$$

and after applying a curl-free Riesz operator, one may discover that (2.4) satisfies the strong null condition. Here D_t and ∇_y are derivatives with respect to Lagrangian coordinates.

Now we give a few more examples of physical systems for which the strong null condition is valid.

For the 2D fully nonlinear wave equations which are considered in [8]:

$$(\partial_t^2 - \Delta)u = N_{\alpha\beta\mu\nu}\partial_\alpha\partial_\beta u \partial_\mu \partial_\nu u, \tag{2.5}$$

where $N_{\alpha\beta\mu\nu}$ satisfies the condition

$$N_{\alpha\beta\mu\nu}X_\alpha X_\beta X_\mu X_\nu = 0$$

for all $X \in \Sigma$, where $\Sigma = \{X \in \mathbb{R}^+ \times \mathbb{R}^2 \colon X_0^2 = X_1^2 + X_2^2\}$. The equations (2.5) satisfy the strong null condition. Moreover, it was shown in [8] that a class of quasilinear wave equations where the null condition is satisfied can be transformed into (2.5).

For ideal magnetohydrodynamic systems:

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = b\cdot\nabla b, \\ \partial_t b + v\cdot\nabla b = b\cdot\nabla v, \\ \nabla\cdot v = 0, \quad \nabla\cdot b = 0. \end{cases} \tag{2.6}$$

Consider the case where the background magnetic field is $e = (1, 0, \dots, 0) \in \mathbb{R}^n$. We introduce the following good unknowns:

$$\Lambda^\pm = v \pm (b - e).$$

Then (2.6) can be rewritten as

$$\begin{cases} \partial_t \Lambda^+ - e\cdot\nabla\Lambda^+ + \Lambda^-\cdot\nabla\Lambda^+ + \nabla p = 0, \\ \partial_t \Lambda^- + e\cdot\nabla\Lambda^- + \Lambda^+\cdot\nabla\Lambda^- + \nabla p = 0, \\ \nabla\cdot\Lambda^+ = 0, \quad \nabla\cdot\Lambda^- = 0. \end{cases}$$

It's obvious now that the strong null condition is satisfied [7].

Inspired by the above examples, we believe there is a large body of physical systems where the strong null condition is satisfied.

Coming back to the system (2.1)–(2.2), following Lei-Sideris-Zhou [31], we call $v + G\omega$ and $G\omega^\perp$ good unknowns. They are similar in spirit to the concept of good unknowns g of Alinhac [5]. One writes the nonlinear terms in the momentum equation as

$$\begin{aligned}
&v \cdot \nabla v - \nabla \cdot (GG^T) \\
&\quad = (v + G\omega) \cdot \nabla v - (G\omega)_j (\nabla_j v + \nabla_j G\omega) - (G\omega^\perp)_j \nabla_j G\omega^\perp \\
&\quad = Q_1(g, \nabla v) + Q_2(G\omega, g(\nabla u)) + Q_2(g, g(\nabla u)).
\end{aligned}$$

We note that the system now is of first-order (if we ignore the viscosity term). It does explain why there is one spatial derivative less on the unknowns in the nonlinear term. Obviously, Q_1 (which is a transport term) must present and thus system (2.1)–(2.2) doesn't explicitly exhibit the strong null structure. One can observe a similar fact for the G-equation in (2.1)–(2.2).

We reformulate the system in order to show the strong null structure explicitly. Since v and $G^\top$ are divergence free, there exist potential functions V and $H = (H_1, H_2)$ such that

$$v = \nabla^\perp V, \quad G^\top = \nabla^\perp H.$$

Then one has the following:

LEMMA 2.3. *For classical solutions, the system* (2.1) *is equivalent to* (1.6):

$$\begin{cases}
\partial_t V - \mu\Delta V - \nabla \cdot H = \nabla^\perp \cdot \nabla \cdot \Delta^{-1}(-\nabla^\perp V \otimes \nabla^\perp V + \nabla^\perp H \otimes \nabla^\perp H), \\
\partial_t H - \nabla V = \nabla^\perp H \nabla V,
\end{cases}$$

and (2.2) *is equivalent to* (1.7):

$$\nabla^\perp \cdot H = \nabla^\perp H_2 \cdot \nabla H_1.$$

Here $\nabla^\perp \cdot \nabla \cdot \Delta^{-1}(\nabla^\perp V \otimes \nabla^\perp V)$ *and* $\nabla^\perp \cdot \nabla \cdot \Delta^{-1}(\nabla^\perp H \otimes \nabla^\perp H)$ *are given by*

$$\begin{cases}
\nabla^\perp \cdot \nabla \cdot \Delta^{-1}(\nabla^\perp V \otimes \nabla^\perp V) = \nabla_i^\perp \nabla_j \Delta^{-1}(\nabla_i^\perp V \nabla_j^\perp V), \\
\nabla^\perp \cdot \nabla \cdot \Delta^{-1}(\nabla^\perp H \otimes \nabla^\perp H) = \nabla_i^\perp \nabla_j \Delta^{-1}(\nabla_i^\perp H \cdot \nabla_j^\perp H).
\end{cases}$$

Before proving the above lemma, let us check first that the above system satisfies the so-called strong null condition. The good quantities here are $V + H \cdot \omega$ and $H \cdot \omega^\perp$. We can calculate that

$$\begin{aligned}
&\nabla_i^\perp V \nabla_j^\perp V - \nabla_i^\perp H \cdot \nabla_j^\perp H \\
&\quad = \big(\nabla_i^\perp V + \nabla_i^\perp H \cdot \omega\big)\nabla_j^\perp V - \nabla_i^\perp H \ \cdot \omega\big(\nabla_j^\perp V + \nabla_j^\perp H \cdot \omega\big) \\
&\quad\quad - \nabla_i^\perp H \ \cdot \omega^\perp \nabla_j^\perp H \ \cdot \omega^\perp.
\end{aligned}$$

The strong null condition has clearly shown up on the right-hand side of the above equation since all good quantities have an extra spatial derivative. The presence of the extra zeroth-order nonlocal Riesz type operator $\nabla^\perp\cdot\nabla\cdot\Delta^{-1}$ in (1.6) is an extra issue that we have to deal with later.

Remark 2.4. At first glance, the resulting system (1.6) seems to be more complicated than the original one (2.1). The nonlinearities of (1.6) have one more derivative than that of (2.1)), which makes (1.6) a fully nonlinear system (in the inviscid case); see also a related formulation in [47]. But the key point is that, together with the use of the modified generalized energy $\mathcal{E}_\kappa$, we can yet apply the ghost weight method along with the strong null condition in this formulation. Moreover, we can avoid the derivative loss in deriving the estimates $X_{\kappa-2}\lesssim E_{\kappa-1}$ and $Y_{\kappa-2}\lesssim E_{\kappa-1}$.

PROOF. We begin by rewriting the first equation of (2.1) as

$$\partial_t v-\mu\Delta v-\nabla\cdot G=-\nabla p-\nabla\cdot(v\otimes v)+\nabla\cdot(GG^\top).$$

Using (V,H) instead of (v,G) and applying $\nabla^\perp\cdot$ to the above equation, one has

$$\Delta(\partial_t V-\mu\Delta V-\nabla\cdot H)=\nabla^\perp\cdot\nabla\cdot(-\nabla^\perp V\otimes\nabla^\perp V+\nabla^\perp H\otimes\nabla^\perp H).$$

Applying Δ^{-1} to the above the equation yields the first equation of (1.6).

For each component of $(2.1)_2$, the same substitution gives

$$\partial_t\nabla_i^\perp H_j-\nabla_j\nabla_i^\perp V=-\nabla_l^\perp V\nabla_l\nabla_i^\perp H_j+\nabla_l\nabla_i^\perp V\nabla_l^\perp H_j.$$

Note that

$$-\nabla_l^\perp V\nabla_l H=\nabla_l V\nabla_l^\perp H,$$

hence,

$$\begin{aligned}\nabla_i^\perp(\partial_t H_j-\nabla_j V)&=-\nabla_l^\perp V\nabla_l\nabla_i^\perp H_j+\nabla_l\nabla_i^\perp V\nabla_l^\perp H_j\\&=\nabla_l V\nabla_l^\perp\nabla_i^\perp H_j+\nabla_l\nabla_i^\perp V\nabla_l^\perp H_j\\&=\nabla_i^\perp(\nabla_l V\nabla_l^\perp H_j),\end{aligned}$$

which implies

$$\partial_t H_j-\nabla_j V=\nabla_l V\nabla_l^\perp H_j.$$

Thus the second equation of (1.6) is obtained.

For (2.2), the same substitution gives

$$\nabla_i^\perp(\nabla^\perp\cdot H)=\nabla_l^\perp H_2\nabla_l\nabla_i^\perp H_1-\nabla_l^\perp H_1\nabla_l\nabla_i^\perp H_2.$$

By the identity

$$-\nabla_l^\perp H_1\nabla_l H_2=\nabla_l H_1\nabla_l^\perp H_2,$$

we deduce that

$$\begin{aligned}\nabla_i^\perp(\nabla^\perp\cdot H)&=\nabla_l^\perp H_2\nabla_l\nabla_i^\perp H_1-\nabla_l^\perp H_1\nabla_l\nabla_i^\perp H_2\\&=\nabla_l^\perp H_2\nabla_l\nabla_i^\perp H_1+\nabla_l H_1\nabla_l^\perp\nabla_i^\perp H_2\\&=\nabla_i^\perp(\nabla_l^\perp H_2\nabla_l H_1),\end{aligned}$$

which implies (1.7).

In all the above argument, the calculation can be reversed if the solution has enough regularity. Hence (2.1) and (2.2) are equivalent to (1.6) and (1.7) for classical solutions. □

2.3 Commutation Properties

Now let us take a look at the various vector fields that play a central role in the proofs. Since the application of the vector field theory is now classical, we sketch the rough ideas and indicate the differences with the classical theory. For related discussions, we refer the reader to [22, 28, 41].

The scaling operator is defined by

$$S = t\partial_t + r\partial_r.$$

Here, due to the scaling of V and H, we will use the modified scaling operator $\widetilde{S}$, which is defined as

$$\widetilde{S} = S - 1.$$

Applying the scaling operator S to (1.6) and (1.7), we get

$$\begin{cases} \partial_t \widetilde{S} V - \mu\Delta(\widetilde{S} - 1)V - \nabla\cdot\widetilde{S}H \\ \quad = \nabla^\perp\cdot\nabla\cdot\Delta^{-1}(-\nabla^\perp\widetilde{S}V\otimes\nabla^\perp V + \nabla^\perp\widetilde{S}H\otimes\nabla^\perp H) \\ \qquad + \nabla^\perp\cdot\nabla\cdot\Delta^{-1}(-\nabla^\perp V\otimes\nabla^\perp\widetilde{S}V + \nabla^\perp H\otimes\nabla^\perp\widetilde{S}H), \\ \partial_t\widetilde{S}H - \nabla\widetilde{S}V = \nabla^\perp\widetilde{S}H\nabla V + \nabla^\perp H\nabla\widetilde{S}V, \end{cases} \tag{2.7}$$

and

$$\nabla^\perp\cdot\widetilde{S}H = \nabla^\perp\widetilde{S}H_2\cdot\nabla H_1 + \nabla^\perp H_2\cdot\nabla\widetilde{S}H_1. \tag{2.8}$$

We see from the above expressions that, when $\mu = 0$, the modified scaling operator commutates well with the inviscid systems, but when $\mu > 0$, there is an extra term -1 coming from the commutation between the viscosity term and the scaling operator. This extra commutator term is troublesome and requires some extra care.

In the two-dimensional case, the rotation operator is defined by

$$\Omega = x^\perp\cdot\nabla = \partial_\theta.$$

Applying the rotation operator to (1.6) and (1.7), we get

$$\begin{cases} \partial_t\widetilde{\Omega}V - \mu\Delta\widetilde{\Omega}V - \nabla\cdot\widetilde{\Omega}H \\ \quad = \nabla^\perp\cdot\nabla\cdot\Delta^{-1}\big(-\nabla^\perp\widetilde{\Omega}V\otimes\nabla^\perp V + \nabla^\perp\widetilde{\Omega}H\otimes\nabla^\perp H\big) \\ \qquad + \nabla^\perp\cdot\nabla\cdot\Delta^{-1}\big(-\nabla^\perp V\otimes\nabla^\perp\widetilde{\Omega}V + \nabla^\perp H\otimes\nabla^\perp\widetilde{\Omega}H\big), \\ \partial_t\widetilde{\Omega}H - \nabla\widetilde{\Omega}V = \nabla^\perp\widetilde{\Omega}H\nabla V + \nabla^\perp H\nabla\widetilde{\Omega}V, \end{cases} \tag{2.9}$$

and

$$\nabla^\perp\cdot\widetilde{\Omega}H = \nabla^\perp\widetilde{\Omega}H_2\cdot\nabla H_1 + \nabla^\perp H_2\cdot\nabla\widetilde{\Omega}H_1, \tag{2.10}$$

where

$$\begin{cases} \widetilde{\Omega} V = \Omega V, \\ \widetilde{\Omega} H = \Omega H - H^{\perp}. \end{cases}$$

Hence, the rotation operator commutates well with the system. In view of this, we will separate the scaling operator from regular derivatives and the rotation operator: Let Γ be any of the following differential operators

$$\Gamma \in \{\partial_t, \partial_1, \partial_2, \widetilde{\Omega}\}.$$

Following the above arguments, repeatedly using (2.7)–(2.8) and (2.9)–(2.10), we have

$$\begin{cases} \partial_t \widetilde{S}^{\alpha} \Gamma^a V - \mu \Delta \sum_{l=0}^{\alpha} C_{\alpha}^{l} (-1)^{\alpha - l} \widetilde{S}^{l} \Gamma^a V - \nabla \cdot \widetilde{S}^{\alpha} \Gamma^a H = f_{\alpha a}^{1}, \\ \partial_t \widetilde{S}^{\alpha} \Gamma^a H - \nabla \widetilde{S}^{\alpha} \Gamma^a V = f_{\alpha a}^{2} \end{cases} \tag{2.11}$$

and

$$\nabla^{\perp} \cdot \widetilde{S}^{\alpha} \Gamma^a H = f_{\alpha a}^{3}, \tag{2.12}$$

where

$$\begin{cases} f_{\alpha a}^{1} = \sum\limits_{\substack{b+c=a \\ \beta+\gamma=\alpha}} C_{\alpha}^{\beta} C_{a}^{b} \nabla^{\perp} \cdot \nabla \cdot \Delta^{-1} (-\nabla^{\perp} \widetilde{S}^{\beta} \Gamma^b V \otimes \nabla^{\perp} \widetilde{S}^{\gamma} \Gamma^c V \\ \qquad\qquad\qquad\qquad + \nabla^{\perp} \widetilde{S}^{\beta} \Gamma^b H \otimes \nabla^{\perp} \widetilde{S}^{\gamma} \Gamma^c H), \\ f_{\alpha a}^{2} = \sum\limits_{\substack{b+c=a \\ \beta+\gamma=\alpha}} C_{\alpha}^{\beta} C_{a}^{b} (\nabla^{\perp} \widetilde{S}^{\beta} \Gamma^b H \nabla \widetilde{S}^{\gamma} \Gamma^c V), \\ f_{\alpha a}^{3} = \sum\limits_{\substack{b+c=a \\ \beta+\gamma=\alpha}} C_{\alpha}^{\beta} C_{a}^{b} (\nabla^{\perp} \widetilde{S}^{\beta} \Gamma^b H_2 \cdot \nabla \widetilde{S}^{\gamma} \Gamma^c H_1). \end{cases} \tag{2.13}$$

Here $\alpha \in \mathbb{N}$ and Γ^a stands for $\Gamma^a = \Gamma^{a_1} \cdots \Gamma^{a_4}$, where a is multi-index $a = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4$. We indicate that the generalized vector field Z used in Section 1.2 refers to

$$Z \in \{\partial_t, \partial_1, \partial_2, \widetilde{\Omega}, \widetilde{S}\}.$$

We also use the abbreviation $\Gamma^k V = \{\Gamma^a V : |a| \leq k\}$ and $\Gamma^k H = \{\Gamma^a H : |a| \leq k\}$. The binomial coefficient C_a^b is given by

$$C_a^b = \frac{a!}{b!(a-b)!}.$$

We remark that the above commutation relation (2.11)-(2.13) is essential in all of the subsequent argument. Schematically, we write the following commutation relationship

$$[\Gamma, \Gamma] = \partial, \quad [\Gamma, S] = \partial.$$

This fact is frequently used implicitly throughout the whole argument.

In order to simplify the presentation, we abbreviate $\widetilde{S}^\alpha \Gamma^a V$ as $V^{(\alpha,a)}$ and abbreviate $\widetilde{S}^\alpha \Gamma^a H$ as $H^{(\alpha,a)}$. Also, we denote $V^{(\alpha,|a|)} = \{V^{(\alpha,b)}: |b| \leq |a|\}$, $H^{(\alpha,|a|)} = \{H^{(\alpha,b)}: |b| \leq |a|\}$. Thus (2.11) and (2.12) can be written as

$$\begin{cases} \partial_t V^{(\alpha,a)} - \mu\Delta \sum_{l=0}^{\alpha} C_\alpha^l (-1)^{\alpha-l} V^{(l,a)} - \nabla \cdot H^{(\alpha,a)} = f_{\alpha a}^1, \\ \partial_t H^{(\alpha,a)} - \nabla V^{(\alpha,a)} = f_{\alpha a}^2, \end{cases} \tag{2.14}$$

and

$$\nabla^\perp \cdot H^{(\alpha,a)} = f_{\alpha a}^3. \tag{2.15}$$

We will also use the notation $f_{ij}^{\alpha a}$ to denote

$$f_{ij}^{\alpha a} = \sum_{\substack{b+c=a \\ \beta+\gamma=\alpha}} C_\alpha^\beta C_a^b \big(\partial_i V^{(\beta,b)} \partial_j V^{(\gamma,c)} - \partial_i H^{(\beta,b)} \cdot \partial_j H^{(\gamma,c)}\big), \tag{2.16}$$

where $1 \leq i, j \leq 2$. Hence, $f_{\alpha a}^1 = R_i^\perp R_j f_{\alpha a}^{ij}$ where $R_i^\perp R_j = \nabla_i^\perp \nabla_j \Delta^{-1}$.

2.4 Some Notations

Now we explain some important concepts and notations used throughout the paper. The spatial derivatives can be decomposed into radial and angular components:

$$\nabla = \omega \partial_r + \frac{\omega^\perp}{r} \partial_\theta, \tag{2.17}$$

where $\partial_r = \omega \cdot \nabla$, $\partial_\theta = x^\perp \cdot \nabla$. This fact plays an important role in the following argument.

We will use Klainerman's generalized energy, which is defined, for $\kappa \geq 1$, by

$$E_\kappa(t) = \sum_{|\alpha|+|a|\leq\kappa} \|U^{(\alpha,a)}(t,\cdot)\|_{L^2}^2,$$

where $U = (V, H)$. Moreover, we introduce the modified generalized energy

$$\mathcal{E}_\kappa(t) = \sum_{|\alpha|+|a|+1\leq\kappa} \|\nabla U^{(\alpha,a)}(t,\cdot)\|_{L^2}^2.$$

Here the word "modified generalized energy" is used to insist on the fact that one of the derivatives has to be a regular derivative. The use of the modified energy $\mathcal{E}_\kappa$ at the highest derivative level is imposed by the ghost weight method and will lead to some difficulties.

We also use the weighted energy norm of Klainerman-Sideris [25]:

$$X_\kappa(t) = \sum_{|\alpha|+|a|+1\leq\kappa} \|\langle r-t\rangle \nabla U^{(\alpha,a)}\|_{L^2}^2,$$

in which we denote $\langle \sigma \rangle = \sqrt{1+\sigma^2}$.

In addition, we introduce a new weighted energy for good quantities $V + H \cdot \omega$ and $H \cdot \omega^{\perp}$:

$$Y_\kappa(t) = \sum_{|\alpha|+|a|+1\le\kappa} \big(\|r(\partial_r V^{(\alpha,a)} + \partial_r H^{(\alpha,a)} \cdot \omega)\|_{L^2}^2 + \|r\partial_r H^{(\alpha,a)} \cdot \omega^{\perp}\|_{L^2}^2\big).$$

The weighted energy Y_κ is used to describe the good decay properties of the good unknowns $V + H \cdot \omega$ and $H \cdot \omega^{\perp}$ near the light cone. We emphasize that we need to treat the derivative loss in what follows when estimating X_κ and Y_κ.

To describe the space of initial data, we introduce (see [41])

$$\Lambda = \{\nabla, \widetilde{\Omega}, r\partial_r - 1\},$$

and

$$H_\Lambda^\kappa = \Big\{(f,g) : \sum_{|a|\le\kappa} \|\Lambda^a f\|_{L^2} + \|\Lambda^a g\|_{L^2} < \infty\Big\}$$

with the norm

$$\|(f,g)\|_{H_\Lambda^\kappa} = \sum_{|a|\le\kappa} \big(\|\Lambda^a f\|_{L^2} + \|\Lambda^a g\|_{L^2}\big),$$

for scalar, vector, or matrix function f and g. Then as in [41], we define

$$H_\Gamma^\kappa(T) = \Big\{(f,g)\colon [0,T) \to \mathbb{R} \times \mathbb{R}^2, (f,g) \in \bigcap_{j=0}^{\kappa} C^j\big([0,T); H_\Lambda^{\kappa-j}\big)\Big\}.$$

Solutions will be constructed in the space $H_\Gamma^\kappa(T)$.

Throughout this paper, we will use $A \lesssim B$ to denote $A \le CB$ for some positive absolute constant C, whose meaning may change from line to line. We remark that, without specification, the constant only depends on κ but never on μ or t.

For the global existence result, we will establish the following a priori estimate:

$$\begin{aligned} &\mathcal{E}_\kappa(t) + E_{\kappa-1}(t) \\ &\quad + \sum_{|\alpha|+|a|\le\kappa-1} \mu \int_0^t \int_{\mathbb{R}^2} |\Delta V^{(\alpha,a)}(\tau)|^2 + |\nabla V^{(\alpha,a)}(\tau)|^2 \, dx \, d\tau \\ &\quad \lesssim \int_0^t \langle\tau\rangle^{-1} (\mathcal{E}_\kappa(\tau) + E_{\kappa-1}(\tau)) E_{\kappa-3}^{1/2}(\tau) d\tau + \mathcal{E}_\kappa(0) + E_{\kappa-1}(0) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &E_{\kappa-3}(t) + \sum_{|\alpha|+|a|\le\kappa-3} \mu \int_0^t \int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}(\tau)|^2 \, dx \, d\tau \\ &\quad \lesssim E_{\kappa-3}(0) + \int_0^t \langle\tau\rangle^{-3/2} E_{\kappa-3}(\tau) E_{\kappa-1}^{1/2}(\tau) d\tau, \end{aligned} \tag{2.19}$$

for $\kappa \ge 12$. Once the above estimates are obtained, the main result holds by a standard continuity method. For the details, one can consult the differential version in [28].

So from now on, our main goal is to prove the two a priori estimates (2.18) and (2.19). In Theorem 1.1, by taking an appropriately large C_0 and small γ, we can assume that $E_{k-3} \ll 1$, which is always assumed in the following argument.

2.5 Energy Estimate Scenario

To see the underlying ideas more clearly in these long computations, we sketch the energy estimates in various scenarios as follows:

The modified energy estimate:

$$\begin{aligned} \mathcal{E}_\kappa + \underbrace{\mathcal{D}_{\kappa+1}}_{\text{modified dissipative energy}} + \underbrace{G_\kappa}_{\text{ghost weight energy}} + \underbrace{\mathit{LinearCommutator}_1}_{\text{due to viscosity and scaling operator}} \\ + \underbrace{\mathit{LinearCommutator}_2}_{\text{due to viscosity and ghost weight}} \lesssim C + \mathit{Nonlinear\text{-}terms}_1. \end{aligned}$$

LinearCommutator$_1$ can be absorbed by $\mathcal{D}_{\kappa+1}$.

LinearCommutator$_2$ is absorbed by $\mathcal{D}_{\kappa+1}$ and D_κ (note that D_κ will be contained in the *standard higher-order energy estimate* $E_{\kappa-1}$).

Nonlinear-terms$_1$ represent derivative loss problems that are present due to the highly fully nonlinear effect, nonlocal effect, and the use of ghost weights. After a long, delicate, integration-by-parts procedure, one can save one derivative. It is important that the null condition be satisfied. Thus one can continue to employ G_k to improve the decay rate to the critical rate. Here one also needs to take care of the derivative loss in dealing with $X_{\kappa-1} + Y_{\kappa-1} \lesssim E_\kappa$.

The standard higher-order energy estimate:

$$\begin{aligned} E_{\kappa-1} + \underbrace{D_\kappa}_{\text{dissipative energy}} + \underbrace{\mathit{LinearCommutator}_3}_{\text{due to viscosity and scaling operator}} \\ \lesssim C + \mathit{Nonlinear\text{-}terms}_2. \end{aligned}$$

LinearCommutator$_3$ is absorbed by D_κ.

Nonlinear-terms$_2$: G_κ is used to improve the decay (note G_κ has been used in the *modified energy estimate* $\mathcal{E}_\kappa$). One also takes care of the derivative loss in derivations of $X_{\kappa-1} + Y_{\kappa-1} \lesssim E_\kappa$.

The lower-order energy estimate:

$$\begin{aligned} E_{\kappa-3} + \underbrace{D_{\kappa-2}}_{\text{dissipative energy}} + \underbrace{\mathit{LinearCommutator}_4}_{\text{due to viscosity and scaling operator}} \\ \lesssim \epsilon^2 + \mathit{Nonlinear\text{-}terms}_3. \end{aligned}$$

LinearCommutator$_4$ is absorbed by $D_{\kappa-2}$.

Nonlinear-terms$_3$ are those that satisfy a strong null condition.

3 Estimates for the Special Quantities

In this section, we are going to estimate the weighted L^2 energies X_κ and Y_κ. The weighted energy X_κ was first introduced by Klainerman and Sideris [25] for proving almost global solutions of three-dimensional quasilinear wave equations and later on used in [31] for proving almost global existence for the two-dimensional incompressible elastodynamic system. The energy Y_κ is new and used here to estimate the good unknowns. Due to the fact that r is not an $\mathcal{A}_2$ weight in two dimensions, the modified one is introduced in [28] to get the global solution of two-dimensional incompressible elastodynamics. Here by transforming the original quasilinear system (2.1) into a fully nonlinear one (1.6), we can simply use the earlier ones introduced by Klainerman-Sideris for X_κ. This advantage is based on the inherent structure of the system, which enables us to simplify the proofs.

3.1 Sobolev-Type Inequalities

The following weighted Sobolev-type inequalities will be used to prove the decay of solutions in the L^∞ norm. A much stronger version of (3.3) appeared in [28]. Since we are able to transform the original system into a fully nonlinear one, the form of (3.3) is enough for us here.

LEMMA 3.1. *For all $f \in H^2(\mathbb{R}^2)$, there holds*

$$r|f(x)|^2 \lesssim \sum_{a=0,1} \big[\|\partial_r \Omega^a f\|_{L^2}^2 + \|\Omega^a f\|_{L^2}^2\big], \tag{3.1}$$

$$r\langle t-r\rangle^2|f(x)|^2 \lesssim \sum_{a=0,1} \big[\|\langle t-r\rangle\partial_r \Omega^a f\|_{L^2}^2 + \|\langle t-r\rangle\Omega^a f\|_{L^2}^2\big], \tag{3.2}$$

$$\langle t\rangle\|f\|_{L^\infty(r\le\langle t\rangle/2)} \lesssim \sum_{|a|\le 2} \|\langle t-r\rangle\partial^a f\|_{L^2}, \tag{3.3}$$

provided the right-hand side is finite.

The proof of this lemma can be found in [31] (the three-dimensional version can be found in [44]); we omit the details here.

3.2 Estimate of the Good Quantities

In this section, we are going to explore the good properties of some special combinations of unknowns. Both the linearities and the nonlinearities will be investigated. The exploration of these special quantities is a prerequisite for the estimate of weighted L^2 energies X_κ and Y_κ. On the other hand, they are also crucial for the energy estimate that will be conducted in the next two sections.

In order to simplify the presentation, we first introduce some notation. Suppose that $(V, H) \in H^\kappa_\Gamma$ solves (1.6) and (1.7). Define

$$\begin{aligned} L_\kappa &= \sum_{|\alpha|+|a|\le k} |U^{(\alpha,a)}|, \\ N_{\kappa+1} &= \sum_{|\alpha|+|a|\le k} \big(t\big|f^1_{\alpha a}\big| + t\big|f^2_{\alpha a}\big| + (t+r)\big|f^3_{\alpha a}\big|\big), \\ \mathcal{N}_{\kappa+2} &= \sum_{|\alpha|+|a|\le k} t\big|\nabla\cdot f^2_{\alpha a}\big|, \end{aligned}$$

where L_κ represents some linear quantity, and $N_{\kappa+1}$ and $\mathcal{N}_{\kappa+2}$ represent some nonlinear quantities.

Remark 3.2. The term $N_{\kappa+1}$ will be used when we multiply systems (2.14) and (2.15) by some t or r factor. The term $\mathcal{N}_{\kappa+2}$ will appear due to the presence of viscosity (see Lemma 3.6).

Remark 3.3. One can also use a stronger version of $N_{\kappa+1}$ by defining

$$N_{\kappa+1} = \sum_{|\alpha|+|a|\le k} \big(t\big|f^1_{\alpha a}\big| + (t+r)\big|f^2_{\alpha a}\big| + (t+r)\big|f^3_{\alpha a}\big|\big).$$

However, one cannot include $r|f^1_{\alpha a}|$ in $N_{\kappa+1}$ since r is not an $\mathcal{A}_2$ weight for a singular integral in two space dimensions.

Now we are going to analyze the linear part of the system and establish several estimates. Before doing so, we need an elementary iteration lemma.

LEMMA 3.4 (Iteration lemma). *Let $\{f_l\}$, $\{g_l\}$, and $\{F_l\}$ be three nonnegative sequences where $0 \le l \le \kappa$. Suppose that*

$$f_0 + g_0 \lesssim F_0,$$

and for all $1 \le l \le \kappa$,

$$f_l + g_l - g_{l-1} \lesssim F_l.$$

Then there holds

$$\sum_{0\le m\le l} (f_m + g_m) \lesssim \sum_{0\le m\le l} F_m$$

for all $0 \le l \le \kappa$.

Remark 3.5. This lemma plays a role in dealing with the commutators between the viscosity term and the scaling operator and will frequently be used throughout the whole paper.

PROOF. We prove the lemma by induction on l. Obviously, the lemma is correct when $l = 0$. Let $1 \le l \le \kappa$. We assume the lemma is correct for $l-1$. This means that

$$\sum_{0\le m\le l-1} (f_m + g_m) \le C \sum_{0\le m\le l-1} F_m. \tag{3.4}$$

On the other hand, we have

$$f_l + g_l - g_{l-1} \le CF_l. \tag{3.5}$$

Multiplying (3.4) by 2 and then adding (3.5), we get

$$2 \sum_{0\le m\le l-2} (f_m + g_m) + f_l + g_l + 2f_{l-1} + g_{l-1} \le 2C \sum_{0\le m\le l-1} F_m + CF_l,$$

which is the required estimate for l. Thus the lemma is proved. □

Now we are ready to state two lemmas for the special linear quantities. These two lemmas are requisite for the estimate of the weighted L^2 norms X_κ and Y_κ.

LEMMA 3.6. *Suppose that* $(V, H) \in H_\Gamma^{\kappa-1}$ *solves* (1.6) *and* (1.7). *Then for all* $|\alpha| + |a| \le \kappa - 3$*, there holds*

$$\begin{aligned}&\big\|r\partial_r V^{(\alpha,a)} + t\nabla\cdot H^{(\alpha,a)}\big\|_{L^2}^2 + \big\|\mu t\Delta V^{(\alpha,a)}\big\|_{L^2}^2\\ &\quad\le \nu X_{|\alpha|+|a|+1} + C\langle\mu\rangle^2\|L_{|\alpha|+|a|+1}\|_{L^2}^2 + C_\nu\mu^2\|L_{|\alpha|+|a|+2}\|_{L^2}^2\\ &\qquad + C\|N_{|\alpha|+|a|+1}\|_{L^2}^2 + C_\nu\mu^2\|\mathcal{N}_{|\alpha|+|a|+2}\|_{L^2}^2,\end{aligned}$$

provided the right-hand side is finite, where ν *can be any positive constant,* C_ν *is a constant that depends only on* α*,* a*, and* ν*, and* C *depends only on* α *and* a*.*

Remark 3.7. While the lemma becomes trivial if there is no viscosity, the viscous version is nontrivial. The viscosity is the main reason we only have an L^2 bound rather than a pointwise bound as in the next lemma.

Remark 3.8. The terms on the left-hand side are of order $|\alpha| + |a| + 1$ except for the viscous term, but on the right-hand side, the order is $|\alpha| + |a| + 2$. This means that we will encounter the problem of losing derivatives in future discussions. If $\mu = 0$, one has no problem with derivative loss.

PROOF. Denote

$$J = \|r\partial_r V^{(\alpha,a)} + t\nabla\cdot H^{(\alpha,a)}\|_{L^2}^2 + \sum_{l=0}^{\alpha}\big(C_\alpha^l\big)^2\|\mu t\Delta V^{(l,a)}\|_{L^2}^2.$$

We first claim that the following fact holds:

$$\begin{aligned}J &\le 6\sum_{l=0}^{\alpha-1}\big(C_\alpha^l\big)^2\|\mu t\Delta V^{(l,a)}\|_{L^2}^2 + \nu\|\langle t-r\rangle\nabla\cdot H^{(\alpha,a)}\|_{L^2}^2\\ &\quad + C\langle\mu\rangle^2\|L_{|\alpha|+|a|+1}\|_{L^2}^2 + C_\nu\mu^2\|L_{|\alpha|+|a|+2}\|_{L^2}^2\\ &\quad + C\|N_{|\alpha|+|a|+1}\|_{L^2}^2 + C_\nu\mu^2\|\mathcal{N}_{|\alpha|+|a|+2}\|_{L^2}^2,\end{aligned} \tag{3.6}$$

where ν can be any positive constant, and C_ν is a constant depending only on α, a, and ν.

Once assertion (3.6) becomes true, noting the assumption $\mu \leq 1$, one can immediately see that Lemma 3.6 is proved by applying Lemma 3.4 to (3.6). Thus it suffices to prove (3.6).

Multiplying the first equation of (2.14) by t and using the scaling operator, one gets

$$r\partial_r V^{(\alpha,a)} + t\nabla \cdot H^{(\alpha,a)} + \mu t\Delta \sum_{l=0}^{\alpha} C_\alpha^l (-1)^{\alpha-l} V^{(l,a)} = SV^{(\alpha,a)} - tf_{\alpha a}^1.$$

Taking the L^2 norm for the above equation, one has

$$\begin{aligned} J &= -2\int_{\mathbb{R}^2} (r\partial_r V^{(\alpha,a)} + t\nabla \cdot H^{(\alpha,a)}) \cdot \mu t\Delta \sum_{l=0}^{\alpha} C_\alpha^l (-1)^{\alpha-l} V^{(l,a)} dx \\ &\quad + \left\| SV^{(\alpha,a)} - tf_{\alpha a}^1 \right\|_{L^2}^2. \end{aligned} \tag{3.7}$$

To prove (3.6), we need to deal with the right-hand side of (3.7).

By separating the highest-order terms from the lower-order ones, (3.7) can be organized as

$$\begin{aligned} J = & \underbrace{-2\int_{\mathbb{R}^2} t\nabla \cdot H^{(\alpha,a)} \cdot \mu t\Delta V^{(\alpha,a)} dx}_{J_1} \underbrace{- 2\int_{\mathbb{R}^2} r\partial_r V^{(\alpha,a)} \cdot \mu t\Delta V^{(\alpha,a)} dx}_{J_2} \\ & \underbrace{\begin{aligned} & - 2\int_{\mathbb{R}^2} (r\partial_r V^{(\alpha,a)} + t\nabla \cdot H^{(\alpha,a)}) \cdot \mu t\Delta \sum_{l=0}^{\alpha-1} C_\alpha^l (-1)^{\alpha-l} V^{(l,a)} dx \\ & + \left\| SV^{(\alpha,a)} - tf_{\alpha a}^1 \right\|_{L^2}^2. \end{aligned}}_{J_3} \end{aligned} \tag{3.8}$$

Here J_3 refers to the lower-order term, and J_1 and J_2 refer to the higher-order terms.

By Hölder's inequality, J_3 can be estimated by

$$\begin{aligned} \frac{1}{2}\left\| r\partial_r V^{(\alpha,a)} + t\nabla \cdot H^{(\alpha,a)} \right\|_{L^2}^2 + 2\sum_{l=0}^{\alpha-1} \left(C_\alpha^l\right)^2 \|\mu t\Delta V^{(l,a)}\|_{L^2}^2 \\ + 2\|SV^{(\alpha,a)}\|_{L^2}^2 + 2\left\| tf_{\alpha a}^1 \right\|_{L^2}^2. \end{aligned}$$

For J_2, one can deduce from integration by parts that

$$J_2 = -2\int_{\mathbb{R}^2} r\partial_r V^{(\alpha,a)} \cdot \mu t\Delta V^{(\alpha,a)} dx = 0.$$

It remains to estimate J_1, which cannot be treated simply by the Cauchy inequality due to an extra t-factor. We will refer to the inherent structure of the systems.

Applying the divergence operator to the second equation of (2.14), one gets

$$\Delta V^{(\alpha,a)} = \partial_t \nabla \cdot H^{(\alpha,a)} - \nabla \cdot f^2_{\alpha a}.$$

Inserting the above expression into J_1 and employing the scaling operator, we have

$$\begin{aligned} J_1 &= -2\int_{\mathbb{R}^2} t\nabla \cdot H^{(\alpha,a)} \cdot \mu\big(t\partial_t \nabla \cdot H^{(\alpha,a)} - t\nabla \cdot f^2_{\alpha a}\big)dx \\ &= -2\int_{\mathbb{R}^2} t\nabla \cdot H^{(\alpha,a)} \cdot \mu\big(-r\partial_r \nabla \cdot H^{(\alpha,a)} + S\nabla \cdot H^{(\alpha,a)} - t\nabla \cdot f^2_{\alpha a}\big)dx. \end{aligned}$$

In view of the fact that

$$2\int_{\mathbb{R}^2} t\nabla \cdot H^{(\alpha,a)} \cdot \mu r\partial_r \nabla \cdot H^{(\alpha,a)} dx = -2\int_{\mathbb{R}^2} \mu t |\nabla \cdot H^{(\alpha,a)}|^2\, dx,$$

we get

$$J_1 \le -2\int_{\mathbb{R}^2} t\nabla \cdot H^{(\alpha,a)} \cdot \mu\big(S\nabla \cdot H^{(\alpha,a)} - t\nabla \cdot f^2_{\alpha a}\big)dx.$$

Now we need to estimate the integral in different regions separately. To do this, define a radial cutoff function $\varphi \in C^\infty(\mathbb{R}^2)$ that satisfies

$$\varphi = \begin{cases} 1 & \text{if } \frac{3}{4} \le r \le \frac{6}{5}, \\ 0 & \text{if } r < \frac{2}{3} \text{ or } r > \frac{5}{4}, \end{cases} \qquad |\nabla \varphi| \lesssim 1.$$

For each fixed $t \ge 1$, let $\varphi^t(x) = \varphi(x/\langle t\rangle)$. Clearly, one has

$$\varphi^t(x) \equiv \begin{cases} 1 & \text{for } \frac{3\langle t\rangle}{4} \le r \le \frac{6\langle t\rangle}{5}, \\ 0 & \text{for } r \le \frac{2\langle t\rangle}{3} \text{ or } r \ge \frac{5\langle t\rangle}{4}, \end{cases}$$

and

$$|\nabla \varphi^t(x)| \lesssim \langle t\rangle^{-1}.$$

Consequently,

$$\begin{aligned} J_1 &\le -2\int_{\mathbb{R}^2} t\nabla \cdot H^{(\alpha,a)} \cdot \mu\big(S\nabla \cdot H^{(\alpha,a)} - t\nabla \cdot f^2_{\alpha a}\big)dx \\ &= \underbrace{-2\int_{\mathbb{R}^2} (1-\varphi^t(x))t\nabla \cdot H^{(\alpha,a)} \cdot \mu\big(S\nabla \cdot H^{(\alpha,a)} - t\nabla \cdot f^2_{\alpha a}\big)dx}_{J_{11}} \\ &\quad \underbrace{-2\int_{\mathbb{R}^2} \varphi^t(x)t\nabla \cdot H^{(\alpha,a)} \cdot \mu\big(S\nabla \cdot H^{(\alpha,a)} - t\nabla \cdot f^2_{\alpha a}\big)dx}_{J_{12}}. \end{aligned}$$

We now estimate J_{11}. Note on the support of $1-\varphi^t(x)$, we have $t \lesssim \langle t-r\rangle$. Thus one can estimate J_{11} as follows:

$$J_{11} \le \nu \|\langle t-r\rangle \nabla \cdot H^{(\alpha,a)}\|^2_{L^2} + \mu^2 C_\nu\big(\|S\nabla \cdot H^{(\alpha,a)}\|^2_{L^2} + \big\|t\nabla \cdot f^2_{\alpha a}\big\|^2_{L^2}\big),$$

where ν can be any positive constant and C_ν is a constant depending on ν.

For J_{12}, employing the first equation of (2.14), we have

$$
\begin{aligned}
J_{12} &= -2\int_{\mathbb{R}^2} \varphi^t(x)t\Big[\partial_t V^{(\alpha,a)} - \mu\Delta\sum_{l=0}^{\alpha} C_\alpha^l(-1)^{\alpha-l}V^{(l,a)} - f_{\alpha a}^1\Big] \\
&\qquad \cdot \mu\big(S\nabla\cdot H^{(\alpha,a)} - t\nabla\cdot f_{\alpha a}^2\big)dx \\
&= \underbrace{-2\int_{\mathbb{R}^2} \varphi^t(x)t\partial_t V^{(\alpha,a)}\cdot\mu\big(S\nabla\cdot H^{(\alpha,a)} - t\nabla\cdot f_{\alpha a}^2\big)dx}_{J_{121}} \\
&\quad \underbrace{\begin{aligned}&+2\int_{\mathbb{R}^2}\varphi^t(x)\Big(\mu t\Delta\sum_{l=0}^{\alpha}C_\alpha^l(-1)^{\alpha-l}V^{(l,a)} + tf_{\alpha a}^1\Big)\\ &\qquad\cdot\mu\big(S\nabla\cdot H^{(\alpha,a)} - t\nabla\cdot f_{\alpha a}^2\big)dx\end{aligned}}_{J_{122}}
\end{aligned}
$$

J_{122} can be directly bounded as follows:

$$
\begin{aligned}
J_{122} &\le \frac{1}{4}\sum_{l=0}^{\alpha}\big(C_\alpha^l\big)^2\|\mu t\Delta V^{(l,a)}\|_{L^2}^2 + 5\mu^2\big\|S\nabla\cdot H^{(\alpha,a)} - t\nabla\cdot f_{\alpha a}^2\big\|_{L^2}^2 \\
&\quad + \big\|tf_{\alpha a}^1\big\|_{L^2}^2 \\
&\le \frac{1}{4}\sum_{l=0}^{\alpha}\big(C_\alpha^l\big)^2\|\mu t\Delta V^{(l,a)}\|_{L^2}^2 + 10\mu^2\|S\nabla\cdot H^{(\alpha,a)}\|_{L^2} \\
&\quad + 10\mu^2\|t\nabla\cdot f_{\alpha a}^2\|_{L^2}^2 + \|tf_{\alpha a}^1\|_{L^2}^2.
\end{aligned}
$$

Finally, we write

$$
\begin{aligned}
J_{121} &= 2\int_{\mathbb{R}^2}\varphi^t(x)\mu\big(r\partial_r V^{(\alpha,a)} - SV^{(\alpha,a)}\big)\cdot\big(S\nabla\cdot H^{(\alpha,a)} - t\nabla\cdot f_{\alpha a}^2\big)dx \\
&= \underbrace{2\int_{\mathbb{R}^2}\varphi^t(x)\mu r\partial_r V^{(\alpha,a)}\cdot\big(S\nabla\cdot H^{(\alpha,a)} - t\nabla\cdot f_{\alpha a}^2\big)dx}_{J_{1211}} \\
&\quad \underbrace{-2\int_{\mathbb{R}^2}\varphi^t(x)\mu SV^{(\alpha,a)}\cdot\big(S\nabla\cdot H^{(\alpha,a)} - t\nabla\cdot f_{\alpha a}^2\big)dx}_{J_{1212}}.
\end{aligned}
$$

J_{1212} can be bounded by

$$
J_{1212} \le 2\|SV^{(\alpha,a)}\|_{L^2}^2 + \mu^2\|S\nabla\cdot H^{(\alpha,a)}\|_{L^2}^2 + \mu^2\big\|t\nabla\cdot f_{\alpha a}^2\big\|_{L^2}^2.
$$

For J_{1211}, note on the support of $\varphi^t(x)$, we have $\langle t\rangle \sim r$. Hence one deduces that

$$\begin{aligned}
J_{1211} &= 2\int_{\mathbb{R}^2} \varphi^t(x)\mu r\partial_r V^{(\alpha,a)} \cdot \big(\nabla\cdot \widetilde{S} H^{(\alpha,a)} - t\nabla\cdot f^2_{\alpha a}\big)dx\\
&= -2\int_{\mathbb{R}^2} \nabla\big(\varphi^t(x)\mu r\partial_r V^{(\alpha,a)}\big) \cdot \big(\widetilde{S} H^{(\alpha,a)} - tf^2_{\alpha a}\big)dx\\
&\le 2\int_{\mathbb{R}^2} \big|\nabla\varphi^t(x)\mu r\partial_r V^{(\alpha,a)} \cdot \big(\widetilde{S} H^{(\alpha,a)} - tf^2_{\alpha a}\big)\big|dx\\
&\quad + 2\int_{\mathbb{R}^2} \varphi^t(x)\mu |\nabla V^{(\alpha,a)}| \cdot \big|\widetilde{S} H^{(\alpha,a)} - tf^2_{\alpha a}\big|dx\\
&\quad + 2\int_{\mathbb{R}^2} \varphi^t(x)\mu r\partial_r \nabla V^{(\alpha,a)} \cdot \big(\widetilde{S} H^{(\alpha,a)} - tf^2_{\alpha a}\big)dx\\
&\le \mu^2\|\nabla V^{(\alpha,a)}\|^2_{L^2} + \frac14\|\mu t\nabla^2 V^{(\alpha,a)}\|^2_{L^2} + C\,\big\|\widetilde{S} H^{(\alpha,a)} - tf^2_{\alpha a}\big\|^2_{L^2}.
\end{aligned}$$

Combining all the above estimates, we conclude by the commutation between the generalized operators that

$$\begin{aligned}
J \,&=\, \big\|r\partial_r V^{(\alpha,a)} + t\nabla\cdot H^{(\alpha,a)}\big\|^2_{L^2} + \sum_{l=0}^{\alpha}\big(C^l_\alpha\big)^2\|\mu t\Delta V^{(l,a)}\|^2_{L^2}\\
&\le \frac12\big\|r\partial_r V^{(\alpha,a)} + t\nabla\cdot H^{(\alpha,a)}\big\|^2_{L^2} + \frac12\|\mu t\Delta V^{(\alpha,a)}\|^2_{L^2}\\
&\quad + 3\sum_{l=0}^{\alpha-1}\big(C^l_\alpha\big)^2\|\mu t\Delta V^{(l,a)}\|^2_{L^2} + \nu\|\langle t-r\rangle\nabla\cdot H^{(\alpha,a)}\|^2_{L^2}\\
&\quad + C\langle\mu\rangle^2\|L_{|\alpha|+|a|+1}\|^2_{L^2} + C_\nu\mu^2\|L_{|\alpha|+|a|+2}\|^2_{L^2}\\
&\quad + C\|N_{|\alpha|+|a|+1}\|^2_{L^2} + C_\nu\mu^2\|\mathcal{N}_{|\alpha|+|a|+2}\|^2_{L^2}.
\end{aligned}$$

Absorbing the first two terms on the right-hand side in the above yields (3.6). Thus the lemma is proved. ☐

We have the following pointwise estimates:

LEMMA 3.9. *Suppose that* $(V, H) \in H^{\kappa-1}_\Gamma$ *solves* (1.6) *and* (1.7). *Then for all* $|\alpha| + |a| \le \kappa - 2$, *there holds*

$$|r(\nabla\cdot H^{(\alpha,a)})\omega + t\nabla V^{(\alpha,a)}| \lesssim L_{|\alpha|+|a|+1} + N_{|\alpha|+|a|+1}, \tag{3.9}$$

$$\begin{aligned}
&|(t\pm r)(\nabla V^{(\alpha,a)} \pm \nabla\cdot H^{(\alpha,a)}\omega)|\\
&\quad\lesssim L_{|\alpha|+|a|+1} + N_{|\alpha|+|a|+1} + \big|r\partial_r V^{(\alpha,a)} + t\nabla\cdot H^{(\alpha,a)}\big|,
\end{aligned}\tag{3.10}$$

$$r\big|\partial_r H^{(\alpha,a)}\cdot\omega^\perp\big| \lesssim L_{|\alpha|+|a|+1} + N_{|\alpha|+|a|+1}, \tag{3.11}$$

$$\begin{aligned}
&r\big|\partial_r V^{(\alpha,a)} + \partial_r H^{(\alpha,a)}\cdot\omega\big|\\
&\quad\lesssim L_{|\alpha|+|a|+1} + N_{|\alpha|+|a|+1} + \big|r\partial_r V^{(\alpha,a)} + t\nabla\cdot H^{(\alpha,a)}\big|.
\end{aligned}\tag{3.12}$$

PROOF. Multiplying the second equation of (2.14) by t and using the scaling operator, we can rearrange the resulting systems as follows:

$$r\partial_r H^{(\alpha,a)} + t\nabla V^{(\alpha,a)} = SH^{(\alpha,a)} - tf^2_{\alpha a}. \tag{3.13}$$

Employing (2.17), one has

$$\begin{aligned}
&r\partial_r H^{(\alpha,a)} + t\nabla V^{(\alpha,a)} \\
&\quad = (r\partial_r H^{(\alpha,a)} \cdot \omega)\omega + (r\partial_r H^{(\alpha,a)} \cdot \omega^\perp)\omega^\perp + t\nabla V^{(\alpha,a)} \\
&\quad = (r\nabla \cdot H^{(\alpha,a)})\omega - (\Omega H^{(\alpha,a)} \cdot \omega^\perp)\omega + (r\nabla^\perp \cdot H^{(\alpha,a)})\omega^\perp \\
&\qquad + (\Omega H^{(\alpha,a)} \cdot \omega)\omega^\perp + t\nabla V^{(\alpha,a)} \\
&\quad = (r\nabla \cdot H^{(\alpha,a)})\omega + t\nabla V^{(\alpha,a)} + rf^3_{\alpha a}\omega^\perp \\
&\qquad - (\Omega H^{(\alpha,a)} \cdot \omega^\perp)\omega + (\Omega H^{(\alpha,a)} \cdot \omega)\omega^\perp.
\end{aligned} \tag{3.14}$$

In view of the relation between S and $\widetilde{S}$, (3.9) is clear from (3.13) and (3.14). Next, note that

$$r\nabla V^{(\alpha,a)} + t(\nabla \cdot H^{(\alpha,a)})\omega = (r\partial_r V^{(\alpha,a)} + t\nabla \cdot H^{(\alpha,a)})\omega + \Omega V\omega^\perp. \tag{3.15}$$

(3.10) is a direct consequence of (3.9) and (3.15).

The estimate of (3.11) follows directly from (2.15) and (2.17). To check (3.12), by analogy with the above proof, we write

$$\begin{aligned}
&r(\partial_r V^{(\alpha,a)} + \partial_r H^{(\alpha,a)} \cdot \omega) \\
&\quad = \omega \cdot \big[r\nabla V^{(\alpha,a)} + (r\partial_r H^{(\alpha,a)} \cdot \omega)\omega\big] \\
&\quad = \omega \cdot \big[r\nabla V^{(\alpha,a)} + r\nabla \cdot H^{(\alpha,a)}\omega - (\Omega H^{(\alpha,a)} \cdot \omega^\perp)\omega\big],
\end{aligned}$$

from which (3.12) follows from (3.10). □

Next we are going to estimate the nonlinearities. The following lemma says that the nonlinearities have the good pointwise decay property near the light cone if we disregard the Riesz transform. This lemma not only is used in the estimate of the weighted L^2 norm in this section, but also plays one of the key roles in the energy estimate in the next section.

LEMMA 3.10. *Let $f^2_{\alpha a}$ and $f^3_{\alpha a}$ denote the nonlinearities in* (2.13). *Then for all $|\alpha| + |a| \le \kappa - 3$, there holds*

$$\big|f^2_{\alpha a}\big| \lesssim \frac{1}{r} \sum_{\substack{|\beta|+|\gamma|\le|\alpha| \\ |b|+|c|\le|a|}} |V^{(|\beta|,|b|+1)}||H^{(|\gamma|,|c|+1)}|, \tag{3.16}$$

$$\big|f^3_{\alpha a}\big| \lesssim \frac{1}{r} \sum_{\substack{|\beta|+|\gamma|\le|\alpha| \\ |b|+|c|\le|a|}} |H^{(|\beta|,|b|+1)}||H^{(|\gamma|,|c|+1)}|, \tag{3.17}$$

$$\big|\nabla \cdot f^2_{\alpha a}\big| \lesssim \frac{1}{r} \sum_{\substack{|\beta|+|\gamma|\le|\alpha| \\ |b|+|c|\le|a|}} |V^{(|\beta|,|b|+2)}||H^{(|\beta|,|c|+2)}|. \tag{3.18}$$

Furthermore, recall the definition of $f^{ij}_{\alpha a}$ in (2.16). *Then there holds*

$$\begin{aligned}\big|f^{ij}_{\alpha a}\big| \lesssim{}& \frac{1}{r} \sum_{\substack{|b|+|c|\le|a| \\ |\beta|+|\gamma|\le|\alpha|}} \big(|V^{(|\beta|,|b|+1)}||V^{(|\gamma|,|c|+1)}| + |H^{(|\beta|,|b|+1)}||H^{|\gamma|,|c|+1}|\big) \\ &+ \sum_{\substack{b+c=a \\ \beta+\gamma=\alpha}} \big[|\partial_r V^{(\beta,b)} + \partial_r H^{(\beta,b)} \cdot \omega|\big(|\nabla V^{(\gamma,c)}| + |\nabla H^{(\gamma,c)}|\big) \\ &\qquad + |\partial_r H^{(\beta,b)} \cdot \omega^\perp \partial_r H^{(\gamma,c)} \cdot \omega^\perp|\big].\end{aligned} \tag{3.19}$$

Recall that the introduction of $f^{ij}_{\alpha a}$ came from $f^1_{\alpha a}$ by dropping the Riesz transforms.

Remark 3.11. Note that all the nonlinearities satisfy the strong null condition, and our estimates always contain one spatial derivative in the good unknowns or gain $\langle t\rangle^{-1}$ near the light cone.

Remark 3.12. In the highest-order energy estimate of the next section, this lemma cannot be used since it causes a derivative loss.

PROOF. Employing (2.17), we write

$$\begin{aligned}f^2_{\alpha a} &= \sum_{\substack{b+c=a \\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a (\nabla^\perp H^{(\beta,b)} \nabla V^{(\gamma,c)}) \\ &= \sum_{\substack{b+c=a \\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \left(\partial_r H^{(\beta,b)} \otimes \omega^\perp - \frac{1}{r}\partial_\theta H^{(\beta,b)} \otimes \omega\right) \\ &\qquad \cdot \left(\omega \partial_r V^{(\gamma,c)} + \frac{\omega^\perp}{r}\partial_\theta V^{(\gamma,c)}\right) \\ &= \frac{1}{r} \sum_{\substack{b+c=a \\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a (\partial_r H^{(\beta,b)} \partial_\theta V^{(\gamma,c)} - \partial_\theta H^{(\beta,b)} \partial_r V^{(\gamma,c)}).\end{aligned}$$

Thus (3.16) is clear from the commutation between ∂_r and $\widetilde{S}$, Γ. Note that (3.17) can be estimated exactly in the same fashion; we omit the details.

To estimate (3.18), we use (2.17) to get that

$$\begin{aligned}
\nabla \cdot f^2_{\alpha a} &= \nabla \cdot \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a (\nabla^\perp H^{(\beta,b)} \nabla V^{(\gamma,c)})\\
&= \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \big(\nabla^\perp_j \nabla_i H^{(\beta,b)}_i \nabla_j V^{(\gamma,c)} + \nabla^\perp_j H^{(\beta,b)}_i \nabla_j \nabla_i V^{(\gamma,c)}\big) =\\
&= \frac{1}{r} \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \big(\partial_r \nabla_i H^{(\beta,b)}_i \partial_\theta V^{(\gamma,c)} - \partial_\theta \nabla_i H^{(\beta,b)}_i \partial_r V^{(\gamma,c)}\big)\\
&\quad + \frac{1}{r} \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \big(\partial_r H^{(\beta,b)}_i \partial_\theta \nabla_i V^{(\gamma,c)} - \partial_\theta H^{(\beta,b)}_i \partial_r \nabla_i V^{(\gamma,c)}\big).
\end{aligned}$$

By the commutation between the generalized operators, (3.18) is clear.

To estimate (3.19), from (2.17) we can deduce that

$$\begin{aligned}
f^{ij}_{\alpha a} = \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \Bigg[&\left(\omega_i \partial_r V^{(\beta,b)} + \frac{1}{r}\omega^\perp_i \Omega V^{(\beta,b)}\right)\\
&\cdot \left(\omega_j \partial_r V^{(\gamma,c)} + \frac{1}{r}\omega^\perp_j \Omega V^{(\gamma,c)}\right)\\
&- \left(\omega_i \partial_r H^{(\beta,b)} + \frac{1}{r}\omega^\perp_i \Omega H^{(\beta,b)}\right)\\
&\cdot \left(\omega_j \partial_r H^{(\gamma,c)} + \frac{1}{r}\omega^\perp_j \Omega H^{(\gamma,c)}\right)\Bigg]\\
= \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \Bigg[&\omega_i\omega_j (\partial_r V^{(\beta,b)} \partial_r V^{(\gamma,c)} - \partial_r H^{(\beta,b)} \cdot \partial_r H^{(\gamma,c)})\\
&+ \frac{1}{r}\omega_i \omega^\perp_j \partial_r V^{(\beta,b)} \Omega V^{(\gamma,c)}\\
&+ \frac{1}{r}\omega^\perp_i \Omega V^{(\beta,b)} \left(\omega_j \partial_r V^{(\gamma,c)} + \frac{1}{r}\omega^\perp_j \Omega V^{(\gamma,c)}\right)\\
&- \frac{1}{r}\omega_i \omega^\perp_j \partial_r H^{(\beta,b)} \cdot \Omega H^{(\gamma,c)}\\
&- \frac{1}{r}\omega^\perp_i \Omega H^{(\beta,b)} \cdot \left(\omega_j \partial_r H^{(\gamma,c)} + \frac{1}{r}\omega^\perp_j \Omega H^{(\gamma,c)}\right)\Bigg].
\end{aligned}$$

Thus (3.19) is obtained by the following grouping:

$$\begin{aligned}&\partial_r V^{(\beta,b)}\partial_r V^{(\gamma,c)} - \partial_r H^{(\beta,b)}\cdot\partial_r H^{(\gamma,c)}\\&\quad = (\partial_r V^{(\beta,b)} + \partial_r H^{(\beta,b)}\cdot\omega)\partial_r V^{(\gamma,c)}\\&\qquad - \partial_r H^{(\beta,b)}\cdot\omega(\partial_r V^{(\gamma,c)} + \partial_r H^{(\gamma,c)}\cdot\omega) - \partial_r H^{(\beta,b)}\cdot\omega^{\perp}\partial_r H^{(\gamma,c)}\cdot\omega^{\perp}.\end{aligned}$$

This completes the proof of the lemma. □

3.3 Estimate of the Weighted L^2 Energy

In what follows, we will show that the weighted energy can be controlled by the generalized energy under the smallness assumptions on lower-order energies.

LEMMA 3.13. *Suppose that* $(V, H) \in H^{\kappa-1}_{\Gamma}$ *solves* (1.6) *and* (1.7). *Then for all* $|\alpha| + |a| \le \kappa - 3$, *there holds*

$$\begin{aligned}&\|N_{|\alpha|+|a|+1} + \mathcal{N}_{|\alpha|+|a|+2}\|^2_{L^2}\\&\quad\lesssim E_{|\alpha|+|a|+2}E_{[(|\alpha|+|a|)/2]+4} + Y_{|\alpha|+|a|+1}E_{[(|\alpha|+|a|)/2]+3}\\&\qquad + E_{|\alpha|+|a|+2}(X_{[(|\alpha|+|a|)/2]+4} + Y_{[(|\alpha|+|a|)/2]+3}).\end{aligned}$$

PROOF. In view of the definition of $N_{|\alpha|+|a|+1}$ and $\mathcal{N}_{|\alpha|+|a|+2}$, it suffices to prove

$$\begin{aligned}&\big\|t\big|f^1_{\alpha a}\big| + t\big|f^2_{\alpha a}\big| + (t+r)\big|f^3_{\alpha a}\big| + t\big|\nabla\cdot f^2_{\alpha a}\big|\big\|^2_{L^2}\\&\quad\lesssim E_{|\alpha|+|a|+2}E_{[(|\alpha|+|a|)/2]+4} + Y_{|\alpha|+|a|+1}E_{[(|\alpha|+|a|)/2]+3}\\&\qquad + E_{|\alpha|+|a|+2}(X_{[(|\alpha|+|a|)/2]+4} + Y_{[(|\alpha|+|a|)/2]+3}).\end{aligned}$$

Let us first treat $\|t|f^2_{\alpha a}| + (t+r)|f^3_{\alpha a}|\|^2_{L^2}$. Recall that $f^2_{\alpha a}$ and $f^3_{\alpha a}$ were defined in (2.13). We need to estimate the norm in different regions separately. When $r \le \langle t\rangle/2$, we have $\langle t\rangle \lesssim \langle t-r\rangle$; thus

$$\begin{aligned}&\big\|t\big|f^2_{\alpha a}\big| + (t+r)\big|f^3_{\alpha a}\big|\big\|^2_{L^2(r\le\langle t\rangle/2)}\\&\quad\lesssim \sum_{\substack{\beta+\gamma=\alpha\\ b+c=a}} \big\|\langle t\rangle|\nabla U^{(\beta,b)}||\nabla U^{(\gamma,c)}|\big\|^2_{L^2(r\le\langle t\rangle/2)}.\end{aligned}$$

By the symmetry between the multi-index b and c and the symmetry between β and γ in the above, we assume $|c| + |\gamma| \le |b| + |\beta|$ without loss of generality. Thus $|\gamma| + |c| \le [(\alpha + |a|)/2]$. Hence thanks to (3.3), the above can be further bounded by

$$\begin{aligned}&\sum_{\substack{\beta+\gamma=\alpha, b+c=a\\ |\gamma|+|c|\le[(\alpha+|a|)/2]}} \|\nabla U^{(\beta,b)}\|^2_{L^2}\|\langle t\rangle\nabla U^{(\gamma,c)}\|^2_{L^\infty(r\le\langle t\rangle/2)}\\&\quad\lesssim E_{|\alpha|+|a|+1}X_{[(|\alpha|+|a|)/2]+3}.\end{aligned}$$

For $r \geq \langle t \rangle/2$, one infers by (3.16), (3.17), and Sobolev embedding that

$$\begin{aligned}
&\big\| t \big| f^2_{\alpha a} \big| + (t+r) \big| f^3_{\alpha a} \big| \big\|^2_{L^2(r \geq \langle t \rangle/2)} \\
&\lesssim \sum_{\substack{|\beta|+|\gamma| \leq |\alpha| \\ |b|+|c| \leq |a|}} \big\| |U^{(|\beta|,|b|+1)}| |U^{(|\gamma|,|c|+1)}| \big\|^2_{L^2(r \geq \langle t \rangle/2)} \\
&\lesssim E_{|\alpha|+|a|+1} E_{[(|\alpha|+|a|)/2]+3}.
\end{aligned}$$

Now we turn our attention to $\|t f^1_{\alpha a}\|_{L^2}$. Recalling that $f^1_{\alpha a}$ is defined in (2.13), by the L^2 boundedness of the Riesz transform, one has

$$\big\| t f^1_{\alpha a} \big\|_{L^2} \lesssim \sum_{1 \leq i,j \leq 2} \big\| t f^{ij}_{\alpha a} \big\|_{L^2},$$

where $f^{ij}_{\alpha a}$ is defined in Lemma 3.10. Hence in the following, we focus our attention on $\|t f^{ij}_{\alpha a}\|_{L^2}$. When $r \leq \langle t \rangle/2$, we can estimate similarly to $\|t f^2_{\alpha a}\|_{L^2(r \leq \langle t \rangle/2)}$ to deduce that

$$\big\| t f^{ij}_{\alpha a} \big\|^2_{L^2(r \leq \langle t \rangle/2)} \lesssim E_{|\alpha|+|a|+1} X_{[(|\alpha|+|a|)/2]+3}.$$

When $r \geq \langle t \rangle/2$, by (3.19), one has

$$\begin{aligned}
&\big\| t f^{ij}_{\alpha a} \big\|^2_{L^2(r \geq \langle t \rangle/2)} \\
&\lesssim \sum_{\substack{|b|+|c| \leq |a| \\ |\beta|+|\gamma| \leq |\alpha|}} \big\| |V^{(|\beta|,|b|+1)}| |V^{(|\gamma|,|c|+1)}| + |H^{(|\beta|,|b|+1)}| |H^{(|\gamma|,|c|+1)}| \big\|^2_{L^2} \\
&\quad + \sum_{\substack{b+c=a \\ \beta+\gamma=\alpha}} \big\| r |(\partial_r V^{(\beta,b)} + \partial_r H^{(\beta,b)} \cdot \omega)| \big(|\nabla V^{(\gamma,c)}| + |\nabla H^{(\gamma,c)}|\big) \big\|^2_{L^2} \\
&\quad + \sum_{\substack{b+c=a \\ \beta+\gamma=\alpha}} \big\| r \partial_r H^{(\beta,b)} \cdot \omega^\perp \partial_r H^{(\gamma,c)} \cdot \omega^\perp \big\|^2_{L^2}.
\end{aligned} \tag{3.20}$$

For the first and third terms on the right-hand side of (3.20), one can use the traditional Sobolev inequality to deduce that they are bounded by

$$E_{|\alpha|+|a|+1} E_{[(|\alpha|+|a|)/2]+3} + Y_{|\alpha|+|a|+1} E_{[(|\alpha|+|a|)/2]+3}.$$

The remaining second terms of (3.20) need further work. Making use of the fact that

$$\begin{aligned}
\partial_\theta (V^{(\alpha,a)} + H^{(\alpha,a)} \cdot \omega) &= \widetilde{\Omega} V^{(\alpha,a)} + \widetilde{\Omega} H^{(\alpha,a)} \cdot \omega \\
&= \widetilde{S}^\alpha \widetilde{\Omega} \Gamma^a V + \widetilde{S}^\alpha \widetilde{\Omega} \Gamma^a H \cdot \omega
\end{aligned}$$

and by (3.1), one gets

$$\begin{aligned}
&\|r(\partial_r V^{(\gamma,c)} + \partial_r H^{(\gamma,c)}\cdot\omega)\|^2_{L^\infty(r\ge\langle t\rangle/2)}\\
&\lesssim \sum_{d=0,1}\big\{\|\partial_r\Omega^d[r(\partial_r V^{(\gamma,c)} + \partial_r H^{(\gamma,c)}\cdot\omega)]\|^2_{L^2}\\
&\qquad\qquad + \|\Omega^d[r(\partial_r V^{(\gamma,c)} + \partial_r H^{(\gamma,c)}\cdot\omega)]\|^2_{L^2}\big\}\\
&\lesssim Y_{|\gamma|+|c|+3} + E_{|\gamma|+|c|+2}.
\end{aligned}$$

This allows us to control the second line of (3.20) as follows:

$$\begin{aligned}
&\sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} \|r(\partial_r V^{(\beta,b)} + \partial_r H^{(\beta,b)}\cdot\omega)(|\nabla V^{(\gamma,c)}| + |\nabla H^{(\gamma,c)}|)\|^2_{L^2(r\ge\langle t\rangle/2)}\\
&\lesssim \sum_{\substack{\beta+\gamma=\alpha, b+c=a\\ |\beta|+|b|\ge|\gamma|+|c|}} \|r(\partial_r V^{(\beta,b)} + \partial_r H^{(\beta,b)}\cdot\omega)\|^2_{L^2}\|\nabla U^{(\gamma,c)}\|^2_{L^\infty}\\
&\quad + \sum_{\substack{\beta+\gamma=\alpha, b+c=a\\ |\beta|+|b|<|\gamma|+|c|}} \|\nabla U^{(\gamma,c)}\|^2_{L^2}\|r(\partial_r V^{(\beta,b)} + \partial_r H^{(\beta,b)}\cdot\omega)\|^2_{L^\infty(r\ge\langle t\rangle/2)}\\
&\lesssim E_{|\alpha|+|a|+1}E_{[(|\alpha|+|a|)/2]+3} + Y_{|\alpha|+|a|+1}E_{[(|\alpha|+|a|)/2]+3}\\
&\quad + E_{|\alpha|+|a|+1}Y_{[(|\alpha|+|a|)/2]+3}.
\end{aligned}$$

Finally, we are going to show that

$$\big\|t\nabla\cdot f^2_{\alpha a}\big\|^2_{L^2} \lesssim E_{|\alpha|+|a|+2}(E_{[(|\alpha|+|a|)/2]+4} + X_{[(|\alpha|+|a|)/2]+4}).$$

For $r \le \langle t\rangle/2$, by (3.3) we have

$$\begin{aligned}
\big\|t\nabla\cdot f^2_{\alpha a}\big\|^2_{L^2(r\le\langle t\rangle/2)} &\lesssim \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} \|\langle t-r\rangle|\nabla^2 U^{(\beta,b)}||\nabla U^{(\gamma,c)}|\|^2_{L^2(r\le\langle t\rangle/2)}\\
&\lesssim E_{|\alpha|+|a|+2}X_{[(|\alpha|+|a|)/2]+4}.
\end{aligned}$$

For $r \ge \langle t\rangle/2$, one deduces by (3.18) that

$$\begin{aligned}
&\big\|t\nabla\cdot f^2_{\alpha a}\big\|^2_{L^2(r\ge\langle t\rangle/2)} \lesssim \big\|r\nabla\cdot f^2_{\alpha a}\big\|^2_{L^2(r\ge\langle t\rangle/2)}\\
&\lesssim \sum_{\substack{|b|+|c|\le|a|\\ |\beta|+|\gamma|\le|\alpha|}} \big\||V^{(|\beta|,|b|+2)}||H^{(|\gamma|,|c|+2)}|\big\|_{L^2(r\ge\langle t\rangle/2)}\\
&\lesssim E_{|\alpha|+|a|+2}E_{[(|\alpha|+|a|)/2]+4}.
\end{aligned}$$

This finishes the proof of the lemma. □

Now, we state a lemma that allows us to estimate the weighted L^2 norms:

LEMMA 3.14. *Suppose that* $(V, H) \in H_\Gamma^{\kappa-1}$ *solves* (1.6) *and* (1.7) *with* $\kappa \geq 12$. *Then there hold*

$$X_{\kappa-4} + Y_{\kappa-4} \lesssim E_{\kappa-3} + Y_{\kappa-4}E_{\kappa-3} + E_{\kappa-3}X_{\kappa-4} + E_{\kappa-3}E_{\kappa-4} \tag{3.21}$$

and

$$\begin{aligned} X_{\kappa-2} + Y_{\kappa-2} \lesssim\ & E_{\kappa-1} + E_{\kappa-1}X_{\kappa-4} + Y_{\kappa-2}E_{\kappa-4} \\ & + E_{\kappa-1}Y_{\kappa-4} + E_{\kappa-1}E_{\kappa-4}. \end{aligned} \tag{3.22}$$

PROOF. For the proof of this lemma, we recall and prove the following simple lemma:

LEMMA 3.15. *For vector* K, *there holds*

$$\|\langle t-r\rangle\nabla K\|_{L^2} \lesssim \|\langle t-r\rangle\nabla\cdot K\|_{L^2} + \|\langle t-r\rangle\nabla^\perp\cdot K\|_{L^2} + \|K\|_{L^2},$$

provided the right-hand side is finite.

PROOF. The proof is rather simple and the version for matrices has appeared in [31]. For completeness we include the proof for vector K. It suffices to prove the lemma for $K \in C_0^2(\mathbb{R}^2)$; the general case can be established by a completion procedure.

For any vector K, we write

$$|\nabla K|^2 = |\nabla\cdot K|^2 + |\nabla^\perp\cdot K|^2 - 2\partial_1 K_1\partial_2 K_2 + \partial_2 K_1\partial_1 K_2.$$

By integration by parts and Young's inequality, we have

$$\begin{aligned} &\|\langle t-r\rangle\nabla K\|_{L^2}^2 - \|\langle t-r\rangle\nabla\cdot K\|_{L^2}^2 - \|(t-r)\nabla^\perp\cdot K\|_{L^2}^2 \\ &\quad = \int_{\mathbb{R}^2} 2\langle t-r\rangle^2[-\partial_1(K_1\partial_2 K_2) + \partial_2(K_1\partial_1 K_2)]dx = \\ &\quad = \int_{\mathbb{R}^2} 4(t-r)[-\omega_1 K_1\partial_2 K_2 + \omega_2 K_1\partial_1 K_2]dx \\ &\quad \leq \frac{1}{2}\|\langle t-r\rangle\nabla K\|_{L^2}^2 + C\|K\|_{L^2}^2. \end{aligned}$$

The lemma then follows from the fact that the first term of the right-hand side can be absorbed by the left-hand side. □

We go back to the proof of Lemma 3.14. First we show that

$$\begin{aligned} & X_{|\alpha|+|a|+1} + Y_{|\alpha|+|a|+1} \\ &\quad \lesssim E_{|\alpha|+|a|+2} + E_{|\alpha|+|a|+2}(X_{[(|\alpha|+|a|)/2]+4} + Y_{[(|\alpha|+|a|)/2]+3}) \\ &\qquad + (X_{|\alpha|+|a|+1} + Y_{|\alpha|+|a|+1})E_{[(|\alpha|+|a|)/2]+3} \\ &\qquad + E_{|\alpha|+|a|+2}E_{[(|\alpha|+|a|)/2]+4}. \end{aligned} \tag{3.23}$$

Actually, by Lemma 3.13, we only need to show

$$\begin{aligned} X_{|\alpha|+|a|+1} + Y_{|\alpha|+|a|+1} \lesssim\ & E_{|\alpha|+|a|+2} + \|N_{|\alpha|+|a|+1}\|_{L^2}^2 \\ & + \|\mathcal{N}_{|\alpha|+|a|+2}\|_{L^2}^2. \end{aligned} \tag{3.24}$$

In view of the fact that

$$\nabla V^{(\alpha,a)} = \frac{1}{2}[\nabla V^{(\alpha,a)} + (\nabla \cdot H^{(\alpha,a)})\omega] + \frac{1}{2}[\nabla V^{(\alpha,a)} - (\nabla \cdot H^{(\alpha,a)})\omega],$$
$$(\nabla \cdot H^{(\alpha,a)})\omega = \frac{1}{2}[\nabla V^{(\alpha,a)} + (\nabla \cdot H^{(\alpha,a)})\omega] - \frac{1}{2}[\nabla V^{(\alpha,a)} - (\nabla \cdot H^{(\alpha,a)})\omega],$$

we deduce that

$$\begin{aligned}&\langle t-r\rangle(|\nabla V^{(\alpha,a)}| + |\nabla \cdot H^{(\alpha,a)}|)\\ &\quad \lesssim \langle t+r\rangle|\nabla V^{(\alpha,a)} + \nabla \cdot H^{(\alpha,a)}\omega| + |\langle t-r\rangle(\nabla V^{(\alpha,a)} - \nabla \cdot H^{(\alpha,a)}\omega)|.\end{aligned}$$

By (3.10), the above can be further bounded by

$$L_{|\alpha|+|a|+1} + N_{|\alpha|+|a|+1} + |r\partial_r V^{(\alpha,a)} + t\nabla \cdot H^{(\alpha,a)}|.$$

Hence by Lemma 3.6 and Lemma 3.15, we have

$$\begin{aligned}&\|\langle t-r\rangle\nabla V^{(\alpha,a)}\|_{L^2}^2 + \|\langle t-r\rangle\nabla H^{(\alpha,a)}\|_{L^2}^2\\ &\quad\lesssim \|\langle t-r\rangle\nabla V^{(\alpha,a)}\|_{L^2}^2 + \|\langle t-r\rangle\nabla \cdot H^{(\alpha,a)}\|_{L^2}^2\\ &\qquad + \|\langle t-r\rangle\nabla^{\perp} \cdot H^{(\alpha,a)}\|_{L^2}^2 + \|H^{(\alpha,a)}\|_{L^2}^2\\ &\quad\lesssim \nu X_{|\alpha|+|a|+1} + \|L_{|\alpha|+|a|+2}\|_{L^2}^2 + C_\nu\|N_{|\alpha|+|a|+1}\|_{L^2}^2 + C_\nu\|\mathcal{N}_{|\alpha|+|a|+2}\|_{L^2}^2\end{aligned}$$

for any positive ν. This further implies that

$$\begin{aligned}X_{|\alpha|+|a|+1} \lesssim{}& \nu X_{|\alpha|+|a|+1} + \|L_{|\alpha|+|a|+2}\|_{L^2}^2 + C_\nu\|N_{|\alpha|+|a|+1}\|_{L^2}^2\\ &+ C_\nu\|\mathcal{N}_{|\alpha|+|a|+2}\|_{L^2}^2.\end{aligned}$$

Taking $\nu > 0$ small enough, the first term on the right-hand side in the above is absorbed by the left-hand side. This yields

$$X_{|\alpha|+|a|+1} \lesssim \|L_{|\alpha|+|a|+2}\|_{L^2}^2 + \|N_{|\alpha|+|a|+1}\|_{L^2}^2 + \|\mathcal{N}_{|\alpha|+|a|+2}\|_{L^2}^2. \tag{3.25}$$

The estimate for $Y_{|\alpha|+|a|+1}$ in (3.24) is obvious from (3.11), (3.12), (3.25), and Lemma 3.6. Thus (3.24) is proved.

Now we turn to the proof of the first inequality in the lemma: Let $\kappa \geq 12$, $|\alpha| + |a| + 1 \leq \kappa - 4$; one has $[(|\alpha| + |a|)/2] + 4 \leq \kappa - 4$. Hence, by (3.23), we have

$$X_{\kappa-4} + Y_{\kappa-4} \lesssim E_{\kappa-3} + Y_{\kappa-4}E_{\kappa-3} + E_{\kappa-3}X_{\kappa-4} + E_{\kappa-3}E_{\kappa-4}.$$

Next, for $|\alpha| + |a| + 1 \leq \kappa - 2$, there holds $[(|a| + |\alpha|)/2] + 4 \leq \kappa - 4$. Hence one can derive from (3.23) that

$$X_{\kappa-2} + Y_{\kappa-2} \lesssim E_{\kappa-1} + E_{\kappa-1}X_{\kappa-4} + Y_{\kappa-2}E_{\kappa-4} + E_{\kappa-1}Y_{\kappa-4} + E_{\kappa-1}E_{\kappa-4}.$$

□

The following lemma gives the control of weighted generalized energies and weighted good quantity energies in terms of Klainerman's generalized ones. Note that we have one derivative loss with respect to similar estimates in [28, 31, 41].

LEMMA 3.16. *Suppose that* $(V, H) \in H_\Gamma^{\kappa-1}$ *solves* (1.6) *and* (1.7) *with* $\kappa \geq 12$, *and suppose* $E_{\kappa-3} \ll 1$. *Then, we have*

$$X_{\kappa-4} + Y_{\kappa-4} \lesssim E_{\kappa-3}, \quad X_{\kappa-2} + Y_{\kappa-2} \lesssim E_{\kappa-1}.$$

Remark 3.17. When $\mu = 0$, we can modify Lemma 3.13 and Lemma 3.14 and finally get a non-derivative-loss version of Lemma 3.16:

$$X_{\kappa-3} + Y_{\kappa-3} \lesssim E_{\kappa-3}, \quad X_{\kappa-1} + Y_{\kappa-1} \lesssim E_{\kappa-1}.$$

PROOF. The first estimate follows from (3.21) and the assumption $E_{\kappa-3} \ll 1$. The second one follows from (3.22), the assumption, and the obtained first estimate. $\square$

3.4 Strengthened L^∞ Estimate for the Good Unknowns

We now complete the decay estimate for the L^∞ of the good unknowns $\partial V + \partial H \cdot \omega$ and $\partial H \cdot \omega^\perp$ near the light cone.

LEMMA 3.18. *Suppose that* $(V, H) \in H_\Gamma^{\kappa-1}$ *solves* (1.6) *and* (1.7) *with* $\kappa \geq 12$ *and suppose* $E_{\kappa-3} \ll 1$. *Then for all* $|\alpha| + |a| \leq \kappa - 7$ *and for* $i = 1, 2$, *we have*

$$\langle t\rangle^{3/2} \big\| |\nabla_i V^{(\alpha,a)} + \nabla_i H^{(\alpha,a)} \cdot \omega| + |\nabla_i H^{(\alpha,a)} \cdot \omega^\perp| \big\|_{L^\infty(r \geq \langle t\rangle/2)} \lesssim E_{\kappa-3}^{1/2}. \tag{3.26}$$

PROOF. In view of (2.17) and (3.1), we only need to show

$$\langle t\rangle^{3/2} \big\| |\partial_r V^{(\alpha,a)} + \partial_r H^{(\alpha,a)} \cdot \omega| + |\partial_r H^{(\alpha,a)} \cdot \omega^\perp| \big\|_{L^\infty(r \geq \langle t\rangle/2)} \lesssim E_{\kappa-3}^{1/2}.$$

Note that

$$\partial_\theta (V^{(\alpha,a)} + H^{(\alpha,a)} \cdot \omega) = \widetilde{\Omega} V^{(\alpha,a)} + \widetilde{\Omega} H^{(\alpha,a)} \cdot \omega.$$

By (3.1) and Lemma 3.16, one gets

$$\begin{aligned}
&\|r^{3/2}(\partial_r V^{(\alpha,a)} + \partial_r H^{(\alpha,a)} \cdot \omega)\|^2_{L^\infty(r \geq \langle t\rangle/2)} \\
&\quad \lesssim \sum_{d=0,1} \Big\{ \big\|\partial_r \Omega^d \big[r\big(\partial_r V^{(\alpha,a)} + \partial_r H^{(\alpha,a)} \cdot \omega\big)\big]\big\|^2_{L^2} \\
&\qquad\qquad + \|\Omega^d \big[r(\partial_r V^{(\alpha,a)} + \partial_r H^{(\alpha,a)} \cdot \omega)\big]\|^2_{L^2} \Big\} \\
&\quad \lesssim \sum_{d=0,1} \Big\{ \big\|r\big(\partial_r^2 \widetilde{\Omega}^d V^{(\alpha,a)} + \partial_r^2 \widetilde{\Omega}^d H^{(\alpha,a)} \cdot \omega\big)\big\|^2_{L^2} \\
&\qquad\qquad + \|\partial_r \widetilde{\Omega}^d V^{(\alpha,a)} + \partial_r \widetilde{\Omega}^d H^{(\alpha,a)} \cdot \omega\|^2_{L^2} \\
&\qquad\qquad + \|r(\partial_r \widetilde{\Omega}^d V^{(\alpha,a)} + \partial_r \widetilde{\Omega}^d H^{(\alpha,a)} \cdot \omega)\|^2_{L^2} \Big\} \\
&\quad \lesssim Y_{|\alpha|+|a|+3} + E_{|\alpha|+|a|+2} \lesssim E_{\kappa-3}.
\end{aligned}$$

The first part of (3.26) is clear from the fact that $r \geq \langle t\rangle/2$. The proof for the remaining part of the inequality is similar. We omit the details. $\square$

4 Energy Estimate

This section is devoted to the energy estimate. We split the proof into three subsections, which correspond to the highest-order modified energy estimate, the highest-order standard energy estimate, and the lower-order standard energy estimate, respectively. Here in this section, both $\mathcal{E}_\kappa$ and E_κ will be called energies. To avoid confusion, we will call E_κ the standard energy and still call $\mathcal{E}_\kappa$ the modified energy.

4.1 Higher-Order Modified Energy Estimate

We first take care of the highest-order modified energy estimate. One needs to be very careful about the derivative loss problem. Ignoring the diffusion, at first glance, we will always lose one derivative in the highest-order modified energy estimate due to the fully nonlinear effect. The nonlocal effect and the application of the ghost weight make this problem even more complicated. Luckily, a delicate analysis of the nonlinearities shows that the system has the requisite symmetry, which is hidden in the Riesz transform. Actually, we can integrate by parts in a way that will produce a Laplacian operator in the worst terms (the worst terms refer to the terms with a derivative loss; the other terms do not have such problems and the null structure is satisfied). Moreover, after gaining the one derivative, the null condition is present again. Then we can take full advantage of this condition by the ghost weight method.

Let $\kappa \geq 12$, $|\alpha| + |a| \leq \kappa - 1$, $\sigma = r - t$, and $q(\sigma) = \arctan\sigma$. We write $e^q = e^{q(\sigma)}$ for simplicity. After applying ∇ to (2.14), we take the L^2 inner product of the first and second equations of the resulting system with $\nabla V^{(\alpha,a)} e^q$ and $\nabla H^{(\alpha,a)} e^q$, respectively, then adding them up, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|\nabla V^{(\alpha,a)}|^2 + |\nabla H^{(\alpha,a)}|^2)e^q\,dx \\
&\quad - \int_{\mathbb{R}^2}\mu\nabla\Delta\sum_{l=0}^{\alpha}C_\alpha^l(-1)^{\alpha-l}V^{(l,a)}\cdot\nabla V^{(\alpha,a)}e^q\,dx \\
&\quad + \frac{1}{2}\sum_{1\leq i\leq 2}\int_{\mathbb{R}^2}\frac{|\nabla_i V^{(\alpha,a)} + \nabla_i H^{(\alpha,a)}\cdot\omega|^2 + |\nabla_i H^{(\alpha,a)}\cdot\omega^\perp|^2}{\langle t-r\rangle^2}e^q\,dx \\
&= \int_{\mathbb{R}^2}\big(\nabla f^1_{\alpha a}\cdot\nabla V^{(\alpha,a)} + \nabla f^2_{\alpha a}:\nabla H^{(\alpha,a)}\big)e^q\,dx = I_1 + I_2,
\end{aligned}\tag{4.1}$$

where

$$\begin{aligned}
I_1 = &\int_{\mathbb{R}^2}\nabla\big[\nabla^\perp\cdot\big(\nabla\cdot\Delta^{-1}\big(-\nabla^\perp V^{(\alpha,a)}\otimes\nabla^\perp V + \nabla^\perp H^{(\alpha,a)}\otimes\nabla^\perp H\big) \\
&\qquad + \nabla^\perp\cdot\nabla\cdot\Delta^{-1}\big(-\nabla^\perp V\otimes\nabla^\perp V^{(\alpha,a)} + \nabla^\perp H\otimes\nabla^\perp H^{(\alpha,a)}\big)\big] \\
&\quad\cdot\nabla V^{(\alpha,a)}e^q\,dx \\
&+ \int_{\mathbb{R}^2}\nabla(\nabla^\perp H^{(\alpha,a)}\nabla V + \nabla^\perp H\nabla V^{(\alpha,a)}):\nabla H^{(\alpha,a)}e^q\,dx,
\end{aligned}$$

and

$$I_2 = \sum_{\substack{\beta+\gamma=\alpha, b+c=a \\ |\beta|+|b|, \\ |\gamma|+|c|<|\alpha|+|a|}} \Big\{ C_\alpha^\beta C_a^b \int_{\mathbb{R}^2} \nabla\big[\nabla^\perp \cdot \nabla \cdot \Delta^{-1}\big(-\nabla^\perp V^{(\beta,b)} \otimes \nabla^\perp V^{(\gamma,c)} + \nabla^\perp H^{(\beta,b)} \otimes \nabla^\perp H^{(\gamma,c)}\big)\big] \cdot \nabla V^{(\alpha,a)} e^q \, dx + C_\alpha^\beta C_a^b \int_{\mathbb{R}^2} \nabla(\nabla^\perp H^{(\beta,b)} \nabla V^{(\gamma,c)}) : \nabla H^{(\alpha,a)} e^q \, dx \Big\}.$$

Here I_1 consists of the terms that contain the highest-order derivatives, namely when all derivatives hit the same factor in the nonlinear term. To avoid notational confusion, we mention that $\nabla_i H^{(\alpha,a)} \cdot \omega$ and $\nabla_i H^{(\alpha,a)} \cdot \omega^\perp$ appearing in the ghost weight energy (4.2) mean $(\nabla_i H^{(\alpha,a)}) \cdot \omega$ and $(\nabla_i H^{(\alpha,a)}) \cdot \omega^\perp$. This notation convention will always be used in the following argument. Also, we define (see the third line of (4.1))

$$G_\kappa(t) := \sum_{|\alpha|+|a|\le\kappa-1} \sum_{1\le i\le 2} \int_{\mathbb{R}^2} \frac{|\nabla_i V^{(\alpha,a)} + \nabla_i H^{(\alpha,a)} \cdot \omega|^2 + |\nabla_i H^{(\alpha,a)} \cdot \omega^\perp|^2}{\langle t-r\rangle^2} e^q \, dx. \tag{4.2}$$

Step 1. Estimate of the highest-order term I_1.

We divide I_1 into five terms:

$$I_1 = I_{11} + I_{12} + I_{13} + I_{14} + I_{15},$$

where

$$\begin{aligned}
I_{11} &= -\int_{\mathbb{R}^2} \nabla\big[\nabla^\perp \cdot \nabla \cdot \Delta^{-1}(\nabla^\perp V^{(\alpha,a)} \otimes \nabla^\perp V)\big] \cdot \nabla V^{(\alpha,a)} e^q \, dx, \\
I_{12} &= \int_{\mathbb{R}^2} \nabla\big[\nabla^\perp \cdot \nabla \cdot \Delta^{-1}(\nabla^\perp H^{(\alpha,a)} \otimes \nabla^\perp H)\big] \cdot \nabla V^{(\alpha,a)} e^q \, dx, \\
I_{13} &= -\int_{\mathbb{R}^2} \nabla\big[\nabla^\perp \cdot \nabla \cdot \Delta^{-1}(\nabla^\perp V \otimes \nabla^\perp V^{(\alpha,a)})\big] \cdot \nabla V^{(\alpha,a)} e^q \, dx, \\
I_{14} &= \int_{\mathbb{R}^2} \nabla\big[\nabla^\perp \cdot \nabla \cdot \Delta^{-1}(\nabla^\perp H \otimes \nabla^\perp H^{(\alpha,a)})\big] \cdot \nabla V^{(\alpha,a)} e^q \, dx, \\
I_{15} &= \int_{\mathbb{R}^2} \nabla\big(\nabla^\perp H^{(\alpha,a)} \nabla V + \nabla^\perp H \nabla V^{(\alpha,a)}\big) : \nabla H^{(\alpha,a)} e^q \, dx.
\end{aligned}$$

Now we transform I_{11} to I_{15} step by step. The goal is take advantage of the symmetric nature of the original system to get rid of the derivative loss that appears. Due to the good property of the original system, it is expected that one can get rid of the derivative loss; however, this requires some lengthy calculations. For I_{11}, we deduce by integration by parts that

$$\begin{aligned}
I_{11} &= -\int_{\mathbb{R}^2} \nabla\nabla^\perp \cdot \nabla \cdot \Delta^{-1}\big(\nabla^\perp V^{(\alpha,a)} \otimes \nabla^\perp V\big) \cdot \nabla V^{(\alpha,a)} e^q \, dx \\
&= -\int_{\mathbb{R}^2} \nabla_k \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp V^{(\alpha,a)} \nabla_j^\perp V\big) \nabla_k V^{(\alpha,a)} e^q \, dx =
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k \nabla_i^\perp V^{(\alpha,a)} \nabla_j^\perp V\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&\quad - \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp V^{(\alpha,a)} \nabla_k \nabla_j^\perp V\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&= \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k V^{(\alpha,a)} \nabla_i^\perp \nabla_j^\perp V\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&\quad - \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k V^{(\alpha,a)} \nabla_j^\perp V\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&\quad - \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp V^{(\alpha,a)} \nabla_k \nabla_j^\perp V\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&= \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k V^{(\alpha,a)} \nabla_i^\perp \nabla_j^\perp V\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}|^2 \nabla_j \big(\nabla_j^\perp V e^q\big) dx \\
&\quad - \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp V^{(\alpha,a)} \nabla_k \nabla_j^\perp V\big) \nabla_k V^{(\alpha,a)} e^q \, dx.
\end{aligned}
$$

Next, for I_{12} we get by integration by parts that

$$
\begin{aligned}
I_{12} &= -\int_{\mathbb{R}^2} \nabla \nabla^\perp \cdot \nabla \cdot \Delta^{-1}\big(\nabla^\perp V \otimes \nabla^\perp V^{(\alpha,a)}\big) \cdot \nabla V^{(\alpha,a)} e^q \, dx \\
&= -\int_{\mathbb{R}^2} \nabla_k \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp V \nabla_j^\perp V^{(\alpha,a)}\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&= -\int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k \nabla_i^\perp V \nabla_j^\perp V^{(\alpha,a)}\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&\quad - \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp V \nabla_k \nabla_j^\perp V^{(\alpha,a)}\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&= -\int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k \nabla_i^\perp V \nabla_j^\perp V^{(\alpha,a)}\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&\quad + \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp \nabla_j^\perp V \nabla_k V^{(\alpha,a)}\big) \nabla_k V^{(\alpha,a)} e^q \, dx.
\end{aligned}
$$

Then for I_{13}, we write

$$
\begin{aligned}
I_{13} &= \int_{\mathbb{R}^2} \nabla \nabla^\perp \cdot \nabla \cdot \Delta^{-1}\big(\nabla^\perp H^{(\alpha,a)} \otimes \nabla^\perp H\big) \cdot \nabla V^{(\alpha,a)} e^q \, dx \\
&= \int_{\mathbb{R}^2} \nabla_k \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp H^{(\alpha,a)} \cdot \nabla_j^\perp H\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&= \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k \nabla_i^\perp H^{(\alpha,a)} \cdot \nabla_j^\perp H\big) \nabla_k V^{(\alpha,a)} e^q \, dx \\
&\quad + \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp H^{(\alpha,a)} \cdot \nabla_k \nabla_j^\perp H\big) \nabla_k V^{(\alpha,a)} e^q \, dx =
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} \nabla_i^{\perp}\nabla_i^{\perp}\nabla_j\Delta^{-1}\big(\nabla_k H^{(\alpha,a)}\cdot\nabla_j^{\perp}H\big)\nabla_k V^{(\alpha,a)}e^q\,dx\\
&\quad - \int_{\mathbb{R}^2} \nabla_i^{\perp}\nabla_j\Delta^{-1}\big(\nabla_k H^{(\alpha,a)}\cdot\nabla_i^{\perp}\nabla_j^{\perp}H\big)\nabla_k V^{(\alpha,a)}e^q\,dx\\
&\quad + \int_{\mathbb{R}^2} \nabla_i^{\perp}\nabla_j\Delta^{-1}\big(\nabla_i^{\perp}H^{(\alpha,a)}\cdot\nabla_k\nabla_j^{\perp}H\big)\nabla_k V^{(\alpha,a)}e^q\,dx\\
&= -\int_{\mathbb{R}^2} \nabla_k H^{(\alpha,a)}\cdot\nabla_j^{\perp}H\,\nabla_j\big(\nabla_k V^{(\alpha,a)}e^q\big)dx\\
&\quad - \int_{\mathbb{R}^2} \nabla_i^{\perp}\nabla_j\Delta^{-1}\big(\nabla_k H^{(\alpha,a)}\cdot\nabla_i^{\perp}\nabla_j^{\perp}H\big)\nabla_k V^{(\alpha,a)}e^q\,dx\\
&\quad + \int_{\mathbb{R}^2} \nabla_i^{\perp}\nabla_j\Delta^{-1}\big(\nabla_i^{\perp}H^{(\alpha,a)}\cdot\nabla_k\nabla_j^{\perp}H\big)\nabla_k V^{(\alpha,a)}e^q\,dx.
\end{aligned}
$$

For I_{14}, we deduce similarly to I_{12} that

$$
\begin{aligned}
I_{14} &= \int_{\mathbb{R}^2} \nabla\nabla^{\perp}\cdot\nabla\cdot\Delta^{-1}\big(\nabla^{\perp}H\otimes\nabla^{\perp}H^{(\alpha,a)}\big)\cdot\nabla V^{(\alpha,a)}e^q\,dx\\
&= \int_{\mathbb{R}^2} \nabla_k\nabla_i^{\perp}\nabla_j\Delta^{-1}\big(\nabla_i^{\perp}H\cdot\nabla_j^{\perp}H^{(\alpha,a)}\big)\nabla_k V^{(\alpha,a)}e^q\,dx\\
&= \int_{\mathbb{R}^2} \nabla_i^{\perp}\nabla_j\Delta^{-1}\big(\nabla_k\nabla_i^{\perp}H\cdot\nabla_j^{\perp}H^{(\alpha,a)}\big)\nabla_k V^{(\alpha,a)}e^q\,dx\\
&\quad + \int_{\mathbb{R}^2} \nabla_i^{\perp}\nabla_j\Delta^{-1}\big(\nabla_i^{\perp}H\cdot\nabla_k\nabla_j^{\perp}H^{(\alpha,a)}\big)\nabla_k V^{(\alpha,a)}e^q\,dx\\
&= \int_{\mathbb{R}^2} \nabla_i^{\perp}\nabla_j\Delta^{-1}\big(\nabla_k\nabla_i^{\perp}H\cdot\nabla_j^{\perp}H^{(\alpha,a)}\big)\nabla_k V^{(\alpha,a)}e^q\,dx\\
&\quad - \int_{\mathbb{R}^2} \nabla_i^{\perp}\nabla_j\Delta^{-1}\big(\nabla_i^{\perp}\nabla_j^{\perp}H\cdot\nabla_k H^{(\alpha,a)}\big)\nabla_k V^{(\alpha,a)}e^q\,dx.
\end{aligned}
$$

For I_{15}, we have

$$
\begin{aligned}
I_{15} &= \int_{\mathbb{R}^2} \nabla_k\big(\nabla_j^{\perp}H_i^{(\alpha,a)}\nabla_j V + \nabla_j^{\perp}H_i\nabla_j V^{(\alpha,a)}\big)\nabla_k H_i^{(\alpha,a)}e^q\,dx\\
&= \int_{\mathbb{R}^2} (\nabla_k\nabla_j^{\perp}H_i^{(\alpha,a)}\nabla_j V + \nabla_j^{\perp}H_i^{(\alpha,a)}\nabla_k\nabla_j V)\nabla_k H_i^{(\alpha,a)}e^q\,dx\\
&\quad + \int_{\mathbb{R}^2} (\nabla_j^{\perp}H_i\nabla_k\nabla_j V^{(\alpha,a)} + \nabla_k\nabla_j^{\perp}H_i\nabla_j V^{(\alpha,a)})\nabla_k H_i^{(\alpha,a)}e^q\,dx\\
&= -\frac{1}{2}\int_{\mathbb{R}^2} |\nabla H^{(\alpha,a)}|^2\nabla_j^{\perp}(\nabla_j V e^q)dx\\
&\quad + \int_{\mathbb{R}^2} \nabla_j^{\perp}H_i\nabla_k\nabla_j V^{(\alpha,a)}\nabla_k H_i^{(\alpha,a)}e^q\,dx\\
&\quad + \int_{\mathbb{R}^2} (\nabla_j^{\perp}H_i^{(\alpha,a)}\nabla_k\nabla_j V + \nabla_k\nabla_j^{\perp}H_i\nabla_j V^{(\alpha,a)})\nabla_k H_i^{(\alpha,a)}e^q\,dx.
\end{aligned}
$$

Inserting the above equalities from I_{11} to I_{15} into I_1, we get

$$
\begin{aligned}
I_1 = &\int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp H^{(\alpha,a)} \cdot \nabla_k \nabla_j^\perp H \\
&\qquad\qquad - \nabla_i^\perp V^{(\alpha,a)} \nabla_k \nabla_j^\perp V\big)\nabla_k V^{(\alpha,a)} e^q \, dx \\
&+ \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k V^{(\alpha,a)} \nabla_i^\perp \nabla_j^\perp V \\
&\qquad\qquad - \nabla_k H^{(\alpha,a)} \cdot \nabla_i^\perp \nabla_j^\perp H\big)\nabla_k V^{(\alpha,a)} e^q \, dx \\
&+ \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k \nabla_i^\perp H \cdot \nabla_j^\perp H^{(\alpha,a)} \\
&\qquad\qquad - \nabla_k \nabla_i^\perp V \nabla_j^\perp V^{(\alpha,a)}\big)\nabla_k V^{(\alpha,a)} e^q \, dx \\
&+ \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp \nabla_j^\perp V \nabla_k V^{(\alpha,a)} \\
&\qquad\qquad - \nabla_i^\perp \nabla_j^\perp H \cdot \nabla_k H^{(\alpha,a)}\big)\nabla_k V^{(\alpha,a)} e^q \, dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}|^2 \nabla_j \big(\nabla_j^\perp V e^q\big) dx \\
&- \frac{1}{2}\int_{\mathbb{R}^2} |\nabla H^{(\alpha,a)}|^2 \nabla_j^\perp \big(\nabla_j V e^q\big) dx \\
&- \int_{\mathbb{R}^2} \nabla_k H^{(\alpha,a)} \cdot \nabla_j^\perp H \nabla_j \big(\nabla_k V^{(\alpha,a)} e^q\big) dx \\
&+ \int_{\mathbb{R}^2} \nabla_j^\perp H_i \nabla_k \nabla_j V^{(\alpha,a)} \nabla_k H_i^{(\alpha,a)} e^q \, dx \\
&+ \int_{\mathbb{R}^2} \big(\nabla_j^\perp H_i^{(\alpha,a)} \nabla_k \nabla_j V \\
&\qquad + \nabla_k \nabla_j^\perp H_i \nabla_j V^{(\alpha,a)}\big) \nabla_k H_i^{(\alpha,a)} e^q \, dx.
\end{aligned}
\tag{4.3}
$$

In view of (4.3), one can see that we have gained one derivative compared with the original expression. We still need to make the strong null condition appear.

We start by treating the first four lines on the right-hand side of (4.3). It is obvious that they have the same structure, so we only treat the first one. By the L^2 boundedness of the Riesz transform, the first term is bounded by

$$
\sum_{1\le i,j,k\le 2} \big\|\nabla_i H^{(\alpha,a)} \cdot \nabla_k \nabla_j H - \nabla_i V^{(\alpha,a)} \nabla_k \nabla_j V\big\|_{L^2} \|\nabla V^{(\alpha,a)}\|_{L^2}. \tag{4.4}
$$

Now, to see the strong null condition, we apply the orthogonal decomposition to the radial and transverse directions:

$$
\begin{aligned}
&\nabla_i H^{(\alpha,a)} \cdot \nabla_k \nabla_j H - \nabla_i V^{(\alpha,a)} \nabla_k \nabla_j V \\
&\quad = \nabla_i H^{(\alpha,a)} \cdot \omega \nabla_k \nabla_j H \cdot \omega + \nabla_i H^{(\alpha,a)} \cdot \omega^\perp \nabla_k \nabla_j H \cdot \omega^\perp \\
&\qquad - \nabla_i V^{(\alpha,a)} \nabla_k \nabla_j V =
\end{aligned}
\tag{4.5}
$$

$$= (\nabla_i V^{(\alpha,a)} + \nabla_i H^{(\alpha,a)} \cdot \omega)\nabla_k \nabla_j H \cdot \omega$$
$$- \nabla_i V^{(\alpha,a)}(\nabla_k \nabla_j V + \nabla_k \nabla_j H \cdot \omega) + \nabla_i H^{(\alpha,a)} \cdot \omega^\perp \nabla_k \nabla_j H \cdot \omega^\perp.$$

Now we are ready to estimate (4.4). When integrating over the domain $\{r \geq \langle t\rangle/2\}$, we use (3.2), Lemma 3.16, and Lemma 3.18 to get

$$\begin{aligned}
&\sum_{1\leq i,j,k\leq 2} \|\nabla_i H^{(\alpha,a)} \cdot \nabla_k \nabla_j H - \nabla_i V^{(\alpha,a)} \nabla_k \nabla_j V\|_{L^2(r\geq \langle t\rangle/2)} \mathcal{E}_\kappa^{1/2} \\
&\lesssim \sum_{1\leq i,j,k\leq 2} \Bigg(\left\| \frac{\nabla_i V^{(\alpha,a)} + \nabla_i H^{(\alpha,a)} \cdot \omega}{\langle r-t\rangle} \right\|_{L^2} \|\langle r-t\rangle \nabla_k \nabla_j H \cdot \omega\|_{L^\infty(r\geq\langle t\rangle/2)} \mathcal{E}_\kappa^{1/2} \\
&\qquad + \|\nabla_i V^{(\alpha,a)}\|_{L^2} \|\nabla_k \nabla_j V + \nabla_k \nabla_j H \cdot \omega\|_{L^\infty(r\geq\langle t\rangle/2)} \mathcal{E}_\kappa^{1/2} \\
&\qquad + \|\nabla_i H^{(\alpha,a)} \cdot \omega^\perp\|_{L^2} \|\nabla_k \nabla_j H \cdot \omega^\perp\|_{L^\infty(r\geq\langle t\rangle/2)} \mathcal{E}_\kappa^{1/2} \Bigg) \\
&\lesssim \eta G_\kappa + C_\eta \langle t\rangle^{-1} \mathcal{E}_\kappa E_{\kappa-3} + \langle t\rangle^{-3/2} \mathcal{E}_\kappa E_{\kappa-3}^{1/2}
\end{aligned} \tag{4.6}$$

for any $\eta > 0$. For the region $\{r \leq \langle t\rangle/2\}$, the bound for (4.4) is easier. By (3.3) and Lemma 3.16, one has

$$\begin{aligned}
&\sum_{1\leq i,j,k\leq 2} \|\nabla_i H^{(\alpha,a)} \cdot \nabla_k \nabla_j H - \nabla_i V^{(\alpha,a)} \nabla_k \nabla_j V\|_{L^2(r\leq\langle t\rangle/2)} \\
&\leq \left\| |\nabla H^{(\alpha,a)}||\nabla^2 H| + |\nabla V^{(\alpha,a)}||\nabla^2 V| \right\|_{L^2(r\leq\langle t\rangle/2)} \\
&\leq \|\nabla H^{(\alpha,a)}\|_{L^2} \|\nabla^2 H\|_{L^\infty(r\leq\langle t\rangle/2)} \\
&\quad + \|\nabla V^{(\alpha,a)}\|_{L^2} \|\nabla^2 V\|_{L^\infty(r\leq\langle t\rangle/2)} \\
&\lesssim \langle t\rangle^{-1} \mathcal{E}_\kappa^{1/2} X_{\kappa-4}^{1/2} \lesssim \langle t\rangle^{-1} \mathcal{E}_\kappa^{1/2} E_{\kappa-3}^{1/2}.
\end{aligned} \tag{4.7}$$

Consequently, inserting (4.6) and (4.7) into (4.4) gives that

$$\begin{aligned}
&\sum_{1\leq i,j,k\leq 2} \left\| \nabla_i H^{(\alpha,a)} \cdot \nabla_k \nabla_j H - \nabla_i V^{(\alpha,a)} \nabla_k \nabla_j V \right\|_{L^2} \|\nabla V^{(\alpha,a)}\|_{L^2} \\
&\quad \lesssim \eta G_\kappa + C_\eta \langle t\rangle^{-1} \mathcal{E}_\kappa E_{\kappa-3}^{1/2}.
\end{aligned}$$

Here we have used the a priori assumption $E_{\kappa-3} \ll 1$.

Then, we estimate the fifth and sixth integrals of (4.3). For these two integrals, the decay is much better. Direct integration by parts shows that they are equal to

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}^2} (|\nabla H^{(\alpha,a)}|^2 + |\nabla V^{(\alpha,a)}|^2) \partial_\theta V \frac{e^q}{r\langle t-r\rangle^2}\,dx \\
&+ \int_{\mathbb{R}^2} \nabla_k H^{(\alpha,a)} \cdot \partial_\theta H \nabla_k V^{(\alpha,a)} \frac{e^q}{r\langle t-r\rangle^2}\,dx.
\end{aligned}$$

By (3.1), the above can be further estimated by

$$\langle t\rangle^{-3/2} \mathcal{E}_\kappa E_{\kappa-3}^{1/2}.$$

Finally, we treat the last two integrals of (4.3). To show the null structure, we employ the orthogonal decomposition into radial and transverse directions to get that

$$\begin{aligned}
&\big(\nabla_j^{\perp} H_i^{(\alpha,a)} \nabla_k \nabla_j V + \nabla_k \nabla_j^{\perp} H_i \nabla_j V^{(\alpha,a)}\big) \nabla_k H_i^{(\alpha,a)} \\
&\quad = \big(\nabla_j^{\perp} H^{(\alpha,a)} \cdot \omega \nabla_k \nabla_j V + \nabla_k \nabla_j^{\perp} H \cdot \omega \nabla_j V^{(\alpha,a)}\big) \nabla_k H^{(\alpha,a)} \cdot \omega \\
&\qquad + \big(\nabla_j^{\perp} H^{(\alpha,a)} \cdot \omega^{\perp} \nabla_k \nabla_j V + \nabla_k \nabla_j^{\perp} H \cdot \omega^{\perp} \nabla_j V^{(\alpha,a)}\big) \nabla_k H^{(\alpha,a)} \cdot \omega^{\perp}.
\end{aligned} \tag{4.8}$$

For the expression inside the parentheses on the first line of the right-hand side of (4.8), one can rewrite it as follows:

$$\begin{aligned}
&\nabla_j^{\perp} H^{(\alpha,a)} \cdot \omega \nabla_k \nabla_j V + \nabla_k \nabla_j^{\perp} H \cdot \omega \nabla_j V^{(\alpha,a)} \\
&\quad = \big(\nabla_j^{\perp} H^{(\alpha,a)} \cdot \omega + \nabla_j^{\perp} V^{(\alpha,a)}\big) \nabla_k \nabla_j V + (\nabla_k \nabla_j^{\perp} H \cdot \omega + \nabla_k \nabla_j^{\perp} V) \nabla_j V^{(\alpha,a)} \\
&\qquad - \nabla_j^{\perp} V^{(\alpha,a)} \nabla_k \nabla_j V - \nabla_k \nabla_j^{\perp} V \nabla_j V^{(\alpha,a)} \\
&\quad = \big(\nabla_j^{\perp} H^{(\alpha,a)} \cdot \omega + \nabla_j^{\perp} V^{(\alpha,a)}\big) \nabla_k \nabla_j V + \big(\nabla_k \nabla_j^{\perp} H \cdot \omega + \nabla_k \nabla_j^{\perp} V\big) \nabla_j V^{(\alpha,a)}.
\end{aligned}$$

Here we have used the fact that

$$\nabla_j^{\perp} V^{(\alpha,a)} \nabla_k \nabla_j V + \nabla_k \nabla_j^{\perp} V \nabla_j V^{(\alpha,a)} = 0.$$

Hence, for the last line of (4.3), by (3.2), Lemma 3.16, and Lemma 3.18, we can estimate the integral over the region $\{r \geq \langle t\rangle/2\}$ by

$$\begin{aligned}
&\big\| \big(\nabla_j^{\perp} H^{(\alpha,a)} \cdot \omega + \nabla_j^{\perp} V^{(\alpha,a)}\big) \nabla_k \nabla_j V \big\|_{L^2(r \geq \langle t\rangle/2)} \big\| \nabla_k H^{(\alpha,a)} \cdot \omega \big\|_{L^2} \\
&\quad + \big\| \big(\nabla_k \nabla_j^{\perp} H \cdot \omega + \nabla_k \nabla_j^{\perp} V\big) \nabla_j V^{(\alpha,a)} \big\|_{L^2(r \geq \langle t\rangle/2)} \| \nabla_k H^{(\alpha,a)} \cdot \omega \|_{L^2} \\
&\quad + \big\| \big(\nabla_j^{\perp} H^{(\alpha,a)} \cdot \omega^{\perp} \nabla_k \nabla_j V + \nabla_k \nabla_j^{\perp} H \cdot \omega^{\perp} \nabla_j V^{(\alpha,a)}\big) \nabla_k H^{(\alpha,a)} \cdot \omega^{\perp} \big\|_{L^1(r \geq \langle t\rangle/2)} \\
&\quad \lesssim \left\| \frac{\nabla_j^{\perp} H^{(\alpha,a)} \cdot \omega + \nabla_j^{\perp} V^{(\alpha,a)}}{\langle t-r\rangle} \right\|_{L^2} \| \langle t-r\rangle \nabla^2 V \|_{L^\infty(r \geq \langle t\rangle/2)} \mathcal{E}_\kappa^{1/2} \\
&\qquad + \| \nabla_k \nabla_j^{\perp} H \cdot \omega + \nabla_k \nabla_j^{\perp} V \|_{L^\infty(r \geq \langle t\rangle/2)} \| \nabla V^{(\alpha,a)} \|_{L^2} \mathcal{E}_\kappa^{1/2} \\
&\qquad + \left\| \frac{\nabla_k^{\perp} H^{(\alpha,a)} \cdot \omega^{\perp}}{\langle t-r\rangle} \right\|_{L^2} \big\| \langle t-r\rangle |\nabla U^{(\alpha,a)}| |\nabla^2 U| \big\|_{L^2(r \geq \langle t\rangle/2)} \\
&\quad \lesssim \eta G_\kappa + C_\eta \langle t\rangle^{-1} \mathcal{E}_\kappa E_{\kappa-3} + \langle t\rangle^{-3/2} \mathcal{E}_\kappa E_{\kappa-3}^{1/2}.
\end{aligned}$$

In the region $\{r \leq \langle t\rangle/2\}$, we can easily estimate the last line of (4.3), similarly to (4.7), to deduce that it is controlled by

$$\langle t\rangle^{-1} \mathcal{E}_\kappa E_{\kappa-3}^{1/2}.$$

Thus we gather the estimates in Step 1 to conclude that

$$I_1 \lesssim \eta G_\kappa + C_\eta \langle t\rangle^{-1} \mathcal{E}_\kappa E_{\kappa-3}^{1/2}.$$

Step 2. Estimate of the lower-order term I_2.

We introduce $\widetilde{f}^{\alpha a}$ given by

$$\widetilde{f}^{\alpha a}_{ijk} = \sum_{\substack{\beta+\gamma=\alpha, b+c=a \\ |\beta|+|b|,|\gamma|+|c|<|\alpha|+|a|}} C_\alpha^\beta C_a^b \nabla_k \big(\nabla_i V^{(\beta,b)} \nabla_j V^{(\gamma,c)} - \nabla_i H^{(\beta,b)} \cdot \nabla_j H^{(\gamma,c)}\big)$$

and

$$\widetilde{f}^{\alpha a}_2 = \sum_{\substack{\beta+\gamma=\alpha, b+c=a \\ |\beta|+|b|,|\gamma|+|c|<|\alpha|+|a|}} C_\alpha^\beta C_a^b \nabla(\nabla^\perp H^{(\beta,b)} \nabla V^{(\gamma,c)}).$$

Using these notations, we can control I_2 from the L^2 boundedness of the Riesz transform by

$$\sum_{ijk} \| \widetilde{f}^{\alpha a}_{ijk} \|_{L^2} \| \nabla V^{(\alpha,a)} \|_{L^2} + \| \widetilde{f}^{\alpha a}_2 \|_{L^2} \| \nabla H^{(\alpha,a)} \|_{L^2}.$$

Thus to estimate I_2, we only need to take care of $\widetilde{f}^{\alpha a}_{ijk}$ and $\widetilde{f}^{\alpha a}_2$.

First we treat $\widetilde{f}^{\alpha a}_{ijk}$. One easily has

$$\begin{aligned} &\nabla_k (\nabla_i V^{(\beta,b)} \nabla_j V^{(\gamma,c)} - \nabla_i H^{(\beta,b)} \cdot \nabla_j H^{(\gamma,c)}) \\ &\quad = \nabla_k \nabla_i V^{(\beta,b)} \nabla_j V^{(\gamma,c)} - \nabla_k \nabla_i H^{(\beta,b)} \cdot \nabla_j H^{(\gamma,c)} \\ &\qquad + \nabla_i V^{(\beta,b)} \nabla_k \nabla_j V^{(\gamma,c)} - \nabla_i H^{(\beta,b)} \cdot \nabla_k \nabla_j H^{(\gamma,c)}. \end{aligned}$$

In view of the fact that the last two lines above are similar, we concentrate only on the first one. To estimate $\| \widetilde{f}^{\alpha a}_{ijk} \|_{L^2}$, we still divide the integral domain $\mathbb{R}^2$ into two different subdomains.

In the region $\{r \le \langle t \rangle/2\}$, we have

$$\| \widetilde{f}^{\alpha a}_{ijk} \|_{L^2(r \le \langle t \rangle/2)} \le \sum_{\substack{\beta+\gamma=\alpha, b+c=a \\ |\beta|+|b|,|\gamma|+|c|<|\alpha|+|a|}} \big\| |\nabla^2 U^{(\beta,b)}| |\nabla U^{(\gamma,c)}| \big\|_{L^2(r \le \langle t \rangle/2)}. \tag{4.9}$$

Here and in what follows, thanks to the fully nonlinear effect of the new formulation, we always have one derivative in the lower-order terms. Thus one has room to use the weighted L^2 norm X_κ even though we are facing the derivative loss $X_{\kappa-2} \lesssim E_{\kappa-1}$.

For (4.9), if $|c|+|\gamma| \le |b|+|\beta|$, then there holds $|b|+|\beta|+2 \le \kappa$, $|\gamma|+|c|+3 \le [(|\alpha|+|a|)/2]+3 \le \kappa-4$. By (3.3) and Lemma 3.16, we have

$$\begin{aligned} &\sum_{\substack{\beta+\gamma=\alpha, b+c=a \\ |\gamma|+|c| \le |\beta|+|b| < |\alpha|+|a|}} \big\| |\nabla^2 U^{(\beta,b)}| |\nabla U^{(\gamma,c)}| \big\|_{L^2(r \le \langle t \rangle/2)} \\ &\lesssim \sum_{\substack{\beta+\gamma=\alpha, b+c=a \\ |\gamma|+|c| \le |\beta|+|b| < |\alpha|+|a|}} \langle t \rangle^{-1} \| \nabla^2 U^{(\beta,b)} \|_{L^2} \| \langle t \rangle \nabla U^{(\gamma,c)} \|_{L^\infty(r \le \langle t \rangle/2)} \lesssim \end{aligned}$$

$$\lesssim \sum_{\substack{\beta+\gamma=\alpha, b+c=a \\ |\gamma|+|c|\le|\beta|+|b|<|\alpha|+|a|}} \langle t\rangle^{-1}\mathcal{E}^{1/2}_{|\beta|+|b|+2} X^{1/2}_{|\gamma|+|c|+3} \lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\kappa} E^{1/2}_{\kappa-3}.$$

If $|b|+|\beta| < |c|+|\gamma|$, then $|\gamma|+|c|+1 \le \kappa$, $|b|+|\beta|+4 \le [(|\alpha|+|a|)/2]+4 \le \kappa-4$. We can similarly obtain

$$\begin{aligned}
&\sum_{\substack{\beta+\gamma=\alpha, b+c=a \\ |\beta|+|b|<|\gamma|+|c|<|\alpha|+|a|}} \big\| |\nabla^2 U^{(\beta,b)}||\nabla U^{(\gamma,c)}| \big\|_{L^2(r\le\langle t\rangle/2)} \\
&\lesssim \sum_{\substack{\beta+\gamma=\alpha, b+c=a \\ |\beta|+|b|<|\gamma|+|c|<|\alpha|+|a|}} \langle t\rangle^{-1}\mathcal{E}^{1/2}_{|\gamma|+|c|+1} X^{1/2}_{|\beta|+|b|+4} \lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\kappa} E^{1/2}_{\kappa-3}.
\end{aligned}$$

Thus we arrive at

$$\|\widetilde{f}^{\alpha a}_{ijk}\|_{L^2(r\le\langle t\rangle/2)} \lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\kappa} E^{1/2}_{\kappa-3}.$$

In the region $\{r \ge \langle t\rangle/2\}$, we need to employ the null structure to get some extra decay in time. A natural idea is to use a variant version of Lemma 3.10; however, this doesn't work due to the derivative loss $Y_{\kappa-2} \lesssim E_{\kappa-1}$. To solve this problem, we will use the ghost weight energy at all derivative levels.

For $\widetilde{f}^{\alpha a}_{ijk}$, we organize similarly to the decomposition (4.5):

$$\begin{aligned}
&\nabla_k\nabla_i V^{(\beta,b)}\nabla_j V^{(\gamma,c)} - \nabla_k\nabla_i H^{(\beta,b)}\cdot\nabla_j H^{(\gamma,c)} \\
&\quad= (\nabla_k\nabla_i V^{(\beta,b)} + \nabla_k\nabla_i H^{(\beta,b)}\cdot\omega)\nabla_j V^{(\gamma,c)} \\
&\qquad - \nabla_k\nabla_i H^{(\beta,b)}\cdot\omega(\nabla_j V^{(\gamma,c)} + \nabla_j H^{(\gamma,c)}\cdot\omega) \\
&\qquad - \nabla_k\nabla_i H^{(\beta,b)}\cdot\omega^{\perp}\nabla_j H^{(\gamma,c)}\cdot\omega^{\perp}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|\widetilde{f}^{\alpha a}_{ijk}\|_{L^2(r\le\langle t\rangle/2)}\|\nabla V^{(\alpha,a)}\|_{L^2} \\
&\le \sum_{\substack{\beta+\gamma=\alpha, b+c=a \\ |\beta|+|b|,|\gamma|+|c|<|\alpha|+|a|}} \Big(\big\|(\nabla_k\nabla_i V^{(\beta,b)} + \nabla_k\nabla_i H^{(\beta,b)}\cdot\omega)\nabla_j V^{(\gamma,c)}\big\|_{L^2(r\le\langle t\rangle/2)}\mathcal{E}^{1/2}_{\kappa} \\
&\quad + \|\nabla_k\nabla_i H^{(\beta,b)}\cdot\omega(\nabla_j V^{(\gamma,c)} + \nabla_j H^{(\gamma,c)}\cdot\omega)\|_{L^2(r\le\langle t\rangle/2)}\mathcal{E}^{1/2}_{\kappa} \\
&\quad + \|\nabla_k\nabla_i H^{(\beta,b)}\cdot\omega^{\perp}\nabla_j H^{(\gamma,c)}\cdot\omega^{\perp}\|_{L^2(r\le\langle t\rangle/2)}\mathcal{E}^{1/2}_{\kappa}\Big).
\end{aligned} \tag{4.10}$$

When $|\gamma|+|c| \leq |\beta|+|b|$, by (3.2), Lemma 3.16, and Lemma 3.18, the right-hand side of (4.10) can be bounded by

$$\begin{aligned}
&\sum_{1\leq i,j,k\leq 2} \sum_{\substack{|\beta|+|b|<|\alpha|+|a| \\ |\gamma|+|c|\leq[(|\alpha|+|a|)/2]}} \\
&\quad \Bigg(\left\| \frac{\nabla_k \nabla_i V^{(\beta,b)} + \nabla_k \nabla_i H^{(\beta,b)} \cdot \omega}{\langle t-r\rangle} \right\|_{L^2} \|\langle t-r\rangle \nabla_j V^{(\gamma,c)}\|_{L^\infty(r\leq\langle t\rangle/2)} \mathcal{E}_k^{1/2} \\
&\qquad + \|\nabla_k \nabla_i H^{(\beta,b)} \cdot \omega\|_{L^2} \|(\nabla_j V^{(\gamma,c)} + v_j H^{(\gamma,c)} \cdot \omega)\|_{L^\infty(r\leq\langle t\rangle/2)} \mathcal{E}_\kappa^{1/2} \\
&\qquad + \|\nabla_k \nabla_i H^{(\beta,b)} \cdot \omega^\perp\|_{L^2} \|\nabla_j H^{(\gamma,c)} \cdot \omega^\perp\|_{L^\infty(r\leq\langle t\rangle/2)} \mathcal{E}_\kappa^{1/2} \Bigg) \\
&\lesssim \eta G_\kappa + C_\eta \langle t\rangle^{-1} \mathcal{E}_\kappa E_{\kappa-3} + \langle t\rangle^{-3/2} \mathcal{E}_\kappa E_{\kappa-3}^{1/2}.
\end{aligned}$$

Repeating the above procedure, we can control the right-hand side of (4.10) in the case $|\beta| + |b| \leq |\gamma| + |c|$ by

$$\eta G_\kappa + C_\eta \langle t\rangle^{-1} \mathcal{E}_\kappa E_{\kappa-3} + \langle t\rangle^{-3/2} \mathcal{E}_\kappa E_{\kappa-3}^{1/2}.$$

Thus, we get

$$\|\widetilde{f}_{ijk}^{\alpha a}\|_{L^2} \|\nabla V^{(\alpha,a)}\|_{L^2} \lesssim \eta G_\kappa + C_\eta \langle t\rangle^{-1} \mathcal{E}_\kappa E_{\kappa-3}^{1/2}.$$

Here we have used the a priori estimate that $E_{\kappa-3} \ll 1$.

We turn our attention to $\|\widetilde{f}_2^{\alpha a}\|_{L^2}$. Since the estimate is similar to $\|\widetilde{f}_{ijk}^{\alpha a}\|_{L^2}$, we only sketch the main line of argument. We still divide the integral domain $\mathbb{R}^2$ into two different parts to estimate them separately. For the integral over the domain $\{r \leq \langle t\rangle/2\}$, the estimate is exactly the same as the one for $\|\widetilde{f}_{ijk}^{\alpha a}\|_{L^2(r\leq\langle t\rangle/2)}$. For the region $\{r \geq \langle t\rangle/2\}$, we still need to make full use of the appropriate null structure. The estimate is similar to one for $\|\widetilde{f}_{ijk}^{\alpha a}\|_{L^2(r\geq\langle t\rangle/2)}$ once the null structure of $\widetilde{f}_2^{\alpha a}$ is present. Hence we only show the strong null structure of $\widetilde{f}_2^{\alpha a}$ below.

Employing the orthogonal decomposition into radial and transverse directions, any term in the sum defining I_2 can be decomposed as

$$\begin{aligned}
&\sum_{\substack{\beta+\gamma=\alpha,b+c=a \\ |\beta|+|b|,|\gamma|+|c|<|\alpha|+|a|}} C_\alpha^\beta C_a^b \nabla_i (\nabla_j^\perp H^{(\beta,b)} \nabla_j V^{(\gamma,c)}) \\
&= \sum_{\substack{\beta+\gamma=\alpha,b+c=a \\ |\beta|+|b|,|\gamma|+|c|<|\alpha|+|a|}} C_\alpha^\beta C_a^b \nabla_i \nabla_j^\perp H^{(\beta,b)} \nabla_j V^{(\gamma,c)} + \nabla_j^\perp H^{(\beta,b)} \nabla_i \nabla_j V^{(\gamma,c)} =
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{\beta+\gamma=\alpha,b+c=a\\|\beta|+|b|,|\gamma|+|c|<|\alpha|+|a|}} \Big[C_\alpha^\beta C_a^b\big(\nabla_i\nabla_j^\perp H^{(\beta,b)}\cdot\omega\nabla_j V^{(\gamma,c)} + \nabla_j^\perp H^{(\beta,b)}\cdot\omega\nabla_i\nabla_j V^{(\gamma,c)}\big)\omega\\
&\qquad + C_\alpha^\beta C_a^b\big(\nabla_i\nabla_j^\perp H^{(\beta,b)}\cdot\omega^\perp\nabla_j V^{(\gamma,c)}\\
&\qquad\qquad + \nabla_j^\perp H^{(\beta,b)}\cdot\omega^\perp\nabla_i\nabla_j V^{(\gamma,c)}\big)\cdot\omega^\perp\Big]\\
&= \sum_{\substack{\beta+\gamma=\alpha,b+c=a\\|\beta|+|b|,|\gamma|+|c|<|\alpha|+|a|}} \Big[C_\alpha^\beta C_a^b\big(\nabla_i\nabla_j^\perp H^{(\beta,b)}\cdot\omega + \nabla_i\nabla_j^\perp V^{(\beta,b)}\big)\nabla_j V^{(\gamma,c)}\\
&\qquad + C_\alpha^\beta C_a^b\big(\nabla_j^\perp H^{(\beta,b)}\cdot\omega + \nabla_j^\perp V^{(\beta,b)}\big)\nabla_i\nabla_j V^{(\gamma,c)}\\
&\qquad + C_\alpha^\beta C_a^b\big(\nabla_i\nabla_j^\perp H^{(\beta,b)}\cdot\omega^\perp\nabla_j V^{(\gamma,c)}\\
&\qquad\qquad + \nabla_j^\perp H^{(\beta,b)}\cdot\omega^\perp\nabla_i\nabla_j V^{(\gamma,c)}\big)\cdot\omega^\perp\Big].
\end{aligned}
$$

Here we have used the fact that

$$
\begin{aligned}
&\sum_{\substack{\beta+\gamma=\alpha,b+c=a\\|\beta|+|b|,|\gamma|+|c|<|\alpha|+|a|}} C_\alpha^\beta C_a^b\nabla_i\nabla_j^\perp V^{(\beta,b)}\nabla_j V^{(\gamma,c)}\\
&\quad + \sum_{\substack{\beta+\gamma=\alpha,b+c=a\\|\beta|+|b|,|\gamma|+|c|<|\alpha|+|a|}} C_\alpha^\beta C_a^b\nabla_i\nabla_j V^{(\gamma,c)}\nabla_j^\perp V^{(\beta,b)} = 0.
\end{aligned}
$$

Thus, we can estimate $\widetilde{f}_2^{\alpha a}$ as $\widetilde{f}_{ijk}^{\alpha a}$ to get that

$$
\|\widetilde{f}_2^{\alpha a}\|_{L^2}\|\nabla H^{(\alpha,a)}\|_{L^2} \lesssim \eta G_\kappa + C_\eta\langle t\rangle^{-1}\mathcal{E}_\kappa E_{\kappa-3}^{1/2}.
$$

Finally, we gather our estimates for (4.1) to derive that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|\nabla V^{(\alpha,a)}|^2 + |\nabla H^{(\alpha,a)}|^2)e^q\,dx\\
&\quad - \int_{\mathbb{R}^2}\mu\nabla\Delta\sum_{l=0}^{\alpha}(-1)^{\alpha-l}V^{(l,a)}\cdot\nabla V^{(\alpha,a)}e^q\,dx\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\frac{|\nabla V^{(\alpha,a)} + \nabla H^{(\alpha,a)}\cdot\omega|^2 + |\nabla H^{(\alpha,a)}\cdot\omega^\perp|^2}{\langle t-r\rangle^2}e^q\,dx\\
&\quad \lesssim \eta G_\kappa + C_\eta\langle t\rangle^{-1}\mathcal{E}_\kappa E_{\kappa-3}^{1/2}.
\end{aligned}
\tag{4.11}
$$

The viscosity terms can be estimated as follows:

$$
\begin{aligned}
&-\int_{\mathbb{R}^2} \mu \nabla\Delta \sum_{l=0}^{\alpha} C_\alpha^l (-1)^{\alpha-l} V^{(l,a)} \cdot \nabla V^{(\alpha,a)} e^q \, dx \\
&= \mu \int_{\mathbb{R}^2} |\Delta V^{(\alpha,a)}|^2 e^q \, dx + \int_{\mathbb{R}^2} \mu\Delta \sum_{l=0}^{\alpha-1} C_\alpha^l (-1)^{\alpha-l} V^{(l,a)} \cdot \Delta V^{(\alpha,a)} e^q \, dx \\
&\quad + \int_{\mathbb{R}^2} \mu\Delta \sum_{l=0}^{\alpha} C_\alpha^l (-1)^{\alpha-l} V^{(l,a)} \nabla V^{(\alpha,a)} \cdot \nabla e^q \, dx \\
&\geq \mu \int_{\mathbb{R}^2} |\Delta V^{(\alpha,a)}|^2 e^q \, dx - \mu \sum_{l=0}^{\alpha-1} \big(C_\alpha^l\big)^2 \int_{\mathbb{R}^2} |\Delta V^{(l,a)}|^2 e^q \, dx \\
&\quad - \frac{1}{4}\mu \int_{\mathbb{R}^2} |\Delta V^{(\alpha,a)}|^2 e^q \, dx - \frac{1}{4}\mu \sum_{l=0}^{\alpha} \big(C_\alpha^l\big)^2 \int_{\mathbb{R}^2} |\Delta V^{(l,a)}|^2 e^q dx \\
&\quad - \mu \int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}|^2 e^q dx \\
&\geq \frac{1}{2}\mu \int_{\mathbb{R}^2} |\Delta V^{(\alpha,a)}|^2 e^q \, dx - 2\mu \sum_{l=0}^{\alpha-1} \big(C_\alpha^l\big)^2 \int_{\mathbb{R}^2} |\Delta V^{(l,a)}|^2 e^q \, dx \\
&\quad - \mu \int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}|^2 e^q dx.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} (|\nabla V^{(\alpha,a)}|^2 + |\nabla H^{(\alpha,a)}|^2) e^q \, dx \\
&\quad + \frac{1}{2}\mu \int |\Delta V^{(\alpha,a)}|^2 e^q \, dx - 2\mu \sum_{l=0}^{\alpha-1} \big(C_\alpha^l\big)^2 \int_{\mathbb{R}^2} |\Delta V^{(l,a)}|^2 e^q \, dx \\
&\quad - \mu \int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}|^2 e^q \, dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} \frac{|\nabla V^{(\alpha,a)} + \nabla H^{(\alpha,a)} \cdot \omega|^2 + |\nabla H^{(\alpha,a)} \cdot \omega^\perp|^2}{\langle t-r\rangle^2} e^q \, dx \\
&\quad \leq \eta G_\kappa + C_\eta \langle t\rangle^{-1} \mathcal{E}_\kappa E_{\kappa-3}^{1/2}.
\end{aligned}
$$

Integrating both sides of the above inequality in time on $[0, t)$, we get

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}(|\nabla V^{(\alpha,a)}(t)|^2 + |\nabla H^{(\alpha,a)}(t)|^2)e^q\,dx \\
&\quad + \frac{1}{2}\mu\int_0^t\int_{\mathbb{R}^2}|\Delta V^{(\alpha,a)}(\tau)|^2 e^q\,dx\,d\tau \\
&\quad - 2\mu\sum_{l=0}^{\alpha-1}\big(C_\alpha^l\big)^2\int_0^t\int_{\mathbb{R}^2}|\Delta V^{(l,a)}(\tau)|^2 e^q\,dx\,d\tau \\
&\quad - \mu\int_0^t\int_{\mathbb{R}^2}|\nabla V^{(\alpha,a)}(\tau)|^2 e^q\,dx\,d\tau \\
&\quad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^2}\frac{|\nabla V^{(\alpha,a)}(\tau) + \nabla H^{(\alpha,a)}(\tau)\cdot\omega|^2 + |\nabla H^{(\alpha,a)}(\tau)\cdot\omega^\perp|^2}{\langle\tau - r\rangle^2}e^q\,dx\,d\tau \\
&\qquad \le \eta\int_0^t G_\kappa(\tau)d\tau + C_\eta\int_0^t\langle\tau\rangle^{-1}\mathcal{E}_\kappa(\tau)E_{\kappa-3}^{\frac{1}{2}}(\tau)d\tau \\
&\qquad\quad + \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla V^{(\alpha,a)}(0)|^2 + |\nabla H^{(\alpha,a)}(0)|^2)e^q\,dx.
\end{aligned}
\tag{4.12}
$$

These two terms on the left of (4.12) will be absorbed by the viscous dissipation coming from the lower orders and the standard energy estimate of the next subsection (see (4.19)).

4.2 Highest-Order Standard Energy Estimate

Now we proceed with the highest-order standard energy estimate. Here we have one less regular derivative to estimate and we will not use the ghost weight. Hence we don't need to handle the commutators between the ghost weight and the viscosity terms, but only handle the commutators between the scaling operator and the viscosity terms. We remark that here the estimate of the nonlinearities is slightly different because of the absence of the extra regular derivative.

Let $\kappa \ge 12$ and $|\alpha| + |a| \le \kappa - 1$, and let us take the L^2 inner product of the first and the second equation of (2.14) with $V^{(\alpha,a)}$ and $H^{(\alpha,a)}$, respectively. Then adding up the resulting equations, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|V^{(\alpha,a)}|^2 + |H^{(\alpha,a)}|^2)dx \\
&\quad - \int_{\mathbb{R}^2}\mu\Delta\sum_{l=0}^{\alpha}C_\alpha^l(-1)^{\alpha-l}V^{(l,a)}\cdot V^{(\alpha,a)}dx \\
&\qquad = \int_{\mathbb{R}^2}f_{\alpha a}^1 V^{(\alpha,a)}dx + \int_{\mathbb{R}^2}f_{\alpha a}^2\cdot H^{(\alpha,a)}dx.
\end{aligned}
\tag{4.13}
$$

Since the estimate for the first term and that for the second one on the right-hand side of (4.13) are very similar, we give the details for only the first one.

It follows easily from the L^2 boundedness of the Riesz transform that

$$\int_{\mathbb{R}^2} f^1_{\alpha a} V^{(\alpha,a)} dx \lesssim \sum_{\substack{\beta+\gamma=\alpha \\ b+c=a}} \sum_{1\le i,j\le 2} \|\nabla_i V^{(\beta,b)} \nabla_j V^{(\gamma,c)} - \nabla_i H^{(\beta,b)} \cdot \nabla_j H^{(\gamma,c)}\|_{L^2} \|V^{(\alpha,a)}\|_{L^2}.$$

In the region $\{r \le \langle t\rangle/2\}$, we have

$$\begin{aligned} &\sum_{\substack{\beta+\gamma=\alpha \\ b+c=a}} \sum_{1\le i,j\le 2} \|\nabla_i V^{(\beta,b)} \nabla_j V^{(\gamma,c)} - \nabla_i H^{(\beta,b)} \cdot \nabla_j H^{(\gamma,c)}\|_{L^2(r\le\langle t\rangle/2)} \\ &\lesssim \sum_{\substack{\beta+\gamma=\alpha \\ b+c=a}} \big\| |\nabla U^{(\beta,b)}| |\nabla U^{(\gamma,c)}| \big\|_{L^2(r\le\langle t\rangle/2)} \\ &\lesssim \sum_{\substack{b+c=a,\beta+\gamma=\alpha \\ |b|+|\beta|\ge|c|+|\gamma|}} \big\| |\nabla U^{(\beta,b)}| |\nabla U^{(\gamma,c)}| \big\|_{L^2(r\le\langle t\rangle/2)}. \end{aligned}$$

Here we used the symmetry between the index in the last inequality. Note that due to the derivative loss $X_{\kappa-2} \lesssim E_{\kappa-1}$ and since $|b| + |\beta| \ge |c| + |\gamma|$, one has $|\gamma| + |c| + 3 \le [(|\alpha| + |a|)/2] + 3 \le \kappa - 4$. By (3.3) and Lemma 3.16, the above quantities can be controlled by

$$\begin{aligned} &\sum_{\substack{b+c=a,\,\beta+\gamma=\alpha \\ |b|+|\beta|\ge|c|+|\gamma|}} \langle t\rangle^{-1} \|\nabla U^{(\beta,b)}\|_{L^2} \|\langle t\rangle \nabla U^{(\gamma,c)}\|_{L^\infty(r\le\langle t\rangle/2)} \\ &\lesssim \sum_{\substack{b+c=a,\,\beta+\gamma=\alpha \\ |b|+|\beta|\ge|c|+|\gamma|}} \langle t\rangle^{-1} \mathcal{E}^{1/2}_{|\beta|+|b|+1} X^{1/2}_{|\gamma|+|c|+3} \\ &\le \langle t\rangle^{-1} \mathcal{E}^{1/2}_{\kappa} X^{1/2}_{\kappa-4} \lesssim \langle t\rangle^{-1} \mathcal{E}^{1/2}_{\kappa} E^{1/2}_{\kappa-3}. \end{aligned}$$

In the region $\{r \ge \langle t\rangle/2\}$, we need to employ the null structure to get extra time decay. An important trick here is that we need to use the appropriate null structure. The situation is similar to the estimate of I_2 in the last subsection. A natural idea is to use Lemma 3.10, but this doesn't work due to the derivative loss $Y_{\kappa-2} \lesssim E_{\kappa-1}$. To solve this problem, we combine the highest-order standard energy estimate and the highest-order modified energy estimate. More precisely, we will use the good term G_κ that comes from the ghost weight energy obtained in the modified energy estimate.

Employing the orthogonal decomposition into radial and transverse directions, we have

$$\begin{aligned}
&\nabla_i V^{(\beta,b)}\nabla_j V^{(\gamma,c)} - \nabla_i H^{(\beta,b)}\cdot\nabla_j H^{(\gamma,c)}\\
&\quad= \nabla_i V^{(\beta,b)}\nabla_j V^{(\gamma,c)} - \nabla_i H^{(\beta,b)}\cdot\omega\nabla_j H^{(\gamma,c)}\cdot\omega\\
&\qquad - \nabla_i H^{(\beta,b)}\cdot\omega^{\perp}\nabla_j H^{(\gamma,c)}\cdot\omega^{\perp}\\
&\quad= (\nabla_i V^{(\beta,b)} + \nabla_i H^{(\beta,b)}\cdot\omega)\nabla_j V^{(\gamma,c)}\\
&\qquad - \nabla_i H^{(\beta,b)}\cdot\omega(\nabla_j V^{(\gamma,c)} + \nabla_j H^{(\gamma,c)}\cdot\omega)\\
&\qquad - \nabla_i H^{(\beta,b)}\cdot\omega^{\perp}\nabla_j H^{(\gamma,c)}\cdot\omega^{\perp}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{\substack{\beta+\gamma=\alpha\\ b+c=a}}\sum_{1\le i,j\le 2}\|\nabla_i V^{(\beta,b)}\nabla_j V^{(\gamma,c)} - \nabla_i H^{(\beta,b)}\cdot\nabla_j H^{(\gamma,c)}\|_{L^2(r\ge\langle t\rangle/2)}\|V^{(\alpha,a)}\|_{L^2}\\
&\quad\lesssim \sum_{\substack{\beta+\gamma=\alpha\\ b+c=a}}\sum_{1\le i\le 2}\big\||\nabla_i H^{(\beta,b)}\cdot\omega + \nabla_i V^{(\beta,b)}||\nabla U^{(\gamma,c)}|\big\|_{L^2(r\ge\langle t\rangle/2)}E_{\kappa-1}^{1/2}\\
&\qquad + \sum_{\substack{\beta+\gamma=\alpha\\ b+c=a}}\sum_{1\le i,j\le 2}\|\nabla_i H^{(\beta,b)}\cdot\omega^{\perp}\nabla_j H^{(\gamma,c)}\cdot\omega^{\perp}\|_{L^2(r\ge\langle t\rangle/2)}E_{\kappa-1}^{1/2}.
\end{aligned}\tag{4.14}$$

In the above inequality, we used the symmetry between the index b and c and the symmetry between β and γ. For (4.14), if $|\beta|+|b|\ge|\gamma|+|c|$, by (3.2), Lemma 3.16, and Lemma 3.18, it can be further bounded by

$$\begin{aligned}
&\sum_{\substack{\beta+\gamma=\alpha, b+c=a\\ |\beta|+|b|\ge|\gamma|+|c|}}\sum_{1\le i\le 2}\left\|\frac{\nabla_i V^{(\beta,b)} + \nabla_i H^{(\beta,b)}\cdot\omega}{\langle t-r\rangle}\right\|_{L^2}\|\langle t-r\rangle\nabla U^{(\gamma,c)}\|_{L^\infty(r\ge\langle t\rangle/2)}E_{\kappa-1}^{1/2}\\
&\quad + \sum_{\substack{\beta+\gamma=\alpha, b+c=a\\ |\beta|+|b|\ge|\gamma|+|c|}}\sum_{1\le i\le 2}\|\nabla H^{(\beta,b)}\|_{L^2}\|\nabla_i H^{(\gamma,c)}\cdot\omega^{\perp}\|_{L^\infty(r\ge\langle t\rangle/2)}E_{\kappa-1}^{1/2}\\
&\qquad\lesssim \eta G_\kappa + C_\eta\langle t\rangle^{-1}E_{\kappa-1}E_{\kappa-3} + \langle t\rangle^{-3/2}\mathcal{E}_\kappa^{1/2}E_{\kappa-1}^{1/2}E_{\kappa-3}^{1/2}.
\end{aligned}$$

If $|\beta|+|b|<|\gamma|+|c|$, we can repeat a similar procedure to deduce that the right-hand side of (4.14) can be bounded by

$$\langle t\rangle^{-3/2}\mathcal{E}_\kappa^{1/2}E_{\kappa-1}^{1/2}E_{\kappa-3}^{1/2}.$$

It then follows by gathering the above estimates that

$$\int_{\mathbb{R}^2} f_{\alpha a}^1 V^{(\alpha,a)}dx \lesssim \eta G_\kappa + C_\eta\langle t\rangle^{-1}(\mathcal{E}_\kappa + E_{\kappa-1})E_{\kappa-3}^{1/2}.$$

The estimate of $\int_{\mathbb{R}^2} f^2_{\alpha a} \cdot H^{(\alpha,a)} dx$ is similar to $\int_{\mathbb{R}^2} f^1_{\alpha a} \cdot V^{(\alpha,a)} dx$. The key point is to explore the appropriate null structure for $f^2_{\alpha a}$. We will prove that

$$
\begin{aligned}
|f^2_{\alpha a}| \lesssim & \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} \left|\nabla_j^{\perp} H^{(\beta,b)} \cdot \omega^{\perp} \nabla_j V^{(\gamma,c)}\right| \\
& + \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} \left|\left(\nabla_j^{\perp} H^{(\beta,b)} \cdot \omega + \nabla_j^{\perp} V^{(\beta,b)}\right) \nabla_j V^{(\gamma,c)} \omega\right|,
\end{aligned} \tag{4.15}
$$

from which we can deduce that

$$
\int_{\mathbb{R}^2} f^2_{\alpha a} H^{(\alpha,a)} dx \lesssim \eta G_\kappa + C_\eta \langle t\rangle^{-1} (\mathcal{E}_\kappa + E_{\kappa-1}) E^{1/2}_{\kappa-3}.
$$

Hence in what follows, we only show (4.15).

Employing the orthogonal decomposition onto radial and transverse directions, we have

$$
\begin{aligned}
f^2_{\alpha a} &= \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \left(\nabla_j^{\perp} H^{(\beta,b)} \nabla_j V^{(\gamma,c)}\right) \\
&= \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \left(\nabla_j^{\perp} H^{(\beta,b)} \cdot \omega \nabla_j V^{(\gamma,c)}\right) \omega \\
&\quad + \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \left(\nabla_j^{\perp} H^{(\beta,b)} \cdot \omega^{\perp} \nabla_j V^{(\gamma,c)}\right) \omega^{\perp}.
\end{aligned}
$$

For the first line on the right-hand side in the above, we rewrite it as as

$$
\begin{aligned}
& \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \left(\nabla_j^{\perp} H^{(\beta,b)} \cdot \omega \nabla_j V^{(\gamma,c)}\right) \omega \\
&= \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \left(\nabla_j^{\perp} H^{(\beta,b)} \cdot \omega + \nabla_j^{\perp} V^{(\beta,b)}\right) \nabla_j V^{(\gamma,c)} \omega.
\end{aligned}
$$

Here we have used the fact that

$$
\sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \nabla_j^{\perp} V^{(\beta,b)} \nabla_j V^{(\gamma,c)} \omega = 0.
$$

This yields (4.15).

Finally, we gather our estimate for (4.13) to derive that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|V^{(\alpha,a)}|^2+|H^{(\alpha,a)}|^2)dx\\
&\quad-\int_{\mathbb{R}^2}\mu\Delta\sum_{l=0}^{\alpha}C_\alpha^l(-1)^{\alpha-l}V^{(l,a)}\cdot V^{(\alpha,a)}dx\\
&\quad\le\eta G_\kappa+C_\eta\langle t\rangle^{-1}(\mathcal{E}_\kappa+E_{\kappa-1})E_{\kappa-3}^{1/2}.
\end{aligned}
\tag{4.16}
$$

We estimate the diffusion terms in (4.16) as follows:

$$
\begin{aligned}
&-\int_{\mathbb{R}^2}\mu\Delta\sum_{l=0}^{\alpha}C_\alpha^l(-1)^{\alpha-l}V^{(l,a)}\cdot V^{(\alpha,a)}dx\\
&=\mu\int_{\mathbb{R}^2}|\nabla V^{(\alpha,a)}|^2\,dx\\
&\quad+\int_{\mathbb{R}^2}\mu\nabla\sum_{l=0}^{\alpha-1}C_\alpha^l(-1)^{\alpha-l}V^{(l,a)}\cdot\nabla V^{(\alpha,a)}dx\\
&\ge\frac{1}{2}\mu\int_{\mathbb{R}^2}|\nabla V^{(\alpha,a)}|^2\,dx-\frac{1}{2}\mu\sum_{l=0}^{\alpha-1}\big(C_\alpha^l\big)^2\int_{\mathbb{R}^2}|\nabla V^{(l,a)}|^2\,dx.
\end{aligned}
\tag{4.17}
$$

Hence we can deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|V^{(\alpha,a)}|^2+|H^{(\alpha,a)}|^2)dx\\
&\quad+\frac{1}{2}\mu\int_{\mathbb{R}^2}|\nabla V^{(\alpha,a)}|^2\,dx-\frac{1}{2}\mu\sum_{l=0}^{\alpha-1}\big(C_\alpha^l\big)^2\int_{\mathbb{R}^2}|\nabla V^{(l,a)}|^2\,dx\\
&\quad\le\eta G_\kappa(t)+C_\eta\langle t\rangle^{-1}(\mathcal{E}_\kappa+E_{\kappa-1})E_{\kappa-3}^{1/2}.
\end{aligned}
$$

Integrating both sides of the above inequality in time over $[0,t)$, we get

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}(|V^{(\alpha,a)}(t)|^2+|H^{(\alpha,a)}(t)|^2)dx\\
&\quad+\frac{1}{2}\mu\int_0^t\int_{\mathbb{R}^2}|\nabla V^{(\alpha,a)}(\tau)|^2\,dx\,d\tau\\
&\quad-\frac{1}{2}\mu\sum_{l=0}^{\alpha-1}\big(C_\alpha^l\big)^2\int_0^t\int_{\mathbb{R}^2}|\nabla V^{(l,a)}(\tau)|^2\,dx\,d\tau\\
&\quad\le\frac{1}{2}\int_{\mathbb{R}^2}(|V^{(\alpha,a)}(0)|^2+|H^{(\alpha,a)}(0)|^2)dx\\
&\qquad+\eta\int_0^t G_\kappa(\tau)d\tau+C_\eta\int_0^t\langle\tau\rangle^{-1}\big(\mathcal{E}_\kappa(\tau)+E_{\kappa-1}(\tau)\big)E_{\kappa-3}^{1/2}(\tau)d\tau.
\end{aligned}
$$

Using Lemma 3.4, we deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^2} (|V^{(\alpha,a)}(t)|^2 + |H^{(\alpha,a)}(t)|^2) dx + \mu \int_0^t \int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}(\tau)|^2 \, dx \, d\tau \\
&\quad \lesssim E_{\kappa-1}(0) + \eta \int_0^t G_\kappa(\tau) d\tau \\
&\qquad + C_\eta \int_0^t \langle \tau \rangle^{-1} \big(\mathcal{E}_\kappa(\tau) + E_{\kappa-1}(\tau) \big) E_{\kappa-3}^{1/2}(\tau) d\tau.
\end{aligned}
\tag{4.18}
$$

Now we are going to combine the highest-order modified energy estimate of the previous subsection with the standard one to deal with the diffusion energy with the negative sign in (4.12). Multiplying (4.18) by $4 \max_{\sigma \in \mathbb{R}} e^{q(\sigma)}$ and then adding (4.12), we get

$$
\begin{aligned}
&\int_{\mathbb{R}^2} (|\nabla V^{(\alpha,a)}(t)|^2 + |\nabla H^{(\alpha,a)}(t)|^2) e^q \, dx \\
&\quad + \int_{\mathbb{R}^2} (|V^{(\alpha,a)}(t)|^2 + |H^{(\alpha,a)}(t)|^2) dx + \mu \int_0^t \int_{\mathbb{R}^2} |\Delta V^{(\alpha,a)}(\tau)|^2 e^q \, dx \, d\tau \\
&\quad - \sum_{l=0}^{\alpha-1} \mu C \int_{\mathbb{R}^2} |\Delta V^{(l,a)}|^2 e^q \, dx + \mu \int_0^t \int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}(\tau)|^2 \, dx \, d\tau \\
&\quad + \sum_{1 \le k \le 2} \int_0^t \int_{\mathbb{R}^2} \frac{|\nabla_k V^{(\alpha,a)}(\tau) + \nabla_k H^{(\alpha,a)}(\tau) \cdot \omega|^2 + |\nabla_k H^{(\alpha,a)}(\tau) \cdot \omega^\perp|^2}{\langle t - r \rangle^2} e^q \, dx \, d\tau \\
&\quad \lesssim \eta \int_0^t G_\kappa(\tau) d\tau \\
&\qquad + C_\eta \int_0^t \langle \tau \rangle^{-1} (\mathcal{E}_\kappa(\tau) + E_{\kappa-1}) E_{\kappa-3}^{1/2}(\tau) d\tau + \mathcal{E}_\kappa(0) + E_{\kappa-1}(0).
\end{aligned}
\tag{4.19}
$$

Summing over all $|\alpha| + |a| \le \kappa - 1$ and using Lemma 3.4 to handle the negative sign diffusion energy on the left-hand side of (4.19), we get that

$$
\begin{aligned}
&\mathcal{E}_\kappa(t) + E_{\kappa-1}(t) + \int_0^t G_\kappa(\tau) d\tau \\
&\quad + \mu \sum_{|\alpha|+|a| \le \kappa-1} \int_0^t \int_{\mathbb{R}^2} |\Delta V^{(\alpha,a)}(\tau)|^2 + |\nabla V^{(\alpha,a)}(\tau)|^2 \, dx \, d\tau \\
&\quad \lesssim \eta \int_0^t G_\kappa(\tau) d\tau + C_\eta \int_0^t \langle \tau \rangle^{-1} (\mathcal{E}_\kappa(\tau) + E_{\kappa-1}(\tau)) E_{\kappa-3}^{1/2}(\tau) d\tau \\
&\qquad + \mathcal{E}_\kappa(0) + E_{\kappa-1}(0).
\end{aligned}
$$

Taking η small enough, we conclude that

$$
\begin{aligned}
&\mathcal{E}_\kappa(t) + E_{\kappa-1}(t) + \int_0^t G_\kappa(\tau)d\tau \\
&\quad + \sum_{|\alpha|+|a|\le\kappa-1} \mu \int_0^t\int_{\mathbb{R}^2} |\Delta V^{(\alpha,a)}(\tau)|^2 + |\nabla V^{(\alpha,a)}(\tau)|^2\,dx\,d\tau \\
&\quad \lesssim \int_0^t \langle\tau\rangle^{-1}(\mathcal{E}_\kappa(\tau) + E_{\kappa-1}(\tau))E_{\kappa-3}^{1/2}(\tau)d\tau + \mathcal{E}_\kappa(0) + E_{\kappa-1}(0).
\end{aligned}
$$

This is the desired a priori estimate (2.18).

4.3 Lower-Order Standard Energy Estimate

In this last subsection, we present the lower-order standard energy estimate. A trick here is that we need to earn the maximum decay in time. In order to achieve this, we are going to take full advantage of the inherent strong null structure.

Let $|\alpha| + |a| \le \kappa - 3$. Taking the L^2 inner product of the first and second equation of (2.14) with $V^{(\alpha,a)}$ and $H^{(\alpha,a)}$, respectively, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|V^{(\alpha,a)}|^2 + |H^{(\alpha,a)}|^2)dx - \int_{\mathbb{R}^2}\mu\Delta\sum_{l=0}^{\alpha} C_\alpha^l(-1)^{\alpha-l}V^{(l,a)}\cdot V^{(\alpha,a)}dx \\
&\quad = \int_{\mathbb{R}^2} f^1_{\alpha a}V^{(\alpha,a)} + f^2_{\alpha a}\cdot H^{(\alpha,a)}dx \\
&\quad \le \|f^{\alpha a}_{ij}\|_{L^2}\|V^{(\alpha,a)}\|_{L^2} + \|f^2_{\alpha a}\|_{L^2}\|H^{(\alpha,a)}\|_{L^2}.
\end{aligned}
$$

We have used the L^2 boundedness of the Riesz transform in the last bound. Here we recall that $f^{\alpha a}_{ij}$ was defined in (2.16).

Now we are going to treat $\|f^{\alpha a}_{ij}\|_{L^2}$. First, we have

$$
\begin{aligned}
&\|f^{\alpha a}_{ij}\|_{L^2(r\le\langle t\rangle/2)} \lesssim \\
&\qquad \sum_{\substack{b+c=a\\ \beta+\gamma=\alpha}} \Big\||\nabla V^{(\beta,b)}||\nabla V^{(\gamma,c)}| + |\nabla H^{(\beta,b)}||\nabla H^{(\gamma,c)}|\Big\|_{L^2(r\le\langle t\rangle/2)}.
\end{aligned}
$$

Since the index (β, b) and (γ, c) in the above quantity are symmetric, we can assume that $|\gamma| + |c| \le |\beta| + |b|$ without loss of generality. Thus $|\gamma| + |c| + 3 \le$

$[(|\alpha| + |a|)/2] + 3 \le \kappa - 4$. In view of (3.3) and Lemma 3.16, we get

$$\begin{aligned}
&\| f_{ij}^{\alpha a} \|_{L^2(r \le \langle t \rangle/2)} \\
&\quad \lesssim \sum_{\substack{b+c=a,\beta+\gamma=\alpha \\ |c|+|\gamma| \le |b|+|\beta|}} \big\| |\nabla U^{(\beta,b)}||\nabla U^{(\gamma,c)}| \big\|_{L^2(r \le \langle t \rangle/2)} \\
&\quad \lesssim \sum_{\substack{b+c=a,\beta+\gamma=\alpha \\ |c|+|\gamma| \le |b|+|\beta|}} \big\| \langle t \rangle^{-2} \langle t - r \rangle^2 |\nabla U^{(\beta,b)}||\nabla U^{(\gamma,c)}| \big\|_{L^2(r \le \langle t \rangle/2)} \\
&\quad \lesssim \sum_{\substack{b+c=a,\beta+\gamma=\alpha \\ |c|+|\gamma| \le |b|+|\beta|}} \langle t \rangle^{-2} X_{|\beta|+|b|+1}^{1/2} X_{|\gamma|+|c|+3}^{1/2} \le \langle t \rangle^{-2} X_{\kappa-2}^{1/2} X_{\kappa-4}^{1/2} \\
&\quad \lesssim \langle t \rangle^{-2} E_{\kappa-1}^{1/2}(t) E_{\kappa-3}^{1/2}(t).
\end{aligned}$$

Moreover, in the region $\{r \ge \langle t \rangle/2\}$, by (3.19), we get

$$\begin{aligned}
&\| f_{\alpha a}^{ij} \|_{L^2(r \ge \langle t \rangle/2)} \\
&\quad \lesssim \Big\| \frac{1}{r} \sum_{\substack{|b|+|c| \le |a| \\ |\beta|+|\gamma| \le |\alpha|}} |V^{(|\beta|,|b|+1)} V^{(|\gamma|,|c|+1)}| + |H^{(|\beta|,|b|+1)}||H^{|\gamma|,|c|+1}| \Big\|_{L^2(r \ge \langle t \rangle/2)} \\
&\qquad + \Big\| \sum_{\substack{b+c=a \\ \beta+\gamma=\alpha}} \big| \partial_r V^{(\beta,b)} + \partial_r H^{(\beta,b)} \cdot \omega \big| (|\nabla V^{(\gamma,c)}| + |\nabla H^{(\gamma,c)}|) \Big\|_{L^2(r \ge \langle t \rangle/2)} \\
&\qquad + \Big\| \sum_{\substack{b+c=a \\ \beta+\gamma=\alpha}} \partial_r H^{(\beta,b)} \cdot \omega^\perp \partial_r H^{(\gamma,c)} \cdot \omega^\perp \Big\|_{L^2(r \ge \langle t \rangle/2)}.
\end{aligned} \tag{4.20}$$

For the first line on the right-hand side of (4.20), by the symmetry between the index (β, b) and (γ, c), we can assume that $|b|+|\beta| \le |c|+|\gamma|$. Thus $|b|+|\beta|+3 \le [(|a| + |\alpha|)/2] + 3 \le \kappa - 4$. By (3.1), the first line can be estimated by

$$\begin{aligned}
&\langle t \rangle^{-1} \sum_{\substack{|b|+|c|+|\beta|+|\gamma| \le |a|+|\alpha| \\ |b|+|\beta| \le |c|+|\gamma|}} \| U^{(|\beta|,|b|+1)} \|_{L^\infty(r \ge \langle t \rangle/2)} \| U^{(|\gamma|,|c|+1)} \|_{L^2} \\
&\quad \lesssim \langle t \rangle^{-3/2} \sum_{\substack{|b|+|c|+|\beta|+|\gamma| \le |a|+|\alpha| \\ |b|+|\beta| \le |c|+|\gamma|}} E_{|c|+|\gamma|+1}^{1/2} E_{|b|+|\beta|+3}^{1/2} \lesssim \langle t \rangle^{-3/2} E_{\kappa-3}^{1/2} E_{\kappa-1}^{1/2}.
\end{aligned}$$

For the second line on the right-hand side of (4.20), if $|b|+|\beta|\geq|c|+|\gamma|$, then by Lemma 3.16, we have

$$
\begin{aligned}
&\Big\|\sum_{\substack{b+c=a,\beta+\gamma=\alpha\\|b|+|\beta|\geq|c|+|\gamma|}}(\partial_r V^{(\beta,b)}+\partial_r H^{(\beta,b)}\cdot\omega)(|\nabla V^{(\gamma,c)}|+|\nabla H^{(\gamma,c)})\Big\|_{L^2(r\geq\langle t\rangle/2)}\\
&\quad\lesssim\langle t\rangle^{-1}\sum_{\substack{b+c=a,\beta+\gamma=\alpha\\|b|+|\beta|\geq|c|+|\gamma|}}\|r(\partial_r V^{(\beta,b)}+\partial_r H^{(\beta,b)}\cdot\omega)\|_{L^2}\|\nabla U^{(\gamma,c)}\|_{L^\infty(r\geq\langle t\rangle/2)}\\
&\quad\lesssim\langle t\rangle^{-\frac{3}{2}}\sum_{\substack{b+c=a,\beta+\gamma=\alpha\\|b|+|\beta|\geq|c|+|\gamma|}}Y^{1/2}_{|\beta|+|b|+1}E^{1/2}_{|\gamma|+c+3}\lesssim\langle t\rangle^{-3/2}E^{1/2}_{\kappa-3}E^{1/2}_{\kappa-1}.
\end{aligned}
$$

Otherwise, if $|b|+|\beta|\leq|c|+|\gamma|$, then by (2.17) and Lemma 3.18, we have

$$
\begin{aligned}
&\sum_{\substack{b+c=a,\beta+\gamma=\alpha\\|b|+|\beta|\leq|c|+|\gamma|}}\big\|(\partial_r V^{(\beta,b)}+\partial_r H^{(\beta,b)}\cdot\omega)(|\nabla V^{(\gamma,c)}|+|\nabla H^{(\gamma,c)})\big\|_{L^2(r\geq\langle t\rangle/2)}\\
&\quad\lesssim\sum_{\substack{b+c=a,\beta+\gamma=\alpha\\|b|+|\beta|\leq|c|+|\gamma|}}\|\partial_r V^{(\beta,b)}+\partial_r H^{(\beta,b)}\cdot\omega\|_{L^\infty(r\geq\langle t\rangle/2)}\|\nabla U^{(\gamma,c)}\|_{L^2}\\
&\quad\lesssim\langle t\rangle^{-3/2}E^{1/2}_{\kappa-3}E^{1/2}_{\kappa-1}.
\end{aligned}
$$

The estimate of the third line of (4.20) can be treated exactly as the second line. Thus we conclude by gathering the estimates that

$$\|f^{\alpha a}_{ij}\|_{L^2}\lesssim\langle t\rangle^{-3/2}E^{1/2}_{\kappa-1}(t)E^{1/2}_{\kappa-3}(t).$$

For $\|f^2_{\alpha a}\|_{L^2}$, we can use the same strategy used for the estimate of $\|f^{\alpha a}_{ij}\|_{L^2}$ to get the same bound. Thus, we gather all the estimates in this subsection to deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|V^{(\alpha,a)}|^2+|H^{(\alpha,a)}|^2)dx\\
&\quad-\int_{\mathbb{R}^2}\mu\Delta\sum_{l=0}^{\alpha}C^l_\alpha(-1)^{\alpha-l}V^{(l,a)}\cdot V^{(\alpha,a)}dx\lesssim\langle t\rangle^{-3/2}E_{\kappa-3}E^{1/2}_{\kappa-1}.
\end{aligned}
$$

For the viscosity terms, by (4.17), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|V^{(\alpha,a)}|^2+|H^{(\alpha,a)}|^2)dx\\
&\quad+\frac{1}{2}\mu\int_{\mathbb{R}^2}|\nabla V^{(\alpha,a)}|^2\,dx-\sum_{l=0}^{\alpha-1}\mu C\int_{\mathbb{R}^2}|\nabla V^{(l,a)}|^2\,dx\lesssim\langle t\rangle^{-3/2}E_{\kappa-3}E^{1/2}_{\kappa-1}.
\end{aligned}
$$

We can integrate in time on $[0,t)$ over the above inequality, then use Lemma 3.4 to absorb the diffusion energy with a negative sign. Finally, summing over $|\alpha|+|a|\leq$

$\kappa - 3$, we get

$$E_{\kappa-3}(t) + \sum_{|\alpha|+|a|\le\kappa-3} \mu \int_0^t \int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}(\tau)|^2 \, dx \, d\tau$$
$$\lesssim E_{\kappa-3}(0) + \int_0^t \langle\tau\rangle^{-3/2} E_{\kappa-3}(\tau) E_{\kappa-1}^{1/2}(\tau) d\tau.$$

This is the desired a priori estimate (2.19).

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Bibliography

[1] Agemi, R. Global existence of nonlinear elastic waves. *Invent. Math.* **142** (2000), no. 2, 225–250. doi:10.1007/s002220000084

[2] Alinhac, S. Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions. II. *Acta Math.* **182** (1999), no. 1, 1–23. doi:10.1007/BF02392822

[3] Alinhac, S. Blowup of small data solutions for a quasilinear wave equation in two space dimensions. *Ann. of Math. (2)* **149** (1999), no. 1, 97–127. doi:10.2307/121020

[4] Alinhac, S. Rank 2 singular solutions for quasilinear wave equations. *Internat. Math. Res. Notices* (2000), no. 18, 955–984. doi:10.1155/S1073792800000507

[5] Alinhac, S. The null condition for quasilinear wave equations in two space dimensions I. *Invent. Math.* **145** (2001), no. 3, 597–618. doi:10.1007/s002220100165

[6] Alinhac, S. The null condition for quasilinear wave equations in two space dimensions. II. *Amer. J. Math.* **123** (2001), no. 6, 1071–1101.

[7] Cai, Y.; Lei, Z. Global well-posedness of the incompressible magnetohydrodynamics. *Arch. Ration. Mech. Anal.* **228** (2018), no. 3, 969–993. doi:10.1007/s00205-017-1210-4

[8] Cai, Y.; Lei, Z.; Masmoudi, N. Global well-posedness for 2D nonlinear wave equations without compact support. *J. Math. Pures Appl. (9)* **114** (2018), 211–34. doi:10.1016/j.matpur.2017.09.011

[9] Chen, Y.; Zhang, P. The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions. *Comm. Partial Differential Equations* **31** (2006), no. 10–12, 1793–1810. doi:10.1080/03605300600858960

[10] Christodoulou, D. Global solutions of nonlinear hyperbolic equations for small initial data. *Comm. Pure Appl. Math.* **39** (1986), no. 2, 267–282. doi:10.1002/cpa.3160390205

[11] DiPerna, R. J.; Lions, P.-L. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98** (1989), no. 3, 511–547. doi:10.1007/BF01393835

[12] Germain, P.; Masmoudi, N.; Shatah, J. Global solutions for the gravity water waves equation in dimension 3. *Ann. of Math. (2)* **175** (2012), no. 2, 691–754. doi:10.4007/annals.2012.175.2.6

[13] He, L.; Xu, L. Global well-posedness for viscoelastic fluid system in bounded domains. *SIAM J. Math. Anal.* **42** (2010), no. 6, 2610–2625. doi:10.1137/10078503X
[14] Hu, X.; Lin, F. Global solutions of two-dimensional incompressible viscoelastic flows with discontinuous initial data. *Comm. Pure Appl. Math.* **69** (2016), no. 2, 372–404. doi:10.1002/cpa.21561
[15] Hu, X.; Masmoudi, N. Global solutions to repulsive Hookean elastodynamics. *Arch. Ration. Mech. Anal.* **223** (2017), no. 1, 543–590. doi:10.1007/s00205-016-1039-2
[16] Hu, X.; Wang, D. Global existence for the multi-dimensional compressible viscoelastic flows. *J. Differential Equations* **250** (2011), no. 2, 1200–1231. doi:10.1016/j.jde.2010.10.017
[17] John, F. Blow-up for quasilinear wave equations in three space dimensions. *Comm. Pure Appl. Math.* **34** (1981), no. 1, 29–51. doi:10.1002/cpa.3160340103
[18] John, F. Formation of singularities in elastic waves. *Trends and applications of pure mathematics to mechanics (Palaiseau, 1983)*, 194–210. Lecture Notes in Physics, 195. Springer, Berlin, 1984. doi:10.1007/3-540-12916-2_58
[19] John, F. Almost global existence of elastic waves of finite amplitude arising from small initial disturbances. *Comm. Pure Appl. Math.* **41** (1988), no. 5, 615–666. doi:10.1002/cpa.3160410507
[20] John, F.; Klainerman, S. Almost global existence to nonlinear wave equations in three space dimensions. *Comm. Pure Appl. Math.* **37** (1984), no. 4, 443–455. doi:10.1002/cpa.3160370403
[21] Kato, T. Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. *J. Functional Analysis.* **9** (1972), 296–305.
[22] Kessenich, P. Global existence with small initial data for three-dimensional incompressible isotropic viscoelastic materials. Preprint, 2009. arXiv:0903.2824 [math.AP]
[23] Klainerman, S. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.* **38** (1985), no. 3, 321–332. doi:10.1002/cpa.3160380305
[24] Klainerman, S. The null condition and global existence to nonlinear wave equations. *Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*, 293–326. Lectures in Applied Mathematics, 23. American Mathematical Society, Providence, R.I., 1986.
[25] Klainerman, S.; Sideris, T. C. On almost global existence for nonrelativistic wave equations in 3D. *Comm. Pure Appl. Math.* **49** (1996), no. 3, 307–321. doi:10.1002/(SICI)1097-0312(199603)49:3<307::AID-CPA4>3.0.CO;2-H
[26] Lei, Z. On 2D viscoelasticity with small strain. *Arch. Ration. Mech. Anal.* **198** (2010), no. 1, 13–37. doi:10.1007/s00205-010-0346-2
[27] Lei, Z. Rotation-strain decomposition for the incompressible viscoelasticity in two dimensions. *Discrete Contin. Dyn. Syst.* **34** (2014), no. 7, 2861–2871. doi:10.3934/dcds.2014.34.2861
[28] Lei, Z. Global well-posedness of incompressible elastodynamics in two dimensions. *Comm. Pure Appl. Math.* **69** (2016), no. 11, 2072–2106. doi:10.1002/cpa.21633
[29] Lei, Z.; Liu, C.; Zhou, Y. Global existence for a 2D incompressible viscoelastic model with small strain. *Commun. Math. Sci.* **5** (2007), no. 3, 595–616.
[30] Lei, Z.; Liu, C.; Zhou, Y. Global solutions for incompressible viscoelastic fluids. *Arch. Ration. Mech. Anal.* **188** (2008), no. 3, 371–398. doi:10.1007/s00205-007-0089-x
[31] Lei, Z.; Sideris, T. C.; Zhou, Y. Almost global existence for 2-D incompressible isotropic elastodynamics. *Trans. Amer. Math. Soc.* **367** (2015), no. 11, 8175–8197. doi:10.1090/tran/6294
[32] Lei, Z.; Wang, F. Uniform bound of the highest energy for the three dimensional incompressible elastodynamics. *Arch. Ration. Mech. Anal.* **216** (2015), no. 2, 593–622. doi:10.1007/s00205-014-0815-0
[33] Lei, Z.; Zhou, Y. Global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit. *SIAM J. Math. Anal.* **37** (2005), no. 3, 797–814. doi:10.1137/040618813
[34] Lin, F. Some analytical issues for elastic complex fluids. *Comm. Pure Appl. Math.* **65** (2012), no. 7, 893–919. doi:10.1002/cpa.21402

[35] Lin, F.-H.; Liu, C.; Zhang, P. On hydrodynamics of viscoelastic fluids. *Comm. Pure Appl. Math.* **58** (2005), no. 11, 1437–1471. doi:10.1002/cpa.20074

[36] Lin, F.; Zhang, P. On the initial-boundary value problem of the incompressible viscoelastic fluid system. *Comm. Pure Appl. Math.* **61** (2008), no. 4, 539–558. doi:10.1002/cpa.20219

[37] Masmoudi, N. Remarks about the inviscid limit of the Navier-Stokes system. *Comm. Math. Phys.* **270** (2007), no. 3, 777–788. doi:10.1007/s00220-006-0171-5

[38] Masmoudi, N. Global existence of weak solutions to the FENE dumbbell model of polymeric flows. *Invent. Math.* **191** (2013), no. 2, 427–500. doi:10.1007/s00222-012-0399-y

[39] Qian, J.; Zhang, Z. Global well-posedness for compressible viscoelastic fluids near equilibrium. *Arch. Ration. Mech. Anal.* **198** (2010), no. 3, 835–868. doi:10.1007/s00205-010-0351-5

[40] Sideris, T. C. The null condition and global existence of nonlinear elastic waves. *Invent. Math.* **123** (1996), no. 2, 323–342. doi:10.1007/s002220050030

[41] Sideris, T. C. Nonresonance and global existence of prestressed nonlinear elastic waves. *Ann. of Math. (2)* **151** (2000), no. 2, 849–874. doi:10.2307/121050

[42] Sideris, T. C.; Thomases, B. Global existence for three-dimensional incompressible isotropic elastodynamics via the incompressible limit. *Comm. Pure Appl. Math.* **58** (2005), no. 6, 750–788. doi:10.1002/cpa.20049

[43] Sideris, T. C.; Thomases, B. Local energy decay for solutions of multi-dimensional isotropic symmetric hyperbolic systems. *J. Hyperbolic Differ. Equ.* **3** (2006), no. 4, 673–690. doi:10.1142/S0219891606000975

[44] Sideris, T. C.; Thomases, B. Global existence for three-dimensional incompressible isotropic elastodynamics. *Comm. Pure Appl. Math.* **60** (2007), no. 12, 1707–1730. doi:10.1002/cpa.20196

[45] Swann H. S. G. The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in $\mathbb{R}^3$. *Trans. Amer. Math. Soc.* **157** (1971), 373–397. doi:10.2307/1995853

[46] Tahvildar-Zadeh, A. S. Relativistic and nonrelativistic elastodynamics with small shear strains. *Ann. Inst. H. Poincaré Phys. Théor.* **69** (1998), no. 3, 275–307.

[47] Wang, X. Global existence for the 2D incompressible isotropic elastodynamics for small initial data. *Ann. Henri Poincaré* **18** (2017), no. 4, 1213–1267. doi:10.1007/s00023-016-0538-x

[48] Zhang, T.; Fang, D. Global existence of strong solution for equations related to the incompressible viscoelastic fluids in the critical L^p framework. *SIAM J. Math. Anal.* **44** (2012), no. 4, 2266–2288. doi:10.1137/110851742

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