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GLOBAL SOLVABILITY FOR SYSTEMS OF NONLINEAR WAVE EQUATIONS WITH MULTIPLE SPEEDS IN TWO SPACE DIMENSIONS

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Abstract. In this paper we deal with systems of nonlinear wave equations in two space dimensions. When the system has common propagation speeds and cubic nonlinearity, the small data global existence result was obtained by Katayama [9], provided that the cubic part of Taylor's expansion for the nonlinearity satisfies the so-called null condition. The aim of this paper is to extend the result to the case where the system has multiple speeds of propagation. To realize this, we make use of a kind of Hardy's inequality given in Lemma 2.2 below, which creates the loss of decay but only with respect to $(1+||x|-c_it|)$. Thus we are able to absorb such a loss by means of the decay estimates in Proposition 4.2 below.

1. Introduction

We consider the Cauchy problem for the following system of nonlinear wave equations:

$$\Box_i u^i \equiv \partial_t^2 u^i - c_i^2 \Delta u^i = F^i(u, \partial u, \partial^2 u) \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \tag{1.1}$$

$$u^i(x,0) = \varepsilon f^i(x), \quad \partial_t u^i(x,0) = \varepsilon g^i(x) \quad \text{for} \quad x \in \mathbb{R}^n \tag{1.2}$$

for $n=2$. Here i runs from 1 to m with m an integer, $u^i = (u^i_1, \dots, u^i_{p_i})$ with $p_i \geq 1$ and $u = {}^t(u^1, \dots, u^m)$. We denote $\partial = (\partial_0, \dots, \partial_n)$ and $\partial^2 =$

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$(\partial_\alpha \partial_\beta)_{\alpha,\beta=0,\dots,n}$, where $\partial_0 = \partial/\partial t$ and $\partial_i = \partial/\partial x_i$ $(i = 1, \dots, n)$. Besides, ε is a small positive parameter and the propagation speeds $c_1, \dots, c_m$ are different from each other; that is,

$$0 < c_1 < c_2 < \cdots < c_m. \tag{1.3}$$

We suppose that the initial data f^i and g^i are smooth functions in their arguments taking values in $\mathbb{R}^{p_i}$, and are compactly supported. We also suppose that the nonlinear term $F^i = F^i(u, v, w)$ is a smooth function in its arguments around the origin taking values in $\mathbb{R}^{p_i}$, where $u \in \mathbb{R}^p$, $v \in \mathbb{R}^{(n+1)p}$, and $w \in \mathbb{R}^{(n+1)^2 p}$, with $p = p_1 + \cdots + p_m$. A basic assumption on F^i is

$$F^i(u, v, w) = O(|u|^q + |v|^q + |w|^q) \quad \text{near} \quad (u, v, w) = 0 \tag{1.4}$$

with some integer q such that $q \geq 2$.

The aim of this paper is to study the problem (1.1) and (1.2) when $n = 2$, and to find a sufficient condition to guarantee the global (in time) existence of a small-amplitude solution for the problem. Before stating our theorem, we recall several known results briefly. In Christodoulou [3] and Klainerman [15], the problem was independently handled for the case where $n = 3$, $q = 2$, and $m = 1$, and the global existence result was proven for sufficiently small ε by different approaches, provided that the quadratic part of Taylor's expansion for $F^1(u, v, w)$ around the origin satisfies the so-called "null condition." A typical example which satisfies the null condition is $|\partial_t u^1|^2 - c_1^2 |\nabla u^1|^2$ with $\nabla = (\partial_1, \dots, \partial_n)$. A generalization of this result for $m \geq 2$ has been done by many authors. When $n = 3$, Kubota and Yokoyama [18] treated the problem assuming an extra condition on F^i, and then Katayama [10] relaxed the assumption on F^i as follows: the quadratic part of it satisfies the null condition and does not include u itself. (For the case where $F^i(u, v, w)$ does not depend explicitly on u itself, see [16], [1], [8], [24], [22], [23], [7], and references therein.) Though the case where F^i may contain u itself in its quadratic part has been studied by [11], [12], and [21], we do not go further in that direction.

Now we turn our attention to the case of $n = 2$. Let $F^i_{(3)}(u, v, w)$ be the cubic part of Taylor's expansion for $F^i(u, v, w)$ around the origin. When $q = 3$ and $m = 1$, Katayama [9] proved the small data global existence result under the assumption that $F^i_{(3)}$ satisfies the null condition (for the case where $n = q = 2$, $m = 1$ and $F^1(u, v, w) \equiv F^1(v, w)$, see Alinhac [2]). Roughly speaking, we prove an analogous result to [9] for $m \geq 2$ in this article. To be more specific, we describe assumptions on F^i in what follows. First of all, we assume $q = 3$ in (1.4). Since we may assume that F^i is linear with respect to the second-order derivatives of u without loss of generality, F^i can be written in the following form:

$$F^i(u,\partial u,\partial^2 u) = \sum_{j=1}^{m}\sum_{\gamma,\delta=0}^{2} H^{ij}_{\gamma\delta}(u,\partial u)\partial_\gamma\partial_\delta u^j + K^i(u,\partial u) \quad (i=1,\dots,m), \tag{1.5}$$

where $H^{ij}_{\gamma\delta}(u,v)$ is a $p_i\times p_j$ matrix-valued function and $K^i(u,v)$ is a p_i vector-valued function (see e.g. Courant and Hilbert [4], Chapter I, Section 7). Therefore (1.4) with $q=3$ implies

$$H^{ij}_{\gamma\delta}(u,v) = O(|u|^2+|v|^2), \quad K^i(u,v) = O(|u|^3+|v|^3) \text{ near } (u,v)=0. \tag{1.6}$$

In order to guarantee the existence of the local solution for the problem, we need to assume

$$H^{ij}_{\gamma\delta} = {}^t H^{ji}_{\gamma\delta} \quad (i,j=1,\dots,m;\ \gamma,\delta=0,1,2). \tag{1.7}$$

Besides, since we consider only small and smooth solutions in this paper, we may suppose

$$H^{ij}_{00}=0, \quad H^{ij}_{\gamma\delta}=H^{ij}_{\delta\gamma} \quad (i,j=1,\dots,m;\ \gamma,\delta=0,1,2) \tag{1.8}$$

(for derivation of the first assumption, see for instance the proof of Theorem 4.1 in [18]).

On one hand, by (1.4) with $q=3$ one can also write F^i as

$$F^i(u,\partial u,\partial^2 u) = F^i_{(3)}(u,\partial u,\partial^2 u) + H^i(u,\partial u,\partial^2 u), \tag{1.9}$$

where

$$H^i(u,\partial u,\partial^2 u) = O(|u|^4+|\partial u|^4+|\partial^2 u|^4) \text{ near } (u,\partial u,\partial^2 u)=(0,0,0). \tag{1.10}$$

We divide $F^i_{(3)}$ into several groups as follows:

$$F^i_{(3)}(u,\partial u,\partial^2 u) = \sum_{j=1}^{m} N^{ij}(u,\partial u^j,\partial^2 u^j) + R^i(u,\partial u,\partial^2 u), \tag{1.11}$$

where N^{ij} is a homogeneous polynomial only in $(u,\partial u^j,\partial^2 u^j)$ of degree 3, while R^i is a homogeneous polynomial in $(u,\partial u,\partial^2 u)$ of degree 3 being explicitly written as

$$R^i(u,\partial u,\partial^2 u) = \sum_{\substack{j,k=1,\dots,m\\ j\neq k}} \sum_{\substack{r=1,\dots,p_j\\ s=1,\dots,p_k}} \sum_{|a|,|b|=1,2} Q^{ijkrs}_{ab}(u,\partial u,\partial^2 u)\partial^a u^j_r \partial^b u^k_s. \tag{1.12}$$

Here $Q_{ab}^{ijkrs}(u, \partial u, \partial^2 u)$ is a homogeneous polynomial of degree 1, and u_r^j $(1 \le r \le p_j)$ denotes the r-th component of u^j. Finally, we suppose that N^{ij} $(i, j = 1, \dots, m)$ satisfies the following condition:

$$N^{ij}(\lambda, (X_\alpha \mu^j)_{\alpha=0,1,2}, (X_\alpha X_\beta \nu^j)_{\alpha,\beta=0,1,2}) \equiv 0 \tag{1.13}$$

holds for any $\lambda \in \mathbb{R}^p$, $\mu^j, \nu^j \in \mathbb{R}^{p_j}$, and $(X_0, X_1, X_2) \in \mathbb{R}^3$ satisfying $X_0^2 = c_j^2(X_1^2 + X_2^2)$. Note that when $m = 1$, the condition (1.13) is equivalent to the null condition which was introduced in [3] and [15].

Under these assumptions, we prove the following.

Theorem 1.1. *Let $n = 2$. We suppose that* (1.5) *through* (1.13) *hold. Then for any $f^i, g^i \in C_0^\infty(\mathbb{R}^2)$, there exists a positive number ε_0 such that the problem* (1.1) *and* (1.2) *admits a unique solution $u \in C^\infty([0,\infty) \times \mathbb{R}^2 : \mathbb{R}^p)$ for $0 < \varepsilon \le \varepsilon_0$.*

When $n = 3$, a suitable L^2 bound for u itself was obtained by [10], which enables one to prove the existence of a global solution. But one can not expect to get such a suitable L^2 bound when $n = 2$, so that in [9] u itself was evaluated in $L^p(\mathbb{R}^2)$ with some $p \in (2, \infty)$ (see also [19]). Then it is necessary to apply the following inequality given by Lemma 3.3 in [9] to the solution: If $u(x,t) = 0$ for $|x| \ge ct + M$ with positive numbers $c, M > 0$, then

$$\|(1 + ||\cdot| - ct|)^{-1} u(\cdot, t)\|_{L^2(\mathbb{R}^2)} \le C \|\partial_r u(\cdot, t)\|_{L^2(\mathbb{R}^2)}. \tag{1.14}$$

However, it does not seem to be useful if $m \ge 2$, because the solution $u^i(x,t)$ of (1.1) does not satisfy $u^i(x,t) = 0$ for $|x| \ge c_i t + M$ $(1 \le i \le m-1)$ in general, even if the initial data vanishes for $|x| \ge M$. To overcome the difficulty, we make use of a variant of Hardy's inequality given in Lemma 2.2 below. Moreover, the inequality leads us to employ the homogeneous Sobolev space to evaluate u itself, instead of $L^p(\mathbb{R}^2)$ used in [9]. In this way, we are able to establish the theorem. We remark that Theorem 1.2 in [9] follows from it, since $R^i(u, v, w)$ disappears when $m = 1$. However, the requirement that (1.13) holds for any $1 \le i, j \le m$ seems to be strong. Indeed, if $F^i(u, v, w) = F^i(v, w)$, then we need (1.13) only for $j = i$ $(1 \le i \le m)$, which was shown in [8]. We hope that the assumption in Theorem 1.1 will be relaxed like this.

This paper is organized as follows. In Section 2 we collect some notation and preliminary estimates. We give basic estimates for $N^{ij}(u, \partial u^j, \partial^2 u^j)$ in Section 3. Section 4 is devoted to deriving weighted L^∞ estimates for the inhomogeneous wave equation. Especially, we give a refined estimate for the first-order derivatives of the solution in Proposition 4.2 below, which improves the decay rate in comparison with former estimates (see for instance [1] and [8]). We establish some estimates for the solution in Sobolev spaces by using the Fourier representation and energy method in Section 5. Finally, in Section 6, we prove Theorem 1.1.

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2. Notation and preliminaries

We introduce the following vector fields:

$$S = t\partial_t + r\partial_r, \quad \Omega = x_1\partial_2 - x_2\partial_1, \tag{2.1}$$

where $r = |x|$ and $r\partial_r = x_1\partial_1 + x_2\partial_2$, and denote $\Gamma = (\Gamma_1, \dots, \Gamma_4) = (\partial_1, \partial_2, \Omega, S)$. Then we have the following commutator relations:

$$[\Gamma_k, \Box_{c_i}] = 0 \quad \text{if} \quad k = 1, 2, 3, \quad [\Gamma_4, \Box_{c_i}] = -2\Box_{c_i} \tag{2.2}$$

for $i = 1, \dots, m$, and

$$\begin{aligned} &[\partial_\alpha, \partial_\beta] = 0 \ (\alpha, \beta = 0, 1, 2), \ [\Omega, \partial_0] = 0, \ [\Omega, \partial_1] = -\partial_2, \ [\Omega, \partial_2] = \partial_1, \\ &[S, \partial_\alpha] = -\partial_\alpha \ (\alpha = 0, 1, 2), \ [S, \Omega] = 0. \end{aligned} \tag{2.3}$$

Here [,] denotes the usual commutator of linear operators.

Let $v(x,t)$ be a smooth function defined on $\mathbb{R}^2 \times [0, T)$ taking values in $\mathbb{R}^p$ with p an integer. For such a function we set

$$|v(x,t)|_k = \sum_{|a| \le k} \sum_{i=1}^{p} |\Gamma^a v_i(x,t)|,$$

where k is a nonnegative integer, $a = (a_1, \dots, a_4)$ is a multi-index, $\Gamma^a = \Gamma_1^{a_1} \cdots \Gamma_4^{a_4}$ and $|a| = a_1 + \cdots + a_4$. Besides, we define

$$\|v(t)\|_k^2 = \int_{\mathbb{R}^2} |v(x,t)|_k^2 dx.$$

Let $u(x,t) = {}^t(u^1(x,t), \dots, u^m(x,t))$ be a function defined on $\mathbb{R}^2 \times [0, T)$ such that each $u^i(x,t)$ is a $\mathbb{R}^{p_i}$-valued smooth function. For such a function we set

$$[u(x,t)]_k = \sum_{i=1}^{m} w_i(|x|, t)|u^i(x,t)|_k, \ \langle u(x,t)\rangle_k = \sum_{i=1}^{m} \eta_i(|x|, t)|u^i(x,t)|_k, \tag{2.4}$$

where w_i and η_i are weight functions defined by

$$w_i(r,t) = (1+r)^{\frac{1}{2}}(1 + |r - c_i t|)^{1+\nu} \quad (1 \le i \le m), \quad w_0(r,t) = (1+r)^{\frac{3}{2}+\nu},$$

$$\eta_i(r,t) = (1+t+r)^{\frac{1}{2}}(1+|r-c_i t|)^\nu \quad (1 \le i \le m),$$
$$\eta_0(r,t) = (1+t+r)^{\frac{1}{2}}(1+r)^\nu$$

for $r \ge 0$, $t \ge 0$, and $0 < \nu < 1/2$. Moreover, we also use the following notation:

$$\mathcal{U}_k(x,t) = \langle u(x,t)\rangle_{k+1} + [\partial u(x,t)]_k, \quad \mathcal{U}_k(t) = \sup_{x\in\mathbb{R}^2} \mathcal{U}_k(x,t). \tag{2.5}$$

Let $c_{m+1} = \min_{1\le i\le m}\{c_i - c_{i-1}\}/3$ with $c_0 = 0$. We see $c_{m+1} > 0$ from (1.3). Then we define

$$\Lambda_i(t) = \{(x,s) \in \mathbb{R}^2 \times [0,t] : \ ||x| - c_i s| \le c_{m+1}s, \ |x| \ge 1\}$$

for $i = 1,\dots,m$, and $\Lambda_0(t)$ is the complementary set of $\bigcup_{i=1}^m \Lambda_i(t)$ in $\mathbb{R}^2 \times [0,t]$. We see from (1.3) that $\Lambda_i \cap \Lambda_j = \emptyset$ if $i \ne j$.

In what follows, we collect some elementary lemmas.

Lemma 2.1. *Let $v \in C_0^\infty(\mathbb{R}^n)$ and $w \in C^1([0,\infty))$. Suppose that $w(r) > 0$ for $r \ge 0$ and that there is a positive constant A such that $|w'(r)| \le Aw(r)$ for $r \ge 0$. Then we have*

$$|x|^{\frac{n-1}{2}} w(|x|)|v(x)| \le C \sum_{|a|\le[\frac{n-1}{2}]+1} (\|w(|\cdot|)\Omega^a v\|_{L^2(\mathbb{R}^n)} + \|w(|\cdot|)\partial_r\Omega^a v\|_{L^2(\mathbb{R}^n)}) \tag{2.6}$$

for $x \in \mathbb{R}^n$, where $\partial_r = \sum_{j=1}^n (x_j/|x|)\partial_j$ and C is a constant independent of x, v, and w.

For the proof, see Proposition 1 in [14].

Lemma 2.2. *For any $R \ge 0$, $0 \le s < 1/2$, and $v \in C_0^\infty(\mathbb{R}^n)$, we have*

$$\left\| \frac{v}{||\cdot| - R|^s} \right\|_{L^2(\mathbb{R}^n)} \le C\|v\|_{\dot H^s(\mathbb{R}^n)}, \tag{2.7}$$

where C is a constant independent of R and v. Here

$$\|v\|_{\dot H^s(\mathbb{R}^n)} = \||\xi|^s \mathcal{F}[v]\|_{L^2(\mathbb{R}^n_\xi)}$$

with $\mathcal{F}[v](\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} v(x)dx$.

For the proof, see Theorem 4.4 and Lemma 4.3 (1) in [5].

Lemma 2.3. *Let κ_1 and κ_2 be positive numbers, $c \ge 0$, and $1 \le p \le \infty$. If $p\kappa_1 \ge n-1$ and $p\kappa_2 > 1$, then we have*

$$\|(1+t+|\cdot|)^{-\kappa_1}(1+|ct-|\cdot||)^{-\kappa_2}\|_{L^p(\mathbb{R}^n)} \le C(1+t)^{-\kappa_1+\frac{n-1}{p}} \quad \text{for} \quad t \ge 0. \tag{2.8}$$

Proof. Since the estimate (2.8) for $p = \infty$ is obvious, we shall assume $1 \le p < \infty$. By the assumption $-p\kappa_1 + n - 1 \le 0$, the left-hand side of (2.8) raised to the power p is estimated by

$$(1+t)^{-p\kappa_1+n-1}\int_{\mathbb{R}^2}(1+t+|x|)^{-(n-1)}(1+|ct-|x||)^{-p\kappa_2}dx.$$

Switching to polar coordinates, we see from $p\kappa_2 > 1$ that the above integral is bounded by some constant; hence, (2.8) holds. This completes the proof. □

At the end of the section, we summarize basic estimates of the solution of the homogeneous wave equation.

Lemma 2.4. *Let $v^i(x,t)$ be a solution of*

$$\partial_t^2 v^i - c_i^2\Delta v^i = 0 \quad in \quad \mathbb{R}^2\times(0,\infty) \tag{2.9}$$

$$v^i(x,0) = \varepsilon\phi^i(x), \quad \partial_t v^i(x,0) = \varepsilon\psi^i(x) \quad for \quad x\in\mathbb{R}^2, \tag{2.10}$$

where $\varepsilon > 0$, and ϕ^i and ψ^i are smooth and compactly supported functions. Then denoting $v = {}^t(v^1,\dots,v^m)$, we have

$$\langle v(x,t)\rangle_0 \le C\varepsilon, \quad [\partial v(x,t)]_0 \le C\varepsilon \quad for \quad (x,t)\in\mathbb{R}^2\times[0,\infty), \tag{2.11}$$

where C is a constant independent of ε. Moreover, for $\rho > 0$ we have

$$\|v(t)\|_{\dot H^\rho} \le C\varepsilon \quad for \quad t \ge 0. \tag{2.12}$$

Proof. First we prove (2.11). As is well known, we have

$$|v^i(x,t)| \le C\varepsilon(1+t+|x|)^{-\frac12}(1+|c_it-|x||)^{-\frac12} \text{ for } (x,t)\in\mathbb{R}^2\times[0,\infty) \tag{2.13}$$

(for the proof, see e.g. Lemma 1 in [6]). Since $0<\nu<1/2$, the first estimate in (2.11) holds. Let $\tilde\Gamma = (\tilde\Gamma_1,\dots,\tilde\Gamma_7)$ be a collection of vector fields defined by

$$\tilde\Gamma_j = \Gamma_j\ (j=1,\dots,4),\ \tilde\Gamma_5 = \partial_t,\ \tilde\Gamma_6 = c_it\partial_1 + \frac{x_1}{c_i}\partial_t,\ \tilde\Gamma_7 = c_it\partial_2 + \frac{x_2}{c_i}\partial_t.$$

Then we have $[\tilde\Gamma_j, \Box_{c_i}] = 0$ $(j\ne 4)$ and $[\tilde\Gamma_4,\Box_{c_i}] = -2\Box_{c_i}$. Therefore (2.13) yields

$$\sum_{|a|\le 1}|\tilde\Gamma^a v^i(x,t)| \le C\varepsilon(1+t+|x|)^{-\frac12}(1+|c_it-|x||)^{-\frac12} \text{ for } (x,t)\in\mathbb{R}^2\times[0,\infty). \tag{2.14}$$

Note that for any C^1 function $w(x,t)$ we have

$$|\partial w(x,t)| \le C(1+|c_it-|x||)^{-1}\sum_{|a|\le1}|\tilde\Gamma^a w(x,t)| \text{ for } (x,t)\in\mathbb{R}^2\times[0,\infty)$$

(for the proof, see e.g. Lemma 3.4 in [9]). Thus the second one in (2.11) is also valid.

Next we prove (2.12). We shall use

$$\||\xi|^{-s}\mathcal{F}[\phi]\|_{L^2(\mathbb{R}^2_\xi)} \le C\|\phi\|_{L^q(\mathbb{R}^2_x)} \quad \text{if} \quad 0 \le s < 1,\ q = 2(s+1)^{-1} \tag{2.15}$$

for the case where $0 < \rho \le 1$. The estimate follows from the Hardy-Littlewood-Sobolev inequality, since $\mathcal{F}[|x|^{-k}](\xi) = C_{n,k}|\xi|^{-(n-k)}$ for $0 < k < n$. Then, recalling the Fourier representation of $v^i(x,t)$,

$$\mathcal{F}[v^i](\xi,t) = \varepsilon\cos(t|\xi|)\mathcal{F}[\phi^i](\xi) + \varepsilon|\xi|^{-1}\sin(t|\xi|)\mathcal{F}[\psi^i](\xi),$$

we obtain (2.12). This completes the proof. □

3. An estimate for the null form

Proposition 3.1. *Assume that* (1.8) *and* (1.13) *hold. Then for* $u = {}^t(u^1,\dots,u^m) \in C^\infty(\mathbb{R}^2\times[0,T) : \mathbb{R}^p)$ *with* $u^i(x,t)$ *a* p_i *vector-valued function and for any nonnegative integer* k*, there is a constant* C_k*, independent of* u *and* T*, such that*

$$|N^{ij}(u,\partial u^j,\partial^2 u^j)(x,t)|_k \le \frac{C_k||x|-c_jt|}{1+|x|+t}\Phi_k(u)(x,t) + \frac{C_k}{1+|x|+t}\Psi_k(u)(x,t) \tag{3.1}$$

for $(x,t) \in \Lambda_j(T)$*, where we have set*

$$\Phi_k(u) = |u|_{[\frac{k}{2}]+1}|\partial u^j|_{[\frac{k}{2}]+1}|\partial u^j|_{k+1} + |\partial u^j|^2_{[\frac{k}{2}]+1}|u|_k,$$
$$\Psi_k(u) = |u|^2_{[\frac{k}{2}]+1}|\partial u^j|_{k+1} + |u|_{[\frac{k}{2}]+1}|\partial u^j|_{[\frac{k}{2}]+1}|u|_{k+1}.$$

Proof. We see from the assumptions that N^{ij} is linear with respect to u. In fact, by (1.13), $N^{ij}(u,0,0) \equiv 0$ for any $u \in \mathbb{R}^p$. Namely, N^{ij} does not include terms which are cubic in u. Moreover, if N^{ij} includes a term which is quadratic in u, it follows from (1.13) that such a term is expressed as $Q^{ij}(u)\Box_j u^j$, where $Q^{ij}(u)$ is a $p_i \times p_j$ matrix. But the assumption (1.8) excludes such a possibility. Therefore, each component N^{ij}_r $(1 \le r \le p_i)$ of N^{ij} can be written as

$$N^{ij}_r(u,\partial u^j,\partial^2 u^j) = \sum_{h,k,l=1}^{p_j}\Big(\sum_{\alpha,\beta,\gamma,\delta=0}^{2} A^{hkl}_{\alpha\beta\gamma\delta}\partial_\alpha u^j_h\partial_\beta u^j_k\partial_\gamma\partial_\delta u^j_l + \sum_{\alpha,\beta,\gamma=0}^{2} B^{hkl}_{\alpha\beta\gamma}\partial_\alpha u^j_h\partial_\beta u^j_k\partial_\gamma u^j_l\Big) \tag{3.2}$$

$$+\sum_{q=1}^{m}\sum_{h=1}^{p_q}\sum_{k,l=1}^{p_j}\Big(\sum_{\alpha,\beta,\gamma=0}^{2} C^{qhkl}_{\alpha\beta\gamma}u^q_h\partial_\alpha u^j_k\partial_\beta\partial_\gamma u^j_l+\sum_{\alpha,\beta=0}^{2} D^{qhkl}_{\alpha\beta}u^q_h\partial_\alpha u^j_k\partial_\beta u^j_l\Big),$$

where $A^{hkl}_{\alpha\beta\gamma\delta}$, $B^{hkl}_{\alpha\beta\gamma}$, $C^{qhkl}_{\alpha\beta\gamma}$, and $D^{qhkl}_{\alpha\beta}$ are constants.

To gain such an additional decay as in (3.1), we introduce the operators

$$R_0=\partial_t+c_j\partial_r,\quad R_k=\partial_k-\frac{x_k}{r}\partial_r\quad (k=1,2).$$

Then by (2.3) and

$$[\partial_l,\partial_r]=\frac{1}{r}R_l\ \ (l=1,2),\quad [\Omega,\partial_r]=0,\quad [S,\partial_r]=-\partial_r, \tag{3.3}$$

one can verify the following.

Lemma 3.1. *For any $l,k=1,2$ and any integer m, we have*

$$[S,R_0]=-R_0,\quad [\Omega,R_0]=0,\quad [\partial_l,R_0]=\frac{c_j}{r}R_l,$$
$$[S,R_k]=-R_k,\quad [\Omega,R_1]=-R_2,\quad [\Omega,R_2]=R_1,$$
$$[\partial_l,R_k]=-\frac{1}{r}\delta_{lk}\partial_r+\frac{x_lx_k}{r^3}\partial_r-\frac{x_k}{r^2}R_l$$

and

$$[S,\frac{1}{r^m}]=-\frac{m}{r^m},\quad [\Omega,\frac{1}{r^m}]=0,\quad [\partial_l,\frac{1}{r^m}]=-\frac{mx_l}{r^{m+2}}.$$

Lemma 3.2. *For a real-valued, smooth function $v(x,t)$ and $1\le j\le m$, we have*

$$|R_\alpha v(x,t)|_k\le\frac{C_k||x|-c_jt|}{1+|x|+t}|\partial_r v(x,t)|_k+\frac{C_k}{1+|x|+t}|v(x,t)|_{k+1}, \tag{3.4}$$

where $(x,t)\in\Lambda_j(T)$ and $\alpha=0,1,2$.

Proof. When $k=0$, we easily find that (3.4) follows from

$$R_0=-\frac{r-c_jt}{t}\partial_r+\frac{1}{t}S\qquad\text{for}\quad t>0, \tag{3.5}$$

$$R_1=-\frac{x_2}{r^2}\Omega,\quad R_2=\frac{x_1}{r^2}\Omega\quad\text{for}\quad r>0, \tag{3.6}$$

since $|x|$ and t are equivalent to $1+t+|x|$ for $(x,t)\in\Lambda_j(T)$. When $k=1$, by Lemma 3.1 and (3.4) with $k=0$, we have

$$|\Gamma R_\alpha v(x,t)|\quad\le\quad C(|R_\alpha\Gamma v(x,t)|+\sum_{\alpha=0}^{2}|R_\alpha v(x,t)|+\frac{1}{r}|\partial_r v(x,t)|)$$

$$\leq \frac{C||x|-c_jt|}{1+|x|+t}|\partial_r v(x,t)|_1 + \frac{C}{1+|x|+t}|v(x,t)|_2.$$

When $k \geq 2$, by Lemma 3.1 we obtain (3.4), inductively. □

End of the proof of Proposition 3.1. For simplicity, we assume $B^{hkl}_{\alpha\beta\gamma} = C^{qhkl}_{\alpha\beta\gamma} = D^{qhkl}_{\alpha\beta} = 0$ in (3.2) (the general case can be treated in a similar fashion). We take $\omega_0 = -c_j$ and $\omega_l = x_l/r$ $(l = 1, 2)$ so that $\omega_0^2 = c_j^2(\omega_1^2+\omega_2^2)$. Choosing $X_\alpha = \omega_\alpha$ $(\alpha = 0, 1, 2)$, $\lambda = u$, $\mu^j = \partial_r u^j$, and $\nu^j = \partial_r^2 u^j$ in (1.13), we get from (3.2)

$$\sum_{h,k,l=1}^{p_j} \sum_{\alpha,\beta,\gamma,\delta=0}^{2} A^{hkl}_{\alpha\beta\gamma\delta}\omega_\alpha\omega_\beta\omega_\gamma\omega_\delta \partial_r u^j_h \partial_r u^j_k \partial_r^2 u^j_l = 0.$$

Therefore, $N^{ij}_r(u, \partial u^j, \partial^2 u^j)$ is rewritten as

$$\sum_{\alpha,\beta,\gamma,\delta=0}^{2} A^{hkl}_{\alpha\beta\gamma\delta}\{R_\alpha u^j_h \partial_\beta u^j_k \partial_\gamma\partial_\delta u^j_l + \omega_\alpha \partial_r u^j_h R_\beta u^j_k \partial_\gamma \partial_\delta u^j_l \tag{3.7}$$
$$+\omega_\alpha\omega_\beta \partial_r u^j_h \partial_r u^j_k R_\gamma \partial_\delta u^j_l + \omega_\alpha\omega_\beta\omega_\gamma \partial_r u^j_h \partial_r u^j_k \partial_r R_\delta u^j_l\},$$

since $R_0 = \partial_0 - \omega_0\partial_r$ and $R_k = \partial_k - \omega_k \partial_r$. We use the following relations derived by (3.3) for the last term:

$$\partial_r R_0 u^j_l = R_0 \partial_r u^j_l, \quad \partial_r R_k u^j_l = R_k \partial_r u^j_l - \frac{1}{r} R_k u^j_l \ (k = 1, 2).$$

Then applying Γ^a to (3.7) and using Lemma 3.2, we get (3.1). (Notice that (1.8) implies $A^{hkl}_{\alpha\beta 00} = 0$.) Thus we complete the proof. □

4. L^∞ ESTIMATES

We consider the following operator associated with the inhomogeneous wave equation whose propagation speed is c_i $(i = 1, \dots, m)$ and data F is in $C(\mathbb{R}^2 \times [0, T))$ $(T > 0)$:

$$L_{c_i}(F)(x,t) = \frac{1}{2\pi c_i}\int_0^t ds \int_{|x-y|<c_i(t-s)} \frac{F(y,s)}{\sqrt{c_i^2(t-s)^2 - |x-y|^2}} dy, \tag{4.1}$$

or

$$L_{c_i}(F)(x,t) = \frac{1}{2\pi}\int_0^t ds \int_0^{t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}} d\rho \int_{|\omega|=1} F(x + c_i\rho\omega, s) dS_\omega, \tag{4.2}$$

where $(x,t) \in \mathbb{R}^2 \times [0, T)$. We prove basic estimates of $L_{c_i}(F)(x,t)$ in Proposition 4.1 and of its spatial derivatives in Proposition 4.2. As an application of these estimates, we derive *a priori* estimates for a solution of (1.1) and (1.2) in Corollary 4.1.

Proposition 4.1. *Let $0 < \kappa < 1/2$ and $\mu > 0$. Then we have*

$$|L_{c_i}(F)(x,t)|(1+|x|+t)^{\frac{1}{2}}(1+|c_i t-|x||)^{\kappa} \le CM_{\kappa}(F)(t), \tag{4.3}$$

for $(x,t) \in \mathbb{R}^2 \times [0,T)$, where we have set

$$M_{\kappa}(F)(t) = \sum_{j=0}^{m} \sup_{(y,s)\in\Lambda_j(t)} \{|y|^{\frac{1}{2}} z_{\kappa+\mu}^{(j)}(|y|,s)|F(y,s)|\}, \tag{4.4}$$

$$z_{\kappa}^{(j)}(\lambda,s) = (1+\lambda+s)^{1+\kappa}(1+|\lambda-c_j s|), \quad c_0 = 0. \tag{4.5}$$

Here C is a constant depending only on μ, κ, and c_j.

Proof. Without loss of generality, we may assume $c_i = 1$. Let $\chi_j(y,s)$ be the characteristic function of $\Lambda_j(t)$. Then it follows that

$$\begin{aligned} |L_1(F)(x,t)| &\le \sum_{j=0}^{m} L_1(\chi_j|F|)(x,t) \qquad (4.6)\\ &\le \sum_{j=0}^{m} \sup_{(y,s)\in\mathbb{R}^2\times[0,t]} \{|y|^{\frac{1}{2}} z_{\kappa+\mu}^{(j)}(|y|,s)\chi_j(y,s)|F(y,s)|\} L_1(F_j)(x,t) \end{aligned}$$

with $F_j(y,s) = [|y|^{\frac{1}{2}} z_{\kappa+\mu}^{(j)}(|y|,s)]^{-1}$. Since $c_j > 0$ if $1 \le j \le m$, Theorem 1.1 in [17] shows the boundedness of $L_1(F_j)(x,t)$ in x and t for such j. Moreover, seeing the proof of the theorem, we find that the same is true also for $j = 0$, once we establish the following:

$$\begin{aligned} J \;&:=\; \int_{-\alpha}^{\alpha} (1+|\frac{\alpha+\beta}{2}|)^{-\frac{1}{2}-\mu}(t+r+\beta)^{-\frac{1}{2}} d\beta \qquad (4.7)\\ &\le\; C(1+\alpha)^{\frac{1}{2}}(1+t+r)^{-\frac{1}{2}} \qquad \text{for} \quad t>0,\ r>0,\ \text{and } 0<\alpha<t+r, \end{aligned}$$

which corresponds to (2.37) in [17]. When $0 < t+r < 1$, we have

$$J \le \int_{-\alpha}^{\alpha} \frac{1}{\sqrt{t+r+\beta}} d\beta \le 2\sqrt{t+r+\alpha},$$

which implies (4.7), since $\alpha < t+r < 1$, while, when $t+r \ge 1$, we have

$$J \le \sqrt{\frac{2}{t+r}} \int_{-\alpha}^{\alpha} (1+|\frac{\alpha+\beta}{2}|)^{-\frac{1}{2}} d\beta;$$

hence, (4.7) holds. Therefore (4.6) implies (4.5). This completes the proof.

Proposition 4.2. *Let* $0 < \nu < 1/2$ *and* $\mu > 0$. *Then we have*

$$|\partial_\ell L_{c_i}(F)(x,t)|(1+|x|)^{\frac{1}{2}}(1+|c_i t-|x||)^{1+\nu} \le C\widetilde{M}_\nu(F)(t) \quad (\ell=1,2) \quad (4.8)$$

for $(x,t) \in \mathbb{R}^2 \times [0,T)$, *where we have set*

$$\widetilde{M}_\nu(F)(t) = \sum_{|a|\le 1} M_\nu(\partial_x^a F)(t) + M_\nu(\Omega F)(t). \quad (4.9)$$

Here C *is a constant depending only on* ν, μ, *and* c_j.

Proof. We assume $c_i = 1$, as before. Since $\partial_\ell \partial L_1(F)(x,t) = L_1(\partial_\ell F)(x,t)$, it follows from Proposition 4.1 that

$$|\partial_\ell L_1(F)(x,t)|(1+r+t)^{\frac{1}{2}}(1+|t-r|)^{\nu} \le CM_\nu(\partial_\ell F)(t) \quad (\ell=1,2);$$

hence, (4.8) holds if $|t-r| \le 2$. In what follows, we assume $|t-r| \ge 2$. We set

$$\begin{aligned} E_1 &= \{(y,s) \in \mathbb{R}^2 \times [0,t): \ |y|+s > t-r, \ |x-y| < t-s\}, \\ E_2 &= \{(y,s) \in \mathbb{R}^2 \times [0,t): \ t-r-1 < |y|+s < (t-r)_+\}, \\ E_3 &= \{(y,s) \in \mathbb{R}^2 \times [0,t): \ |y|+s < (t-r-1)_+\}. \end{aligned}$$

Clearly, E_2 and E_3 are the empty set when $t-r<0$, and

$$\overline{E_1} \cup \overline{E_2} \cup \overline{E_3} = \{(y,s) \in \mathbb{R}^2 \times [0,t): \ |x-y| < t-s\}.$$

According to this decomposition, we define

$$P_j(F)(x,t) = \frac{1}{2\pi}\iint_{E_j} \frac{F(y,s)}{\sqrt{(t-s)^2-|x-y|^2}} dy\,ds \quad (j=1,2,3), \quad (4.10)$$

so that $\partial_\ell L_1(F)(x,t) = \sum_{j=1}^3 P_j(\partial_\ell F)(x,t)$.

Firstly we deal with $P_1(\partial_\ell F)(x,t)$. Following the computation made in Section 4 of [8], we find that

$$|P_1(\partial_\ell F)(x,t)| \le \widetilde{M}_\nu(F)(t)\sum_{k=0}^5 I_k, \quad (4.11)$$

where we have set

$$\begin{aligned} I_1 &= \sum_{j=0}^m \iint_{D_1} \frac{\lambda^{\frac{1}{2}}}{z^{(j)}_{\nu+\mu}(\lambda,s)} d\lambda\,ds \int_{-\varphi}^{\varphi} K_1(\lambda,\psi;r,t-s)d\psi, \\ I_2 &= \sum_{j=0}^m \int_{D_2'} \frac{\lambda^{\frac{1}{2}}}{z^{(j)}_{\nu+\mu}(\lambda,s)} d\sigma \int_0^1 K_2(\lambda,\tau;r,t-s)d\tau, \end{aligned}$$

$$I_3 = \sum_{j=0}^{m} \iint_{D_2} \frac{1}{\lambda^{\frac{1}{2}} z^{(j)}_{\nu+\mu}(\lambda, s)} d\lambda\, ds \int_0^1 K_2(\lambda, \tau; r, t-s) d\tau,$$

$$I_4 = \sum_{j=0}^{m} \iint_{D_2} \frac{\lambda^{\frac{1}{2}}}{z^{(j)}_{\nu+\mu}(\lambda, s)} d\lambda\, ds \int_0^1 |\partial_\lambda K_2(\lambda, \tau; r, t-s)| d\tau,$$

$$I_5 = \sum_{j=0}^{m} \iint_{D_2} \frac{\lambda^{\frac{1}{2}}}{z^{(j)}_{\nu+\mu}(\lambda, s)} d\lambda\, ds \int_0^1 |(\partial_\lambda \Psi \cdot K_2)(\lambda, \tau; r, t-s)| d\tau.$$

Here we have used the following notation:

$$K_1(\lambda, \psi; r, t) = (2\pi)^{-1}\{t^2 - r^2 - \lambda^2 + 2r\lambda\cos\psi\}^{-\frac{1}{2}},$$

$$K_2(\lambda, \tau; r, t) = (2\pi)^{-1}\{2r\lambda\tau(1-\tau)(2-(1-\cos\varphi)\tau)\}^{-\frac{1}{2}},$$

$$\varphi(\lambda; r, t) = \arccos\left[\frac{r^2+\lambda^2-t^2}{2r\lambda}\right],$$

$$\Psi(\lambda, \tau; r, t) = \arccos[1-(1-\cos\varphi(\lambda; r, t))\tau],$$

$$D_1 = \{(\lambda, s) \in (0,\infty)\times(0,t) :\ \lambda_- < \lambda \le \lambda_- + \delta \text{ or } \lambda_+ - \delta \le \lambda < \lambda_+\},$$

$$D_2 = \{(\lambda, s) \in (0,\infty)\times(0,t) :\ \lambda_- + \delta \le \lambda \le \lambda_+ - \delta\},$$

$$D_2' = \{(\lambda, s) \in (0,\infty)\times(0,t) :\ \lambda = \lambda_- + \delta \text{ or } \lambda = \lambda_+ - \delta\}$$

with $\lambda_- = |t-s-r|$, $\lambda_+ = t-s+r$, and $\delta = \min\{r, 1/2\}$. To evaluate I_k in the above, we shall use the following estimates.

Lemma 4.1. *Let $(\lambda, s) \in D_1 \cup D_2$. Then we have*

$$\int_{-\varphi}^{\varphi} K_1\, d\psi = 2\int_0^1 K_2\, d\tau \le \frac{C}{(r\lambda)^{\frac{1}{2}}} \log\left[2 + \frac{r\lambda}{(\lambda-\lambda_-)(\lambda_+ + \lambda)} H(t-s-r)\right], \tag{4.12}$$

$$\int_0^1 |\partial_\lambda K_2| d\tau \le \frac{C}{(r\lambda)^{\frac{1}{2}}(\lambda+s+r-t)}, \tag{4.13}$$

$$\int_0^1 |\partial_\lambda \Psi \cdot K_2| d\tau \le \frac{C}{(r\lambda)^{\frac{1}{2}}}\left(\frac{1}{\sqrt{(\lambda_+-\lambda)(\lambda-\lambda_-)}} + \frac{1}{\sqrt{\lambda^2-\lambda_-^2}}\right), \tag{4.14}$$

where $H(s) = 1$ for $s > 0$ and $H(s) = 0$ otherwise.

Proof. For the proof of (4.12) and (4.13), see for instance Proposition 5.3 in [1]. Here we prove only (4.14). Putting

$$P := \cos\varphi(\lambda; r, t-s) = \frac{r^2+\lambda^2-(t-s)^2}{2r\lambda},$$

we have $\Psi = \arccos(1+P\tau-\tau)$. Therefore we have

$$\partial_\lambda \Psi = \frac{-\sqrt{\tau}}{\sqrt{(1-P)(2+P\tau-\tau)}} \frac{\lambda^2+(t-s)^2-r^2}{2r\lambda^2}.$$

From the identity $\lambda^2+(t-s)^2-r^2 = \lambda(\lambda+t-s-r)+(t-s-r)(\lambda_+-\lambda)$, we have

$$\begin{aligned} |\lambda^2+(t-s)^2-r^2| &\le \lambda(\lambda+\lambda_-)+\lambda_-(\lambda_+-\lambda) \\ &\le \lambda\sqrt{\lambda+\lambda_+}(\sqrt{\lambda+\lambda_-}+\sqrt{\lambda_+-\lambda}) \end{aligned}$$

for $\lambda_- \le \lambda \le \lambda_+$. Therefore we see that the left-hand side of (4.14) is estimated by

$$\frac{1}{\sqrt{2r\lambda}} \frac{\sqrt{\lambda+\lambda_+}(\sqrt{\lambda+\lambda_-}+\sqrt{\lambda_+-\lambda})}{4\pi r\lambda\sqrt{1-P}} \int_0^1 \frac{1}{\sqrt{1-\tau}(2+P\tau-\tau)} d\tau. \tag{4.15}$$

Notice that the τ-integral is estimated by $C/\sqrt{1+P}$. Indeed, when $0 \le P < 1$, it suffices to see that it is just bounded. On the other hand, when $-1 < P \le 0$, by changing the variables by $\sigma = \sqrt{1-\tau}$, we have

$$\begin{aligned} \tau - \text{integral} &= 2\int_0^1 \frac{1}{(1+P)+(1-P)\sigma^2} d\sigma \\ &= \frac{2}{\sqrt{(1-P)(1+P)}} \arctan\sqrt{\frac{1-P}{1+P}} \le \frac{\pi}{\sqrt{1+P}}, \end{aligned}$$

since $\sqrt{1-P} \ge 1$ for $-1 < P \le 0$. By means of

$$\sqrt{1-P}\sqrt{1+P} = \sqrt{1-P^2} = (2r\lambda)^{-1}\sqrt{\lambda_+^2-\lambda^2}\sqrt{\lambda^2-\lambda_-^2}, \tag{4.16}$$

we obtain (4.14). This completes the proof. □

Now we start the proof of the following estimate for I_k:

$$I_k \le C(1+r)^{-\frac{1}{2}}(1+|t-r|)^{-(1+\nu)} \quad (k=1,\ldots,5). \tag{4.17}$$

First we evaluate I_1. Notice that when $t-s-r>0$ and $\lambda > \lambda_+ - \delta$, we have

$$\log\Big[2+\frac{r\lambda}{(\lambda-\lambda_-)(\lambda_++\lambda)}\Big] \le \log 3,$$

since $\lambda - \lambda_- > r$. Besides, for any $\rho > 0$ satisfying $\rho \le \min\{1/2, \mu/2\}$, we have

$$\log\Big[2+\frac{r\lambda}{(\lambda-\lambda_-)(\lambda_++\lambda)}\Big] \le C_\rho\Big[1+\Big(\frac{\lambda_-}{\lambda-\lambda_-}\Big)^\rho\Big], \tag{4.18}$$

for $t-s-r>0$, $\lambda>\lambda_-$. Moreover, we note that $z_\nu^{(j)}(\lambda,s)$ is equivalent to $z_\nu^{(j)}(\lambda_+,s)$ (respectively $z_\nu^{(j)}(\lambda_-,s)$) for $\lambda_+-\delta<\lambda<\lambda_+$ (respectively $\lambda_-<\lambda<\lambda_-+\delta$). Hence by (4.12), we get

$$I_1 \le Cr^{-\frac{1}{2}}\sum_{j=0}^{m}[A_{1,j}+A_{2,j}+A_{3,j}], \tag{4.19}$$

where for $0\le j\le m$ we have set

$$\begin{aligned}
A_{1,j} &= \int_0^t\int_{\lambda_+-\delta}^{\lambda_+}\frac{1}{z_{\nu+\mu}^{(j)}(\lambda_+,s)}d\lambda\,ds,\\
A_{2,j} &= \int_0^{(t-r)_+}\int_{\lambda_-}^{\lambda_-+\delta}\frac{1}{z_{\nu+\mu}^{(j)}(\lambda_-,s)}\Big[1+\Big(\frac{\lambda_-}{\lambda-\lambda_-}\Big)^{\rho}\Big]d\lambda\,ds,\\
A_{3,j} &= \int_{(t-r)_+}^{t}\int_{\lambda_-}^{\lambda_-+\delta}\frac{1}{z_{\nu+\mu}^{(j)}(\lambda_-,s)}d\lambda\,ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
A_{1,j} &\le \frac{C\delta}{(1+t+r)^{1+\nu}}\int_{-\infty}^{\infty}\frac{1}{(1+|(c_j+1)s-t-r|)^{1+\mu}}ds \\
&\le C\delta(1+t+r)^{-(1+\nu)}.
\end{aligned} \tag{4.20}$$

Since the λ integral in $A_{2,j}$ is evaluated by $C(\delta+\delta^{1-\rho}\lambda_-{}^{\rho})\le C\delta^{1-\rho}(1+|t-r|)^{\rho}$ for $0<s<t-r$ and $0<\rho<1$, we get

$$\begin{aligned}
A_{2,j} &\le \frac{C\delta^{1-\rho}}{(1+|t-r|)^{1+\nu}}\int_{-\infty}^{\infty}\frac{1}{(1+|(c_j+1)s-t+r|)^{1+\frac{\mu}{2}}}ds \\
&\le C\delta^{1-\rho}(1+|t-r|)^{-(1+\nu)},
\end{aligned} \tag{4.21}$$

since we took $\rho\le\mu/2$. When $s>(t-r)_+$, we have

$$s+\lambda_-=2s-t+r\ge|t-r|,\quad s+\lambda_-\ge C|c_js-\lambda_-|=C|(c_j-1)s+t-r|$$

with $C^{-1}=\max\{c_j,1\}$; hence,

$$\begin{aligned}
z_{\nu+\mu}^{(j)}(\lambda_-,s) &\ge C(1+|t-r|)^{1+\nu}(1+|(c_j-1)s+t-r|)^{1+\mu} && \text{if } c_j\ne1,\\
z_{\nu+\mu}^{(j)}(\lambda_-,s) &\ge C(1+|t-r|)^{1+\nu}(1+2s-t+r)^{1+\mu} && \text{if } c_j=1.
\end{aligned}$$

Therefore, we get

$$A_{3,j}\le C\delta(1+|t-r|)^{-(1+\nu)}. \tag{4.22}$$

Summing up (4.20), (4.21), and (4.22), we see from (4.19) that I_1 is estimated by $C\delta^{1-\rho}r^{-\frac{1}{2}}(1+|t-r|)^{-(1+\nu)}$. This bound implies (4.17) for $k=1$, since $0<\rho\le 1/2$ and $\delta/r\le 2/(1+r)$.

In the following, we assume $r\ge 1/2$ so that $\delta=1/2$, because D_2 is the empty set when $0<r<1/2$. Since $\lambda=\lambda_-+(1/2)$ or $\lambda=\lambda_+-(1/2)$ for $(\lambda,s)\in D_2^{'}$, we get (4.17) for $k=2$ analogously to the previous argument.

Next we evaluate I_3. Note that $\lambda\ge 1/2$ if $(\lambda,s)\in D_2$ and that

$$\log\left[2+\frac{r\lambda}{(\lambda-\lambda_-)(\lambda_++\lambda)}\right]\le C_\rho(1+\lambda)^\rho$$

for $\lambda\ge\lambda_-+(1/2)$ and $0<\rho\le\mu/2$. Therefore we get from (4.12)

$$r^{\frac{1}{2}}I_3\le C\sum_{j=0}^{m}\iint_{D_2}\frac{d\lambda\,ds}{(1+\lambda)z^{(j)}_{\nu+(\mu/2)}(\lambda,s)}\le C\sum_{j=0}^{m}A_{3,j}, \tag{4.23}$$

where we have set

$$\begin{aligned} A_{3,j} &= \iint_{D_2}\frac{d\lambda\,ds}{(1+s+\lambda)^{2+\nu}(1+|c_js-\lambda|)^{1+\frac{\mu}{2}}} \quad \text{if } 1\le j\le m,\\ A_{3,j} &= \iint_{D_2}\frac{d\lambda\,ds}{(1+s+\lambda)^{1+\nu}(1+\lambda)^{2+\frac{\mu}{2}}} \quad \text{if } j=0. \end{aligned}$$

When $1\le j\le m$, changing the variables by

$$\alpha=\lambda+s \quad\text{and}\quad \beta=\lambda-s, \tag{4.24}$$

we have

$$A_{3,j}\le\frac{1}{2}\int_{|t-r|}^{t+r}\frac{1}{(1+\alpha)^{2+\nu}}d\alpha\int_{r-t}^{\alpha}\frac{1}{(1+|\psi_j(\alpha,\beta)|)^{1+\frac{\mu}{2}}}d\beta\le C(1+|t-r|)^{-(1+\nu)},$$

where

$$2\psi_j(\alpha,\beta)=(c_j+1)\beta-(c_j-1)\alpha. \tag{4.25}$$

On the other hand, when $j=0$, we have

$$A_{3,0}\le\frac{C}{(1+|t-r|)^{1+\nu}}\iint_{D_2}\frac{1}{(1+\lambda)^{2+\frac{\mu}{2}}}d\lambda\,ds\le\frac{C}{(1+|t-r|)^{1+\nu}},$$

since $s+\lambda\ge|t-r|$ for $(\lambda,s)\in D_2$. Therefore (4.17) holds for $k=3$.

Next we evaluate I_4. Since $\lambda+s+r-t\ge 1/2$ for $\lambda\ge\lambda_-+(1/2)$, we get from (4.13)

$$r^{\frac{1}{2}}I_4\le C\sum_{j=0}^{m}\iint_{D_2}\frac{d\lambda\,ds}{z^{(j)}_{\nu+\mu}(\lambda,s)(\lambda+s+r-t+1)} \tag{4.26}$$

$$\leq C\int_{|t-r|}^{t+r}\frac{d\alpha}{(\alpha-t+r+1)(1+\alpha)^{1+\nu+\frac{\mu}{2}}}\int_{r-t}^{\alpha}\frac{d\beta}{(1+|\psi_j(\alpha,\beta)|)^{1+\frac{\mu}{2}}}$$
$$\leq \frac{C}{(1+|t-r|)^{1+\nu}}\int_{|t-r|}^{\infty}\Big(\frac{1}{(\alpha-t+r+1)^{1+\frac{\mu}{2}}}+\frac{1}{(1+\alpha)^{1+\frac{\mu}{2}}}\Big)d\alpha,$$

which yields (4.17) for $k=4$.

Next we evaluate I_5. It follows from (4.14) that

$$r^{\frac{1}{2}}I_5 \leq C\sum_{j=0}^{m}(A_{5,j}+B_{5,j}+C_{5,j}),$$

where for $0\leq j\leq m$ we have set

$$A_{5,j}=\iint_{D_2}\frac{d\lambda\, ds}{z_{\nu+\mu}^{(j)}(\lambda,s)\sqrt{t-s+r-\lambda+1}\sqrt{\lambda-t+s+r+1}},$$
$$B_{5,j}=\iint_{D_2}\frac{d\lambda\, ds}{z_{\nu+\mu}^{(j)}(\lambda,s)\sqrt{t-s+r-\lambda+1}\sqrt{\lambda+t-s-r+1}},$$
$$C_{5,j}=\iint_{D_2}\frac{d\lambda\, ds}{z_{\nu+\mu}^{(j)}(\lambda,s)\sqrt{\lambda-t+s+r+1}\sqrt{\lambda+t-s-r+1}}.$$

Changing the variables by (4.24), we have

$$A_{5,j}\leq \frac{1}{2}\int_{|t-r|}^{t+r}\frac{d\alpha}{(1+\alpha)^{1+\nu}\sqrt{t+r-\alpha}\sqrt{\alpha-t+r}}\int_{r-t}^{\alpha}\frac{d\beta}{(1+|\psi_j(\alpha,\beta)|)^{1+\mu}}$$
$$\leq C(1+|t-r|)^{-(1+\nu)}\int_{t-r}^{t+r}\frac{d\alpha}{\sqrt{t+r-\alpha}\sqrt{\alpha-t+r}}=C\pi(1+|t-r|)^{-(1+\nu)}.$$

Moreover, changing the variables first by (4.24) and then $\sigma=\psi_j(\alpha,\beta)$, we get

$$B_{5,j}\leq \frac{1}{c_j+1}\int_{|t-r|}^{t+r}\frac{d\alpha}{(1+\alpha)^{1+\nu+\frac{\mu}{2}}\sqrt{t+r-\alpha+1}}$$
$$\times\int_{\beta_j}^{\alpha}\frac{d\sigma}{(1+|\sigma|)^{1+\rho}\sqrt{1+\frac{2}{c_j+1}(\sigma-\beta_j)}},$$

where $2\beta_j=(1-c_j)\alpha+(1+c_j)(r-t)$ and ρ is taken such that $0<\rho<\mu/2$. It has been shown in Lemma 3.13 in [18] that the σ integral in the above is estimated by $C(1+|\beta_j|)^{-\frac{1}{2}}$. Therefore, if $c_j\neq 1$, then we have

$$(1+|t-r|)^{1+\nu}B_{5,j}\leq C\int_{|t-r|}^{t+r}\frac{d\alpha}{\sqrt{t+r-\alpha+1}(1+|\beta_j|)^{\frac{1}{2}+\frac{\mu}{2}}}\leq C.$$

While, if $c_j = 1$, then we get

$$(1+|t-r|)^{\frac{1}{2}} B_{5,j} \le C \int_{|t-r|}^{t+r} \frac{d\alpha}{(1+\alpha)^{1+\nu+\frac{\mu}{2}} \sqrt{t+r-\alpha+1}} \le C(1+|t-r|)^{-(\frac{1}{2}+\nu)}.$$

Thus we find that $B_{5,j}$ has the same bound as $A_{5,j}$. Since we can deal with $C_{5,j}$ similarly, we obtain (4.17) for all $k = 1, \dots, 5$ in conclusion.

Secondly we deal with $P_2(\partial_\ell F)(x,t)$ for $t - r \ge 2$. Switching to polar coordinates,

$$x = (r\cos\theta, r\sin\theta), \quad y = \lambda\xi = (\lambda\cos(\theta+\psi), \lambda\sin(\theta+\psi)), \tag{4.27}$$

we get

$$P_2(\partial_\ell F)(x,t) = \int_0^{t-r} \int_{(\lambda_- -1)_+}^{\lambda_-} \int_{-\pi}^{\pi} \lambda \partial_\ell F(\lambda\xi, s) K_1(\lambda, \psi; r, t-s) d\psi \, d\lambda \, ds. \tag{4.28}$$

From the following estimates for K_1,

$$\int_{-\pi}^{\pi} K_1(\lambda, \psi; r, t-s) d\psi \le \frac{C}{\sqrt{(\lambda+\lambda_-)(\lambda_+ - \lambda)}} \log\Big[2 + \frac{r\lambda}{(\lambda_- - \lambda)(\lambda_+ + \lambda)}\Big],$$

where $0 < s < t-r$ and $0 < \lambda < \lambda_-$ (for the proof, see e.g. Proposition 5.2 in [1]), we get

$$\int_{-\pi}^{\pi} K_1(\lambda, \psi; r, t-s) d\psi \le \frac{C}{\sqrt{\lambda}\sqrt{r+1}\sqrt{\lambda_- - \lambda}} \Big(\frac{t-r}{\lambda_- - \lambda}\Big)^{\rho}, \tag{4.29}$$

if $0 < s < t-r$, $\lambda_- - 1 \le \lambda < \lambda_-$ and $0 < \rho < \min\{1/2, \mu/2\}$, because we have

$$\frac{1}{\sqrt{\lambda_+ - \lambda}} \le \frac{\sqrt{2}}{\sqrt{r+1}\sqrt{\lambda_- - \lambda}}$$

for such s and λ. Therefore we obtain from (4.28)

$$\sqrt{r+1}|P_2(\partial_\ell F)(x,t)| \le C|t-r|^{\rho} \widetilde{M}_\nu(F)(t) \sum_{j=0}^{m} A_{6,j}, \tag{4.30}$$

where we have set

$$A_{6,j} = \int_0^{t-r} \int_{t-r-s-1}^{t-s-r} \frac{d\lambda \, ds}{z_{\nu+\mu}^{(j)}(\lambda, s)(\lambda_- - \lambda)^{\rho+\frac{1}{2}}}.$$

Changing the variables by (4.24), we see that $A_{6,j}$ is bounded by

$$C\int_{t-r-1}^{t-r}\frac{d\alpha}{(1+\alpha)^{1+\nu+\frac{\mu}{2}}(t-r-\alpha)^{\rho+\frac{1}{2}}}\leq C(1+|t-r|)^{-(1+\nu+\frac{\mu}{2})},$$

since $0<\rho<1/2$ and $t-r>2$. Thus we get

$$|P_2(\partial_\ell F)(x,t)|\leq C\widetilde{M}_\nu(F)(t)(r+1)^{-\frac{1}{2}}(1+|t-r|)^{-(1+\nu)}. \tag{4.31}$$

Thirdly we deal with $P_3(\partial_\ell F)(x,t)$ for $t-r\geq 2$. Making the integration by parts in y and switching to polar coordinates as in (4.27), we get

$$P_3(\partial_\ell F)(x,t)=\int_0^{t-r-1}\int_0^{t-s-r-1}\int_{-\pi}^{\pi}\lambda F(\lambda\xi,s)K_3(\lambda,\psi;x,t-s)d\psi\,d\lambda\,ds$$
$$+\int_0^{t-r-1}\int_{-\pi}^{\pi}\lambda\xi_\ell F(\lambda\xi,s)K_1(\lambda,\psi;r,t-s)\Big|_{\lambda=t-s-r-1}d\psi\,ds, \tag{4.32}$$

where we have set

$$K_3(\lambda,\psi;x,t)=\frac{-(x_\ell-\lambda\xi_\ell)}{2\pi(t^2-r^2-\lambda^2+2r\lambda\cos\psi)^{\frac{3}{2}}}.$$

We see from (4.29) that the second term on the right-hand side of (4.32) is bounded by

$$\frac{C|t-r|^\rho\widetilde{M}_\nu(F)(t)}{\sqrt{r+1}}\sum_{j=0}^{m}\int_0^{t-r-1}\frac{1}{z_{\nu+\mu}^{(j)}(\lambda,s)(\lambda_--\lambda)^{\frac{1}{2}+\rho}}\Big|_{\lambda=t-s-r-1}ds \tag{4.33}$$
$$\leq C\widetilde{M}_\nu(F)(t)(r+1)^{-\frac{1}{2}}(1+|t-r|)^{-(1+\nu)},$$

since $t-r\geq 2$ and $\rho<\mu/2$. Suppose we have found

$$\int_{-\pi}^{\pi}|K_3(\lambda,\psi;r,t-s)|d\psi\leq\frac{C}{(\lambda_--\lambda)\sqrt{(\lambda_-+\lambda)(\lambda_+-\lambda)}}. \tag{4.34}$$

Then, noting $\lambda_+-\lambda\geq 2r+1$ for $\lambda<t-s-r-1$, we see that the first term on the right-hand side of (4.32) is estimated by $C\widetilde{M}_\nu(F)(t)/\sqrt{r+1}$ times

$$\int_0^{t-r-1}\int_{(\frac{t-r}{2}-s)_+}^{t-s-r-1}\frac{d\lambda\,ds}{z_{\nu+\mu}^{(j)}(\lambda,s)(\lambda_--\lambda)}+\int_0^{\frac{t-r}{2}}\int_0^{\frac{t-r}{2}-s}\frac{\sqrt{\lambda}d\lambda\,ds}{z_{\nu+\mu}^{(j)}(\lambda,s)(\lambda_--\lambda)^{\frac{3}{2}}}$$
$$\leq\frac{C}{(1+|t-r|)^{1+\nu}}\int_{\frac{t-r}{2}}^{t-r}\frac{d\alpha}{(1+\alpha)^{\frac{\mu}{2}}(t-r-\alpha+1)}\int_{r-t}^{\alpha}\frac{d\beta}{(1+|\psi_j(\alpha,\beta)|)^{1+\frac{\mu}{2}}}$$
$$+\frac{C}{(1+|t-r|)^{\frac{3}{2}}}\int_0^{\frac{t-r}{2}}\frac{d\alpha}{(1+\alpha)^{\frac{1}{2}+\nu}}\int_{r-t}^{\alpha}\frac{d\beta}{(1+|\psi_j(\alpha,\beta)|)^{1+\mu}}$$

$\le C(1+|t-r|)^{-(1+\nu)}$,

since $\nu < 1/2$. Thus we get (4.8) from (4.11), (4.17), (4.31), and (4.33). It remains to show (4.34). Since

$$|K_3(\lambda,\psi;x,t-s)| \le \frac{r+\lambda}{\sqrt{(t-s)^2-(r+\lambda)^2}}\frac{1}{(-2r\lambda P)(1-P^{-1}\cos\psi)}$$

with $P=(r^2+\lambda^2-(t-s)^2)/2r\lambda$ (<-1), and

$$\int_{-\pi}^{\pi}\frac{1}{1+a\cos\psi}d\psi = \frac{2\pi}{\sqrt{1-a^2}} \quad \text{for } |a|<1,$$

we get

$$\int_{-\pi}^{\pi}|K_3(\lambda,\psi;r,t-s)|d\psi \le \frac{\sqrt{r+\lambda}}{\sqrt{\lambda_- -\lambda}}\frac{1}{(-2r\lambda P)\sqrt{1-P^{-2}}}.$$

By (4.16) we obtain (4.34). This completes the proof of Proposition 4.2. □

Corollary 4.1. *Let $0<\nu<1/2$, $\mu>0$, and $q\ge 0$. If $u\in C^\infty(\mathbb{R}^2\times[0,T))$ is a solution of* (1.1) *and* (1.2)*, then we have*

$$\langle u(x,t)\rangle_{k+1}(1+|x|+t)^{-q} \le C_0\varepsilon + C_0\sum_{i=1}^m M_{\nu-q}(|F^i|_{k+1})(t), \tag{4.35}$$

$$[\partial_\ell u(x,t)]_k(1+|x|+t)^{-q} \le C_0\varepsilon + C_0\sum_{i=1}^m M_{\nu-q}(|F^i|_{k+1})(t) \quad (\ell=1,2) \tag{4.36}$$

for $(x,t)\in\mathbb{R}^2\times[0,T)$ and a nonnegative integer k, where C_0 is a constant depending only on μ, ν, k, q, and c_j. Besides, here and later on as well we abbreviate $F^i(u,\partial u,\partial^2 u)$ as F^i.

Proof. We see from (2.2) that $u(x,t)$ satisfies

$$\Box_{c_i}\Gamma^a u^i = \sum_{|b|\le|a|}C_{a,b}\Gamma^b F^i \tag{4.37}$$

with some constant $C_{a,b}$; hence, we have

$$\Gamma^a u^i(x,t) = v^i(x,t) + \sum_{|b|\le|a|}C_{a,b}L_{c_i}(\Gamma^b F^i)(x,t), \tag{4.38}$$

where $v^i(x,t)$ is the solution of (2.9) and (2.10) with ϕ^i and ψ^i determined by f^i and g^i in (1.2) suitably. Therefore, by (2.11) we obtain

$$\langle\Gamma^a u^i(x,t)\rangle_0 \le C\varepsilon + C\sum_{|b|\le|a|}\langle L_{c_i}(\Gamma^b F^i)(x,t)\rangle_0, \tag{4.39}$$

$$[\partial_\ell \Gamma^a u^i(x,t)]_0 \leq C\varepsilon + C \sum_{|b|\leq|a|} [\partial_\ell L_{c_i}(\Gamma^b F^i)(x,t)]_0 \quad (\ell = 1,2) \tag{4.40}$$

for $(x,t) \in \mathbb{R}^2 \times [0,T)$. Since $s + |y| \leq C(t + |x|)$ if $|x - y| \leq c_i(t-s)$, we get the following estimates for $q \geq 0$, in view of the proof of Propositions 4.1 and 4.2:

$$\langle L_{c_i}(F)(x,t)\rangle_0 (1 + |x| + t)^{-q} \leq C M_{\nu-q}(F)(t), \tag{4.41}$$

$$[\partial_\ell L_{c_i}(F)(x,t)]_0 (1 + |x| + t)^{-q} \leq C \widetilde{M}_{\nu-q}(F)(t) \quad (\ell = 1,2) \tag{4.42}$$

for $(x,t) \in \mathbb{R}^2 \times [0,T)$. Thus, recalling (4.9), we see that (4.39) with $|a| \leq k+1$ and (4.40) with $|a| \leq k$ give (4.35) and (4.36) respectively. This completes the proof. □

5. L^2 ESTIMATES

In this section we shall derive an energy estimate. To be more specific, we introduce

$$D_k(t) = \|\partial u(t)\|_k + \|\partial\partial_x u(t)\|_k + (1+t)^{-\frac{1}{p}} \|u(t)\|_X, \tag{5.1}$$

$$\|u(t)\|_X = \sum_{|a|\leq k} \|\Gamma^a u(t)\|_{\dot{H}^\rho}, \quad p = \frac{2}{1-\rho}, \tag{5.2}$$

where $t \geq 0$, k is a nonnegative integer, $\partial_x = (\partial_1, \partial_2)$, and $0 < \rho < 1/4$. We put

$$C_* := \max_{\substack{i,l=1,\ldots,m \\ q,k=0,1,2}} \sum_{|a|\leq 1} \sum_{j=1}^{m} \sum_{p=1}^{p_j} \sup_{|u|+|\partial u|\leq 1} \left| \frac{\partial H^{il}_{qk}}{\partial(\partial^a u^j_p)}(u, \partial u) \right|. \tag{5.3}$$

Then our purpose of this section is formulated as follows.

Proposition 5.1. *We suppose that* (1.5) *through* (1.13) *hold. Let* $u \in C^\infty(\mathbb{R}^2 \times [0,T))$ *be a solution of* (1.1) *and* (1.2). *Suppose that* $1/4 < \nu < 1/2$, $0 < \rho < 1/4$, *and that*

$$C_* \mathcal{U}_0(t) \leq \frac{c_1^2}{4m}, \quad \mathcal{U}_{[\frac{k}{2}]+1}(t) \leq 1 \quad \textit{for} \quad 0 \leq t < T. \tag{5.4}$$

Then there exists a constant $C_1 > 0$ *independent of* T *and* ε *such that*

$$D_k(t) \leq C\varepsilon(1+t)^{C_1(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2} \quad \textit{for} \quad 0 \leq t < T. \tag{5.5}$$

Proof. First we estimate $\|u(t)\|_X$. By (4.37) we have the Fourier representation of $\Gamma^a u^i$. Evaluating it in $\dot{H}^\rho$, we see from (2.12) and (2.15) with $s = 1 - \rho$ that

$$\|u(t)\|_X \leq C\varepsilon + C \sum_{|a| \leq k} \int_0^t \|\Gamma^a F(s)\|_{L^q}\, ds, \tag{5.6}$$

where $1/q = (1/p) + (1/2)$ with $p = 2/(1-\rho)$. Thus it suffices to estimate $\|\Gamma^a F(s)\|_{L^q}$. Suppose we have found

$$\begin{aligned} |\Gamma^a F^i(x,s)| \leq C(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2 [&\eta_l(|x|,s)^{-2}(|\partial u(x,s)|_k + |\partial\partial_x u(x,s)|_k) \\ &+ \eta_l(|x|,s)^{-3}|u(x,s)|_k] \end{aligned} \tag{5.7}$$

for $(x,s) \in \Lambda_l(t)$ $(0 \leq l \leq m)$ and $|a| \leq k$, where $\eta_l(\lambda, s)$ is given in (2.4). Then for any r with $1 \leq r \leq 2$, (5.7) and Lemma 2.2 yield

$$\begin{aligned} \|\Gamma^a F^i(s)\|_{L^r} \leq C(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2 \sum_{l=0}^m [&\|\eta_l(|\cdot|,s)^{-2}\|_{L^{r^*}}(\|\partial u(s)\|_k + \|\partial\partial_x u(s)\|_k) \\ &+ \|\eta_l(|\cdot|,s)^{-3}|c_l s - |\cdot||^\rho\|_{L^{r^*}}\|u(s)\|_X], \end{aligned}$$

where $1/r = (1/r^*) + (1/2)$. Since $r^* \geq 2$, $\nu > 1/4$, and $0 < \rho < 1/4$, we have $r^*(2\nu) > 1$ and $r^*(3\nu - \rho) > 1$. Therefore, the application of Lemma 2.3 gives

$$\|\eta_l(|\cdot|,s)^{-2}\|_{L^{r^*}} \leq C(1+s)^{-1+\frac{1}{r^*}}, \ \ \|\eta_l(|\cdot|,s)^{-3}|c_l s - |\cdot||^\rho\|_{L^{r^*}} \leq C(1+s)^{-\frac{3}{2}+\frac{1}{r^*}}.$$

Noting $1/p < 1/2$ for $\rho > 0$, we get

$$\|\Gamma^a F^i(s)\|_{L^r} \ \ \leq \ \ C(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2(1+s)^{-1+\frac{1}{r^*}} D_k(s). \tag{5.8}$$

Using this estimate with $r = q$ and $r^* = p$, we get from (5.6)

$$(1+t)^{-\frac{1}{p}}\|u(t)\|_X \leq C\left[\varepsilon + (\mathcal{U}_{[\frac{k}{2}]+1}(t))^2 \int_0^t (1+s)^{-1} D_k(s) ds\right]. \tag{5.9}$$

Now we prove (5.7). In the following, we always assume that $(x,s) \in \Lambda_l(t)$ with $0 \leq l \leq m$ and $|a| \leq k$. Then (2.4) implies

$$\begin{aligned} |u^j(x,s)|_k &\leq C\mathcal{U}_k(t)\eta_l(|x|,s)^{-1}, \\ |\partial u^j(x,s)|_k &\leq C\mathcal{U}_k(t)\eta_l(|x|,s)^{-1}(1 + |c_l s - |x||)^{-1}, \\ |\partial u^j(x,s)|_k &\leq C\mathcal{U}_k(t)\eta_l(|x|,s)^{-2} \qquad \text{if} \quad j \neq l, \end{aligned}$$

where $1 \le j \le m$ and C is independent of x and s. According to (1.9) and (1.11), we write

$$F^i = \sum_{j=1}^{m} \widetilde{N}^{ij} + \widetilde{R}^i + G^i + H^i, \tag{5.10}$$

where we have set

$$\begin{aligned} \widetilde{N}^{ij}(u, \partial u^j, \partial^2 u^j) &= N^{ij}(u, \partial u^j, \partial^2 u^j) - N^{ij}(0, \partial u^j, \partial^2 u^j), \\ \widetilde{R}^i(u, \partial u, \partial^2 u) &= R^i(u, \partial u, \partial^2 u) - R^i(0, \partial u, \partial^2 u), \\ G^i(\partial u, \partial^2 u) &= N^{ij}(0, \partial u^j, \partial^2 u^j) + R^i(0, \partial u, \partial^2 u). \end{aligned}$$

By (1.10) and the fact that G^i is a cubic polynomial in $(\partial u, \partial^2 u)$, we get

$$\begin{aligned} |\Gamma^a H^i(x,s)| &\le C(\mathcal{U}_{[\frac{k}{2}]+1}(t))^3 \eta_l(|x|,s)^{-3}(|u(x,s)|_k \\ &\qquad + |\partial u(x,s)|_k + |\partial\partial_x u(x,s)|_k), \\ |\Gamma^a G^i(x,s)| &\le C(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2 \eta_l(|x|,s)^{-2}(|\partial u(x,s)|_k + |\partial\partial_x u(x,s)|_k). \end{aligned}$$

Besides, we see from (1.12) that $\Gamma^a \widetilde{R}^i(x,s)$ is estimated by the right-hand side of (5.7), since for $j \ne h$ we have

$$\sum_{|a|,|b|=1,2} |\partial^a u^j(x,s)|_{[\frac{k}{2}]} |\partial^b u^h(x,s)|_{[\frac{k}{2}]} \le C(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2 \eta_l(|x|,s)^{-3}.$$

Moreover, when $j \ne l$, it is easy to see that $\Gamma^a \widetilde{N}^{ij}(x,s)$ is bounded by

$$C(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2 \eta_l(|x|,s)^{-3}(|u(x,s)|_k + |\partial u(x,s)|_k + |\partial\partial_x u(x,s)|_k).$$

Thus it remains to evaluate $\Gamma^a \widetilde{N}^{ij}(x,s)$ for the case of $j = l$. It follows from (3.2) that

$$\widetilde{N}^{ij}_r = \sum_{q=1}^{m} \sum_{h=1}^{p_q} \sum_{k,l=1}^{p_j} \Big(\sum_{\alpha,\beta,\gamma=0}^{2} C^{qhkl}_{\alpha\beta\gamma} u^q_h \partial_\alpha u^j_k \partial_\beta \partial_\gamma u^j_l + \sum_{\alpha,\beta=0}^{2} D^{qhkl}_{\alpha\beta} u^q_h \partial_\alpha u^j_k \partial_\beta u^j_l \Big).$$

By (1.13) we can rearrange the above expression of $\widetilde{N}^{ij}_r$ so that each term involves the operator $R_\alpha = \partial_\alpha - \omega_\alpha \partial_r$ as in (3.7). Employing Lemma 3.2 together with the trivial estimate $|R_\alpha v(x,s)|_k \le C|\partial v(x,t)|_k$, we find that $\Gamma^a \widetilde{N}^{ij}(x,s)$ is estimated by the right-hand side of (5.7). Therefore we have shown (5.7); hence, (5.9) holds.

Next we estimate $\|\partial u(t)\|_k$. Using (5.8) with $r = 2$ and $r^* = \infty$, we easily have

$$\|\partial u(t)\|_k \le C\Big[\varepsilon + (\mathcal{U}_{[\frac{k}{2}]+1}(t))^2 \int_0^t (1+s)^{-1} D_k(s)\,ds\Big]. \tag{5.11}$$

Finally we estimate $\|\partial\partial_x u(t)\|_k$, by making use of the following.

Lemma 5.1. *We suppose that the assumptions in Proposition* 5.1 *are fulfilled. For* $v = {}^t(v^1, \dots, v^m) \in C^\infty(\mathbb{R}^2 \times [0,T) : \mathbb{R}^p)$ *with* $v^i(x,t)$ *a* p_i *vector-valued function, we set*

$$\|v(t)\|_E^2 = \sum_{i=1}^m \int_{\mathbb{R}^2} \Big\{ |\partial_t v^i(x,t)|^2 + c_i^2 |\nabla v^i(x,t)|^2 + \sum_{l=1}^m \sum_{q,k=1}^2 H_{qk}^{il}(u,\partial u)\partial_q v^i \cdot \partial_k v^l(x,t) \Big\} dx.$$

Then for $0 \le t < T$ *we have*

$$\frac{d}{dt}\|v(t)\|_E \le C \max_{\substack{i,l=1,\dots,m \\ q,k=0,1,2}} \|\partial(H_{qk}^{il}(u,\partial u)(t)\|_{L^\infty} \|v(t)\|_E + \sum_{i=1}^m C[\|Q^i(v)(t)\|_{L^2} + \|K^i(u,\partial u)(t)\|_{L^2}], \tag{5.12}$$

where we have set

$$Q^i(v) = \Box_{c_i} v^i - \sum_{l=1}^m \sum_{\gamma,\delta=0}^2 H_{qk}^{il}(u,\partial u)\partial_\gamma \partial_\delta v^l - K^i(u,\partial u). \tag{5.13}$$

Proof. By (1.7) we have

$$H_{\gamma\delta}^{il}(u,\partial u) w^l \cdot w^i = w^l \cdot {}^t H_{\gamma\delta}^{il}(u,\partial u) w^i = H_{\gamma\delta}^{li}(u,\partial u) w^i \cdot w^l,$$

where $a \cdot b$ means the inner product of $a, b \in \mathbb{R}^{p_i}$. Therefore direct computation yields

$$\frac{d}{dt}\|v(t)\|_E^2 = \sum_{i=1}^m \int_{\mathbb{R}^2} [J^i(v)(x,t) + 2(Q^i(v) + 2K^i(u,\partial u)) \cdot \partial_t v^i(x,t)] dx,$$

where we have set

$$J_i(v) = -2 \sum_{l=1}^m \sum_{q=1}^2 \sum_{k=0}^2 \partial_q (H_{qk}^{il}(u,\partial u)) \partial_k v^i \cdot \partial_t v^l + \sum_{l=1}^m \sum_{q,k=1}^2 \partial_t (H_{qk}^{il}(u,\partial u)) \partial_q v^i \cdot \partial_k v^l.$$

Thus, to prove (5.12), it suffices to show that $\|v(t)\|_E$ is equivalent to $\|\partial v(t)\|_{L^2}$. From (1.6) and (5.4) with (5.3) we have $|H_{qk}^{il}(u,\partial u)(x,t)| \le$

$c_1^2/4m^2$. Recalling (1.3), we see that the desired assertion is valid. This completes the proof. □

By (1.6) it is easy to see that

$$\max_{\substack{i,l=1,\dots,m\\ q,k=0,1,2}} \|\partial(H^{il}_{qk}(u,\partial u))(x,t)\|_{L^\infty_x} \le C(1+t)^{-1}(\mathcal{U}_1(t))^2.$$

Moreover, we have

$$\|K^i(u,\partial u)(t)\|_{L^2} \le C(1+t)^{-1}(\mathcal{U}_0(t))^2 D_0(t),$$

since the cubic term in $K^i(u,\partial u)$ is supposed to be linear in u by (1.13) and (1.12) (see also the beginning of the proof of Proposition 3.1), while for such terms as u^4 we can use Lemma 2.2 as in the proof of (5.8). Now, taking $v = \Gamma^a\partial_x u$ ($|a| \le k$) in (5.12), we get

$$\frac{d}{dt}\|\Gamma^a\partial_x u(t)\|_E \le C(1+t)^{-1}(\mathcal{U}_1(t))^2 D_k(t) + \sum_{i=1}^m C\|Q^i(\Gamma^a\partial_x u)(t)\|_{L^2}, \tag{5.14}$$

because of the equivalence between $\|v(t)\|_E$ and $\|\partial v(t)\|_{L^2}$. Therefore, it remains to estimate $\|Q^i(\Gamma^a\partial_x u)(t)\|_{L^2}$. In view of (5.13) and (4.37) we have

$$\begin{aligned} Q_i(\Gamma^a\partial_x u) &= \Box_{c_i}\Gamma^a\partial_x u^i - F^i(u,\partial u,\partial^2\Gamma^a\partial_x u)\\ &= \Gamma^a\partial_x F^i(u,\partial u,\partial^2 u) - F^i(u,\partial u,\partial^2\Gamma^a\partial_x u) + \sum_{|b|\le|a|} C_{a,b}\Gamma^b F^i(u,\partial u,\partial^2 u); \end{aligned}$$

hence, such terms containing $\Gamma^a\partial_x\partial^2 u$ are canceled out, modulo lower-order terms. Thus, we find that $\|Q_i(\Gamma^a\partial_x u)(t)\|_{L^2}$ is estimated by $C(1+t)^{-1}\cdot$ $\cdot(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2 D_k(t)$. This means

$$\|\partial\partial_x u(t)\|_k \le C\left[\varepsilon + (\mathcal{U}_{[\frac{k}{2}]+1}(t))^2\int_0^t (1+s)^{-1}D_k(s)ds\right]. \tag{5.15}$$

Summing up (5.9), (5.11), and (5.15), we obtain (5.5) by the Gronwall inequality. □

6. Proof of Theorem 1.1

To show the solution exists globally in time, we are going to estimate $\mathcal{U}_k(t)$.

Lemma 6.1. *Suppose that* (1.9) *through* (1.13) *hold. Let* $1/4 < \nu < 1/2$, $0 < \rho < 1/4$, *k be a nonnegative integer, and* $q \ge 0$. *If we take* $\mu \ge 0$ *and* $\theta > 0$ *so that*

$$\nu + \mu + \theta \le \frac{1}{2}, \tag{6.1}$$

then for any $u = {}^t(u^1, \dots, u^m) \in C^\infty(\mathbb{R}^2 \times [0,T) : \mathbb{R}^p)$ *with* $u^i(x,t)$ *a* p_i *vector-valued function satisfying*

$$\mathcal{U}_{[\frac{k+1}{2}]}(t) \le 1 \quad \text{for} \ \ 0 \le t < T, \tag{6.2}$$

there is a constant C_2*, independent of* u *and* T*, such that*

$$|y|^{\frac{1}{2}} z^{(l)}_{\nu+\mu-q}(|y|, s)|F^i(u, \partial u, \partial^2 u)(y,s)|_k \tag{6.3}$$

$$\le C_2(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2(1+s)^{-\theta} D_{k+2}(s) + C_2(\mathcal{U}_{[\frac{k+1}{2}]}(t))^2[(1+|y|+s)^{-q}\langle u(y,s)\rangle_k$$

$$+ C_3(q)(1+|y|+s)^{-\frac{1}{2}-q}[\partial u(y,s)]_{k+1}] \tag{6.4}$$

for $(y,s) \in \Lambda_l(t)$ $(0 \le l \le m)$*, where* $C_3(q)$ *is a constant such that* $C_3(q) = 0$ *for* $q \ge 1/2$*, and* $C_3(q) = 1$ *otherwise.*

Proof. For simplicity, we shall use the following notation:

$$w_+(\lambda, s) = 1 + \lambda + s, \quad w_c(\lambda, s) = 1 + |\lambda - cs|.$$

Then for $j = 1, \dots, m$ and $l = 0, 1, \dots, m$, we have

$$|u^j(y,s)|_k \le \langle u(y,s)\rangle_k w_+(|y|,s)^{-\frac{1}{2}} w_{c_j}(|y|,s)^{-\nu} \quad \text{for} \quad (y,s) \in \Lambda_j(t),$$

$$|u^j(y,s)|_k \le C\langle u(y,s)\rangle_k w_+(|y|,s)^{-\frac{1}{2}-\nu} \qquad \text{for} \quad (y,s) \in \Lambda_l(t),\ l \ne j,$$

and

$$|\partial u^j(y,s)|_k \le C[\partial u(y,s)]_k w_+(|y|,s)^{-\frac{1}{2}} w_{c_j}(|y|,s)^{-1-\nu} \quad \text{for} \quad (y,s) \in \Lambda_j(t),$$

$$|\partial u^j(y,s)|_k \le C[\partial u(y,s)]_k w_+(|y|,s)^{-1-\nu}(1+|y|)^{-\frac{1}{2}} \text{ for } (y,s) \in \Lambda_l(t),\ l \ne j,$$

where C is independent of y and s. In the following, we assume that $(y,s) \in \Lambda_l(t)$. As before, we write $F^i(y,s) = \sum_{j=1}^m N^{ij}(y,s) + R^i(y,s) + H^i(y,s)$.

First we consider $N^{ij}(y,s)$. When $l = j$, it follows from Proposition 3.1 that

$$\begin{aligned}
&|y|^{\frac{1}{2}} z^{(j)}_{\nu+\mu-q}(|y|,s)|N^{ij}(y,s)|_k \\
\le\ & C(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2[w_+(|y|,s)^{-1+\nu+\mu-q} w_{c_j}(|y|,s)^{1-2\nu}|y|^{\frac{1}{2}}|\partial u^j(y,s)|_{k+1} \\
&\quad + w_+(|y|,s)^{-1+\nu+\mu-q} w_{c_j}(|y|,s)^{-2\nu}|y|^{\frac{1}{2}}|u(y,s)|_{k+1}] \\
\le\ & C(\mathcal{U}_{[\frac{k}{2}]+1}(t))^2\big[w_+(|y|,s)^{-\theta}|y|^{\frac{1}{2}}|\partial u(y,s)|_{k+1} \\
&\quad + w_+(|y|,s)^{-\frac{1}{2}-\theta}\langle |y| - c_j s\rangle^{-\rho}|y|^{\frac{1}{2}}|u(y,s)|_{k+1}\big],
\end{aligned} \tag{6.5}$$

by (6.1), $q \ge 0$, $\nu > 1/4$, and $\rho < 2\nu$, where we put $\langle\lambda\rangle = \sqrt{1+\lambda^2}$. Applying Lemmas 2.1 and 2.2 with $n = 2$ together with (5.1), we have

$$|y|^{\frac{1}{2}}|\partial u(y,s)|_{k+1} \le C D_{k+2}(s),$$

$$\langle |y|-c_js\rangle^{-\rho}|y|^{\frac{1}{2}}|u(y,s)|_{k+1}\le C(1+s)^{\frac{1}{p}}D_{k+2}(s).$$

Since $1/p=(1-\rho)/2<1/2$, we see that $|y|^{\frac{1}{2}}z^{(j)}_{\nu+\mu-q}(|y|,s)|N^{ij}(y,s)|_k$ is estimated by the first term of right-hand side in (6.3). On the other hand, when $l\neq j$, we have

$$\begin{aligned}
&|y|^{\frac{1}{2}}z^{(l)}_{\nu+\mu-q}(|y|,s)|N^{ij}(y,s)|_k \qquad (6.6)\\
\le\;& C(\mathcal{U}_{[\frac{k+1}{2}]}(t))^2\big[w_+(|y|,s)^{-1+\mu-q}w_{c_l}(|y|,s)^{1-\nu}|y|^{\frac{1}{2}}|\partial u(y,s)|_{k+1}\\
&\quad+w_+(|y|,s)^{-2-\nu+\mu-q}w_{c_l}(|y|,s)^{1-\nu}\langle u(y,s)\rangle_k\big]\\
\le\;& C(\mathcal{U}_{[\frac{k+1}{2}]}(t))^2\{(1+s)^{-\theta}D_{k+2}(s)+(1+|y|+s)^{-q}\langle u(y,s)\rangle_k\},
\end{aligned}$$

since $N^{ij}(y,s)$ is linear in u, and $\theta+\mu<1/4<\nu$.

Next we consider $R^i(y,s)$. We see from (1.12) that

$$\begin{aligned}
&|y|^{\frac{1}{2}}z^{(l)}_{\nu+\mu-q}(|y|,s)|R^i(y,s)|_k\\
&\le C(\mathcal{U}_{[\frac{k+1}{2}]}(t))^2w_+(|y|,s)^{-1+\mu-q}w_{c_l}(|y|,s)^{-2\nu}(\langle u(y,s)\rangle_k+[\partial u(y,s)]_{k+1})\\
&\le C(\mathcal{U}_{[\frac{k+1}{2}]}(t))^2\{(1+|y|+s)^{-q}\langle u(y,s)\rangle_k+(1+|y|+s)^{-\frac{1}{2}-q}[\partial u(y,s)]_{k+1}\},
\end{aligned}$$

by $0<\mu<1/2$. Moreover, if $q\ge 1/2$, then we have

$$\begin{aligned}
&|y|^{\frac{1}{2}}z^{(l)}_{\nu+\mu-q}(|y|,s)|R^i(y,s)|_k\\
\le\;& C(\mathcal{U}_{[\frac{k+1}{2}]}(t))^2[w_+(|y|,s)^{\nu+\mu-q}w_{c_l}(|y|,s)^{-2\nu}|y|^{\frac{1}{2}}|\partial u(y,s)|_{k+1}\\
&+w_+(|y|,s)^{-1+\mu-q}w_{c_l}(|y|,s)^{-2\nu}\langle u(y,s)\rangle_k]\\
\le\;& C(\mathcal{U}_{[\frac{k+1}{2}]}(t))^2\{(1+s)^{-\theta}D_{k+2}(s)+(1+|y|+s)^{-q}\langle u(y,s)\rangle_k\}.
\end{aligned}$$

Finally, we consider $H^i(y,s)$. Then we have

$$\begin{aligned}
&|y|^{\frac{1}{2}}z^{(l)}_{\nu+\mu-q}(|y|,s)|H^i(y,s)|_k\le C(\mathcal{U}_{[\frac{k+1}{2}]}(t))^3w_+(|y|,s)^{-\frac{1}{2}+\nu+\mu-q}\\
&\qquad\times w_{c_l}(|y|,s)^{1-3\nu}(|y|^{\frac{1}{2}}|\partial u(y,s)|_{k+1}+\langle u(y,s)\rangle_k)\\
&\le C(\mathcal{U}_{[\frac{k+1}{2}]}(t))^2\{(1+s)^{-\theta}D_{k+2}(s)+(1+|y|+s)^{-q}\langle u(y,s)\rangle_k\},
\end{aligned}$$

by (6.1) and $\nu>1/3$. Thus we obtain (6.3), and the proof is completed. □

Corollary 6.1. *Let the assumptions of Lemma* 6.1 *be fulfilled, and* C_0 *and* C_2 *be the constants in Corollary* 4.1 *and Lemma* 6.1, *respectively. Let* $u\in C^\infty(\mathbb{R}^2\times[0,T))$ *be a solution of* (1.1) *and* (1.2). *If we assume*

$$C_0C_2\mathcal{U}_{[\frac{k+2}{2}]}(t)\le\frac{1}{2},\quad \mathcal{U}_{[\frac{k+2}{2}]}(t)\le 1\quad\text{for}\ \ 0\le t<T, \qquad (6.7)$$

then we have

$$\langle u(x,t)\rangle_{k+1}(1+|x|+t)^{-q} \le C_0\varepsilon + C_4(\mathcal{U}_{[\frac{k+1}{2}]+1}(t))^2\big[(1+s)^{-\theta}D_{k+3}(s) + C_3(q)(1+|y|+s)^{-\frac{1}{2}-q}[\partial u(y,s)]_{k+2}\big], \tag{6.8}$$

and

$$[\partial_\ell u(x,t)]_k(1+|x|+t)^{-q} \le C_0\varepsilon + C_4(\mathcal{U}_{[\frac{k+1}{2}]+1}(t))^2\big[(1+s)^{-\theta}D_{k+3}(s) + C_3(q)(1+|y|+s)^{-\frac{1}{2}-q}[\partial u(y,s)]_{k+2}\big] \tag{6.9}$$

for $(x,t) \in \mathbb{R}^2 \times [0,T)$ *and* $\ell = 1,2$, *where* C_4 *is a constant independent of* T *and* ε.

Proof. First we prove (6.8), by using (4.35), (4.4), and (6.3) with $q = 0$. Thanks to (6.7), the term $\sup_{(y,s)\in\mathbb{R}^2\times[0,t]}(1+|y|+s)^{-q}\langle u(y,s)\rangle_{k+1}$ on the right-hand side can be absorbed by the left. Thus we get (6.8). Moreover, (6.9) follows from (4.36), (6.3), and (6.8). □

End of the Proof of Theorem 1.1. Let $N \ge 6$ and put

$$B_0 := \min\Big\{1, \frac{C_1\theta}{2}, \frac{1}{2C_0C_2}\Big\},$$

where C_0 is the number in Corollary 4.1, C_1 in Proposition 5.1, and C_2 and θ in Lemma 6.1. By the local existence theorem (see for instance [18], also [13]), for any initial data $f^i, g^i \in C_0^\infty(\mathbb{R}^2)$, there are positive constants ε_N and T_N such that for any ε with $0 < \varepsilon \le \varepsilon_N$ there exists a smooth solution $u(x,t)$ of (1.1) and (1.2) in $(x,t) \in \mathbb{R}^2 \times [0,T_N]$. Moreover, we have

$$\lim_{\varepsilon\to 0}\max_{0\le t\le T_N}(\|u(t)\|_{H^N} + \|\partial_t u(t)\|_{H^{N-1}}) = 0, \tag{6.10}$$

where $\|u\|_{H^k} = \|\langle\xi\rangle^k\mathcal{F}[u]\|_{L^2_\xi}$. Hence it follows from Sobolev's inequality and the finite speed of propagation that there is $\varepsilon_* = \varepsilon_*(N,\nu,T_N,B_0) > 0$ such that for $0 < \varepsilon \le \varepsilon_*$

$$\mathcal{U}_N(t) \le B_0 \quad \text{for} \quad 0 \le t \le T, \tag{6.11}$$

where $T = T_N$. Since $[(N+5)/2]+1 \le N$ for $N \ge 6$, we have from Proposition 5.1

$$D_{N+5}(t) \le C\varepsilon(1+t)^{\frac{\theta}{2}} \quad \text{for} \quad 0 \le t < T, \tag{6.12}$$

where C is independent of T and ε. Therefore, we get from Corollary 6.1

$$\Big(\langle u(x,t)\rangle_{k+1} + \sum_{\ell=1}^{2}[\partial_\ell u(x,t)]_k\Big)(1+|x|+t)^{-q} \tag{6.13}$$

$$\leq C\varepsilon + C(\mathcal{U}_{[\frac{k+1}{2}]+1}(t))^2 \left[\varepsilon + C_3(q)(1+|y|+s)^{-\frac{1}{2}-q}[\partial u(y,s)]_{k+2}\right],$$

for $(x,t) \in \mathbb{R}^2 \times [0,T)$ and $k \leq N+2$.

Hence, if we could show that for $(x,t) \in \mathbb{R}^2 \times [0,T)$, $q \geq 0$, and $k \leq N+2$

$$\begin{aligned} &[\partial_t u(x,t)]_k (1+|x|+t)^{-q} \\ &\leq C\varepsilon + C(\mathcal{U}_{[\frac{k+1}{2}]+1}(t))^2 \left[\varepsilon + C_3(q)(1+|y|+s)^{-\frac{1}{2}-q}[\partial u(y,s)]_{k+2}\right] \end{aligned} \tag{6.14}$$

holds, then there is a constant B_1, independent of T and ε, such that

$$\mathcal{U}_N(t) \leq B_1\varepsilon(1+(\mathcal{U}_N(t))^2) \quad \text{for} \quad 0 \leq t < T. \tag{6.15}$$

In fact, (6.13) and (6.14) with $k = N+2$ and $q = 1/2$ yield

$$(1+|x|+t)^{-\frac{1}{2}}[\partial u(x,t)]_{N+2} \leq C\varepsilon + C\varepsilon(\mathcal{U}_N(t))^2, \tag{6.16}$$

since $C_3(q) = 0$ for $q \geq 1/2$ and $[(N+3)/2]+1 \leq N$. Substituting (6.16) into (6.13) and (6.14) with $k = N$ and $q = 0$, we get (6.15).

Once we get (6.15), we are able to show the existence of the global solution of (1.1) and (1.2) for $0 < \varepsilon \leq \varepsilon_0$ from the standard argument (see also, for instance, Theorem 2.2 or its Corollary 2 of [20]), where ε_0 is supposed to satisfy

$$0 < \varepsilon_0 \leq \varepsilon_*, \quad 2B_1\varepsilon_0 \leq B_0, \quad 2(2B_1\varepsilon_0)^2 \leq 1.$$

Thus it remains to prove (6.14). We put $T_0 = \min\{T, 1\} (\leq 1)$. Then for $(x,t) \in \mathbb{R}^2 \times [0,T_0)$ we have (6.14) from Sobolev's inequality and (6.12), due to the finite speed of the propagation. While, if $(x,t) \in \mathbb{R}^2 \times [T_0, T)$, then we use the operator $S = t\partial_t + r\partial_r$ and see that

$$w_i(r,t)|\partial_t u^i(x,t)|_k \leq C\sum_{\ell=1}^{2}[\partial_l u(x,t)]_k + C\langle u(x,t)\rangle_{k+1}$$

since $t \geq 1$. This means that (6.14) follows from (6.13); hence, (6.15) holds. This completes the proof of Theorem 1.1. □

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