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The Existence of Global Solutions to Systems of Quasilinear Wave Equations with Quadratic Nonlinearities in 2-Dimensional Space

By

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Abstract. We deal with systems of quasilinear wave equations which contain quadratic nonlinearities in 2-dimensional space. We have already known that such the system has a smooth solution till the time $t_0 = C\varepsilon^{-2}$ for sufficiently small $\varepsilon > 0$, where ε is the size of initial data. In this paper, we shall show that if quadratic and cubic nonlinearities satisfy so-called *Null-condition*, then the smooth solution exists globally in time. In the proof of the theorem, we use the Alinhac ghost weight energy.

Key Words and Phrases. Null-form, Multiple speeds, Global existence.

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1. Introduction

Let us consider the initial value problem

$$\Box_i u^i \equiv \partial_t^2 u^i - c_i^2 \Delta u^i = F^i(\partial u, \partial^2 u) \qquad \text{in } \boldsymbol{R}^2 \times (0,\infty), \tag{1.1}$$

$$u^i(x,0) = \varepsilon f^i(x), \quad \partial_t u^i(x,0) = \varepsilon g^i(x) \qquad \text{in } \boldsymbol{R}^2, \tag{1.2}$$

where $i = 1,\dots,m$ and $u(x,t) = {}^t(u^1(x,t),\dots,u^m(x,t))$. We denote $\partial = (\partial_0, \partial_1, \partial_2)$ with $\partial_0 = \partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$ $(j = 1,2)$. Let $\varepsilon > 0$ be a small parameter and assume $f^i, g^i \in C_0^\infty(\boldsymbol{R}^2)$ and $\mathrm{supp}\{f^i\}, \mathrm{supp}\{g^i\} \subset \{x \in \boldsymbol{R}^2 : |x| \le M\}$ for some positive constant M. We also assume that the propagation speeds of (1.1) are positive and distinct, namely we assume

$$0 < c_1 < c_2 < \cdots < c_m. \tag{1.3}$$

As for the nonlinearity F^i, we assume that

$$F^i(\partial u, \partial^2 u) = \sum_{l=1}^{m} \sum_{\alpha,\beta=0}^{2} A_l^{i,\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u^l + B^i(\partial u), \tag{1.4}$$

where $A_l^{i,\alpha\beta}(\lambda), B^i(\lambda) \in C^\infty(\boldsymbol{R}^{3m})$ and

$$A_l^{i,\alpha\beta}(\lambda) = O(|\lambda|), \qquad B^i(\lambda) = O(|\lambda|^2) \tag{1.5}$$

near the origin. In order to derive an energy estimate, we assume that

$$A_l^{i,\alpha\beta}(\lambda) = A_i^{l,\alpha\beta}(\lambda) = A_i^{l,\beta\alpha}(\lambda) \qquad (\lambda \in \boldsymbol{R}^{3m}) \tag{1.6}$$

holds for each $i, l = 1, \dots, m$ and $\alpha, \beta = 0, 1, 2$. Furthermore, for the sake of simplicity, we assume that

$$|A_l^{i,\alpha\beta}(\lambda)| < \frac{(\min\{1, c_1, \dots, c_m\})^2}{2m} \qquad (\lambda \in \boldsymbol{R}^{3m}). \tag{1.7}$$

Actually, this constitutes no additional restriction, since we shall only deal with solutions, for which ∂u stays close to the origin. Note that if $u(x,t) = {}^t(u^1(x,t), \dots, u^m(x,t))$ is a solution to (1.1) and (1.2), then by (1.3), (1.4) and (1.5), we have for any $i = 1, \dots, m$

$$u^i(x,t) = 0 \qquad \text{for } |x| \geq c_m t + M. \tag{1.8}$$

For the proof, see Theorem 4a in F. John [12].

Our purpose in this paper is to estimate the *lifespan* T_ε of the solution to (1.1) and (1.2). Here T_ε is defined by the supremum of all T for which there exists a solution u to (1.1) and (1.2) in $(C^\infty(\boldsymbol{R}^2 \times [0,T)))^m$. Many works with respect to this problem have been obtained in the last several years. Denote

$$F^i(\partial u, \partial^2 u) = F_2^i(\partial u, \partial^2 u) + F_3^i(\partial u, \partial^2 u) + H^i(\partial u, \partial^2 u),$$

where F_p^i is the homogeneous p-th order terms $(p = 2, 3)$ and $H^i(\partial u, \partial^2 u) = O(|\partial u|^4 + |\partial^2 u|^4)$. Then we know the following. If $B^i \equiv 0$ and $F_2^i \not\equiv 0$, then $T_\varepsilon \geq C_1/\varepsilon^2$ holds for sufficiently small ε. Besides if $B^i \equiv 0$ and $F_2^i \equiv 0$, then $T_\varepsilon \geq \exp(C_2/\varepsilon^2)$ holds for sufficiently small ε. Here C_1 and C_2 are constants depending on f^i, g^i, F^i and c_i. These were proved by L. Hörmander [6] when $m = 1$ and by [8] and [9] when $m \geq 2$. Furthermore, if $B^i \equiv 0$, $F_2^i \equiv 0$ and $F_3^i \equiv 0$, then $T_\varepsilon = \infty$ for sufficiently small ε. This was proved by [6] when $m = 1$ and by [10] and [11] when $m \geq 2$. In general, we cannot expect to improve the above estimate of T_ε, because of counter results, for example S. Alinhac [1] or [7]. However, we can obtain better estimate of T_ε, in the case where the nonlinearities F^i satisfy an algebraic condition, which is so called *Null-condition*. In order to state the Null-condition precisely, we introduce another representation of F^i. By (1.5), we can write $A_l^{i,\alpha\beta}$ and B^i as

$$A_l^{i,\alpha\beta}(\partial u) = \sum_{j=1}^{m} \sum_{\gamma=0}^{2} a_{jl}^{i,\alpha\beta\gamma} \partial_\gamma u^j + \sum_{j,k=1}^{m} \sum_{\gamma,\delta=0}^{2} c_{jkl}^{i,\alpha\beta\gamma\delta} \partial_\gamma u^j \partial_\delta u^k + O(|\partial u|^3), \tag{1.9}$$

$$\begin{aligned} B^i(\partial u) = &\sum_{j,l=1}^{m} \sum_{\alpha,\beta=0}^{2} b_{jl}^{i,\alpha\beta} \partial_\alpha u^j \partial_\beta u^l \\ &+ \sum_{j,k,l=1}^{m} \sum_{\alpha,\beta,\gamma=0}^{2} d_{jkl}^{i,\alpha\beta\gamma} \partial_\alpha u^j \partial_\beta u^k \partial_\gamma u^l + O(|\partial u|^4) \end{aligned} \tag{1.10}$$

near the origin. Here $a_{jl}^{i,\alpha\beta\gamma}$, $b_{jl}^{i,\alpha\beta}$, $c_{jkl}^{i,\alpha\beta\gamma\delta}$ and $d_{jkl}^{i,\alpha\beta\gamma}$ are constants. Then, for $X = (X_0, X_1, X_2) \in \boldsymbol{R}^3$, we define functions $\Phi(X) = (\Phi_j^i(X))_{i,j=1,\dots,m}$, $\Psi(X) = (\Psi_j^i(X))_{i,j=1,\dots,m}$, $\Theta(X) = (\Theta_j^i(X))_{i,j=1,\dots,m}$ and $\Xi(X) = (\Xi_j^i(X))_{i,j=1,\dots,m}$ by

$$\Phi_j^i(X) = \sum_{\alpha,\beta,\gamma=0}^{2} a_{jj}^{i,\alpha\beta\gamma} X_\alpha X_\beta X_\gamma, \tag{1.11}$$

$$\Psi_j^i(X) = \sum_{\alpha,\beta=0}^{2} b_{jj}^{i,\alpha\beta} X_\alpha X_\beta, \tag{1.12}$$

$$\Theta_j^i(X) = \sum_{\alpha,\beta,\gamma,\delta=0}^{2} c_{jjj}^{i,\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma X_\delta, \tag{1.13}$$

$$\Xi_j^i(X) = \sum_{\alpha,\beta,\gamma=0}^{2} d_{jjj}^{i,\alpha\beta\gamma} X_\alpha X_\beta X_\gamma. \tag{1.14}$$

Furthermore, for a function $\phi(X) = (\phi_j^i(X))_{i,j=1,\dots,m}$, if

$$\phi_j^i(X) \equiv 0 \qquad \text{for } X_0^2 = c_j^2(X_1^2 + X_2^2) \tag{1.15}$$

holds for each $i, j = 1, \dots, m$, we denote $\phi \approx 0$ and we say that the corresponding nonlinearity satisfies *Strong Null-condition*. On the other hand, if (1.15) holds when $j = i$ $(i = 1, \dots, m)$, we denote $\phi \sim 0$ and we say that the corresponding nonlinearity satisfies *Standard Null-condition*. Note that the Strong Null-condition is stronger than the Standard Null-condition, while there is no difference between two of them when $m = 1$.

We know the following. When $m = 1$, S. Alinhac [2] and [3] showed that if $B^1(\partial u) \equiv 0$, $\Phi \sim 0$ and $\Theta \sim 0$, then $T_\varepsilon = \infty$ for sufficiently small $\varepsilon > 0$ and also that if $B^1(\partial u) \equiv 0$, $\Phi \sim 0$ and $\Theta \not\sim 0$, then

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log T_\varepsilon = C \tag{1.16}$$

holds, where C is a positive constant depending on f^1, g^1, Θ and c_1. When $m = 1$, P. Godin [5] showed that if $A_1^{1,\alpha\beta}(\partial u) \equiv 0$, $\Psi \sim 0$ and $\Xi \sim 0$, then $T_\varepsilon = \infty$ for sufficiently small ε. He also showed that if $A_1^{1,\alpha\beta}(\partial u) \equiv 0$, $\Psi \sim 0$ and $\Xi \not\sim 0$, then

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log T_\varepsilon \geq C \tag{1.17}$$

holds, where C is a positive constant depending on f^1, g^1, Ξ and c_1. On the other hand, when $m \geq 2$, the author showed in [10] and [11] with H. Kubo that if $B^i(\partial u) \equiv 0$, $F_2^i(\partial u, \partial^2 u) \equiv 0$ $(i = 1, \dots, m)$ and $\Theta \sim 0$, then $T_\varepsilon = \infty$ for sufficiently small $\varepsilon > 0$. The author also showed in [9] that when $m \geq 2$,

$B^i(\partial u) \equiv 0$ and $\Phi \not\approx 0$, there is an example of radially symmetric solution to (1.1) and (1.2) whose lifespan satisfies

$$\lim_{\varepsilon \to 0} \varepsilon\sqrt{T_\varepsilon} = C$$

for some constant $C > 0$.

The purpose of this paper is to analyse the case where $m \geq 2$, $F_2^i \not\equiv 0$, $\Phi \approx 0$, $\Psi \approx 0$, $\Theta \approx 0$ and $\Xi \approx 0$. We have the following.

Theorem 1.1. *Let T_ε be the lifespan of the solution to* (1.1) *and* (1.2). *Assume that* (1.3), (1.4), (1.5), (1.6), (1.7), (1.9) *and* (1.10) *hold and $\Phi \approx 0$, $\Psi \approx 0$, $\Theta \approx 0$ and $\Xi \approx 0$. Assume also that $a_{jl}^{i,\alpha\beta\gamma} = b_{jl}^{i,\alpha\beta} = 0$ if $j \neq l$. Then there is an $\varepsilon_0 > 0$ such that $T_\varepsilon = \infty$ holds for $\varepsilon \in (0, \varepsilon_0)$.*

We have two typical forms of the nonlinearity F^i satisfying the Strong Null-condition; *Null-form* and *Nonresonance-form*. The Null-form consists of polynomials of $\partial^a u^j$ $(a = 1, 2)$ for which $\Phi_j^i(X)$, $\Psi_j^i(X)$, $\Theta_j^i(X)$ and $\Xi_j^i(X)$ vanish on the hyperplane $X_0^2 = c_j^2(X_1^2 + X_2^2)$, while the Nonresonance-form consists of terms which contain products of waves whose propagation speeds are different. For futher consideration, we write F^i as

$$F^i(\partial u, \partial^2 u) = \sum_{j=1}^{m} N_j^i(\partial u^j, \partial^2 u^j) + R^i(\partial u, \partial^2 u) + H^i(\partial u, \partial^2 u),$$

where N_j^i stands for the Null-form and R^i stands for the Nonresonance-form. We also use a notation $N_j^i = N_{j,2}^i + N_{j,3}^i$ with

$$(1.18) \quad N_{j,2}^i(\partial u^j, \partial^2 u^j) = \sum_{\alpha,\beta,\gamma=0}^{2} a_{jj}^{i,\alpha\beta\gamma}\partial_\gamma u^j \partial_\alpha\partial_\beta u^j + \sum_{\alpha,\beta=0}^{2} b_{jj}^{i,\alpha\beta}\partial_\alpha u^j \partial_\beta u^j,$$

$$(1.19) \quad N_{j,3}^i(\partial u^j, \partial^2 u^j) = \sum_{\alpha,\beta,\gamma,\delta=0}^{2} c_{jjj}^{i,\alpha\beta\gamma\delta}\partial_\gamma u^j \partial_\delta u^j \partial_\alpha\partial_\beta u^j + \sum_{\alpha,\beta,\gamma=0}^{2} d_{jjj}^{i,\alpha\beta\gamma}\partial_\alpha u^j \partial_\beta u^j \partial_\gamma u^j.$$

It follows from the assumption in Theorem 1.1 that R^i is cubic with respect to ∂u and $\partial^2 u$ and also that we can write $A_l^{i,\alpha\beta}$ as

$$A_l^{i,\alpha\beta}(\partial u) = \sum_{\gamma=0}^{2} a_{ll}^{i,\alpha\beta\gamma}\partial_\gamma u^l + O(|\partial u|^2).$$

Furthermore (1.6) implies that for any $\alpha, \beta = 0, 1, 2$ and $i, l = 1, \ldots, m$

$$\sum_{\gamma=0}^{2} a_{ll}^{i,\alpha\beta\gamma}\lambda_\gamma^l = \sum_{\gamma=0}^{2} a_{ii}^{l,\alpha\beta\gamma}\lambda_\gamma^i$$

holds for any $\lambda = (\lambda_\gamma^l)_{l=1,\dots,m,\gamma=0,1,2}$. This means that for any $\alpha, \beta, \gamma = 0, 1, 2$,

$$a_{ll}^{i,\alpha\beta\gamma} = 0 \qquad \text{if } l \neq i. \tag{1.20}$$

In the proof of Theorem 1.1, we derive an *a priori* estimate of the solution and for this purpose we use the method of *Ghost weight energy* which was introduced by S. Alinhac in [2]. It makes enable us to use the advantage of the Null-form in the energy argument. In [2], Alinhac used the Klainerman type embedding inequality to bound the weighted L^∞ norm of the solution. The Klainerman inequality holds for any function u possessing suitable regularity. But it does not give enough decay estimates to ∂u. Thus, Alinhac could not deal with semilinear terms B^1 in F^1. However, in the present paper, we successfully deal with the semilinear term B^i. This is due to the L^∞-L^∞ estimate of the solution to the wave equation, which appears in Proposition 3.2 below. It was developed in [11].

This paper is organized as follows. In Section 2, we introduce some notation and we also describe a lemma which implies Theorem 1.1. Section 3 is devoted to the weighted L^2-L^∞ estimate of the solution. In Section 4, we carry out the L^2 estimate of the solution, by using the method of Ghost weight energy. In the final section, we complete the proof of the main lemma stated in the section 2.

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2. Preliminary for the proof of Theorem 1.1

In this section, we explain notation and discribe a lemma to show Theorem 1.1. First of all, we introduce the following vector fields:

$$\Omega = x_1\partial_2 - x_2\partial_1, \qquad S = t\partial_t + r\partial_r, \tag{2.1}$$

where $r = |x|$ and $r\partial_r = x_1\partial_1 + x_2\partial_2$, and we denote

$$\Gamma = (\Gamma_0, \dots, \Gamma_4) = (\partial_0, \partial_1, \partial_2, \Omega, S).$$

We can easily verify the following commutator relations:

$$[\Gamma_\sigma, \Box_i] = -2\delta_{4\sigma}\Box_i \qquad \text{for } \sigma = 0, \dots, 4,\ i = 1, \dots, m, \tag{2.2}$$

$$[\partial_\alpha, \partial_\beta] = 0 \quad (\alpha, \beta = 0, 1, 2), \qquad [S, \partial_\alpha] = -\partial_\alpha \quad (\alpha = 0, 1, 2),$$

$$[\Omega, \partial_1] = -\partial_2, \qquad [\Omega, \partial_2] = \partial_1, \qquad [\Omega, \partial_0] = 0, \qquad [S, \Omega] = 0.$$

Here $[\,,]$ denotes the usual commutator of linear operators and $\delta_{\alpha\beta}$ is the Kronecker delta.

Let $v(x,t) = {}^t(v^1(x,t), \dots, v^m(x,t))$ be a vector valued function defined in $\boldsymbol{R}^2 \times [0, T)$. Then we set

$$|v(x,t)|_k = \sum_{|a|\le k}\sum_{i=1}^{m} |\Gamma^a v^i(x,t)|,$$

$$|v(t)|_k = \sup_{x\in \boldsymbol{R}^2} |v(x,t)|_k,$$

$$[v(x,t)]_k = \sum_{|a|\le k}\sum_{i=1}^{m} |w^i(|x|,t)\Gamma^a v^i(x,t)|,$$

$$[v(t)]_k = \sup_{x\in \boldsymbol{R}^2} [v(x,t)]_k,$$

$$\langle v(x,t)\rangle_k = \sum_{|a|\le k}\sum_{i=1}^{m} (1+|x|+t)^{1/2}|\Gamma^a v^i(x,t)|,$$

$$\langle v(t)\rangle_k = \sup_{x\in \boldsymbol{R}^2} \langle v(x,t)\rangle_k,$$

$$\|v(t)\|_k = \sum_{|a|\le k}\sum_{i=1}^{m} \|\Gamma^a v^i(\cdot,t)\|_{L^2},$$

where k is a nonnegative integer, $a = (a_0, \dots, a_4)$ is a multi-index, $|a| = a_0 + \cdots + a_4$ and

$$w^i(r,t) = (1+r)^{1/2}(1+|r-c_i t|), \qquad i = 0, 1, \dots, m$$

with $c_0 = 0$.

We also use the notation

$$\|v\|_{k,t} = \sup_{0\le s<t} \|v(s)\|_k, \qquad \langle v\rangle_{k,t} = \sup_{0\le s<t} \langle v(s)\rangle_k,$$

$$[v]_{k,t} = \sup_{0\le s<t} [v(s)]_k, \qquad |v|_{k,t} = \sup_{0\le s<t} |v(s)|_k.$$

Then, we find that the following lemma implies Theorem 1.1.

Lemma 2.1. *Let T and J be positive constants and let $u(x,t) = {}^t(u^1(x,t), \dots, u^m(x,t)) \in (C^\infty(\boldsymbol{R}^2 \times [0,T)))^m$ be a solution to (1.1) and (1.2). Choose an integer k as $k \ge 8$. Then, there exist positive constants K and $\varepsilon_0(J)$ such that if*

$$[\partial u]_{k,T} + \langle u\rangle_{k+1,T} \le J\varepsilon \tag{2.3}$$

holds for some $\varepsilon \in (0, \varepsilon_0(J))$, then

$$[\partial u]_{k,T} + \langle u\rangle_{k+1,T} \le K\varepsilon \tag{2.4}$$

holds for the same ε. Here K is independent of ε and J.

Proof of Theorem 1.1. We show that Lemma 2.1 implies Theorem 1.1 by contradiction. Let $k \geq 8$ be an integer. By the local existence theorem (see K. Kubota and K. Yokoyama [15] or Theorem 2.2 in A. Majda [16] for example), we find that there are positive constants ε_k and T_k such that for any $\varepsilon \in (0, \varepsilon_k)$ there exists a smooth solution $u(x,t)$ to (1.1) and (1.2) in $\boldsymbol{R}^2 \times [0, T_k)$. Let $L = L_k(f,g) > 0$ be a constant satisfying $[\partial u(0)]_k + \langle u(0) \rangle_{k+1} \leq L\varepsilon$ for any $\varepsilon \in (0, \varepsilon_k)$ and set

$$J_0 = 2 \max\{K, L\}, \tag{2.5}$$

where K is the one determined in Lemma 2.1. Then, we can define positive constants τ_ε by

$$\tau_\varepsilon = \sup\{t > 0 : t < T_\varepsilon \text{ and } [\partial u]_{k,t} + \langle u \rangle_{k+1,t} \leq J_0 \varepsilon\} \ (\leq T_\varepsilon)$$

for any $\varepsilon \in (0, \varepsilon_k)$. Set $\varepsilon_0 = \min\{\varepsilon_0(J_0), \varepsilon_k\} > 0$. If we assume that there exists an $\hat{\varepsilon} \in (0, \varepsilon_0)$ satisfying $\tau_{\hat{\varepsilon}} < \infty$, then either $\tau_{\hat{\varepsilon}} = T_{\hat{\varepsilon}}$ or

$$[\partial u]_{k,\tau_{\hat{\varepsilon}}} + \langle u \rangle_{k+1,\tau_{\hat{\varepsilon}}} = J_0 \hat{\varepsilon} \tag{2.6}$$

holds. However, using (1.6) and (1.7), we can show $\tau_{\hat{\varepsilon}} < T_{\hat{\varepsilon}}$. (For the detail, see the proof of Lemma 2.1 in [9].) Therefore, we find that (2.6) holds. On the other hand, since $\hat{\varepsilon} < \varepsilon_0(J_0)$ and $\tau_{\hat{\varepsilon}} < T_{\hat{\varepsilon}}$, Lemma 2.1 implies that

$$[\partial u(t)]_k + \langle u(t) \rangle_{k+1} \leq K\hat{\varepsilon} \leq \frac{J_0}{2}\hat{\varepsilon} \qquad \text{for } 0 \leq t < \tau_{\hat{\varepsilon}}.$$

This contradicts (2.6). Therefore $\tau_\varepsilon = \infty$ for any $\varepsilon \in (0, \varepsilon_0)$. This completes the proof of Theorem 1.1.

In the following sections, we shall show Lemma 2.1. For this purpose, we prepare a proposition about the dacay rate of the Null-form in the light cone. Set $c_* = \min_{1 \leq i \leq m}\{c_i - c_{i-1}\}/3$ with $c_0 = 0$. We see $c_* > 0$ from (1.3). Moreover, we define

$$\Lambda_i(T) = \{(x,t) \in \boldsymbol{R}^2 \times [0,T) : |\,|x| - c_i t| \leq c_* t\}, \qquad i = 1, \dots, m \tag{2.7}$$

and

$$\Lambda_0(T) = \boldsymbol{R}^2 \times [0,T) \Big\backslash \bigcup_{i=1}^m \Lambda_i(T).$$

By the definition, $\Lambda_i(T) \cap \Lambda_j(T) = \phi$ holds for any $T > 0$, if $i \neq j$ and hence there exists a positive constant C such that

$$\frac{1}{C}(|x|+t) \le ||x| - c_j t| \le C(|x|+t) \tag{2.8}$$

holds for $(x,t) \in \Lambda_i(T)$, if $i \neq j$.

Owing to the strong Null-condition, we can show that the Null-form N_j^i has a good decay property in $\Lambda_j(T)$. In this argument, the following operators

$$Z^j = (Z_1^j, Z_2^j) \equiv \frac{c_j t - |x|}{t}(\partial_1, \partial_2) + \frac{(x_1, x_2)S + (-x_2, x_1)\Omega}{|x|t} \tag{2.9}$$

play an impotant role. These were introduced in [2]. It follows from (2.9) that

$$Z_\alpha^j = c_j \partial_\alpha + \frac{x_\alpha}{|x|}\partial_0 \qquad (\alpha = 1, 2) \tag{2.10}$$

and

$$|Z^j v(x,t)| \le C\left(\frac{||x| - c_j t|}{t}|\partial v(x,t)|_0 + \frac{1}{t}|v(x,t)|_1\right) \tag{2.11}$$

hold. Then we have the following.

Proposition 2.1. *Let $T > 1$ be a constant and k a positive integer. Also let $u(x,t) = {}^t(u^1(x,t), \dots, u^m(x,t)) \in (C^\infty(\boldsymbol{R}^2 \times [0,T)))^m$ be a solution to* (1.1) *and* (1.2) *and let $N_{j,2}^i$ and $N_{j,3}^i$ be functions defined in* (1.18) *and* (1.19). *Assume that $\Phi \approx 0$, $\Psi \approx 0$, $\Theta \approx 0$ and $\Xi \approx 0$ hold. Then, there exists a positive constant C_k independent of T such that*

$$|N_{j,2}^i(\partial u^j(x,t), \partial^2 u^j(x,t))|_k \le C_k \sum_{|b+c| \le k+1} |Z^j \Gamma^b u^j(x,t)|\,|\Gamma^c \partial u^j(x,t)|, \tag{2.12}$$

$$\left|\sum_{\alpha,\beta=0}^{2} b_{jj}^{i,\alpha\beta}\partial_\alpha u^j(x,t)\partial_\beta u^j(x,t)\right|_k \le C_k \sum_{|b+c| \le k} |Z^j \Gamma^b u^j(x,t)|\,|\Gamma^c \partial u^j(x,t)|, \tag{2.12}$$

$$\begin{aligned} &|N_{j,3}^i(\partial u^j(x,t), \partial^2 u^j(x,t))|_k \\ &\quad \le C_k \sum_{|b+c+d| \le k+1} |Z^j \Gamma^b u^j(x,t)|\,|\Gamma^c \partial u^j(x,t)|\,|\Gamma^d \partial u^j(x,t)| \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &\sum_{|a| \le k} |\Gamma^a \{a_{jj}^{i,\alpha\beta\gamma}\partial_\gamma u^j \partial_\alpha \partial_\beta u^j(x,t)\} - a_{jj}^{i,\alpha\beta\gamma}\partial_\gamma u^j \partial_\alpha \partial_\beta \Gamma^a u^j(x,t)| \\ &\quad \le C_k \sum_{\substack{|b+c| \le k+1 \\ |b|, |c| \le k}} |Z^j \Gamma^b u^j(x,t)|\,|\Gamma^c \partial u^j(x,t)| \end{aligned} \tag{2.15}$$

hold for $(x,t) \in \Lambda_j(T) \cap \{(y,s) : s \geq 1\}$, $j = 1, \dots, m$. *Moreover, we have*

$$(2.16) \quad |N^i_{j,2}(\partial u^j(x,t), \partial^2 u^j(x,t))|_k \leq \frac{C_k||x| - c_j t|}{1+|x|+t} |\partial u^j(x,t)|_{[(k+1)/2]} |\partial u^j(x,t)|_{k+1} + \frac{C_k}{1+|x|+t} P^j_k(x,t),$$

$$(2.17) \quad |N^i_{j,3}(\partial u^j(x,t), \partial^2 u^j(x,t))|_k \leq \frac{C_k||x| - c_j t|}{1+|x|+t} |\partial u^j(x,t)|^2_{[(k+1)/2]} |\partial u^j(x,t)|_{k+1} + \frac{C_k}{1+|x|+t} |\partial u^j(x,t)|_{[(k+1)/2]} P^j_k(x,t)$$

and

$$(2.18) \quad \sum_{|a| \leq k} |\Gamma^a\{a^{i,\alpha\beta\gamma}_{jj} \partial_\gamma u^j \partial_\alpha \partial_\beta u^j(x,t)\} - a^{i,\alpha\beta\gamma}_{jj} \partial_\gamma u^j \partial_\alpha \partial_\beta \Gamma^a u^j(x,t)| \leq \frac{C_k||x| - c_j t|}{1+|x|+t} |\partial u^j(x,t)|_{[(k+1)/2]} |\partial u^j(x,t)|_k + \frac{C_k}{1+|x|+t} Q^j_k(x,t)$$

hold for $(x,t) \in \Lambda_j(T) \cap \{(y,s) : s \geq 1\}$, $j = 1, \dots, m$. *Here we have set*

$$P^j_k(x,t) = |u^j(x,t)|_{[(k+1)/2]} |\partial u^j(x,t)|_{k+1} + |\partial u^j(x,t)|_{[(k+1)/2]} |u^j(x,t)|_{k+1},$$

$$Q^j_k(x,t) = |u^j(x,t)|_{[(k+1)/2]} |\partial u^j(x,t)|_k + |\partial u^j(x,t)|_{[(k+1)/2]} |u^j(x,t)|_{k+1}.$$

Proof. Since $N^i_{j,3}$ and $\sum_{\alpha,\beta=0}^2 b^{i,\alpha\beta}_{jj} \partial_\alpha u^j \partial_\beta u^j$ can be treated by the same manner as $N^i_{j,2}$, we concentrate on the proof of (2.12), (2.15), (2.16) and (2.18). Moreover, for the sake of simplicity, we assume that $N^i_{j,2}$ is quasilinear, namely we assume $b^{i,\alpha\beta}_{jj} = 0$ for any $i, j = 1, \dots, m$ and $\alpha, \beta = 0, 1, 2$ through the proof of the proposition. At first, we prove the following.

Lemma 2.2. *Fix* $j \in \{1, \dots, m\}$, *let* T *be a positive constant and set*

$$N(\partial v(x,t), \partial^2 w(x,t)) = \sum_{\alpha,\beta,\gamma=0}^2 h^{\alpha\beta\gamma} \partial_\gamma v(x,t) \partial_\alpha \partial_\beta w(x,t)$$

for $v, w \in C^\infty(\boldsymbol{R}^2 \times [0,T))$, *where* $h^{\alpha\beta\gamma}$ *are constants. Assume that*

$$(2.19) \quad \sum_{\alpha,\beta,\gamma=0}^2 h^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma \equiv 0 \qquad \text{for } X_0^2 = c_j^2(X_1^2 + X_2^2).$$

Then,

$$(2.20)\quad |N(\partial v(x,t), \partial^2 w(x,t))| \le C(|Z^j v(x,t)|\,|\partial^2 w(x,t)| + |\partial v(x,t)|\,|Z^j \partial w(x,t)|)$$

holds.

Proof. It follows from (2.19) that

$$(2.21)\quad N(\partial v(x,t), \partial^2 w(x,t))$$
$$= \sum_{\alpha,\beta,\gamma=0}^{2} \left(h^{\alpha\beta\gamma} \partial_\gamma v(x,t) \partial_\alpha \partial_\beta w(x,t) + h^{\alpha\beta\gamma} \frac{\omega_\alpha}{c_j} \frac{\omega_\beta}{c_j} \frac{\omega_\gamma}{c_j} \partial_0 v(x,t) \partial_0^2 w(x,t) \right)$$
$$= \sum_{\alpha,\beta,\gamma=0}^{2} h^{\alpha\beta\gamma} \Bigg\{ \partial_\gamma v(x,t) \left(\partial_\alpha + \frac{\omega_\alpha}{c_j} \partial_0 \right) \partial_\beta w(x,t)$$
$$- \partial_\gamma v(x,t) \frac{\omega_\alpha}{c_j} \partial_0 \left(\partial_\beta + \frac{\omega_\beta}{c_j} \partial_0 \right) w(x,t)$$
$$+ \left(\partial_\gamma + \frac{\omega_\gamma}{c_j} \partial_0 \right) v(x,t) \frac{\omega_\alpha}{c_j} \frac{\omega_\beta}{c_j} \partial_0^2 w(x,t) \Bigg\},$$

where $(\omega_0, \omega_1, \omega_2) = (-c_j, x_1/|x|, x_2/|x|)$. Therefore, (2.2), (2.10) and (2.21) imply (2.20).

Next we show the following.

Lemma 2.3. *Let Γ be an operator belonging to $(\Gamma_0, \dots, \Gamma_4)$. Then, for any $i, j = 1, \dots, m$, there exists a function $\hat{N}^i_{j,2}(\partial u^j, \partial^2 u^j)$ satisfying the Strong Null-condition and*

$$(2.22)\quad \Gamma N^i_{j,2}(\partial u^j, \partial^2 u^j) = N^i_{j,2}(\partial \Gamma u^j, \partial^2 u^j) + N^i_{j,2}(\partial u^j, \partial^2 \Gamma u^j) + \hat{N}^i_{j,2}(\partial u^j, \partial^2 u^j).$$

Proof. By (2.2), we find that

$$\partial_\delta N^i_{j,2}(\partial u^j, \partial^2 u^j) = N^i_{j,2}(\partial(\partial_\delta u^j), \partial^2 u^j) + N^i_{j,2}(\partial u^j, \partial^2(\partial_\delta u^j)) \qquad (\delta = 0, 1, 2)$$

and

$$S N^i_{j,2}(\partial u^j, \partial^2 u^j) = N^i_{j,2}(\partial(S u^j), \partial^2 u^j) + N^i_{j,2}(\partial u^j, \partial^2(S u^j)) - 3 N^i_{j,2}(\partial u^j, \partial^2 u^j),$$

which imply (2.22) for the case $\Gamma = \partial_\delta$ $(\delta = 0, 1, 2)$ and $\Gamma = S$. When $\Gamma = \Omega$, it follows from (2.2) that

$$\Omega N^i_{j,2}(\partial u^j, \partial^2 u^j) = N^i_{j,2}(\partial(\Omega u^j), \partial^2 u^j) + N^i_{j,2}(\partial u^j, \partial^2(\Omega u^j)) + \hat{N}^i_{j,2}(\partial u^j, \partial^2 u^j), \tag{2.23}$$

where we have set

$$\begin{aligned}\hat{N}^i_{j,2}(\partial u^j, \partial^2 u^j) = &\sum_{\alpha,\beta=0}^{2} (a^{i,\alpha\beta 2}_{jj}\partial_1 u^j - a^{i,\alpha\beta 1}_{jj}\partial_2 u^j)\partial_\alpha\partial_\beta u^j \\ &+ \sum_{\beta,\gamma=0}^{2} (a^{i,2\beta\gamma}_{jj}\partial_1\partial_\beta u^j - a^{i,1\beta\gamma}_{jj}\partial_2\partial_\beta u^j)\partial_\gamma u^j \\ &+ \sum_{\alpha,\gamma=0}^{2} (a^{i,\alpha 2\gamma}_{jj}\partial_\alpha\partial_1 u^j - a^{i,\alpha 1\gamma}_{jj}\partial_\alpha\partial_2 u^j)\partial_\gamma u^j.\end{aligned} \tag{2.24}$$

Hence, our task is to show that $\hat{N}^i_{j,2}(\partial u^j, \partial^2 u^j)$ satisfies the Strong Null-condition, namely we need to show for any $i, j = 1, \dots, m$

$$\begin{aligned}&(a^{i,002}_{jj} + a^{i,020}_{jj} + a^{i,200}_{jj})X_0^2 X_1 - (a^{i,001}_{jj} + a^{i,010}_{jj} + a^{i,100}_{jj})X_0^2 X_2 \\ &+ (a^{i,102}_{jj} + a^{i,012}_{jj} + a^{i,210}_{jj} + a^{i,201}_{jj} + a^{i,120}_{jj} + a^{i,021}_{jj})X_0 X_1^2 \\ &- (a^{i,021}_{jj} + a^{i,201}_{jj} + a^{i,120}_{jj} + a^{i,102}_{jj} + a^{i,210}_{jj} + a^{i,012}_{jj})X_0 X_2^2 \\ &+ 2(a^{i,022}_{jj} + a^{i,202}_{jj} + a^{i,220}_{jj} - a^{i,011}_{jj} - a^{i,101}_{jj} - a^{i,110}_{jj})X_0 X_1 X_2 \\ &+ (a^{i,112}_{jj} + a^{i,211}_{jj} + a^{i,121}_{jj})X_1^3 \\ &+ \{2(a^{i,122}_{jj} + a^{i,212}_{jj} + a^{i,221}_{jj}) - 3a^{i,111}_{jj}\}X_1^2 X_2 \\ &- \{2(a^{i,211}_{jj} + a^{i,121}_{jj} + a^{i,112}_{jj}) - 3a^{i,222}_{jj}\}X_1 X_2^2 \\ &- (a^{i,221}_{jj} + a^{i,212}_{jj} + a^{i,122}_{jj})X_2^3 \\ = {}&0 \qquad \text{for } X_0^2 = c_j^2(X_1^2 + X_2^2).\end{aligned} \tag{2.25}$$

We find that the following lemma immediately leads to (2.25).

Lemma 2.4. *Assume that $\Phi \approx 0$. Then we have*

$$\begin{aligned}a^{i,110}_{jj} + a^{i,101}_{jj} + a^{i,011}_{jj} &= a^{i,220}_{jj} + a^{i,202}_{jj} + a^{i,022}_{jj} = -c_j^2 a^{i,000}_{jj} \\ a^{i,102}_{jj} + a^{i,012}_{jj} + a^{i,210}_{jj} &+ a^{i,201}_{jj} + a^{i,120}_{jj} + a^{i,021}_{jj} = 0 \\ a^{i,112}_{jj} + a^{i,121}_{jj} + a^{i,211}_{jj} &= -c_j^2(a^{i,200}_{jj} + a^{i,020}_{jj} + a^{i,002}_{jj}) = a^{i,222}_{jj} \\ a^{i,221}_{jj} + a^{i,212}_{jj} + a^{i,122}_{jj} &= -c_j^2(a^{i,100}_{jj} + a^{i,010}_{jj} + a^{i,001}_{jj}) = a^{i,111}_{jj}.\end{aligned} \tag{2.26}$$

Proof. It follows from the Strong Null- condition that

$$
\begin{aligned}
(2.27)\qquad \Phi_j^i(X) &= a_{jj}^{i,000}X_0^3 + (a_{jj}^{i,100} + a_{jj}^{i,010} + a_{jj}^{i,001})X_0^2X_1 \\
&\quad + (a_{jj}^{i,200} + a_{jj}^{i,020} + a_{jj}^{i,002})X_0^2X_2 \\
&\quad + (a_{jj}^{i,110} + a_{jj}^{i,101} + a_{jj}^{i,011})X_0X_1^2 + (a_{jj}^{i,220} + a_{jj}^{i,202} + a_{jj}^{i,022})X_0X_2^2 \\
&\quad + (a_{jj}^{i,012} + a_{jj}^{i,021} + a_{jj}^{i,102} + a_{jj}^{i,201} + a_{jj}^{i,120} + a_{jj}^{i,210})X_0X_1X_2 \\
&\quad + a_{jj}^{i,111}X_1^3 + (a_{jj}^{i,112} + a_{jj}^{i,121} + a_{jj}^{i,211})X_1^2X_2 \\
&\quad + (a_{jj}^{i,122} + a_{jj}^{i,212} + a_{jj}^{i,221})X_1X_2^2 + a_{jj}^{i,222}X_2^3 \\
&= 0 \qquad \text{for } X_0^2 = c_j^2(X_1^2 + X_2^2).
\end{aligned}
$$

Substituting $X_0^2 = c_j^2(X_1^2 + X_2^2)$ to (2.27), we have

$$
\begin{aligned}
(2.28)\qquad &\{c_j^2a_{jj}^{i,000}(X_1^2 + X_2^2) + (a_{jj}^{i,110} + a_{jj}^{i,101} + a_{jj}^{i,011})X_1^2 \\
&\quad + (a_{jj}^{i,220} + a_{jj}^{i,202} + a_{jj}^{i,022})X_2^2 \\
&\quad + (a_{jj}^{i,012} + a_{jj}^{i,021} + a_{jj}^{i,102} + a_{jj}^{i,201} + a_{jj}^{i,120} + a_{jj}^{i,210})X_1X_2\}X_0 \\
&\quad + c_j^2(a_{jj}^{i,100} + a_{jj}^{i,010} + a_{jj}^{i,001})(X_1^2 + X_2^2)X_1 \\
&\quad + c_j^2(a_{jj}^{i,200} + a_{jj}^{i,020} + a_{jj}^{i,002})(X_1^2 + X_2^2)X_2 \\
&\quad + a_{jj}^{i,111}X_1^3 + (a_{jj}^{i,112} + a_{jj}^{i,121} + a_{jj}^{i,211})X_1^2X_2 \\
&\quad + (a_{jj}^{i,122} + a_{jj}^{i,212} + a_{jj}^{i,221})X_1X_2^2 + a_{jj}^{i,222}X_2^3 \\
&= 0.
\end{aligned}
$$

Choosing (X_0, X_1, X_2) suitably in (2.28), we obtain (2.26).

Now we complete the proof of Proposition 2.1. By Lemma 2.3, we have

$$
\begin{aligned}
(2.29)\qquad &|N_{j,2}^i(\partial u^j(x,t), \partial^2 u^j(x,t))|_k \\
&\quad \le \sum_{|b+c| \le k} |\hat{N}_{j,2,b,c}^i(\partial(\Gamma^b u^j(x,t)), \partial^2(\Gamma^c u^j(x,t)))|,
\end{aligned}
$$

where each $\hat{N}_{j,2,b,c}^i$ satisfies the Strong Null-condition. Hence, by (2.2), (2.29) and Lemma 2.2, we obtain (2.12). Similarly, by Lemma 2.3, we have

$$\sum_{|a|\le k} |\Gamma^a\{a_{jj}^{i,\alpha\beta\gamma}\partial_\gamma u^j\partial_\alpha\partial_\beta u^j(x,t)\} - a_{jj}^{i,\alpha\beta\gamma}\partial_\gamma u^j\partial_\alpha\partial_\beta\Gamma^a u^j(x,t)| \le \sum_{\substack{|b+c|\le k\\ |c|<k}} |\hat{N}^i_{j,2,b,c}(\partial(\Gamma^b u^j(x,t)), \partial^2(\Gamma^c u^j(x,t)))|. \tag{2.30}$$

Hence, by (2.30) and Lemma 2.2, we obtain (2.15). Moreover, (2.11), (2.12) and (2.15) imply (2.16) and (2.18).

3. L^2-L^∞ estimates

In this section, we devote ourselves to proving the following.

Proposition 3.1. *Let k be a positive integer, δ a positive and small constant and T a positive constant. Also let $u(x,t) = {}^t(u^1(x,t),\dots,u^m(x,t)) \in (C^\infty(\boldsymbol{R}^2\times[0,T)))^m$ be a solution to* (1.1) *and* (1.2)*. Assume that*

$$[\partial u]_{[(k+6)/2],T} + \langle u\rangle_{[(k+4)/2],T} \le \delta. \tag{3.1}$$

Then there exist positive constants $C_0 = C_0(f,g,k)$, $C_1 = C_1(\delta,k)$ and ξ such that

$$[\partial u(t)]_k + \langle u(t)\rangle_{k+1} \le C_0\varepsilon + C_1([\partial u]_{[(k+6)/2],t} + \langle u\rangle_{[(k+4)/2],t}) \sup_{0\le s<t}\{(1+s)^{-\xi}\|\partial u(s)\|_{k+8}\} \tag{3.2}$$

holds for $0\le t<T$.

Proof. First of all, by (1.8) and the identity $\partial_0 = S/t - x\cdot\nabla/t$ ($\nabla = (\partial_1,\partial_2)$), we have

$$|\partial_0 u(x,t)|_k \le C|\nabla u(x,t)|_k + \frac{C}{1+t}|u(x,t)|_{k+1} \qquad \text{in } \boldsymbol{R}^2\times[0,T). \tag{3.3}$$

Therefore it follows from (3.3) that we only need to show (3.2) with ∂ replaced by ∂_j $(j=1,2)$ in the left-hand side. Moreover, we can write the solution u^i of (1.1) and (1.2) as $u^i = \varepsilon u_0^i + L_i(F^i)$. Here u_0^i is the solution of

$$\Box_i u_0^i = 0 \qquad \text{in } \boldsymbol{R}^2\times(0,\infty),$$

$$u_0^i(x,0) = f^i(x), \quad \partial_0 u_0^i(x,0) = g^i(x) \qquad \text{in } \boldsymbol{R}^2,$$

while $L_i(F)$ is the solution of

$$\Box_i L_i(F) = F \qquad \text{in } \boldsymbol{R}^2\times(0,T),$$

$$L_i(F)(x,0) = \partial_0 L_i(F)(x,0) = 0 \qquad \text{in } \boldsymbol{R}^2$$

for $F \in C^\infty(\boldsymbol{R}^2 \times [0,T))$. Since we know that

$$[\partial u_0(t)]_k + \langle u_0(t)\rangle_{k+1} \le C_0(f^1,\dots,f^m,g^1,\dots,g^m,k) \tag{3.4}$$

for $0 \le t < \infty$ with $u_0 = {}^t(u_0^1,\dots,u_0^m)$ (for the proof, see [4]), our task is reduced to show

$$\begin{aligned}&[\nabla L_i(F^i)(t)]_k + \langle L_i(F^i)(t)\rangle_{k+1}\\ &\quad \le C_0\varepsilon + C_1([\partial u]_{[(k+6)/2],t} + \langle u\rangle_{[(k+4)/2],t}) \sup_{0\le s<t}\{(1+s)^{-\xi}\|\partial u(s)\|_{k+8}\}\end{aligned} \tag{3.5}$$

for $0 \le t < T$. Here and hereafter, we interpret $[\nabla L_i(F)(t)]_k$ as

$$[\nabla L_i(F)(t)]_k = \sup_{x\in\boldsymbol{R}^2}[\nabla L_i(F)(x,t)]_k = \sup_{x\in\boldsymbol{R}^2}\sum_{|a|\le k}|w^i(|x|,t)\Gamma^a\nabla L_i(F)(x,t)|$$

for $i = 1,\dots,m$. To prove (3.5), the following propositions are important.

Proposition 3.2. *Let k be a positive integer, T a positive constant and F a function belonging to $C^\infty(\boldsymbol{R}^2 \times [0,T))$. Then for any $\mu > 0$ there exists a positive constant $C = C(\mu)$ such that*

$$\langle L_i(F)(x,t)\rangle_k \le CM^{(i)}_{\mu,k}(F)(x,t) \tag{3.6}$$

and

$$[\nabla L_i(F)(x,t)]_k \le CM^{(i)}_{\mu,k+1}(F)(x,t) \tag{3.7}$$

hold for each $i = 1,\dots,m$, where we have set

$$\begin{aligned}&M^{(i)}_{\mu,k}(F)(x,t)\\ &\quad = \sum_{j=0}^m \sup_{\substack{(y,s)\in\\ D^i(x,t)\cap\Lambda_j(t)}}\{|y|^{1/2}(1+|y|+s)^{1+\mu}(1+||y|-c_js|)|F(y,s)|_k\}\end{aligned} \tag{3.8}$$

and

$$D^i(x,t) = \{(y,s)\in\boldsymbol{R}^2\times[0,t] : |y-x| \le c_i(t-s)\}. \tag{3.9}$$

Proposition 3.3. *If $u \in C^2(\boldsymbol{R}^2 \times [0,T))$ is a function satisfying $\|u\|_{2,T} < \infty$, then*

$$|x|^{1/2}|u(x,t)| \le C\|u(t)\|_2 \tag{3.10}$$

holds for $0 \le t < T$. Here C is a numerical constant.

We omit the proof of these propositions. For the details of Proposition 3.2, see Proposition 4.2 in [11] and Proposition 3.2 in [9]. On the other hand, for Proposition 3.3, see Theorem 1 (iii) and (iv) in S. Klainerman [14].

Now we show (3.5). It is led by the case $p=0$ in the following proposition.

Proposition 3.4. *Choose positive constants k, T and δ as in Proposition* 3.1. *Also choose μ as $0<\mu<1/6$. Let $u(x,t)={}^t(u^1(x,t),\dots,u^m(x,t))\in(C^\infty(\boldsymbol{R}^2\times[0,T)))^m$ be a solution to* (1.1) *and* (1.2). *Assume* (3.1). *Then, there exists a positive constant ξ such that if*

$$[\partial u]_{[(k+6-2p)/2],T}+\langle u\rangle_{[(k+4-2p)/2],T}\le\delta, \tag{3.11}$$

then

$$\begin{aligned}(1+|x|+t)^{(-1/2+\mu)p}&([\nabla L_i(F^i)]_k+\langle L_i(F^i)\rangle_{k+1})\\ &\le C([\partial u]_{[(k+6-2p)/2],t}+(1-\delta_{2p})\langle u\rangle_{[(k+4-2p)/2],t})\\ &\quad\times\left(\varepsilon+\sup_{0\le s<t}\{(1+s)^{-\xi}\|\partial u(s)\|_{k+8-2p}\}\right)\end{aligned} \tag{3.12}$$

holds for $0\le t<T$ and $p=0,1,2$. Here, δ_{ab} means the Kronecker delta.

Proof. At First, we show (3.12) with $p=2$. By (1.5), (3.6), (3.7), (3.10), (3.11) and the fact

$$1+|y|+s\le C(1+|x|+t)\qquad\text{for }(y,s)\in D^i(x,t), \tag{3.13}$$

we have

$$\begin{aligned}(1+|x|+t)^{-1+2\mu}&([\nabla L_i(F^i)(x,t)]_k+\langle L_i(F^i)(x,t)\rangle_{k+1})\\ &\le C\sum_{j=0}^m\sup_{\substack{(y,s)\in\\ D^i(x,t)\cap\Lambda_j(t)}}\{|y|^{1/2}(1+|y|+s)^{3\mu}(1+||y|-c_js|)\\ &\qquad\qquad\times|\partial u(y,s)|_{[(k+2)/2]}|\partial u(y,s)|_{k+2}\}\\ &\le C[\partial u]_{[(k+2)/2],t}\sup_{0\le s<t}\{(1+s)^{-1/2+3\mu}\|\partial u(s)\|_{k+4}\},\end{aligned}$$

which implies (3.12) with $p=2$, if we choose ξ as $0<\xi<1/2-3\mu$.

Next, we show (3.12) with $p=1$. By the same way to the case $p=2$, we have

$$(3.14)\qquad (1+|x|+t)^{-1/2+\mu}([\nabla L_i(F^i)(x,t)]_k + \langle L_i(F^i)(x,t)\rangle_{k+1})$$
$$\le C\sum_{j=0}^{m} \sup_{\substack{(y,s)\in \\ D^i(x,t)\cap \Lambda_j(t)}} \Bigg\{ |y|^{1/2}(1+|y|+s)^{1/2+2\mu}(1+||y|-c_js|)$$
$$\times \left(\sum_{l=1}^{m} |N^i_{l,2}(y,s)|_{k+1} + |F^i_3(y,s)|_{k+1} + |H^i(y,s)|_{k+1}\right)\Bigg\}.$$

When $0 \le s \le 1$, it follows from (1.8) and (3.10) that

$$(3.15)\qquad |y|^{1/2}(1+|y|+s)^{1/2+2\mu}(1+||y|-c_js|)$$
$$\times \left(\sum_{l=1}^{m} |N^i_{l,2}(y,s)|_{k+1} + |F^i_3(y,s)|_{k+1} + |H^i(y,s)|_{k+1}\right)$$
$$\le C[\partial u]_{[(k+2)/2],t}\|\partial u(s)\|_{k+4}$$
$$\le C[\partial u]_{[(k+2)/2],t}(1+s)^{-\xi}\|\partial u(s)\|_{k+4}$$

holds for any $\xi \in \boldsymbol{R}$. Hence we consider the case $s \ge 1$. Firstly we estimate $N^i_{l,2}$. If $l = j$, by (2.16), (3.3), (3.4) and (3.12) with $p = 2$, we have for $(y,s) \in D^i(x,t) \cap \Lambda_j(t) \cap \{(z,\tau) : \tau \ge 1\}$

$$(3.16)\qquad |y|^{1/2}(1+|y|+s)^{1/2+2\mu}(1+||y|-c_js|)|N^i_{j,2}(y,s)|_{k+1}$$
$$\le C|y|^{1/2}(1+|y|+s)^{-1/2+2\mu}$$
$$\times \{(1+||y|-c_js|)^2|\partial u^j(y,s)|_{[(k+2)/2]}|\partial u^j(y,s)|_{k+2}$$
$$+ (1+||y|-c_js|)(|\partial u^j(y,s)|_{[(k+2)/2]}|u^j(y,s)|_{k+2}$$
$$+ |u^j(y,s)|_{[(k+2)/2]}|\partial u^j(y,s)|_{k+2})\}$$
$$\le C([\partial u]_{[(k+2)/2],t} + \langle u\rangle_{[(k+2)/2],t})$$
$$\times \{\varepsilon + (1+|y|+s)^{-1+2\mu}([\nabla L_j(F^j)(y,s)]_{k+2} + \langle L_j(F^j)(y,s)\rangle_{k+3})\}$$
$$\le C([\partial u]_{[(k+2)/2],t} + \langle u\rangle_{[(k+2)/2],t})$$
$$\times \left(\varepsilon + [\partial u]_{[(k+4)/2],t} \sup_{0\le s<t}\{(1+s)^{-\xi}\|\partial u(s)\|_{k+6}\}\right).$$

On the other hand, if $l \ne j$, by (2.8), (3.3), (3.4) and (3.12) with $p = 2$, we have for $(y,s) \in D^i(x,t) \cap \Lambda_j(t) \cap \{(z,\tau) : \tau \ge 1\}$

$$(3.17)\qquad |y|^{1/2}(1+|y|+s)^{1/2+2\mu}(1+||y|-c_js|)|N^i_{l,2}(y,s)|_{k+1}$$
$$\le C[\partial u]_{[(k+2)/2],t}\{\varepsilon+(1+|y|+s)^{-1+2\mu}([\nabla L_l(F^l)(y,s)]_{k+2}$$
$$+\langle L_l(F^l)(y,s)\rangle_{k+3})\}$$
$$\le C[\partial u]_{[(k+2)/2],t}\left(\varepsilon+[\partial u]_{[(k+4)/2],t}\sup_{0\le s<t}\{(1+s)^{-\xi}\|\partial u(s)\|_{k+6}\}\right).$$

Secondly we estimate F^i_3 and H^i. By (1.5), (3.10) and (3.11), we have for $(y,s)\in D^i(x,t)\cap \Lambda_j(t)\cap\{(z,\tau):\tau\ge 1\}$

$$(3.18)\qquad |y|^{1/2}(1+|y|+s)^{1/2+2\mu}(1+||y|-c_js|)(|F^i_3(y,s)|_{k+1}+|H^i(y,s)|_{k+1})$$
$$\le C|y|^{1/2}(1+|y|+s)^{-1/2+2\mu}[\partial u(y,s)]^2_{[(k+2)/2]}|\partial u(y,s)|_{k+2}$$
$$\le C[\partial u]^2_{[(k+2)/2],t}\sup_{0\le s<t}\{(1+s)^{-\xi}\|\partial u(s)\|_{k+4}\}.$$

Therefore, it follows from (3.11), (3.14), (3.15), (3.16), (3.17) and (3.18) that (3.12) holds for $p=1$.

Finally we show (3.12) with $p=0$. By (3.6) and (3.7), we have

$$(3.19)\qquad [\nabla L_i(F^i)(x,t)]_k+\langle L_i(F^i)(x,t)\rangle_{k+1}$$
$$\le C\sum_{j=0}^m \sup_{\substack{(y,s)\in\\ D^i(x,t)\cap\Lambda_j(t)}}\Bigg\{|y|^{1/2}(1+|y|+s)^{1+\mu}(1+||y|-c_js|)$$
$$\times\left(\sum_{l=1}^m|N^i_l(y,s)|_{k+1}+|R^i(y,s)|_{k+1}+|H^i(y,s)|_{k+1}\right)\Bigg\}.$$

When $0\le s\le 1$, it follows from (1.8) and (3.10) that

$$(3.20)\qquad |y|^{1/2}(1+|y|+s)^{1+\mu}(1+||y|-c_js|)$$
$$\times\left(\sum_{l=1}^m|N^i_l(y,s)|_{k+1}+|R^i(y,s)|_{k+1}+|H^i(y,s)|_{k+1}\right)$$
$$\le C[\partial u]_{[(k+2)/2],t}\|\partial u(s)\|_{k+4}$$
$$\le C[\partial u]_{[(k+2)/2],t}(1+s)^{-\xi}\|\partial u(s)\|_{k+4}$$

holds for any $\xi\in\boldsymbol{R}$. Hence we consider the case $s\ge 1$. Firstly we estimate N^i_l. If $l=j$, by (2.16), (2.17), (3.3), (3.4) and (3.12) with $p=1$, we have for $(y,s)\in D^i(x,t)\cap\Lambda_j(t)\cap\{(z,\tau):\tau\ge 1\}$

$$(3.21)\quad |y|^{1/2}(1+|y|+s)^{1+\mu}(1+||y|-c_js|)|N_j^i(y,s)|_{k+1}$$
$$\le C|y|^{1/2}(1+|y|+s)^{\mu}$$
$$\times\{(1+||y|-c_js|)^2|\partial u^j(y,s)|_{[(k+2)/2]}|\partial u^j(y,s)|_{k+2}$$
$$+(1+||y|-c_js|)(|\partial u^j(y,s)|_{[(k+2)/2]}|u^j(y,s)|_{k+2}$$
$$+|u^j(y,s)|_{[(k+2)/2]}|\partial u^j(y,s)|_{k+2})\}$$
$$\le C([\partial u]_{[(k+2)/2],t}+\langle u\rangle_{[(k+2)/2],t})$$
$$\times(\varepsilon+(1+|y|+s)^{-1/2+\mu}([\nabla L_j(F^j)(y,s)]_{k+2}+\langle L_j(F^j)(y,s)\rangle_{k+3}))$$
$$\le C([\partial u]_{[(k+2)/2],t}+\langle u\rangle_{[(k+2)/2],t})$$
$$\times\left(\varepsilon+([\partial u]_{[(k+6)/2],t}+\langle u\rangle_{[(k+4)/2],t})\sup_{0\le s<t}\{(1+s)^{-\xi}\|\partial u(s)\|_{k+8}\}\right).$$

On the other hand, if $l\neq j$, by (2.8), (3.3), (3.4) and (3.12) with $p=1$, we have for $(y,s)\in D^i(x,t)\cap\Lambda_j(t)\cap\{(z,\tau):\tau\ge 1\}$

$$(3.22)\quad |y|^{1/2}(1+|y|+s)^{1+\mu}(1+||y|-c_js|)|N_l^i(y,s)|_{k+1}$$
$$\le C[\partial u]_{[(k+2)/2],t}$$
$$\times\{\varepsilon+(1+|y|+s)^{-1/2+\mu}([\nabla L_l(F^l)(y,s)]_{k+2}+\langle L_l(F^l)(y,s)\rangle_{k+3})\}$$
$$\le C[\partial u]_{[(k+2)/2],t}$$
$$\times\left(\varepsilon+([\partial u]_{[(k+6)/2],t}+\langle u\rangle_{[(k+4)/2],t})\sup_{0\le s<t}\{(1+s)^{-\xi}\|\partial u(s)\|_{k+8}\}\right).$$

Secondly we consider R^i. By (3.3), (3.4), (3.12) with $p=2$ and the definition of R^i, we have for $(y,s)\in D^i(x,t)\cap\Lambda_j(t)\cap\{(z,\tau):\tau\ge 1\}$

$$(3.23)\quad |y|^{1/2}(1+|y|+s)^{1+\mu}(1+||y|-c_js|)|R^i(y,s)|_{k+1}$$
$$\le C[\partial u]^2_{[(k+2)/2],t}$$
$$\times\left(\varepsilon+(1+|y|+s)^{-1+\mu}\sum_{j=1}^m([\nabla L_i(F^i)(y,s)]_{k+2}+\langle L_i(F^i)\rangle_{k+3})\right)$$
$$\le C[\partial u]^2_{[(k+2)/2],t}\left(\varepsilon+[\partial u]_{[(k+4)/2],t}\sup_{0\le s<t}\{(1+s)^{-\xi}\|\partial u(s)\|_{k+6}\}\right).$$

Thirdly we consider H^i. By (3.10) and (3.11), we have for $(y,s)\in D^i(x,t)\cap\Lambda_j(t)\cap\{(z,\tau):\tau\ge 1\}$

$$(3.24)\qquad \begin{aligned} &|y|^{1/2}(1+|y|+s)^{1+\mu}(1+||y|-c_js|)|H^i(y,s)|_{k+1} \\ &\le C[\partial u]^3_{[(k+2)/2],t}|y|^{1/2}(1+|y|+s)^{-1/2+\mu}|\partial u(y,s)|_{k+2} \\ &\le C[\partial u]^3_{[(k+2)/2],t}\sup_{0\le s<t}\{(1+s)^{-\xi}\|\partial u(s)\|_{k+4}\}. \end{aligned}$$

Therefore, it follows from (3.11), (3.19), (3.20), (3.21), (3.22), (3.23) and (3.24) that (3.12) holds for $p=0$. This completes the proof of Proposition 3.4.

4. Energy estimate

In this section, we describe and prove an energy inequality.

Proposition 4.1. *Let k be a positive integer, δ a positive and small constant, T a positive constant and $u(x,t)={}^t(u^1(x,t),\dots,u^m(x,t))\in(C^\infty(\boldsymbol{R}^2\times[0,T)))^m$ a solution to* (1.1) *and* (1.2). *Assume that*

$$(4.1)\qquad [\partial u]_{[(k+1)/2],T}+\langle u\rangle_{[(k+1)/2]+1,T}\le\delta$$

holds. Then, there exist positive constants C_2 and C_3 such that

$$(4.2)\qquad \|\partial u(t)\|_k\le C_2\varepsilon(1+t)^{C_3([\partial u]_{[(k+1)/2],T}+\langle u\rangle_{[(k+1)/2]+1,T})}$$

holds for $0\le t<T$.

Proof. At first, we define a function $q\in C^1(\boldsymbol{R})$ by

$$(4.3)\qquad q(\lambda)=\int_{-\infty}^{\lambda}\frac{1}{(1+|\rho|)^2}\,d\rho$$

and set

$$p_i(x,t)=q(|x|-c_it)\qquad\text{and}\qquad \dot p_i(x,t)=q'(|x|-c_it)$$

for $i=1,\dots,m$. We immediately find that

$$(4.4)\qquad 0\le p_i(x,t)\le 2\qquad\text{and}\qquad \dot p_i(x,t)=\frac{1}{(1+||x|-c_it|)^2}$$

for any $(x,t)\in\boldsymbol{R}^2\times(0,\infty)$. For a vector function $v(x,t)={}^t(v^1(x,t),\dots,v^m(x,t))$, we introduce energy functions

$$(4.5)\qquad \begin{aligned} E_0(v(t))&=\left(\sum_{i=1}^m\int_{\boldsymbol{R}^2}(|\partial_0 v^i(x,t)|^2+c_i^2|\nabla v^i(x,t)|^2)dx\right)^{1/2}, \\ E_k(v(t))&=\sum_{|a|\le k}E_0(\Gamma^a v(t)), \end{aligned}$$

$$(4.6)\qquad W_0(v(t)) = \left(\sum_{i=1}^{m}\int_{\boldsymbol{R}^2} e^{p_i(x,t)}(|\partial_0 v^i(x,t)|^2 + c_i^2|\nabla v^i(x,t)|^2)dx\right)^{1/2},$$

$$W_k(v(t)) = \sum_{|a|\le k} W_0(\Gamma^a v(t)).$$

It follows from (4.4) that

$$(4.7)\qquad \frac{1}{\hat{C}}\|\partial v(t)\|_k \le \frac{1}{\bar{C}}E_k(v(t)) \le W_k(v(t)) \le \bar{C}E_k(v(t)) \le \hat{C}\|\partial v(t)\|_k$$

for some positive constants $\hat{C}$ and $\bar{C}$. This implies that our task for the proof of (4.2) is to show

$$(4.8)\qquad W_k(u(t)) \le C\varepsilon(1+t)^{C([\partial u]_{[(k+1)/2],T} + \langle u\rangle_{[(k+1)/2]+1,T})}$$

for $0 \le t < T$. We prove (4.8) by the ghost weight energy method. It follows from (1.1), (1.4) and (2.2) that

$$(4.9)\quad G_0^{i,a} \equiv \partial_0^2\Gamma^a u^i - c_i^2\Delta\Gamma^a u^i - \sum_{l=1}^{m}\sum_{\alpha,\beta=0}^{2} A_l^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^l$$

$$= \sum_{l=1}^{m}\sum_{\alpha,\beta=0}^{2}\left\{\sum_{b\le a} C_b\Gamma^b(A_l^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u^l + B^i(\partial u)) - A_l^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^l\right\}$$

for any multi-index a. Here, C_b is a constant which is determined by the commutation relation (2.2). Note that $C_a = 1$. Hence, multiplying $\partial_0\Gamma^a u^i \exp(p_i)$ to (4.9), we obtain

$$(4.10)\quad e^{p_i}\partial_0\Gamma^a u^i G_0^{i,a}$$

$$= \frac{1}{2}\partial_0\{e^{p_i}(|\partial_0\Gamma^a u^i|^2 + c_i^2|\nabla\Gamma^a u^i|^2)\} - c_i^2\sum_{\alpha=1}^{2}\partial_\alpha(e^{p_i}\partial_0\Gamma^a u^i\partial_\alpha\Gamma^a u^i)$$

$$+ e^{p_i}\left(\frac{c_i}{2}\dot{p}_i|Z^i\Gamma^a u^i|^2 - \partial_0\Gamma^a u^i\sum_{l=1}^{m}\sum_{\alpha,\beta=0}^{2}A_l^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^l\right)$$

$$= G_1^{i,a},$$

where we have set

$$(4.11)\quad G_1^{i,a} \equiv e^{p_i}\partial_0\Gamma^a u^i\sum_{l=1}^{m}\sum_{\alpha,\beta=0}^{2}$$

$$\times\left\{\sum_{b\le a} C_b\Gamma^b(A_l^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u^l + B^i(\partial u)) - A_l^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^l\right\}.$$

Furthermore by (1.6), we find that

$$\sum_{i,l=1}^{m}\sum_{\alpha,\beta=0}^{2} e^{p_i}\partial_0\Gamma^a u^i A_l^{i,\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^l$$

$$= \sum_{i,l=1}^{m}\sum_{\alpha,\beta=0}^{2}\Bigg[\frac{1}{2}\partial_0\Bigg\{e^{p_i}\Bigg(A_l^{i,00}(\partial u)\partial_0\Gamma^a u^i\partial_0\Gamma^a u^l - \sum_{\alpha,\beta=1}^{2}A_l^{i,\alpha\beta}(\partial u)\partial_\alpha\Gamma^a u^i\partial_\beta\Gamma^a u^l\Bigg)\Bigg\}$$

$$+\sum_{\alpha=1}^{2}\partial_\alpha\Bigg\{e^{p_i}\Bigg(A_l^{i,\alpha0}(\partial u)\partial_0\Gamma^a u^i\partial_0\Gamma^a u^l + \sum_{\beta=1}^{2}A_l^{i,\alpha\beta}(\partial u)\partial_0\Gamma^a u^i\partial_\beta\Gamma^a u^l\Bigg)\Bigg\}$$

$$+\frac{\dot{p}_i}{2}e^{p_i}\Bigg(c_iA_l^{i,00}(\partial u)\partial_0\Gamma^a u^i\partial_0\Gamma^a u^l - 2\sum_{\alpha=1}^{2}\omega_\alpha A_l^{i,\alpha0}(\partial u)\partial_0\Gamma^a u^i\partial_0\Gamma^a u^l$$

$$-2\sum_{\alpha,\beta=1}^{2}\omega_\alpha A_l^{i,\alpha\beta}(\partial u)\partial_0\Gamma^a u^i\partial_\beta\Gamma^a u^l - c_i\sum_{\alpha,\beta=1}^{2}A_l^{i,\alpha\beta}(\partial u)\partial_\alpha\Gamma^a u^i\partial_\beta\Gamma^a u^l\Bigg)$$

$$-\frac{1}{2}e^{p_i}\Bigg(\partial_0(A_l^{i,00}(\partial u))\partial_0\Gamma^a u^i\partial_0\Gamma^a u^l + 2\sum_{\alpha=1}^{2}\partial_\alpha(A_l^{i,\alpha0}(\partial u))\partial_0\Gamma^a u^i\partial_0\Gamma^a u^l$$

$$+2\sum_{\alpha,\beta=1}^{2}\partial_\alpha(A_l^{i,\alpha\beta}(\partial u))\partial_0\Gamma^a u^i\partial_\beta\Gamma^a u^l - \sum_{\alpha,\beta=1}^{2}\partial_0(A_l^{i,\alpha\beta}(\partial u))\partial_\alpha\Gamma^a u^i\partial_\beta\Gamma^a u^l\Bigg)\Bigg].$$

Here we have set $\omega_\alpha = x_\alpha/|x|(\alpha = 1,2)$. This impies

$$(4.12)\qquad \sum_{i=1}^{m} e^{p_i}\partial_0\Gamma^a u^i G_0^{i,a}$$

$$= \sum_{i=1}^{m}\Bigg[\frac{1}{2}\partial_0\Bigg\{e^{p_i}\Bigg(|\partial_0\Gamma^a u^i|^2 + c_i^2|\nabla\Gamma^a u^i|^2 - \sum_{l=1}^{m}A_l^{i,00}(\partial u)\partial_0\Gamma^a u^i\partial_0\Gamma^a u^l + \sum_{l=1}^{m}\sum_{\alpha,\beta=1}^{2}A_l^{i,\alpha\beta}(\partial u)\partial_\alpha\Gamma^a u^i\partial_\beta\Gamma^a u^l\Bigg)\Bigg\}$$

$$- \sum_{\alpha=1}^{2} \partial_\alpha \Bigg\{ e^{p_i} \Bigg(c_i^2 \partial_0 \Gamma^a u^i \partial_\alpha \Gamma^a u^i + \sum_{l=1}^{m} A_l^{i,\alpha 0}(\partial u) \partial_0 \Gamma^a u^i \partial_0 \Gamma^a u^l$$

$$+ \sum_{l=1}^{m} \sum_{\beta=1}^{2} A_l^{i,\alpha\beta}(\partial u) \partial_0 \Gamma^a u^i \partial_\beta \Gamma^a u^l \Bigg) \Bigg\}$$

$$+ \frac{c_i \dot{p}_i}{2} e^{p_i} |Z^i \Gamma^a u^i|^2 + G_2^{i,a} + G_3^{i,a} \Bigg],$$

where we have set

$$G_2^{i,a} \equiv -\frac{\dot{p}_i}{2} e^{p_i} \sum_{l=1}^{m} \Bigg\{ c_i A_l^{i,00}(\partial u) \partial_0 \Gamma^a u^i \partial_0 \Gamma^a u^l \tag{4.13}$$

$$- 2 \sum_{\alpha=1}^{2} \omega_\alpha A_l^{i,\alpha 0}(\partial u) \partial_0 \Gamma^a u^i \partial_0 \Gamma^a u^l$$

$$- 2 \sum_{\alpha,\beta=1}^{2} \omega_\alpha A_l^{i,\alpha\beta}(\partial u) \partial_0 \Gamma^a u^i \partial_\beta \Gamma^a u^l$$

$$- c_i \sum_{\alpha,\beta=1}^{2} A_l^{i,\alpha\beta}(\partial u) \partial_\alpha \Gamma^a u^i \partial_\beta \Gamma^a u^l \Bigg\},$$

$$G_3^{i,a} \equiv \frac{1}{2} e^{p_i} \sum_{l=1}^{m} \Bigg\{ \partial_0 (A_l^{i,00}(\partial u)) \partial_0 \Gamma^a u^i \partial_0 \Gamma^a u^l \tag{4.14}$$

$$+ 2 \sum_{\alpha=1}^{2} \partial_\alpha (A_l^{i,\alpha 0}(\partial u)) \partial_0 \Gamma^a u^i \partial_0 \Gamma^a u^l$$

$$+ 2 \sum_{\alpha,\beta=1}^{2} \partial_\alpha (A_l^{i,\alpha\beta}(\partial u)) \partial_0 \Gamma^a u^i \partial_\beta \Gamma^a u^l$$

$$- \sum_{\alpha,\beta=1}^{2} \partial_0 (A_l^{i,\alpha\beta}(\partial u)) \partial_\alpha \Gamma^a u^i \partial_\beta \Gamma^a u^l \Bigg\}.$$

By (1.7), (4.6), (4.12) and the divergence theorem, we have

$$W_k(u(t))^2 + \sum_{j=1}^{m} \sum_{|a| \le k} \int_0^t \int_{\boldsymbol{R}^2} \dot{p}_j e^{p_j} |Z^j \Gamma^a u^j(x,s)|^2 dx ds \tag{4.15}$$

$$\le C W_k(u(0))^2 + C \int_0^t \int_{\boldsymbol{R}^2} G_4(x,s) dx ds,$$

for $0 \le t < T$. Here we have set

$$G_4(x,t) = \sum_{i=1}^{m} \sum_{|a| \le k} (|G_1^{i,a}(x,t)| + |G_2^{i,a}(x,t)| + |G_3^{i,a}(x,t)|). \tag{4.16}$$

If we can show

$$\begin{aligned} C\int_{\boldsymbol{R}^2} G_4(x,s)dx \le \frac{1}{2}\sum_{j=1}^{m}\sum_{|a|\le k}\int_{\boldsymbol{R}^2} \dot{p}_j e^{p_j}|Z^j\Gamma^a u^j(x,s)|^2 dx \\ + \frac{C}{1+s}([\partial u]_{[(k+1)/2],T} + \langle u\rangle_{[(k+1)/2]+1,T}) \\ \times \sum_{j=1}^{m}\int_{\boldsymbol{R}^2} e^{p_j}|\partial u^j(x,s)|_k^2 dx \end{aligned} \tag{4.17}$$

for $0 \le t < T$, then we have (4.8). In fact, it follows from (4.15) and (4.17) that

$$\begin{aligned} W_k(u(t))^2 \le CW_k(u(0))^2 \\ + C\int_0^t \frac{1}{1+s}([\partial u]_{[(k+1)/2],T} + \langle u\rangle_{[(k+1)/2]+1,T}) W_k(u(s))^2 ds. \end{aligned} \tag{4.18}$$

Hence (1.2), (4.18) and the Gronwall inequality imply (4.8). Therefore we aim at proving (4.17) in the rest of this section. If $0 \le t \le 1$, (1.5) immediately yields (4.17). Hence we have only to consider the case $1 \le t < T$.

Firstly, we estimate $G_1^{i,a}$. By (1.20), (2.8), (2.11), (2.12), (2.13), (2.15), (4.1), (4.4), (4.11), the Schwarz inequality and the fact that

$$\left|\sum_{b\le a}\Gamma^b\{(F_3^i + H^i)(\partial u, \partial^2 u)\} - (F_3^i + H^i)(\partial u, \partial^2\Gamma^b u)\right| \le C|\partial u|_{[(|a|+1)/2]}|\partial u|_{|a|-1},$$

we have for $1 \le t < T$

$$\begin{aligned} &\int_{\boldsymbol{R}^2} |G_1^{i,a}(x,t)|dx \\ &\le \sum_{\alpha,\beta,\gamma=0}^{2}\int_{(x,t)\in\Lambda_i(T)} |\Gamma^a(a_{ii}^{i,\alpha\beta\gamma}\partial_\gamma u^i\partial_\alpha\partial_\beta u^i) \\ &\quad - a_{ii}^{i,\alpha\beta\gamma}\partial_\gamma u^i\partial_\alpha\partial_\beta\Gamma^a u^i|e^{p_i}|\partial_0\Gamma^a u^i|dx \\ &\quad + \sum_{b<a}\sum_{\alpha,\beta,\gamma=0}^{2}\int_{(x,t)\in\Lambda_i(T)} |C_b\Gamma^b(a_{ii}^{i,\alpha\beta\gamma}\partial_\gamma u^i\partial_\alpha\partial_\beta u^i)|e^{p_i}|\partial_0\Gamma^a u^i|dx \end{aligned} \tag{4.19}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\sum_{b\le a}\sum_{\alpha,\beta=0}^{2}\int_{(x,t)\in\Lambda_j(T)}|C_b\Gamma^b(b_{jj}^{i,\alpha\beta}\partial_\alpha u^j\partial_\beta u^j)|e^{p_i}|\partial_0\Gamma^a u^i|dx\\
&+\sum_{j=1}^{m}\sum_{b\le a}\int_{(x,t)\in\Lambda_j^c(T)}|C_b\Gamma^b\{N_{j,2}^i(\partial u^j,\partial^2 u^j)\}\\
&-N_{j,2}^i(\partial u^j,\partial^2\Gamma^a u^j)|e^{p_i}|\partial_0\Gamma^a u^i|dx\\
&+\sum_{b\le a}\int_{\boldsymbol{R}^2}|C_b\Gamma^b\{F_3^i(\partial u,\partial^2 u)\}-F_3^i(\partial u,\partial^2\Gamma^a u)|e^{p_i}|\partial_0\Gamma^a u^i|dx\\
&+\sum_{b\le a}\int_{\boldsymbol{R}^2}|C_b\Gamma^b\{H^i(\partial u,\partial^2 u)\}-H^i(\partial u,\partial^2\Gamma^a u)|e^{p_i}|\partial_0\Gamma^a u^i|dx\\
\le{}& C\sum_{j=1}^{m}\int_{(x,t)\in\Lambda_j(T)}\biggl(\frac{|\,|x|-c_jt|}{1+|x|+t}|\partial u^j|_{[(k+1)/2]}\\
&\qquad+\frac{1}{1+|x|+t}|u^j|_{[(k+1)/2]+1}\biggr)e^{p_i}|\partial u^i|_k|\partial u^j|_k dx\\
&+C\sum_{j=1}^{m}\sum_{|b|\le k}\int_{(x,t)\in\Lambda_j(T)}|\partial u^j|_{[(k+1)/2]}|Z^j\Gamma^b u^j|e^{p_i}|\partial u^i|_k dx\\
&+C\sum_{j=1}^{m}\int_{\boldsymbol{R}^2}e^{p_i}\biggl(\frac{1}{1+t}[\partial u]_{[(k+1)/2],T}+|\partial u(x,t)|_{[(k+1)/2]}^2\biggr)|\partial u^j|_k^2 dx\\
\le{}&\eta\sum_{j=1}^{m}\sum_{|b|\le k}\int_{\boldsymbol{R}^2}\frac{e^{p_j}}{(1+|\,|x|-c_jt|)^2}|Z^j\Gamma^b u^j|^2 dx\\
&+\frac{C_\eta}{1+t}([\partial u]_{[(k+1)/2],T}+\langle u\rangle_{[(k+1)/2]+1,T})\sum_{j=1}^{m}\int_{\boldsymbol{R}^2}e^{p_j}|\partial u^j|_k^2 dx\\
\le{}&\eta\sum_{j=1}^{m}\sum_{|b|\le k}\int_{\boldsymbol{R}^2}\dot{p}_j e^{p_j}|Z^j\Gamma^b u^j|^2 dx\\
&+\frac{C_\eta}{1+t}([\partial u]_{[(k+1)/2],T}+\langle u\rangle_{[(k+1)/2]+1,T})\sum_{j=1}^{m}\int_{\boldsymbol{R}^2}e^{p_j}|\partial u^j|_k^2 dx
\end{aligned}
$$

for any constant $\eta > 0$ and multi-index a with $|a| \le k$. Next we estimate $G_2^{i,a}$. By (1.20), (2.8), (2.11), (4.1), (4.13), the Schwarz inequality and the Strong Null-condition, we have for $1 \le t < T$

$$
\begin{aligned}
(4.20)\qquad & \int_{\boldsymbol{R}^2} |G_2^{i,a}|dx \\
& \le \int_{(x,t)\in\Lambda_i(T)} \left| \frac{\dot p_i}{2} e^{p_i} \sum_{\gamma=0}^{2} \left\{ c_i a_{ii}^{i,00\gamma} \partial_\gamma u^i (\partial_0 \Gamma^a u^i)^2 \right.\right. \\
& \quad - 2\sum_{\alpha=1}^{2} \omega_\alpha a_{ii}^{i,\alpha 0\gamma} \partial_\gamma u^i (\partial_0 \Gamma^a u^i)^2 - \sum_{\alpha,\beta=1}^{2} (2\omega_\alpha a_{ii}^{i,\alpha\beta\gamma} \partial_\gamma u^i \partial_0 \Gamma^a u^i \partial_\beta \Gamma^a u^i \\
& \quad \left.\left. + c_i a_{ii}^{i,\alpha\beta\gamma} \partial_\gamma u^i \partial_\alpha \Gamma^a u^i \partial_\beta \Gamma^a u^i) \right\} \right| dx \\
& \quad + C\left(\int_{(x,t)\in\Lambda_i^c(T)} \dot p_i e^{p_i} |\partial u^i(x,t)|_0 |\partial u^i(x,t)|_k^2 dx \right. \\
& \quad \left. + \int_{\boldsymbol{R}^2} \dot p_i e^{p_i} |\partial u(x,t)|_0^2 |\partial u^i(x,t)|_k^2 dx \right) \\
& \le \int_{(x,t)\in\Lambda_i(T)} \left| \frac{\dot p_i}{2} e^{p_i} \left\{ -c_i \sum_{\alpha,\beta,\gamma=0}^{2} a_{ii}^{i,\alpha\beta\gamma} \frac{\omega_\alpha\omega_\beta\omega_\gamma}{c_i^3} \partial_0 u^i (\partial_0 \Gamma^a u^i)^2 \right.\right. \\
& \quad + \sum_{\gamma=1}^{2} a_{ii}^{i,00\gamma} (c_i\partial_\gamma + \omega_\gamma\partial_0) u^i (\partial_0 \Gamma^a u^i)^2 \\
& \quad - 2\sum_{\alpha,\gamma=1}^{2} \frac{\omega_\alpha}{c_i} a_{ii}^{i,\alpha 0\gamma} (c_i\partial_\gamma + \omega_\gamma\partial_0) u^i (\partial_0 \Gamma^a u^i)^2 \\
& \quad - \sum_{\alpha,\beta=0}^{2} \frac{1}{c_i} a_{ii}^{i,\alpha\beta 0} \partial_0 u^i (c_i\partial_\alpha + \omega_\alpha\partial_0)\Gamma^a u^i (c_i\partial_\beta + \omega_\beta\partial_0)\Gamma^a u^i \\
& \quad - \sum_{\alpha,\beta,\gamma=1}^{2} \left(2\frac{\omega_\alpha}{c_i} a_{ii}^{i,\alpha\beta\gamma} (c_i\partial_\gamma + \omega_\gamma\partial_0) u^i \partial_0 \Gamma^a u^i \partial_\beta \Gamma^a u^i \right. \\
& \quad - 2\frac{\omega_\alpha\omega_\gamma}{c_i^2} a_{ii}^{i,\alpha\beta\gamma} \partial_0 u^i \partial_0 \Gamma^a u^i (c_i\partial_\beta + \omega_\beta\partial_0)\Gamma^a u^i \\
& \quad + \frac{\omega_\alpha\omega_\beta}{c_i^2} a_{ii}^{i,\alpha\beta\gamma} (c_i\partial_\gamma + \omega_\gamma\partial_0) u^i (\partial_0 \Gamma^a u^i)^2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\omega_\beta}{c_i} a_{ii}^{i,\alpha\beta\gamma} \partial_\gamma u^i (c_i \partial_\alpha + \omega_\alpha \partial_0) \Gamma^a u^i \partial_0 \Gamma^a u^i \\
&+ a_{ii}^{i,\alpha\beta\gamma} \partial_\gamma u^i \partial_\alpha \Gamma^a u^i (c_i \partial_\beta + \omega_\beta \partial_0) \Gamma^a u^i \Big) \Big\} \Big| dx \\
&+ \frac{C}{1+t} [\partial u]_{0,T} \int_{\boldsymbol{R}^2} \dot{p}_i e^{p_i} |\partial u^i|_k^2 dx \\
\le{}& C \int_{(x,t) \in \Lambda_i(T)} \dot{p}_i e^{p_i} (|Z^i u^i| \, |\partial \Gamma^a u^i|^2 + |\partial u^i| \, |\partial \Gamma^a u^i| \, |Z^i \Gamma^a u^i|) dx \\
&+ \frac{C}{1+t} [\partial u]_{0,T} \int_{\boldsymbol{R}^2} e^{p_i} |\partial u^i|_k^2 dx \\
\le{}& \eta \sum_{|a| \le k} \int_{\boldsymbol{R}^2} \dot{p}_i e^{p_i} |Z^i \Gamma^a u^i|^2 dx \\
&+ \frac{C_\eta}{1+t} ([\partial u]_{0,T} + \langle u \rangle_{1,T}) \int_{\boldsymbol{R}^2} e^{p_i} |\partial u^i|_k^2 dx,
\end{aligned}
$$

where $\omega_0 = -c_i$. By the similar argument, we obtain

$$
(4.21) \quad \int_{\boldsymbol{R}^2} |G_3^{i,a}| dx \le \eta \sum_{|a| \le k} \int_{\boldsymbol{R}^2} \dot{p}_i e^{p_i} |Z^i \Gamma^a u^i|^2 dx + \frac{C_\eta}{1+t} [\partial u]_{1,T} \int_{\boldsymbol{R}^2} e^{p_i} |\partial u^i|_k^2 dx.
$$

Combining (4.19), (4.20) and (4.21), we have (4.17), if we take $\eta > 0$ sufficiently small. This completes the proof of Proposition 4.1.

5. Proof of Lemma 2.1

Finally we show Lemma 2.1. Take an integer k so that $k \ge 8$ and hence $[(k+9)/2] \le k$. Let δ be a positive and small constant. Assume (2.3). Then, if we choose $\varepsilon_1 > 0$ as $\varepsilon_1 \le \delta / J$, we have

$$
\begin{aligned}
[\partial u]_{[(k+6)/2],T} + \langle u \rangle_{[(k+4)/2],T} &\le [\partial u]_{[(k+9)/2],T} + \langle u \rangle_{[(k+9)/2]+1,T} \\
&\le [\partial u]_{k,T} + \langle u \rangle_{k+1,T} \le J\varepsilon \le \delta
\end{aligned}
$$

for $0 < \varepsilon < \varepsilon_1$. Then we have (3.1) and (4.1) for which k is replaced by $k+8$. Hence it follows from Propositions 3.1 and 4.1 that

$$
\begin{aligned}
&[\partial u(t)]_k + \langle u(t) \rangle_{k+1} \\
&\quad \le C_0 \varepsilon + C_1 ([\partial u]_{[(k+6)/2],t} + \langle u \rangle_{[(k+4)/2],t}) \sup_{0 \le s < t} \{(1+s)^{-\xi} \|\partial u(s)\|_{k+8}\}
\end{aligned}
$$

$$\leq C_0\varepsilon + C_1C_2J\varepsilon^2 \sup_{0\leq s<t}\{(1+s)^{C_3J\varepsilon-\xi}\}$$
$$\leq 2C_0\varepsilon$$

for $\varepsilon \in (0, \varepsilon_2)$, if we choose ε_2 as

$$0 < \varepsilon_2 \leq \min\left\{\frac{C_0}{C_1C_2J}, \frac{\xi}{C_3J}\right\}. \tag{5.1}$$

Here, ξ is the constant determined in Proposition 3.1. Therefore, we find that Lemma 2.1 is true, if we take

$$\varepsilon_0(J) = \min\{\varepsilon_1, \varepsilon_2\} \qquad \text{and} \qquad K = 2C_0.$$

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