Weighted L^∞ and L^1 Estimates for Solutions to the Classical Wave Equation in Three Space Dimensions

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1. Introduction

The aim of this paper is to present some new estimates, which we consider of independent interest, necessary to extend our previous work [1] on "semiglobal existence" to nonlinear wave equations in three space dimensions from the spherical symmetric case, considered there, to the general case. The extension will appear shortly in a joint paper with F. John [4].

We start with a study of the reduced initial value problem

$$\begin{aligned} &\Box u = 0, \\ &u(x,0)=0, \qquad u_t(x,0)=g(x), \end{aligned} \tag{1.1}$$

where $\Box$ denotes the D'Alembertian $\partial_t^2 - D_1^2 - D_2^2 - D_3^2$ of the four-dimensional Minkovski space-time and ∂_t, D_1, D_2, D_3 the partial derivatives with respect to the variables t and $x=(x_1, x_2, x_3)$. The solution $u=u(x,t)$ of (1.1) can be expressed in the simple form

$$u(x,t) = \frac{1}{4\pi t} \iint_{|x-y|=t} g(y)\, dS_y, \tag{1.2}$$

where dS_y is the area element of the sphere $|y-x|=t$. Throughout this paper we shall assume g to be smooth and compactly supported in $\mathbb{R}^3$; however both conditions can be appropriately relaxed.

The following well-known estimates are immediate consequences of the closed formula (1.2):

$$\begin{aligned} &\text{(i)} \quad |u(x,t)| \le c\frac{1}{t}\int |Dg(y)|\, dy, && x\in\mathbb{R}^3,\ t>0, \\ &\text{(ii)} \quad \|u(t)\|_{L^1} \le Ct\|g\|_{L^1}, && t \ge 0, \end{aligned} \tag{1.3}$$

where $\| \ \|_{L^1}$ denotes the usual L^1 norm in $\mathbb{R}^3$ and $|Dg| = \sum_{i=1}^3 |D_i g|$. The inequality (i) can be somewhat refined by

$$\text{(i}'\text{)} \quad |u(x,t)| \le C\frac{1}{t}\int_{|y-x|\ge t} |Dg(y)|\, dy,$$

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whence, in particular,

$$\text{(i}''\text{)}\quad |u(x,t)| \leqq C\frac{1}{t(t-|x|)^k}\||y|^k Dg\|_{L^1},$$

with

$$\||y|^k Dg\|_{L^1} = \int |y|^k |Dg(y)|\, dy,$$

for every positive integer k, $0 \leqq |x| < t$. Though (i″) seems sharper than (i), in practice it does not help much; when applied to nonlinear problems the gain in powers of $1/(t-|x|)$ is more than compensated by the loss in powers of $|y|$. On the other hand, if g is spherical symmetric, i.e., $g(x) = g(r)$ with $r = |x|$, we can express u in the form (see [1])

$$u(r,t) = \frac{1}{2r}\int_{|r-t|}^{r+t} \lambda g(\lambda)\, d\lambda$$

from which we can easily derive (see [1])

$$\text{(1.4)}\qquad \text{(i)}\quad |u(r,t)| \leqq \frac{1}{2r|r-t|}\|g\|_{L^1} \quad \text{for all} \quad r \neq 0, t.$$

Also,

$$\text{(ii)}\quad \int_0^\infty r|u(r,t)|\, dr \leqq \int_0^\infty \lambda^2 |g(\lambda)|\, d\lambda = \|g\|_{L^1(\mathbf{R}^3)},$$

$$\text{(iii)}\quad \int_0^\infty r|u_t(r,t)|\, dr \leqq \int_0^\infty \lambda |g(\lambda)|\, d\lambda,$$

$$\text{(iii}_2\text{)}\quad \int_0^\infty r|u_r(r,t)|\, dr \leqq \int_0^\infty \lambda |g(\lambda)|\, d\lambda + \tfrac{1}{2}\int_0^\infty \lambda \log\frac{\lambda+t}{|\lambda-t|}|g(\lambda)|\, d\lambda.$$

One aim of this paper is to generalize the estimates (1.4) to the nonspherical symmetric case. The lack of spherical symmetry is best measured by the angular momentum operators

$$\text{(1.5)}\qquad \Omega_1 = x_2D_3 - x_3D_2, \quad \Omega_2 = x_3D_1 - x_1D_3, \quad \Omega_3 = x_1D_2 - x_2D_1,$$

which have the remarkable property of commuting with $\square$,

$$[\square, \Omega_i] = 0 \quad \text{for} \quad i = 1, 2, 3.$$

These operators are intimately connected to the radiation operators

$$\text{(1.6)}\qquad L_i = D_i - \sum_{j=1}^{3} \frac{x_i x_j}{|x|^2} D_j, \qquad i = 1, 2, 3,$$

which have played a major role in the recent fundamental work of F. John [2].

Indeed, introducing $R_i = |x|L_i$ and $X_i = x_i/|x|$ for $i = 1, 2, 3$, we have

$$R = -X \times \Omega. \tag{1.7}$$

In particular, (1.7) shows that for any given solution u of (1.1) the vector Ru must have the same asymptotic properties as those of u. Together with (1.3) (i″) this remark gives a very simple interpretation for the improved uniform decay properties of $L_1 u$, $L_2 u$, $L_3 u$ which were derived and used in [2]. Our main results are included in Theorems 1, 2 and 3; Theorem 3 is the most important for applications to nonlinear problems.

THEOREM 1. *Consider $u = u(x, t)$ to be a solution of* (1.1). *Then,*

$$\text{(i)} \quad |u(x,t)| \leqq C \frac{1}{|x|\,||x|-t|}(\|g\|_{L^1} + \|\Omega g\|_{L^1} + \|\Omega^2 g\|_{L^1})$$

for all $x \neq 0$, $|x| \neq t$,

$$\text{(ii)} \quad \int \frac{1}{|x|} |u(x,t)|\, dx \leqq \|g\|_{L^1},$$

$$\text{(iii)} \quad \int \frac{1}{|x|} |\nabla u(x,t)|\, dx \leqq C \int \frac{1}{|x|}\left(1 + \log \frac{|x|+t}{||x|-t|}\right)(|\Omega g(x)| + |g(x)|)\, dx,$$

where $\nabla u = (u_t, u_{x_1}, u_{x_2}, u_{x_3})$ *and* $t \geqq 0$.

Here, and elsewhere in this paper, $\Omega^k g = (\Omega_{i_1} \cdots \Omega_{i_k} g)_{i_1, \ldots, i_k = 1,2,3}$ for every $k \geqq 0$.

Remark 1. The inequality (i) can also be expressed in the form

$$\text{(i}'\text{)} \quad \int_{|X|=1} |u(rX, t)|\, dS_X \leqq \frac{1}{r|r-t|} \|g\|_{L^1}$$

for all $r \geqq 0$, $r \neq t$, or, sharper,

$$\text{(ii}'\text{)} \quad \int_{|X|=1} |u(rX, t)|\, dS_X \leqq \tfrac{1}{2} \int_{A \leqq |y| \leqq B} \frac{1}{|y|} |g(y)|\, dy,$$

where $A = |r - t|$, $B = r + t$.

In fact, (i) follows immediately from (i′) and the classical Sobolev inequality on the sphere $|X| = 1$ (see Lemma 1).

Remark 2. To remove the singularities in (i) we observe that, according to (1.3)(i),

$$|u(x, t)| \leqq C\frac{1}{1+t}\int (|g(y)|+|Dg(y)|+|D^2g(y)|)\, dy$$

which, together with (i), yields

$$\text{(i}''\text{)}\quad |u(x, t)| \leqq \frac{1}{(1+|x|)(1+||x|-t|)}\sum_{i=0}^{2}(\|D^i g\|_{L^1}+\|\Omega^i g\|_{L^1})$$

for any $x \in \mathbb{R}^3$, $t > 0$.

Remark 3. As in (1.3)(i'') we can sharpen (i'') so that it reflects the fact that the solutions to (1.1) decay faster in the interior of their domain of propagation:

$$\text{(i}'''\text{)}\quad |u(x, t)| \leqq C\frac{1}{(1+|x|)(1+||x|-t|)^{1+p}} \times \sum_{i=0}^{2}(\|(|y|+1)^p D^i g\|_{L^1}+\|(|y|+1)^p\Omega^i g\|_{L^1})$$

for any $p \geqq 0$, $x \in \mathbb{R}^3$, $t \geqq 0$. As a consequence of (i''') we derive

$$\frac{1}{(1+|x|)^p}|u(x, t)| \leqq C\frac{1}{(1+t)^{1+p}}$$

uniformly for $x \in \mathbb{R}^3$, $t \geqq 0$.

The estimates (ii), (iii) show that the derivatives of u behave better, for large t, than u itself. Though, somewhat less transparent, this also holds true in the sup norm, a fact which is crucial in the proof of Theorem 3(i).

THEOREM 2. *Let $u = u(x, t)$ be a solution of* (1.1); *then for all $r, t \geqq 0$*

$$\int_{|X|=1} |\nabla u(rX, t)|\, dS_X \leqq C\frac{1}{r}\left(1+\frac{A}{r}\right)\int_{A \leqq |y| \leqq B}\frac{1}{|y|^2}(|g|+|\Omega g|+|\Omega^2 g|)\, dy + C\frac{1}{r}\left[A^{-1}\int_{|y|=A}|g(y)|\, dSy + B^{-1}\int_{|y|=B}|g(y)|\, dSy\right],$$

where $A = |r-t|$, $B = r+t$.

The proofs of both theorems are based on the following "polar expression" of formula (1.2) used by F. John in his appendix to [2]:

$$u(x, t) = \frac{1}{2r}\int_{|r-t|}^{r+t}\lambda j_g(x, \lambda, Q)\, d\lambda, \tag{1.8}$$

where $r=|x|$, $Q=(\lambda^2+r^2-t^2)/2\lambda r$ and $j_g(x,\lambda,q)$ is the average of g on the circle of intersection between the cone $y\cdot x=q|y|\,|x|$ with the sphere $|y|=\lambda$, i.e.,

$$j_g(x,\lambda,q)=j_g\left(\frac{x}{|x|},\lambda,q\right)=\frac{1}{2\pi}\int_{y\cdot x=q|y||x|,\,|y|=\lambda} g(y)\,d\phi \tag{1.9}$$

for $x\neq 0$, $|q|\leqq 1$, ϕ being the angular measure on the circle. The formulas (1.8), (1.9) follow easily from (1.2) by introducing spherical coordinates θ,ϕ on the sphere $|y-x|=t$ with the polar axis pointing in the direction from x to 0 and introducing the new variable of integration

$$\lambda^2=r^2+t^2-2rt\cos\theta. \tag{1.10}$$

In the last section of this paper we shall apply Theorems 1 and 2 to prove a theorem concerning the inhomogeneous problem

$$\Box u=g,\qquad u=u_t=0\quad\text{at}\quad t=0, \tag{1.11}$$

where g is assumed to be a smooth function of the arguments x,t compactly supported in $x\in\mathbb{R}^3$ for each fixed t. We define the following weighted norms for g:

$$\begin{aligned} M(g)&=\sup_{s\geqq 0}\int_{\mathbb{R}^3}(1+|y|)(1+||y|-s|)|g(y,s)|\frac{1}{1+|y|}\,dy,\\ N(g)&=\sup_{s\geqq 0}\left(\int_{\mathbb{R}^3}(1+|y|)^2(1+||y|-s|)^2|g(y,s)|^2\,dy\right)^{1/2}, \end{aligned}\tag{1.12}$$

and also

$$\begin{aligned} M_k(g)&=\sum_{|\alpha|+|\beta|\leqq k} M(D^\alpha\Omega^\beta g),\\ N_k(g)&=\sum_{|\alpha|+|\beta|\leqq k} N(D^\alpha\Omega^\beta g), \end{aligned}\tag{1.13}$$

where, for any given multi-indices α,β, $D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}$ and $\Omega^\beta=\Omega_1^{\beta_1}\Omega_2^{\beta_2}\Omega_3^{\beta_3}$.

Given this notation we have (compare it with [2], Appendix):

THEOREM 3. *The solution $u(x,t)$ of (1.11) verifies the estimates*

(i) $|\nabla u(x,t)|\leqq C\dfrac{\log(1+t)}{(1+|x|)(1+||x|-t|)}M_6(g)$

for all $x\in\mathbb{R}^3$, $t\geqq 0$,

(ii) $\displaystyle\int\frac{1}{(|x|+1)}|\nabla u(x,t)|\,dx\leqq C\log(1+t)\cdot N_1(g)$

for all $t\geqq 0$ and C a positive constant.

Remark. The estimates (i), (ii) of Theorem 3 are, in general, invalid if one replaces ∇u by u itself. However, if in (1.11), g has the form $g = D_i h$ for some $i = 0, 1, 2, 3$ with $D_0 = \partial_t$ and h a smooth function compactly supported in x, we have

(i') $$|u(x,t)| \leqq C \frac{\log(1+t)}{(1+|x|)(1+||x|-t|)} M_6(h),$$

(ii'') $$\int \frac{1}{1+|x|} |u(x,t)|\, dx \leqq C \log(1+t) N_1(h).$$

Before ending the introduction we make a few more remarks about the radiation operators L_1, L_2, L_3 and $L_0 = \partial_t + \sum_{i=1}^3 (x_i/|x|) D_i$ which were considered in [2]. We introduce the "Lorentz operators"

$$\Lambda_i = x_i \partial_t + t D_i, \qquad i = 1, 2, 3, \tag{1.14}$$

and the dilation operators (see [3])

$$\Lambda_0 = t\partial_t + \sum_{i=1}^3 x_i D_i. \tag{1.15}$$

Like the angular momentum operators Ω_i, the Λ_i operators commute with $\Box$ while $[\Lambda_0, \Box] = -\Box$. On the other hand, we can write L_0, L_1, L_2, L_3 as linear combinations of Λ_0, Λ_1, Λ_2, Λ_3,

$$\begin{aligned} L_0 &= \frac{1}{t+|x|}\left(\sum_{i=1}^3 \frac{x_i}{|x|} \Lambda_i + \Lambda_0 \right), \\ L_i &= \frac{1}{t}\left(\Lambda_i - \sum_{i=1}^3 \frac{x_i x_j}{|x|^2} \Lambda_j \right) \end{aligned} \tag{1.16}$$

for $i = 1, 2, 3$. The formulas (1.13) and (1.7) together with the commutation properties of $\Omega_i, \Lambda_i, \Lambda_0$ give a very simple, quantitative explanation of the improved decay properties of $L_i u$, $i = 0, 1, 2, 3$, where u is a solution of (1.1), in both L^2 and L^∞ norms.

2. Proof of Theorem 1

The proof of (i) follows quite easily from (1.8). Indeed,

$$|u(x,t)| \leqq \frac{1}{2r|r-t|} \int_0^\infty \lambda^2 \sup_{|Y|=1} |g(\lambda Y)|\, d\lambda. \tag{2.1}$$

On the other hand,

$$\sup_{|Y|=1} |g(\lambda Y)| \leqq C\left(\int_{|Y|=1} |g(\lambda Y)|\, dS_Y + \int_{|Y|=1} |\Omega g(\lambda Y)|\, dS_Y + \int_{|Y|=1} |\Omega^2 g(\lambda Y)|\, dS_Y \right), \tag{2.2}$$

which is an immediate consequence of the following form of the Sobolev inequality on spheres.

LEMMA 1. *Consider f to be a smooth function defined on $|Y|=1$. We have*

$$\sup_{|Y|=1} |f(Y)| \leqq C(\|f\|_{L^1(S)} + \|\Omega f\|_{L^1(S)} + \|\Omega^2 f\|_{L^1(S)}), \tag{2.3}$$

where $\|\ \|_{L^1(S)}$ is the L^1 norm on the sphere $S = \{Y \in \mathbb{R}^3 \,|\, |Y| = 1\}$.

Proof of Lemma 1: It suffices to prove (2.3) for $Y \in S$ in a neighborhood of the great circle $Y_1 = 0$. Introducing polar coordinates $Y_1 = \cos\alpha$, $Y_2 = \sin\alpha\cos\beta$, $Y_3 = \sin\alpha\sin\beta$, we have $\partial_\beta = \Omega_1$ and $\partial_\alpha = -\sin\beta\,\Omega_2 + \cos\beta\,\Omega_3$ and the proof follows that of the classical Sobolev inequality.

In the proof of (ii) and (iii) we shall need the following

LEMMA 2. *Let g be a smooth function with compact support and let $j_g(x, \lambda, q)$ be defined by (1.9). We have*

$$\int_{|X|=1} j_g(X, \lambda, q)\, dS_X = \int_{|X|=1} g(\lambda X)\, dS_X \tag{2.4}$$

for every $\lambda > 0$, $|q| \leqq 1$.

Proof: The lemma follows from the invariance, with respect to rotations of the measure on S, induced by the linear continuous functional $g \to \int_{|X|=1} j_g(X, 1, q)\, dS_X$.

The proof of (ii) of the theorem is now easily deduced. By (1.8) and Lemma 2,

$$\begin{aligned} \int \frac{1}{|x|} |u(x,t)|\, dx &= \int_0^\infty r\, dr \int_{|x|=1} |u(rX, t)|\, dS_X \\ &\leqq \tfrac{1}{2} \int_0^\infty dr \int_{|X|=1} dS_X \int_{|r-t|}^{r+t} \lambda |j_g(X, \lambda, Q)|\, d\lambda \\ &\leqq \tfrac{1}{2} \int_0^\infty dr \int_{|r-t|}^{r+t} \lambda\, d\lambda \int_{|X|=1} |g(\lambda X)|\, dS_X \end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\int_0^\infty \lambda\, d\lambda \int_{|X|=1} |g(\lambda X)|\, dS_X \int_{|\lambda-t|}^{\lambda+t} dr\\
&\leq \int_0^\infty \lambda^2\, d\lambda \int_{|X|=1} |g(\lambda x)|\, dS_X\\
&= |g|_{L^1(\mathbf{R}^3)},
\end{aligned}
$$

which proves (ii).

It remains to prove (iii). According to formula (1.8) we have, for $i=1,2,3$,

$$
D_i u(x,t) = \frac{1}{2r}\int_{|r-t|}^{r+t} \lambda\, j_{D_i g}(x,\lambda,Q)\, d\lambda. \tag{2.5}
$$

$Q=(\lambda^2+r^2-t^2)/2\lambda r$, $r=|x|\neq 0$. We now split the derivatives $D_i = D_{y_i} = \partial/\partial y_i$, $i=1,2,3$, into their radial component $D_{|y|} = D_\lambda = \sum_{j=1}^3 y_j/|y| \cdot D_{y_j}$, and the angular components

$$
L_i = D_{y_i} - \sum_{j=1}^{3} \frac{y_i y_j}{|y|^2} D_{y_j} = \frac{1}{\lambda} R_i,
$$

$$
D_i g(y) = L_i g(y) + Y_i D_\lambda g, \tag{2.6}
$$

where

$$
Y_i = \frac{y_i}{|y|}, \qquad |y| = \lambda.
$$

Accordingly, we obtain the following important decomposition of $j_{D_i g}$ (see also [2], Appendix):

$$
j_{D_i g}(x,\lambda,Q) = j_{L_i g} + D_\lambda j_{Y_i g} - Q_\lambda \frac{d}{dq} j_{Y_i g} \tag{2.7}
$$

with $Q_\lambda = D_\lambda Q = (t^2+\lambda^2-r^2)/2\lambda^2 r$, and, as a consequence,

$$
D_i u = u_1 + u_2 + u_3, \tag{2.8}
$$

where, with $X_i = x_i/r$,

$$
\begin{aligned}
u_1(x,t) &= \frac{1}{2r}\int_{|r-t|}^{r+t} j_{R_i g}(x,\lambda,Q)\, d\lambda,\\
u_2(x,t) &= \frac{1}{2r}\int_{|r-t|}^{r+t} j_{Y_i g}(x,\lambda,Q)\, d\lambda\\
&\quad + \frac{1}{2r}[(r+t)X_i(g(r+t)X) - (\pm|r-t|X_i)g(\pm|r-t|X)],^1\\
u_3(x,t) &= -\frac{1}{2r}\int_{|r-t|}^{r+t} \lambda Q_\lambda \frac{d}{dq} j_{Y_i g}(x,\lambda,Q)\, d\lambda.
\end{aligned}
$$

[1] Plus or minus sign depending on whether $r \geqq t$ or $r < t$.

Similarly, we have

$$u_t(x,t) = u_1' + u_2', \tag{2.9}$$

where

$$u_1' = \frac{1}{2r}[(r+t)X_i g((r+t)X) - (\mp|r-t|X_i)g(\pm|r-t|X)],$$

$$u_2' = \frac{1}{2r}\int_{|r-t|}^{r+t} \lambda Q_t \frac{d}{dq} j_g(x,\lambda,Q)\,d\lambda,$$

for all $|x| = r \neq 0$ and $Q_t = -t/\lambda r$.

As in the proof of part (ii) of the theorem, we find

$$\int \frac{1}{|x|}|u_1(x,t)|\,dx \leqq \int \frac{1}{|x|}|R_i g|\,dx \tag{2.10}$$

and,

$$\begin{aligned}\int \frac{1}{|x|}|u_2(x,t)|\,dx &\leqq \int \frac{1}{|x|}|g(x)|\,dx + \tfrac{1}{2}\int_0^\infty (r+t)\,dr \int_{|X|=1} |g((r+t)X)|\,dS_X \\ &\quad + \tfrac{1}{2}\int_0^\infty (r-t)\,dr \int_{|X|=1} |g((r-t)X)|\,dS_X \\ &\leqq 2\int \frac{1}{|x|}|g(x)|\,dx,\end{aligned} \tag{2.11}$$

$$\int \frac{1}{|x|}|u_1'(x,t)|\,dx \leqq \int \frac{1}{|x|}|g(x)|\,dx. \tag{2.12}$$

It only remains to estimate u_3 and u_2' in (2.8), respectively (2.9). To do this we need the following (see [2], Appendix):

LEMMA 3. *Consider g as above; then*

$$\left|\frac{d}{dq} j_g(x,\lambda,q)\right| \leqq \frac{\lambda}{(1-q^2)^{1/2}}|j_{Lg}(x,\lambda,q)| \tag{2.13}$$

for all $x \neq 0$, $q \neq \pm 1$ *and* $j_{Lg} = (j_{L_1 g}, j_{L_2 g}, j_{L_3 g})$.

Proof: We start by verifying the formula

$$\frac{d}{dq} j_g(x,\lambda,q) = \frac{\lambda}{1-q^2}\sum_{i=1}^{3} X_i j_{L_i g}(x,\lambda,q). \tag{2.14}$$

Indeed, performing a rotation of x, it is enough to verify (2.14) for $x=E_1=(1,0,0)$. Thus,

$$j_g(E_1,\lambda,q)=\frac{1}{2\pi}\int_0^{2\pi} g(\lambda q,\lambda(1-q^2)^{1/2}\cos\phi,\lambda(1-q^2)^{1/2}\sin\phi)\,d\phi.$$

Hence,

$$\begin{aligned}\frac{d}{dq}j_g(E_1,\lambda,q)&=\frac{1}{2\pi}\int_0^{2\pi}\lambda\Big(D_1g-\frac{q}{(1-q^2)^{1/2}}\cos\phi D_2g-\frac{q}{(1-q^2)^{1/2}}\sin\phi D_3g\Big)\,d\phi\\&=\frac{1}{2\pi}\int_0^{2\pi}\lambda\left(D_1g-\frac{q}{1-q^2}Y_2D_2g-\frac{q}{1-q^2}Y_3D_3g\right)d\phi\\&=\frac{\lambda}{1-q^2}j_{L_1g}(E_1,\lambda,q).\end{aligned}$$

On the other hand, since $\sum_{i=1}^3 Y_iL_i\equiv 0$, we can rewrite (2.14) as

$$\frac{d}{dq}j_g(x,\lambda,q)=\frac{\lambda}{1-q^2}\sum_{i=1}^3 (X_i-qY_i)j_{L_ig}(x,\lambda,q)$$

and, since $X\cdot Y=q$, $|x|=|Y|=1$ we have $|X_i-qY_i|\leqq|X-qY|=(1-q^2)^{1/2}$ for all $i=1,2,3$, which proves the lemma.

We now proceed to estimate u_3 and u_2'. From the definition of $u_3(x,t)$ in (2.8) we have, applying first Lemma 3 and then Lemma 2,

$$\begin{aligned}\int\frac{1}{|x|}|u_3(x,t)|\,dx&\leqq\tfrac12\int_0^\infty dr\int_{|X|=1}dS_X\int_{|r-t|}^{r+t}|\lambda Q_\lambda|\left|\frac{d}{dq}j_{Y_ig}(X,\lambda,O)\right|d\lambda\\&\leqq\tfrac12\int_0^\infty dr\int_{|r-t|}^{r+t}\frac{|\lambda Q_\lambda|}{(1-Q^2)^{1/2}}\,d\lambda\\&\quad\times\left(\int_{|X|=1}|Rg(\lambda X)|\,dS_X+\int_{|X|=1}|g(\lambda X)|\,dS_X\right).\end{aligned}\tag{2.15}$$

Since,

$$\frac{|\lambda Q_\lambda|}{(1-Q^2)^{1/2}}\leqq 2\lambda t\frac{1}{((\lambda+r-t)(r+t-\lambda)(\lambda+t-r)(\lambda+r+t))^{1/2}},$$

we have

$$\begin{aligned}&\int\frac{1}{|x|}|u_3(x,t)|\,dx\\&\quad\leqq\int_0^\infty\lambda tI(\lambda,t)\,d\lambda\left(\int_{|X|=1}|Rg(\lambda X)|\,dS_X+\int_{|X|=1}|g(\lambda X)|\,dS_X\right),\end{aligned}\tag{2.16}$$

where

$$I(\lambda, t) = \int_{|\lambda+t|}^{\lambda+t} \frac{dr}{((\lambda+r-t)(r+t-\lambda)(\lambda+t-r)(\lambda+r+t))^{1/2}} \tag{2.17}$$
$$= \frac{1}{2}\int_{\max(\lambda,t)}^{\lambda+t} \frac{dp}{((p(p-t)(p-\lambda)(\lambda+t-p))^{1/2}}$$

with $2p = \lambda + r + t$.

Similarly, from (2.9),

$$\int \frac{1}{|x|}|u_2'(x, t)|\, dx \leqq \int_0^\infty \lambda t I(\lambda, t)\, d\lambda \int_{|X|=1} |Rg(\lambda X)|\, dS_X. \tag{2.18}$$

On the other hand, the following lemma holds.

LEMMA 4. *$I = I(\lambda, t)$ can be estimated by*

$$I(\lambda, t) \leqq C\frac{1}{t}\left(1+\log\left(1+\left(\frac{\min(\lambda, t)}{|\lambda-t|}\right)^{1/2}\right)\right) \tag{2.19}$$

for all $\lambda, t \geqq 0, \lambda \neq t$.

Together with (2.17), (2.18) and (2.10)–(2.12) we thus conclude the proof of part (iii) of the theorem.

Proof of Lemma 4: From (2.17) we have

$$I(\lambda, t) \leqq \frac{1}{2t^{1/2}} \int_{\max(\lambda,t)}^{\lambda+t} \frac{dp}{((p-t)(p-\lambda)(\lambda+t-p))^{1/2}}. \tag{2.20}$$

We shall distinguish now between the following cases.

Case 1°. $0 \leqq \lambda \leqq \frac{1}{2}t$ or $\lambda \geqq 2t$.

Case 2°. $\frac{1}{2}t \leqq \lambda \leqq 2t, \lambda \neq t$.

Assume we are in the first case. If $0 \leqq \lambda \leqq \frac{1}{2}t$, then,

$$I(\lambda, t) \leqq \frac{1}{2t^{1/2}} \int_t^{\lambda+t} \frac{dp}{((p-t)(p-\lambda)(\lambda+t-p))^{1/2}} \tag{2.21}$$
$$\leqq C\frac{1}{t}\int_t^{\lambda+t} \frac{dp}{((p-t)(\lambda+t-p))^{1/2}}.$$

Introducing $\sigma=((p-t))^{1/2}/\lambda^{1/2}$, we have

$$I(\lambda,t)\leqq C\frac{1}{t}\int_0^1\frac{d\sigma}{(1-\sigma^2)^{1/2}}$$

(2.22)

$$\leqq C\frac{1}{t}\cdot 2\pi.$$

If $\lambda\geqq 2t$, then

$$I(\lambda,t)\leqq\frac{1}{2t^{1/2}}\int_\lambda^{\lambda+t}\frac{dp}{((p-t)(p-\lambda)(\lambda+t-p))^{1/2}}$$

(2.23)

$$\leqq C\frac{1}{t}\int_\lambda^{\lambda+t}\frac{dp}{((p-\lambda)(\lambda+t-p))^{1/2}}$$

$$=C\frac{1}{t}\int_0^1\frac{d\sigma}{(1-\sigma^2)^{1/2}}=C\frac{1}{t}2\pi.$$

Hence, $I(\lambda,t)\leqq C1/t$ for case 1°. On the other hand, in the second case, performing the change of variables $\sigma=((p-\max(\lambda,t))/|\lambda-t|)^{1/2}$, for the integral in (2.21) we find

(2.24)
$$I(\lambda,t)<\frac{1}{2(t\lambda)^{1/2}}\int_0^A\frac{du}{\left((1+u^2)\left(1-\frac{1}{A^2}u^2\right)\right)^{1/2}},$$

where $A=(\min(t,\lambda)/|t-\lambda|)^{1/2}$. Taking $\alpha=u/A$ we obtain

$$I(\lambda,t)\leqq\frac{1}{2(t\lambda)^{1/2}}\int_0^1\frac{A}{(1+\alpha^2A^2)^{1/2}}\frac{d\alpha}{(1-\alpha^2)^{1/2}}$$

(2.25)

$$\leqq C\frac{1}{t}\log(1+A),$$

which completes the proof of Lemma 4.

Remark. At the end of this section we derive an L^2-estimate which might be of some interest. With the same assumptions as those of Theorem 1 we have

(2.26)
$$\int|u(x,t)|^2\,dx\leqq C\int(1+|y|)^2\log^2(1+|y|)|g(y)|^2\,dy.$$

The proof is similar to that of part (ii) of Theorem 1. By virtue of (1.8), (1.9), Cauchy–Schwartz inequality and Lemma 2, we derive

$$\begin{aligned}\int |u(x,t)|^2\,dx &= \int_0^\infty r^2\,dr \int_{|X|=1} |u(rX,t)|^2\,dS_X \\ &= \tfrac{1}{4}\int_0^\infty dr \int_{|X|=1} dS_X \left[\int_{|r-t|}^{r+t} \lambda j_g(X,\lambda,Q)\,d\lambda\right] \qquad (2.27) \\ &\leqq CI(r,t)\int_{\mathbf{R}^3} (1+|y|)^2 \log^2(1+|y|)|g(y)|^2\,dy,\end{aligned}$$

where

$$\begin{aligned}I(r,t) &= \int_0^\infty dr \int_{|r-t|}^{r+t} \frac{1}{(1+\lambda)^2 \log^2(1+\lambda)}\,d\lambda \\ &= \int_{|r-t|}^{r+t} \frac{\min(\lambda,t)}{(1+\lambda)^2\log^2(1+\lambda)}\,d\lambda \leqq C\end{aligned}$$

for every $r, t \geqq 0$. Together with (2.27) this proves the assertion.

3. Proof of Theorem 2

As in the proof of part (iii) of Theorem 2 we shall make use of the decompositions (2.8), (2.9). We shall also need the following modifications of Lemma 2, 3.

Let

$$S_{(x,\lambda,q)} = \{y \in \mathbf{R}^3/|y| = \lambda\,;\; y\cdot x \geqq q|y||x|\}$$

for any $x \neq 0$, $\lambda \geqq 0 \leqq q \leqq 1$. Given a function on $\mathbb{R}^3$ we define

$$J_f^+(x,\lambda,q) = \frac{1}{A(x,\lambda,q)} \int_{S_{(x,\lambda,q)}} f(y)\,dS_y,$$

where $A(x,\lambda,q)$ is the area of $S_{(x,\lambda,q)}$. Denoting by $\Delta_S = \Omega_1^2+\Omega_2^2+\Omega_3^2 = R_1^2+R_2^2+R_3^2$ the Laplace–Beltrami operator of the unit sphere S, we have the following Green's identity on $|y| = \lambda$;

$$\int_{S_{x,\lambda,q}} \Delta_S f(y)\,dS_y = -(2\pi)(1-q^2)\frac{d}{dq} j_f(x,\lambda,q) \qquad (3.1)$$

which, for $q \geqq 0$, yields

LEMMA 3. *Given g as in Lemma 3,*

$$\left|\frac{d}{dq} j_g(x,\lambda,q)\right| \leqq J^+_{|\Delta_S g|}(x,\lambda,q) \qquad (3.2)$$

for all $x \neq 0$, $q \geqq 0$, $\lambda \geqq 0$.

Following the same proof as that of Lemma 2 we deduce

LEMMA 2'. *Given g as above, $q \geqq 0$,*

$$\int_{|X|=1} J_g^+(X, \lambda, q)\, dS_X = \int_{|X|=1} g(\lambda X)\, dS_X. \tag{3.3}$$

Using both these lemmas, we obtain

$$\int_{|X|=1} \left|\frac{d}{dq} j_g(X, \lambda, q)\right| dS_X \leqq \int_{|X|=1} |\Delta_S g(\lambda X)|\, dS_X. \tag{3.4}$$

The proof of Theorem 2 as follows now easily. Indeed, from (2.8),

$$\int_{|X|=1} |u_1(rX, t)|\, dS_X \leqq \frac{1}{2r} \int_{A \leqq |y| \leqq B} \frac{1}{|y|^2} |Rg(y)|\, dy, \tag{3.5}$$

$$\begin{aligned} \int_{|X|=1} |u_2(rX, t)|\, dS_X \leqq \frac{1}{2r} \int_{A \leqq |y| \leqq B} \frac{1}{|y|^2} |g(y)|\, dy \\ + \frac{1}{2r}\left[A^{-1} \int_{|y|=A} |g(y)|\, dS_y \right. \\ \left. + B^{-1} \int_{|y|=B} |g(y)|\, dS_y \right], \end{aligned} \tag{3.6}$$

$$\int_{|X|=1} |u_3(rX, t)|\, dS_X \leqq \frac{1}{2r} \cdot \int_{A \leqq |y| \leqq B} \frac{1}{|y|^2} |\lambda Q_\lambda|\, |\Delta_S g(y)|\, dy, \tag{3.7}$$

where

$$\lambda = |y|, \qquad \lambda Q_\lambda = \frac{t^2 + \lambda^2 - r^2}{2\lambda r}.$$

On the other hand,

$$\lambda Q_\lambda = Q - \frac{(r-t)(r+t)}{\lambda r}.$$

Hence, for $|r - t| \leqq \lambda \leqq r + t$,

$$\begin{aligned} |\lambda Q_\lambda| &\leqq 1 + \frac{|r-t|}{\lambda r} \leqq 1 + \frac{r+t}{r} \\ &\leqq 3 + \frac{|r-t|}{r}, \end{aligned} \tag{3.8}$$

i.e.,

$$\int_{|X|=1} |u_3(rX, t)|\, dS_X \leqq \frac{1}{2r} \int_{A \leqq |y| \leqq B} \frac{1}{|y|^2} \left(1 + \frac{A}{r}\right) |\Delta_S g(y)|\, dy. \tag{3.9}$$

The inequalities (3.5), (3.6), (3.9) prove Theorem 2 for the spatial derivatives of u. The estimates of the time derivative follow in identical manner from (2.9).

4. Proof of Theorem 3

By Duhamel's principle the solution to the inhomogeneous Cauchy problem

$$\Box u = g(x, t), \qquad u = u_t = 0 \quad \text{at} \quad t = 0 \tag{4.1}$$

can be expressed in the form

$$u(x, t) = \int_0^t U^s(x, t-s)\, ds, \tag{4.2}$$

where $U^s(x, t)$ is the solution to the homogeneous problem

$$\Box U = 0, \qquad U(x, 0) = 0, \quad U_t(x, 0) = g(x, s).$$

Taking the gradient ∇, with respect to x, t, in (4.2) we deduce

$$\nabla u(x, t) = \int_0^t [\nabla_{(x,s')} U^s(x, s')]|_{s'=t-s}\, ds. \tag{4.3}$$

To prove part (i) of Theorem 3 we apply Theorem 2 to (4.3). Thus, for all $r, t > 0$,

$$\int_{|x|=1} |\nabla u(rX, t)|\, dS_X$$

$$\leq C\frac{1}{r}\int_0^t \left(1 + \frac{A}{r}\right) ds \int_{A\leq|y|\leq B} (|g(y, s)| + |\Omega g(y, s)| + |\Omega^2 g(y, s)|)\frac{1}{|y|^2}\, dy$$
$$+ C\frac{1}{r}\left(\int_0^t A^{-1}\, ds \int_{|y|=A} |g(y, s)|\, dS_y + \int_0^t B^{-1}\, ds \int_{|y|=B} |g(y, s)|\, dS_y\right), \tag{4.4}$$

where $A = |r + s - t|$, $B = r + t - s$.

We now make use of the norms $M_k(g)$ introduced by (1.12), (1.13) and of the following straightforward

LEMMA 5. *Assume h is a smooth, compactly supported function in $\mathbb{R}^3$. Then,*

(a) $\displaystyle\int \frac{1}{|y|}|h(y)|\, dy \leq C \int |Dh(y)|\, dy,$

(b) $\displaystyle\int \frac{1}{|y|^2}|h(y)|\, dy \leq C \int \frac{1}{(1+|y|)^2}(|h(y)| + |Dh(y)| + |D^2 h(y)|)\, dy,$

(c) $\displaystyle\int_{|Y|=1} |h(AY)|\, dS_Y \leq C \int_A^\infty \int_{|Y|=1} |Dh(\lambda Y)|\, dS_Y,$

for all $A>0$. Consequently,

$$\int_{|x|=1} |\nabla u(rX, t)|\, dS_X \leqq C\frac{1}{r} M_4(g)(I_1+I_2+I_4), \tag{4.5}$$

where C is a positive constant and

$$I_1 = I_1(r,t) = \int_0^t \left(1+\frac{A}{r}\right) \max_{A\leqq\lambda\leqq B} \frac{1}{(1+\lambda)^2(1+|\lambda-s|)}\, ds,$$

$$I_2 = I_2(r,t) = \int_0^t \frac{1}{(1+A)(1+|A-s|)}\, ds,$$

$$I_3 = I_3(r,t) = \int_0^t \frac{1}{(1+B)(1+|B-s|)}\, ds,$$

with $A=|r+s-t|$, $B=r+t-s$.

If $r=|x|\geqq\frac{1}{2}$, part (i) of the theorem is an immediate consequence of the following lemma.

LEMMA 6. *Given $A=|r+s-t|$, $B=r+t-s$ and I_1, I_2, I_3 defined above, we have, for all $r\geqq\frac{1}{2}$, $t\geqq 0$ and C a positive constant,*

(a) $I_1(r,t) \leqq C\dfrac{\log(1+t)}{1+|r-t|}$,

(b) $I_2(r,t) \leqq C\dfrac{\log(1+t)}{1+|r-t|}$,

(c) $I_3(r,t) \leqq C\dfrac{\log(1+t)}{1+r+t}$.

Indeed, if $r\geqq\frac{1}{2}$ we derive, from (4.5),

$$\int_{|x|=1} |\nabla u(rX, t)| \leqq CM_4(g)\frac{\log(1+t)}{(1+r)(1+|r-t|}, $$

where C is a positive constant. On the other hand, using Lemma 1 and the commutation properties of the Ω's with $\square$ we conclude that

$$|\nabla u(x,t)| \leqq CM_6(g)\frac{\log(1+t)}{(1+r)(1+|r-t|)} \tag{4.6}$$

for all $r=|x|\geqq\frac{1}{2}$, $t\geqq 0$ and C a positive constant.

If $0\leqq|x|\leqq\frac{1}{2}$ we use, instead of Theorem 2, formula (1.2) of the introduction.

Applying it to (4.3) we derive

$$u(x,t)=\frac{1}{4\pi}\int_0^t (t-s)\,ds\int_{|\xi|=1} g(x+(t-s)\xi,s)\,dS_\xi, \tag{4.7}$$

and by virtue of Lemma 5 and the notation (1.12), (1.13) we infer that

$$|\nabla u(x,t)|\leqq CM_2(g)\int_0^t \frac{1}{(t-s+\frac{1}{2})(|t-2s|+\frac{1}{2})}\,ds,$$

or, since $t-s-\frac{1}{2}\leqq|y|\leqq t-s+\frac{1}{2}$,

$$\begin{aligned}|\nabla u(x,t)|&\leqq CM_2(g)\int_0^t \frac{1}{(t-s+\frac{1}{2})(|t-2s|+\frac{1}{2})}\,ds\\ &\leqq C\frac{\log(1+t)}{1+t}\cdot M_2(g)\\ &\leqq C\frac{\log(1+t)}{1+|t-|x||}M_2(g).\end{aligned}$$

Together with (4.6) this proves Theorem 3(i).

Proof of Lemma 6: We start with a proof of (b). Assume $r\geqq t$; then, $1+|A-s|=1+r-t$ and thus

$$\begin{aligned}I_2(r,t)&=\frac{1}{1+r-t}\int_0^t\frac{ds}{1+r+s-t}\\ &=\frac{1}{1+r-t}\,\frac{\log(1+r)}{1+r-t}\\ &\leqq\frac{\log(1+t)}{1+r-t}.\end{aligned} \tag{4.8}$$

If $0\leqq r<t$,

$$\begin{aligned}I_2&=\int_0^{(t-r)/2}\frac{1}{1+t-r-s}\,\frac{1}{1+t-r-2s}\,ds\\ &\quad+\int_{(t-r)/2}^{t-r}\frac{1}{1+t-r-s}\,\frac{1}{2s-t+r}\,ds\\ &\quad+\int_{t-r}^{t}\frac{1}{1+s+r-t}\,\frac{1}{1+t-r}\,ds\\ &\leqq C\frac{\log(1+t)}{1+t-r},\end{aligned} \tag{4.9}$$

which together with (4.8) proves (b). The proof of (c) follows exactly the same lines. To prove (a) we first remark that

$$
\begin{aligned}
(4.10)\qquad \int_0^t \max_{A\le\lambda\le B} \frac{1}{(1+\lambda)^2(1+|\lambda-s|)}\,ds &\le \int_0^t \frac{1}{1+A} \max_{A\le\lambda\le B} \frac{1}{(1+\lambda)(1+|\lambda-s|)}\,ds \\
&\le \int_0^t \frac{1}{1+A}\,\frac{1}{1+s}\,ds \le C\frac{\log(1+t)}{1+|t-r|}.
\end{aligned}
$$

Thus, it remains to estimate

$$H=\int_0^t \frac{1}{r}\max \frac{1}{(1+\lambda)(1+|\lambda-s|)}\,ds.$$

Assume $3r\le t$. Then,

$$(4.11)\qquad 0\le \tfrac{1}{2}(t-r)\le \tfrac{1}{2}(t+r)\le t-r\le t.$$

Accordingly we split up H into $H=H_1+H_2+H_3$, where

$$
\begin{aligned}
H_1&=\frac{1}{r}\int_0^{(t-r)/2} ds \sup_{A\le\lambda\le B} \frac{1}{(1+\lambda)(1+\lambda-s)},\\
H_2&=\frac{1}{r}\int_{(t-r)/2}^{(t+r)/2} ds \sup_{A\le\lambda\le B} \frac{1}{(1+\lambda)(1+|\lambda-s|)},\\
H_3&=\frac{1}{r}\int_{(t+r)/2}^{t} ds \sup_{A\le\lambda\le B} \frac{1}{(1+\lambda)(1+s-\lambda)}.
\end{aligned}
$$

Thus, we verify easily that

$$
\begin{aligned}
(4.12)\qquad H_1&\le \frac{1}{r}\int_0^{(t-r)/2} \frac{1}{1+t-r-s}\,\frac{1}{1+t-r-2s}\,ds\\
&\le C\frac{1}{r}\,\frac{\log(1+t-r)}{1+t-r},
\end{aligned}
$$

$$(4.13)\qquad H_2\le \frac{1}{r}\int_{(t-r)/2}^{(t+r)/2} \frac{1}{1+s}\,ds\le C\frac{1}{1+t-r},$$

$$
\begin{aligned}
H_3&\le \frac{1}{r}\int_{(t+r)/2}^{t} \frac{1}{2+s}\,ds \sup_{A\le\lambda\le B}\left(\frac{1}{1+\lambda}+\frac{1}{1+s-\lambda}\right)\\
(4.14)\qquad &\le \frac{1}{r}\int_{(t+r)/2}^{t} \frac{1}{2+s}\left(\frac{1}{1+|r+s-t|}+\frac{1}{1+2s-(r+t)}\right) ds\\
&\le C\frac{1}{r}\,\frac{\log(1+t)}{1+t-r}.
\end{aligned}
$$

Hence, for every $r, t \geqq 0, 3r \leqq t$,

$$H(r,t) \leqq C\left(1+\frac{1}{r}\right)\frac{\log(1+t)}{1+t-r}. \tag{4.15}$$

On the other hand, for $3r \geqq t$ and $r \geqq \frac{1}{2}$ we have $1+|r-t| \leqq 4r$. Hence,

$$\begin{aligned} H(r,t) &\leqq C\frac{1}{1+|r-t|}\int_0^t \max_{A\leqq\lambda\leqq B}\frac{1}{(1+\lambda)(1+|\lambda-s|)}\,ds \\ &\leqq C\frac{1}{1+|r-t|}\int_0^t \frac{1}{1+s}\,dS \leqq C\frac{\log(1+t)}{1+|r-t|}, \end{aligned} \tag{4.16}$$

which, together with (4.15), (4.10) concludes the proof of part (i) of Theorem 3.

Proof of part (ii): This is a straightforward consequence of Theorem 1 (iii) applied to (4.3). Indeed, for all $t \geqq 0$,

$$\begin{aligned} &\int \frac{1}{|x|+1}|\nabla u(x,t)|\,dx \\ &\leqq \int \frac{1}{|x|}|\nabla u(x,t)|\,dx \\ &\leqq \int_0^t ds \int_{\mathbb{R}^3}\frac{1}{|x|}\left(1+\log\frac{|x|+t-s}{||x|-t+s|}\right)(|\Omega g(x,s)|+|g(x,s)|)\,dx. \end{aligned} \tag{4.17}$$

Applying the Cauchy–Schwartz inequality and the notation (1.12), (1.13), we derive

$$\int \frac{1}{(1+|x|)}|\nabla u(x,t)|\,dx \leqq CJ(t)N_1(g), \tag{4.18}$$

where C is a positive constant and

$$\begin{aligned} J(t) &= \int_0^t ds\left(\int_0^\infty \frac{(1+\log(\lambda+t-s)/|\lambda+s-t|)^2}{(1+\lambda)^2(1+|\lambda-s|)^2}\,d\lambda\right)^{1/2} \\ &= \int_0^t I(t,s)\,ds, \end{aligned} \tag{4.19}$$

where

$$I(t,s) = \left(\int_0^\infty \frac{(1+\log((\lambda+s)/|\lambda-s|))^2}{(1+\lambda)^2(1+|\lambda+s-t|)^2}\right)^{1/2} \tag{4.20}$$

Splitting up the interval of integration in (4.20) into $\lambda \in [s-\frac{1}{2}, s+\frac{1}{2}]$ and $\lambda \in R_+ \setminus [s-\frac{1}{2}, s+\frac{1}{2}]$ and using, for the second part, the inequalities

$$\log \frac{\lambda+s}{|\lambda-s|} \leqq \log\left(1+2\frac{\min(\lambda, s)}{|\lambda-s|}\right) \leqq 2\frac{\lambda}{|\lambda-s|} \tag{4.21}$$

as well as

$$\frac{1}{(1+\lambda)(1+|\lambda+s-t|)} \leqq \frac{1}{1+t-s}\left(\frac{1}{1+\lambda}+\frac{1}{1+|\lambda+s-t|}\right), \tag{4.22}$$

$$\frac{1}{|\lambda-s|(1+|\lambda+s-t|)} \leqq \frac{1}{1+|t-2s|}\left(\frac{1}{|\lambda-s|}+\frac{1}{1+|\lambda+s-t|}\right), \tag{4.23}$$

we infer that, for all $0 \leqq s \leqq t$,

$$I(t, s) \leqq C\left[\frac{1}{2+t-s}+\frac{1}{1+|t-2s|}\right]. \tag{4.24}$$

Hence, $J(t) \leqq C \log(1+t)$ which, together with (4.18), concludes the proof of Theorem 3.

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