Global Well-Posedness of Incompressible Elastodynamics in Two Dimensions

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Abstract

We prove that for sufficiently small initial displacements in some weighted Sobolev space, the Cauchy problem of the systems of incompressible isotropic Hookean elastodynamics in two space dimensions admits a uniqueness global classical solution. © 2016 Wiley Periodicals, Inc.

1 Introduction

This paper considers the global existence of classical solutions to the Cauchy problem in incompressible nonlinear elastodynamics. The elastic body is assumed to be homogeneous, isotropic, and hyperelastic. The systems of equations describing the motion exhibit a nonlocal nature when one solves the pressure by inverting a Laplacian. The linearized system turns out to be of wave type. We exploit the fact that the nonlinearities in the systems of incompressible isotropic Hookean elastodynamics are *inherently strong linearly degenerate* and automatically satisfy a *strong null condition*.[1] We prove the global existence of classical solutions to this Cauchy problem in the two-dimensional case for small initial displacements in a certain weighted Sobolev space.

To place our result in context, we review a few highlights from the existence theory of nonlinear wave equations and elastodynamics. If the spatial dimension is no bigger than three, the global existence of these equations hinges on two basic assumptions (see [37]): the initial data being sufficiently small and the nonlinearities satisfying a type of null condition. The absence of either of these conditions may lead to the finite-time blowup of solutions. In particular, for 3D compressible elastodynamics, John [14] proved the formation of finite-time singularities for arbitrarily small spherically symmetric displacements without the null condition. On the other hand, Tahvildar-Zadeh [42] proved the formation of singularities for large displacements. For nonlinear wave equations with sufficiently small initial

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[1] The general case will be treated in a forthcoming paper using similar ideas as those in this article.

data but without the null condition, the finite-time blowup was shown by John [13] and Alinhac [4] in 3D, and Alinhac [2, 3, 6] in 2D. We remark that the 2D case is highly nontrivial. See also some related classical results by Sideris [34, 35].

From now on, we will always assume that the initial data is sufficiently small in certain weighted Sobolev space since we are concerned with the long-time behavior of nonlinear elastodynamics and nonlinear wave equations. The first nontrivial long-time existence result may be the one by John and Klainerman [16] where the almost global existence theory is obtained for the 3D quasilinear scalar wave equation. In the seminal work [19], Klainerman introduced the vector field theory and the so-called generalized energy method based on the scaling, rotation, and Lorentz invariant properties of the wave operator, providing a general framework for studying nonlinear wave equations. Then in [20], Klainerman proved the global existence of classical solutions for 3D scalar quasilinear wave equations under the assumptions that the nonlinearity satisfies the null condition. This landmark work was also obtained independently by Christodoulou [8] using a conformal mapping method. We also mention that John [15] established the almost global existence theory for 3D compressible elastodynamics via an L^1-L^∞ estimate.

The generalized energy method of Klainerman can be adapted to prove almost global existence for certain nonrelativistic systems of 3D nonlinear wave equations by using Klainerman-Sideris's weighted L^2 energy estimate, which only involves the scaling and rotation invariance of the system, as was first done in [21]. This approach was subsequently developed to obtain the global existence under the null condition in [41]; see also [44] for a different method. Of particular importance is that Sideris [37] (independently by Agemi [1], see also an earlier result [36]) proved the global existence of classical solutions to the 3D compressible elastodynamics under a null condition. For 3D incompressible elastodynamics, the Hookean part of the system is inherently degenerate and satisfies a null condition. The global existence was established by Sideris and Thomases in [38, 40]. We would like to point out that a unified treatment for obtaining weighted L^2 estimates (of second-order derivatives of unknowns) for certain 3D hyperbolic systems appeared in [39].

As pointed out in [27], the existence question is more delicate in the 2D case because, even under the assumption of the null condition, quadratic nonlinearities have at most critical time decay. A series of articles considered the case of cubically nonlinear equations satisfying the null condition; see, for example, [11, 17]. Alinhac [5] was the first to establish the global existence for the 2D scalar wave equation with null bilinear forms. His argument combines vector fields with what he called the ghost energy method. We emphasize that Alinhac took the advantage of the Lorentz invariance of the system. In particular, at the time of this writing, under null conditions, the global well-posedness of the following problems (for which Lorentz invariance is not available) still remains widely open:

- nonlinear nonrelativistic wave systems in two dimensions and
- systems of nonlinear elastodynamics in two dimensions.

The first nontrivial long-time existence result concerning the above two problems is the recent work by Lei, Sideris, and Zhou [27], in which the authors proved the almost global existence for incompressible isotropic elastodynamics in 2D by formulating the system in eulerian coordinates. Their proof is based on Klainerman's generalized energy method, enhanced by Alinhac's ghost energy method and Klainerman-Sideris's weighted L^2 energy method. This was the first time that Alinhac's ghost weight method was applied to the case where the Lorentz invariance is absent and the system is nonlocal. A novel observation is the treatment of the term involving pressure, which is shown to enjoy a null structure. Unfortunately, at present, it seems hard for us to improve the result in [27] to be a global one under the framework there.

In this paper, we prove the global existence of the incompressible elasticity in two dimensions. To better illustrate our ideas and to make the presentation as simple as possible, we will only focus on the typical Hookean case and treat the general case in another paper using similar ideas. Our proof here is a more structural than one requiring involved technical tools. We first formulate the system in Lagrangian coordinates. Let p denote the pressure and X the flow map. Let $Y(t,y) = X(t,y) - y$. The first observation is that the main part of the main nonlinearity $\nabla\Gamma^\alpha p$ always contains a term of $(\partial_t^2 - \Delta)\Gamma^\alpha Y$ or $(\partial_t + \partial_r) D\Gamma^\alpha Y$. Here D denotes a space or time derivative and Γ is a vector field that is defined in Section 2, above (2.11). This gives us the so-called *strong null condition*[2] (we suggest using this terminology). When we perform the highest-order generalized energy estimate, at first glance we always lose one derivative if we bound the pressure term in L^2 (see Sections 4 and 5).

A natural way to avoid this difficulty may be to introduce the new unknowns

$$U^\alpha = \Gamma^\alpha Y + (\nabla X)^{-\mathsf{T}} \nabla (-\Delta)^{-1} \nabla \cdot [(\nabla X)^{\mathsf{T}} \Gamma^\alpha Y]$$

to symmetrize the system. Unfortunately, this idea leads to complicated calculations, and more essentially, it may not work at all. But fortunately the inherent strong null structure of nonlinearities helps us to obtain a kind of estimate in which we gain one derivative when estimating the L^2 norm of $\nabla\Gamma^\alpha p$ (see Section 5). The price we pay here is to have a smaller decay rate in time, which will be overcome by applying Alinhac's ghost weight as in [1]. For the lower-order energy estimate, our observation is that instead of estimating the L^2 norm of $D\Gamma^\alpha Y$, we turn to estimate its divergence-free part and curl-free part. The divergence-free part is not a main problem. When estimating the curl-free part of $\Gamma^\alpha Y$, the strong linearly degenerate structure is present once again by appropriately rewriting the system in the form of (6.1). Then the generalized energy estimate for the curl-free part of $\Gamma^\alpha Y$ can be carried out with a subcritical time decay, which is $\langle t\rangle^{-3/2}$ (it is still not clear to us whether the similar estimate is true in eulerian coordinates).

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[2] When the usual *null condition* is satisfied by a quasi-linear wave systems, the nonlinearities contain the term of $(\partial_t + \partial_r)\Gamma^\alpha Y$ in general.

As in [21, 37], we need to estimate a kind of weighted L^2 generalized energy norm. A technical difficulty here is to control the weighted generalized L^2 energy $\mathcal{X}_\kappa$ in terms of the generalized energy $\mathcal{E}_\kappa$. For this purpose, we will have to estimate the r-weighted null form of $r(\partial_t^2 - \Delta)Y$, which does not seem true since r is not in the $\mathcal{A}_p$ class of a zero-order Riesz operator for $p = 2$ in two space dimensions (see Lemma 3.3 for details). We overcome this difficulty by considering a variety of the weighted L^2 generalized energy of Klainerman-Sideris [21] (see its definition (2.13) in Section 2) so that we only need to estimate $\langle t\rangle(\partial_t^2 - \Delta)Y$. A trick here is that instead of a gain of one derivative as in performing the highest-order energy estimate in Section 5, we use the strong null condition to gain a suitable decay rate in time.

Before ending this introduction, let us mention some related works on viscoelasticity where there is viscosity in the momentum equation. The global well-posedness with near-equilibrium states in 2D is first obtained in [30]. The 3D case was obtained independently in [7] (see also the thesis [22]) and [26]. The initial boundary value problem is considered in [31], and the compressible case can be found in [12, 33]. For more results near equilibrium, readers are referred to the nice review paper by Lin [29] and other works in [10, 28, 45, 46] as references. In [24] a class of large solutions in two space dimensions is established via the strain-rotation decomposition (which is based on earlier results in [23] and [25]). In all of these works, the initial data is restricted by the viscosity parameter. The work [18] was the first to establish global existence for 3D viscoelastic materials uniformly in the viscosity parameter. We also mention that Hao and Wang recently established the local a priori estimate for the free boundary incompressible elastodynamics in [9].

We will give a self-contained presentation for the whole proof. The remaining part of this paper is organized as follows: In Section 2 we will formulate the system of incompressible elastodynamics in Lagrangian coordinates and present its basic properties; then we introduce some notation and state the main result of this paper. We will outline the main steps of the proof at the end of this section. In Section 3 we will prove some weighted Sobolev imbedding inequalities, the weighted L^∞ estimate, and a refined Sobolev inequality. We also give the estimate for good derivatives and lay down a preliminary step for estimating the weighted generalized L^2 energy. Then we will explore the estimate for the strong null form in Section 4, and at the end of that section we give the estimate for the weighted L^2 energy. In Section 5 we present the highest-order generalized energy estimate. Then we perform the lower-order generalized energy estimate in Section 6.

REMARK 1.1. After posting this article on arXiv (see arXiv:1402.6605), Xuecheng Wang informed the author that he could give another proof of the main result that, as being claimed in Wang's paper [43], can improve the understanding of the behavior of solutions in different coordinates using a different approach and from the point of view of frequency space.

2 Equation and Its Basic Properties

In the incompressible case, the equations of elastodynamics are in general more conveniently written as a first-order system with constraints in eulerian coordinates (see, for instance, [27, 38, 40]). But we will formulate the system in Lagrangian coordinates below.

For any given smooth flow map $X(t,y)$, we call it incompressible if

$$\int_{\Omega} dy = \int_{\Omega_t} dX, \quad \Omega_t = \{X(t,y) \mid y \in \Omega\},$$

for any smooth bounded connected domain Ω. Clearly, the incompressibility is equivalent to

$$\det(\nabla X) \equiv 1.$$

Denote

$$X(t,y) = y + Y(t,y). \tag{2.1}$$

Then we have

$$\nabla \cdot Y = -\det(\nabla Y). \tag{2.2}$$

Moreover, a simple calculation shows that

$$(\nabla X)^{-1} = \begin{pmatrix} 1 + \partial_2 Y^2 & -\partial_1 Y^2 \\ -\partial_2 Y^1 & 1 + \partial_1 Y^1 \end{pmatrix} = (\nabla \cdot X) I - (\nabla X)^{\mathsf{T}}. \tag{2.3}$$

We remark that throughout this paper we use the following convention:

$$(\nabla Y)_{ij} = \frac{\partial Y^i}{\partial y^j}.$$

We often use the following notation:

$$\omega = \frac{y}{r}, \quad r = |y|, \quad \omega^{\perp} = \big(-\omega_2, \omega_1\big), \quad \nabla^{\perp} = \big(-\partial_2, \partial_1\big).$$

For homogeneous, isotropic, and hyperelastic materials, the motion of the elastic fluids is determined by the following Lagrangian functional of flow maps:

$$\begin{aligned} \mathcal{L}(X; T, \Omega) = \int_0^T \int_{\Omega} \Big(& \frac{1}{2} |\partial_t X(t,y)|^2 - W(\nabla X(t,y)) \\ & + p(t,y)[\det(\nabla X(t,y)) - 1] \Big) dy\, dt. \end{aligned} \tag{2.4}$$

Here $W \in C^{\infty}(GL_2, \mathbb{R}_+)$ is the strain energy function, which depends only on $F = \nabla X$, and $p(t,y)$ is a Lagrangian multiplier that is used to force the flow maps to be incompressible. We say that $X(t, y)$ is a critical point of $\mathcal{L}$ if for a given $T \in (0, \infty)$ and bounded smooth connected domain Ω, there holds

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{L}(X^\epsilon; T, \Omega) = 0$$

for all $p \in C^1(\mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R})$ and any one-parameter family of incompressible flow maps $X^\epsilon \in C^1(\mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2)$ with $\frac{d}{dt}\big|_{\epsilon=0} X^\epsilon(t, y) = Z(t, y)$, $X^0(t, y) = X(t, y)$, and $Z(0, y) = Z(T, y) \equiv 0$ for all $y \in \Omega$, $Z(t, y) \equiv 0$ for all $t \in [0, T]$, and $y \in \partial\Omega$.

Let us focus on the simplest case, i.e., the Hookean case, in which the strain energy functional is simply given by

$$W(\nabla X) = \frac{1}{2}|\nabla X|^2.$$

Clearly, the Euler-Lagrangian equation of (2.4) takes

$$\begin{cases} \partial_t^2 Y - \Delta Y = -(\nabla X)^{-\top} \nabla p, \\ \nabla \cdot Y = -\det(\nabla Y). \end{cases} \tag{2.5}$$

Let us take a look at the invariance groups of system (2.5). Suppose that $X(t, y)$ is a critical point of $\mathcal{L}$ in (2.4). Clearly, $\widetilde{X}(t, y)$ are also critical points of $\mathcal{L}$ in (2.4), which are defined either by

$$\widetilde{X}(t, y) = Q^\top X(t, Qy) \quad \forall Q = e^{\lambda A},\ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.6}$$

or by

$$\widetilde{X}(t, y) = \lambda^{-1} X(\lambda t, \lambda y) \tag{2.7}$$

for all $\lambda > 0$. As a result, one also has

$$\begin{cases} \partial_t^2 \widetilde{Y} - \Delta \widetilde{Y} = -(\nabla \widetilde{X})^{-\top} \nabla \widetilde{p}, \\ \nabla \cdot \widetilde{Y} = -\det(\nabla \widetilde{Y}), \end{cases} \tag{2.8}$$

where $\widetilde{p}(t, y) = p(t, Qy)$ if $\widetilde{X}$ is defined by (2.6) and $\widetilde{p}(t, y) = p(\lambda t, \lambda y)$ if $\widetilde{X}$ is defined by (2.7).

Let us first take $\widetilde{X}$ as defined in (2.7). Differentiating (2.8) with respect to λ and then taking $\lambda = 1$, one has

$$\begin{cases} (\nabla X)^\top (\partial_t^2 - \Delta)(S-1)Y + [\nabla(S-1)Y]^\top (\partial_t^2 - \Delta)Y = -\nabla S p, \\ \nabla \cdot (S-1)Y = \partial_1(S-1)Y^2 \partial_2 Y^1 + \partial_1 Y^2 \partial_2 (S-1) Y^1 \\ \qquad\qquad - \partial_1 Y^1 \partial_2 (S-1) Y^2 - \partial_1 (S-1) Y^1 \partial_2 Y^2. \end{cases} \tag{2.9}$$

Here S denotes the scaling operator:

$$S = t\partial_t + y^j \partial_j,$$

and throughout this paper, we use Einstein's convention for repeated indices.

Similarly, denote the rotation operator by

$$\Omega = I\partial_\theta + A, \quad \partial_\theta = y^1\partial_2 - y^2\partial_1,$$

where A is given in (2.6). Let $\widetilde{X}$ be defined in (2.6). Differentiating (2.8) with respect to λ and then taking $\lambda \to 0$, one has

$$\begin{cases} (\nabla X)^{\mathsf{T}}(\partial_t^2\Omega Y - \Delta\Omega Y) + (\nabla\Omega Y)^{\mathsf{T}}(\partial_t^2 Y - \Delta Y) = -\nabla\partial_\theta p, \\ \nabla\cdot\Omega Y = \partial_1\Omega Y^2\partial_2 Y^1 + \partial_1 Y^2\partial_2\Omega Y^1 - \partial_1 Y^1\partial_2\Omega Y^2 \\ \qquad - \partial_1\Omega Y^1\partial_2 Y^2. \end{cases} \tag{2.10}$$

For any vector Y and scalar p, we make the following conventions:

$$\begin{cases} \widetilde{\Omega} Y \triangleq \partial_\theta Y + AY, \ \ \widetilde{\Omega} p \triangleq \partial_\theta p, \ \ \widetilde{\Omega} Y^j = (\widetilde{\Omega} Y)^j, \\ \widetilde{S} Y \triangleq (S-1)Y, \ \ \widetilde{S} p \triangleq Sp, \ \ \widetilde{S} Y^j = (\widetilde{S} Y)^j. \end{cases}$$

Let Γ be any operator of the following set:

$$\{\partial_t, \partial_1, \partial_2, \widetilde{\Omega}, \widetilde{S}\}.$$

Then for any multi-index $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}^{\mathsf{T}} \in \mathbb{N}^5$, using similar arguments as in (2.9) and (2.10), we have (see the Appendix)

$$\begin{aligned} \partial_t^2\Gamma^\alpha Y - \Delta\Gamma^\alpha Y = &-(\nabla X)^{-\mathsf{T}}\nabla\Gamma^\alpha p \\ &- \sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\neq\alpha}} C_\alpha^\beta (\nabla X)^{-\mathsf{T}}(\nabla\Gamma^\beta Y)^{\mathsf{T}}\big(\partial_t^2 - \Delta\big)\Gamma^\gamma Y. \end{aligned} \tag{2.11}$$

together with the constraint

$$\nabla\cdot\Gamma^\alpha Y = \sum_{\beta+\gamma=\alpha} C_\alpha^\beta\big(\partial_1\Gamma^\beta Y^2\partial_2\Gamma^\gamma Y^1 - \partial_1\Gamma^\gamma Y^1\partial_2\Gamma^\beta Y^2\big). \tag{2.12}$$

Here the binomial coefficient C_α^β is given by

$$C_\alpha^\beta = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

The structural identity (2.12) will be of extreme importance in our proof.

Throughout this paper, we use the notation D for space-time derivatives:

$$D = (\partial_t, \partial_1, \partial_2).$$

We use $\langle a\rangle$ to denote

$$\langle a\rangle = \sqrt{1+a^2}$$

and $[a]$ to denote the biggest integer that is no more than a:

$$[a] = \text{biggest integer that is no more than } a.$$

We often use the following abbreviations:

$$\|\Gamma^{\le|\alpha|} f\| = \sum_{|\beta|\le|\alpha|} \|\Gamma^\beta f\|.$$

We need to use Klainerman's generalized energy, which is defined by

$$\mathcal{E}_\kappa = \sum_{|\alpha|\leq \kappa-1} \|D\Gamma^\alpha Y\|_{L^2}^2.$$

We define the following weighted L^2 generalized energy by

$$\mathcal{X}_\kappa = \sum_{|\alpha|\leq\kappa-2}\Big(\int_{r\leq 2\langle t\rangle}\langle t-r\rangle^2|D^2\Gamma^\alpha Y|^2\,dy + \int_{r>2\langle t\rangle}\langle t\rangle^2|D^2\Gamma^\alpha Y|^2\,dy\Big), \tag{2.13}$$

which is a modification of the original one by Klainerman-Sideris in [21]. To describe the space of the initial data, we follow Sideris [37] and introduce

$$\Lambda = \{\nabla, r\partial_r, \Omega\}$$

and

$$H_\Lambda^\kappa = \Big\{(f,g)\ \Big|\ \sum_{|\alpha|\leq\kappa-1}(\|\Lambda^\alpha f\|_{L^2} + \|\nabla\Lambda^\alpha f\|_{L^2} + \|\Lambda^\alpha g\|_{L^2}) < \infty\Big\}.$$

Then as in [37], we define

$$\begin{aligned} H_\Gamma^\kappa(T) = \big\{Y : [0,T)\times\mathbb{R}^2 \to \mathbb{R}^2\ \big|\ &\Gamma^\alpha Y \in L^\infty([0,T);L^2(\mathbb{R}^2)),\\ &\partial_t\Gamma^\alpha Y, \nabla\Gamma^\alpha Y \in L^\infty([0,T);L^2(\mathbb{R}^2))\ \forall|\alpha|\leq\kappa-1\big\} \end{aligned} \tag{2.14}$$

with the norm

$$\sup_{0\leq t<T}\mathcal{E}_\kappa^{1/2}(Y).$$

We are ready to state the main theorem of this paper.

THEOREM 2.1. *Let $W(F) = \frac{1}{2}|F|^2$ be an isotropic Hookean strain energy function. Let $M_0 > 0$ and $0 < \delta < \frac{1}{8}$ be two given constants and $(Y_0, v_0) \in H_\Lambda^\kappa$ with $\kappa \geq 10$. Suppose that Y_0 satisfies the structural constraint* (2.2) *at $t = 0$ and*

$$\begin{aligned} \mathcal{E}_\kappa^{1/2}(0) &= \sum_{|\alpha|\leq\kappa-1}\big(\|\nabla\Lambda^\alpha Y_0\|_{L^2} + \|\Lambda^\alpha v_0\|_{L^2}\big) \leq M_0,\\ \mathcal{E}_{\kappa-2}^{1/2}(0) &= \sum_{|\alpha|\leq\kappa-3}\big(\|\nabla\Lambda^\alpha Y_0\|_{L^2} + \|\Lambda^\alpha v_0\|_{L^2}\big) \leq \epsilon. \end{aligned}$$

There exists a positive constant $\epsilon_0 < e^{-M_0}$ that depends only on κ, M_0, and δ such that, if $\epsilon \leq \epsilon_0$, then the system of incompressible Hookean elastodynamics (2.5) *with initial data*

$$Y(0,y) = Y_0(y), \quad \partial_t Y(0,y) = v_0(y),$$

has a unique global classical solution such that

$$\mathcal{E}_{\kappa-2}^{1/2}(t) \leq \epsilon\exp\big\{C_0^2 M_0\big\}, \quad \mathcal{E}_\kappa^{1/2}(t) \leq C_0 M_0(1+t)^\delta,$$

for some $C_0 > 1$ uniformly in t.

The main strategy of the proof is as follows: For initial data satisfying the constraints in Theorem 2.1, we will prove that

$$\mathcal{E}_\kappa'(t) \le \frac{C_0}{4}\langle t\rangle^{-1}\mathcal{E}_\kappa(t)\mathcal{E}_{\kappa-2}^{1/2}(t), \tag{2.15}$$

which is given in (5.6) and

$$\mathcal{E}_{\kappa-2}'(t) \le \frac{C_0}{4}\langle t\rangle^{-3/2}\mathcal{E}_{\kappa-2}(t)\mathcal{E}_{\kappa}^{1/2}(t), \tag{2.16}$$

which is given in (6.2) and (6.3) for all $t \ge 0$ and some absolute positive constant C_0 depending only on κ. Once the above differential inequalities are proved, it is easy to show that the bounds for $\mathcal{E}_{\kappa-2}^{1/2}$ and $\mathcal{E}_\kappa^{1/2}$ given in the theorem hold true for all $t \ge 0$ by taking an appropriate small ϵ_0, which yields the global existence result and completes the proof of Theorem 2.1.

Indeed, one may just take

$$\epsilon_0 = C_0^{-1}\delta\exp\{-C_0^2 M_0\}.$$

Note that

$$\mathcal{E}_{\kappa-2}^{1/2}(0) \le \epsilon \le \epsilon_0, \quad \mathcal{E}_\kappa^{1/2}(0) \le M_0.$$

By continuity, there exists a positive time $T < \infty$ such that the bounds of $\mathcal{E}_{\kappa-2}^{1/2}(t)$ and $\mathcal{E}_\kappa^{1/2}(t)$ in Theorem 2.1 are true for $t \in [0, T]$:

$$\mathcal{E}_{\kappa-2}^{1/2}(t) \le \epsilon\exp\left\{C_0^2 M_0\right\} \le C_0^{-1}\delta, \quad \mathcal{E}_\kappa^{1/2}(t) \le C_0 M_0(1+t)^\delta. \tag{2.17}$$

We claim that (2.17) is true for all $t \in [0,\infty)$. We prove this claim by contradiction. Suppose that $T_{\max} \in [T,\infty)$ is the largest time such that (2.17) is true on $[0, T_{\max}]$. We are going to deduce a consequence that contradicts the assumption on $T_{\max} < \infty$. Keep in mind that now we have both (2.17) and the differential inequalities (2.15)–(2.16) in hand for $t \in [0, T_{\max}]$. By using (2.17) and the first differential inequality (2.15), one has

$$\mathcal{E}_\kappa(t) \le \mathcal{E}_\kappa(0)\exp\left\{\frac{\delta}{4}\int_0^t \langle t\rangle^{-1}\,dt\right\} = M_0^2(1+t)^{\delta/2}, \quad 0 \le t \le T_{\max}.$$

Similarly, by using (2.17) and the second differential inequality (2.16), one has

$$\begin{aligned}\mathcal{E}_{\kappa-2}(t) &\le \mathcal{E}_{\kappa-2}(0)\exp\left\{\frac{C_0^2 M_0}{4}\int_0^t \langle s\rangle^{\delta-\frac{3}{2}}\,ds\right\}\\ &< \epsilon^2\exp\left\{C_0^2 M_0\right\}, \quad 0 \le t \le T_{\max}.\end{aligned}$$

Consequently, we have proved that, by taking

$$\epsilon_0 = C_0^{-1}\delta\exp\{-C_0^2 M_0\},$$

one has

$$\mathcal{E}_{\kappa-2}(t) < \epsilon^2\exp\{C_0^2 M_0\}, \quad \mathcal{E}_\kappa(t) < M_0^2(1+t)^{2\delta}, \quad 0 \le t \le T_{\max}.$$

The above inequalities show that (2.17) can still be true for $t \in [0, T_{\max} + \epsilon']$ for some $\epsilon' > 0$. This contracts to the assumption on $T_{\max}$. Hence we in fact proved the a priori bounds (2.17) on $[0, \infty)$, which is stated in Theorem 2.1. Moreover, we have

$$\mathcal{E}_{\kappa-2}^{1/2} \leq \epsilon \exp\{C_0^2 M_0\} \leq C_0^{-1}\delta.$$

So from now on our main goal is going to show the two a priori differential inequalities (2.15)–(2.16). The highest-order one will be done in Section 5, and the lower-order one will be done in Section 6. By taking an appropriately large C_0 and an appropriately small δ, we can assume that $\mathcal{E}_{\kappa-2}^{1/2} \ll 1$, which is always assumed in the remainder of this paper. We often use the fact that $\|\nabla X\|_{L^\infty} \leq 3$ since $\|\nabla X - I\|_{L^\infty} \lesssim \mathcal{E}_{\kappa-2}^{1/2}$. Similarly, by (2.3), the above is also applied for $(\nabla X)^{-1}$.

3 Preliminaries

In this section we derive several weighted L^∞-L^2 types of decay in time estimates. We remark that the idea of part of the proofs basically appeared in the earlier work [27] and the references therein. The new weighted L^2 energy $\mathcal{X}$ makes the proofs slightly different. For a self-contained presentation, we still include their proofs below.

We shall need to apply the Littlewood-Paley theory. Let ϕ be a smooth function supported in $\{\tau \in \mathbb{R}^+ : \frac{3}{4} \leq \tau \leq \frac{8}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}} \phi(2^{-j}\tau) = 1.$$

For $f \in L(\mathbb{R}^2)$, we set

$$\Delta_j f = \mathcal{F}^{-1}(\phi(2^{-j}|\xi|)\mathcal{F}(f)) \quad \text{and} \quad S_j f = \sum_{\substack{-\infty < k \leq j-1,\\ k\in\mathbb{Z}}} \Delta_k f.$$

Here $\mathcal{F}$ denotes the usual Fourier transformation in the y-variable and $\mathcal{F}^{-1}$ the inverse Fourier transformation.

The following lemma takes care of the decay properties of the L^∞ norm of the derivatives of unknowns. The 3D version of some of this kind of estimates has already appeared in the work of Klainerman [19], Klainerman-Sideris [21], Sideris [37], and the references therein. See also [27] for some 2D cases. It shows that the L^∞ norm of the derivatives of unknowns will decay in time at least as $\langle t\rangle^{-1/2}$. This can be improved a little bit to get an extra factor $\langle t - r\rangle^{-1/2}$ near the light cone region $\langle t\rangle/2 \leq r \leq 3\langle t\rangle/2$. By a refined Sobolev imbedding inequality, one can even improve the decay rate in time to be $\langle t\rangle^{-1}\ln^{1/2}(e + t)$ in the space-time region away from the light cone (we remark that in this paper we don't need the full strength of the estimate in (3.2)). This will be used to break the criticality of the lower-order generalized energy estimate in the space-time region $|r - t| \geq \langle t\rangle/2$.

It also shows that the lack of Lorentz invariance only leads to a loss of time decay of $\ln^{1/2}(e+t)$ in (3.2).

LEMMA 3.1. *Let $t \geq 4$. Then there holds*

$$\langle r\rangle^{1/2}|D\Gamma^\alpha Y| \lesssim \|D\Gamma^{\leq 2}\Gamma^\alpha Y\|_{L^2} \leq \mathcal{E}^{1/2}_{|\alpha|+3}. \tag{3.1}$$

Moreover, for $r \leq 2\langle t\rangle/3$, or $r \geq 5\langle t\rangle/4$, there holds

$$t|D\Gamma^\alpha Y| \lesssim \big(\mathcal{E}^{1/2}_{|\alpha|+1} + \mathcal{X}^{1/2}_{|\alpha|+3}\big)\ln^{1/2}(e+t). \tag{3.2}$$

For $\langle t\rangle/3 \leq r \leq 5\langle t\rangle/2$, there holds

$$\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}|D\Gamma^\alpha Y| \lesssim \mathcal{E}^{1/2}_{|\alpha|+2} + \mathcal{X}^{1/2}_{|\alpha|+3}. \tag{3.3}$$

PROOF. First of all, by the Sobolev imbedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, (3.1) is automatically true for $r \leq 1$. By the Sobolev imbedding on the sphere $H^1(\mathbb{S}^1) \hookrightarrow L^\infty(\mathbb{S}^1)$, one has

$$|f(r\omega)|^2 \lesssim \sum_{|\beta|\leq 1}\int_{\mathbb{S}^1}|\Omega^\beta f(r\omega)|^2\,d\sigma.$$

Consequently, we have

$$\begin{aligned}
r|f(r\omega)|^2 &\lesssim \sum_{|\beta|\leq 1}\int_{\mathbb{S}^1} r|\Omega^\beta f(r\omega)|^2\,d\sigma\\
&= -\sum_{|\beta|\leq 1}\int_{\mathbb{S}^1} d\sigma \int_r^\infty r\partial_\rho[|\Omega^\beta f(\rho\omega)|^2]d\rho\\
&= -\sum_{|\beta|\leq 1}\int_{\mathbb{S}^1} d\sigma \int_r^\infty r2\Omega^\beta f(\rho\omega)\partial_\rho\Omega^\beta f(\rho\omega)d\rho\\
&\lesssim \sum_{|\beta|\leq 1}\int_{\mathbb{S}^1}\int_r^\infty |\Omega^\beta f(\rho\omega)||\partial_\rho\Omega^\beta f(\rho\omega)|\rho\,d\rho\,d\sigma\\
&\lesssim \sum_{|\beta|\leq 1}\|\partial_r\Omega^\beta f\|_{L^2}\|\Omega^\beta f\|_{L^2}.
\end{aligned}$$

Then (3.1) follows by taking $f = D\Gamma^\alpha Y$ in the above estimate.

Next, by the well-known Bernstein inequality

$$\|\Delta_j f\|_{L^\infty} \lesssim 2^j\|\Delta_j f\|_{L^2}, \quad \|S_j f\|_{L^\infty} \lesssim 2^j\|S_j f\|_{L^2},$$

one has

$$\begin{aligned}\|f\|_{L^\infty} &= \|\sum_j \Delta_j f\|_{L^\infty}\\ &\lesssim 2^{-N}\|S_{-N}f\|_{L^2} + \sum_{-N\le j\le N} 2^j\|\Delta_j f\|_{L^2} + \sum_{j\ge N+1} 2^j\|\Delta_j f\|_{L^2}\\ &\lesssim 2^{-N}\|f\|_{L^2} + \sqrt{2N}\|\nabla f\|_{L^2} + 2^{-N}\|\nabla^2 f\|_{L^2}.\end{aligned}$$

Choosing $N = \ln(e+t)/\ln 2$, one has

$$\|f\|_{L^\infty} \lesssim \|\nabla f\|_{L^2}\ln^{1/2}(e+t) + \frac{1}{1+t}\big(\|f\|_{L^2} + \|\nabla^2 f\|_{L^2}\big). \tag{3.4}$$

Let us first choose a radial cutoff function $\varphi \in C_0^\infty(\mathbb{R}^2)$ that satisfies

$$\varphi = \begin{cases} 1 & \text{if } \frac34 \le r \le \frac65,\\ 0 & \text{if } r < \frac23 \text{ or } r > \frac54, \end{cases} \quad |\nabla\varphi| \le 100.$$

For each fixed $t \ge 4$, let $\varphi^t(y) = \varphi(y/\langle t\rangle)$. Clearly, one has

$$\varphi^t(y) \equiv 1 \ \text{ for } \frac{3\langle t\rangle}{4} \le r \le \frac{6\langle t\rangle}{5}, \quad \varphi^t(y) \equiv 0 \ \text{ for } r \le \frac{2\langle t\rangle}{3} \text{ or } r \ge \frac{5\langle t\rangle}{4},$$

and

$$|\nabla\varphi^t(y)| \le 100\langle t\rangle^{-1}.$$

Consequently, for $r \le 2\langle t\rangle/3$ or $r \ge 5\langle t\rangle/4$, by using (3.4), one has

$$\begin{aligned} t|f| &\lesssim \langle t\rangle\|(1-\varphi^t)f\|_{L^\infty}\\ &\lesssim \langle t\rangle\|\nabla[(1-\varphi^t)f]\|_{L^2}\ln^{1/2}\big(e+t\big) + \|(1-\varphi^t)f\|_{L^2}\\ &\quad + \|\nabla^2[(1-\varphi^t)f]\|_{L^2}\\ &\lesssim \big(\|f\|_{L^2} + \langle t\rangle\|(1-\varphi^t)\nabla f\|_{L^2}\big)\ln^{1/2}(e+t)\\ &\quad + \|f\|_{L^2} + \|(1-\varphi^t)\nabla^2 f\|_{L^2} + \langle t\rangle^{-1}\|\mathbb{1}_{\operatorname{supp}\varphi^t}\nabla f\|_{L^2}. \end{aligned}$$

Here we use $\mathbb{1}_\Omega$ to denote the characteristic function of Ω. Note that the weight in the definition of $\mathcal{X}_{|\alpha|}(Y)$ is equivalent to $\langle t\rangle$ on the support of $1-\varphi^t(y)$. Hence we have

$$\begin{aligned} t|D\Gamma^\alpha Y| &\lesssim \big(\|D\Gamma^\alpha Y\|_{L^2} + \|\langle t\rangle(1-\varphi^t)\nabla D\Gamma^\alpha Y\|_{L^2}\big)\ln^{1/2}(e+t)\\ &\quad + \|D\Gamma^\alpha Y\|_{L^2} + \|(1-\varphi^t)\nabla^2 D\Gamma^\alpha Y\|_{L^2}\\ &\quad + \langle t\rangle^{-1}\|\mathbb{1}_{\operatorname{supp}\varphi^t}\nabla D\Gamma^\alpha Y\|_{L^2}\\ &\lesssim \big(\mathcal{E}_{|\alpha|+1}^{1/2} + \mathcal{X}_{|\alpha|+3}^{1/2}\big)\ln^{1/2}\big(e+t\big). \end{aligned}$$

This completes the proof of (3.2).

It remains to prove (3.3). Notice that $r \geq 1$ for $t \geq 4$. Similarly as in proving (3.1), we calculate that

$$\begin{aligned}
r\langle t-r\rangle|f(r\omega)|^2 &\lesssim r\langle t-r\rangle \sum_{|\beta|\leq 1}\int_{\mathbb{S}^1} |\Omega^\beta f(r\omega)|^2\, d\sigma\\
&= -\sum_{|\beta|\leq 1}\int_{\mathbb{S}^1} d\sigma \int_r^\infty r\partial_\rho[\langle t-\rho\rangle|\Omega^\beta f(\rho\omega)|^2]d\rho\\
&\lesssim \sum_{|\beta|\leq 1}\int_{\mathbb{S}^1} d\sigma \int_r^\infty \big[\langle t-\rho\rangle|\Omega^\beta f(\rho\omega)||\partial_r\Omega^\beta f(\rho\omega)| + |\Omega^\beta f(\rho\omega)|^2\big]\rho\, d\rho\\
&\lesssim \sum_{|\beta|\leq 1}\int_{\mathbb{S}^1} d\sigma \int_r^\infty \big[\langle t-\rho\rangle^2|\partial_r\Omega^\beta f(\rho\omega)|^2 + |\Omega^\beta f(\rho\omega)|^2\big]\rho\, d\rho\\
&= \sum_{|\beta|\leq 1}\big[\|\langle t-r\rangle\partial_r\Omega^\beta f\|_{L^2} + \|\Omega^\beta f\|_{L^2}\big]^2.
\end{aligned}$$

Slightly changing the definition of φ^t and then taking $f = \varphi^t D\Gamma^\alpha Y$ in the above inequality, one has

$$\begin{aligned}
r\langle t-r\rangle|\varphi^t D\Gamma^\alpha Y|^2 &\lesssim \sum_{|\beta|\leq 1}\big(\|\langle t-r\rangle\partial_r\Omega^\beta[\varphi^t D\Gamma^\alpha Y]\|_{L^2} + \|\Omega^\beta D\Gamma^\alpha Y\|_{L^2}\big)^2\\
&\lesssim \sum_{|\beta|\leq 1}\big\|\varphi^t\langle t-r\rangle\partial_r\Omega^\beta[D\Gamma^\alpha Y]\big\|_{L^2}^2\\
&\quad + \sum_{|\beta|\leq 1}\big\|\partial_r\varphi^t\langle t-r\rangle\Omega^\beta[D\Gamma^\alpha Y]\big\|_{L^2}^2 + \mathcal{E}_{|\alpha|+2}
\end{aligned}$$

which yields (3.3). Here we used the fact that Ω commutes with φ^t due to the symmetry of φ^t. □

Now let us study the decay properties of the second-order derivatives of unknowns in the L^∞ norm. The following lemma shows that away from the light cone, the second derivatives of unknowns decay in time like $\langle t\rangle^{-1}$. But near the light cone, the decay rate is only $\langle t\rangle^{-1/2}$, with an extra factor $\langle t-r\rangle^{-1}$. We emphasis that the 3D version has already appeared in [21].

LEMMA 3.2. *Let $t \geq 4$. Then for $r \leq 2\langle t\rangle/3$ or $r \geq 5\langle t\rangle/4$, there hold*

$$t|D^2\Gamma^\alpha Y| \lesssim \mathcal{X}_{|\alpha|+4}^{1/2}. \tag{3.5}$$

For $r \leq 5\langle t\rangle/2$, there holds

$$\langle r\rangle^{1/2}\langle t-r\rangle|D^2\Gamma^\alpha Y| \lesssim \mathcal{X}_{|\alpha|+4}^{1/2} + \mathcal{E}_{|\alpha|+3}^{1/2}. \tag{3.6}$$

PROOF. Let us use the cutoff function φ^t in Lemma 3.1. Note that the weight in the definition of $\mathcal{X}^{1/2}_{|\alpha|+4}(Y)$ is equivalent to $\langle t\rangle$ on the support of $1-\varphi^t(x)$. Thus, by applying the simple Sobolev imbedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned}
t|D^2\Gamma^\alpha Y| &\lesssim t\|(1-\varphi^t(y))D^2\Gamma^\alpha Y\|_{L^2} + t\|(1-\varphi^t(y))\nabla^2 D^2\Gamma^\alpha Y\|_{L^2}\\
&\quad + t\|\nabla(1-\varphi^t(y))\nabla D^2\Gamma^\alpha Y\|_{L^2} + t\|\nabla^2(1-\varphi^t(y))D^2\Gamma^\alpha Y\|_{L^2}\\
&\lesssim \mathcal{X}^{1/2}_{|\alpha|+4} + \langle t\rangle^{-1}\mathcal{X}^{1/2}_{|\alpha|+3} + \langle t\rangle^{-2}\mathcal{X}^{1/2}_{|\alpha|+2}\\
&\lesssim \mathcal{X}^{1/2}_{|\alpha|+4}.
\end{aligned}$$

This proves (3.5).

Next, let us prove (3.6). For $r \le 1$, (3.6) is an immediate consequence of (3.5). We consider the case when $r \ge 1$. Similarly as in proving (3.3), one has

$$\begin{aligned}
&r\langle t-r\rangle^2|f(r\omega)|^2\\
&\quad\lesssim r\langle t-r\rangle^2 \sum_{|\beta|\le 1}\int_{\mathbb{S}^1} |\Omega^\beta f(r\omega)|^2\,d\sigma\\
&\quad= -\sum_{|\beta|\le 1}\int_{\mathbb{S}^1} d\sigma \int_r^\infty r\partial_\rho[\langle t-\rho\rangle^2|\Omega^\beta f(\rho\omega)|^2]d\rho\\
&\quad\lesssim \sum_{|\beta|\le 1}\int_{\mathbb{S}^1} d\sigma \int_r^\infty \big[\langle t-\rho\rangle^2|\Omega^\beta f(\rho\omega)|\big|\partial_r\Omega^\beta f(\rho)\big| + \langle t-\rho\rangle|\Omega^\beta f(\rho\omega)|^2\big]\rho\,d\rho\\
&\quad\lesssim \sum_{|\beta|\le 1}\int_{\mathbb{S}^1} d\sigma \int_r^\infty \big[\langle t-\rho\rangle^2\big|\partial_r\Omega^\beta f(\rho\omega)\big|^2 + \langle t-\rho\rangle^2|\Omega^\beta f(\rho\omega)|^2\big]\rho\,d\rho\\
&\quad= \sum_{|\beta|\le 1}\big[\|\langle t-r\rangle\partial_r\Omega^\beta f\|_{L^2} + \|\langle t-r\rangle\Omega^\beta f\|_{L^2}\big]^2.
\end{aligned}$$

Now let us choose another cutoff function $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^2)$ that is radial and satisfies

$$\widetilde{\varphi} = \begin{cases} 1 & \text{if } r \le \frac{5}{2},\\ 0 & \text{if } r > 3,\end{cases} \quad |\nabla\widetilde{\varphi}| \le 3.$$

For each fixed $t \ge 4$, let $\widetilde{\varphi}^t(y) = \widetilde{\varphi}(y/\langle t\rangle)$. Clearly, one has

$$\widetilde{\varphi}^t(y) \equiv 1 \ \text{ for } r \le \frac{5\langle t\rangle}{2}, \quad \widetilde{\varphi}^t(y) \equiv 0 \ \text{ for } r \ge 3\langle t\rangle, \quad \text{and} \quad |\nabla\widetilde{\varphi}^t(y)| \le 3\langle t\rangle^{-1}.$$

Taking $f = \widetilde{\varphi}^t D^2\Gamma^\alpha Y$, one has

$$\begin{aligned}
&r\langle t-r\rangle^2|\widetilde{\varphi}^t D^2\Gamma^\alpha Y|^2\\
&\quad\lesssim \sum_{|\beta|\le 1}\big\|\langle t-r\rangle\widetilde{\varphi}^t\partial_r\Omega^\beta D^2\Gamma^\alpha Y\big\|_{L^2}^2+\sum_{|\beta|\le 1}\big\|\langle t-r\rangle\partial_r\widetilde{\varphi}^t\Omega^\beta D^2\Gamma^\alpha Y\big\|_{L^2}^2\\
&\qquad+\sum_{|\beta|\le 1}\|\langle t-r\rangle\widetilde{\varphi}^t\Omega^\beta D^2\Gamma^\alpha Y\|_{L^2}^2\\
&\quad\lesssim \|\langle t-r\rangle\widetilde{\varphi}^t D^2\Gamma^{|\alpha|+2}Y\|_{L^2}^2+\big\|\mathbb{1}_{\operatorname{supp}\widetilde{\varphi}^t}D^2\Gamma^{|\alpha|+1}Y\big\|_{L^2}^2,
\end{aligned}$$

which gives (3.6) for $r\ge 1$. □

The next lemma gives a preliminary estimate for the weighted L^2 generalized energy norm $\mathcal{X}_\kappa$. We remark that the definition of $\mathcal{X}_\kappa$ here is different from the original one that appeared in [21,37], where similar results are obtained in 3D. For a self-contained presentation, we still include the detailed proof below.

LEMMA 3.3. *There holds*

$$\mathcal{X}_2^{1/2}\lesssim \mathcal{E}_2^{1/2}+\langle t\rangle\big\|\big(\partial_t^2-\Delta\big)Y\big\|_{L^2}.$$

PROOF. First of all, one may use the decomposition of the gradient operator

$$\nabla=\omega\partial_r+\frac{\omega^\perp}{r}\partial_\theta$$

and the expression of the Laplacian in polar coordinates

$$\Delta=\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\theta^2$$

to derive that

$$\big|\Delta Y-\partial_r^2 Y\big|\le\frac{|\partial_r Y|}{r}+\frac{|\partial_\theta^2 Y|}{r^2}\lesssim\frac{|\nabla Y|+|\nabla\Omega Y|}{r}. \tag{3.7}$$

Let us further write

$$\begin{aligned}
(t^2-r^2)\Delta Y&=-t^2\big(\partial_t^2-\Delta\big)Y-r^2\big(\Delta Y-\partial_r^2Y\big)+t^2\partial_t^2Y-r^2\partial_r^2Y\\
&=-t^2\big(\partial_t^2-\Delta\big)Y-r^2\big(\Delta Y-\partial_r^2Y\big)\\
&\quad+(t\partial_t-r\partial_r)(t\partial_t+r\partial_r-1)Y.
\end{aligned}$$

Hence, using (3.7), one has

$$\begin{aligned}
|(t-r)\Delta Y|&\lesssim t\big|\big(\partial_t^2-\Delta\big)Y\big|+r\big|\Delta Y-\partial_r^2Y\big|+|DSY|+|DY|\\
&\lesssim|D\Gamma Y|+|DY|+t\big|\big(\partial_t^2-\Delta\big)Y\big|.
\end{aligned}\tag{3.8}$$

We remark that in (3.8), r can be even larger than $2\langle t\rangle$.

Next, using (3.8) and integration by parts, one immediately has

$$\begin{aligned}\|(t-r)\partial_i\partial_j Y\|_{L^2}^2 &= \int (t-r)^2\partial_i\partial_j Y\partial_i\partial_j Y\,dy\\ &= 2\int (t-r)\omega_i\partial_j Y\partial_i\partial_j Y\,dy - \int (t-r)^2\partial_j Y\Delta\partial_j Y\,dy\\ &= 2\int (t-r)\omega_i\partial_j Y\partial_i\partial_j Y\,dy - 2\int (t-r)\omega_j\partial_j Y\Delta Y\,dy\\ &\quad + \int (t-r)^2\Delta Y\Delta Y\,dy\\ &\le 10\|\nabla Y\|_{L^2}^2 + \frac{1}{2}\|(t-r)\partial_i\partial_j Y\|_{L^2}^2 + \int (t-r)^2\Delta Y\Delta Y\,dy.\end{aligned}$$

Using (3.8), one obtains that

$$\|(t-r)\partial_i\partial_j Y\|_{L^2} \lesssim \mathcal{E}_2^{1/2} + \langle t\rangle\big\|\big(\partial_t^2-\Delta\big)Y\big\|_{L^2}. \tag{3.9}$$

To estimate $|(t-r)\partial_t\nabla Y|$, let us first write

$$\begin{aligned}(t-r)\partial_t\partial_r Y &= -(\partial_t-\partial_r)(t\partial_t + r\partial_r - 1)Y + t\partial_t^2 Y - r\partial_r^2 Y\\ &= -(\partial_t-\partial_r)(t\partial_t + r\partial_r - 1)Y + t\big(\partial_t^2-\Delta\big)Y\\ &\quad + (t-r)\Delta Y + r\big(\Delta Y - \partial_r^2 Y\big).\end{aligned}$$

Then using (3.7) and (3.8), one has

$$|(t-r)\partial_t\partial_r Y| \lesssim |D\Gamma Y| + |DY| + t\big|\big(\partial_t^2-\Delta\big)Y\big|. \tag{3.10}$$

Consequently, we have

$$\begin{aligned}&|(t-r)\partial_t\partial_j Y|\\ &\quad = \big|(t-r)\partial_t\omega_j\partial_r Y + (t-r)r^{-1}\omega_j^{\perp}\partial_t\partial_\theta Y\big|\\ &\quad \le |(t-r)\omega_j\partial_t\partial_r Y| + \big|r^{-1}(t\partial_t + r\partial_r)\partial_\theta Y - \partial_\theta(\partial_t+\partial_r)Y\big|\\ &\quad \lesssim |(t-r)\omega_j\partial_t\partial_r Y| + r^{-1}|\partial_\theta SY| + |\partial_\theta\partial_t Y| + |\partial_\theta(\omega\cdot\nabla Y)|\\ &\quad \lesssim |(t-r)\omega_j\partial_t\partial_r Y| + r^{-1}|\partial_\theta SY| + |\partial_\theta\partial_t Y| + |\omega^{\perp}\cdot\nabla Y| + |\omega\cdot\partial_\theta\nabla Y|\\ &\quad \lesssim |D\Gamma Y| + |DY| + t|(\partial_t^2-\Delta)Y|,\end{aligned}$$

which gives that

$$\|(t-r)\partial_t\partial_j Y\|_{L^2} \lesssim \mathcal{E}_2^{1/2} + \langle t\rangle\big\|\big(\partial_t^2-\Delta\big)Y\big\|_{L^2}. \tag{3.11}$$

Finally, using (3.8), one has

$$\begin{aligned}\big|(t-r)\partial_t^2 Y\big| &\le \big|(t-r)\big(\partial_t^2-\Delta\big)Y\big| + |(t-r)\Delta Y|\\ &\lesssim |(t-r)\big(\partial_t^2-\Delta\big)Y| + |D\Gamma Y| + |DY| + t\big|\big(\partial_t^2-\Delta\big)Y\big|\\ &\lesssim |D\Gamma Y| + |DY| + \langle t\rangle\big|\big(\partial_t^2-\Delta\big)Y\big|\end{aligned}$$

if $r \le 2\langle t\rangle$, and

$$\langle t\rangle\big|\partial_t^2 Y\big| \le \langle t\rangle\big|\big(\partial_t^2-\Delta\big)Y\big| + \langle t\rangle|\Delta Y|$$

if $r > 2\langle t\rangle$. Hence, by (3.9), we have

$$\begin{aligned}
&\int_{r\le 2\langle t\rangle} \langle t-r\rangle^2\big|\partial_t^2 Y\big|^2\,dy + \int_{r>2\langle t\rangle} \langle t\rangle^2\big|\partial_t^2 Y\big|^2\,dy\\
&\lesssim \langle t\rangle^2\big\|\big(\partial_t^2-\Delta\big)Y\big\|_{L^2}^2 + \mathcal{E}_2^{1/2}(Y) + \int_{r>2\langle t\rangle}\langle t\rangle^2|\Delta Y|^2\,dy \qquad (3.12)\\
&\lesssim \langle t\rangle^2\big\|\big(\partial_t^2-\Delta\big)Y\big\|_{L^2}^2 + \mathcal{E}_2.
\end{aligned}$$

Then the lemma follows from (3.9), (3.11), and (3.12). □

At the end of this section, let us show the estimate for good derivatives $\omega_j\partial_t+\partial_j$ (see some related results in [27]).

LEMMA 3.4. *For $\langle t\rangle/3 \le r \le 5\langle t\rangle/2$, there holds*

$$\langle t\rangle|\omega_j\partial_t DY + \partial_j DY| \lesssim |DY| + |D\Gamma Y| + t\big|\big(\partial_t^2-\Delta\big)Y\big|.$$

PROOF. First, let us calculate that

$$\begin{aligned}
t(\partial_t+\partial_r)(\partial_t-\partial_r)Y &= t\big(\partial_t^2-\Delta\big)Y + t\big(\Delta-\partial_r^2\big)Y\\
&= t\big(\partial_t^2-\Delta\big)Y + \frac{t}{r}\left(\partial_r Y + \frac{\partial_\theta^2 Y}{r}\right).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&t|(\partial_t+\partial_r)(\partial_t-\partial_r)Y|\\
&\le t\big|\big(\partial_t^2-\Delta\big)Y\big| + \frac{t}{r}\left(|\nabla Y| + \frac{|(x_i\partial_j - x_j\partial_i)\partial_\theta Y|}{r}\right) \qquad (3.13)\\
&\lesssim t\big|\big(\partial_t^2-\Delta\big)Y\big| + \frac{t}{r}(|\nabla Y| + |\nabla\Omega Y|).
\end{aligned}$$

Next, using (3.7), (3.8), (3.10), and (3.13), we calculate that

$$\begin{aligned}
&t|(\partial_t+\partial_r)\partial_r Y|\\
&\le \frac{t}{2}|(\partial_t+\partial_r)(\partial_t-\partial_r)Y| + \frac{t}{2}|(\partial_t+\partial_r)(\partial_t+\partial_r)Y|\\
&\lesssim \frac{t}{2}|(\partial_t+\partial_r)(\partial_t-\partial_r)Y| + \frac{1}{2}|S(\partial_t+\partial_r)Y| + \frac{1}{2}|(t-r)\partial_r(\partial_t+\partial_r)Y|\\
&\lesssim \frac{t}{2}|(\partial_t+\partial_r)(\partial_t-\partial_r)Y| + \frac{1}{2}|S(\partial_t+\partial_r)Y|\\
&\quad + \frac{1}{2}\big|(t-r)\partial_{tr}^2 Y\big| + \frac{1}{2}\big|(t-r)\big(\Delta-\partial_r^2\big)Y\big| + \frac{1}{2}|(t-r)\Delta Y|\\
&\lesssim t\big|\big(\partial_t^2-\Delta\big)Y\big| + \left(1+\frac{t}{r}\right)(|\nabla Y| + |\nabla\Gamma Y|).
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&t|(\omega_j\partial_t + \partial_j)\partial_k Y| \\
&\quad = \left|t\omega_j(\partial_t + \partial_r)\partial_k Y + \frac{t}{r}\omega_j^{\perp}\partial_\theta\partial_k Y\right| \\
&\quad \lesssim |t\omega_j(\partial_t + \partial_r)\omega_k\partial_r Y| + \left|\frac{t}{r}\frac{\partial_\theta}{r}\partial_r Y\right| + \frac{t}{r}(|\partial_\theta\partial_k Y| + |\partial_r\partial_\theta Y|) \\
&\quad \lesssim t\left|\left(\partial_t^2 - \Delta\right)Y\right| + \left(1 + \frac{t}{r}\right)(|DY| + |D\Gamma Y|).
\end{aligned}
\tag{3.14}
$$

Finally, let us calculate that

$$
\begin{aligned}
t|(\partial_t + \partial_r)\partial_t Y| &\le t|(\partial_t + \partial_r)(\partial_t - \partial_r)Y| + t|(\partial_t + \partial_r)\partial_r Y| \\
&\le t|(\partial_t + \partial_r)(\partial_t - \partial_r)Y| + t|\omega_j\omega_k(\omega_j\partial_t + \partial_j)\partial_k Y| \\
&\quad + t|\omega_j\partial_j\omega_k\partial_k Y|,
\end{aligned}
$$

which together with (3.13) and (3.14) gives that

$$
\begin{aligned}
&t|(\omega_j\partial_t + \partial_j)\partial_t Y| \\
&\quad \le t|(\partial_t + \partial_r)\partial_t Y| + \left|r^{-1}\partial_\theta\partial_t Y\right| \\
&\quad \lesssim t\left|\left(\partial_t^2 - \Delta\right)Y\right| + \left(1 + \frac{t}{r}\right)(|\nabla Y| + |\nabla\Gamma Y|).
\end{aligned}
\tag{3.15}
$$

Then the lemma follows from (3.14) and (3.15). □

4 Estimate of the L^2 Weighted Norm

Now we are going to estimate the L^2 weighted generalized energy $\mathcal{X}_\kappa$. First of all, we prove the following lemma, which says that the L^2 norms of $\nabla\Gamma^\alpha p$ and $\|(\partial_t^2 - \Delta)\Gamma^\alpha Y\|_{L^2}$, which involve the $(|\alpha| + 2)^{\text{th}}$-order derivatives of unknowns, can be bounded by certain matters that only involve $(|\alpha| + 1)^{\text{th}}$-order derivatives of unknowns. This surprising result is based on the inherent special structures of nonlinearities in the system.

LEMMA 4.1. *Suppose that $\|\nabla Y\|_{L^\infty} \le \delta$ for some absolutely positive constant $\delta < 1$. Then there holds*

$$
\|\nabla\Gamma^\alpha p\|_{L^2} + \left\|\left(\partial_t^2 - \Delta\right)\Gamma^\alpha Y\right\|_{L^2} \lesssim \Pi(|\alpha| + 2)
$$

provided that δ is small enough, where

$$
\begin{aligned}
\Pi(|\alpha| + 2) \lesssim &\sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\ne\alpha}} \left\||\nabla\Gamma^\beta Y|\left|\left(\partial_t^2 - \Delta\right)\Gamma^\gamma Y\right|\right\|_{L^2} \\
&+ \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}} \Pi_1 + \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|\ge|\gamma|}} \Pi_2,
\end{aligned}
\tag{4.1}
$$

where Π_1 *and* Π_2 *are given by*

$$\Pi_1 = \big\|(-\Delta)^{-1/2}\nabla\cdot\big(\partial_t\Gamma^\beta Y^1\nabla^\perp\partial_t\Gamma^\gamma Y^2 - \partial_j\Gamma^\beta Y^1\nabla^\perp\partial_j\Gamma^\gamma Y^2\big)\big\|_{L^2}, \tag{4.2}$$

and

$$\Pi_2 = \big\|(-\Delta)^{-1/2}\nabla\cdot\big(\partial_t\Gamma^\beta Y^2\nabla^\perp\partial_t\Gamma^\gamma Y^1 - \partial_j\Gamma^\beta Y^2\nabla^\perp\partial_j\Gamma^\gamma Y^1\big)\big\|_{L^2}. \tag{4.3}$$

PROOF. By (2.11), one has

$$-\nabla\Gamma^\alpha p = (\nabla X)^{\mathsf T}\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y + \sum_{\beta+\gamma=\alpha,\ \gamma\neq\alpha} C_\alpha^\beta(\nabla\Gamma^\beta Y)^{\mathsf T}\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y.$$

Applying the divergence operator to the above equation and then applying the operator $\nabla(-\Delta)^{-1}$, we obtain that

$$\begin{aligned}\nabla\Gamma^\alpha p = \sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\neq\alpha}} C_\alpha^\beta\big\{&\nabla(-\Delta)^{-1}\nabla\cdot\big[(\nabla\Gamma^\beta Y)^{\mathsf T}\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y\big)\big]\\ &+\nabla(-\Delta)^{-1}\nabla\cdot\big[(\nabla Y)^{\mathsf T}\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\big]\\ &+\nabla(-\Delta)^{-1}\nabla\cdot\big[\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\big]\big\}.\end{aligned}$$

Hence, using the fact that the Riesz operator is bounded in L^2, one has

$$\begin{aligned}\|\nabla\Gamma^\alpha p\|_{L^2} \lesssim \sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\neq\alpha}} &\big\|(\nabla\Gamma^\beta Y)^{\mathsf T}\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y\|_{L^2}\\ &+\big\|(\nabla Y)^{\mathsf T}\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\big\|_{L^2}\\ &+\big\|\nabla(-\Delta)^{-1}\nabla\cdot\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\big\|_{L^2}.\end{aligned} \tag{4.4}$$

Here we kept the Riesz operator in the last quantity on the right-hand side of the above estimate, which needs additional treatments using the fantastic inherent structures of the system.

First of all, using (2.12), we compute that

$$
\begin{aligned}
&\nabla\cdot\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\\
&\quad=\big(\partial_t^2-\Delta\big)\sum_{\beta+\gamma=\alpha}C_\alpha^\beta\big[\partial_1\Gamma^\beta Y^2\partial_2\Gamma^\gamma Y^1-\partial_1\Gamma^\gamma Y^1\partial_2\Gamma^\beta Y^2\big]\\
&\quad=\sum_{\beta+\gamma=\alpha}C_\alpha^\beta\big\{\big[\partial_1\big(\partial_t^2-\Delta\big)\Gamma^\beta Y^2\partial_2\Gamma^\gamma Y^1-\partial_1\Gamma^\gamma Y^1\partial_2\big(\partial_t^2-\Delta\big)\Gamma^\beta Y^2\big]\\
&\qquad+\sum_{\beta+\gamma=\alpha}\big[\partial_1\Gamma^\beta Y^2\partial_2\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y^1-\partial_1\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y^1\partial_2\Gamma^\beta Y^2\big]\\
&\qquad+2\sum_{\beta+\gamma=\alpha}\big[\partial_1\partial_t\Gamma^\beta Y^2\partial_2\partial_t\Gamma^\gamma Y^1-\partial_1\partial_t\Gamma^\gamma Y^1\partial_2\partial_t\Gamma^\beta Y^2\big]\\
&\qquad-2\sum_{\beta+\gamma=\alpha}\big[\partial_1\partial_j\Gamma^\beta Y^2\partial_2\partial_j\Gamma^\gamma Y^1-\partial_1\partial_j\Gamma^\gamma Y^1\partial_2\partial_j\Gamma^\beta Y^2\big]\big\}.
\end{aligned}
\tag{4.5}
$$

Noting the inherent cancellation relation, one sees that the first two terms on the right-hand side of (4.5) can be reorganized to be

$$
\nabla\cdot\sum_{\beta+\gamma=\alpha}C_\alpha^\beta\big[\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y^1\nabla^\perp\Gamma^\beta Y^2-\big(\partial_t^2-\Delta\big)\Gamma^\beta Y^2\nabla^\perp\Gamma^\gamma Y^1\big].
\tag{4.6}
$$

We still need to take care of the last two terms on the right-hand side of (4.5). We first divide them into three parts:

$$A_{11}+A_{12}+A_{13},$$

where A_{11} is the portion of the summation in which $|\beta|=|\gamma|$:

$$
\begin{aligned}
A_{11}&=2\sum_{\substack{\beta+\gamma=\alpha,\\|\beta|=|\gamma|}}C_\alpha^\beta\big[\partial_1\partial_t\Gamma^\beta Y^2\partial_2\partial_t\Gamma^\gamma Y^1-\partial_1\partial_t\Gamma^\gamma Y^1\partial_2\partial_t\Gamma^\beta Y^2\big]\\
&\quad-2\sum_{\substack{\beta+\gamma=\alpha,\\|\beta|=|\gamma|}}C_\alpha^\beta\big[\partial_1\partial_j\Gamma^\beta Y^2\partial_2\partial_j\Gamma^\gamma Y^1-\partial_1\partial_j\Gamma^\gamma Y^1\partial_2\partial_j\Gamma^\beta Y^2\big]\\
&=-2\nabla\cdot\sum_{\substack{\beta+\gamma=\alpha,\\|\beta|=|\gamma|}}C_\alpha^\beta\big[\partial_t\Gamma^\beta Y^2\nabla^\perp\partial_t\Gamma^\gamma Y^1-\partial_j\Gamma^\beta Y^2\nabla^\perp\partial_j\Gamma^\gamma Y^1\big],
\end{aligned}
\tag{4.7}
$$

A_{12} is responsible for the terms involving time derivatives in the summation when $|\beta|\neq|\gamma|$:

$$
\begin{aligned}
A_{12}=2\Big(&\sum_{\substack{\beta+\gamma=\alpha,\\|\beta|>|\gamma|}}+\sum_{\substack{\beta+\gamma=\alpha,\\|\beta|<|\gamma|}}\Big)C_\alpha^\beta\partial_1\partial_t\Gamma^\beta Y^2\partial_2\partial_t\Gamma^\gamma Y^1\\
&-2\Big(\sum_{\substack{\beta+\gamma=\alpha,\\|\beta|>|\gamma|}}+\sum_{\substack{\beta+\gamma=\alpha,\\|\beta|<|\gamma|}}\Big)C_\alpha^\beta\partial_1\partial_t\Gamma^\gamma Y^1\partial_2\partial_t\Gamma^\beta Y^2,
\end{aligned}
$$

and A_{13} is the portion in the summation when $|\beta| \neq |\gamma|$ that only involves spatial derivatives:

$$
\begin{aligned}
A_{13} = {} & 2\Big(\sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}} + \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|<|\gamma|}} \Big) C_\alpha^\beta \partial_1 \partial_j \Gamma^\gamma Y^1 \partial_2 \partial_j \Gamma^\beta Y^2 \\
& - 2\Big(\sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}} + \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|<|\gamma|}} \Big) C_\alpha^\beta \partial_1 \partial_j \Gamma^\beta Y^2 \partial_2 \partial_j \Gamma^\gamma Y^1 .
\end{aligned}
$$

Note that if $|\alpha|$ is odd, then $A_{11} = 0$.

By symmetry of β and γ, we can rewrite A_{12} and A_{13} as

$$
\begin{aligned}
A_{12} = {} & 2 \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}} C_\alpha^\beta \big(\partial_1 \partial_t \Gamma^\beta Y^2 \partial_2 \partial_t \Gamma^\gamma Y^1 + \partial_1 \partial_t \Gamma^\gamma Y^2 \partial_2 \partial_t \Gamma^\beta Y^1 \big) \\
& - 2 \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}} C_\alpha^\beta \big(\partial_1 \partial_t \Gamma^\gamma Y^1 \partial_2 \partial_t \Gamma^\beta Y^2 + \partial_1 \partial_t \Gamma^\beta Y^1 \partial_2 \partial_t \Gamma^\gamma Y^2 \big),
\end{aligned}
$$

and

$$
\begin{aligned}
A_{13} = {} & 2 \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}} C_\alpha^\beta \big(\partial_1 \partial_j \Gamma^\gamma Y^1 \partial_2 \partial_j \Gamma^\beta Y^2 + \partial_1 \partial_j \Gamma^\beta Y^1 \partial_2 \partial_j \Gamma^\gamma Y^2 \big) \\
& - 2 \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}} C_\alpha^\beta \big(\partial_1 \partial_j \Gamma^\beta Y^2 \partial_2 \partial_j \Gamma^\gamma Y^1 + \partial_1 \partial_j \Gamma^\gamma Y^2 \partial_2 \partial_j \Gamma^\beta Y^1 \big).
\end{aligned}
$$

By merging the first and third terms, and the second and the last terms respectively, we further rewrite A_{12} and A_{13} as follows:

$$
A_{12} = 2\nabla \cdot \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}} C_\alpha^\beta \big(-\partial_t \Gamma^\beta Y^2 \nabla^\perp \partial_t \Gamma^\gamma Y^1 + \partial_t \Gamma^\beta Y^1 \nabla^\perp \partial_t \Gamma^\gamma Y^2 \big)
$$

and

$$
A_{13} = 2\nabla \cdot \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}} C_\alpha^\beta \big(\partial_j \Gamma^\beta Y^2 \nabla^\perp \partial_j \Gamma^\gamma Y^1 - \partial_j \Gamma^\beta Y^1 \nabla^\perp \partial_j \Gamma^\gamma Y^2 \big).
$$

Now it is clear that we may add up the above two identities and figure out the contribution of A_{12} and A_{13} to (4.6), which is

$$
\begin{aligned}
&A_{12}+A_{13}\\
&\quad = 2\nabla\cdot \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}} C_\alpha^\beta\Big[\big(\partial_t\Gamma^\beta Y^1\nabla^\perp\partial_t\Gamma^\gamma Y^2-\partial_j\Gamma^\beta Y^1\nabla^\perp\partial_j\Gamma^\gamma Y^2\big)\\
&\qquad\qquad + \big(\partial_j\Gamma^\beta Y^2\nabla^\perp\partial_j\Gamma^\gamma Y^1-\partial_t\Gamma^\beta Y^2\nabla^\perp\partial_t\Gamma^\gamma Y^1\big)\Big].
\end{aligned} \tag{4.8}
$$

Let us insert (4.6), (4.7), and (4.8) into (4.5) to derive that

$$
\begin{aligned}
&\big\|\nabla(-\Delta)\nabla\cdot\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\big\|_{L^2}\\
&\quad\lesssim \sum_{\beta+\gamma=\alpha}\big\|\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y^1\nabla^\perp\Gamma^\beta Y^2-\big(\partial_t^2-\Delta\big)\Gamma^\beta Y^2\nabla^\perp\Gamma^\gamma Y^1\big\|_{L^2}\\
&\qquad + \sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|>|\gamma|}}\Pi_1+\sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|\ge|\gamma|}}\Pi_2.
\end{aligned} \tag{4.9}
$$

Here Π_1 and Π_2 are given in (4.2) and (4.3). We emphasize that in the expressions for Π_1 and Π_2 we still kept the zeroth-order Riesz operator. A crude estimate by removing them directly is not enough to take full advantage of the structure of the system, which may only lead to an almost global existence result and recovers what we already proved in [27] by a different method.

Now let us insert (4.9) into (4.4) to derive that

$$
\|\nabla\Gamma^\alpha p\|_{L^2}\lesssim\|\nabla Y\|_{L^\infty}\big\|\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\big\|_{L^2}+\Pi(|\alpha|+2), \tag{4.10}
$$

where $\Pi(|\alpha|+2)$ is given in (4.1). Using equations (2.11) and (4.10), one has

$$
\begin{aligned}
&\big\|\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\big\|_{L^2}\\
&\quad\lesssim\|(\nabla X)^{-\mathsf{T}}\|_{L^\infty}\Big(\|\nabla\Gamma^\alpha p\|_{L^2}+\sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\neq\alpha}}\big\|(\nabla\Gamma^\beta Y)^{\mathsf{T}}\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y\big\|_{L^2}\Big)\\
&\quad\lesssim\|\nabla Y\|_{L^\infty}\big\|\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\big\|_{L^2}+\Pi(|\alpha|+2),
\end{aligned}
$$

which gives that

$$
\big\|\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\big\|_{L^2}\lesssim\Pi(|\alpha|+2) \tag{4.11}
$$

provided that $\|\nabla Y\|_{L^\infty}$ is appropriately smaller than an absolute positive constant $\delta<1$. Inserting (4.11) into (4.10), one also has

$$
\|\nabla\Gamma^\alpha p\|_{L^2}\lesssim\Pi(|\alpha|+2).
$$

We have proved the lemma. □

In the next lemma, we will use Lemma 4.1 to estimate the main source of nonlinearities in (2.11) by carefully dealing with the last two terms in (4.1). On the right-hand sides of those estimates that we are going to prove, we did not gain derivatives since both sides are of the same order. But what we gain is the time decay rate. In Section 5 where we perform the higher-order energy estimate, we will deal with the last two terms in (4.1) once again, in a different way. The purpose there is to gain one derivative, with the price of slowing down the decay rate in time.

LEMMA 4.2. *Suppose that $\kappa \geq 10$. There exists $\delta > 0$ such that if $\mathcal{E}_{\kappa-2} \leq \delta$, then there hold*

$$\langle t\rangle \|\nabla\Gamma^{\leq\kappa-4} p\|_{L^2} + \langle t\rangle \big\|\big(\partial_t^2 - \Delta\big)\Gamma^{\leq\kappa-4} Y\big\|_{L^2} \lesssim \mathcal{E}_{\kappa-2}^{1/2}\big(\mathcal{E}_{\kappa-2}^{1/2} + \mathcal{X}_{\kappa-2}^{1/2}\big)$$

and

$$\langle t\rangle \|\nabla\Gamma^{\leq\kappa-2} p\|_{L^2} + \langle t\rangle \big\|\big(\partial_t^2 - \Delta\big)\Gamma^{\leq\kappa-2} Y\big\|_{L^2} \lesssim \mathcal{E}_{\kappa}^{1/2}\big(\mathcal{E}_{\kappa-2}^{1/2} + \mathcal{X}_{\kappa-2}^{1/2}\big).$$

PROOF. We need to deal with the two quantities in (4.2) and (4.3). They can be estimated in a similar way. Below we only present the estimate for Π_1 in (4.2). We first deal with the integrals away from the light cone. Let ϕ^t be as defined in Lemma 3.1. It is easy to obtain the following first step estimate:

$$\begin{aligned}\sum_{\substack{\beta+\gamma=\alpha,\\ |\gamma|\leq[|\alpha|/2]}} &\Big\|\Big(\big|\partial_t\Gamma^{\beta}Y^1\nabla^{\perp}\partial_t\Gamma^{\gamma}Y^2 - \partial_j\Gamma^{\beta}Y^1\nabla^{\perp}\partial_j\Gamma^{\gamma}Y^2\big| \\ &+ \big|\partial_t\Gamma^{\beta}Y^2\nabla^{\perp}\partial_t\Gamma^{\gamma}Y^1 - \partial_j\Gamma^{\beta}Y^2\nabla^{\perp}\partial_j\Gamma^{\gamma}Y^1\big|\Big)(1-\phi^t)\Big\|_{L^2} \\ \lesssim \sum_{\substack{\beta+\gamma=\alpha,\\ |\gamma|\leq[|\alpha|/2]}} &\|D\Gamma^{\beta}Y\|_{L^2}\big\|\mathbb{1}_{\mathrm{supp}(1-\varphi^t)}D^2\Gamma^{\gamma}Y\big\|_{L^\infty}.\end{aligned}$$

Using Lemma 3.2, the above is bounded by

$$\langle t\rangle^{-1}\mathcal{E}_{|\alpha|+1}^{1/2}\mathcal{X}_{[|\alpha|/2]+4}^{1/2}.$$

Hence the estimate (4.1) is improved to be

$$\begin{aligned}\Pi(|\alpha|+2) &\lesssim \langle t\rangle^{-1}\mathcal{E}_{|\alpha|+1}^{1/2}\mathcal{X}_{[|\alpha|/2]+4}^{1/2} \\ &+ \sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\neq\alpha}} \big\||\nabla\Gamma^{\beta}Y|\big|\big(\partial_t^2-\Delta\big)\Gamma^{\gamma}Y\big|\big\|_{L^2} \\ &+ \sum_{\substack{\beta+\gamma=\alpha,\\ |\gamma|\leq[|\alpha|/2]}} \big(\Pi_1(\varphi^t)+\Pi_2(\varphi^t)\big),\end{aligned} \tag{4.12}$$

where

$$\Pi_1(\varphi^t) = \big\|(-\Delta)^{-\frac{1}{2}}\nabla\cdot\big[\varphi^t\big(\partial_t\Gamma^{\beta}Y^1\nabla^{\perp}\partial_t\Gamma^{\gamma}Y^2 - \partial_j\Gamma^{\beta}Y^1\nabla^{\perp}\partial_j\Gamma^{\gamma}Y^2\big)\big]\big\|_{L^2}$$

and

$$\Pi_2(\varphi^t) = \big\|(-\Delta)^{-1/2}\nabla\cdot\big[\varphi^t\big(\partial_t\Gamma^\beta Y^2\nabla^\perp\partial_t\Gamma^\gamma Y^1 - \partial_j\Gamma^\beta Y^2\nabla^\perp\partial_j\Gamma^\gamma Y^1\big)\big]\big\|_{L^2}.$$

We will still need to use this estimate in Section 5.

Now let us deal with the third line of (4.12). First of all, we have

$$\begin{aligned}
&\nabla\cdot\big(\varphi^t\partial_t\Gamma^\beta Y^1\nabla^\perp\partial_t\Gamma^\gamma Y^2 - \varphi^t\partial_j\Gamma^\beta Y^1\nabla^\perp\partial_j\Gamma^\gamma Y^2\big)\\
&\quad = \nabla\cdot\big(\varphi^t\partial_t\Gamma^\beta Y^1\nabla^\perp\big[\omega_j(\omega_j\partial_t + \partial_j)\Gamma^\gamma Y^2\big]\\
&\qquad\qquad - \varphi^t\partial_j\Gamma^\beta Y^1\nabla^\perp(\omega_j\partial_t + \partial_j)\Gamma^\gamma Y^2\big)\\
&\qquad + \nabla\cdot\big(\varphi^t\partial_j\Gamma^\beta Y^1\nabla^\perp\big(\omega_j\partial_t\Gamma^\gamma Y^2\big) - \varphi^t\partial_t\Gamma^\beta Y^1\nabla^\perp\big[\omega_j\partial_j\Gamma^\gamma Y^2\big]\big).
\end{aligned}$$

The last line on the right-hand side of the above equality can be rewritten as

$$\nabla^\perp\cdot\big(\omega_j\partial_t\Gamma^\gamma Y^2\nabla\big(\varphi^t\partial_j\Gamma^\beta Y^1\big) - \omega_j\partial_j\Gamma^\gamma Y^2\nabla\big(\varphi^t\partial_t\Gamma^\beta Y^1\big)\big),$$

which can be further reorganized as follows:

$$\begin{aligned}
&\nabla^\perp\cdot\big((\omega_j\partial_t + \partial_j)\Gamma^\gamma Y^2\nabla\big(\varphi^t\partial_j\Gamma^\beta Y^1\big) - \partial_j\Gamma^\gamma Y^2\nabla\varphi^t(\omega_j\partial_t + \partial_j)\Gamma^\beta Y^1\\
&\qquad - \partial_j\Gamma^\gamma Y^2\varphi^t(\omega_j\partial_t + \partial_j)\nabla\Gamma^\beta Y^1\big)\\
&\quad = \nabla\cdot\big(\varphi^t\partial_j\Gamma^\beta Y^1\nabla^\perp(\omega_j\partial_t + \partial_j)\Gamma^\gamma Y^2\big)\\
&\qquad - \nabla^\perp\cdot\big(\partial_j\Gamma^\gamma Y^2\varphi^t(\omega_j\partial_t + \partial_j)\nabla\Gamma^\beta Y^1\big)\\
&\qquad - \nabla^\perp\cdot\big(\partial_j\Gamma^\gamma Y^2\nabla\varphi^t(\omega_j\partial_t + \partial_j)\Gamma^\beta Y^1\big).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\nabla\cdot\big(\varphi^t\partial_t\Gamma^\beta Y^1\nabla^\perp\partial_t\Gamma^\gamma Y^2 - \varphi^t\partial_j\Gamma^\beta Y^1\nabla^\perp\partial_j\Gamma^\gamma Y^2\big)\\
&\quad = \nabla\cdot\big(\varphi^t\partial_t\Gamma^\beta Y^1\nabla^\perp[\omega_j(\omega_j\partial_t + \partial_j)\Gamma^\gamma Y^2]\\
&\qquad\qquad - \varphi^t\partial_j\Gamma^\beta Y^1\nabla^\perp(\omega_j\partial_t + \partial_j)\Gamma^\gamma Y^2\\
&\qquad\qquad + \varphi^t\partial_j\Gamma^\beta Y^1\nabla^\perp(\omega_j\partial_t + \partial_j)\Gamma^\gamma Y^2\big)\\
&\qquad - \nabla^\perp\cdot\big(\partial_j\Gamma^\gamma Y^2\varphi^t(\omega_j\partial_t + \partial_j)\nabla\Gamma^\beta Y^1\\
&\qquad\qquad + \partial_j\Gamma^\gamma Y^2\nabla\varphi^t(\omega_j\partial_t + \partial_j)\Gamma^\beta Y^1\big).
\end{aligned}\tag{4.13}$$

The expression in (4.13) will be rewritten in a different form in Section 5 for a different purpose.

Now we are ready to estimate the third line on the right-hand side of (4.12) as follows:

$$\begin{aligned}
&\big\|(-\Delta)^{-1/2}\nabla\cdot\big[\varphi^t\big(\partial_t\Gamma^\beta Y^1\nabla^\perp\partial_t\Gamma^\gamma Y^2 - \partial_j\Gamma^\beta Y^1\nabla^\perp\partial_j\Gamma^\gamma Y^2\big)\big]\big\|_{L^2}\\
&\quad \lesssim \|D\Gamma^\beta Y\|_{L^2}\big\|\mathbb{1}_{\mathrm{supp}\,\varphi^t}(\omega_j\partial_t + \partial_j)\nabla\Gamma^\gamma Y\big\|_{L^\infty}\\
&\qquad + \big\|\mathbb{1}_{\mathrm{supp}\,\varphi^t}D\Gamma^\gamma Y\big\|_{L^\infty}\big\|\mathbb{1}_{\mathrm{supp}\,\varphi^t}(\omega_j\partial_t + \partial_j)\nabla\Gamma^\beta Y\big\|_{L^2}\\
&\qquad + \langle t\rangle^{-1}\big\|\mathbb{1}_{\mathrm{supp}\,\varphi^t}D\Gamma^\gamma Y\big\|_{L^\infty}\|D\Gamma^\beta Y\|_{L^2}.
\end{aligned}$$

Note that the last term in the above is due to the commutation between ∇ and good derivatives.

We first use Lemma 3.4 to bound the quantities on the right-hand side of the above estimate by

$$
\begin{aligned}
&\langle t\rangle^{-1}\|D\Gamma^{\beta}Y\|_{L^2}\big(\big\|\mathbb{1}_{\operatorname{supp}\varphi^t}\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\gamma}Y\big\|_{L^\infty}+\big\|\mathbb{1}_{\operatorname{supp}\varphi^t}D\Gamma^{\le|\gamma|+1}Y\big\|_{L^\infty}\big)\\
&\quad+\langle t\rangle^{-1}\big\|\mathbb{1}_{\operatorname{supp}\varphi^t}D\Gamma^{\gamma}Y\big\|_{L^\infty}\big(\big\|\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\beta}Y\big\|_{L^2}+\|D\Gamma^{\le|\beta|+1}Y\|_{L^2}\big)\\
&\quad+\langle t\rangle^{-1}\big\|\mathbb{1}_{\operatorname{supp}\varphi^t}D\Gamma^{\gamma}Y\big\|_{L^\infty}\|D\Gamma^{\beta}Y\|_{L^2}.
\end{aligned}
$$

Consequently, noting $|\gamma|\le[|\alpha|/2]$ and using (3.1) in Lemma 3.1 (slightly changing its proof by using the cutoff function φ^t to keep the wave operator $\partial_t^2-\Delta$), one can estimate the third line on the right-hand side of (4.12) as follows:

$$
\begin{aligned}
\sum_{\substack{\beta+\gamma=\alpha,\\ |\gamma|\le[|\alpha|/2]}}\Pi_1(\varphi^t)&\lesssim\langle t\rangle^{-3/2}\mathcal{E}^{1/2}_{[|\alpha|/2]+4}\mathcal{E}^{1/2}_{|\alpha|+2}\\
&\quad+\langle t\rangle^{-3/2}\mathcal{E}^{1/2}_{|\alpha|+1}\big\|\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\le[|\alpha|/2]+2}Y\big\|_{L^2}\\
&\quad+\langle t\rangle^{-3/2}\mathcal{E}^{1/2}_{[|\alpha|/2]+3}\big\|\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\le|\alpha|}Y\big\|_{L^2}.
\end{aligned}
\tag{4.14}
$$

As we have already mentioned, $\Pi_2(\varphi^t)$ in the last line on the right-hand side of (4.12) can be bounded similarly as $\Pi_2(\varphi^t)$.

It remains to estimate the second line in (4.12). Using the Sobolev imbedding $H^2\hookrightarrow L^\infty$, we have

$$
\begin{aligned}
&\sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\ne\alpha}}\big\||\nabla\Gamma^{\beta}Y|\big|\big(\partial_t^2-\Delta\big)\Gamma^{\gamma}Y\big|\big\|_{L^2}\\
&\lesssim\sum_{\substack{\beta+\gamma=\alpha,\\ |\gamma|\le[|\alpha|/2]}}\|\nabla\Gamma^{\beta}Y\|_{L^2}\big\|\big(\partial_t^2-\Delta\big)\Gamma^{\gamma}Y\big\|_{L^\infty}\\
&\quad+\sum_{\substack{\beta+\gamma=\alpha,\\ [|\alpha|/2]<|\gamma|<|\alpha|}}\|\nabla\Gamma^{\beta}Y\|_{L^\infty}\big\|\big(\partial_t^2-\Delta\big)\Gamma^{\gamma}Y\big\|_{L^2}\\
&\lesssim\langle t\rangle^{-1}\mathcal{E}^{1/2}_{|\alpha|+1}\big\|\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\le[|\alpha|/2]+2}Y\big\|_{L^2}\\
&\quad+\langle t\rangle^{-1}\mathcal{E}^{1/2}_{[|\alpha|/2]+3}\big\|\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\le|\alpha|-1}Y\big\|_{L^2}.
\end{aligned}
\tag{4.15}
$$

We now insert (4.15) and (4.14) into (4.12) to obtain that

$$
\begin{aligned}
\Pi(|\alpha|+2)&\lesssim\langle t\rangle^{-1}\mathcal{E}^{1/2}_{|\alpha|+2}\big(\mathcal{E}^{1/2}_{[|\alpha|/2]+4}+\mathcal{X}^{1/2}_{[|\alpha|/2]+4}\big)\\
&\quad+\langle t\rangle^{-1}\mathcal{E}^{1/2}_{|\alpha|+1}\big\|\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\le[|\alpha|/2]+2}Y\big\|_{L^2}\\
&\quad+\langle t\rangle^{-1}\mathcal{E}^{1/2}_{[|\alpha|/2]+3}\big\|\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\le|\alpha|}Y\big\|_{L^2}.
\end{aligned}
\tag{4.16}
$$

Now for $\kappa \geq 10$ and $|\alpha| \leq \kappa - 4$, one has $|\alpha| + 2 \leq \kappa - 2$ and $[|\alpha|/2] + 4 \leq \kappa - 3$. Hence, by (4.16), we have

$$\begin{aligned}\Pi(\kappa-2) &\lesssim \langle t\rangle^{-1}\mathcal{E}_{\kappa-2}^{1/2}\big(\mathcal{E}_{\kappa-3}^{1/2} + \mathcal{X}_{\kappa-3}^{1/2}\big)\\ &\quad + \langle t\rangle^{-1}\mathcal{E}_{\kappa-3}^{1/2}\big\|\langle t\rangle\big(\partial_t^2 - \Delta\big)\Gamma^{\leq\kappa-4}Y\big\|_{L^2}.\end{aligned}$$

Inserting the above inequality into the estimate in Lemma 4.1 and noting that $\mathcal{E}_{\kappa-2}(Y) \leq \delta$, one has

$$\langle t\rangle\|\nabla\Gamma^{\leq\kappa-4}p\|_{L^2} + \langle t\rangle\big\|\big(\partial_t^2 - \Delta\big)\Gamma^{\leq\kappa-4}Y\big\|_{L^2} \lesssim \mathcal{E}_{\kappa-2}^{1/2}\big(\mathcal{E}_{\kappa-2}^{1/2} + \mathcal{X}_{\kappa-2}^{1/2}\big). \tag{4.17}$$

This proves the first estimate in Lemma 4.2.

Next, for $|\alpha| \leq \kappa - 2$, there holds $[|\alpha|/2] + 4 \leq \kappa - 2$. Hence, one can derive from (4.16) that

$$\begin{aligned}\Pi(\kappa) &\lesssim \langle t\rangle^{-1}\mathcal{E}_{\kappa}^{1/2}\big(\mathcal{E}_{\kappa-2}^{1/2} + \mathcal{X}_{\kappa-2}^{1/2}\big)\\ &\quad + \langle t\rangle^{-1}\mathcal{E}_{\kappa}^{1/2}\big\|\langle t\rangle\big(\partial_t^2 - \Delta\big)\Gamma^{\leq\kappa-4}Y\big\|_{L^2}\\ &\quad + \langle t\rangle^{-1}\mathcal{E}_{\kappa-2}^{1/2}\big\|\langle t\rangle\big(\partial_t^2 - \Delta\big)\Gamma^{\leq\kappa-2}Y\big\|_{L^2},\end{aligned}$$

which, by combining (4.17) and Lemma 4.1, gives the second estimate in the lemma. □

We are ready to state the following lemma:

LEMMA 4.3. *Suppose that* $\kappa \geq 10$. *There exists* $\delta > 0$ *such that if* $\mathcal{E}_{\kappa-2} \leq \delta$, *then there hold*

$$\mathcal{X}_{\kappa-2} \lesssim \mathcal{E}_{\kappa-2}, \quad \mathcal{X}_{\kappa} \lesssim \mathcal{E}_{\kappa}.$$

PROOF. Applying Lemma 3.3 and Lemma 4.2, one has

$$\begin{aligned}\mathcal{X}_{\kappa-2}^{1/2} &\lesssim \mathcal{E}_{\kappa-2}^{1/2} + \langle t\rangle\big\|\big(\partial_t^2 - \Delta\big)\Gamma^{\leq\kappa-4}Y\big\|_{L^2}\\ &\lesssim \mathcal{E}_{\kappa-2}^{1/2} + \mathcal{E}_{\kappa-2}^{1/2}\big(\mathcal{E}_{\kappa-2}^{1/2} + \mathcal{X}_{\kappa-2}^{1/2}\big),\end{aligned}$$

which gives the first estimate of the lemma by noting the assumption.

Next, applying Lemma 3.3 and Lemma 4.2 once more, one has

$$\begin{aligned}\mathcal{X}_{\kappa}^{1/2} &\lesssim \mathcal{E}_{\kappa}^{1/2} + \langle t\rangle\big\|\big(\partial_t^2 - \Delta\big)\Gamma^{\leq\kappa-2}Y\big\|_{L^2}\\ &\lesssim \mathcal{E}_{\kappa}^{1/2} + \mathcal{E}_{\kappa}^{1/2}\big(\mathcal{E}_{\kappa-2}^{1/2} + \mathcal{X}_{\kappa-2}^{1/2}\big).\end{aligned}$$

Then the second estimate of the lemma follows from the first one and the assumption. □

5 Higher-Order Energy Estimate

This section is devoted to the higher-order generalized energy estimate. We will see that the ghost weight method introduced by Alinhac in [5] plays an important role.

Let $\kappa \geq 10$ and $|\alpha| \leq \kappa - 1$. Let $\sigma = t - r$ and $q(\sigma) = \arctan\sigma$. Taking the L^2 inner product of (2.11) with $e^{-q(\sigma)}\partial_t\Gamma^\alpha Y$ and using integration by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\int e^{-q(\sigma)}(|\partial_t\Gamma^\alpha Y|^2 + |\nabla\Gamma^\alpha Y|^2)dy\\
&= -\int \frac{e^{-q(\sigma)}}{1+\sigma^2}(|\partial_t\Gamma^\alpha Y|^2 + |\nabla\Gamma^\alpha Y|^2)dy\\
&\quad + 2\int e^{-q(\sigma)}\partial_t\Gamma^\alpha Y\cdot\big(\partial_t^2 - \Delta\big)\Gamma^\alpha Y\,dy - 2\int \partial_j e^{-q(\sigma)}\partial_j\Gamma^\alpha Y\,\partial_t\Gamma^\alpha Y\,dy\\
&= -\sum_j\int \frac{e^{-q(\sigma)}}{1+\sigma^2}\big|(\omega_j\partial_t + \partial_j)\Gamma^\alpha Y\big|^2\,dy\\
&\quad - 2\int e^{-q(\sigma)}\partial_t\Gamma^\alpha Y\cdot\big[(\nabla X)^{-\mathsf{T}}\nabla\Gamma^\alpha p\big]dy\\
&\quad - 2\int e^{-q(\sigma)}\partial_t\Gamma^\alpha Y\cdot\sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\neq\alpha}} C_\alpha^\beta(\nabla X)^{-\mathsf{T}}(\nabla\Gamma^\beta Y)^{\mathsf{T}}\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y\,dy,
\end{aligned}$$

which gives that

$$\begin{aligned}
&\frac{d}{dt}\int e^{-q(\sigma)}\big(\big|\partial_t\Gamma^\alpha Y\big|^2 + |\nabla\Gamma^\alpha Y|^2\big)dy\\
&\qquad + \sum_j\int\frac{e^{-q(\sigma)}}{1+\sigma^2}|(\omega_j\partial_t+\partial_j)\Gamma^\alpha Y|^2\,dy\\
&\qquad \lesssim \mathcal{E}_\kappa^{1/2}\|(\nabla X)^{-\mathsf{T}}\|_{L^\infty}\\
&\qquad\quad \cdot\Big(\|\nabla\Gamma^\alpha p\|_{L^2} + \sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\neq\alpha}}\big\|(\nabla\Gamma^\beta Y)^{\mathsf{T}}\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y\big\|_{L^2}\Big),
\end{aligned}\tag{5.1}$$

We will use the simple estimate $\|(\nabla X)^{-\mathsf{T}}\|_{L^\infty} \leq 4$. At the first glance, we will always lose one derivative since $\|\nabla\Gamma^\alpha p\|_{L^2}$ contains the $|\alpha| + 2$ derivatives of Y. Fortunately, we may modify the proof for Lemma 4.2 so that we have similar estimates but gain one derivative and at the same time lose $\langle t\rangle^{-1/2}$ decay rate. Moreover, whenever we lose $\langle t\rangle^{-1/2}$ decay rate, we have a good derivative $\omega_j\partial_t + \partial_j$. Then the ghost weight method of Alinhac enables us to take the advantage of the null structure of nonlinearities when we perform the highest-order energy estimate. We emphasize that all of those calculations are based on the physical structures of the system.

We still use the estimate in (4.12), but we need to refine the last line of (4.13) as follows:

$$
\begin{aligned}
&\nabla^{\perp}\cdot\big(\partial_j\Gamma^{\gamma}Y^2\phi^t(\omega_j\partial_t+\partial_j)\nabla\Gamma^{\beta}Y^1+\partial_j\Gamma^{\gamma}Y^2\nabla\phi^t(\omega_j\partial_t+\partial_j)\Gamma^{\beta}Y^1\big)\\
&\quad=\nabla^{\perp}\cdot\big(\partial_j\Gamma^{\gamma}Y^2\nabla\big[\phi^t(\omega_j\partial_t+\partial_j)\Gamma^{\beta}Y^1\big]\big)\\
&\qquad+\nabla^{\perp}\cdot\big(\partial_j\Gamma^{\gamma}Y^2\phi^t\nabla\omega_j\partial_t\Gamma^{\beta}Y^1\big)\\
&\quad=\nabla\cdot\big(\big[\phi^t(\omega_j\partial_t+\partial_j)\Gamma^{\beta}Y^1\big]\nabla^{\perp}\partial_j\Gamma^{\gamma}Y^2\big)\\
&\qquad+\nabla^{\perp}\cdot\big(\partial_j\Gamma^{\gamma}Y^2\phi^t\nabla\omega_j\partial_t\Gamma^{\beta}Y^1\big).
\end{aligned}\tag{5.2}
$$

Replacing the last line in (4.13) by (5.2), one has

$$
\begin{aligned}
&\nabla\cdot\big(\phi^t\partial_t\Gamma^{\beta}Y^1\nabla^{\perp}\partial_t\Gamma^{\gamma}Y^2-\phi^t\partial_j\Gamma^{\beta}Y^1\nabla^{\perp}\partial_j\Gamma^{\gamma}Y^2\big)\\
&\quad=\nabla\cdot\Big(\phi^t\partial_t\Gamma^{\beta}Y^1\nabla^{\perp}\big[\omega_j(\omega_j\partial_t+\partial_j)\Gamma^{\gamma}Y^2\big]\\
&\qquad\quad-\phi^t\partial_j\Gamma^{\beta}Y^1\nabla^{\perp}(\omega_j\partial_t+\partial_j)\Gamma^{\gamma}Y^2\\
&\qquad\quad+\phi^t\partial_j\Gamma^{\beta}Y^1\nabla^{\perp}(\omega_j\partial_t+\partial_j)\Gamma^{\gamma}Y^2\\
&\qquad\quad-\big[\phi^t(\omega_j\partial_t+\partial_j)\Gamma^{\beta}Y^1\big]\nabla^{\perp}\partial_j\Gamma^{\gamma}Y^2\Big)\\
&\qquad-\nabla^{\perp}\cdot\big(\partial_j\Gamma^{\gamma}Y^2\phi^t\nabla\omega_j\partial_t\Gamma^{\beta}Y^1\big).
\end{aligned}\tag{5.3}
$$

Using (5.3), we may re-estimate $\Pi_1(\varphi^t)$ in (4.12) as follows:

$$
\begin{aligned}
&\big\|(-\Delta)^{-1/2}\nabla\cdot\big[\varphi^t\big(\partial_t\Gamma^{\beta}Y^1\nabla^{\perp}\partial_t\Gamma^{\gamma}Y^2-\partial_j\Gamma^{\beta}Y^1\nabla^{\perp}\partial_j\Gamma^{\gamma}Y^2\big)\big]\big\|_{L^2}\\
&\quad\lesssim\|D\Gamma^{\beta}Y\|_{L^2}\big\|\mathbb{1}_{\operatorname{supp}\varphi^t}(\omega_j\partial_t+\partial_j)\nabla\Gamma^{\gamma}Y\big\|_{L^\infty}\\
&\qquad+\langle t\rangle^{-1/2}\big\|\mathbb{1}_{\operatorname{supp}\varphi^t}\langle r\rangle^{1/2}\langle t-r\rangle D^2\Gamma^{\gamma}Y\big\|_{L^\infty}\big\|\langle t-r\rangle^{-1}(\omega_j\partial_t+\partial_j)\Gamma^{\beta}Y\big\|_{L^2}\\
&\qquad+\langle t\rangle^{-1}\big\|\mathbb{1}_{\operatorname{supp}\varphi^t}D\Gamma^{\gamma}Y\big\|_{L^\infty}\|D\Gamma^{\beta}Y\|_{L^2}.
\end{aligned}
$$

Again, the last term in the above inequality is due to the commutation of ∇ with good derivatives. Using Lemma 3.4 and Lemma 3.2, we can bound the above by

$$
\begin{aligned}
&\langle t\rangle^{-1}\|D\Gamma^{\beta}Y\|_{L^2}\big(\big\|\mathbb{1}_{\operatorname{supp}\varphi^t}\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\gamma}Y\big\|_{L^\infty}+\big\|\mathbb{1}_{\operatorname{supp}\varphi^t}D\Gamma^{\le|\gamma|+1}Y\big\|_{L^\infty}\big)\\
&\quad+\langle t\rangle^{-1/2}(\mathcal{X}_{|\gamma|+4}+\mathcal{E}_{|\gamma|+3})^{1/2}\big\|\langle t-r\rangle^{-1}(\omega_j\partial_t+\partial_j)\Gamma^{\beta}Y\big\|_{L^2}\\
&\quad+\langle t\rangle^{-1}\|D\Gamma^{\gamma}Y\|_{L^\infty}\|D\Gamma^{\beta}Y\|_{L^2}.
\end{aligned}
$$

Notice that $|\gamma|\le[|\alpha|/2]$. We further use Lemma 3.1 (again, we need modify the proof slightly by adding a cutoff function φ^t to keep the wave operator $\partial_t^2-\Delta$) and Lemma 4.3 to bound the above by

$$
\begin{aligned}
&\langle t\rangle^{-1/2}\mathcal{E}_{|\gamma|+4}^{1/2}\big\|\langle t-r\rangle^{-1}(\omega_j\partial_t+\partial_j)\Gamma^{\beta}Y\big\|_{L^2}\\
&\quad+\langle t\rangle^{-\frac{3}{2}}\mathcal{E}_{|\alpha|+1}^{1/2}\big\|\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\le[|\alpha|/2]+2}Y\big\|_{L^2}+\langle t\rangle^{-3/2}\mathcal{E}_{[|\alpha|/2]+4}^{1/2}\mathcal{E}_{|\alpha|+1}^{1/2}.
\end{aligned}
$$

Since $[|\alpha|/2] + 4 \le \kappa - 2$, we can use Lemma 4.2 and Lemma 4.3 to bound the above quantities by

$$\langle t\rangle^{-1/2}\mathcal{E}_{\kappa-2}^{1/2}\big\|\langle t-r\rangle^{-1}(\omega_j\partial_t+\partial_j)\Gamma^\beta Y\big\|_{L^2} + \langle t\rangle^{-3/2}\mathcal{E}_{\kappa-2}^{1/2}\mathcal{E}_\kappa^{1/2}. \tag{5.4}$$

Similarly, the last line in (4.12) can also be bounded by the quantity in (5.4). Consequently, we can derive by inserting (5.4) into (4.12) that

$$\begin{aligned}\|\nabla\Gamma^\alpha p\|_{L^2} \lesssim &\sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\ne\alpha}} \big\||\nabla\Gamma^\beta Y|\big|(\partial_t^2-\Delta)\Gamma^\gamma Y\big|\big\|_{L^2}\\ &+\langle t\rangle^{-1/2}\mathcal{E}_{\kappa-2}^{1/2}\big\|\langle t-r\rangle^{-1}(\omega_j\partial_t+\partial_j)\Gamma^\beta Y\big\|_{L^2}\\ &+\langle t\rangle^{-1}\mathcal{E}_{\kappa-2}^{1/2}\mathcal{E}_\kappa^{1/2}.\end{aligned}$$

Inserting the above estimate into (5.1), we have

$$\begin{aligned}&\frac{d}{dt}\int e^{-q(\sigma)}\big(|\partial_t\Gamma^\alpha Y|^2+|\nabla\Gamma^\alpha Y|^2\big)dy\\ &\quad+\int\frac{e^{-q(\sigma)}}{1+\sigma^2}\big(|\omega\partial_t\Gamma^\alpha Y+\nabla\Gamma^\alpha Y|^2\big)dy\\ &\quad\lesssim \mathcal{E}_\kappa^{1/2}\sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\ne\alpha}}\big\|(\nabla\Gamma^\beta Y)^{\mathsf T}(\partial_t^2-\Delta)\Gamma^\gamma Y\big\|_{L^2}\\ &\qquad+\langle t\rangle^{-1/2}\mathcal{E}_{\kappa-2}^{1/2}\mathcal{E}_\kappa^{1/2}\big\|\langle t-r\rangle^{-1}(\omega_j\partial_t+\partial_j)\Gamma^\beta Y\big\|_{L^2}+\langle t\rangle^{-1}\mathcal{E}_{\kappa-2}^{1/2}\mathcal{E}_\kappa.\end{aligned} \tag{5.5}$$

Now let us estimate the remaining terms in (5.5). Using Lemma 3.1 and Lemma 4.2, it is easy to derive that

$$\begin{aligned}&\sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\ne\alpha}}\big\|(\nabla\Gamma^\beta Y)^{\mathsf T}(\partial_t^2-\Delta)\Gamma^\gamma Y\big\|_{L^2}\\ &\lesssim \sum_{\substack{\beta+\gamma=\alpha,\\ |\gamma|\le[|\alpha|/2]}}\|\nabla\Gamma^\beta Y\|_{L^2}\big\|(\partial_t^2-\Delta)\Gamma^\gamma Y\big\|_{L^\infty}\\ &\quad+\sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|\le[|\alpha|/2],\\ \gamma\ne\alpha}}\|\nabla\Gamma^\beta Y\|_{L^\infty}\big\|(\partial_t^2-\Delta)\Gamma^\gamma Y\big\|_{L^2}\\ &\lesssim \mathcal{E}_\kappa^{1/2}\big\|(\partial_t^2-\Delta)\Gamma^{\le\kappa-4}Y\big\|_{L^2}+\langle t\rangle^{-1/2}\mathcal{E}_{\kappa-2}^{1/2}\big\|(\partial_t^2-\Delta)\Gamma^{\le\kappa-2}Y\big\|_{L^2}\\ &\lesssim \langle t\rangle^{-1}\mathcal{E}_\kappa^{1/2}\mathcal{E}_{\kappa-2}.\end{aligned}$$

Inserting the above estimates into (5.5) and using the Cauchy inequality, we have

$$\frac{d}{dt}\sum_{|\alpha|\le\kappa-1}\int e^{-q(\sigma)}\big(|\partial_t\Gamma^\alpha Y|^2+|\nabla\Gamma^\alpha Y|^2\big)dy\lesssim\langle t\rangle^{-1}\mathcal{E}_\kappa\mathcal{E}_{\kappa-2}^{1/2}. \tag{5.6}$$

Here we used $\mathcal{E}_{\kappa-2} \le 1$. This gives the first differential inequality (2.15) at the end of Section 2.

6 Lower-Order Energy Estimate

In this section we perform the lower-order energy estimate. Let $|\alpha| \le \kappa - 3$. We rewrite (2.11) as

$$(\nabla X)^{\mathsf{T}}\big(\partial_t^2 - \Delta\big)\Gamma^\alpha Y + \nabla\Gamma^\alpha p = -\sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\neq\alpha}} C_\alpha^\beta (\nabla\Gamma^\beta Y)^{\mathsf{T}}\big(\partial_t^2 - \Delta\big)\Gamma^\gamma Y.$$

Applying the curl operator to the above equation, one has

$$\begin{aligned}\big(\partial_t^2 - \Delta\big)\nabla^\perp\cdot\Gamma^\alpha Y = -\sum_{\substack{\beta+\gamma=\alpha,\\ \gamma\neq\alpha}} & C_\alpha^\beta\big\{\nabla^\perp\cdot\big[(\nabla\Gamma^\beta Y)^{\mathsf{T}}\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y\big]\\ &- \nabla^\perp\cdot\big[(\nabla Y)^{\mathsf{T}}\big(\partial_t^2-\Delta\big)\Gamma^\alpha Y\big]\big\}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\big(\partial_t^2 - \Delta\big)(-\Delta)^{-1/2}\nabla^\perp\cdot\Gamma^\alpha Y &= \\ \sum_{\beta+\gamma=\alpha} C_\alpha^\beta(-\Delta)^{-1/2}&\nabla^\perp\cdot\big[(\nabla\Gamma^\beta Y)^{\mathsf{T}}\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y\big].\end{aligned} \tag{6.1}$$

Multiplying (6.1) by $\partial_t(-\Delta)^{-1/2}\nabla^\perp\cdot\Gamma^\alpha Y$ and then integrating over $\mathbb{R}^2$, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\big(|\partial_t(-\Delta)^{-1/2}\nabla^\perp\cdot\Gamma^\alpha Y|^2 + |\nabla(-\Delta)^{-1/2}\nabla^\perp\cdot\Gamma^\alpha Y|^2\big)dy\\ &\quad\lesssim \sum_{\beta+\gamma=\alpha}\|\partial_t(-\Delta)^{-1/2}\nabla^\perp\cdot\Gamma^\alpha Y\|_{L^2}\big\||\nabla\Gamma^\beta Y|\big|\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y\big|\big\|_{L^2}\\ &\quad\lesssim \mathcal{E}_{\kappa-2}^{1/2}\sum_{\beta+\gamma=\alpha}\big\||\nabla\Gamma^\beta Y|\big|\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y\big|\big\|_{L^2}.\end{aligned}$$

Let us first use Lemma 3.1, Lemma 4.2, and Lemma 4.3 to estimate that

$$\begin{aligned}&\sum_{\substack{\beta+\gamma=\alpha,\\ |\beta|\le[|\alpha|/2]}}\big\||\nabla\Gamma^\beta Y|\big|\big(\partial_t^2-\Delta\big)\Gamma^\gamma Y\big|\big\|_{L^2}\\ &\qquad\lesssim \|\nabla\Gamma^{\le[|\alpha|/2]}Y\|_{L^\infty}\big\|\big(\partial_t^2-\Delta\big)\Gamma^{\le|\alpha|}Y\big\|_{L^2}\\ &\qquad\lesssim \langle t\rangle^{-3/2}\mathcal{E}_{\kappa-2}^{1/2}\big\|\langle t\rangle\big(\partial_t^2-\Delta\big)\Gamma^{\le\kappa-2}Y\big\|_{L^2}\lesssim \langle t\rangle^{-3/2}\mathcal{E}_{\kappa-2}\mathcal{E}_\kappa^{1/2}.\end{aligned}$$

Similarly, by $[|\alpha|/2]+4\le\kappa-2$, one also has

$$\begin{aligned}
&\sum_{\substack{\beta+\gamma=\alpha,\\ |\gamma|\le[|\alpha|/2]}} \big\||\nabla\Gamma^\beta Y|\big|(\partial_t^2-\Delta)\Gamma^\gamma Y\big|\big\|_{L^2}\\
&\lesssim \|(1-\varphi^t)\nabla\Gamma^{\le\kappa-3}Y\|_{L^\infty}\big\|(1-\varphi^t)(\partial_t^2-\Delta)\Gamma^{\le[|\alpha|/2]}Y\big\|_{L^2}\\
&\quad+\|\varphi^t\nabla\Gamma^{\le\kappa-3}Y\|_{L^2}\big\|\varphi^t(\partial_t^2-\Delta)\Gamma^{\le[|\alpha|/2]}Y\big\|_{L^\infty}\\
&\lesssim \langle t\rangle^{-3/2}\mathcal{E}_\kappa^{1/2}\big\|\langle t\rangle(1-\varphi^t)(\partial_t^2-\Delta)\Gamma^{\le[|\alpha|/2]}Y\big\|_{L^2}\\
&\quad+\langle t\rangle^{-3/2}\mathcal{E}_{\kappa-2}^{1/2}\big\|\langle t\rangle\varphi^t(\partial_t^2-\Delta)\Gamma^{\le[|\alpha|/2]+2}Y\big\|_{L^2}\\
&\lesssim \langle t\rangle^{-3/2}\mathcal{E}_{\kappa-2}\mathcal{E}_\kappa^{1/2}.
\end{aligned}$$

Hence, we have

$$\frac{d}{dt}\sum_{|\alpha|\le\kappa-3}\int|D(-\Delta)^{-1/2}\nabla^\perp\cdot\Gamma^\alpha Y|^2\,dy\lesssim\langle t\rangle^{-3/2}\mathcal{E}_\kappa^{1/2}\mathcal{E}_{\kappa-2}. \tag{6.2}$$

Now let us estimate $(-\Delta)^{-1/2}\nabla\cdot D\Gamma^\alpha Y$. Using (2.12), one has

$$\begin{aligned}
&(-\Delta)^{-1/2}\nabla\cdot D\Gamma^\alpha Y\\
&\quad=(-\Delta)^{-1/2}D\sum_{\beta+\gamma=\alpha}C_\alpha^\beta\big[\partial_1\Gamma^\beta Y^2\partial_2\Gamma^\gamma Y^1-\partial_1\Gamma^\gamma Y^1\partial_2\Gamma^\beta Y^2\big]\\
&\quad=(-\Delta)^{-1/2}\sum_{\beta+\gamma=\alpha}C_\alpha^\beta\big[\partial_1 D\Gamma^\beta Y^2\partial_2\Gamma^\gamma Y^1-\partial_1\Gamma^\gamma Y^1\partial_2 D\Gamma^\beta Y^2\big]\\
&\qquad+(-\Delta)^{-1/2}\sum_{\beta+\gamma=\alpha}C_\alpha^\beta\big[\partial_1\Gamma^\beta Y^2\partial_2 D\Gamma^\gamma Y^1-\partial_1 D\Gamma^\gamma Y^1\partial_2\Gamma^\beta Y^2\big]\\
&\quad=(-\Delta)^{-1/2}\nabla^\perp\cdot\sum_{\beta+\gamma=\alpha}C_\alpha^\beta[D\Gamma^\beta Y^2\nabla\Gamma^\gamma Y^1-\nabla\Gamma^\beta Y^2 D\Gamma^\gamma Y^1].
\end{aligned}$$

Hence, one has

$$\begin{aligned}
\|(-\Delta)^{-1/2}\nabla\cdot D\Gamma^\alpha Y\|_{L^2}&\lesssim\sum_{\beta+\gamma=\alpha}\||D\Gamma^\beta Y||D\Gamma^\gamma Y|\|_{L^2}\\
&\lesssim\|D\Gamma^{\le|\alpha|}Y\|_{L^2}\|D\Gamma^{\le[|\alpha|/2]}\|_{L^\infty}\\
&\lesssim\mathcal{E}_{\kappa-2}.
\end{aligned} \tag{6.3}$$

Hence, we see that $\|(-\Delta)^{-1/2}\nabla^\perp\cdot D\Gamma^{\le\kappa-3}Y\|_{L^2}$ is equivalent to $\|D\Gamma^{\le\kappa-3}Y\|_{L^2}$ since

$$\big|\|D\Gamma^{\le\kappa-3}Y\|_{L^2}^2-\|(-\Delta)^{-1/2}\nabla^\perp\cdot D\Gamma^{\le\kappa-3}Y\|_{L^2}^2\big|\lesssim\mathcal{E}_{\kappa-3}^2\lesssim\epsilon^2\mathcal{E}_{\kappa-2}.$$

Then we can replace all $\mathcal{E}_{\kappa-2}$ appearing throughout this paper by $\|(-\Delta)^{-1/2}\nabla^{\perp}\cdot D\Gamma^{\leq\kappa-3}Y\|_{L^2}^2$ without changing the final result. Then (6.2) gives the second differential inequality (2.16) at the end of Section 2.

Appendix

In this appendix we explain how to obtain (2.11) and (2.12). Let

$$\Gamma_i \in \{\partial_t, \partial_1, \partial_2, \widetilde{\Omega}, \widetilde{S}\}, \quad i = 1, \ldots, 5.$$

Recall that we have defined the scaling and rotation groups in Section 2 so that their generators are $\widetilde{S}$ and $\widetilde{\Omega}$. Similarly, we can define translation groups so that their generators are ∂_t, ∂_1, and ∂_2.

Now for each multi-index α and

$$\Gamma^{\alpha} = \Gamma_1^{\alpha_1} \cdots \widetilde{S}^{\alpha_j} \cdots \Gamma_5^{\alpha_5},$$

we can naturally define the group T_α such that

$$\frac{d^{\alpha_1}}{d\lambda_1^{\alpha_1}} \cdots \frac{d^{\alpha_5}}{d\lambda_5^{\alpha_5}} T_\alpha X\Big|_{(\lambda_1,\ldots,\lambda_5)=e_j} = \Gamma^{\alpha} X. \tag{A.1}$$

Here e_j is the unit vector in $\mathbb{R}^5$ whose j^{th} component is 1. Indeed, $T_\alpha X$ can be defined as follows:

$$T_\alpha X = (T_{\lambda_1})^{\alpha_1} \cdots (T_{\lambda_5})^{\alpha_5} X,$$

where each group T_{λ_j} has a generator Γ_j. A similar definition is applied to $T_\alpha p$.

Due to the invariance property of the system, one has the fact that $(T_\alpha X, T_\alpha p)$ is still a solution of the system (2.5). Consequently, we have

$$\begin{cases} (\nabla T_\alpha X)^{\mathsf{T}}(\partial_t^2 T_\alpha Y - \Delta T_\alpha Y) = -\nabla T_\alpha p, \\ \nabla \cdot T_\alpha Y = -\det(\nabla T_\alpha Y). \end{cases}$$

Clearly, differentiating the above equations with respect to the λ_j's and then using (A.1), one deduces (2.11) and (2.12).

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