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Decay of Fourier modes of solutions to the dissipative surface quasi-geostrophic equations on a finite domain

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ABSTRACT

We consider the two dimensional dissipative surface quasi-geostrophic equation on the unit square with mixed boundary conditions. Under some suitable assumptions on the initial stream function, we obtain existence and uniqueness of solutions in the form of a fast converging trigonometric series. We prove that the Fourier coefficients of solutions have a non-uniform decay: in one direction the decay is exponential and along the other direction it is only power like. We establish global wellposedness for arbitrary large initial data.

1. Introduction and the formulation of the main results

In this paper we are concerned with the following Cauchy problem for the 2D dissipative surface quasi-geostrophic (SQG) equation

$$\begin{cases} \frac{\partial\theta}{\partial t} + u\cdot\nabla\theta = -\nu(-\Delta)^{\gamma/2}\theta, & (t,x,y)\in\mathbb{R}^+\times\Omega; \\ \theta(0,x,y) = \theta_0(x,y), & (x,y)\in\Omega, \end{cases} \tag{1.1}$$

where $\nu > 0$, $\gamma \in (0,2]$ are fixed parameters. The parameter ν is usually called the viscosity coefficient and it controls the strength of the dissipation term. The unknown function $\theta = \theta(t,x): \mathbb{R}^+\times\Omega\to\mathbb{R}$ represents the potential temperature in geostrophic flows and its distribution indicates the temperature at different locations on the earth surface (see [5,26]). The vector-valued function $u = u(t,x,y): \Omega\times\mathbb{R}^+\to\mathbb{R}^2$ is called the velocity and it is expressed in terms of the stream function:

$$u = (u_1,u_2) = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\right). \tag{1.2}$$

The stream function $\psi: \mathbb{R}^+\times\Omega\to\mathbb{R}$ is then related to the so-called "potential temperature" θ by the following nonlocal differentiation:

$$(-\Delta)^{1/2}\psi = \theta. \tag{1.3}$$

Here if the set Ω is the whole of $\mathbb{R}^2$ or the torus, then the fractional Laplacian $(-\Delta)^\alpha$ for any $\alpha\in\mathbb{R}$ is defined by the Fourier transform:

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\hat{f}(\xi), \quad \xi\in\mathbb{R}^2, \tag{1.4}$$

where $\hat{f}$ is the Fourier transform (or Fourier coefficient in the periodic case) of f. In the physical space, the fractional Laplacian has an integral representation by a singular kernel of power law form. If the set Ω is a bounded domain in $\mathbb{R}^2$, then one has to consider (1.1) with appropriate boundary conditions, e.g. one can fix the value of the solution for the whole exterior. In this paper we shall take Ω to be the square in the positive quadrant of the plane with side lengths equal to π. We will impose mixed boundary conditions for the stream function ψ (see below). By using the special boundary conditions, we will express ψ, θ and u in terms of convergent trigonometric series. The fractional Laplacian $(-\Delta)^\alpha$ can then be defined the same way as in (1.4).

The SQG system is a very important model in geostrophy and has been widely used in the study of atmosphere and oceanic flows (cf. [26]). It is derived from the 3D Euler equation and temperature equation in Boussinesq approximation set in a strongly rotating half-space. For SQG the cases when the set $\Omega = \mathbb{R}^2$ or the torus have been widely studied in the literature. The case $\nu = 0$ is called inviscid

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SQG since no dissipation is present. When $\nu > 0$, the cases $\gamma > 1$, $\gamma = 1$ and $\gamma < 1$ are called subcritical, critical and supercritical respectively. The inviscid SQG is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency (see [5,26]). It is an outstanding open problem whether smooth initial data would blow up in finite time. The dissipative SQG (i.e. $\nu > 0$) has been studied intensively. In the subcritical case the global wellposedness result for initial data in certain Sobolev spaces is well-known (cf. [6,2,14,16,24,25] and references therein). In the critical case the global wellposedness of SQG was recently settled by Kiselev, Nazarov and Volberg [17] with C^∞ periodic initial data and by Caffarelli, Vasseur [7] in the whole space case with L^2 initial data (see also the extension by Dong and Du [13]). The problem of global regularity or finite-time blow-up for large initial data in the supercritical case is still open. However some partial results are available (cf. [8,9]) and blowup can occur in a few related models in which u is not divergence free (cf. [1,15,20,19,4] and references therein).

The main purpose of this paper is study the SQG Eq. (1.1) on the two dimensional square with mixed boundary conditions. This is a generalization of our earlier works (see [10,12,11]) where we consider two dimensional Navier–stokes systems with special boundary conditions. The objective is to study the quantitative decay of Fourier coefficients of the solutions depending on the geometry of the underlying domain. As we shall see, in our case the Fourier coefficients will display a rather non-uniform behavior: in the horizontal direction they decay exponentially while in the vertical direction they decay only power-like. This is perhaps a bit surprising as opposed to the torus case where the Fourier coefficients decay uniformly exponentially in time. On the other hand one can attribute this non-uniform decay pattern to the fact that our solution tries to accommodate the mixed boundary conditions. We now formulate more precisely our boundary conditions.

1.1. The mixed boundary conditions for the stream function

Consider the 2D SQG inside the two dimensional square Ω whose sides are equal to π. On the vertical two sides, we assume the stream function ψ and the potential temperature vanishes, i.e.

$$\psi(t,0,y) = \psi(t,\pi,y) = 0, \quad \forall\, t \geqslant 0,\ 0 \leqslant y \leqslant \pi, \tag{1.5}$$
$$\theta(t,0,y) = \theta(t,\pi,y) = 0, \quad \forall\, t \geqslant 0,\ 0 \leqslant y \leqslant \pi. \tag{1.6}$$

On the horizontal two sides, we assume a Neumann type boundary condition, i.e.

$$\frac{\partial\psi}{\partial y}(t,x,0) = \frac{\partial\psi}{\partial y}(t,x,\pi) = 0, \quad \forall\, t \geqslant 0,\ 0 \leqslant x \leqslant \pi, \tag{1.7}$$
$$\frac{\partial\theta}{\partial y}(t,x,0) = \frac{\partial\theta}{\partial y}(t,x,\pi) = 0, \quad \forall\, t \geqslant 0,\ 0 \leqslant x \leqslant \pi. \tag{1.8}$$

The Cauchy problem (1.1)–(1.3) together with the boundary conditions (1.5)–(1.8) then form a complete system for which we shall construct our solutions. Remark that the boundary conditions (1.5), (1.7) in general do not preserve the L^2_x norm of θ, i.e. the quantity

$$\int_0^\pi \int_0^\pi \theta(t,x,y)^2 dxdy$$

may grow in time. This is in sharp contrast to the usual case when $\Omega = \mathbb{R}^2$ or the torus, where one can prove that the L^2_x norm (more generally L^p_x norm, see [3]) of θ does not increase in time. As will become clear soon, the absence of L^2_x conservation is an obstruction to global wellposedness even when the dissipation is subcritical (i.e. $1 < \gamma \leqslant 2$).

We shall consider solutions of SQG with finite L^2_x norm. Therefore by (1.5)–(1.7) the stream function ψ can be expanded into Fourier series:

$$\psi(t,x,y) = \sum_{m\geqslant 1, n\geqslant 0} h(t,m,n)\sin mx \cos ny. \tag{1.9}$$

We assume that this series and all series below like (1.9) converge fast enough so that formal operations like differentiations are possible. The formal operation can be easily justified once we prove the decay of the corresponding Fourier coefficients. By (1.3) and (1.4), we obtain

$$\theta(t,x,y) = \sum_{m\geqslant 1, n\geqslant 0} (m^2+n^2)^{\frac{1}{2}} \cdot h(t,m,n) \cdot \sin mx \cos ny. \tag{1.10}$$

Therefore

$$\frac{\partial\theta}{\partial x}(t,x,y) = \sum_{m\geqslant 1, n\geqslant 0} (m^2+n^2)^{\frac{1}{2}} \cdot m \cdot h(t,m,n) \cdot \cos mx \cos ny, \tag{1.11}$$

$$\frac{\partial\theta}{\partial y}(t,x,y) = \sum_{m\geqslant 1, n\geqslant 0} (m^2+n^2)^{\frac{1}{2}} \cdot (-n) \cdot \sin mx \sin ny. \tag{1.12}$$

For the expansion for the velocities, we use (1.2) to obtain

$$u_1(t,x,y) = \sum_{m\geqslant 1, n\geqslant 0} n \cdot h(t,m,n) \cdot \sin mx \sin ny, \tag{1.13}$$
$$u_2(t,x,y) = \sum_{m\geqslant 1, n\geqslant 0} m \cdot h(t,m,n) \cdot \cos mx \cos ny. \tag{1.14}$$

Note that in the expansion for $\frac{\partial\theta}{\partial y}$ and u_1 the effective summation actually only extends over $n \geqslant 1$ since the term $n = 0$ vanishes. We shall make use of the expansions (1.9)–(1.14) to derive an ODE system for the Fourier coefficients $h(t,m,n)$, $m \geqslant 1$, $n \geqslant 0$.

1.2. System of ODE for h(t,m,n)

By (1.10) and (1.4), we get

$$\begin{aligned}\left(\frac{\partial\theta}{\partial t} + \nu(-\Delta)^{\gamma/2}\theta\right)(t,x,y) = \sum_{m\geqslant 1, n\geqslant 0} \Big(&(m^2+n^2)^{\frac{1}{2}} \cdot \dot{h}(t,m,n) \\ &+ \nu \cdot (m^2+n^2)^{\frac{\gamma+1}{2}} \cdot h(t,m,n)\Big) \\ &\times \sin mx \cos ny.\end{aligned} \tag{1.15}$$

Here $\dot{h}$ denotes time differentiation.

By using (1.11) and (1.13), we have

$$\left(u_1\cdot\frac{\partial\theta}{\partial x}\right)(t,x,y)=\left(\sum_{m'\geqslant 1,n'\geqslant 0} n'\cdot h(t,m',n')\cdot\sin m'x\sin n'y\right)$$
$$\cdot\left(\sum_{m''\geqslant 1,n''\geqslant 0}((m'')^2+(n'')^2)^{\frac{1}{2}}\cdot m''\cdot h(t,m'',n'')\right.$$
$$\left.\cdot\cos m''x\cos n''y\right)$$
$$=\frac{1}{4}\sum_{\substack{m'\geqslant 1,m''\geqslant 1\\ n'\geqslant 0,n''\geqslant 0}} n'\cdot((m'')^2$$
$$+(n'')^2)^{\frac{1}{2}}\cdot m''\cdot h(t,m',n')\cdot h(t,m'',n'')$$
$$\cdot(\sin(m'+m'')x+\sin(m'-m'')x)$$
$$\cdot(\sin(n'+n'')y+\sin(n'-n'')y), \tag{1.16}$$

where we have used the trigonometric formula

$$\sin\alpha\cos\beta=\frac{1}{2}\sin(\alpha+\beta)+\frac{1}{2}\sin(\alpha-\beta).$$

Similarly by using (1.12) and (1.14) we compute

$$\left(u_2\cdot\frac{\partial\theta}{\partial y}\right)(t,x,y)=\frac{1}{4}\sum_{\substack{m'\geqslant 1,m''\geqslant 1\\ n'\geqslant 0,n''\geqslant 0}} m'\cdot((m'')^2$$
$$+(n'')^2)^{\frac{1}{2}}\cdot(-n'')\cdot h(t,m',n')\cdot h(t,m'',n'')$$
$$\cdot(\sin(m'+m'')x-\sin(m'-m'')x)$$
$$\cdot(\sin(n'+n'')y-\sin(n'-n'')y), \tag{1.17}$$

Adding together (1.16) and (1.17) and grouping the sums, we obtain

$$u_1\cdot\frac{\partial\theta}{\partial x}+u_2\cdot\frac{\partial\theta}{\partial y}=\sum_{m\geqslant 1,n\geqslant 1}\sin mx\sin ny$$
$$\cdot\left(\frac{1}{4}\sum_{\substack{m'\geqslant 1,n'\geqslant 0,m''\geqslant 1,n''\geqslant 0\\ m'\pm m''=\pm m\\ n'\pm n''=\pm n}} h(t,m',n')h(t,m'',n'')\right.$$
$$\cdot((m'')^2+(n'')^2)^{\frac{1}{2}}$$
$$\left.\cdot\left(m''n'\cdot(-1)^{v_1(m',m'',m)+v_1(n',n'',n)}-m'n''\cdot(-1)^{v_2(m',m'',m)+v_2(n',n'',n)}\right)\right). \tag{1.18}$$

Here v_1, v_2 are integer functions defined as

$$v_1(m',m'',m)=\begin{cases}1, & \text{if } m'-m''=-m,\\ 0, & \text{otherwise.}\end{cases}$$

$$v_2(m',m'',m)=\begin{cases}1, & \text{if } m'-m''=m,\\ 0, & \text{otherwise.}\end{cases}$$

The formulae (1.15) and (1.18) are not enough for giving the equations for $h(t,m,n)$ and an additional step is needed. We remark that the sequence $\{\cos nx, n\geqslant 0\}$ is an orthogonal basis in the Hilbert space $L^2([0,\pi],dx)$. Therefore we can write

$$\sin nx=\sum_{\tilde{n}\geqslant 0}\Gamma(n,\tilde{n})\cos\tilde{n}x, \tag{1.19}$$

where

$$\Gamma(n,\tilde{n})=\begin{cases}\frac{4n}{(n^2-\tilde{n}^2)\pi}, & \text{if } n-\tilde{n} \text{ is odd, } n\geqslant 1,\tilde{n}\geqslant 1,\\ \frac{2}{n\pi}, & \text{if } n\geqslant 1,\tilde{n}=0 \text{ and } n \text{ is odd,}\\ 0, & \text{otherwise.}\end{cases} \tag{1.20}$$

Consider again the formula (1.18). In the product $\sin mx\cdot\sin ny$ we replace the second factor by the series

$$u_1\cdot\frac{\partial\theta}{\partial x}+u_2\cdot\frac{\partial\theta}{\partial y}=\sum_{m\geqslant 1,\tilde{n}\geqslant 0}\sin mx\cos\tilde{n}y\sum_{n\geqslant 1}\Gamma(n,\tilde{n})\left(\frac{1}{4}\sum_{\substack{m'\geqslant 1,n'\geqslant 0,m''\geqslant 1,n''\geqslant 0\\ m'\pm m''=\pm m\\ n'\pm n''=\pm n}} h(t,m',n')h(t,m'',n'')\right.$$
$$\left.\cdot((m'')^2+(n'')^2)^{\frac{1}{2}}\cdot\left(m''n'\cdot(-1)^{v_1(m',m'',m)+v_1(n',n'',n)}-m'n''\cdot(-1)^{v_2(m',m'',m)+v_2(n',n'',n)}\right)\right)$$
$$=\sum_{m\geqslant 1,n\geqslant 0}\sin mx\cos ny\sum_{k\geqslant 1}\Gamma(k,n)\left(\frac{1}{4}\sum_{\substack{m'\geqslant 1,n'\geqslant 0,m''\geqslant 1,n''\geqslant 0\\ m'\pm m''=\pm m\\ n'\pm n''=\pm k}} h(t,m',n')h(t,m'',n'')\cdot((m'')^2\right.$$
$$\left.+(n'')^2)^{\frac{1}{2}}\cdot\left(m''n'\cdot(-1)^{v_1(m',m'',m)+v_1(n',n'',k)}-m'n''\cdot(-1)^{v_2(m',m'',m)+v_2(n',n'',k)}\right)\right). \tag{1.21}$$

(1.19) and this gives us a series w.r.t. $\sin mx \cdot \cos ny$, The equality between the coefficients of the series of this new expression and Eq.(1.15) will give us the needed system of equations for $h(t,m,n)$. Collecting the formulae 1.1,1.15,1.21, we obtain

$$\dot{h}(t,m,n) + \frac{1}{(m^2+n^2)^{\frac{1}{2}}} N(t,m,n) = -\nu \cdot (m^2+n^2)^{\frac{\gamma}{2}} h(t,m,n), \quad \forall\, m \geqslant 1,\ n \geqslant 0. \tag{1.22}$$

Here for $m \geqslant 1$, $n \geqslant 0$,

$$N(t,m,n) = \sum_{k\geqslant 1} \Gamma(k,n) \left(\frac{1}{4} \sum_{\substack{m' \geqslant 1, n' \geqslant 0, m'' \geqslant 1, n'' \geqslant 0 \\ m' \pm m'' = \pm m \\ n' \pm n'' = \pm k}} h(t,m',n') h(t,m'',n'') \cdot ((m'')^2 + (n'')^2)^{\frac{1}{2}} \cdot \left(m''n' \cdot (-1)^{v_1(m',m'',m)+v_1(n',n'',k)} - m'n'' \cdot (-1)^{v_2(m',m'',m)+v_2(n',n'',k)} \right) \right). \tag{1.23}$$

The RHS of (1.22) describes the influence of viscosity. The infinite system of Eq. (1.22) is our basic ODE system for the coefficients $h(t,m,n)$, $m \geqslant 1$, $n \geqslant 0$. We now state the main results of this paper.

Theorem 1.1 (*Wellposedness and mixed decay*). *Let $\nu > 0$ and $1 < \gamma \leqslant 2$. Let $h(0,m,n)$ satisfy the inequalities*

$$|h(0,m,n)| \leqslant \frac{D_0}{m^\alpha (n+1)^\beta} \cdot \frac{1}{(m^2+n^2)^{\frac{1}{2}}}, \quad \forall\, m \geqslant 1,\ n \geqslant 0, \tag{1.24}$$

and $\alpha > 2$, $2 < \beta < 3$, $D_0 > 0$. Then there exists a time $T = T(D_0,\alpha,\beta,\nu,\gamma) > 0$, a constant $D_1 = D_1(D_0,\alpha,\beta,\nu,\gamma) > 0$, such that (1.23) has a unique solution $h(t,m,n)$ which satisfies for all $0 \leqslant t < T$ the inequalities

$$|h(t,m,n)| \leqslant \frac{D_1}{m^\alpha (n+1)^\beta} \cdot e^{-\frac{m}{2}\nu t} \cdot \frac{1}{(m^2+n^2)^{\frac{1}{2}}}, \quad \forall\, m \geqslant 1,\ n \geqslant 0. \tag{1.25}$$

In fact $h(t,m,n)$ satisfies an even stronger inequality. For any $0 < t_0 < T$, there exists a constant $D_2 = D_2(D_0,\alpha,\beta,\nu,\gamma,t_0) > 0$ such that for any $t_0 \leqslant t < T$, we have

$$|h(t,m,n)| \leqslant \frac{D_2}{(n+1)^{3+\gamma}} e^{-\frac{m}{10}\nu t}, \quad \forall\, m \geqslant 1,\ n \geqslant 0. \tag{1.26}$$

Finally if D_0 is sufficiently small, then the corresponding solution is global and the estimates (1.25) and (1.26) hold for $T = +\infty$.

Table 1
Power-law decay rates of the Fourier modes: the decay in n is given by b_1 and the decay in m is given by b_2.

t	$D_0 = 1$				$D_0 = 10$			
	$K = 20$		$K = 30$		$K = 20$		$K = 30$	
	b_1	b_2	b_1	b_2	b_1	b_2	b_1	b_2
0.00	2.0	2.5	2.1	2.5	2.0	2.5	2.1	2.5
0.05	3.2	3.9	3.6	5.0	2.3	3.9	2.6	5.0
0.10	3.7	5.3	4.1	7.5	2.6	5.3	2.7	7.4
0.15	4.1	6.8	4.3	9.9	2.8	6.5	2.8	9.7
0.20	4.4	8.2	4.4	12.4	2.9	7.9	2.9	12.1
0.30	4.6	10.9	4.6	16.9	2.9	9.7	2.9	13.3
0.40	4.6	13.6	4.5	20.6	3.0	10.2	3.0	14.1
0.50	4.5	16.1	4.3	23.7	3.0	10.7	3.0	14.9
0.60	4.4	18.3	4.2	26.2	2.9	11.2	2.9	15.9

Remark 1.2. The decay assumption (1.24) is a bit unusual: namely the decay in the n-direction is only power-like and limited to a certain range (n^{-2},n^{-3}) (if we count the factor $1/(m^2+n^2)$ then it is (n^{-3}, n^{-4})). The upper bound $\beta < 3$ is actually not necessary for the initial data. The main point is that only the regularity (n^{-2}, n^{-3}) is propagated by the nonlinear flow (see (1.25)). Such a slow decay is connected with our special boundary condition and the switching of Fourier basis when we make the nonlinear estimates (see (1.19)). It is possible that one can work with other types of solutions (say weak solutions) and push down further the regularity assumption in (1.24). We shall not dwell on this issue here.

Remark 1.3. The inequality (1.26) shows that the smoothing effect of the fractional Laplacian is non-uniform: in the horizontal direction the solution is infinitely smooth while in the vertical direction it has finite number of derivatives on the boundary. By Theorem 1.1 and especially the decay estimate (1.25) and (1.26), we obtain a classical solution to (1.1) satisfying aforementioned boundary conditions (1.5) and (1.7). In particular it is not difficult to check that θ has strong derivatives up to the boundary and therefore the boundary conditions (1.5) and (1.7) hold in the usual sense.

Theorem 1.1 is a bit unsatisfactory since it is a local result. Our next theorem establishes global wellposedness for arbitrary large initial data.

Theorem 1.4 (*Global wellposedness for large data*). *Let $\nu > 0$ and $1 < \gamma \leqslant 2$ be the same as in Theorem 1.1. Let $h(0,m,n)$ satisfy (1.24). Then the corresponding solution constructed in Theorem 1.1 exists globally.*

The proof of Theorem 1.4 relies on an important Hopf-type maximum principle (Proposition 4.1 in Section 4) which gives a priori control of L^∞ norm of the solution. The main work needed to continue the solution is to upgrade this L^∞ estimate to Sobolev estimates needed in Theorem 1.1. The proof of Theorem 1.4 is given in Section 4.

Remark 1.5. With some small modifications, our method here applies to the 2D Navier–Stokes system considered in [12] and establishes global wellposedness for arbitrary large initial data. It is worthwhile pointing out the difference between 2D Navier–Stokes equation and the 2D surface quasi-gestrophic equation. In vorticity formulation, the 2D Navier–Stokes equation takes the form

$$\partial_t \omega + (u \cdot \nabla)\omega = \Delta\omega,$$

where $u = \nabla^\perp \psi$ and $\Delta\psi = \omega$. The potential temperature θ in (1.1) is analogous to the vorticity ω. Formally speaking, the advection velocity u in (1.1) scales like θ (up to a Riesz transform). On the other hand, in 2D Navier–stokes the

velocity u scales like $(-\Delta)^{-\frac{1}{2}}\omega$ which is more "smoothing" than the quasi-gestrophic case. This is the main reason why in general the analysis of the quasi-geostrophic equation is more difficult than the 2D Navier–Stokes equation.

2. Numerical simulations

To verify and illustrate the conclusions of Theorem 1.1, we have computed the Fourier modes $h(t,m,n)$ and $f(t,m)$ in a numerical experiment where the initial values were set to

$$h(0,m,n) = \frac{D_0}{m^\alpha (n+1)^\beta} \cdot \frac{1}{(m^2+n^2)^{\frac{1}{2}}}, \qquad \forall\, m \geqslant 1, n \geqslant 0 \tag{2.1}$$

with $\alpha = 2.5$ and $\beta = 2.5$ and some $D_0 > 0$, so that the assumptions (1.24) are valid. We set $\nu = 1$ and $\gamma = 1.5$.

In our numerical solution of the ODE system (1.22) we restricted the indices m,n to a finite interval $1 \leqslant m \leqslant K$, $0 \leqslant n \leqslant K$ (which is a Galerkin approximation to the infinite system (1.22)). Then we solved the resulting finite-dimensional system numerically by the classical Runge–Kutta method. To test the accuracy we have changed the Galerkin size parameter K and the time step Δt in the Runge–Kutta scheme several times to make sure that our results remained stable.

After computing the Fourier modes $h(t, m, n)$ we estimated their decay rates in m and n by approximating their logarithms

$$\tilde{h}(t,m,n) = \log|h(t,m,n)|$$

by two linear functions:

$$\tilde{h}(t,m,n) = a_1 - b_1 n \quad (\text{when } m \text{ is kept fixed}),$$
$$\tilde{h}(t,m,n) = a_2 - b_2 m \quad (\text{when } n \text{ is kept fixed}).$$

The slopes b_1 and b_2 represent the powers of the decay rates of $h(t, m, n)$ in n and m, respectively. Since the value of b_1 depends on m and the value of b_2 depends on n, we averaged the values of b_1 over $m = 1, 2, 3$ and averaged the values of b_2 over $n = 0, 1, 2$.

Table 1 shows how the computed values of b_1 and b_2 change in time. We see that the decay rates in m (given by b_2) increase steadily (in the case $D_0 = 1$ it appears that b_2 grows linearly in t, while in the case $D_0 = 10$ the growth is less regular), indicating that the true decay becomes faster than any power function, which is consistent with the exponential bound (1.25). On the other hand, the decay rate in n (the value of b_1), after a short initial growth, stabilizes and in fact starts decreasing. Interestingly, for $D_0 = 1$ the value of b_1 stabilizes near $3 + \gamma = 4.5$, which agrees with the stronger bound (1.26).

Our results support the conclusions and conjectures stated in Theorem 1.1: the decay of the Fourier modes in m is indeed much faster than the decay in n. Actually, the former is faster than any power function, while the latter remains power-like.

3. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. Let $h(0, m, n)$ satisfy (1.24). The system (1.22) in the integral form can be written as

$$h(t,m,n) = e^{-\nu(m^2+n^2)^{\frac{\gamma}{2}}t} h(0,m,n) - \int_0^t \frac{1}{(m^2+n^2)^{\frac{1}{2}}} e^{-\nu(m^2+n^2)^{\frac{\gamma}{2}}s} N(t-s,m,n)ds, \quad \forall\, m \geqslant 1,\ n \geqslant 0. \tag{3.1}$$

Note here the terms $N(t,m,n)$, $m \geqslant 1$, $n \geqslant 0$ are nonlinear functionals of $h(t,m,n)$, $m \geqslant 1$, $n \geqslant 1$. We then seek a solution of (3.1) by iterations. Define the iterates

$$h^{(1)}(t,m,n) = e^{-(m^2+n^2)^{\gamma/2}\nu t} h(0,m,n), \quad \forall\, m \geqslant 1,\ n \geqslant 0, \tag{3.2}$$

and for $j \geqslant 2$

$$h^{(j)}(t,m,n) = h^{(1)}(t,m,n) - \int_0^t \frac{1}{(m^2+n^2)^{\frac{1}{2}}} e^{-\nu(m^2+n^2)^{\frac{\gamma}{2}}s} N^{(j-1)} \times (t-s,m,n)ds, \quad \forall\, m \geqslant 1,\ n \geqslant 0. \tag{3.3}$$

Here for $m \geqslant 1$, $n \geqslant 0$,

$$N^{(j-1)}(t,m,n) = \sum_{k \geqslant 1} \Gamma(k,n) \left(\frac{1}{4} \sum_{\substack{m' \geqslant 1, n' \geqslant 0, m'' \geqslant 1, n'' \geqslant 0 \\ m' \pm m'' = \pm m \\ n' \pm n'' = \pm k}} h^{(j-1)}(t,m',n') h^{(j-1)} (t,m'',n'') \cdot ((m'')^2 + (n'')^2)^{\frac{1}{2}} \cdot \left(m''n' \cdot (-1)^{\nu_1(m',m'',m)+\nu_1(n',n'',k)} - m'n'' \cdot (-1)^{\nu_2(m',m'',m)+\nu_2(n',n'',k)} \right) \right). \tag{3.4}$$

We begin with the following two lemmas which were proved in [12].

Lemma 3.1. *Let $0 < \tilde{\alpha} < \infty$, $n \geqslant 0$. There exists a constant $C_1 > 0$ depending only on $\tilde{\alpha}$, such that*

$$\sum_{\substack{k \neq n \\ k \geqslant 1}} |\Gamma(k,n)| \cdot \frac{1}{k^{\tilde{\alpha}}} \leqslant \begin{cases} C_1, & \text{if } 0 \leqslant n \leqslant 4 \\ C_1 \cdot \frac{\log n}{n^{\tilde{\alpha}}}, & \text{if } 0 < \tilde{\alpha} \leqslant 2, n \geqslant 5 \\ C_1 \cdot \frac{1}{n^2}, & \text{if } \tilde{\alpha} > 2, n \geqslant 5. \end{cases} \tag{3.5}$$

Here $\Gamma(k,n)$ is defined in (1.20).

Proof 1. See [12]. □

Lemma 3.2. *Let $\alpha_1 > 1$, $\alpha_2 > 1$. Let $k \geqslant 1$ be an integer. There is a constant $C_2 > 0$ depending only on (α_1,α_2) such that*

$$\sum_{\substack{k_1 \geqslant 1,\ k_2 \geqslant 1 \\ |k_1 \pm k_2| = k}} \frac{1}{k_1^{\alpha_1}} \cdot \frac{1}{k_2^{\alpha_2}} \leqslant C_2 \cdot \frac{1}{k^{\alpha_3}}, \tag{3.6}$$

where $\alpha_3 = \min\{\alpha_1, \alpha_2\}$.

Proof 2. See [12]. □

Lemma 3.3. *Let $\alpha > 2$, $\beta > 2$. Let $m \geqslant 1$, $k \geqslant 1$ be integers. There exists a constant $C_3 > 0$ depending only on (α,β) such that*

$$\sum_{\substack{m' \geqslant 1, n' \geqslant 0, m'' \geqslant 1, n'' \geqslant 0 \\ |m' \pm m''| = m \\ |n' \pm n''| = k}} \frac{1}{(m')^{\alpha}(n'+1)^{\beta}} \cdot \frac{1}{(m'')^{\alpha}(n''+1)^{\beta}} \cdot \frac{m'n''}{\left((m')^2 + (n')^2\right)^{\frac{1}{2}}} \leqslant C_3 \cdot \frac{1}{m^{\alpha}k^{\beta-1}}, \tag{3.7}$$

$$\sum_{\substack{m' \geqslant 1, n' \geqslant 0, m'' \geqslant 1, n'' \geqslant 0 \\ |m' \pm m''| = m \\ |n' \pm n''| = k}} \frac{1}{(m')^{\alpha}(n'+1)^{\beta}} \cdot \frac{1}{(m'')^{\alpha}(n''+1)^{\beta}} \cdot \frac{m''n'}{\left((m')^2 + (n')^2\right)^{\frac{1}{2}}} \leqslant C_3 \cdot \frac{1}{m^{\alpha-1}k^{\beta}}. \tag{3.8}$$

Proof of Lemma 3.3. We first deal with (3.7). By Lemma 3.2,

$$\begin{aligned}
&|\text{LHS of}(3.7)| \\
&\leqslant \sum_{\substack{m' \geqslant 1, n' \geqslant 0, m'' \geqslant 1, n'' \geqslant 0 \\ |m' \pm m''| = m \\ |n' \pm n''| = k}} \frac{1}{(m')^{\alpha-1}(n'+1)^{\beta}} \cdot \frac{1}{(m'')^{\alpha}(n''+1)^{\beta-1}} \cdot \frac{1}{((m')^2+(n')^2)^{\frac{1}{2}}} \\
&\leqslant \sum_{\substack{m', m'' \geqslant 1 \\ |m' \pm m''| = m}} \frac{1}{(m')^{\alpha}(m'')^{\alpha}} \cdot \sum_{\substack{n', n'' \geqslant 0 \\ |n' \pm n''| = k}} \frac{1}{(n'+1)^{\beta}(n''+1)^{\beta-1}} \\
&\leqslant C_3 \cdot \frac{1}{m^{\alpha}k^{\beta-1}}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&|\text{LHS of}(3.8)| \\
&\leqslant \sum_{\substack{m' \geqslant 1, n' \geqslant 0, m'' \geqslant 1, n'' \geqslant 0 \\ |m' \pm m''| = m \\ |n' \pm n''| = k}} \frac{1}{(m')^{\alpha}(n'+1)^{\beta-1}} \cdot \frac{1}{(m'')^{\alpha-1}(n''+1)^{\beta}} \cdot \frac{1}{\left((m')^2+(n')^2\right)^{\frac{1}{2}}} \\
&\leqslant \sum_{\substack{m', m'' \geqslant 1 \\ |m' \pm m''| = m}} \frac{1}{(m')^{\alpha}(m'')^{\alpha-1}} \cdot \sum_{\substack{n', n'' \geqslant 0 \\ |n' \pm n''| = k}} \frac{1}{(n'+1)^{\beta}(n''+1)^{\beta}} \\
&\leqslant C_3 \cdot \frac{1}{m^{\alpha-1}k^{\beta}}. \quad \square
\end{aligned}$$

Now for any $\alpha > 2$, $\beta > 2$, $T > 0$, we introduce the Banach space $X_{\alpha,\beta,T}$ consisting of continuous functions $\tilde{h}(t) = (\tilde{h}(t,m,n))_{m \geqslant 1, n \geqslant 0}$, endowed with the norm

$$\|\tilde{h}\|_{X_{\alpha,\beta,T}} := \sup_{0 \leqslant t \leqslant T} \sup_{m \geqslant 1, n \geqslant 0} |\tilde{h}(t,m,n)| \cdot (m^2+n^2)^{\frac{1}{2}} \cdot m^{\alpha}(n+1)^{\beta} \cdot e^{\frac{1}{2}m\nu t},$$

We will prove Theorem 1.1 by a contraction argument in the space $X_{\alpha,\beta,T}$ for some sufficiently small $T > 0$.

Proof of Theorem 1. Let $\nu > 0$, $1 < \gamma \leqslant 2$. Define the iterations according to (3.2) and (3.3). We first show that if T is sufficiently small depending on $(D_0, \alpha, \beta, \nu, \gamma)$, then

$$\|h^{(j)}\|_{X_{\alpha,\beta,T}} \leqslant 2D_0, \quad \forall\, j \geqslant 1. \tag{3.9}$$

By (1.24) and (3.2), we have

$$\|h^{(1)}\|_{X_{\alpha,\beta,T}} \leqslant \sup_{\substack{m \geqslant 1, n \geqslant 0 \\ 0 \leqslant t \leqslant T}} D_0 \cdot e^{-\left((m^2+n^2)^{\frac{\gamma}{2}} - \frac{m}{2}\right)\nu t} \leqslant D_0,$$

where the last inequality follows from the fact that $m \geqslant 1$, $n \geqslant 0$ and $1 < \gamma \leqslant 2$. Assume (3.9) holds for $1 \leqslant j < j_0$, $j_0 \geqslant 2$. Then for $j = j_0$, $n \geqslant 0$, we have
Now remark that if $|m' \pm m''| = m$, then $m' + m'' \geqslant m$. Therefore by Lemmas 3.1–3.3, we have

$$\begin{aligned}
|\text{RHS of } (3.10)| &\leqslant D_0^2 \cdot e^{-\frac{1}{2}m\nu t} \cdot \sum_{k \geqslant 1} |\Gamma(k,n)| \cdot C_3 \cdot \left(\frac{1}{m^{\alpha-1}k^{\beta}} + \frac{1}{m^{\alpha}k^{\beta-1}}\right) \\
&\leqslant C_4 \cdot e^{-\frac{1}{2}m\nu t} \cdot \left(\frac{1}{m^{\alpha-1}(n+1)^{\beta}} + \frac{\log(n+5)}{m^{\alpha}(n+1)^{\beta-1}}\right),
\end{aligned}$$

$$\begin{aligned}
|N^{(j_0-1)}(t,m,n)| &\leqslant \frac{1}{4}\sum_{k \geqslant 1} |\Gamma(k,n)| \left(\sum_{\substack{m' \geqslant 1, n' \geqslant 0, m'' \geqslant 1, n'' \geqslant 0 \\ m' \pm m'' = \pm m \\ n' \pm n'' = \pm k}} |h^{(j_0-1)}(t,m',n')| \cdot |h^{(j_0-1)}(t,m'',n'')| \cdot ((m'')^2 + (n'')^2)^{\frac{1}{2}} \cdot (m''n' + m'n'') \right) \\
&\leqslant D_0^2 \sum_{k \geqslant 1} |\Gamma(k,n)| \cdot \left(\sum_{\substack{m' \geqslant 1, n' \geqslant 0, m'' \geqslant 1, n'' \geqslant 0 \\ m' \pm m'' = \pm m \\ n' \pm n'' = \pm k}} e^{-\frac{1}{2}(m'+m'')\nu t} \cdot \frac{1}{((m')^2+(n')^2)^{\frac{1}{2}}} \cdot \frac{1}{(m')^{\alpha}(n'+1)^{\beta}} \cdot \frac{m''n' + m'n''}{(m'')^{\alpha} \cdot (n''+1)^{\beta}} \right).
\end{aligned} \tag{3.10}$$

where C_3, C_4 are constants depending only on (D_0, α, β). Substituting this estimate into (3.3), we get

$$\begin{aligned}|h^{(j_0)}(t,m,n)| &\leqslant e^{-\nu(m^2+n^2)^{\frac{\gamma}{2}}t}\cdot\frac{1}{(m^2+n^2)^{\frac{1}{2}}}\cdot\frac{D_0}{m^\alpha(n+1)^\beta}\\ &\quad+\frac{1}{(m^2+n^2)^{\frac{1}{2}}}\cdot C_4\cdot\int_0^t e^{-(m^2+n^2)^{\frac{\gamma}{2}}\nu s}\\ &\quad\cdot e^{-\frac{1}{2}\nu m(t-s)}ds\\ &\quad\cdot\left(\frac{1}{m^{\alpha-1}(n+1)^\beta}+\frac{\log(n+5)}{m^\alpha(n+1)^{\beta-1}}\right)\\ &\leqslant e^{-\nu(m^2+n^2)^{\frac{\gamma}{2}}t}\cdot\frac{1}{(m^2+n^2)^{\frac{1}{2}}}\cdot\frac{D_0}{m^\alpha(n+1)^\beta}\\ &\quad+\frac{1}{(m^2+n^2)^{\frac{1}{2}}}\cdot C_4\cdot e^{-\frac{1}{2}\nu mt}\\ &\quad\times\int_0^t e^{-\nu((m^2+n^2)^{\frac{\gamma}{2}}-\frac{1}{2}m)s}ds\\ &\quad\cdot\left(\frac{1}{m^{\alpha-1}(n+1)^\beta}+\frac{\log(n+5)}{m^\alpha(n+1)^{\beta-1}}\right).\end{aligned}\tag{3.11}$$

Now since $m\geqslant 1$, $n\geqslant 0$, $1<\gamma\leqslant 2$, we have

$$\frac{1}{2}(m^2+n^2)^{\frac{\gamma}{2}}\geqslant\frac{1}{2}m,$$

and therefore

$$\int_0^t e^{-\nu((m^2+n^2)^{\frac{\gamma}{2}}-\frac{1}{2}m)s}ds\leqslant\int_0^t e^{-\frac{\nu}{2}(m^2+n^2)^{\frac{\gamma}{2}}s}ds\leqslant\frac{2}{\nu}\cdot\frac{1-e^{-\frac{\nu}{2}(m^2+n^2)^{\frac{\gamma}{2}}t}}{(m^2+n^2)^{\frac{\gamma}{2}}}.$$

Plugging this estimate into the RHS of (3.11), we obtain

$$\begin{aligned}|\text{RHS of (3.11)}| &\leqslant e^{-\nu(m^2+n^2)^{\frac{\gamma}{2}}t}\cdot\frac{1}{(m^2+n^2)^{\frac{1}{2}}}\cdot\frac{D_0}{m^\alpha(n+1)^\beta}\\ &\quad+e^{-\frac{1}{2}\nu mt}\cdot\frac{1}{(m^2+n^2)^{\frac{1}{2}}}\cdot C_4\cdot\frac{2}{\nu}\cdot\frac{1}{(m^2+n^2)^{\frac{\gamma+1}{2}}}\\ &\quad\cdot\frac{1-e^{-\frac{\nu}{2}(m^2+n^2)^{\frac{\gamma}{2}}t}}{\left((m^2+n^2)^{\frac{\gamma}{2}}\cdot\frac{\nu}{2}t\right)^{\frac{\gamma-1}{\gamma}}}\cdot\left(\frac{\nu}{2}\right)^{\frac{\gamma-1}{\gamma}}\cdot t^{\frac{\gamma-1}{\gamma}}\\ &\quad\cdot\left(\frac{1}{m^{\alpha-1}(n+1)^\beta}+\frac{\log(n+5)}{m^\alpha(n+1)^{\beta-1}}\right).\end{aligned}\tag{3.12}$$

Now remark that since $1<\gamma\leqslant 2$,

$$\sup_{x>0}\frac{1-e^{-x}}{x^{\frac{\gamma-1}{\gamma}}}\leqslant C_5<\infty,$$

where C_5 is a constant depending only on γ. Also it is clear that

$$\begin{aligned}&\left(\frac{\nu}{2}\right)^{-\frac{1}{\gamma}}\cdot\frac{1}{(m^2+n^2)^{\frac{\gamma+1}{2}}}\cdot\left(\frac{1}{m^{\alpha-1}(n+1)^\beta}+\frac{\log(n+5)}{m^\alpha(n+1)^{\beta-1}}\right)\\ &\quad\leqslant C_6\cdot\frac{1}{m^\alpha}\cdot\frac{1}{(n+1)^\beta},\end{aligned}$$

where C_6 is another constant depending only on (D_0, α, β, γ, ν). Substituting the above two estimates into the RHS of (3.12), we obtain

$$\begin{aligned}|h^{(j_0)}(t,m,n)| &\leqslant e^{-\frac{1}{2}\nu mt}\cdot\frac{1}{(m^2+n^2)^{\frac{1}{2}}}\cdot\frac{1}{m^\alpha(n+1)^\beta}\cdot(D_0+t^{\frac{\gamma-1}{\gamma}}\\ &\quad\cdot C_4\cdot C_5\cdot C_6).\end{aligned}$$

Let $T\leqslant\left(\frac{D_0}{C_4\cdot C_5\cdot C_6}\right)^{\frac{\gamma}{\gamma-1}}$. Then for $0<t<T$ we get

$$|h^{(j_0)}(t,m,n)|\leqslant e^{-\frac{1}{2}\nu mt}\cdot\frac{1}{(m^2+n^2)^{\frac{1}{2}}}\cdot\frac{1}{m^\alpha(n+1)^\beta}\cdot 2D_0,\tag{3.13}$$

which implies that (3.9) holds for $j=j_0$. This finishes the induction step and (3.9) is proved for all $j\geqslant 1$. By essentially repeating the above estimates, we also obtain strong contraction of the sequence $h^{(j)}(t)$. Namely there exists $T_0=T_0(D_0,\alpha,\beta,\nu,\gamma)>0$ and a constant $0<\theta<1$, such that if $T\leqslant T_0$ then

$$\|h^{(j+1)}-h^{(j)}\|_{X_{\alpha,\beta,T}}\leqslant\theta\cdot\|h^{(j)}-h^{(j-1)}\|_{X_{\alpha,\beta,T}},\quad\forall\, j\geqslant 2.$$

This shows that $(h^{(j)}(t))$ is Cauchy in $X_{\alpha,\beta,T}$ and hence we have shown the existence and uniqueness of a solution to (3.1) in $X_{\alpha,\beta,T}$. Consequently (1.25) holds with $D_1=2D_0$. We still have to show (1.26). We shall establish this by a bootstrap argument. Without loss of generality assume $t_0<\frac{T}{100}$. Denote $\hat{h}(t)=h(t+\frac{t_0}{2})$. Then $\hat{h}(t)$ solves (3.1) with $h(\frac{t_0}{2})$ as initial data. For $0\leqslant t<T-\frac{t_0}{2}$, by (1.25), we get

$$\begin{aligned}|\hat{h}(t,m,n)| &= \left|h\left(t+\frac{t_0}{2},m,n\right)\right|\\ &\leqslant\frac{D_1}{m^\alpha(n+1)^\beta}\cdot e^{-\frac{1}{2}\nu mt}\cdot e^{-\frac{1}{4}\nu mt_0}\cdot\frac{1}{(m^2+n^2)^{\frac{1}{2}}}\\ &\leqslant D_1\cdot e^{-\frac{1}{2}\nu mt}\cdot\frac{1}{(n+1)^{\beta+1}}\cdot\frac{e^{-\frac{1}{4}\nu mt_0}}{m^\alpha}.\end{aligned}$$

Since $m\geqslant 1$ and $t_0>0$, we have

$$D_1\cdot\sup_{m\geqslant 1}\frac{e^{-\frac{1}{4}\nu mt_0}}{m^\alpha}\cdot m^{10}\leqslant D_3<\infty,$$

where D_3 is a constant depending only on (D_0,α,β,ν, t_0). Therefore we obtain

$$|\hat{h}(t,m,n)|\leqslant D_3\cdot e^{-\frac{1}{2}\nu mt}\cdot e^{-\frac{1}{4}\nu mt_0}\cdot\frac{1}{(n+1)^{\beta+1}m^{10}},\quad\forall\, m\geqslant 1, n\geqslant 0.\tag{3.14}$$

Denote $\widehat{N}(t)$ as the expression in (1.23) with $h(t)$ now replaced by $\hat{h}(t)$. Then by Lemmas 3.1–3.3 and (3.14), we have for any $0\leqslant s<T-\frac{t_0}{2}$,

$$|\widehat{N}(s,m,n)|\leqslant e^{-\frac{1}{2}\nu ms-\frac{1}{4}\nu mt_0}\cdot D_4\cdot\frac{\log(n+5)}{(n+1)^{\beta-1}},$$

where D_4 is another constant depending only on (D_0,α,β,ν,γ,t_0). Plugging this estimate into the RHS of (3.1) and using again (3.14) for $\hat{h}(0,m,n)$, we obtain for $t\geqslant\frac{t_0}{100}$,

$$
\begin{aligned}
|\hat{h}(t,m,n)| &\leqslant e^{-(m^2+n^2)^{\frac{\gamma}{2}}\nu t}|\hat{h}(0,m,n)| + e^{-\frac{1}{4}m\nu t_0}\cdot D_4\cdot\frac{\log(n+5)}{(n+1)^{\beta}} \\
&\quad\cdot e^{-\frac{1}{2}m\nu t}\cdot\int_0^t e^{-((m^2+n^2)^{\frac{\gamma}{2}}-\frac{m}{2})\nu s}ds \\
&\leqslant e^{-(m^2+n^2)^{\frac{\gamma}{2}}\nu t}\cdot D_3\cdot e^{-\frac{1}{4}m\nu t_0}\cdot\frac{1}{(n+1)^{\beta+1}m^{10}} + 10^{\gamma} \\
&\quad\cdot e^{-\frac{1}{4}m\nu t_0-\frac{1}{2}m\nu t}\cdot D_4\cdot\frac{\log(n+5)}{(n+1)^{\beta+\gamma}} \\
&\leqslant D_5\cdot e^{-\frac{1}{2}m\nu t-\frac{1}{4}m\nu t_0}\cdot\frac{\log(n+5)}{(n+1)^{\beta+\gamma}},
\end{aligned}
$$

where D_5 depends only on $(D_0,\alpha,\beta,\nu,\gamma,t_0)$. Compare this bound with (3.14), we have a better estimate of $\hat{h}(t,m,n)$ with the decay in n improved from $n^{-(\beta+1)}$ to $n^{-(\beta+\gamma)}\cdot\log(n+5)$. Iterating the above process once more and noting that (3.5) only produces a decay of n^{-2} for $\tilde{\alpha}>2$, we obtain for any $\frac{t_0}{100}\leqslant s<T-\frac{t_0}{2}$,

$$
|\widehat{N}(s,m,n)| \leqslant e^{-\frac{1}{2}m\nu s-\frac{1}{7}m\nu t_0}\cdot D_6\cdot\frac{1}{(n+1)^2},
$$

and consequently

$$
|\hat{h}(t,m,n)| \leqslant D_7\cdot e^{-\frac{1}{2}m\nu t-\frac{1}{8}m\nu t_0}\cdot(n+1)^{-3-\gamma}.
$$

Here again D_6, D_7 are constants depending only on $(D_0,\alpha,\beta,\nu,\gamma,t_0)$. Hence (1.26) holds. Finally we remark that the small data result follows along the same lines as in the proof of the local contraction argument. We omit the standard details. The theorem is proved. □

4. Global wellposedness for large data

To upgrade Theorem 1.1 to a global result, we need a priori control of the solution. For this purpose we shall first prove a maximum principle which gives control of L^∞-norm of the solution. The situation here is slightly nonstandard due the boundary condition (1.5)–(1.8) and the nonlocal fractional Laplacian $(-\Delta)^{\frac{\gamma}{2}}$ (when $\gamma\neq 2$).

To this end, let $1<\gamma\leqslant 2$ and $\nu>0$ be the same as in Theorem 1.1. Let $\overline{\Omega}=[0,\pi]\times[0,\pi]$. Let $T_1>0$ be arbitrary but fixed. Suppose a function $\phi=\phi(t,x,y)$ has the expansion

$$
\phi(t,x,y)=\sum_{m\geqslant 1,n\geqslant 0} g(t,m,n)\sin mx\cos ny,\ \forall\ 0\leqslant t\leqslant T, \tag{4.1}
$$

where for some constants $A_1>0$, $\beta_1>0$,

$$
|g(t,m,n)|\leqslant\frac{A_1}{(n+1)^{2+\gamma}}e^{-\beta_1 m},\quad \forall\ m\geqslant 1,\ n\geqslant 0. \tag{4.2}
$$

Assume $\phi\in C_t^1C_{x,y}^2([0,T]\times\overline{\Omega})$ solves the equation

$$
\partial_t\phi+a(t,x,y)\cdot\nabla\phi=-\nu(-\Delta)^{\frac{\gamma}{2}}\phi, \tag{4.3}
$$

where $a\in C_t^0C_{x,y}^0([0,T]\times\overline{\Omega})$ and

$$
\left((-\Delta)^{\frac{\gamma}{2}}\phi\right)(t,x,y)=\sum_{m\geqslant 1,n\geqslant 0} g(t,m,n)(m^2+n^2)^{\frac{\gamma}{2}}\sin mx\cos ny. \tag{4.4}
$$

Then we have

Proposition 4.1 (*Maximum principle*). *For any $0\leqslant t\leqslant T_1$, we have*

$$
\|\phi(t,\cdot,\cdot)\|_{L^\infty_{x,y}(\overline{\Omega})}\leqslant\|\phi(0,\cdot,\cdot)\|_{L^\infty_{x,y}(\overline{\Omega})}. \tag{4.5}
$$

Proof 3. By (4.1) and (4.2), we can regard ϕ as a periodic function with period 2π. Observe that

$$
\begin{aligned}
\phi(t,x,y) &= -\phi(t,2\pi-x,y),\quad \phi(t,x,y) \\
&= \phi(t,x,2\pi-y).
\end{aligned} \tag{4.6}
$$

Therefore it is easy to check that the function $\phi_2(t,x,y)=\phi(t,x,y)^2$ must achieve its maximum in $\overline{\Omega}$.

If $1<\gamma<2$, then by (4.4) and Proposition 2.2 from [3], we have (suppressing the t-dependence for the moment)

$$
\left((-\Delta)^{\frac{\gamma}{2}}\phi_2\right)(x,y)=C_\gamma\sum_{\mathbf{b}\in\mathbb{Z}^2}PV\int_{\mathbb{T}^2}\frac{\phi_2(x,y)-\phi_2(x',y')}{|(x,y)-(x',y')-2\pi\mathbf{b}|^{2+\gamma}}dx'dy', \tag{4.7}
$$

where C_γ is a constant depending on γ and PV is the principal value. In particular if (x^*,y^*) is a maximum point of ϕ_2, then by (4.7)

$$
\left((-\Delta)^{\frac{\gamma}{2}}\phi_2\right)(x^*,y^*)\geqslant 0. \tag{4.8}
$$

It is easy to check that (4.8) also holds when $\gamma=2$.

By Proposition 2.3 from [3], we have

$$
\phi(x^*,y^*)\left((-\Delta)^{\frac{\gamma}{2}}\phi\right)(x^*,y^*)\geqslant\left((-\Delta)^{\frac{\gamma}{2}}\phi_2\right)(x^*,y^*)\geqslant 0. \tag{4.9}
$$

Let $\epsilon>0$ and consider

$$
\phi_\epsilon(t,x,y)=\phi(t,x,y)^2-\epsilon t.
$$

For $0\leqslant t\leqslant T_1$, $0\leqslant x,y\leqslant 2\pi$, by (4.6) we can assume ϕ_ϵ attains its maximum at $(t^*,x^*,y^*)\in[0,T_1]\times\overline{\Omega}$. If $0<t^*\leqslant T_1$, then by (4.9)

$$
\begin{aligned}
(\partial_t\phi_\epsilon)(t^*,x^*,y^*) &= -\epsilon-2\nu\phi(t^*,x^*,y^*)\left((-\Delta)^{\frac{\gamma}{2}}\phi\right)(t^*,x^*,y^*) \\
&\leqslant -\epsilon<0
\end{aligned}
$$

which is clearly impossible.

Therefore we get for any $(x,y)\in\overline{\Omega}$, $0\leqslant t\leqslant T_1$,

$$
\phi(t,x,y)^2-\epsilon t\leqslant\|\phi(0,\cdot,\cdot)\|^2_{L^\infty_{x,y}(\overline{\Omega})}.
$$

Sending $\epsilon\to 0$ yields (4.5). □

We are now ready to complete the

Proof of Theorem 1.4. In this proof, to simplify the presentation we denote by $X\pm$ any quantity of the form $X\pm\epsilon$ for any $\epsilon>0$. We use $X\lesssim Y$ whenever $X\leqslant CY$ for some constant $C>0$.

By Theorem 1.1, to continue the local solution it suffices for us to show

$$
\sup_{0<t<T}\sup_{m\geqslant 1,n\geqslant 0}\left|h(t,m,n)m^{2+}(n+1)^{2+}(m^2+n^2)^{\frac{1}{2}}\right|\leqslant C(T), \tag{4.10}
$$

for any $T>0$. Here $C(T)$ is some finite constant depending on T. We shall adopt this notation for the rest of the proof. Without loss of generality we may assume the initial data satisfies

$$|h(0,m,n)| \leqslant \frac{B_1}{(n+1)^{3+\gamma}} e^{-\beta_1 m}, \quad \forall\, m \geqslant 1, n \geqslant 0, \tag{4.11}$$

where $B_1 > 0$, $\beta_1 > 0$. Otherwise we can choose t_0 sufficiently small and replace $h(0,m,n)$ by $h(t_0,m,n)$ and do a simple shift in time.

Let $0 < \beta_2 < \gamma - 1$ and denote $\Lambda = (-\Delta)^{\frac{1}{2}}$. To show (4.10), it is enough for us to prove

$$\||\partial_x|^k \theta(t)\|_{L^2_{x,y}(\Omega)} \leqslant C(T), \tag{4.12}$$

and

$$\|\Lambda^{2+\beta_2}\theta(t)\|_{L^2_{x,y}(\Omega)} \leqslant C(T), \tag{4.13}$$

where $k \geqslant k_0(\beta)$ and $k_0(\beta)$ is an integer sufficiently large depending on β.

We shall only prove (4.13). The proof of (4.12) is similar and actually simpler. In the mild formulation, we write (1.1) as

$$\theta(t) = e^{-\nu\Lambda^\gamma t}\theta(0) + \int_0^t e^{-\nu\Lambda^\gamma(t-s)}(u(s)\cdot\nabla\theta(s))ds. \tag{4.14}$$

By using Proposition 4.1, we have for some $0 < \delta < 1$ (note that $0 < \beta_2 < \gamma - 1$ and actually $\delta = (1+\beta)/\gamma$), that

$$\begin{aligned}
&\|\Lambda^{1+\beta_2}\theta(t)\|_{L^2_{x,y}} \\
&\lesssim \|\Lambda^{1+\beta_2}\theta(0)\|_{L^2_{x,y}} + \int_0^t (t-s)^{-\delta}\|u(s)\cdot\nabla\theta(s)\|_{L^2_{x,y}}ds \\
&\lesssim \|\Lambda^{1+\beta_2}\theta(0)\|_{L^2_{x,y}} + \int_0^t (t-s)^{-\delta}\|u(s)\|_{L^{\infty-}_{x,y}}\|\nabla\theta(s)\|_{L^{2+}_{x,y}}ds \\
&\lesssim \|\Lambda^{1+\beta_2}\theta(0)\|_{L^2_{x,y}} + \|\theta(0)\|_{L^\infty_{x,y}}\int_0^t (t-s)^{-\delta}\|\Lambda^{1+\beta_2}\theta(s)\|_{L^2_{x,y}}ds.
\end{aligned}$$

By (4.11) and a simple Gronwall-type argument, we get

$$\sup_{0<t<T}\|\Lambda^{1+\beta_2}\theta(t)\|_{L^2_{x,y}} \leqslant C(T). \tag{4.15}$$

It follows that

$$\sup_{0<t<T}\|u(t)\|_{L^\infty_{x,y}} \leqslant C(T). \tag{4.16}$$

By using (4.16) and a similar argument to the derivation of (4.15), we then obtain

$$\|\Lambda^{2+\beta_2}\theta(t)\|_{L^2_{x,y}} \leqslant C(T).$$

The estimate (4.12) can be proved in a similar manner. We omit the details □

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