Finite Time Singularities and Global Well-posedness for Fractal Burgers Equations

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ABSTRACT. Burgers equations with fractional dissipation on $\mathbb{R}\times\mathbb{R}^+$ or on $\mathbb{S}^1\times\mathbb{R}^+$ are studied. In the supercritical dissipative case, we show that with very generic initial data, the equation is locally well-posed and its solution develops gradient blow-up in finite time. In the critical dissipative case, the equation is globally well-posed with arbitrary initial data in $H^{1/2}$. Finally, in the subcritical dissipative case, we prove that with initial data in the scaling-invariant Lebesgue space, the equation is globally well-posed. Moreover, the solution is spatial analytic and has optimal Gevrey regularity in the time variable.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction We study in this paper Burgers equations with a fractional dissipative term (also called *fractal Burgers equations*)

$$\begin{cases} u_t - uu_x + \kappa(-\Delta)^{\gamma/2}u = 0 & \text{on } \mathbb{R}\times\mathbb{R}^+ \text{ (or } \mathbb{S}^1\times\mathbb{R}^+), \\ u(0,x) = u_0(x) & \text{in } \mathbb{R} \text{ (or } \mathbb{S}^1), \end{cases} \tag{1.1}$$

where $\kappa \geq 0$, $\gamma \in (0,2]$ are fixed parameters and the fractional Laplacian $(-\Delta)^{\gamma/2}$ is defined by the Fourier transform:

$$\widehat{(-\Delta)^{\gamma/2}u}(\xi) = |\xi|^\gamma \hat{u}(\xi).$$

This equation with $\kappa = 0$ or $\kappa > 0$, $\gamma = 2$ has been studied extensively in the last fifty years. When $\kappa = 0$, (1.1) becomes the inviscid Burgers equation. The

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local well-posedness and finite-time blowup of it are well-known. When $\kappa > 0$ and $\gamma = 2$, the equation is the classical viscous Burgers equation, and the global well-poseness of it is well known. In this paper we shall focus on the situation when $\kappa > 0$ and $0 < \gamma < 2$, i.e., the fractal Burgers equation. The cases $\gamma > 1$, $\gamma = 1$ and $\gamma < 1$ are called subcritical, critical and supercritical dissipative fractal Burgers equations, respectively. We are interested in the well-posedness and the finite-time blowup in these cases.

This problem was recently studied in [1] and [15]. In [1], the authors proved the finite-time blowup for (1.1) on $\mathbb{R} \times \mathbb{R}^+$ with supercritical dissipation under certain convexity and oddness conditions on the initial data, while in [15] the authors considered the equation on $\mathbb{S}^1 \times \mathbb{R}^+$. In the supercritical case, after obtaining some fairly precise pointwise estimates by using time splitting, they showed the finite-time blowup provided that the initial data is odd and bounded below by some piecewise linear envelop function. The global well-posedness and spatial analyticity for (1.1) are also proved there in the critical and subcritical cases.

However, it appears to us that some arguments in [15] cannot be easily extended to the equations on $\mathbb{R} \times \mathbb{R}^+$, or solutions without the oddness assumption.

In this paper, we consider the equation on both $\mathbb{R} \times \mathbb{R}^+$ and $\mathbb{S}^1 \times \mathbb{R}^+$. Our results are summarized as follows. In all three cases, we prove that the equation is locally well-posed in certain critical Lebesgue or Sobolev spaces. In the supercritical case, we show that with very generic initial data the solution develops gradient blow-up in finite time. In the critical case, the equation is globally well-posed with arbitrary initial data in $H^{1/2}$. In the subcritical case, we prove that the solution is global in time, analytic in the space variables with optimal radius of convergence. We also show that the solution has Gevrey regularity of order γ in the time variable.

The proof of the finite-time blowup in the supercritical case (Theorem 1.2), which is the main part of this paper, is mainly inspired by [6] (see also [7]). Our proof is much simpler than the proofs in [1] and [15]. Roughly speaking, we consider the evolution of an integral of the solution multiplied by a properly chosen weight. The integral satisfies some ordinary differential inequality, and it would blow up in finite time under very generic conditions on the initial data. This then implies the blowup of the first spatial derivative of the solution, due to the Beale-Kato-Majda criterion [2].

This interesting method was first used in [6] to study the blowup phenomenon of a one-dimensional transport equation with non-local velocity. Later it was adapted to prove the formation of finite-time singularity for a class of generalized quasi-geostrophic equations with radial initial data [12].

A perhaps surprising new observation here is that both singular and regular weights can be used to prove blow up (see Remark 2.2). With this method, we are able to deal with more general initial data than those in [1] and [15]. In particular, we don't require any positivity or oddness of the initial data (see Theorem 1.2).

1.2. Main results. Next we shall state the main results of this paper. The first theorem is about the local well-posedness, the small-data global well-posedness and the smoothing effect in the supercritical case.

Theorem 1.1 (Local well-posedness in the supercritical case). *Suppose that $\gamma \in (0,1)$ and $u_0 \in H^{3/2-\gamma}$. Then there exists $T_1 > 0$ such that the initial value problem of* (1.1) *has a unique solution*

$$u(t,x) \in C([0,T_1);H^{3/2-\gamma}) \cap L^2(0,T_1;H^{3/2-\gamma/2}).$$

satisfying

$$\sup_{0<t<T_1} t^{\beta/\gamma}\|u(t,\cdot)\|_{\dot{H}^{3/2-\gamma+\beta}} < \infty, \tag{1.2}$$

for any $\beta \geq 0$ and

$$\lim_{t\to 0} t^{\beta/\gamma}\|u(t,\cdot)\|_{\dot{H}^{3/2-\gamma+\beta}} = 0, \tag{1.3}$$

for any $\beta > 0$. Moreover, the maximal time of existence T_1 can be infinity provided that $\|u_0\|_{\dot{H}^{3/2-\gamma}} \leq \delta\kappa$, for some $\delta > 0$ depending only on γ.

The next theorem contains the main result of this paper.

Theorem 1.2 (Finite-time blowup in the supercritical case). *Suppose that $\gamma \in (0,1)$ and $u_0 \in H^{3/2-\gamma} \cap L^\infty$. Then the solution u in Theorem 1.1 develops gradient blowup in finite time provided that there exists a constant $\delta \in (0, 1-\gamma)$ and $C = C(\gamma,\delta)$ such that*

$$\int_{-1}^{1} (u_0(x) - u_0(0))\,\mathrm{sgn}(x)(|x|^{-\delta} - 1)\,\mathrm{d}x > C(\kappa\|u_0\|_{L^\infty})^{1/2}. \tag{1.4}$$

Remark 1.3. The condition $u_0 \in H^{3/2-\gamma}$ can be dropped as long as there exists a local solution. By using a scaling argument, from the proof of Theorem 1.2 one can easily see that the condition (1.4) can be relaxed. For example, we only need

$$\int_{-a}^{a} (u_0(x) - u_0(x_0))\,\mathrm{sgn}(x)(|x|^{-\delta} - a^{-\delta})\,\mathrm{d}x > C(\kappa\|u_0\|_{L^\infty})^{1/2},$$

for some $a > 0$, $x_0 \in \mathbb{R}$ and $\delta \in (0, 1-\gamma)$. Note that we don't require any positivity of u_0 in any interval.

For the critical dissipative Burgers equation, we have the following global result.

Theorem 1.4 (Global well-posedness and analyticity in the critical case). *Suppose that $\gamma = 1$ and $u_0 \in H^{1/2}$. Then the initial value problem* (1.1) *has a unique (smooth) global solution*

$$u(t,x) \in C([0,\infty);H^{1/2}) \cap L^2_{\mathrm{loc}}(0,\infty;H^1), \tag{1.5}$$

and for any $\varepsilon \in (0,1)$ and any $\beta \geq \varepsilon$, we have the following decay estimate

$$\sup_{1 \leq t < \infty} t^{\beta-\varepsilon} \|u(t,\cdot)\|_{\dot{H}^{1/2+\beta}} < \infty. \tag{1.6}$$

Furthermore, the unique global solution u is spatial analytic: for any $t_0 > 0$, there exists a constant $C = C(t_0) > 0$ such that

$$\|D_x^k u(t)\|_{L_t^\infty L_x^\infty([t_0,\infty)\times R)} \leq C^{k+1} \cdot k!, \quad \forall\, k \geq 0. \tag{1.7}$$

For the proof of the global wellposedness part in Theorem 1.4, we refer the reader to [14], [8] and [15] for the periodic case, and [10] for the non-periodic case. It is worth noting that in the periodic case, inhomogeneous Sobolev norms of the solution decay exponentially in time; see, for example, [8]. We give the proof of analyticity in Section 4.

In the subcritical case, we shall show that (1.1) is globally well-posed. The solution is spatial analytic and has Gevrey regularity of order γ in t. To state the precise results, we first fix some notation. For $q \in (1/(\gamma-1), \infty]$, $T \in (0,\infty]$, let $\alpha = 1 - 1/\gamma - 1/(q\gamma)$. We define the space

$$K_q = \left\{ u \mid t^\alpha u \in BC([0,T), L_x^q),\ \lim_{t\to 0} t^\alpha \|u\|_{L_x^q} = 0 \right\}.$$

Introduce the Banach spaces,

$$X_{q,T}^0 = BC([0,T), L_x^{1/(\gamma-1)}) \cap K_q,$$

with norm

$$\|u\|_{X_{q,T}^0} = \max\left\{ \|u\|_{L_t^\infty L_x^{1/(\gamma-1)}([0,T)\times\mathbb{R})}, \|t^\alpha u\|_{L_t^\infty L_x^q((0,T)\times\mathbb{R})} \right\},$$

and

$$\begin{aligned} X_{q,T}^k = X_{q,T}^{k-1} &\cap \left\{ u \mid t^{k/\gamma} D^k u \in BC((0,T), L_x^{1/(\gamma-1)}) \right\} \\ &\cap \left\{ u \mid t^{k/\gamma+\alpha} D^k u \in BC((0,T), L_x^q) \right\}, \end{aligned}$$

with norm

$$\|u\|_{X_{q,T}^k} = \max\{ \|u\|_{X_{q,T}^{k-1}}, \|u\|_{Y_{q,T}^k} \},$$

where $\|\cdot\|_{Y_{q,T}^k}$ is a semi-norm defined by

$$\|u\|_{Y_{q,T}^k} = \max\left\{ \|t^{k/\gamma} D^k u\|_{L_t^\infty L_x^{1/(\gamma-1)}((0,T)\times\mathbb{R})}, \|t^{k/\gamma+\alpha} D^k u\|_{L_t^\infty L_x^q((0,T)\times\mathbb{R})} \right\}.$$

Theorem 1.5 (Global well-posedness and analyticity in the subcritical case). *Suppose that* $\gamma \in (1,2]$, $q \in (1/(\gamma-1), \infty]$ *and* $u_0 \in L^{1/(\gamma-1)}$. *Then the initial value problem for* (1.1) *has a unique mild solution* $u(t,x)$ *in* $X^k_{q,\infty}$ *for any integer* $k \geq 0$, *and we have the estimate*

$$\|u\|_{X^k_{q,\infty}} \leq C^{k+1}k^k, \tag{1.8}$$

with a constant C *independent of* k. *In particular, the following decay in time estimate holds for any integer* $k \geq 0$,

$$\|D^k u(t,\cdot)\|_{L^q_x} \leq C^k t^{-k/\gamma-\alpha} k^k.$$

Moreover, u *is a Gevrey function of order* γ *in the time variable, i.e.,*

$$\|\partial_t^m D_x^k u\|_{L^\infty_x(R)} \leq C^{k+m+1}(\gamma m + k)^{\gamma m + k} \cdot t^{-m-(k+\gamma-1)/\gamma}, \quad \forall\, t > 0.$$

Although we are only dealing with the fractal Burgers equations in this paper, we stress that our method is also applicable to more general equations including, for instance, the following fractal one-dimensional conservation law

$$u_t + (F(u))_x + \kappa(-\Delta)^{\gamma/2}u = 0,$$

where $F(\cdot)$ is a C^1 strictly concave function.

2. Finite-Time Blowup in the Supercritical Case

The proof of Theorem 1.1 is by now standard. Instead of repeating the existing arguments, we refer the readers to [16] for the proof of the local well-posedness part, and [9] for the proof of the smoothing effect. This section is devoted to the proof of Theorem 1.2.

For $\delta \in (0,1)$, we define a singular weight on the real line with compact support:

$$w(x) = \begin{cases} \operatorname{sgn}(x)(|x|^{-\delta} - 1) & |x| \in (0,1), \\ 0 & |x| \geq 1. \end{cases} \tag{2.1}$$

Recall the integral representation of the fractional Laplace

$$\Lambda^\gamma f(x) = C_\gamma \mathrm{P.V.}\int_{\mathbb{R}} \frac{f(x) - f(y)}{|x-y|^{1+\gamma}}\, dy,$$

where C_γ is some constant depending on γ. To control the dissipation term, we need the following pointwise estimate.

Lemma 2.1.
(i) *For any x satisfying $|x| \geq 1$, we have*

$$|\Lambda^\gamma w(x)| \leq C|x|^{-2-\gamma}. \tag{2.2}$$

(ii) *For any x satisfying $|x| \in (0,1)$, we have*

$$|\Lambda^\gamma w(x)| \leq C|x|^{-\delta-\gamma}. \tag{2.3}$$

In particular, if $\delta \in (0, 1-\gamma)$, then

$$\int_0^\infty |\Lambda^\gamma w \, \mathrm{d}x| \leq C.$$

Proof. Note that $\Lambda^\gamma w$ is an odd function. It suffices to consider $x > 0$. First we prove part (i). Since w is supported on $[-1, 1]$ and odd, it is easy to see that for $x \geq 1$,

$$\Lambda^\gamma w(x) = -\int_0^1 (r^{-\delta} - 1)((x-r)^{-1-\gamma} - (x+r)^{-1-\gamma}) \, \mathrm{d}r.$$

Since the second factor inside the integral sign is decreasing in x, we have

$$|\Lambda^\gamma w(x)| \leq \int_0^1 (r^{-\delta} - 1)((1-r)^{-1-\gamma} - (1+r)^{-1-\gamma}) \, \mathrm{d}r \leq C. \tag{2.4}$$

The last inequality is because $\gamma \in (0,1)$. For $x \geq 2$, by the mean value theorem,

$$|\Lambda^\gamma w(x)| \leq C \int_0^1 (r^{-\delta} - 1) r x^{-2-\gamma} \, \mathrm{d}r \leq C x^{-2-\gamma}. \tag{2.5}$$

The inequality above together with (2.4) proves (2.2).

Next we turn to the proof of part (ii). By the definition of w, we have

$$\Lambda^\gamma w(x) = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4,$$

where

$$\begin{aligned}
\mathrm{I}_1 &= \int_{-x}^{1-x} |r|^{-1-\gamma}(x^{-\delta} - (x+r)^{-\delta}) \, \mathrm{d}r, \\
\mathrm{I}_2 &= \int_{-x-1}^{-x} |r|^{-1-\gamma}(x^{-\delta} + |x+r|^{-\delta} - 2) \, \mathrm{d}r, \\
\mathrm{I}_3 &= \int_{1-x}^{\infty} r^{-1-\gamma}(x^{-\delta} - 1) \, \mathrm{d}r, \\
\mathrm{I}_4 &= \int_{-\infty}^{-x-1} |r|^{-1-\gamma}(1 - x^{-\delta}) \, \mathrm{d}r.
\end{aligned}$$

Simply,

$$(2.6)\qquad |\mathrm{I}_3 + \mathrm{I}_4| = \gamma^{-1}(x^{-\delta} - 1)((1-x)^{-\gamma} - (1+x)^{-\gamma}) \le C.$$

Moreover, by the mean value theorem, a rather direct computation gives

$$|\mathrm{I}_1| \le \left(\int_{-x}^{-x/2} + \int_{-x/2}^{x/2} + \int_{x/2}^{1-x}\right)|r|^{-1-\gamma}\,|(x^{-\delta} - (x+r)^{-\delta}|\,\mathrm{d}r \le Cx^{-\delta-\gamma},$$
$$|\mathrm{I}_2| \le \left(\int_{-x-1}^{-3x/2} + \int_{-3x/2}^{-x}\right)|r|^{-1-\gamma}\,|x+r|^{-\delta}\,\mathrm{d}r + Cx^{-\delta-\gamma} \le Cx^{-\delta-\gamma}.$$

These estimates together with (2.6) yield (2.3), thus finishing the proof of the lemma. □

Now we are ready to prove Theorem 1.2. We prove by contradiction. Assume the smooth solution u in Theorem 1.1 exists globally. To illustrate the idea, we first consider the case when u_0 is an odd function. It is then clear that $u(t,\cdot)$ is an odd function for any $t > 0$ and $u(t,0) \equiv 0$. After multiplying both sides of the first equation of (1.1) by w with $\delta \in (1-\gamma)$ and integrating in x, we obtain

$$(2.7)\quad \frac{d}{dt}\int_0^1 u(t,\cdot)w\,\mathrm{d}x - \frac{1}{2}\int_0^1 (u^2(t,\cdot))_x w\,\mathrm{d}x + \kappa\int_0^\infty \Lambda^\gamma u(t,\cdot)w\,\mathrm{d}x = 0.$$

To shift the fractional derivative from u to w in the last integral, we write

$$\begin{aligned}(2.8)\quad \int_0^\infty \Lambda^\gamma u(t,\cdot)w\,\mathrm{d}x &= \frac{1}{2}\int_{\mathbb{R}} \Lambda^\gamma u(t,\cdot)w\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{u(t,x)-u(t,y)}{|x-y|^{1+\gamma}}w(x)\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{u(t,x)-u(t,y)}{|x-y|^{1+\gamma}}(w(x)-w(y))\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} u(t,x)\frac{w(x)-w(y)}{|x-y|^{1+\gamma}}\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^\infty u(t,\cdot)\Lambda^\gamma w\,\mathrm{d}x.\end{aligned}$$

By using (2.8) and integrating by parts in the second integral of (2.7), we obtain for $L(t) := \int_0^1 u(t,\cdot)w\,\mathrm{d}x$,

$$\frac{d}{dt}L(t) = \frac{\delta}{2}\int_0^1 u^2(t,x)x^{-\delta-1} - \kappa\int_0^\infty u(t,\cdot)\Lambda^\gamma w\,\mathrm{d}x.$$

Due to Hölder's inequality, Lemma 2.1 and the maximum principle, it holds that

$$\frac{d}{dt}L(t) \geq c(\delta)(L(t))^2 - C\kappa\|u_0\|_{L^\infty}, \tag{2.9}$$

for some positive constants $c(\delta)$ and $C = C(\delta, \gamma)$. This implies that $L(t)$ blows up at some finite time provided that

$$c(\delta)(L(0))^2 - C\kappa\|u_0\|_{L^\infty} > 0.$$

Now since

$$L(t) \leq C \sup_{x\in(0,1)} \frac{u(t,x)}{x} \leq C\|u_x(t,\cdot)\|_{L^\infty},$$

we conclude that $\|u_x\|_{L^\infty}$ also blows up in finite time, which gives a contradiction.

Next we consider the general case. Again, we prove by contradiction and assume the smooth solution u is global. We shall use the idea of Lagrangian coordinates. Let $y(t)$ be the solution of the ordinary differential equation

$$y'(t) = -u(t,y)$$

with initial condition $y(0) = 0$. Since u is bounded and Lipschitz in y, $y(t)$ is well-defined for any $t \geq 0$. Let

$$v(t,x) = u(t, x + y(t)).$$

It is easily seen that v is smooth and satisfies

$$\begin{cases} v_t - (v(t,x) - v(t,0))v_x + \kappa\Lambda^\gamma v = 0 \quad t > 0, \\ v(0,x) = u_0(x). \end{cases} \tag{2.10}$$

From (2.10), we get

$$\begin{aligned}(v(t,x) - v(t,0))_t &- (v(t,x) - v(t,0))v_x(t,x) \\ &+ \kappa[(\Lambda^\gamma v)(t,x) - (\Lambda^\gamma v)(t,0)] = 0.\end{aligned} \tag{2.11}$$

We choose the same weight function ω as in (2.1). Multiply both sides of (2.11) by ω and integrate with respect to x. By noting that ω is odd so that the integral of $(\Lambda^\gamma v)(t,0)$ term in (2.11) is zero, we get

$$\frac{d}{dt}L(t) = \delta\left(\int_{-1}^{0} + \int_{0}^{1}\right)(v(t,x) - v(t,0))v_x\omega\,dx - \kappa\int_{-\infty}^{\infty} v(t,\cdot)\Lambda^\gamma w\,dx,$$

where

$$L(t) = \int_{-1}^{1} (v(t,x) - v(t,0))\omega \, dx.$$

Now we integrate by parts in the first two terms on the right-hand side and estimate the third term on the right-hand side as before to get

$$\frac{d}{dt}L(t) \geq c(\delta)(L(t))^2 - C\kappa \|u_0\|_{L^\infty}, \tag{2.12}$$

for some positive constants $c(\delta)$ and $C = C(\delta, \gamma)$. As before, this gives a contradiction provided that (1.4) holds, and the theorem is proved.

Remark 2.2. Instead of choosing the compact weight

$$w(x) = \operatorname{sgn}(x)(|x|^{-\delta} - 1) \cdot \chi_{|x|\leq 1},$$

we could also choose

$$w(x) = \operatorname{sgn}(x)\exp\{-|x|\}, \quad \text{or} \quad \operatorname{sgn}(x)(1 - |x|) \cdot \chi_{|x|\leq 1}.$$

In this case one can show that

$$|\Lambda^\gamma w(x)| \leq \text{Const} \cdot |x|^{-\gamma} \qquad \text{as } |x| \to 0$$

and

$$|\Lambda^\gamma w(x)| \leq \text{Const} \cdot |x|^{-2-\gamma} \quad \text{as } |x| \to \infty.$$

This would give $\Lambda^\gamma w \in L^1(\mathbb{R})$ and yield almost immediately the blow up of u_x in finite time. We leave the details to the interested readers.

Remark 2.3. It is also interesting to point out that our argument also covers the periodic boundary condition case with little modification (see [15] for a different proof for odd solutions).

3. Proof of Theorem 1.5

We begin by noting that the proof of (1.8) is almost the same as the proof of a corresponding theorem in [13] (see also [11]). We omit the proof and just mention that the idea there is to use so-called fractional bootstrapping. In what follows we show that the solution u is a Gevrey function of order γ in the time variable, i.e., for some constant $C_1 > 0$,

$$\|t^{m+(k+\gamma-1)/\gamma}\partial_t^m \partial_x^k u\|_{L^\infty_{t,x}([0,\infty)\times\mathbb{R})} \leq C_1^{\gamma m+k+1}(\gamma m + k)^{\gamma m+k}, \quad \forall\, m, k \geq 0. \tag{3.1}$$

To prove this, without loss of generality assume $\kappa = 1$ in (1.1) and we have

$$\begin{aligned}\partial_t^m \partial_x^k u &= \partial_t^{m-1}\partial_x^{k+1}\left(\frac{u^2}{2}\right) - \partial_t^{m-1}\partial_x^k(-\Delta)^{\gamma/2}u \\ &= \frac{1}{2}\sum_{j=0}^{m-1}\sum_{\ell=0}^{k+1}\binom{m-1}{j}\binom{k+1}{\ell}(\partial_t^j\partial_x^\ell u)\cdot(\partial_t^{m-1-j}\partial_x^{k+1-\ell}u) \\ &\qquad - (-\Delta)^{\gamma/2}\partial_t^{m-1}\partial_x^k u.\end{aligned}$$

To bound the last term, we use the following interpolation inequality which can be easily proved by using standard Littlewood-Paley decomposition:

$$\|(-\Delta)^{\gamma/2}f\|_{L_x^\infty} \le \text{Const}\cdot \|f\|_{L_x^\infty}^{1-\gamma/2}\|\Delta f\|_{L_x^\infty}^{\gamma/2}.$$

To this end, define

$$C(m,k) = \|t^{m+(k+\gamma-1)/\gamma}\partial_t^m\partial_x^k u\|_{L_{t,x}^\infty([0,\infty)\times\mathbb{R})}. \tag{3.2}$$

With the help of the interpolation inequality above, we have for some constant $C_1 > 0$ independent of (m, k),

$$\begin{aligned}C(m,k) &\le \\ &\le C_1\sum_{j=0}^{m-1}\sum_{\ell=0}^{k+1}\binom{m-1}{j}\binom{k+1}{\ell}C(j,\ell)\cdot C(m-1-j,\ k+1-\ell) \\ &\qquad + C_1C(m-1,\ k)^{1-\gamma/2}C(m-1,\ k+2)^{\gamma/2}.\end{aligned} \tag{3.3}$$

To bound $C(m,k)$ we need the following lemma.

Lemma 3.1 (Interpolation inequality for the Gamma function). *Let $0 \le \gamma \le 2$ and $x_0 > 1$. Denote by $\Gamma(x)$ the usual Gamma function. Then there exists a constant $C = C(x_0,\gamma) > 0$ such that*

$$\Gamma(x-1)^{1-\gamma/2}\Gamma(x+1)^{\gamma/2} \le C\cdot\Gamma(x+\gamma-1), \quad \forall\, x \ge x_0.$$

Proof. This is an easy application of the following Stirling formula which holds for any real $z \ge 1$:

$$\Gamma(z) = \sqrt{2\pi z}\left(\frac{z}{e}\right)^z\left(1+O\left(\frac{1}{z}\right)\right). \qquad \square$$

The next lemma is slightly technical in nature. It shows that the Gamma function satisfies some nice combinatorial relations.

Lemma 3.2. *Let* $1 < \gamma \le 2$. *There exists a constant* $C = C(\gamma) > 0$ *such that for any* $m \ge 1$, $k \ge 0$,

$$\sum_{j=0}^{m-1}\sum_{\ell=0}^{k+1}\binom{m-1}{j}\binom{k+1}{\ell}\Gamma(\gamma(j+1)+\ell-1)\Gamma(\gamma(m-j)+k-\ell) \le C\Gamma(\gamma(m+1)+k-1). \tag{3.4}$$

Proof. By using the definition of $\Gamma(x)$:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t, \quad \mathrm{Re}(x) > 0,$$

we infer that the left-hand side of (3.4) is equal to

$$\int_0^\infty\int_0^\infty e^{-(x+y)}\sum_{j=0}^{m-1}\sum_{\ell=0}^{k+1}\binom{m-1}{j}\binom{k+1}{\ell}x^{\gamma(j+1)+\ell-2}y^{\gamma(m-j)+k-\ell-1}\,\mathrm{d}x\,\mathrm{d}y.$$

By change of variables $t = x + y$, $s = x/t$, we get

$$\begin{aligned}
(\text{L.H.S.}) &= \int_0^\infty e^{-t}t^{\gamma(m+1)+k-2}\,\mathrm{d}t\cdot\int_0^1\sum_{j=0}^{m-1}\sum_{\ell=0}^{k+1}\binom{m-1}{j}\binom{k+1}{\ell}\\
&\qquad\times x^{\gamma(j+1)+\ell-2}(1-x)^{\gamma(m-j)+k-\ell-1}\,\mathrm{d}x\\
&= \Gamma(\gamma(m+1)+k-1)\int_0^1(x^\gamma+(1-x)^\gamma)^{m-1}x^{\gamma-2}(1-x)^{\gamma-2}\,\mathrm{d}x\\
&\le \Gamma(\gamma(m+1)+k-1)\int_0^1 x^{\gamma-2}(1-x)^{\gamma-2}\,\mathrm{d}x.
\end{aligned}$$

Clearly, it suffices for us to take

$$C = C(\gamma) = \int_0^1 x^{\gamma-2}(1-x)^{\gamma-2}\,\mathrm{d}x.$$

The integral converges since $1 < \gamma \le 2$. The lemma is proved. $\square$

We are now ready to prove the following theorem.

Theorem 3.3. *Let* $C(m,k)$ *be defined as in* (3.2) *which satisfy* (3.3). *Then there exists a constant* $C_2 > 0$ *such that for any* m, $k \ge 0$,

$$C(m,k) \le C_2^{m+k+1}\Gamma(\gamma(m+1)+k-1).$$

Remark 3.4. By easy asymptotics, Theorem 3.3 immediately gives us the desired bound (3.1) on $C(m,k)$.

Proof of Theorem 3.3. We prove by induction on m. Let $m \geq 1$. Our inductive hypothesis is,

$$C(r,s) \leq C_3 C_4^r C_5^s \Gamma(\gamma(r+1)+s-1), \quad \forall\, 0 \leq r < m,\ s \geq 0,$$

where C_3, C_4, C_5 are some sufficiently large constants to be specified during the course of proof. By (1.8) the case $r = 0$, $s \geq 0$ is obviously true by choosing C_3 and C_5 to be sufficiently large. We now justify our inductive hypothesis at step m. Denote by the letter C_1 the generic constants which may vary from line to line. By Lemma 3.1 and Lemma 3.2 we get

$$\begin{aligned} C(m,k) &\leq \Gamma(\gamma(m+1)+k-1) \cdot C_1 \cdot C_3 \cdot C_4^m \cdot C_5^k \cdot \frac{C_3 C_5 + C_5^\gamma}{C_4} \\ &\leq \Gamma(\gamma(m+1)+k-1) \cdot C_3 \cdot C_4^m \cdot C_5^k, \end{aligned}$$

where we choose C_4 such that $C_4 \geq C_1 C_3 C_5 + C_1 C_5^\gamma$. The theorem is proved. □

4. PROOF OF THE ANALYTICITY IN THE CRITICAL CASE

The proof of the analyticity in the critical case (see Theorem 1.4) seems to be out of reach for the usual mild solution approach. We shall follow the Fourier method used in [15] for the periodic boundary condition case to establish spatial analyticity. To deal with the whole space case, some minor changes are needed. We detail the arguments below for the sake of completeness.

In what follows we shall present the a priori estimates and leave the interested reader to verify the standard limiting arguments. To this end, denote by $\hat{u}(k,t)$ the Fourier transform of $u(x,t)$, it is easy to see that

$$\partial_t \hat{u}(k,t) = -\frac{ik}{2} \int \hat{u}(k-k',t)\hat{u}(k',t)\,\mathrm{d}k' - |k|\hat{u}(k,t).$$

Let $\xi(k,t) = \hat{u}(k,t)e^{(1/2)|k|t}$. The equation for ξ is given by

$$\partial_t \xi(k,t) = -\frac{ik}{2} \int e^{-\gamma_{k,k'}t} \xi(k-k',\, t)\xi(k',t)\,\mathrm{d}k' - \frac{|k|}{2}\xi(k,t),$$

where $\gamma_{k,k'} = \frac{1}{2}(|k'| + |k-k'| - |k|)$. Observe that

$$0 \leq \gamma_{k,k'} \leq \min\{|k'|, |k-k'|\}.$$

Now consider

$$Y(t) = \int |\xi(k,t)|^2 (1+k^4)\,\mathrm{d}k.$$

Then,

$$\begin{aligned}\frac{dY(t)}{dt} &= \mathrm{Re}\Big(-i\int k^5(e^{-\gamma_{k,k'}t}-1)\xi(k-k',t)\xi(k',t)\xi(-k,t)\,\mathrm{d}k'\,\mathrm{d}k\Big)\\ &\quad+\mathrm{Re}\Big(-i\int k(e^{-\gamma_{k,k'}t}-1)\xi(k-k',t)\xi(k',t)\xi(-k,t)\,\mathrm{d}k'\,\mathrm{d}k\Big)\\ &\quad+\mathrm{Re}\Big(-i\int (k+k^5)\xi(k-k',t)\xi(k',t)\xi(-k,t)\,\mathrm{d}k'\,\mathrm{d}k\Big)\\ &\quad-\int|\xi(k,t)|^2\,|k|(1+k^4)\,\mathrm{d}k\\ &=: \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4.\end{aligned}$$

Next we estimate I_1, I_2 and I_3 separately.

Estimate of I_3: By symmetrizing the sum, we have

$$\begin{aligned}\mathrm{I}_3 &\le 10\int_{|k-k'|\le|k'|\le|k|}|k^5-(k')^5-(k-k')^5|\\ &\qquad\qquad\times|\xi(k-k',t)\xi(k',t)\xi(-k,t)|\,\mathrm{d}k'\,\mathrm{d}k\\ &\le 100\int(k-k')^2|k'|\,|k|^2\,|\xi(k-k',t)\xi(k',t)\xi(-k,t)|\,\mathrm{d}k'\,\mathrm{d}k\\ &\le 1000\,Y^{3/2}.\end{aligned}$$

Estimate of I_2: This is rather easy and we have

$$\mathrm{I}_2\le 100\,Y^{3/2}.$$

Estimate of I_1: We have

$$\begin{aligned}\mathrm{I}_1 &\le t\int\min\{|k'|,|k-k'|\}|k|^5\,|\xi(k-k',t)\xi(k',t)\xi(-k,t)|\,\mathrm{d}k'\,\mathrm{d}k\\ &\le 100t\int_{|k'|\le|k-k'|\le|k|}|k|^{5/2}\,|k-k'|^{5/2}\,|k'|\\ &\qquad\qquad\times|\xi(k-k',t)\xi(k',t)\xi(-k,t)|\,\mathrm{d}k'\,\mathrm{d}k\\ &\le 500t\,Y^{1/2}\int|k|^5\,|\xi(k)|^2\,\mathrm{d}k.\end{aligned}$$

Collecting all the estimates, we have

$$\frac{d}{dt}Y(t)\le 2000\,Y(t)^{3/2}+(1000t\,Y(t)^{3/2}-1)\int|k|^5\,|\xi(k)|^2\,\mathrm{d}k.$$

Now note that $Y(0)=\|u(0)\|_{H^2}$. Then by Gronwall $Y(t)$ is finite for a short time T depending only on $\|u(0)\|_{H^2}$. By (1.6) we have uniform control of $\|u(t)\|_{H^2}$ on the time interval $[\varepsilon,\infty)$ for any $\varepsilon>0$. Then we can apply this argument to any moment $t_0\ge\varepsilon$ and choose T to be uniform. This immediately gives us (1.7).

Remark 4.1. It is worth noting that by following the arguments in Section 3, it can be shown that in the critical case the solution has Gevrey regularity of order $1 + \varepsilon$ in the time variable for any $\varepsilon > 0$. At present, it is not clear to us whether the solution is also analytic in the time variable.

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REFERENCES

[1] N. ALIBAUD, J. DRONIOU, and J. VOVELLE, *Occurrence and non-appearance of shocks in fractal Burgers equations*, J. Hyperbolic Differ. Equ. **4** (2007), 479–499, http://dx.doi.org/10.1142/S0219891607001227. MR 2339805 (2008i:35198)

[2] J.T. BEALE, T. KATO, and A. MAJDA, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys. **94** (1984), 61–66, http://dx.doi.org/10.1007/BF01212349. MR 763762 (85j:35154)

[3] P. BILER, T. FUNAKI, and W.A. WOYCZYNSKI, *Fractal Burgers equations*, J. Differential Equations **148** (1998), 9–46, http://dx.doi.org/10.1006/jdeq.1998.3458. MR 1637513 (99g:35111)

[4] P. CONSTANTIN, P.D. LAX, and A. MAJDA, *A simple one-dimensional model for the three-dimensional vorticity equation*, Comm. Pure Appl. Math. **38** (1985), 715–724, http://dx.doi.org/10.1002/cpa.3160380605. MR 812343 (87a:76037)

[5] P. CONSTANTIN, A.J. MAJDA, and E. TABAK, *Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar*, Nonlinearity **7** (1994), 1495–1533, http://dx.doi.org/10.1088/0951-7715/7/6/001. MR 1304437 (95i:76107)

[6] A. CÓRDOBA, D. CÓRDOBA, and M.A. FONTELOS, *Formation of singularities for a transport equation with nonlocal velocity*, Ann. of Math. (2) **162** (2005), 1377–1389. MR 2179734 (2007b:35011)

[7] ———, *Integral inequalities for the Hilbert transform applied to a nonlocal transport equation*, J. Math. Pures Appl. (9) **86** (2006), 529–540. MR 2281451 (2007k:35040) (English, with English and French summaries)

[8] H. DONG, *Higher regularity for the critical and super-critical dissipative quasi-geostrophic equations, preprint* (2006), arXiv:math.AP/0701826.

[9] ———, *Well-posedness for a transport equation with nonlocal velocity*, J. Funct. Anal. **255** (2008), 3070–3097. MR 2464570

[10] H. DONG and D. DU, *Global well-posedness and a decay estimate for the critical dissipative quasi-geostrophic equation in the whole space*, Discrete Contin. Dyn. Syst. **21** (2008), 1095–1101, http://dx.doi.org/10.3934/dcds.2009.23.765. MR 2399451 (2009b:76175)

[11] H. DONG and D. LI, *Spatial analyticity of the solutions to the subcritical dissipative quasi-geostrophic equations*, Arch. Ration. Mech. Anal. **189** (2008), 131–158, http://dx.doi.org/10.1007/s00205-007-0100-6. MR 2403602 (2009c:35185)

[12] ______, *Finite time singularities for a class of generalized surface quasi-geostrophic equations*, Proc. Amer. Math. Soc. **136** (2008), 2555–2563, http://dx.doi.org/10.1090/S0002-9939-08-09328-3. MR 2390526 (2009c:35361)

[13] ______, *Optimal local smoothing and analyticity rate estimates for the generalized Navier-Stokes equations*, Comm. Math. Sci. **7** (2009), 67–80.

[14] A. KISELEV, F. NAZAROV, and A. VOLBERG, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation*, Invent. Math. **167** (2007), 445–453, http://dx.doi.org/10.1007/s00222-006-0020-3. MR 2276260 (2008f:35308)

[15] A. KISELEV, F. NAZAROV, and R. SHTERENBERG, *Blow up and regularity for Burgers equation with fractional dissipation*, arXiv:0804.3549 (preprint).

[16] H. MIURA, *Dissipative quasi-geostrophic equation for large initial data in the critical Sobolev space*, Comm. Math. Phys. **267** (2006), 141–157, http://dx.doi.org/10.1007/s00220-006-0023-3. MR 2238907 (2007j:35174)

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