

REMARKS OF GLOBAL WELLPOSEDNESS OF LIQUID CRYSTAL FLOWS AND HEAT FLOWS OF HARMONIC MAPS IN TWO DIMENSIONS

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ABSTRACT. We consider the Cauchy problem to the two-dimensional incompressible liquid crystal equation and the heat flows of the harmonic maps equation. Under a natural geometric angle condition, we give a new proof of the global wellposedness of smooth solutions for a class of large initial data in energy space. This result was originally obtained by Ding-Lin and Lin-Lin-Wang. Our main technical tool is a rigidity theorem which gives the coercivity of the harmonic energy under a certain angle condition. Our proof is based on a frequency localization argument combined with the concentration-compactness approach which can be of independent interest.

1. INTRODUCTION

We consider the following hydrodynamic system modeling the flow of nematic liquid crystal materials in two dimensions (see, for instance, [5, 7, 8]):

$$(1.1) \quad \begin{cases} u_t + (u \cdot \nabla)u + \nabla p = \Delta u - \nabla \cdot (\nabla d \otimes \nabla d), \\ d_t + (u \cdot \nabla)d = \Delta d + |\nabla d|^2 d, \\ \nabla \cdot u = 0, \quad |d| = 1, \end{cases}$$

where u is the velocity field, p is the scalar pressure and $d = (d_1, d_2, d_3)$ is the unit-vector ($|d| = \sqrt{d_1^2 + d_2^2 + d_3^2} = 1$) on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ representing the macroscopic molecular orientation of the liquid crystal materials. Here the i^{th} component of $\nabla \cdot (\nabla d \otimes \nabla d)$ is given by $\nabla_j(\nabla_i d \cdot \nabla_j d) = \sum_{j=1}^3 \nabla_j(\nabla_i d \cdot \nabla_j d)$. We apply Einstein's summation convention over repeated indices throughout this paper. For simplicity, we have set all the positive constants in the system to be one. We are interested in the Cauchy problem of (1.1) with the initial data

$$(1.2) \quad u(0, x) = u_0(x), \quad d(0, x) = d_0(x), \quad \lim_{|x| \rightarrow \infty} d_0(x) = \hat{n} \in \mathbb{S}^2,$$

where \hat{n} is a given unit-vector.

The above system (1.1) is a simplified version of the Ericksen-Leslie model for the hydrodynamics of nematic liquid crystals [5, 7]. The mathematical analysis of the liquid crystal flows was initiated by Lin and Liu in [8, 9]. In [15], Wang proved the global existence of strong solutions for rough initial data with sufficiently small BMO^{-1} norm (for u_0 and ∇d_0). See also [10] for a small data global existence result in 3D. For any bounded smooth domain in \mathbb{R}^2 , Lin, Lin and Wang [11] obtained the

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existence of global weak solutions which are smooth everywhere except on finitely many time slices (see also [6]). The uniqueness of weak solutions in two dimensions was studied in [12, 16]. Very recently, a family of non-trivial two-dimensional exact large solutions was constructed in [4].

In this paper, we are concerned with the global existence of large classical solutions to (1.1). As always with these types of problems, to extend the smooth local solution globally in time one needs to obtain certain a priori estimates. By the regularity theory in [11],¹ the smooth local solution (u, d) to (1.1) can be continued past any time $T > 0$ provided that we have

$$(1.3) \quad \int_0^T \left(\|u(t, \cdot)\|_{L^4}^4 + \|\nabla d(t, \cdot)\|_{L^4}^4 \right) dt < \infty.$$

The basic energy inequality associated with (1.1) is the following:

$$(1.4) \quad \begin{aligned} & \frac{1}{2} (\|u(t, \cdot)\|_{L^2}^2 + \|\nabla d(t, \cdot)\|_{L^2}^2) + \int_0^t \left(\|\nabla u(s, \cdot)\|_{L^2}^2 + \|\Delta d(s, \cdot) + |\nabla d|^2 d(s, \cdot)\|_{L^2}^2 \right) ds \\ & \leq \frac{1}{2} (\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2), \quad \forall t \geq 0. \end{aligned}$$

By (1.3)–(1.4) and the Gagliardo-Nirenberg inequality

$$\|u\|_{L^4(\mathbb{R}^2)} \leq C \cdot \|u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \cdot \|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}},$$

it is obvious that

$$\begin{aligned} \|u\|_{L^4_{t,x}([0,T] \times \mathbb{R}^2)} & \leq C \cdot \|u\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R}^2)}^{\frac{1}{2}} \cdot \|\nabla u\|_{L^2_{t,x}([0,T] \times \mathbb{R}^2)}^{\frac{1}{2}} \\ & \leq C \cdot (\|u_0\|_{L^2} + \|\nabla d_0\|_{L^2}) < \infty, \quad \text{for any } T > 0. \end{aligned}$$

Hence the non-blowup criteria (1.3) can be sharpened to

$$(1.5) \quad \int_0^T \|\nabla d(t, \cdot)\|_{L^4}^4 dt < \infty.$$

Note that in (1.1) if we take $u \equiv 0$ we obtain the usual heat flows of harmonic maps. Condition (1.5) asserts that the regularity of the whole liquid crystal system (1.1) is in some sense governed by the pure heat harmonic flow part. For smooth solutions, the main obstruction in obtaining the a priori estimate (1.5) is that the harmonic energy $E_{harm}(t) = \|\Delta d(t) + |\nabla d(t)|^2 d(t)\|_2^2$ in (1.4) is not coercive. In particular, it yields no useful lower bounds for general maps $d : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ (a trivial example is the constant map). A natural idea is to introduce some geometric constraints and work with a set of “restricted maps” such that the resulting harmonic energy is coercive. This idea was first used in the seminal work of Ding-Lin in [3] when they were studying a generalization of the Eells-Sampson theorem. Inspired by [3], we follow this line of thought and establish global wellposedness for a family of initial data under a geometric angle condition. Roughly speaking, this condition says that the image of the initial orientation vector d_0 is strictly contained in a hemisphere. The set of such maps is invariant under the dynamics of (1.1) thanks to the maximum principle.

¹Although the main results in [11] are stated for the bounded domains in \mathbb{R}^2 , it is not difficult to check that the arguments there carry over to the \mathbb{R}^2 case.

Theorem 1.1. Denote $e_3 = (0, 0, 1) \in \mathbb{S}^2$. Let $u_0, \nabla d_0 \in L^2(\mathbb{R}^2)$ with $|d_0| = 1$, $d_0 - e_3 \in L^2(\mathbb{R}^2)$ satisfy the geometric angle condition

$$(1.6) \quad \inf_{x \in \mathbb{R}^2} d_{03} > 0,$$

where d_{03} is the third component of d_0 . Then there exists a unique global smooth solution (d, u) to the incompressible liquid crystal equations (1.1) with the initial data (1.2). Moreover, one has

$$(1.7) \quad \int_0^\infty \left(\|\Delta d(t, \cdot)\|_{L^2}^2 + \|\nabla d(t, \cdot)\|_{L^4}^4 + \|\nabla d(t, \cdot)\|_{L^2}^2 \right) dt < \infty.$$

Remark 1.2. We emphasize that the result in Theorem 1.1 (and also Theorem 1.7 below) can be obtained by using the method of [11].² Our proof here is based on a frequency localization argument combined with the concentration-compactness approach which can be of independent interest.

Remark 1.3. In Theorem 1.1, the choice of e_3 is for convenience only. In general one can choose any reference vector $\hat{n} \in \mathbb{S}^2$ and require that the image of d be contained in a hemisphere around \hat{n} . We should also point it out that the geometric angle condition (1.6) may be necessary due to some counterexamples in heat flows of harmonic maps. In [1] the authors proved the global wellposedness of large solutions to the heat flows of harmonic maps for a class of initial data with symmetry:

$$d_0(x) = \begin{pmatrix} x_1 r^{-1} \sin \psi_0(r) \\ x_2 r^{-1} \sin \psi_0(r) \\ \cos \psi_0(r) \end{pmatrix},$$

where

$$\psi_0(0) = 0, \quad \psi_0(R) < \pi \text{ for all } R > 0.$$

Finite time singularities are also shown to exist in the case where $\psi_0(R) > \pi$ for some $R > 0$ (see [2] for more details).

Remark 1.4. As was shown in the previous remark, there exists a family of blowup solutions for the heat flows of harmonic maps when the geometric angle condition is violated. These blowup examples furnish a trivial ($u \equiv 0$) set of counterexamples for the liquid crystal system (1.1). A very interesting open question is whether there are finite time singularities for the incompressible liquid crystal flows in two dimensions with finite energy and non-trivial velocity. We note that such non-trivial counterexamples (if they exist) must be non-radial since the incompressible constraint in the momentum equation of (1.1) destroys the spherical symmetry.

By standard local theory (cf. [11] and [13, 14]) there exists a local smooth solution to (1.1) with the initial data (1.2) under the assumptions stated in Theorem 1.1. Moreover, by the weak-strong uniqueness (cf. [12] and [16]), such a solution is also unique. For simplicity of presentation we will refrain from proving such local wellposedness results and only work with smooth local solutions. The main work then is to prove the a priori estimate (1.5). For this we will invoke the geometric angle assumption and prove that the corresponding harmonic energy is coercive. A crucial step is to establish the rigidity of approximate harmonic maps (cf. (2.6)) under the geometric angle condition. We state it as the following:

²We are grateful to Professor Fanghua Lin for pointing this out to us.

Theorem 1.5. *Let $\epsilon_0 > 0, C_0 > 0$. There exists a positive constant $\delta_0 = \delta_0(\epsilon_0, C_0)$ such that the following holds:*

If $d : \mathbb{R}^2 \rightarrow \mathbb{S}^2, \nabla d \in H^1(\mathbb{R}^2)$ with $\|\nabla d\|_{L^2} \leq C_0$ and $d_3 \geq \epsilon_0$, then

$$\|\nabla d\|_{L^4}^4 \leq (1 - \delta_0)\|\Delta d\|_{L^2}^2.$$

Consequently for such maps the associated harmonic energy is coercive, i.e.

$$(1.8) \quad \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 \geq \frac{\delta_0}{2} \left(\|\Delta d\|_{L^2}^2 + \|\nabla d\|_{L^4}^4 \right).$$

Remark 1.6. Applying the method of [3] one can even obtain a stronger rigidity theorem. We thank Professor Fanghua Lin for pointing this out. Our approach here is based on frequency localization combined with concentration-compactness.

Note that our rigidity Theorem 1.5 is purely “static” and has nothing to do with the velocity field of the fluid dynamics. As such it is stated for any map $d : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ under general assumptions. By using essentially the same arguments, we also have a similar result for the heat flows of harmonic maps:

$$(1.9) \quad d_t = \Delta d + |\nabla d|^2 d,$$

where d is still a unit-vector on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ (see [3] for a generalization of Eells-Sampson’s theorem). We state the result as the following:

Theorem 1.7. *Let $\nabla d_0 \in L^2(\mathbb{R}^2)$ with $|d_0| = 1, d_0 - e_3 \in L^2(\mathbb{R}^2)$ and $\inf_{x \in \mathbb{R}^2} d_{03} > 0$. Then there exists a unique global smooth solution to the heat flows of harmonic maps (1.9) with the initial data $d(0, x) = d_0(x)$. Moreover, (1.7) is satisfied.*

The proof of Theorem 1.7 will be omitted since it is essentially a repetition of the proof of Theorem 1.1.

We close this introduction by setting up some notation and conventions which will be used in this paper.

Notation. For any two quantities A and B , we use $A \lesssim B$ (resp. $A \gtrsim B$) to denote the inequality $A \leq CB$ (resp. $A \geq CB$) for a generic positive constant C . The dependence of C on other parameters or constants is usually clear from the context, and we will often suppress this dependence. The value of C may change from line to line. For any function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, we use $\|f\|_{L^p}$ or sometimes $\|f\|_p$ to denote the usual Lebesgue L^p norm of a function for $1 \leq p \leq \infty$. We use $L_t^p L_x^r$ to denote the space-time norm

$$\|f\|_{L_t^p L_x^r} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |f(t, x)|^r dx \right)^{\frac{p}{r}} dt \right)^{\frac{1}{p}},$$

with the usual modifications when p or r is infinity, or when the domain $\mathbb{R} \times \mathbb{R}^2$ is replaced by a space-time slab. When $p = r$ we abbreviate $L_t^p L_x^r$ by $L_{t,x}^p$ or L_{tx}^p .

Occasionally we will need to use the fractional Laplacian operator $|\nabla|^s, s > 0$, which is defined via Fourier transform as

$$\mathcal{F}(|\nabla|^s f)(\xi) = |\xi|^s (\mathcal{F}f)(\xi), \quad \xi \in \mathbb{R}^2.$$

The homogeneous Sobolev norm \dot{H}^s for any $s > 0$ is defined as $\|f\|_{\dot{H}^s} = \| |\nabla|^s f \|_2$ or, more explicitly,

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^2} |\xi|^{2s} |(\mathcal{F}f)(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

We will need to use the Littlewood-Paley (LP) frequency projection operators. For simplicity we shall fix the notation on \mathbb{R}^2 , but it is straightforward to define everything in \mathbb{R}^d for any $d \geq 1$. To fix the notation let $\phi \in C_0^\infty(\mathbb{R}^2)$ and satisfy

$$0 \leq \phi \leq 1, \quad \phi(x) = 1 \text{ for } |x| \leq 1, \quad \phi(x) = 0 \text{ for } |x| \geq 2.$$

For two real positive numbers $\alpha < \beta$, define the frequency localized (LP) projection operator $P_{\alpha < \cdot < \beta}$ by

$$P_{\alpha < \cdot < \beta} f = \mathcal{F}^{-1}([\phi(\beta^{-1}\xi) - \phi(\alpha^{-1}\xi)]\mathcal{F}(f)).$$

Here \mathcal{F} and \mathcal{F}^{-1} denote the Fourier transform and its inverse transform, respectively. Similarly, the operators $P_{< \alpha}$ and $P_{> \beta}$ are defined by

$$P_{< \beta} f = \mathcal{F}^{-1}(\phi(\beta^{-1}\xi)\mathcal{F}(f))$$

and

$$P_{> \alpha} f = \mathcal{F}^{-1}([1 - \phi(\alpha^{-1}\xi)]\mathcal{F}(f)).$$

We recall the following Bernstein estimates: for any $1 \leq p \leq q \leq \infty$ and dyadic $N > 0$,

$$\|P_{< N} f\|_{L_x^q(\mathbb{R}^2)} \lesssim N^{\frac{2}{p} - \frac{2}{q}} \|f\|_{L_x^p(\mathbb{R}^2)}.$$

2. RIGIDITY

In this section we prove our rigidity Theorem 1.5. Our proof is purely analytic and uses in a quantitative way the geometric angle condition. The key ingredients of the proof are a frequency localization argument and a concentration-compactness procedure.

We begin with the following simple lemma which locks the non-trivial L^2 weak limit of an L^2 bounded sequence of functions whose frequency is essentially localized to a (large) ring. The proof is stated for \mathbb{R}^2 , but it naturally generalizes to \mathbb{R}^d for any dimension $d \geq 1$.

Lemma 2.1 (Non-evacuation of L^2 -norm). *Let C_0 and α be two positive constants, and N be a dyadic integer. Suppose that $f_n : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a sequence of functions such that*

$$\|f_n\|_{L^2} \leq C_0, \quad \forall n,$$

and

$$(2.1) \quad \|P_{\frac{1}{N} < \cdot < N} f_n\|_{L^\infty} \geq \alpha > 0, \quad \forall n.$$

Then one can find a subsequence (which we still label as f_n) and centers $x_n \in \mathbb{R}^2$ such that

$$f_n(x_n - \cdot) \rightharpoonup f \quad \text{weakly in } L^2$$

and

$$\|f\|_{L^2} \gtrsim 1.$$

Proof. By (2.1), we can find $x_n \in \mathbb{R}^2$ such that

$$|(P_{\frac{1}{N} < \cdot < N} f_n)(x_n)| \geq \frac{\alpha}{2}, \quad \forall n.$$

Or more precisely, for $\psi = \mathcal{F}^{-1}(\phi(N^{-1}\xi) - f(N\xi)) \in \mathcal{S}(\mathbb{R}^2)$, one has

$$(2.2) \quad \left| \int_{\mathbb{R}^2} \psi(y) f_n(x_n - y) dy \right| \geq \frac{\alpha}{2}, \quad \forall n.$$

Since by assumption the sequence $f_n(x_n - \cdot)$ is uniformly bounded in L^2 , there exists a subsequence (we still denote it by $f_n(x_n - \cdot)$) which converges weakly in L^2 to some function f . Clearly, by taking the limit in (2.2) and using the Cauchy-Schwartz inequality, one concludes that $\|f\|_{L^2} \gtrsim 1$. □

Our next lemma allows us to remove the translational degrees of freedom in studying the rigidity property of approximate harmonic maps from \mathbb{R}^2 to \mathbb{S}^2 under the geometric angle condition. By removing the translational degrees of freedom (and quotienting out other possible non-compact group actions), we can restore the compactness in the same spirit as the usual concentration-compactness procedure.

Lemma 2.2 (Removing translational invariance). *Let C_0 and α be two positive constants. Suppose a sequence of maps $d_n : \mathbb{R}^2 \rightarrow \mathbb{S}^2$, $n \geq 1$, satisfies the following conditions:*

- $\|\nabla d_n\|_{L^2} + \|\nabla d_n\|_{\dot{H}^1} \leq C_0 < \infty, \quad \forall n,$
- $\|\nabla d_n\|_{L^4}^2 \geq \alpha > 0, \quad \forall n.$

Then, upon a subsequence in n if necessary, we can find a sequence of points $x_n \in \mathbb{R}^2$ such that

$$|\nabla d_n(x_n - \cdot)|^2 \rightharpoonup f, \quad \text{weakly in } L^2$$

and $\|f\|_{L^2} \gtrsim 1$.

Proof. Let N be a dyadic integer which will be taken sufficiently large in the course of the proof. Our main idea is to localize the ∇d_n -sequence within the frequency window $[1/N, N]$. After that we apply Lemma 2.1 to find the non-trivial weak limit.

We first deal with the high frequency piece. By frequency localization, we have

$$P_{>N}((\nabla d_n)^2) = P_{>N}(P_{>N/8} \nabla d_n \cdot P_{>N/8} \nabla d_n) + 2P_{>N}(P_{\leq N/8} \nabla d_n \cdot P_{>N/8} \nabla d_n).$$

By Sobolev embedding, we then have

$$(2.3) \quad \begin{aligned} \|P_{>N}((\nabla d_n)^2)\|_{L^2} &\lesssim \|P_{>N/8} \nabla d_n\|_{L^4} \|\nabla d_n\|_{L^4} \\ &\lesssim \| |\nabla|^{\frac{1}{2}} P_{>N/8} \nabla d_n \|_{L^2} \cdot C_0 \\ &\lesssim N^{-1/2} \|\Delta d_n\|_{L^2} \cdot C_0 \\ &\lesssim N^{-1/2} C_0^2 \leq \alpha/10, \end{aligned}$$

if we take N large enough.

Similarly for the low frequency part, we use Bernstein’s inequality to get

$$(2.4) \quad \|P_{<\frac{1}{N}}((\nabla d_n)^2)\|_{L^2} \leq N^{-1} \|(\nabla d_n)^2\|_{L^1} \leq \alpha/10,$$

where again we need to take N large enough.

Now using (2.3) and (2.4) and the assumption $\|(\nabla d_n)^2\|_{L^2} \geq \alpha$, we obtain

$$\|P_{\frac{1}{N} < \cdot < N}((\nabla d_n)^2)\|_{L^2} \geq \frac{\alpha}{2}.$$

On the other hand, by Hölder, one has

$$\begin{aligned} & \|P_{\frac{1}{N} < \cdot < N}((\nabla d_n)^2)\|_{L^2} \\ & \lesssim \|(\nabla d_n)^2\|_{L^1}^{\frac{1}{2}} \| \|P_{\frac{1}{N} < \cdot < N}((\nabla d_n)^2)\|_{L^\infty}^{\frac{1}{2}} \\ & \lesssim \|P_{\frac{1}{N} < \cdot < N}((\nabla d_n)^2)\|_{L^\infty}^{\frac{1}{2}}. \end{aligned}$$

Obviously

$$\|P_{\frac{1}{N} < \cdot < N}((\nabla d_n)^2)\|_{L^\infty} \gtrsim 1.$$

We can then apply Lemma 2.1 to the sequence $f_n = |\nabla d_n|^2$ to get the result. \square

Now we are ready to prove the rigidity Theorem 1.5 under the geometric angle condition $d_3 \geq \epsilon_0$.

Proof of Theorem 1.5. First of all, using a scaling argument $d(x) \rightarrow d(\lambda x)$ with $\lambda = \|\Delta d\|_{L^2}^{-1}$, we may assume

$$\|\Delta d\|_{L^2} = 1.$$

It then suffices to show that there exists $\delta_0 = \delta_0(\epsilon_0, C_0) > 0$ such that

$$(2.5) \quad \|\nabla d\|_{L^4}^4 \leq 1 - \delta_0.$$

Assume (2.5) does not hold. Then we can find a sequence $d_n : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ such that

$$\|\Delta d_n\|_{L^2} = 1, \quad \|\nabla d_n\|_{L^2} \leq C_0, \quad d_{n3} \geq \epsilon_0,$$

but

$$\|\nabla d_n\|_{L^4} \nearrow 1, \text{ as } n \rightarrow \infty.$$

Denote

$$(2.6) \quad g_n = \Delta d_n + |\nabla d_n|^2 d_n.$$

It is easy to compute that

$$\begin{aligned} \|g_n\|_{L^2}^2 &= \|\Delta d_n\|_{L^2}^2 + \|\nabla d_n\|_{L^4}^4 + 2 \int |\nabla d_n|^2 \Delta d_n \cdot d_n dx \\ (2.7) \quad &= 1 - \|\nabla d_n\|_{L^4}^4. \end{aligned}$$

Hence

$$\|g_n\|_{L^2} \searrow 0 \text{ as } n \rightarrow \infty.$$

Applying Lemma 2.2 and performing a simple transform to eliminate the translation invariance if necessary, we conclude that

$$|\nabla d_n|^2 \rightharpoonup g(x), \text{ weakly in } L^2 \text{ as } n \rightarrow \infty$$

for some $g(x) \geq 0$ and $\|g\|_{L^2} \gtrsim 1$. Consequently, by taking the limit $n \rightarrow \infty$ for the equation of d_{n3} in (2.6), we have

$$\Delta d_3^* + g(x)\epsilon_0 \leq 0,$$

where d_3^* is the weak limit of the sequence (upon subsequence if necessary) d_{n3} which satisfies

$$\|\Delta d_3^*\|_{L^2} \leq 1, \quad \|\nabla d_3^*\|_{L^2} \leq C_0.$$

Now d_3^* is superharmonic on \mathbb{R}^2 and is bounded. Obviously d_3^* must be a constant map. But since $\Delta d_3^* + g(x)\epsilon_0 \leq 0$, one concludes that $g(x)$ must be identically 0. This contradicts the fact that $\|g\|_{L^2} \gtrsim 1$.

We have obtained the desired contradiction, and therefore (2.5) holds. Finally, (1.8) follows from (2.5) and the same computation (with d_n replaced by d) as in (2.6) and (2.7).

This concludes the proof of Theorem 1.5. □

3. PROOF OF THEOREM 1.1

As was already mentioned in the introduction, we only need to establish the a priori estimate (1.5). By using the maximum principle (applied to the third component of d_3), we have

$$\inf_{x \in \mathbb{R}^2} d_3(t, x) \geq \inf_{x \in \mathbb{R}^2} d_{03}, \quad \forall t > 0.$$

One can apply the rigidity Theorem 1.5 to conclude that

$$(3.1) \quad \|\Delta d(t) + |\nabla d(t)|^2 d(t)\|_{L^2}^2 \geq \frac{\delta_0}{2} \left(\|\Delta d(t)\|_{L^2}^2 + \|\nabla d(t)\|_{L^4}^4 \right), \quad \forall t > 0,$$

where $\delta_0 > 0$ is the same constant as in Theorem 1.5.

Consequently, the basic energy law (1.4) yields

$$(3.2) \quad \begin{aligned} \frac{1}{2} \|\nabla d(t)\|_{L^2}^2 + \frac{\delta_0}{2} \int_0^T (\|\Delta d(t)\|_{L^2}^2 + \|\nabla d(t)\|_{L^4}^4) dt \\ \leq \frac{1}{2} (\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2), \quad \forall t \geq 0. \end{aligned}$$

This clearly settles (1.5) and the first two terms in the estimate (1.7). It remains for us to prove the estimate

$$(3.3) \quad \int_0^\infty \|\nabla d(t)\|_{L^2}^2 dt < \infty.$$

By performing the basic energy estimate for the equation of $d - e_3$ and using the interpolation inequalities, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|d - e_3\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 \\ & \leq \int |\nabla d|^2 d (d - e_3) dx \leq \|\nabla d\|_{L^4}^2 \|d - e_3\|_{L^2} \\ & \lesssim \|\nabla d\|_{L^2} \|\Delta d\|_{L^2} \|d - e_3\|_{L^2} \\ & \leq \frac{1}{2} \|\nabla d\|_{L^2}^2 + C \|\Delta d\|_{L^2}^2 \|d - e_3\|_{L^2}^2, \end{aligned}$$

where $C > 0$ is an absolute constant. A Gronwall in time argument then gives

$$\|d(t, \cdot) - e_3\|_{L^2}^2 + \int_0^t \|\nabla d\|_{L^2}^2 ds \lesssim \|d_0 - e_3\|_{L^2}^2 \exp \left(\int_0^t \|\Delta d\|_{L^2}^2 ds \right), \quad \forall t \geq 0.$$

Noting (3.2), one concludes that

$$\int_0^t \|\nabla d\|_{L^2}^2 ds < \infty, \quad \forall t \geq 0.$$

This obviously implies (3.3).

This concludes the proof of Theorem 1.1. Finally, we point out that a further higher order energy estimate also implies that any H^s norm of $(u, \nabla d)$ is uniformly bounded in time for any $s > 0$.

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