Multilinear Space-Time Estimates and Applications to Local Existence Theory for Nonlinear Wave Equations

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Abstract

We prove a quadrilinear integral estimate in space-time for solutions of the homogeneous wave equation on $\mathbb{R}^{1+2}$. This estimate is a generalization of a previously known bilinear L^2 estimate, and it arises naturally in the study of the local regularity properties of a hyperbolic model equation connected with wave maps from Minkowski space $\mathbb{R}^{1+2}$ into a sphere. The scale invariant data space for this equation is $L^2(\mathbb{R}^2)$, and we prove local well-posedness for data in $H^s(\mathbb{R}^2)$ for all $s > 1/4$. In space dimension three and higher, the same equation has previously been studied by Klainerman and Machedon. Using a recently proved $L_t^1(L_x^\infty)$ bilinear estimate for solutions of the homogeneous wave equation, we obtain a simpler proof of their result, and we also extend it to the full system from which the model equation was derived.

The main new idea introduced by Klainerman and Machedon in their work on the aforementioned model equation was to estimate a Picard iterate using information not just from the preceding iterate, but from *two* previous iterates. This procedure leads to integrals of quadrilinear expressions involving functions in certain "hyperbolic" Sobolev spaces which are adapted to the wave operator. Klainerman and Machedon estimated these expressions by reducing them to trilinear and bilinear L^2 estimates in space-time for solutions of the homogeneous wave equation. Here we show that this reduction is impossible in the two-dimensional case, so the problem is of a genuinely quadrilinear nature.

A general framework for proving local well-posedness for nonlinear wave equations based on estimates in space-time Sobolev norms is developed, refining and unifying earlier results of this type.

Contents

Introduction

Introduction

On a general level, this dissertation deals with the problem of local existence and uniqueness for a system of nonlinear wave equations of the form

$$\square u = F(u, \partial u) \qquad (t,x) \in \mathbb{R}^{1+n}$$
$$u\big|_{t=0} = f \in H^s, \quad \partial_t u\big|_{t=0} = g \in H^{s-1},$$

where $\square = -\partial_t^2 + \Delta$ is the standard d'Alembertian, ∂u is the space-time gradient of u, and F is a smooth (possibly vector-valued) function with $F(0) = 0$. Given F, we want to determine the lower bound of the range of Sobolev exponents s for the data space such that the Cauchy problem is locally well-posed. Associated to F there is a real number s_c, the *critical exponent*, such that the homogeneous data space $\dot{H}^{s_c} \times \dot{H}^{s_c - 1}$ is invariant under the natural scaling law of the equation. One can then show by a scaling argument that for $s < s_c$ there is no local well-posedness of the above system.

Take the case where F is quadratic in ∂u, i.e., $F = \Gamma(u) Q(\partial u, \partial u)$, where Q is some quadratic form on $\mathbb{R}^{1+n}$ and Γ is a smooth function. For such F it turns out that, in low space dimensions, one cannot expect local well-posedness for all $s > s_c$ unless Q has a *null structure*, which roughly means that it exhibits cancellations on the light cone in Fourier space.

A trend which has emerged in recent years is to study the above Cauchy problem via an iteration argument in certain weighted space-time Sobolev norms which are intimately connected with the wave operator $\square$, and the manner in which estimates on $F(u, \partial u)$ in such norms imply local existence for the above system, at least for small-norm data, is well-known. In this dissertation we carry this program further, proving some quite general results which guarantee strong local well-posedness of the above system for arbitrarily large data, subject to some simple mapping properties of the nonlinearity F. By strong local well-posedness we mean local existence, uniqueness, persistence of higher regularity and stability with respect to perturbations of the initial data.

This is done in chapter four, theorems 14 and 15. In that chapter we also show that theorem 14 can be brought to bear upon some well-known local existence results, thus obtaining new proofs of the classical local existence theorem for hyperbolic equations and the sharp local existence theorem of Ponce and Sideris. We also apply our theorem to the wave map equation in local coordinates in space dimension two and higher, thereby improving the small-norm local existence results proved in [13] and [18] to strong local well-posedness, and also disposing of the assumption of real-analyticity of the Christoffel symbols of the target manifold, which was made in those papers. The latter improvement relies on the stability of our space-time norms under nonlinear maps, a fact proved in chapter three, theorem 11.

In the fifth and last chapter, we apply the general theory developed in chapter four to the following system in two space dimensions:

$$\Box u^I = a^I_{JK} Q(u^J, u^K) \qquad (t,x) \in \mathbb{R}^{1+2}$$
$$u\big|_{t=0} = f \in H^s, \quad D^{-1}\partial_t u\big|_{t=0} = g \in H^s,$$

where $D = \sqrt{-\Delta}$, the a^I_{JK}'s are constants and Q is the bilinear operator given by

$$Q(u,v) = \sum_{i=1}^{n} \partial_i (R_0 R_i u \cdot v - u \cdot R_0 R_i v),$$

with $R_0 = D^{-1}\partial_t$ and $R_i = D^{-1}\partial_i$. This system arises as a hyperbolic model problem for a coordinate-free formulation of the wave map equation in the case where the target manifold is a Lie group endowed with a bi-invariant metric; see chapter five for details and some references. The critical exponent for this problem is $s_c = (n-2)/2$.

In space dimensions $n \geq 3$, Klainerman and Machedon [14] proved local existence for $s > s_c$ for this model problem. The new idea introduced in that paper is that in order to estimate the k-th Picard iterate for s close to s_c, they use the information not only from the previous iterate, but from the *two* previous iterates. This procedure leads to integrals of quadrilinear expressions involving functions in certain "hyperbolic" Sobolev spaces which are adapted to the wave operator. Klainerman and Machedon estimated these integrals by reducing them to trilinear and bilinear L^2 estimates in space-time for solutions of the homogeneous wave equation. Here we show that this reduction is impossible in the two-dimensional case when s is close to the critical exponent $s_c = 0$, so one is stuck with a quadrilinear expression. The 2D problem, which has not been studied before, is therefore much harder, and we do not yet know how to get well-posedness below $s = 1/4$.

The idea of using two previous iterates still works in 2D if $s > 1/4$, and we prove well-posedness in this range. The proof of the latter result relies on a bilinear L^2 estimate for solutions of the homogeneous wave equation proved in [18].

Although we do not know how to get the optimal result in 2D, we do prove the boundedness of the quadrilinear integral in the important special case where all four functions correspond to solutions of the homogeneous wave equation. This estimate, which can be said to be the main result of the dissertation, is described below.

In chapter five we also obtain a simplified proof of the 3D result of Klainerman and Machedon [14]. The fact which makes life easier in dimension three and higher is the availability of bilinear $L^1_t(L^\infty_x)$ estimates. In dimension two no such estimate is true. Moreover, we extend the 3D result to the full system of equations from which the above model problem was derived.

The theme of the second chapter is space-time estimates of multilinear expressions involving solutions of the homogeneous wave equation, in terms of homogeneous Sobolev norms of the Cauchy data. The new result we prove here is the estimate

$$\left| \int_{\mathbb{R}^{1+2}} D^{-a} D_-(u_1 u_2) \cdot u_3 u_4 \, dt \, dx \right| \lesssim \|f_1\|_{\dot{H}^{2-a}} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2} ,$$

where $3/4 < a < 1$, the u_j are solutions of the homogeneous wave equation on $\mathbb{R}^{1+2}$ with Cauchy data $u_j\big|_{t=0} = f_j$, $\partial_t u_j\big|_{t=0} = 0$, and D_- is the multiplier with Fourier symbol[1] $\big||\tau| - |\xi|\big|$. The important point about this inequality is the asymmetry: all the regularity is concentrated on one of the functions. This inequality is essentially what one needs to conclude that the second non-trivial Picard iterate of the 2D hyperbolic model problem mentioned above is in $C(\mathbb{R}, H^s)$ for any $s > 0$. By the trivial Picard iterate we understand the solution of the homogeneous wave equation with the given data.

The above estimate generalizes the inequality

$$\left\| D_-^{1/2}(u_1 u_2) \right\|_{L^2(\mathbb{R}^{1+2})} \lesssim \|f_1\|_{\dot{H}^1} \|f_2\|_{L^2} ,$$

which was proved in [18], but it cannot itself be proved by a reduction to bilinear L^2 estimates via the Cauchy-Schwarz inequality, since the inequality

$$\left\| D^{-1} D_-^{1/2}(u_1 u_2) \right\|_{L^2(\mathbb{R}^{1+2})} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2}$$

fails to hold.

We also prove some variations of previously known bilinear estimates which are needed in subsequent chapters.

In what follows, we briefly describe the contents of the remaining chapters of the thesis.

In the first chapter, we state the standard existence and uniqueness theorem for the linear wave equation on $\mathbb{R}^{1+n}$ with Cauchy data at time $t = 0$ belonging to the space $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$, and we present a proof based on the calculus for Hilbert space-valued functions, the relevant facts of which we briefly review. In particular, we recall the definition of the integral of a Hilbert space-valued function, which is also used to some extent in chapter three.

Chapter three deals with Sobolev spaces adapted to the wave operator, and how these spaces relate to solutions of the linear wave equation. We define $H^{s,\theta}$ to be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^{1+n})$ with respect to the norm

$$\|u\|_{s,\theta} = \left\| \Lambda^s \Lambda_-^\theta u \right\|_{L^2} ,$$

where Λ and Λ_- are smooth, inhomogeneous versions of the multipliers D and D_-, respectively. For $\theta > 1/2$, this space embeds in $C(\mathbb{R}, H^s)$.

[1] τ and ξ are the Fourier variables corresponding to t and x, respectively.

Spaces of this type first appeared in [22] in the study of propagation of singularities for hyperbolic equations, and for the wave equation they were first used in [10]. Similar spaces for the KdV equation were used in [1] and [7].

We characterize the elements of $H^{s,\theta}$, for $\theta > 1/2$, in terms of H^s-valued integrals on the real line, and from this we deduce the simple but useful principle that a multilinear $L_t^q(L_x^r)$ estimate involving solutions of the homogeneous wave equation with H^s initial data implies a corresponding estimate for elements of $H^{s,\theta}$.

It is well-known that $H^{s,\theta}$ is an algebra when $s > n/2$ and $\theta > 1/2$. Here we extend this result, proving that this space is stable under the mapping $u \mapsto f(u)$ for any smooth f leaving the origin fixed. The proof is inspired by an idea from [22].

Next, we define the space $\mathcal{X}^{s,\theta} = \{u : u \in H^{s,\theta}, \partial_t u \in H^{s-1,\theta}\}$, with the obvious norm. This is the basic space in which we obtain solutions to nonlinear wave equations, and it differs somewhat from similar spaces used in earlier works on well-posedness for nonlinear wave equations; see the remark on p. 51. The remainder of the chapter is devoted to an investigation of the relation between this space and solutions of the linear wave equation.

Consider the Cauchy problem for the linear wave equation:

$$\begin{gathered}\square u = F \qquad (t,x) \in [0,T] \times \mathbb{R}^n \\ u\big|_{t=0} = f \in H^s, \quad \partial_t u\big|_{t=0} = g \in H^{s-1}.\end{gathered}$$

By the linear theory expounded in chapter one, this problem admits a unique solution $u \in C([0,T], H^s) \cap C([0,T], H^{s-1})$. For $0 < T < 1$ and $\theta > 1/2$, we prove the existence of an extension u_T of u to all of $\mathbb{R}^{1+n}$ such that

$$\|u_T\|_{s,\theta} + \|\partial_t u_T\|_{s-1,\theta} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}} + T^{\varepsilon/2} \|F\|_{s-1,\theta+\varepsilon-1}$$

for some $\varepsilon > 0$ independent of T.

This estimate is the analog of the energy inequality in the setting of the $H^{s,\theta}$ spaces, and it refines an earlier result in [13] where the time T was fixed, say $T = 1$. It has of course been known that letting T tend to 0 should produce some decay on the right hand side of the above inequality (see [19, Remark 1.8]), but a proof of this fact has not appeared before. It should be noted that the proof depends on our new definition of the basic space-time norms, cf. the remark on p. 51.

We also give sufficient conditions for a semi-norm $\|\cdot\|$, defined on some subspace of $\mathcal{S}'(\mathbb{R}^{1+n})$ containing $\mathcal{S}(\mathbb{R}^{1+n})$, to satisfy the estimate

$$\|u_T\| \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}} + T^{\varepsilon/2} \left(\|F\|_{s-1,\theta+\varepsilon-1} + \left\|\Lambda^{-1}\Lambda_-^{-1+\varepsilon} F\right\| \right),$$

with u_T as above; see theorem 16, p. 80.

Chapter 1

The Linear Wave Equation

In this chapter we recall the basic local existence properties of the linear wave equation. Since our point of view is that of L^2 theory, it is natural to use the calculus of Hilbert space-valued functions. For easy reference we review the pertinent facts, including the Hilbert space-valued integral, which will be used on numerous occasions in this dissertation. For us, the relevant Hilbert space is, unsurprisingly, the standard L^2 Sobolev space H^s.

We consider the Cauchy problem for the linear wave equation:

$$\square u = F \qquad (t,x) \in \mathbb{R}^{1+n} \tag{1.1a}$$

$$u\big|_{t=0} = f, \quad \partial_t u\big|_{t=0} = g, \tag{1.1b}$$

where $\square = -\partial_t^2 + \Delta$ and Δ is the Laplacian in the space variable x. We will also use frequently the operator $D = \sqrt{-\Delta}$.

1.1 Existence and Uniqueness

For data whose regularity is measured in L^2 Sobolev spaces, we have the following basic existence and uniqueness statement.

Proposition 1. *Assuming $f \in H^s, g \in H^{s-1}$ and $F \in L^1_{\mathrm{loc}}(\mathbb{R}, H^{s-1})$, there is a unique solution u of* (1.1) *such that*

$$u \in C(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-1}). \tag{1.2}$$

Moreover, the solution is given by the formula

$$u(t) = \cos(tD) \cdot f + D^{-1}\sin(tD) \cdot g - \int_0^t D^{-1} \sin\big((t-t')D\big) \cdot F(t')\,dt', \tag{1.3}$$

and it satisfies the energy inequality*:*

$$\begin{aligned} &\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} \\ &\qquad \lesssim \|f\|_{H^s} + (1+t)\,\|g\|_{H^{s-1}} + (1+t) \int_0^t \|F(t')\|_{H^{s-1}}\,dt' \end{aligned} \tag{1.4}$$

for all $t \geq 0$.

Remarks. (i) Formula (1.3) is derived, formally, by applying the Fourier transform in the space variable x, thereby converting the PDE problem (1.1) to an ODE problem in time t:

$$-\partial_t^2 \widehat{u}(t,\xi) - |\xi|^2 \, \widehat{u}(t,\xi) = \widehat{F}(t,\xi)$$
$$\widehat{u}(0,\xi) = \widehat{f}(\xi), \quad \partial_t \widehat{u}(0,\xi) = \widehat{g}(\xi).$$

By standard linear ODE theory and Duhamel's principle, we get

$$\widehat{u}(t,\xi) = \cos(t\,|\xi|)\widehat{f}(\xi) + |\xi|^{-1} \sin(t\,|\xi|)\widehat{g}(\xi) - \int_0^t |\xi|^{-1} \sin\big((t-t')\,|\xi|\big)\widehat{F}(t',\xi)\, dt'.$$

Applying the inverse Fourier transform then gives (1.3). This formal argument can be made into a rigorous one without much difficulty. In fact, this is how we prove uniqueness.

(ii) When we say that u solves (1.1a), we mean in the sense of distributions on $\mathbb{R}^{1+n}$. However, any $u \in C(\mathbb{R}, H^s)$ is an element of $\mathcal{D}'(\mathbb{R}^{1+n})$ by means of the bilinear pairing

$$\langle\, u, \phi \,\rangle = \int_{-\infty}^{\infty} \langle\, u(t), \phi(t) \,\rangle \; dt \quad \text{for} \quad \phi \in C_c^\infty(\mathbb{R}^{1+n}).$$

Indeed, if (ϕ_j) is a sequence of smooth functions on $\mathbb{R}^{1+n}$ which are all supported in a cube $[-a,a]^{1+n}$, and if $\partial^\alpha \phi_j \to 0$ uniformly for every multi-index α, then

$$\begin{aligned} |\langle\, u, \phi_j \,\rangle| &\leq \int_{-\infty}^{\infty} |\langle\, u(t), \phi_j(t) \,\rangle| \; dt \\ &\leq \int_{-\infty}^{\infty} \|u(t)\|_{H^s} \, \|\phi_j(t)\|_{H^{-s}} \; dt \\ &\leq \left(\sup_{|t| \leq a} \|\phi_j(t)\|_{H^N} \right) \int_{-a}^{a} \|u(t)\|_{H^s} \; dt \end{aligned}$$

for some positive integer N, and we have

$$\|\phi_j(t)\|_{H^N} \lesssim \sum_{|\alpha| \leq N} \|\partial_x^\alpha \phi_j(t)\|_{L^2} \longrightarrow 0$$

uniformly in t as $j \to \infty$. A simple modification of this argument shows that if $u \in L^\infty(\mathbb{R}, H^s)$, then $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ with the above pairing.

(iii) If u has the regularity (1.2), then the distribution derivative $\partial_t u$ agrees with the strong H^{s-1} derivative. To prove this, let v be the strong H^{s-1} derivative of u, so that

$$\left\| \frac{1}{h}\big(u(t+h) - u(t)\big) - v \right\|_{H^{s-1}} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0.$$

Given a test function ϕ, we must show that $\langle v, \phi \rangle = -\langle u, \partial_t \phi \rangle$, i.e.,

$$\int_{-\infty}^{\infty} \langle v(t), \phi(t) \rangle \, dt = -\int_{-\infty}^{\infty} \langle u(t), \partial_t \phi(t) \rangle \, dt.$$

Define $A(t) = \langle u(t), \phi(t) \rangle$. We claim that $A \in C^1(\mathbb{R})$ and that $A'(t) = \langle v(t), \phi(t) \rangle + \langle u(t), \partial_t \phi(t) \rangle$. Integrating this in t gives the relation we want. The claim is easily proved. We have

$$\begin{aligned} &|A(t+h) - A(t)| \\ &\quad \le |\langle u(t+h) - u(t), \phi(t+h) \rangle| + |\langle u(t), \phi(t+h) - \phi(t) \rangle| \\ &\quad \le \|u(t+h) - u(t)\|_{H^s} \|\phi(t+h)\|_{H^{-s}} \\ &\qquad + \|u(t)\|_{H^s} \|\phi(t+h) - \phi(t)\|_{H^{-s}}, \end{aligned}$$

and using the H^s continuity of u and the fact that ϕ is C_c^∞, we see that the limit of this as $h \to 0$ equals 0. In fact, the mapping $t \mapsto \phi(t)$ is in $C^\infty(\mathbb{R}, H^\sigma)$ for all $\sigma \in \mathbb{R}$. Clearly, it suffices to show that it is in $C^1(\mathbb{R}, H^N)$ for all positive integers N. But since ϕ and all its derivatives are uniformly continuous and $\frac{1}{h}\big(\partial^\alpha \phi(t+h) - \partial^\alpha \phi(t)\big)$ converges uniformly to $\partial_t \partial^\alpha \phi(t)$ for every α, this is immediate. Proving the rest of the claim is now an easy exercise, and we leave this to the interested reader.

(iv) The conclusion of the previous remark still holds under the weaker assumption that u is strongly H^{s-1} differentiable for almost every t, with derivative in $L^1_{\text{loc}}(\mathbb{R}, H^{s-1})$. The same proof works, except that now A is a.e. differentiable, and A' is in $L^1_{\text{loc}}(\mathbb{R})$. In fact, at every t for which the strong H^{s-1} derivative $v(t)$ exists, the proof goes through to show that $A'(t) = \langle v(t), \phi(t) \rangle + \langle u(t), \partial_t \phi(t) \rangle$.

(v) The regularity statement (1.2) is equivalent to

$$(u, \partial_t u) \in C(\mathbb{R}, H^s) \times C(\mathbb{R}, H^{s-1}), \tag{1.5}$$

where $\partial_t u$ is taken in the sense of $\mathcal{D}'(\mathbb{R}^{1+n})$. That (1.2) implies (1.5) is the content of remark (iii), and the converse follows from the fact that if $u \in C(\mathbb{R}, H^\sigma)$ and the distribution derivative $\partial_t u$ is in $C(\mathbb{R}, H^\sigma)$, then this derivative is in fact a strong H^σ derivative, whence $u \in C^1(\mathbb{R}, H^\sigma)$. This fact is proved in proposition 2 below.

Proposition 2. *Assume $u \in C(\mathbb{R}, H^\sigma), \sigma \in \mathbb{R}$.*

(a) If the distribution derivative $\partial_t u \in L^1_{\text{loc}}(\mathbb{R}, H^\sigma)$, then $\partial_t u(t)$ is the strong H^σ derivative of u for a.e. t.

(b) If we strengthen the hypothesis in (a) to $\partial_t u \in C(\mathbb{R}, H^\sigma)$, then the same conclusion holds for every t.

The proof requires the following lemma.

Lemma 1. *(a) If $u \in \mathcal{D}'(\mathbb{R})$ and the distribution derivative u' vanishes, then u is a constant.*

(b) If $u \in C(\mathbb{R}, H^\sigma)$ and the distribution derivative $\partial_t u$ vanishes, then $u(t) = u(0)$ for every t.

Proof. To prove (a), fix $\phi \in C_c^\infty(\mathbb{R})$ such that $\int \phi = 1$. Then if u is indeed a constant, we should have $u = \langle u, \phi \rangle$, i.e.,

$$\langle u - \langle u, \phi \rangle, \psi \rangle = 0 \quad \text{for all } \psi \in C_c^\infty.$$

To prove this, it would be enough to show that the left hand side equals $\langle u, \theta' \rangle$ for some $\theta \in C_c^\infty$, since by assumption this vanishes. But we have $\langle u - \langle u, \phi \rangle, \psi \rangle = \langle u, \psi \rangle - \langle u, \phi \rangle \int \psi = \left\langle u, \psi - \left(\int \psi \right) \phi \right\rangle$, and if we set $\theta(t) = \int_{-\infty}^t \psi - \left(\int \psi \right) \int_{-\infty}^t \phi$, then it is easy to see that $\theta \in C_c^\infty$ and $\theta' = \psi - \left(\int \psi \right) \phi$. This concludes the proof of part (a).

To prove (b), we fix a test function $\phi \in C_c^\infty(\mathbb{R}^n)$. We want to show that $\langle u(t), \phi \rangle$ is independent of t. Call this quantity $U(t)$. Then $U \in C(\mathbb{R})$, hence in $\mathcal{D}'(\mathbb{R})$, and by part (a) it suffices to show that the distribution derivative U' vanishes. But for any $\psi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} \langle U', \psi \rangle = - \langle U, \psi' \rangle = \int_{-\infty}^{\infty} \langle u(t), \phi \rangle \, \psi'(t) \, dt &= \int_{-\infty}^{\infty} \langle u(t), \psi'(t) \phi \rangle \, dt \\ &= - \langle u, \partial_t \theta \rangle = \langle \partial_t u, \theta \rangle = 0, \end{aligned}$$

where $\theta(t, x) = \psi(t)\phi(x)$. □

Proof of proposition 2. If we can show that

$$u(t) = u(0) + \int_0^t \partial_t u(t') \, dt' \quad \text{for all } t, \tag{1.6}$$

then the conclusion follows from the Hilbert space version of the fundamental theorem of calculus, which is proved below. Denote by $v(t)$ the quantity on the right hand side of (1.6). We want to use lemma 1 to conclude that $u(t) = v(t)$ for every t. By the fundamental theorem of calculus, $v \in C(\mathbb{R}, H^\sigma)$ and the strong H^σ derivative $v'(t)$ exists for a.e. t and equals $\partial_t u(t)$. Therefore, by remark (iv), the distribution derivatives $\partial_t u$ and $\partial_t v$ are equal, so lemma 1 guarantees that $u(t) - v(t) = 0$ for every t. □

We now turn to the proof of proposition 1. First we prove that u defined by the formula (1.3) is a solution of (1.1) with regularity (1.2). Consider first $u(t) = D^{-1} \sin(tD) \cdot g$. We claim that $u \in \bigcap_{j=0}^\infty C^j(\mathbb{R}, H^{s-j})$ and that

$$\begin{aligned} \partial_t u(t) &= \cos(tD) \cdot g, \\ \partial_t^2 u(t) &= -D \sin(tD) \cdot g, \\ \partial_t^3 u(t) &= -D^2 \cos(tD) \cdot g \end{aligned}$$

and so on. In other words, the ordinary chain rule applies. Assuming this, we have $-\partial_t^2 u = D^2 u = -\Delta u$, so u is a solution of (1.1) with $f = 0$.

The claim follows immediately from a simple observation:

Lemma 2. *Assume that ϕ is a C^∞ function on $\mathbb{R}$ such that ϕ and ϕ' are bounded, and consider the multiplier $D^j\phi(tD)$ acting on functions on $\mathbb{R}^n$:*

$$D^j\phi(tD)\cdot f = \mathcal{F}^{-1}\left(|\xi|^j\,\phi(t\,|\xi|)\widehat{f}(\xi)\right)$$

whenever this is well-defined. Here $j \geq -1$ is an integer. If $j = -1$ we assume in addition that $\phi(0) = 0$.

With these assumptions, $D^j\phi(tD)$ is a bounded mapping from H^s to H^{s-j} for any $s \in \mathbb{R}$. Moreover, if $f \in H^s$, then

$$D^j\phi(tD)\cdot f \in C(\mathbb{R}, H^{s-j}) \cap C^1(\mathbb{R}, H^{s-j-1}),$$

and the chain rule applies, i.e.,

$$\partial_t\left(D^j\phi(tD)\cdot f\right) = D^{j+1}\phi'(tD)\cdot f.$$

Proof. Let us do the case $j = -1$, since it is more difficult. We have

$$\begin{aligned}
\left\|D^{-1}\phi(tD)\cdot f\right\|_{H^{s+1}}^2 &= \int \left(1+|\xi|^2\right)^{s+1}\left||\xi|^{-1}\,\phi(t\,|\xi|)\widehat{f}(\xi)\right|^2 d\xi \\
&\leq 2t^2\,\|\phi'\|_{L^\infty}^2 \int_{|\xi|<1}\left(1+|\xi|^2\right)^s\left|\widehat{f}(\xi)\right|^2 d\xi \\
&\quad + 4\,\|\phi\|_{L^\infty}^2 \int_{|\xi|\geq 1}\left(1+|\xi|^2\right)^s\left|\widehat{f}(\xi)\right|^2 d\xi,
\end{aligned}$$

where we used the mean value theorem to estimate $|\phi(r)|\,/r \leq \|\phi'\|_{L^\infty}$ for $r > 0$. This proves boundedness of the operator.

Set $u(t) = D^{-1}\phi(tD)\cdot f$. To prove continuity, we use again the mean value theorem on the low frequency part, obtaining

$$\begin{aligned}
&\|u(t+h) - u(t)\|_{H^{s+1}}^2 \\
&\quad = \int \left(1+|\xi|^2\right)^{s+1}|\xi|^{-2}\left|\phi\big((t+h)\,|\xi|\big) - \phi\big(t\,|\xi|\big)\right|^2\left|\widehat{f}(\xi)\right|^2 d\xi \\
&\quad \leq 2h^2\,\|\phi'\|_{L^\infty}^2 \int_{|\xi|<1}\left(1+|\xi|^2\right)^s\left|\widehat{f}(\xi)\right|^2 d\xi \\
&\qquad + 4\int_{|\xi|\geq 1}\left(1+|\xi|^2\right)^s\left|\phi\big((t+h)\,|\xi|\big) - \phi\big(t\,|\xi|\big)\right|^2\left|\widehat{f}(\xi)\right|^2 d\xi.
\end{aligned}$$

By the dominated convergence theorem, the last integral vanishes in the limit $h \to 0$. Finally, to prove differentiability, we write

$$\begin{aligned}
&\left\|\frac{1}{h}\big(u(t+h) - u(t)\big) - \phi'(tD)\cdot f\right\|_{H^s}^2 \\
&\quad = \int \left(1+|\xi|^2\right)^s\left|\frac{\phi\big((t+h)\,|\xi|\big) - \phi\big(t\,|\xi|\big)}{h\,|\xi|} - \phi'(t\,|\xi|)\right|^2\left|\widehat{f}(\xi)\right|^2 d\xi.
\end{aligned}$$

By the dominated convergence theorem, this converges to 0 as $h \to 0$. □

The term $\cos(tD) \cdot f$ in (1.3) is treated in the same way, so what remains is the inhomogeneous part of the solution, namely

$$u(t) = -\int_0^t D^{-1} \sin\big((t-t')D\big) \cdot F(t')\, dt'.$$

Note that this is a Hilbert space-valued integral. Since elements of H^s for $s < 0$ are in general not functions, it may not be possible to evaluate this integral pointwise. We interrupt the proof of proposition 1.1 in order to recall the relevant facts from the theory of integration of functions with values in a separable Hilbert space. For the conclusion of the proof, see section 1.4.

1.2 Hilbert space integrals

Let $(X, \mathcal{M}, \mu)$ be a measure space and H a separable Hilbert space. We say that a map $f : X \to H$ is *measurable* if it is $(\mathcal{M}, \mathcal{B}_H)$-measurable, where $\mathcal{B}_H$ is the Borel σ-algebra of H.

Proposition 3. *A map $f : X \to H$ is measurable iff $\phi \circ f$ is $(\mathcal{M}, \mathcal{B}_\mathbb{R})$-measurable for all $\phi \in C(H, \mathbb{R})$.*

Proof. Assume that $\phi \circ f$ is $(\mathcal{M}, \mathcal{B}_\mathbb{R})$-measurable for all $\phi \in C(H, \mathbb{R})$. Fix a ball $U = \{y \in H : \|y - y_0\| < R\}$, and set $\phi(y) = \|y - y_0\|$ for $y \in H$. Then ϕ is continuous and $U = \phi^{-1}([0, R))$, whence $f^{-1}(U) = (\phi \circ f)^{-1}([0, R)) \in \mathcal{M}$. Since the open balls in H generate $\mathcal{B}_H$, this proves that f is measurable. The converse is trivial. □

If $f : X \to H$ is measurable, then $x \mapsto \|f(x)\|$ is $(\mathcal{M}, \mathcal{B}_\mathbb{R})$-measurable, since $y \mapsto \|y\|$ is continuous, and we define $\|f\|_{L^p} = \left(\int \|f(x)\|^p \, d\mu(x)\right)^{1/p}$ for $1 \le p < \infty$ and $\|f\|_{L^\infty} = \text{ess sup}_{x \in X} \|f(x)\|$. Then we let $L^p = L^p(X, H)$ be the vector space of measurable maps $f : X \to H$ with $\|f\|_{L^p} < \infty$. If we identify maps which are equal a.e., then $\|\cdot\|_{L^p}$ is a norm on L^p, and L^p is a Banach space. (The standard proof works.)

By a *simple function* we mean a function $f : X \to H$ of the form $f = \sum_1^n y_j \chi_{E_j}$, where $y_j \in H$, $E_j \in \mathcal{M}$, $\mu(E_j) < \infty$ and χ_E denotes the characteristic function of a set E. It is obvious how to define $\int f(x)\, d\mu(x)$ when f is a simple function. The next lemma allows us to pass to the limit and define the integral of any $f \in L^1$.

Proposition 4. *If $f \in L^1$, there is a sequence (f_j) of simple functions such that $\|f_j - f\|_{L^1} \to 0$.*

Proof. Fix a dense sequence (y_n) in H. We may assume $y_n \neq 0$ for all n. For $\varepsilon > 0$ set $B_n^\varepsilon = \{y \in H : \|y - y_n\| < \varepsilon \|y_n\|\}$. It is readily verified that $\cup_{n=1}^\infty B_n^\varepsilon = H \setminus \{0\}$ for $0 < \varepsilon < 1$. Now set $A_{nj} = B_n^{1/j} \setminus \bigcup_{m=1}^{n-1} B_m^{1/j}$ and $E_{nj} = f^{-1}(A_{nj})$. Then for all j we have $\{x : f(x) \neq 0\} = \bigcup_{n=1}^\infty E_{nj}$, and the union is disjoint. Define $f_j = \sum_{n=1}^{N(j)} y_n \chi_{E_{nj}}$, where $N(j)$ is a positive integer to be determined. We claim that

$$\|f_j(x) - f(x)\| \le \frac{2}{j} \|f(x)\| \quad \text{for} \quad x \in \operatorname{supp}(f_j), j \ge 2.. \tag{1.7}$$

This would give

$$\int \|f_j(x) - f(x)\| \, d\mu(x) \le \frac{2}{j} \|f\|_{L^1} + \int\limits_{\bigcup_{n>N(j)} E_{nj}} \|f(x)\| \, d\mu(x),$$

and since $f \in L^1$, the last integral can be made as small as we like by choosing $N(j)$ sufficiently large.

To prove (1.7), note that if $x \in E_{nj}$ with $1 \le n \le N(j)$, then $f_j(x) = y_n$ and $f(x) \in B_n^{1/j}$, whence $\|f_j(x) - f(x)\| < 1/j \|y_n\|$. But by the triangle inequality, this implies $\|y_n\| \le 2 \|f(x)\|$ if $j \ge 2$. □

Thus, the simple functions form a dense subspace of the complete space L^1, so it follows immediately that there is a unique bounded linear operator $\int : L^1 \to H$ such that $\int y\chi_E = \mu(E)y$ for $y \in H$ and $E \in \mathcal{M}$ with $\mu(E) < \infty$, and we write $\int f = \int f(x)\, d\mu(x)$ for $f \in L^1$. This operator is usually called the *Bochner integral.*

Theorem 1. *The Bochner integral has the following properties:*

(a) $\left\| \int f \right\| \le \|f\|_{L^1}$ *for all* $f \in L^1$.

(b) If $H = \mathbb{C} = \mathbb{R}^2$ *with the standard norm, then the Bochner integral coincides with the standard integral.*

(c) (The dominated convergence theorem) Assume that (f_n) *is a sequence in* L^1 *converging a.e. to* f*, and that there is a* $g \in L^1(X, \mathbb{R})$ *such that* $\|f_n(x)\| \le g(x)$ *for a.e.* x*. Then* $f \in L^1$ *and* $\int f_n \to \int f$.

(d) If H' *is separable Hilbert space,* T *is a bounded linear operator from* H *to* H' *and* $f \in L^1(X, H)$*, then* $Tf \in L^1(X, H')$*, and* $\int Tf = T \int f$.

Proof. Properties (a),(b) and (d) are obvious for simple functions, and the general statements follow by simple limiting arguments. To prove (c), first note that by redefining f_n and f on a set of measure zero, we may assume that $f_n(x) \to f(x)$ for every x. This changes nothing on the level of L^1, since we identify elements of L^1 which are equal a.e. By proposition 3, f is measurable. Indeed, if $\phi \in C(H, \mathbb{R})$, then $\phi \circ f_n(x) \to \phi \circ f(x)$ for every x, so $\phi \circ f$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable. It now follows from the scalar version of the dominated convergence theorem that $\int \|f_n - f\| \to 0$. By part (a), this implies $\int f_n \to \int f$. □

This concludes our discussion of the Bochner integral.

1.3 More Hilbert space calculus

In the remainder of this chapter, the measure space will be $\mathbb{R}$ equipped with Lebesgue measure dt. We let $L^1_{\text{loc}}(\mathbb{R}, H)$ be the space of measurable functions $f : \mathbb{R} \to H$ such that $\int_K \|f(t)\|\, dt < \infty$ for every compact subset K of $\mathbb{R}$.

The Fundamental Theorem of Calculus. *Let $f \in L^1_{\text{loc}}(\mathbb{R}, H)$, where H is a separable Hilbert space.*

(a) If $F : \mathbb{R} \to H$ is defined by

$$F(t) = \int_0^t f(s)\, ds,$$

then $F \in C(\mathbb{R}, H)$ and $F'(t) = f(t)$ for a.e. t. Moreover, if f is continuous, then $F \in C^1(\mathbb{R}, H)$ and $F' = f$.

(b) Conversely, if $F \in C(\mathbb{R}, H)$ has derivative $f(t)$ for a.e. t, then

$$F(t) - F(0) = \int_0^t f(s)\, ds$$

for all t.

Proof. We first show that part (b) follows from part (a). Indeed, if we set $G(t) = F(0) + \int_0^t f(s)\, ds$, then $G \in C(\mathbb{R}, H)$ and $G'(t) = f(t)$ for a.e. t, by part (a). Therefore, $F - G$ is a continuous map from $\mathbb{R}$ to H whose derivative vanishes a.e., whence

$$t \longmapsto \phi(t) = \|F(t) - G(t)\|^2 = \langle\, F(t) - G(t), F(t) - G(t)\,\rangle$$

is a continuous function on $\mathbb{R}$ whose derivative vanishes a.e. By the scalar version of the theorem, ϕ is therefore a constant, and this constant must be 0, since $\phi(0) = 0$.

We now prove part (a). Continuity of F follows from the dominated convergence theorem. To prove differentiability, note that

$$\left\| \frac{1}{h}\big(F(t+h) - F(t)\big) - f(t) \right\| = \left\| \frac{1}{h} \int_t^{t+h} \big(f(s) - f(t)\big)\, ds \right\| \le \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|\, ds$$

for every $h \neq 0$. Thus, it suffices to show that for a.e. t,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|\, ds = 0. \tag{1.8}$$

If f is continuous, a direct application of the scalar version of the fundamental theorem gives (1.8) for every t. In general this will not work, since the variable

t appears not only in the limits of integration, but in the integrand as well. To avoid this problem, we fix a dense sequence (y_n) in H. By the scalar version of the theorem, for each n there is a set $E_n \subseteq \mathbb{R}$ of measure zero such that for every $t \in \mathbb{R} \setminus E_n$,

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \|f(s) - y_n\| \, ds = \|f(t) - y_n\| .$$

Set $E = \bigcup_1^\infty E_n$. Then E has measure zero, and for every $t \in \mathbb{R} \setminus E$ and every n we have

$$\limsup_{h\to 0} \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\| \, ds \leq 2 \|f(t) - y_n\| .$$

Since the points y_n form a dense set in H, this proves (1.8). □

Proposition 5. *(Differentiation under the integral sign) Let H and H' be separable Hilbert spaces such that $H \subseteq H'$ and the inclusion map is bounded. Assume that $f : \mathbb{R}^2 \to H$ has the following properties:*

(a) $f(s, \cdot) \in C(\mathbb{R}, H) \cap C^1(\mathbb{R}, H')$ for all s.

(b) $f(\cdot, t)$ is measurable for all t.

(c) For every compact interval $[a, b] \subseteq \mathbb{R}$, there are $g, k \in L^1_{\text{loc}}(\mathbb{R})$ such that

$$\|f(s,t)\|_H \leq g(s), \quad \|\partial_t f(s,t)\|_{H'} \leq k(s) \tag{1.9}$$

for all $t \in [a, b]$ and all $s \in \mathbb{R}$.

Then the map $u : \mathbb{R} \to H$ defined by

$$u(t) = \int_0^t f(s,t)\, ds$$

is in $C(\mathbb{R}, H)$, and

$$u'(t) = f(t,t) + \int_0^t \partial_t f(s,t)\, ds \quad \textit{for a.e. } t. \tag{1.10}$$

Moreover, if $t \mapsto f(t,t)$ is a continuous map from $\mathbb{R}$ to H', then (1.10) *holds for every t.*

Proof. It suffices to prove this for $t \in [-N, N]$, with N an arbitrary positive integer, and then we may assume that (1.9) holds for $t \in [-N-1, N+1]$. The continuity of u follows easily from the dominated convergence theorem. To prove differentiability a.e., we write

$$\begin{aligned}
&\frac{1}{h}\big(u(t+h)-u(t)\big) - f(t,t) - \int_0^t \partial_t f(s,t)\,ds \\
&\qquad = \int_0^{t+h}\left(\frac{f(s,t+h)-f(s,t)}{h} - \partial_t f(s,t)\right)ds + \int_t^{t+h}\partial_t f(s,t)\,ds \\
&\qquad\quad + \frac{1}{h}\int_t^{t+h}\big(f(s,t)-f(t,t)\big)\,ds \\
&\qquad = I_1 + I_2 + I_3.
\end{aligned}$$

We have to show that $\|I_j\|_{H'} \to 0$ as $h \to 0$ for $j = 1,2,3$.

Since the integrand of I_1 converges pointwise to 0 in H' as $h \to 0$, the dominated convergence theorem guarantees that $I_1 \to 0$, provided that the integrand is bounded uniformly in h by a non-negative locally integrable function. But by the fundamental theorem of calculus,

$$\frac{f(s,t+h)-f(s,t)}{h} - \partial_t f(s,t) = \frac{1}{h}\int_t^{t+h}\big(\partial_t f(s,t') - \partial_t f(s,t)\big)\,dt',$$

and the H' norm of this is bounded by $\frac{1}{h}\int_t^{t+h} 2k(s)\,dt' = 2k(s)$. This proves that $\lim_{h\to 0}\|I_1\|_{H'} = 0$ for every t.

The dominated convergence theorem also shows that $\lim_{h\to 0}\|I_2\|_{H'} = 0$ for every t, since (1.9) gives a uniform bound on the integrand.

Finally, to prove that $\lim_{h\to 0}\|I_3\|_{H'} = 0$ for a.e. t, we fix a dense sequence (t_n) in $\mathbb{R}$, and write

$$\begin{aligned}
\|I_3\|_{H'} &\le \frac{1}{h}\int_t^{t+h}\|f(s,t)-f(s,t_n)\|_{H'}\,ds \\
&\quad + \left\|\frac{1}{h}\int_t^{t+h} f(s,t_n)\,ds - f(t,t_n)\right\|_{H'} + \|f(t,t_n)-f(t,t)\|_{H'} \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

By the fundamental theorem of calculus, for each n there is a set E_n of measure zero such that $\lim_{h\to 0} J_2 = 0$ for $t \in \mathbb{R}\setminus E_n$. Let $E = \bigcup_{n=1}^{\infty} E_n$. Since $f(s,t) - f(s,t_n) = \int_{t_n}^t \partial_t f(s,t')\,dt'$, we have $J_1 \le \frac{1}{h}\int_t^{t+h}|t-t_n|\,k(s)\,ds$, and by the fundamental theorem, $\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} k(s)\,ds = k(t)$ for $t \in \mathbb{R}\setminus F$, where F has measure zero. We conclude that

$$\limsup_{h\to 0}\|I_3\|_{H'} \le |t-t_n|\,k(t) + \|f(t,t_n)-f(t,t)\|_{H'}$$

for $t \in \mathbb{R}\setminus(E\cup F)$ and all n, and since the t_n are dense in $\mathbb{R}$ and f is continuous in its second argument, this shows that $\lim_{h\to 0}\|I_3\|_{H'} = 0$ a.e.

If $t \mapsto f(t,t)$ is in $C(\mathbb{R},H')$, then the function defined by the right hand side of (1.10)—call it $v(t)$—is in $C(\mathbb{R},H')$. Since $v = u'$ a.e., part (b) of the fundamental theorem implies that $u(t) = u(0) + \int_0^t v(s)\,ds$, and then part (a) tells us that $u'(t) = v(t)$ for every t. □

1.4 Conclusion of proof of proposition 1

Recall that we are considering the term $u(t) = -\int_0^t D^{-1}\sin\big((t-t')D\big)\cdot F(t')\,dt'$. Lemma 2 and its proof show that the integrand satisfies the hypotheses of the last proposition, with $H = H^s$ and $H' = H^{s-1}$. Therefore, u has the regularity (1.2), and

$$u'(t) = -\int_0^t \cos\big((t-t')D\big)\cdot F(t')\,dt'$$

for all t. By the same argument, u' is a.e. differentiable, and

$$\begin{aligned} u''(t) &= -F(t) - \int_0^t -D\sin\big((t-t')D\big)\cdot F(t')\,dt' \\ &= -F(t) - \Delta\int_0^t D^{-1}\sin\big((t-t')D\big)\cdot F(t')\,dt' \end{aligned}$$

for a.e. t. Here we used Theorem 1, part (d), and the fact that Δ is a bounded linear map from H^s to H^{s-2}. By remarks (iii) and (iv), the distribution derivatives $\partial_t u$ and $\partial_t^2 u$ agree with u' and u'', respectively, and it follows that (1.1) is satisfied with $f = g = 0$. This completes the proof of the existence part of proposition 1. The energy inequality (1.4) follows easily from the formula (1.3), since the proof of lemma 2 shows that the operator norm of $D^j\phi(tD)$ is bounded by $2(1+|t|)\max\big(\|\phi\|_{L^\infty}, \|\phi'\|_{L^\infty}\big)$ if $j = -1$, and by $\|\phi\|_{L^\infty}$ if $j \geq 0$.

To prove uniqueness, it suffices, by linearity, to prove that if u is a solution of (1.1) with f, g and F all identically zero, and if u satisfies the regularity assumption (1.2), then u vanishes. With these assumptions, $\partial_t^2 u = \Delta u$, and since

$$\Delta u \in C(\mathbb{R}, H^{s-2}) \cap C^1(\mathbb{R}, H^{s-3}),$$

it follows from proposition 2 that

$$\partial_t u \in C(\mathbb{R}, H^{s-1}) \cap C^1(\mathbb{R}, H^{s-2}).$$

In fact, by taking time derivatives of the equation and applying proposition 2 repeatedly, one finds that $u \in \bigcap_{j=1}^\infty C^j(\mathbb{R}, H^{s-j})$. Applying the Fourier transform in the space variable x, it now follows that $t \mapsto \widehat{u(t)}(\xi)$ satisfies the ODE initial value problem in remark (i) for every $\xi \in \mathbb{R}^n$, with f, g and F vanishing. We conclude that $\widehat{u(t)}(\xi) = 0$ for all $(t,\xi) \in \mathbb{R}^{1+n}$, so u vanishes.

Chapter 2

Space-Time Estimates for the Wave Equation

The principal tool for proving existence theorems for nonlinear wave equations is multilinear space-time estimates for solutions of the homogeneous wave equation. The main new result proved in this chapter is a sharp quadrilinear integral estimate in two space dimensions. This estimate, which we prove in section 2.3, is in some sense a generalization of the bilinear L^2 estimate (2.10) proved in [18], but the estimate we prove is genuinely quadrilinear; it cannot be proved by a reduction to bilinear estimates.

In this chapter we also prove some variations of previously known bilinear estimates, which will be needed in subsequent chapters. This is done in section 2.2. First, however, we recall the linear estimates.

2.1 The linear estimates

In space dimension $n \geq 2$, the solution of the Cauchy problem

$$\Box u = 0 \qquad (t,x) \in \mathbb{R}^{1+n} \tag{2.1a}$$

$$u\big|_{t=0} = f \in \mathcal{S}(\mathbb{R}^n), \quad \partial_t u\big|_{t=0} = 0 \tag{2.1b}$$

satisfies the mixed norm estimate

$$\|u\|_{L_t^q(L_x^r)} \lesssim \|f\|_{\dot{H}^s} \tag{2.2}$$

iff

$$s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}, \quad \frac{2}{\min(1, \gamma(r))} \leq q \leq \infty, \quad \text{and} \quad 2 \leq r < \infty, \tag{2.3}$$

where $\gamma(r) = (n-1)\left(\frac{1}{2} - \frac{1}{r}\right)$ and

$$\|u\|_{L_t^q(L_x^r)} = \left(\int_{\mathbb{R}} \|u(t,\cdot)\|_{L^q(\mathbb{R}^n)}^r \, dt \right)^{1/r}$$

For the proof, and further references, see [4], [5].

Remarks. (i) The assumption that f is in the Schwartz class can be removed by the following density argument. Assume (2.2) to be true for data in the Schwartz class. Let $f \in \dot{H}^s$, and let u be the unique solution of (2.1) such that

$$u \in C(\mathbb{R}; \dot{H}^s) \cap C^1(\mathbb{R}; \dot{H}^{s-1}).$$

Then we want to show that the distribution u is a function in $L_t^q(L_x^r)$ satisfying (2.2). First recall that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\dot{H}^\sigma(\mathbb{R}^n)$ for all $\sigma > -\frac{n}{2}$. For by definition, the Fourier transform $\mathcal{F}$ maps $\dot{H}^\sigma$ isometrically onto $L^2(|\xi|^{2\sigma}\, d\xi)$, and the latter space contains $\mathcal{S}$ as a dense subset precisely when $\sigma > -\frac{n}{2}$. But $s > 0$, so there is a sequence (f_j) in $\mathcal{S}$ such that $f_j \to f$ in $\dot{H}^s$. Let u_j solve (2.1) with f replaced by f_j. By assumption, u_j satisfies (2.2), so (u_j) is a Cauchy sequence in the complete space $L_t^q(L_x^r)$, and hence converges in this space to some function $\widetilde{u}$ which satisfies (2.2) with f on the right hand side. It therefore remains to show that u and $\widetilde{u}$ are one and the same distribution. But $u_j \to \widetilde{u}$ in the sense of distributions on $\mathbb{R}^{1+n}$, so it suffices to show that the same is true for u. For every test function $\phi \in C_c^\infty(\mathbb{R}^{1+n})$ we have

$$\begin{aligned}\int_{\mathbb{R}} &\langle u_j(t) - u(t), \phi(t) \rangle \, dt \\ &\le \int_{\mathbb{R}} \|u_j(t) - u(t)\|_{\dot{H}^s} \|\phi(t)\|_{\dot{H}^{-s}} \, dt \le \|f_j - f\|_{\dot{H}^s} \int_{\mathbb{R}} \|\phi(t)\|_{\dot{H}^{-s}} \, dt,\end{aligned}$$

and since $s < \frac{n}{2}$, the last integral is bounded.

(ii) The assumption $\partial_t u\big|_{t=0} = 0$ is made simply for ease of notation, and implies no loss of generality, since all the estimates we state in this chapter for solutions of the wave equation hold, more generally, for the *half wave* operators $f \mapsto e^{\pm itD} f$. The unique solution

$$u \in C(\mathbb{R}; \dot{H}^s) \cap C^1(\mathbb{R}; \dot{H}^{s-1})$$

of (2.1a) with initial conditions

$$u\big|_{t=0} = f \in \dot{H}^s, \quad \partial_t u\big|_{t=0} = g \in \dot{H}^{s-1},$$

is given by

$$u = \frac{1}{2}\left(e^{itD} f + e^{-itD} f\right) + \frac{1}{2i}\left(e^{itD} D^{-1} g - e^{-itD} D^{-1} g\right).$$

Hence, if $\left\|e^{\pm itD} f\right\|_{L_t^q(L_x^r)} \lesssim \|f\|_{\dot{H}^s}$, then $\|u\|_{L_t^q(L_x^r)} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}}$.

2.2 Bilinear estimates

Since q, $r \geq 2$, we can interpret (2.2) as a bilinear estimate. That is, if u is the solution of (2.1) and v satisfies

(2.4a) $$\Box v = 0 \qquad (t,x) \in \mathbb{R}^{1+n}$$

(2.4b) $$v\big|_{t=0} = g \in \mathcal{S}(\mathbb{R}^n), \quad \partial_t v\big|_{t=0} = 0,$$

then by Hölder's inequality and (2.2),

(2.5) $$\|uv\|_{L_t^{q/2}(L_x^{r/2})} \lesssim \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^s},$$

or equivalently,

$$\left\|D^{-s}u \cdot D^{-s}v\right\|_{L_t^{q/2}(L_x^{r/2})} \lesssim \|f\|_{L^2} \|g\|_{L^2}.$$

Once the estimates are written like this, it is natural to ask for which $a > 0$ we have

$$\left\|D^{-a}\left(D^{-s+a/2} \cdot uD^{-s+a/2}v\right)\right\|_{L_t^{q/2}(L_x^{r/2})} \lesssim \|f\|_{L^2} \|g\|_{L^2}.$$

The first such result, for the case $q = r = 4$, was obtained by Klainerman and Machedon [15].

Theorem 2. ***(Klainerman-Machedon)*** *If $n \geq 3$, then*

(2.6) $$\left\|D_+^{-a}(uv)\right\|_{L^2} \lesssim \|f\|_{\dot{H}^{(n-1)/4-a/2}} \|g\|_{\dot{H}^{(n-1)/4-a/2}} \quad \textit{for} \quad 0 \leq a < \frac{n-2}{2},$$

where D_+^γ is the multiplier with symbol $(|\tau| + |\xi|)^\gamma$.

Recently, Klainerman and Tataru [19] proved:

Theorem 3. ***(Klainerman-Tataru)*** *If $n \geq 2$, (2.3) holds and*

$$(2/q, \gamma(r)) \neq (1,1), \quad \textit{where} \quad \gamma(r) = (n-1)\left(\frac{1}{2} - \frac{1}{r}\right),$$

then

(2.7) $$\left\|D_+^{-a}(uv)\right\|_{L_t^{q/2}(L_x^{r/2})} \lesssim \|f\|_{\dot{H}^{s-a/2}} \|g\|_{\dot{H}^{s-a/2}}$$

for $0 \leq a < 1 - \frac{2}{r}$.

It should be remarked that the estimate (2.7) is optimal on the line $\gamma = 2/q$ in the $(1/q, 1/r)$-plane, in the sense that the estimate fails if $a > 1 - 2/r$. If $\gamma > 2/q$, on the other hand, one can obtain a better result by using the Sobolev inequality. We will not need this, and hence ignore it.

Even more generally, one can ask what are the possible estimates of the form

(2.8) $$\left\|D_+^\alpha D_-^\beta(uv)\right\|_{L_t^{q/2}(L_x^{r/2})} \lesssim \|f\|_{\dot{H}^{s_1}} \|g\|_{\dot{H}^{s_2}},$$

where D_-^γ is the multiplier whose symbol is $\big||\tau| - |\xi|\big|^\gamma$. Estimates of this type come up naturally in connection with bilinear null forms, as we shall see in chapters four and five. Again, the first result of this type was proved in [15].

Theorem 4. ***(Klainerman-Machedon)*** *In dimension $n = 2$,*

$$\left\| D_+^{-a} D_-^{1/4}(uv) \right\|_{L^2} \lesssim \|f\|_{\dot{H}^{3/8-a/2}} \|g\|_{\dot{H}^{3/8-a/2}} \tag{2.9}$$

for $0 \le a < \frac{1}{4}$.

Subsequently, Klainerman and Selberg [18] proved:

Theorem 5. ***(Klainerman-Selberg)*** *In any dimension $n \ge 2$,*

$$\left\| D_-^{1/2}(uv) \right\|_{L^2} \lesssim \|f\|_{\dot{H}^{n/2}} \|g\|_{L^2} . \tag{2.10}$$

Very recently, Klainerman and Foschi [9] have completely settled the question of which are the possible estimates of type (2.8) for $(q, r) = (4, 4)$ and $(q, r) = (\infty, 4)$. For other exponents q and r, this question remains open. We refer to [8] for more details.

The bilinear estimates that will be used in this dissertation are (2.6), (2.7), (2.9) and (2.10). For our applications, however, it is essential to replace the space-time differentiation operator D_+ in (2.6), (2.7) and (2.9) by the operator D, which acts only in the space variables, see theorem 6 below. The rest of this section is devoted to proving that the modified estimates are true.

We write $u = u_+ + u_-$, where $u_\pm = e^{\pm itD} f/2$. Similarly, $v = v_+ + v_-$. It suffices to prove the estimates with uv replaced by u_+v_+ and u_+v_-. Since $|\tau| \le |\xi|$ on the support of $\widehat{u_+v_-}(\tau, \xi)$, the operators D_+ and D are essentially the same when applied to u_+v_-, so in this case there is nothing to prove. Hence, it suffices to prove the estimates for u_+v_+. In fact, for (2.7), it will not be necessary to make this decomposition of uv, since the proof reduces everything to linear estimates, and in a linear estimate like (2.2), u_+ and u_- are completely equivalent, by simply negating of the sign of t.

We will prove the following.

Theorem 6. *(a) If $n \ge 3$ and $0 \le a < (n-1)/2$, then*

$$\left\| D^{-a}(u_+v_+) \right\|_{L^2} \lesssim \|f\|_{\dot{H}^{(n-1)/4-a/2}} \|g\|_{\dot{H}^{(n-1)/4-a/2}} . \tag{2.11}$$

(b) If $n = 2$ and $0 \le a < 1/2$, then

$$\left\| D^{-a} D_-^{1/4}(u_+v_+) \right\|_{L^2} \lesssim \|f\|_{\dot{H}^{3/8-a/2}} \|g\|_{\dot{H}^{3/8-a/2}} . \tag{2.12}$$

(c) If $n \ge 2$, $0 \le a < 1 - 2/r$, (2.3) holds and $(2/q, \gamma) \ne (1, 1)$, then

$$\left\| D^{-a}(uv) \right\|_{L_t^{q/2}(L_x^{r/2})} \lesssim \|f\|_{\dot{H}^{s-a/2}} \|g\|_{\dot{H}^{s-a/2}} . \tag{2.13}$$

Notice that in the first two of the above inequalities, the upper bounds for a are larger than in (2.6) and (2.9). This reflects the fact that the estimates for u_+v_+ are less delicate than the ones for u_+v_-.

The proof of (2.7) given in [19] can be modified in a simple way to give (2.13). Instead of doing a Littlewood-Paley decomposition in both time and space, we just decompose space. We include the details, partly for the sake of completeness, but also because the dyadic method employed can be used to prove (2.11) and (2.12).

2.2.1 Review of the dyadic method

We fix a function $\beta \in C_c^\infty(\mathbb{R}^n)$ with the properties

(i) β takes values in $[0,1]$

(ii) β is supported in the spherical shell $\mathcal{C} = \{\xi : 1/2 \le |\xi| \le 2\}$

(iii) $\sum_{j\in\mathbb{Z}} \beta(\xi/2^j) = 1$ for all $\xi \neq 0$.

Then for any $F \in \mathcal{S}'$ we define

$$\Delta_j F = \mathcal{F}^{-1}\left(\beta\left(\frac{\xi}{2^j}\right)\widehat{F}(\xi)\right) = \Phi_j * F,$$

where $\Phi_j = 2^{jn}\left(\mathcal{F}^{-1}\beta\right)(2^j\cdot)$. The sequence $(\Delta_j F)$, which we call the *dyadic decomposition* of F, has the following properties:

(i) $\mathcal{F}(\Delta_j F)$ is supported in $2^j\mathcal{C}$

(ii) If $F \in L^p$, $1 \le p \le \infty$, then $\|\Delta_j F\|_{L^p} \le \left\|\mathcal{F}^{-1}\beta\right\|_{L^1} \|F\|_{L^p}$.

(iii) If $\widehat{F} \in L^1$, then $F = \sum_{j\in\mathbb{Z}} \Delta_j F$ pointwise on $\mathbb{R}^n$.

(iv) If $F \in \dot{H}^\sigma$, for any $\sigma \in \mathbb{R}$, then $F = \sum_{j\in\mathbb{Z}} \Delta_j F$ in the sense of $\dot{H}^\sigma$.

(v) The norms $\|\cdot\|_{\dot{H}^\sigma}$ and $\left(\sum_{j\in\mathbb{Z}} 2^{2j\sigma} \|\Delta_j(\cdot)\|_{L^2}^2\right)^{1/2}$ are equivalent.

The proofs of the last two properties rely, of course, on Plancherel's theorem. However, the utility of the dyadic decomposition is not limited to norms based on L^2. The next lemma is the tool we need to relate the dyadic decomposition to L^p norms for general p.

Lemma 3. *Assume* $0 < r_1 < r_2$, *and let*

$$\widetilde{\mathcal{C}} = \{\xi : r_1 \le |\xi| \le r_2\}.$$

If $F \in L^p$, $1 \le p \le \infty$, *and* $\widehat{F}$ *is supported in* $\lambda\widetilde{\mathcal{C}}$ *for some* $\lambda > 0$, *then for any* $\theta \in \mathbb{R}$,

$$C^{-1}\lambda^\theta \|F\|_{L^p} \le \left\|D^\theta F\right\|_{L^p} \le C\lambda^\theta \|F\|_{L^p},$$

where C *is a constant depending only on* θ, p, n *and* $\widetilde{\mathcal{C}}$.

Proof. Pick a function $\phi \in C_c^\infty\left(\{\xi : r_1/2 \le |\xi| \le 2r_2\}\right)$ such that $\phi(\xi) = 1$ for all $\xi \in \widetilde{\mathcal{C}}$. Then

$$|\xi|^\theta \widehat{F} = \lambda^\theta \left|\lambda^{-1}\xi\right|^\theta \phi(\lambda^{-1}\xi)\widehat{F}$$

in the sense of tempered distributions, so that

$$D^\theta F = \lambda^{\theta+n}\left(D^\theta\psi\right)(\lambda\,\cdot) * F,$$

where $\psi = \mathcal{F}^{-1}\phi$. Hence it follows from Young's inequality that

$$\left\|D^\theta F\right\|_{L^p} \le C\lambda^\theta \left\|D^\theta\psi\right\|_{L^1} \|F\|_{L^p}.$$

This proves the second inequality, and the proof of the first is similar. □

Definition 1. *Write $\mathbb{R}^n$ as an almost disjoint union of cubes,*

$$\mathbb{R}^n = \bigcup_{\mu \in \mathbb{Z}^n} (\mu + Q), \quad Q = [-1/2, 1/2]^n,$$

and fix a function $\psi \in C_c^\infty$ such that

$$0 \leq \psi \leq 1, \quad \psi\big|_Q = 1, \quad \operatorname{supp} \psi \subseteq Q^*,$$

where $Q^* = [-1,1]^n$ is the double of Q. For any $\mu \in \mathbb{Z}^n$ set

$$\phi_\mu(\xi) = \frac{\psi(\xi - \mu)}{\sum_{\nu \in \mathbb{Z}^n} \psi(\xi - \nu)},$$

so that

$$\operatorname{supp} \phi_\mu \subseteq \mu + Q^*, \quad \sum_{\mu \in \mathbb{Z}^n} \phi_\mu(\xi) \equiv 1.$$

Now define the operator $\Omega_\mu : \mathcal{S}' \to \mathcal{S}'$ by

$$\widehat{\Omega_\mu F}(\xi) = \phi_\mu(\xi) \widehat{F}(\xi).$$

Proposition 6. *Let T be a bilinear operator given by*

$$\widehat{T(f,g)}(\xi) = \int_{\mathbb{R}^n} \kappa(\xi - \eta, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta)\, d\eta,$$

where κ is homogeneous of degree γ and

$$\int |\xi|^{-a} \left| \kappa(\xi - \eta, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta) \right| d\xi\, d\eta < \infty \tag{2.14}$$

for all $0 \leq a < A$ and $f, g \in \mathcal{S}$. Fix $\rho = \pm 1$, and define

$$u(t) = e^{itD} f, \quad v(t) = e^{\rho itD} g \quad \text{for} \quad f, g \in \mathcal{S}.$$

Let $1 \leq q, r \leq \infty$, set $s = \frac{\gamma}{2} + \frac{n}{2} - \frac{n}{2r} - \frac{1}{2q}$ and assume that

$$\|T(u,v)\|_{L_t^q(L_x^r)} \lesssim \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^s} \tag{2.15}$$

and

$$\begin{aligned} &\|\Delta_0 T(\Omega_\mu u, \Omega_\nu v)\|_{L_t^q(L_x^r)} \\ &\quad \lesssim |\mu|^{s - A/2} |\nu|^{s - A/2} \Bigg(\sum_{|\alpha| \leq C} \|\Omega_{\mu+\alpha} f\|_{L^2} \Bigg) \Bigg(\sum_{|\beta| \leq C} \|\Omega_{\nu+\beta} g\|_{L^2} \Bigg) \end{aligned} \tag{2.16}$$

for all $f, g \in \mathcal{S}$ and all $\mu, \nu \in \mathbb{Z}^n$ such that

$$|\mu + \nu| \leq 2(1 + \sqrt{n}) \quad \text{and} \quad |\mu|, |\nu| \geq 8(1 + \sqrt{n}).$$

Then

$$\left\| D^{-a} T(u,v) \right\|_{L_t^q(L_x^r)} \lesssim \|f\|_{\dot{H}^{s-a/2}} \|g\|_{\dot{H}^{s-a/2}} \tag{2.17}$$

for all $f, g \in \mathcal{S}$ *and* $0 \le a < A$. *Moreover, if* $(q,r) \neq (\infty,\infty)$ *and*

$$T : \dot{H}^{s-a/2} \times \dot{H}^{s-a/2} \longrightarrow \dot{H}^{2s-n/2-\gamma-a} \quad \textit{for} \quad 0 \le a < A, \tag{2.18}$$

then (2.17) *holds for all* $f, g \in \dot{H}^{s-a/2}$, $0 \le a < A$.

Remark. The condition (2.16) may look complicated, but what it says is essentially the following: If $\widehat{f}$ and $\widehat{g}$ are supported in diametrically opposite cubes of side length 1 and at distance R from the origin, then $T(u,v)$ satisfies the estimate

$$\|T(u,v)\|_{L_t^q(L_x^r)} \lesssim R^{-A} \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^s} .$$

Proof of proposition 6. Note that the condition $2s = \gamma + n - n/r - 1/q$ comes from scaling. We write

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f, \quad g = \sum_{k \in \mathbb{Z}} \Delta_k g,$$

and set $u_j = e^{itD} \Delta_j f$ and $v_k = e^{\rho itD} \Delta_k g$, so that

$$u = \sum_{j \in \mathbb{Z}} u_j, \quad v = \sum_{k \in \mathbb{Z}} v_k. \tag{2.19}$$

We will assume that $a > 0$, since the case $a = 0$ is just (2.15). Since (2.14) holds, it follows from the dominated convergence theorem that

$$D^{-a} T(u,v) = \sum_{j,k \in \mathbb{Z}} D^{-a} T(u_j, v_k)$$

pointwise on $\mathbb{R}^{1+n}$. Hence, by Minkowski's integral inequality,

$$\left\| D^{-a} T(u,v) \right\|_{L_t^q(L_x^r)} \le \sum_{j,k \in \mathbb{Z}} \left\| D^{-a} T(u_j, v_k) \right\|_{L_t^q(L_x^r)} .$$

It is natural to consider separately the terms for which $|j-k| \le 2$ and those for which $|j-k| > 2$. We call these terms *diagonal* and *off-diagonal*, respectively.

The off-diagonal case

Let us assume $j - k > 2$, since the case $k - j > 2$ is treated in exactly the same way. It is easily seen that

$$\operatorname{supp} \widehat{T(u_j, v_k)} \subseteq 2^j \widetilde{\mathcal{C}},$$

where

$$\widetilde{\mathcal{C}} = \{ \xi : 1/4 \le |\xi| \le 4 \} .$$

Hence, by lemma 3 and (2.15),

$$\begin{aligned}\left\|D^{-a}T(u_j,v_k)\right\|_{L_t^q(L_x^r)} &\lesssim 2^{-aj}\left\|T(u_j,v_k)\right\|_{L_t^q(L_x^r)}\\ &\lesssim 2^{-aj}\left\|\Delta_j f\right\|_{\dot H^s}\left\|\Delta_k g\right\|_{\dot H^s}\\ &\lesssim 2^{-a(j-k)/2}\left\|\Delta_j f\right\|_{\dot H^{s-a/2}}\left\|\Delta_k g\right\|_{\dot H^{s-a/2}}.\end{aligned}$$

Summing and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\sum_{j-k>2}\left\|D^{-a}T(u_j,v_k)\right\|_{L_t^q(L_x^r)}\\ &\quad\lesssim \sum_{l>2}2^{-al/2}\sum_{j\in\mathbb{Z}}\left\|\Delta_j f\right\|_{\dot H^{s-a/2}}\left\|\Delta_{j-l} g\right\|_{\dot H^{s-a/2}}\\ &\quad\lesssim \left\|f\right\|_{\dot H^{s-a/2}}\left\|g\right\|_{\dot H^{s-a/2}}.\end{aligned}$$

This concludes the proof in the off-diagonal case.

The diagonal case

This case is more delicate, and requires a refined decomposition in frequency space. We start by noting that

$$D^{-a}T(u_j,v_k)=\sum_{l\le\max(j,k)+2}\Delta_l D^{-a}T(u_j,v_k)$$

pointwise on $\mathbb{R}^{1+n}$. Since D^{-a} and Δ_l commute, it follows from Minkowski's integral inequality and lemma 3 that

$$\begin{aligned}&\sum_{|j-k|\le2}\left\|D^{-a}T(u_j,v_k)\right\|_{L_t^q(L_x^r)}\\ &\quad\le \sum_{|j-k|\le2}\ \sum_{l\le\max(j,k)+2}\left\|D^{-a}\Delta_l T(u_j,v_k)\right\|_{L_t^q(L_x^r)}\\ &\quad\lesssim \sum_{|j-k|\le2}\ \sum_{l\le\max(j,k)+2}2^{-al}\left\|\Delta_l T(u_j,v_k)\right\|_{L_t^q(L_x^r)}.\end{aligned}$$

Now if we could prove that

$$\left\|\Delta_l T(u_j,v_k)\right\|_{L_t^q(L_x^r)}\lesssim 2^{-A(j-l)}\left\|\Delta_j f\right\|_{\dot H^s}\left\|\Delta_k g\right\|_{\dot H^s}, \tag{2.20}$$

then it would follow that

$$\begin{aligned}&\sum_{|j-k|\le2}\ \sum_{l\le\max(j,k)+2}2^{-al}\left\|\Delta_l T(u_j,v_k)\right\|_{L_t^q(L_x^r)}\\ &\quad\lesssim \sum_{|j-k|\le2}\ \sum_{l\le j+4}2^{-(A-a)(j-l)}\left\|\Delta_j f\right\|_{\dot H^{s-a/2}}\left\|\Delta_k g\right\|_{\dot H^{s-a/2}}\\ &\quad= \sum_{i\ge-4}2^{-(A-a)i}\sum_{|m|\le2}\sum_{j\in\mathbb{Z}}\left\|\Delta_j f\right\|_{\dot H^{s-a/2}}\left\|\Delta_{j+m} g\right\|_{\dot H^{s-a/2}}\\ &\quad\lesssim \left\|f\right\|_{\dot H^{s-a/2}}\left\|g\right\|_{\dot H^{s-a/2}}\end{aligned}$$

for $0 \le a < A$, and this would complete the proof of the diagonal case.

By scaling, it suffices to prove (2.20) for $l = 0$. Indeed, setting $\lambda = 2^l$, it is easily checked that

$$\Delta_l T\big(u_j(t), v_k(t)\big)(x) = \lambda^{\gamma+2n} \Delta_0 T\big(e^{i\lambda tD}\Delta_{j-l}F, e^{\rho i\lambda tD}\Delta_{k-l}G\big)(\lambda x),$$

where $\widehat{F}(\xi) = \widehat{f}(\lambda\xi)$ and $\widehat{G}(\eta) = \widehat{g}(\lambda\eta)$. Assuming that (2.20) holds for $l = 0$, we therefore get

$$\begin{aligned}
&\|\Delta_l T(u_j, v_k)\|_{L_t^q(L_x^r)} \\
&\quad = \lambda^{\gamma+2n-n/r-1/q} \left\|\Delta_0 T\big(e^{itD}\Delta_{j-l}F, e^{\rho itD}\Delta_{k-l}G\big)(x)\right\|_{L_t^q(L_x^r)} \\
&\quad \lesssim \lambda^{\gamma+2n-n/r-1/q} 2^{-A(j-l)} \|\Delta_{j-l}F\|_{\dot{H}^s} \|\Delta_{k-l}G\|_{\dot{H}^s} \\
&\quad = \lambda^{\gamma+2n-n/r-1/q-n-2s} 2^{-A(j-l)} \|\Delta_j f\|_{\dot{H}^s} \|\Delta_k g\|_{\dot{H}^s} \\
&\quad = 2^{-A(j-l)} \|\Delta_j f\|_{\dot{H}^s} \|\Delta_k g\|_{\dot{H}^s},
\end{aligned}$$

since $2s = \gamma + n - n/r - 1/q$.

It only remains to prove (2.20) for $l = 0$. If $2^j < 2^7(1+\sqrt{n})$, we can simply estimate

$$\|\Delta_0 T(u_j, v_k)\|_{L_t^q(L_x^r)} \lesssim \|T(u_j, v_k)\|_{L_t^q(L_x^r)}$$

and apply (2.15).

Henceforth we assume $2^j \ge 2^7(1+\sqrt{n})$. If $u_j(t)$ is at frequency ξ and $v_k(t)$ at frequency η, then $\Delta_0 T\big(u_j(t), v_k(t)\big)$ is at frequency $\xi + \eta = O(1)$. Thus ξ and η will only interact if they are in opposite cubes of side length $O(1)$. It therefore makes sense to decompose ξ-space and η-space into unit cubes.

We write

$$\Delta_0 T(u_j, v_k) = \sum_{\mu,\nu\in\mathbb{Z}^n} \Delta_0 T(\Omega_\mu u_j, \Omega_\nu v_k),$$

and conclude that

$$\|\Delta_0 T(u_j, v_k)\|_{L_t^q(L_x^r)} \lesssim \sum_{\mu,\nu\in\mathbb{Z}^n} \|\Delta_0 T(\Omega_\mu u_j, \Omega_\nu v_k)\|_{L_t^q(L_x^r)}. \tag{2.21}$$

But since

$$\xi \in \mu + Q^*, \quad \eta \in \nu + Q^*, \quad \xi + \eta = O(1) \quad \Longrightarrow \quad \mu + \nu = O(1),$$

the sum can be restricted to the set of μ, ν such that $|\mu + \nu| \le C$ for some C depending only on n. In fact, $C = 2(1+\sqrt{n})$ will suffice. Since we are assuming $2^j \ge 2^7(1+\sqrt{n})$ and $|j - k| \le 2$, it is easily checked that

$$|\mu|, |\nu| \ge 2^3(1+\sqrt{n}),$$

and that

$$\mu \notin 2^j\widetilde{\mathcal{C}} \Longrightarrow \Omega_\mu \Delta_j f = 0, \quad \nu \notin 2^k\widetilde{\mathcal{C}} \Longrightarrow \Omega_\nu \Delta_k g = 0. \tag{2.22}$$

Hence, by (2.21), (2.16) and the Cauchy-Schwarz inequality,

$$\begin{aligned}\|\Delta_0 T(u_j, v_k)\|_{L_t^q(L_x^r)} &\lesssim 2^{-Aj} \sum_{|\beta|\le C} \sum_{\alpha} \|\Omega_\alpha \Delta_j f\|_{\dot H^s} \|\Omega_{-\alpha+\beta}\Delta_k g\|_{\dot H^s} \\ &\lesssim 2^{-Aj} \|\Delta_j f\|_{\dot H^s} \|\Delta_k g\|_{\dot H^s},\end{aligned}$$

where α, $\beta \in \mathbb{Z}^n$ and C depends only on n.

Assume

$$T : \dot H^{s-a/2} \times \dot H^{s-a/2} \longrightarrow \dot H^{2s-n/2-\gamma-a}$$

for $0 \le a < A$. Then since

$$2s - n/2 - \gamma - a = n/2 - n/r - 1/q - a,$$

and since D^θ is bounded from $\dot H^\sigma$ to $\dot H^{\sigma-\theta}$ iff $\sigma - \theta < \frac{n}{2}$, it follows that

$$D^{-a}T : \dot H^{s-a/2} \times \dot H^{s-a/2} \longrightarrow \dot H^{n/2-n/r-1/q}$$

for $0 \le a < A$. Pick two sequences (f_j) and (g_j) in $\mathcal{S}$ which approximate f and g in $\dot H^{s-a/2}$, and let $u_j = e^{itD} f_j$ and $v_j = e^{\rho itD} g_j$. Then $\big(D^{-a}T(u_j, v_j)\big)$ is a Cauchy sequence in $L_t^{q/2}(L_x^{r/2})$ and therefore converges in this space to some function F. To show that $F = D^{-a}T(u, v)$ in $\mathcal{D}'(\mathbb{R}^{1+n})$, it then suffices to show that $D^{-a}T(u_j, v_j) \to D^{-a}T(u, v)$ in $\mathcal{D}'(\mathbb{R}^{1+n})$. But using the boundedness and bilinearity of $D^{-a}T$, we get

$$\begin{aligned}&\int_{\mathbb{R}} \langle D^{-a}T(u_j, v_j) - D^{-a}T(u, v), \phi(t)\rangle \, dt \\ &\quad\le \int_{\mathbb{R}} \big\|D^{-a}T(u_j, v_j) - D^{-a}T(u, v)\big\|_{\dot H^{n/2-n/r-1/q}} \|\phi(t)\|_{\dot H^{n/r+1/q-n/2}} \, dt \\ &\quad\le \int_{\mathbb{R}} \|u_j(t) - u(t)\|_{\dot H^{s-a/2}} \|v(t)\|_{\dot H^{s-a/2}} \|\phi(t)\|_{\dot H^{n/r+1/q-n/2}} \, dt \\ &\qquad + \|u_j(t)\|_{\dot H^{s-a/2}} \int_{\mathbb{R}} \|v_j(t) - v(t)\|_{\dot H^{s-a/2}} \|\phi(t)\|_{\dot H^{n/r+1/q-n/2}} \, dt \\ &\quad\le \|f_j - f\|_{\dot H^{s-a/2}} \|g\|_{\dot H^{s-a/2}} \int_{\mathbb{R}} \|\phi(t)\|_{\dot H^{n/r+1/q-n/2}} \, dt \\ &\qquad + \|f_j\|_{\dot H^{s-a/2}} \|g_j - g\|_{\dot H^{s-a/2}} \int_{\mathbb{R}} \|\phi(t)\|_{\dot H^{n/r+1/q-n/2}} \, dt,\end{aligned}$$

and since we assume $n/r + 1/q - n/2 > -n/2$, the last integral is bounded. □

2.2.2 Proof of (2.13)

By proposition 6 and (2.5), it suffices to prove that

$$\begin{aligned}&\|\Omega_\mu u \cdot \Omega_\nu v\|_{L_t^{q/2}(L_x^{r/2})} \\ &\qquad \lesssim \big(|\mu|\,|\nu|\big)^{s+1/r-1/2} \Big(\sum_{|\alpha|\le C} \|\Omega_{\mu+\alpha} f\|_{L^2}\Big)\Big(\sum_{|\beta|\le C} \|\Omega_{\nu+\beta} g\|_{L^2}\Big)\end{aligned} \tag{2.23}$$

for some constant C. We use the following variation of the linear space-time estimate (2.2), proved in [19]:

$$\left\| e^{\pm itD}\Omega_\mu f \right\|_{L_t^q(L_x^r)} \le C_{q,r,n} \, |\mu|^{s+1/r-1/2} \, \|f\|_{L^2} \quad \text{for} \quad \mu \neq 0. \tag{2.24}$$

Since

$$\Omega_\mu \Omega_\nu = 0 \quad \text{if} \quad |\mu - \nu| > C,$$

where C depends only on n, the estimate (2.24) implies that

$$\begin{aligned} \left\| e^{\pm itD}\Omega_\mu f \right\|_{L_t^q(L_x^r)} &= \left\| e^{\pm itD}\Omega_\mu \Big(\sum_{|\alpha|\le C} \Omega_{\mu+\alpha} f \Big) \right\|_{L_t^q(L_x^r)} \\ &\le \sum_{|\alpha|\le C} \left\| e^{\pm itD}\Omega_\mu \Omega_{\mu+\alpha} f \right\|_{L_t^q(L_x^r)} \\ &\lesssim |\mu|^{s+1/r-1/2} \sum_{|\alpha|\le C} \|\Omega_{\mu+\alpha} f\|_{L^2} . \end{aligned}$$

Hence, (2.23) follows after an application of Hölder's inequality.

Note that (2.18) follows from the fact that if

$$0 \le \sigma, \, \theta < \frac{n}{2} \quad \text{and} \quad \sigma + \theta > 0,$$

then the multiplication operator

$$m : \mathcal{S} \times \mathcal{S} \to \mathcal{S}, \quad m(f,g) = fg$$

extends to a bounded operator

$$\widetilde{m} : \dot{H}^\sigma \times \dot{H}^\theta \to \dot{H}^{\sigma+\theta-n/2}.$$

2.2.3 Proof of (2.11)

In this case $A = (n-1)/2$ and with notation as in proposition 6, $2s = (n-1)/2$, so it suffices to prove that

$$\left\| \Omega_\mu e^{itD} f \cdot \Omega_\nu e^{itD} g \right\|_{L^2} \lesssim \|\Omega_\mu f\|_{L^2} \, \|\Omega_\nu g\|_{L^2}$$

for all μ, $\nu \in \mathbb{Z}^n$ such that

$$|\mu + \nu| \le 2(1+\sqrt{n}) \quad \text{and} \quad |\mu|, \, |\nu| \ge 8(1+\sqrt{n}).$$

The space-time Fourier transform of $\Omega_\mu e^{itD} f \cdot \Omega_\nu e^{itD} g$ evaluated at (τ, ξ) is

$$2\pi \int \delta\big(\tau - |\eta| - |\xi - \eta|\big) \phi_\mu(\xi - \eta) \widehat{f}(\xi - \eta) \phi_\nu(\eta) \widehat{g}(\eta) \, d\eta,$$

so by Plancherel's theorem and the Cauchy-Schwarz inequality, it is enough to show that

$$\sup_{\tau,\xi} \int \chi_{\mu+Q^*}(\xi - \eta) \chi_{\nu+Q^*}(\eta) \delta\big(\tau - |\eta| - |\xi - \eta|\big) \, d\eta < \infty.$$

But the conditions on μ and ν imply that if $\xi - \eta \in \mu + Q^*$ and $\eta \in \nu + Q^*$, then

$$|\xi - \eta| \sim |\eta| \sim |\mu| \sim |\nu| \quad \text{and} \quad |\xi - \eta| + |\eta| \geq 2\,|\xi|\,.$$

Consequently, the ellipsoid

$$E = \big\{\eta : \tau = |\xi - \eta| + |\eta|\big\}$$

approximates a sphere of radius $\sim \tau$, and

$$\begin{aligned} &\int \chi_{\mu+Q^*}(\xi - \eta)\chi_{\nu+Q^*}(\eta)\delta\big(\tau - |\eta| - |\xi - \eta|\big)\,d\eta \\ &\qquad = \int_E \chi_{\mu+Q^*}(\xi - \eta)\chi_{\nu+Q^*}(\eta)\left(\frac{|\xi - \eta|\,|\eta|}{\tau^2 - |\xi|^2}\right)^{\frac{1}{2}} dA(\eta) \\ &\qquad \sim \int_E \chi_{\mu+Q^*}(\xi - \eta)\chi_{\nu+Q^*}(\eta)\,dA(\eta), \end{aligned}$$

where dA is surface measure on E. For the calculation required to go from the Dirac delta to surface measure, we refer to section 2.2.5 below.

The last integral is the surface area of the intersection of a unit cube with a convex surface, and is therefore uniformly bounded.

2.2.4 Proof of (2.12)

In this case, the symbol of T is

$$\kappa(\xi, \eta) = \big(|\xi| + |\eta| - |\xi + \eta|\big)^{1/4},$$

so with notation as in proposition 6, we have $\gamma = 1/4$, and hence $2s = 3/4$. Since $A = 1/2$, we conclude that $s - A/2 = 1/8$. We must therefore prove

$$\left\|T(\Omega_\mu e^{itD} f, \Omega_\nu e^{itD} g)\right\|_{L^2} \lesssim |\mu|^{1/8}\,|\nu|^{1/8}\,\|\Omega_\mu f\|_{L^2}\,\|\Omega_\nu g\|_{L^2}$$

for all μ, $\nu \in \mathbb{Z}^n$ such that

$$|\mu + \nu| \leq 2(1 + \sqrt{n}) \quad \text{and} \quad |\mu|,\, |\nu| \geq 8(1 + \sqrt{n}).$$

It suffices to show that

$$\int \kappa^2(\xi - \eta, \eta)\chi_{\mu+Q^*}(\xi - \eta)\chi_{\nu+Q^*}(\eta)\delta\big(\tau - |\eta| - |\xi - \eta|\big)\,d\eta \lesssim |\mu|^{1/4}\,|\nu|^{1/4}$$

uniformly in τ, ξ. But

$$\begin{aligned} &\int \kappa^2(\xi - \eta, \eta)\chi_{\mu+Q^*}(\xi - \eta)\chi_{\nu+Q^*}(\eta)\delta\big(\tau - |\eta| - |\xi - \eta|\big)\,d\eta \\ &\qquad = \int_E \chi_{\mu+Q^*}(\xi - \eta)\chi_{\nu+Q^*}(\eta)\left(\frac{|\xi - \eta|\,|\eta|}{\tau + |\xi|}\right)^{\frac{1}{2}} dA(\eta) \\ &\qquad \sim |\mu|^{1/4}\,|\nu|^{1/4}\int_E \chi_{\mu+Q^*}(\xi - \eta)\chi_{\nu+Q^*}(\eta)\,dA(\eta), \end{aligned}$$

and the last integral is uniformly bounded.

Since $\kappa(\xi,\eta) \le 2\,|\xi|^{1/8}\,|\eta|^{1/8}$, it follows that

$$T : \dot H^{3/8-a/2} \times \dot H^{3/8-a/2} \longrightarrow \dot H^{-1/2-a}$$

for $0 \le a < 1/2$.

2.2.5 Further remarks

Let us show that

$$\int f(\eta)\delta\big(\phi(\eta)\big)\,d\eta = \int_{\tau=|\xi-\eta|\mp|\eta|} f(\eta)\frac{|\xi-\eta|^{1/2}\,|\eta|^{1/2}}{\left|\tau^2-|\xi|^2\right|^{1/2}}dA(\eta),$$

where dA is surface measure on the hypersurface $\{\eta \in \mathbb{R}^n : \tau = |\xi-\eta| \mp |\eta|\}$. We have

$$\int f(\eta)\delta\big(\tau \pm |\eta| - |\xi-\eta|\big)\,d\eta = \int_{\tau=|\xi-\eta|\mp|\eta|} f(\eta)\frac{dA(\eta)}{|\nabla_\eta(\tau - |\xi-\eta| \pm |\eta|)|}.$$

Moreover,

$$\nabla_\eta(\tau - |\xi-\eta| \pm |\eta|) = \frac{\xi-\eta}{|\xi-\eta|} \pm \frac{\eta}{|\eta|},$$

and a straightforward calculation yields

$$\left|\frac{\xi-\eta}{|\xi-\eta|} + \frac{\eta}{|\eta|}\right|^2 = \frac{|\xi|^2 - \big||\xi-\eta|-|\eta|\big|^2}{|\xi-\eta|\,|\eta|},$$
$$\left|\frac{\xi-\eta}{|\xi-\eta|} - \frac{\eta}{|\eta|}\right|^2 = \frac{(|\xi-\eta|+|\eta|)^2 - |\xi|^2}{|\xi-\eta|\,|\eta|}.$$

From the proof of proposition 6, it is clear that we can replace the assumption (2.15) by certain dyadic estimates. First, we need the off-diagonal dyadic estimate

$$\|T(u_j,v_k)\|_{L_t^q(L_x^r)} \lesssim \|\Delta_j f\|_{\dot H^s}\,\|\Delta_k g\|_{\dot H^s} \quad \text{for} \quad |j-k| > 2.$$

If $q = r = 2$, this estimate follows if we can show that

$$\int \frac{\kappa^2(\xi-\eta,\eta)}{|\xi-\eta|^{2s}\,|\eta|^{2s}}\chi_{2^j\mathcal{C}}(\xi-\eta)\chi_{2^k\mathcal{C}}(\eta)\delta\big(\tau \pm |\eta| - |\xi-\eta|\big)\,d\eta$$

is uniformly bounded for all τ, ξ and all j, $k \in \mathbb{Z}$ satisfying the off-diagonal condition $|j-k| > 2$. But this condition implies that the above integral is bounded by

$$\int_{|\xi-\eta|+|\eta|\le C|\xi|} \frac{\kappa^2(\xi-\eta,\eta)}{|\xi-\eta|^{2s}\,|\eta|^{2s}}\delta\big(\tau \pm |\eta| - |\xi-\eta|\big)\,d\eta. \tag{2.25}$$

Second, we need the low frequency diagonal estimate

$$\|\Delta_0 T(u_j, v_k)\|_{L_t^q(L_x^r)} \lesssim \|\Delta_j f\|_{\dot{H}^s} \|\Delta_k g\|_{\dot{H}^s} \quad \text{for} \quad j, k \le C,$$

where C depends on n. But in this case we also have $|\xi - \eta| + |\eta| \le C|\xi|$, so again the estimate can be reduced to bounding (2.25).

For all the operators T that we are interested in, the integral (2.25) is easy to bound by explicit calculation, even when we choose the $+$ sign in the delta function, which of course corresponds to the estimate for u_+v_-. See [18], [9]. We are simplifying somewhat here, since $|\xi - \eta|$ and $|\eta|$ are not always taken to the same power, cf. (2.10).

Thus, the off-diagonal estimate is in principle the easy part, and the interesting part is the diagonal estimate (2.16). In this case we need to bound

$$\begin{aligned}
&\chi_C(\xi) \int \kappa^2(\xi - \eta, \eta) \chi_{\mu + Q^*}(\xi - \eta) \chi_{\nu + Q^*}(\eta) \delta\big(\tau \pm |\eta| - |\xi - \eta|\big)\, d\eta \\
&= \chi_C(\xi) \int_{\tau = |\xi - \eta| \mp |\eta|} \kappa^2(\xi - \eta, \eta) \chi_{\mu + Q^*}(\xi - \eta) \chi_{\nu + Q^*}(\eta) \frac{|\xi - \eta|^{1/2} |\eta|^{1/2}}{\left|\tau^2 - |\xi|^2\right|^{1/2}} dA(\eta)
\end{aligned}$$

in terms of powers of $|\mu|$, $|\nu|$ for all μ, $\nu \in \mathbb{Z}^n$ such that

$$|\mu + \nu| \le 2(1 + \sqrt{n}) \quad \text{and} \quad |\mu|,\ |\nu| \ge 8(1 + \sqrt{n}).$$

Let us denote by $I = I(\tau, \xi)$ the above integral. If we are looking at u_+v_+, then

$$\chi_{\mu + Q^*}(\xi - \eta) \chi_{\nu + Q^*}(\eta) \frac{|\xi - \eta|^{1/2} |\eta|^{1/2}}{\left|\tau^2 - |\xi|^2\right|^{1/2}} \sim \chi_{\mu + Q^*}(\xi - \eta) \chi_{\nu + Q^*}(\eta),$$

and as we have seen, the estimate is then easy. In the case u_+v_-, on the other hand,

$$\begin{aligned}
&\chi_C(\xi) \chi_{\mu + Q^*}(\xi - \eta) \chi_{\nu + Q^*}(\eta) \frac{|\xi - \eta|^{1/2} |\eta|^{1/2}}{\left|\tau^2 - |\xi|^2\right|^{1/2}} \\
&\qquad \sim \chi_C(\xi) \chi_{\mu + Q^*}(\xi - \eta) \chi_{\nu + Q^*}(\eta) \frac{|\mu|^{1/2} |\nu|^{1/2}}{(|\xi| - |\tau|)^{1/2}},
\end{aligned}$$

and the factor $(|\xi| - |\tau|)^{-1}$ is not bounded, so unless there is some cancellation from the symbol κ, we cannot bound the integral. In this case one must use a further dyadic decomposition w.r.t. the angle between $\xi - \eta$ and η. See [23]. For the estimate (2.12), however, we have sufficient cancellation from κ, so the proof is in principle the same for u_+v_- as the one we gave for u_+v_+. The only difference is that for u_+v_- we must take $A = 1/4$, since then $I \le |\mu|^{1/2} |\nu|^{1/2}$, as opposed to $I \le |\mu|^{1/4} |\nu|^{1/4}$ for u_+v_+.

2.3 A quadrilinear estimate

In this section we aim to prove the following.

Theorem 7. *Let u_j, $1 \le j \le 4$, be solutions of $\Box u_j = 0$ on $\mathbb{R}^{1+2}$, with Cauchy data $u_j\big|_{t=0} = f_j$, $\partial_t u_j\big|_{t=0} = 0$. Then the estimate*

$$\left| \int_{\mathbb{R}^{1+2}} D^{-a} D_-(u_1 u_2) \cdot u_3 u_4 \, dt \, dx \right| \lesssim \|f_1\|_{\dot H^{2-a}} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2} \tag{2.26}$$

holds for $3/4 < a < 1$.

The key point is that all the regularity is concentrated on u_1. This makes the proof much harder than in the case where the regularity is evenly distributed among the four functions. This asymmetry makes it impossible to prove the inequality by a reduction to bilinear L^2 estimates.

To use the full force of the cancellations coming from the operator D_-, we must combine the next lemma with a special change of variables which was first used by Klainerman and Machedon in [15]. The following lemma is a sharper version of similar results proved in [15] and [18].

Lemma 4. *If η_1 and η_2 are two points on the ellipse $|\xi - \eta| + |\eta| = \tau$ in $\mathbb{R}^2$, where $\tau > |\xi|$, $\xi \in \mathbb{R}^2$, then*

$$|\eta_1 - \eta_2| - \big||\eta_1| - |\eta_2|\big| \lesssim (\tau - |\xi|)^{1/2} \min\big\{|\eta_1|^{1/2}, |\eta_2|^{1/2}\big\}. \tag{2.27}$$

Moreover, if $|\eta_1| \ge 2|\eta_2|$, then

$$|\eta_1 - \eta_2| - \big||\eta_1| - |\eta_2|\big| \lesssim \tau - |\xi|. \tag{2.28}$$

Proof. We first show that it suffices to prove

$$|\eta_1||\eta_2| - \eta_1 \cdot \eta_2 \lesssim (\tau - |\xi|) \max\big\{|\eta_1|, |\eta_2|\big\}. \tag{2.29}$$

This estimate would imply

$$\begin{aligned} |\eta_1 - \eta_2| - \big||\eta_1| - |\eta_2|\big| &\lesssim \frac{|\eta_1||\eta_2| - \eta_1 \cdot \eta_2}{|\eta_1 - \eta_2|} \\ &\lesssim \frac{(\tau - |\xi|)\max\big\{|\eta_1|, |\eta_2|\big\}}{|\eta_1 - \eta_2|}. \end{aligned} \tag{2.30}$$

If $|\eta_1| \ge 2|\eta_2|$ or $|\eta_2| \ge 2|\eta_1|$, then $|\eta_1 - \eta_2| \sim \max\big\{|\eta_1|, |\eta_2|\big\}$, so (2.28) holds. Next, assume $|\eta_1| \sim |\eta_2|$. Then if $|\eta_1 - \eta_2| \le (\tau - |\xi|)^{1/2}|\eta_1|^{1/2}$, (2.27) is obviously satisfied, so we may assume $|\eta_1 - \eta_2| \ge (\tau - |\xi|)^{1/2}|\eta_1|^{1/2}$, which combined with (2.30) implies (2.27).

It suffices to prove (2.29) for $\xi = (1,0)$, $\tau > 1$. Set $r_j = |\eta_j|$, $\omega_j = \eta_j / |\eta_j|$ and $y_j = \xi \cdot \omega_j$ for $j = 1, 2$. A simple calculation reveals that $r_j = \frac{\tau^2 - 1}{2(\tau - y_j)}$, whence

$$r_1 r_2 (1 - \omega_1 \cdot \omega_2) = (\tau^2 - 1) \max\{r_1, r_2\} \frac{1 - y_1 y_2 \pm (1 - y_1^2)^{1/2}(1 - y_2^2)^{1/2}}{2\big(\tau - \min\{y_1, y_2\}\big)}.$$

Clearly,

$$1 - y_1y_2 + (1-y_1^2)^{1/2}(1-y_2^2)^{1/2} \lesssim \tau - \min\{y_1, y_2\},$$

and this completes the proof. □

The first step in proving (2.26) is to observe that we may assume $\widehat{f}_j(\xi) \geq 0$ for $1 \leq j \leq 4$. Indeed, if $u = e^{itD}f$ and $v = e^{\pm itD}g$, then

$$\widehat{uv}(\tau,\xi) = \int \delta\bigl(\tau \pm |\eta| - |\xi-\eta|\bigr)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta,$$

and $\delta\bigl(\tau \pm |\eta| - |\xi-\eta|\bigr)\,d\eta$ is a positive measure on the hypersurface given by $\pm|\eta| + |\xi-\eta| = \tau$. With this assumption, the integral in (2.26) is non-negative, so we can forget about the absolute value from now on.

Notation. *If T is a bilinear operator with Fourier symbol κ, i.e.,*

$$\widehat{T(u,v)}(\tau,\xi) = \int \kappa(\tau-\lambda, \xi-\eta, \lambda, \eta)\widehat{u}(\tau-\lambda, \xi-\eta)\widehat{v}(\lambda,\eta)\,d\tau\,d\xi\,d\lambda\,d\eta,$$

and if A is a subset of $\mathbb{R}^{2+2n}$, we let T_A be the operator whose symbol is $\kappa\chi_A$.

Define $A = \bigl\{(\tau,\xi,\lambda,\eta) : |\eta|/2 \leq |\xi| \leq 2|\eta|\bigr\}$, and write

$$u_3u_4 = (u_3u_4)_A + (u_3u_4)_{A^c}.$$

The first term on the right is the diagonal part, the second term the off-diagonal part. We first show that the estimate for the off-diagonal part can be reduced directly to bilinear estimates. By Plancherel's theorem and Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^{1+2}} D^{-a}D_-(u_1u_2)\cdot(u_3u_4)_{A^c}\,dt\,dx \\
&\quad= \int_{\mathbb{R}^{1+2}} D_-^{3/2-a}(u_1u_2)\cdot D^{-a}D_-^{a-1/2}(u_3u_4)_{A^c}\,dt\,dx \\
&\quad\leq \left\|D_-^{3/2-a}(u_1u_2)\right\|_{L^2}\left\|D_-^{a-1/2}(D^{-a/2}u_3D^{-a/2}u_4)_{A^c}\right\|_{L^2}.
\end{aligned}$$

Using the fact that

$$\Bigl|\bigl||\xi-\eta| \pm |\eta|\bigr| - |\xi|\Bigr| \leq 2\min\{|\xi-\eta|, |\eta|\},$$

we get

$$\begin{aligned}
\left\|D_-^{3/2-a}(u_1u_2)\right\|_{L^2} &\lesssim \left\|D_-^{1/2}(D^{1-a}u_1\cdot u_2)\right\|_{L^2}, \\
\left\|D_-^{a-1/2}(D^{-a/2}u_3D^{-a/2}u_4)_{A^c}\right\|_{L^2} &\lesssim \left\|D_-^{1/4}(D^{-3/8}u_3\cdot D^{-3/8}u_4)\right\|_{L^2}.
\end{aligned}$$

Now apply theorems 4 and 5.

Proving the estimate for the diagonal part is much harder, and we cannot use Cauchy-Schwarz as above to reduce the estimate to bilinear estimates, since there is no L^2 estimate for $D^{-a}D_-^{a-1/2}(u_3u_4)_A$. Instead, we use the method of dyadic decomposition. Since this was considered in detail in the previous section, we can afford to be a bit less careful this time around. Thus, in the sum $(u_3u_4)_A = \sum_{|j-k|\le 2}(\Delta_j u_3 \Delta_k u_4)_A$ we only consider the terms where $j = k$. Furthermore, we have

$$\Delta_j u_3 \Delta_j u_4 = \sum_{l \le j+2} \Delta_l(\Delta_j u_3 \Delta_j u_4).$$

Note that if $|l - j| \le C$, then

$$D^{-a}\Delta_l(\Delta_j u_3 \Delta_j u_4)$$

is comparable to

$$\Delta_l(D^{-a/2}\Delta_j u_3 D^{-a/2}\Delta_j u_4),$$

so we can use the same proof as in the off-diagonal case. We may therefore assume $l \ll j$.

Next, we write $u_1u_2 = \sum_{p,q} \Delta_p u_1 \Delta_q u_2$. Since the regularity is concentrated on u_1, the worst possible case is clearly $p \le q$. Again we simplify and assume $p = q$ in the diagonal case, whereas $p \ll q$ in the off-diagonal case. Thus, we want to prove

$$\sum_j \sum_{l \ll j} \sum_p \int_{\mathbb{R}^{1+2}} D^{-a}D_-(\Delta_p u_1 \Delta_p u_2) \cdot \Delta_l(\Delta_j u_3 \Delta_j u_4)\, dt\, dx \lesssim \|f_1\|_{\dot{H}^{2-a}} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2} \tag{2.31}$$

and

$$\sum_j \sum_{l \ll j} \sum_q \sum_{p \ll q} \int_{\mathbb{R}^{1+2}} D^{-a}D_-(\Delta_p u_1 \Delta_q u_2) \cdot \Delta_l(\Delta_j u_3 \Delta_j u_4)\, dt\, dx \lesssim \|f_1\|_{\dot{H}^{2-a}} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}. \tag{2.32}$$

Observe that in (2.32) we must have $|q - l| \le C$. We therefore assume $q = l$ in (2.32). It should now be obvious that (2.31) and (2.32) follow from the dyadic estimates

$$\int_{\mathbb{R}^{1+2}} D_-(\Delta_p u_1 \Delta_p u_2) \cdot \Delta_l(\Delta_j u_3 \Delta_j u_4)\, dt\, dx \lesssim 2^l 2^p \|\Delta_p f_1\|_{L^2} \|\Delta_p f_2\|_{L^2} \|\Delta_j f_3\|_{L^2} \|\Delta_j f_4\|_{L^2}, \tag{2.33}$$

where we assume $l \ll j$, $l \le p + 3$, and

$$\int_{\mathbb{R}^{1+2}} D_-(\Delta_p u_1 \Delta_l u_2) \cdot \Delta_l(\Delta_j u_3 \Delta_j u_4)\, dt\, dx \lesssim 2^{3l/4} 2^{5p/4} \|\Delta_p f_1\|_{L^2} \|\Delta_l f_2\|_{L^2} \|\Delta_j f_3\|_{L^2} \|\Delta_j f_4\|_{L^2}, \tag{2.34}$$

where $p \ll l \ll j$. By scaling, we may assume $l = 0$

To avoid cumbersome notation, we suppress the indices and assume $f_1 = f_2 = f_3 = f_4 = f$, and we set $u = (u_+ + u_-)/2$, where $u_\pm = e^{\pm itD} f$. This does not restrict the generality of our proof, as the reader can easily convince himself. In other words, the indices can be put back in without any modifications to the proof.

Observe that $u_+ u_+$ and $u_+ u_-$ have almost disjoint supports in Fourier space. The first product we will say is of type $++$, the second of type $+-$. Thus in (2.33) and (2.34) the products are either both of type $++$ or both of type $+-$. There are no interactions between products of different types. Before we start the proofs, we remark that the estimate (2.34) only makes sense for products of type $+-$, since in the $++$ case, simple geometric considerations show that we would necessarily have $|p - j| \le C$, contradicting the assumption $p \ll j$.

2.3.1 Proof of (2.33) for products of type $++$

In this case we must necessarily have $|p - j| \le C$, so we may as well assume $p = j$. By a further decomposition in unit cubes, we see that it suffices to prove

$$\int_{\mathbb{R}^{1+2}} D_-(\Omega_\mu u_+ \cdot \Omega_{-\mu} u_+) \cdot (\Omega_\nu u_+ \cdot \Omega_{-\nu} u_+)\, dt\, dx \\ \lesssim |\mu| \, \|\Omega_\mu f\|_{L^2} \|\Omega_{-\mu} f\|_{L^2} \|\Omega_\nu f\|_{L^2} \|\Omega_{-\nu} f\|_{L^2} \,,$$

where $|\mu| \sim |\nu| \gg 1$. Applying Cauchy-Schwarz, this reduces to proving

$$\left\| D_-^{1/2} (\Omega_\mu u_+ \cdot \Omega_{-\mu} u_+) \right\|_{L^2} \lesssim |\mu|^{1/2} \|\Omega_\mu f\|_{L^2} \|\Omega_{-\mu} f\|_{L^2}$$

for all $\mu \in \mathbb{Z}^2$, $|\mu| \gg 1$. By the usual Cauchy-Schwarz argument, this can be reduced to proving that

$$(\tau - |\xi|) \int \chi_{\mu + Q^*}(\eta) \chi_{\mu + Q^*}(\xi - \eta) \delta\big(\tau - |\eta| - |\xi - \eta|\big)\, d\eta \lesssim |\mu| \,.$$

But this is easy; see section 2.2.5.

2.3.2 Proof of (2.33) for products of type $+-$

Decomposing in unit cubes, we conclude that it suffices to prove

$$\int_{\mathbb{R}^{1+2}} D_-(\Omega_\mu u_+ \cdot \Omega_\nu u_-) \cdot (\Omega_\kappa u_+ \cdot \Omega_{-\kappa} u_-)\, dt\, dx \\ \lesssim |\mu| \, \|\Omega_\mu f\|_{L^2} \|\Omega_\nu f\|_{L^2} \|\Omega_\kappa f\|_{L^2} \|\Omega_{-\kappa} f\|_{L^2} \,, \tag{2.35}$$

where $|\kappa| \gg 1$ and either $|\mu|$, $|\nu| \le C$ or $|\mu| \gg 1$ and $\nu = -\mu$. If the first alternative holds, there is no orthogonality, but summing is not a problem, since there is only a fixed, finite number of terms. Denote by I the integral in (2.35). A calculation shows that I equals

$$\int \big(|\xi| - \big||\eta_1| - |\xi-\eta_1|\big|\big)\,\delta\big(|\xi-\eta_2| - |\eta_2| - |\xi-\eta_1| + |\eta_1|\big) \\ \times \phi_\mu \widehat{f}(\xi-\eta_1)\phi_\nu \widehat{f}(\eta_1)\phi_\kappa \widehat{f}(-\eta_2)\phi_{-\kappa}\widehat{f}(\eta_2-\xi)\,d\eta_1\,d\eta_2\,d\xi.$$

Performing the linear change of variables (see [15], [18])

$$(\xi,\eta_1,\eta_2) \longrightarrow (\xi-\eta_1-\eta_2, -\eta_2, -\eta_1), \tag{2.36}$$

we find that I equals

$$\int \big(|\xi-\eta_1-\eta_2| - \big||\eta_2| - |\xi-\eta_1|\big|\big) \\ \times \delta\big(\tau - |\eta_1| - |\xi-\eta_1|\big)\delta\big(\tau - |\eta_2| - |\xi-\eta_2|\big) \\ \times \phi_\mu \widehat{f}(\xi-\eta_1)\phi_\nu \widehat{f}(-\eta_2)\phi_\kappa \widehat{f}(\eta_1)\phi_{-\kappa}\widehat{f}(\eta_2-\xi)\,d\eta_1\,d\eta_2\,d\xi.$$

By lemma 4,

$$|\xi-\eta_1-\eta_2| - \big||\eta_2| - |\xi-\eta_1|\big| \lesssim (\tau - |\xi|)^{1/2}\,|\mu|^{1/2},$$

so applying Cauchy-Schwarz first w.r.t. η_1, η_2 and then w.r.t. τ, ξ, we find that I is bounded by

$$|\mu|^{1/2}\, I_{\mu,\kappa}^{1/2} I_{-\nu,\kappa}^{1/2} \left\|\Omega_\mu f\right\|_{L^2} \left\|\Omega_\nu f\right\|_{L^2} \left\|\Omega_\kappa f\right\|_{L^2} \left\|\Omega_{-\kappa} f\right\|_{L^2},$$

where

$$I_{\mu,\lambda} = \sup_{\tau,\xi} (\tau - |\xi|)^{1/2} \int \chi_{\mu+Q^*}(\xi-\eta)\chi_{\lambda+Q^*}(\eta)\delta\big(\tau - |\eta| - |\xi-\eta|\big)\,d\eta \\ = \sup_{\tau,\xi} \int_{\tau=|\xi-\eta|+|\eta|} \chi_{\mu+Q^*}(\xi-\eta)\chi_{\lambda+Q^*}(\eta)\frac{|\xi-\eta|^{1/2}\,|\eta|^{1/2}}{(\tau+|\xi|)^{1/2}}\,dA(\eta).$$

Clearly, $I_{\mu,\lambda} \lesssim \min\big\{|\mu|^{1/2}, |\lambda|^{1/2}\big\}$, and this finishes the proof.

2.3.3 Proof of (2.34) for products of type $+-$

In this case we only decompose the second product in unit cubes. Thus, we must show

$$\int_{\mathbb{R}^{1+2}} D_-(\Delta_p u_+ \Delta_0 u_-)\cdot(\Omega_\mu u_+ \Omega_{-\mu} u_-)\,dt\,dx \\ \lesssim 2^{5p/4} \left\|\Delta_p f\right\|_{L^2} \left\|\Delta_0 f\right\|_{L^2} \left\|\Omega_\mu f\right\|_{L^2} \left\|\Omega_{-\mu} f\right\|_{L^2} \tag{2.37}$$

for $p \ll 0$ and $|\mu| \gg 1$. The integral I in (2.37) equals

$$\int \big(|\xi| - \big||\eta_1| - |\xi - \eta_1|\big|\big)\, \delta\big(|\xi - \eta_2| - |\eta_2| - |\xi - \eta_1| + |\eta_1|\big) \\ \times \beta(\eta_1/2^p)\widehat{f}(\eta_1)\beta\widehat{f}(\xi - \eta_1)\phi_\mu \widehat{f}(-\eta_2)\phi_{-\mu}\widehat{f}(\eta_2 - \xi)\, d\eta_1\, d\eta_2\, d\xi.$$

Performing the linear change of variables (2.36), we find that I equals

$$\int \big(|\xi - \eta_1 - \eta_2| - \big||\eta_2| - |\xi - \eta_1|\big|\big) \\ \times \delta\big(\tau - |\eta_1| - |\xi - \eta_1|\big)\delta\big(\tau - |\eta_2| - |\xi - \eta_2|\big) \\ \times \beta(-\eta_2/2^p)\widehat{f}(-\eta_2)\beta\widehat{f}(\xi - \eta_1)\phi_\mu \widehat{f}(\eta_1)\phi_{-\mu}\widehat{f}(\eta_2 - \xi)\, d\eta_1\, d\eta_2\, d\xi.$$

By lemma 4,

$$|\xi - \eta_1 - \eta_2| - \big||\eta_2| - |\xi - \eta_1|\big| \lesssim (\tau - |\xi|)^{1/2} 2^{p/2}$$

and proceeding as in the previous section, we find that I is bounded by

$$2^{p/2} J_{p,\mu}^{1/2} J_{0,\mu}^{1/2} \left\|\Delta_p f\right\|_{L^2} \left\|\Delta_0 f\right\|_{L^2} \left\|\Omega_\mu f\right\|_{L^2} \left\|\Omega_{-\mu} f\right\|_{L^2},$$

where

$$J_{k,\mu} = \sup_{\tau,\xi} (\tau - |\xi|)^{1/2} \int \chi_{2^k \mathcal{C}}(\xi - \eta)\chi_{\mu + Q^*}(\eta)\delta\big(\tau - |\eta| - |\xi - \eta|\big)\, d\eta \\ = \sup_{\tau,\xi} \int_{\tau = |\xi - \eta| + |\eta|} \chi_{2^k \mathcal{C}}(\xi - \eta)\chi_{\mu + Q^*}(\eta) \frac{|\xi - \eta|^{1/2} |\eta|^{1/2}}{(\tau + |\xi|)^{1/2}} dA(\eta).$$

Assuming $2^k \ll |\mu|$, it is clear that $J_{k,\mu} \lesssim 2^{3k/2}$. This finishes the proof of theorem 7.

Chapter 3

Hyperbolic Sobolev Spaces

In this chapter we define the basic spaces in which we will obtain solutions of nonlinear wave equations. These spaces arise from L^2 Sobolev norms in space-time, with weights adapted to the wave operator. We prove the basic principle that a multilinear space-time estimate involving solutions of the homogeneous wave equation implies a corresponding estimate for elements of the Sobolev spaces referred to above, although these spaces themselves do not contain any non-trivial solution of the homogeneous wave equation. Applying this principle to the estimates from chapter two, we derive a number of useful estimates. We then discuss the algebra property of these spaces, extending this to general nonlinear functions. The remainder of the chapter is devoted to proving a suitable version of the energy inequality for these spaces, a basic ingredient in the well-posedness theorems of chapter four.

3.1 The space $H^{s,\theta}$

For $s, \theta \in \mathbb{R}$, we let $H^{s,\theta}$ be the space of all $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ for which $\widehat{u}$ is a tempered function such that

$$(\tau, \xi) \longmapsto \langle \xi \rangle^s \, w_-^\theta(\tau, \xi) \widehat{u}(\tau, \xi) \tag{3.1}$$

is in $L^2(\mathbb{R}^{1+n})$, where $\langle \xi \rangle = 1 + |\xi|$ and $w_-(\tau, \xi) = 1 + \big| |\tau| - |\xi| \big|$. We denote the L^2 norm of this function by $\|u\|_{s,\theta}$. Then $\|\cdot\|_{s,\theta}$ is a norm on $H^{s,\theta}$. An alternative definition is

$$H^{s,\theta} = \left\{ u \in \mathcal{S}' : \Lambda^s \Lambda_-^\theta u \in L^2 \right\} \tag{3.2}$$

with norm $\|u\|'_{s,\theta} = \left\|\Lambda^s \Lambda_-^\theta u\right\|_{L^2}$, where Λ^s and Λ_-^θ are continuous linear maps of $\mathcal{S}'$ to itself, defined by

$$\widehat{\Lambda^s u} = (1 + |\xi|^2)^{s/2} \widehat{u}$$

$$\widehat{\Lambda_-^\theta u} = \left(1 + \frac{\left(\tau^2 - |\xi|^2\right)^2}{1 + \tau^2 + |\xi|^2}\right)^{\theta/2} \widehat{u}.$$

In fact, since $\Lambda^{-s} = (\Lambda^s)^{-1}$ and $\Lambda_-^{-\theta} = \left(\Lambda_-^\theta\right)^{-1}$, these maps are isomorphisms. Since there are constants $C_1, C_2 > 0$ such that

$$C_1 \langle\xi\rangle \le (1 + |\xi|^2)^{1/2} \le C_2 \langle\xi\rangle \tag{3.3a}$$

$$C_1 w_-(\tau, \xi) \le \left(1 + \frac{\left(\tau^2 - |\xi|^2\right)^2}{1 + \tau^2 + |\xi|^2}\right)^{1/2} \le C_2 w_-(\tau, \xi), \tag{3.3b}$$

the two definitions are equivalent, as are the norms $\|\cdot\|_{s,\theta}$ and $\|\cdot\|'_{s,\theta}$. We will use both these norms, depending on whether we are working in physical or frequency space, but since they are equivalent, it should cause no confusion to denote both of them by $\|\cdot\|_{s,\theta}$.

Since $\mathcal{S}$ is dense in L^2 and $\Lambda^s \Lambda_-^\theta$ maps $\mathcal{S}$ onto itself, the definition (3.2) shows immediately that $\mathcal{S}$ is dense in $H^{s,\theta}$.

For later use we also define

$$\widehat{\Lambda_+^s u} = \left(1 + \tau^2 + |\xi|^2\right)^{s/2} \widehat{u}.$$

Like Λ^s and Λ_-^θ, this map is an isomorphism of both $\mathcal{S}$ and $\mathcal{S}'$.

3.2 An integral representation

Given $u \in H^{s,\theta}$, there is a unique decomposition

$$u = u_+ + u_-$$

such that $\widehat{u_+}$ is supported in $[0, \infty) \times \mathbb{R}^n$ and $\widehat{u_-}$ in $(-\infty, 0] \times \mathbb{R}^n$. Obviously, $u_\pm \in H^{s,\theta}$, and $\|u\|_{s,\theta}^2 = \|u_+\|_{s,\theta}^2 + \|u_-\|_{s,\theta}^2$.

When $\theta > 1/2$, we have the following useful characterization of $H^{s,\theta}$.

Proposition 7. *If* $\theta > 1/2$, *then:*

(a) $H^{s,\theta} \subseteq C_b(\mathbb{R}, H^s)$, *in the sense that any tempered distribution* $u \in H^{s,\theta}$ *has a unique representative* $t \mapsto u(t)$ *in* $C_b(\mathbb{R}, H^s)$. *Moreover,*

$$\|u(t)\|_{H^s} \le C \|u\|_{s,\theta} \quad \textit{for all } t \in \mathbb{R}, \tag{3.4}$$

where C *depends only on* θ.

(b) *$u \in H^{s,\theta}$ iff there exist $f_+, f_- \in L^2(\mathbb{R}, H^s)$ such that*

$$\widehat{f_+(\rho)}(\xi) = 0 \quad \textit{for} \quad |\xi| < -\rho,$$
$$\widehat{f_-(\rho)}(\xi) = 0 \quad \textit{for} \quad |\xi| < \rho$$

and

$$u_\pm(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it(\rho \pm D)} f_\pm(\rho)}{(1+|\rho|)^\theta} \, d\rho.$$

Moreover, $\|u_\pm\|_{s,\theta} = \|f_\pm\|_{L^2(\mathbb{R},H^s)}$.

Proof. We start by proving that a tempered distribution can have at most one representative in $C_b(\mathbb{R}, H^s)$. First note that by remark (ii) on p. 2, any element of $C_b(\mathbb{R}, H^s)$ is a tempered distribution. Now assume that $u, v \in C_b(\mathbb{R}, H^s)$ are equal in the sense of distributions. We have to show that $u(t) = v(t)$ for all t.

Fix $\phi \in \mathcal{S}(\mathbb{R}^n)$ and define $f : \mathbb{R} \to \mathbb{C}$ by

$$f(t) = \langle \, u(t) - v(t), \phi \, \rangle \quad \text{for} \quad t \in \mathbb{R}.$$

Then f is continuous, and for every $\psi \in \mathcal{S}(\mathbb{R})$,

$$\int f(t)\psi(t) \, dt = \int \langle \, u(t) - v(t), \psi(t)\phi \, \rangle \, dt = \langle \, u - v, \zeta \, \rangle = 0,$$

where $\zeta \in \mathcal{S}(\mathbb{R}^{1+n})$ is given by $\zeta(t,x) = \psi(t)\phi(x)$. Hence, $f(t) = 0$ for all t.

The existence statement of part (a) follows from part (b), which also implies, using theorem 1(a) and the Cauchy-Schwarz inequality, that

$$\begin{aligned}
\|u(t)\|_{H^s} &\le \int \frac{\|f_+(\rho)\|_{H^s}}{(1+|\rho|)^\theta} \, d\rho + \int \frac{\|f_-(\rho)\|_{H^s}}{(1+|\rho|)^\theta} \, d\rho \\
&\le C \left(\|f_+\|_{L^2(\mathbb{R},H^s)} + \|f_-\|_{L^2(\mathbb{R},H^s)} \right) \\
&\le C \, \|u\|_{s,\theta} \, ,
\end{aligned}$$

where the last constant equals $4 \left(\int (1+|\rho|)^{-2\theta} \, d\rho \right)^{1/2}$.

We now proceed to prove part (b). Under the isometry

$$u \longleftrightarrow F = \mathcal{F}(\Lambda^s \Lambda_-^\theta u), \quad H^{s,\theta} \longleftrightarrow L^2,$$

u_+ and u_- correspond to $F_+ = \chi_{[0,\infty)\times\mathbb{R}^n} F$ and $F_- = \chi_{(-\infty,0]\times\mathbb{R}^n} F$.

We define another isometry

$$F_\pm \longleftrightarrow f_\pm, \quad L^2(\mathbb{R}^{1+n}) \longleftrightarrow L^2(\mathbb{R}, H^s),$$

where

$$\begin{aligned}
(1+|\xi|)^s \, \widehat{f_+(\rho)}(\xi) &= F_+(\rho + |\xi| \, , \xi) \\
(1+|\xi|)^s \, \widehat{f_-(\rho)}(\xi) &= F_-(\rho - |\xi| \, , \xi).
\end{aligned} \tag{3.5}$$

It is a straightforward exercise in measure theory to prove that, after redefining $f_\pm$ on a set of measure zero if necessary, $f_\pm$ is in $L^2(\mathbb{R}, H^s)$ iff $F_\pm$ is in $L^2(\mathbb{R}^{1+n})$. We omit the details. Thus, $u \in H^{s,\theta}$ iff $f_+, f_- \in L^2(\mathbb{R}, H^s)$, and we set

$$v_+(t) = \frac{1}{2\pi} \int \frac{e^{it(\rho+D)} f_+(\rho)}{(1+|\rho|)^\theta}\, d\rho, \quad v_-(t) = \frac{1}{2\pi} \int \frac{e^{it(\rho-D)} f_-(\rho)}{(1+|\rho|)^\theta}\, d\rho$$

for $t \in \mathbb{R}$. By the dominated convergence theorem, $v_+, v_- \in C(\mathbb{R}, H^s)$, and since the H^s norm of $v_\pm(t)$ is bounded uniformly in t, v_+ and v_- are tempered distributions. We will prove that $u_+ = v_+$ in the sense of distributions. The proof that $u_- = v_-$ is similar, and will be omitted.

Since the bilinear pairing $\langle \cdot, \cdot \rangle : H^s \times H^{-s} \to \mathbb{C}$ is continuous, it follows from theorem 1(d) and Fubini's theorem that

$$\begin{aligned}
\langle v_+, \phi \rangle &= \int \langle v_+(t), \phi(t) \rangle \, dt \\
&= \iint \left\langle \frac{e^{it(\rho+D)} f_+(\rho)}{2\pi(1+|\rho|)^\theta}, \phi(t) \right\rangle d\rho\, dt \\
&= \iiint \frac{e^{it(\rho+|\xi|)} \widehat{f_+(\rho)}(\xi)}{2\pi(1+|\rho|)^\theta} \mathcal{F}^{-1}\big(\phi(t)\big)(\xi)\, d\xi\, d\rho\, dt \\
&= \iiint \frac{1}{2\pi} e^{it(\rho+|\xi|)} \widehat{u_+}(\rho+|\xi|, \xi) \mathcal{F}^{-1}\big(\phi(t)\big)(\xi)\, d\xi\, d\rho\, dt \\
&= \iint \widehat{u_+}(\tau,\xi) \frac{1}{2\pi} \int e^{it\tau} \mathcal{F}^{-1}\big(\phi(t)\big)(\xi)\, dt\, d\tau\, d\xi \\
&= \iint \widehat{u_+}(\tau,\xi) \mathcal{F}^{-1}\big(\phi\big)(\tau,\xi)\, d\tau\, d\xi \\
&= \langle u_+, \phi \rangle
\end{aligned}$$

for every $\phi \in \mathcal{S}(\mathbb{R}^{1+n})$. This concludes the proof. □

3.3 Space-time estimates

In this section we prove a highly useful corollary to proposition 7, namely the fact that a multilinear space-time estimate involving solutions of the linear wave equation with data in H^s in many cases implies a corresponding estimate for elements of $H^{s,\theta}$ with $\theta > 1/2$. We then apply this result to the estimates of chapter 2, thereby immediately deriving a number of important inequalities which are collected in theorems 8 and 9. Other estimates are then deduced from these, and we also include a well-known bilinear estimate based on the Cauchy-Schwarz inequality, see proposition 10 below.

Proposition 8. *Assume that* $T : H^{s_1}(\mathbb{R}^n) \times \cdots \times H^{s_k}(\mathbb{R}^n) \longrightarrow H^\sigma(\mathbb{R}^n)$ *is k-linear, and let* $\theta > 1/2$.

(a) If

$$\left\|T(e^{\lambda_1 itD}f_1,\dots,e^{\lambda_k itD}f_k)\right\|_{L^q_t(L^r_x)} \le C\left\|f_1\right\|_{H^{s_1}}\cdots\left\|f_k\right\|_{H^{s_k}}, \tag{3.6}$$

where λ is a fixed k-tuple in $\{-1,1\}^k$, then

$$\left\|T(u_1,\dots,u_k)\right\|_{L^q_t(L^r_x)} \le C\left\|u_1\right\|_{s_1,\theta}\cdots\left\|u_k\right\|_{s_k,\theta} \tag{3.7}$$

for all $(u_1,\dots,u_k)\in H^{s_1,\theta}\times\cdots\times H^{s_k,\theta}$ such that

$$\operatorname{supp}\widehat{u_j} \subseteq \begin{cases} [0,\infty)\times\mathbb{R}^n & \textit{if}\quad \lambda_j = 1, \\ (-\infty,0]\times\mathbb{R}^n & \textit{if}\quad \lambda_j=-1. \end{cases} \tag{3.8}$$

(b) If (3.6) *holds for all $\lambda\in\{-1,1\}^k$, then* (3.7) *holds for all*

$$(u_1,\dots,u_k)\in H^{s_1,\theta}\times\cdots\times H^{s_k,\theta}.$$

Proof. By proposition 7(b) and the condition (3.8), which is equivalent to

$$u_j = \begin{cases} u_{j+} & \text{if}\quad \lambda_j=1, \\ u_{j-} & \text{if}\quad \lambda_j=-1, \end{cases}$$

there exists $f_j\in L^2(\mathbb{R},H^{s_j})$ such that

$$u_j = \int_{-\infty}^{\infty}\frac{e^{it\rho}e^{\lambda_j itD}f_j(\rho)}{(1+|\rho|)^\theta}\,d\rho \quad \text{for}\quad 1\le j\le k,$$

and we have $\left\|u_j\right\|_{s_j,\theta} = \left\|f_j\right\|_{L^2(\mathbb{R},H^{s_j})}$.

Theorem 1(d) yields

$$\begin{aligned} &T(u_1,\dots,u_k) \\ &\quad = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{e^{it(\rho_1+\cdots+\rho_k)}T\big(e^{\lambda_1 itD}f_1(\rho_1),\dots,e^{\lambda_k itD}f_k(\rho_k)\big)}{(1+|\rho_1|)^\theta\cdots(1+|\rho_k|)^\theta}d\rho_1\cdots d\rho_k, \end{aligned}$$

and it follows from Minkowski's integral inequality, (3.6) and the Cauchy-Schwarz inequality that

$$\begin{aligned} &\left\|T(u_1,\dots,u_k)\right\|_{L^q_t(L^r_x)} \\ &\quad \le \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{C\left\|f_1(\rho_1)\right\|_{H^{s_1}}\cdots\left\|f_k(\rho_k)\right\|_{H^{s_k}}}{(1+|\rho_1|)^\theta\cdots(1+|\rho_k|)^\theta}d\rho_1\cdots d\rho_k \\ &\quad \le C\left\|f_1\right\|_{L^2(\mathbb{R},H^{s_1})}\cdots\left\|f_k\right\|_{L^2(\mathbb{R},H^{s_k})}. \end{aligned}$$

This concludes the proof of part (a), and to prove part (b) we simply write $u_j = u_{j+}+u_{j-}$, use the multilinearity of T, and apply part (a). ☐

Applying this proposition to the space-time estimates (2.2), (2.13), (2.11) and (2.6), we obtain the following result.

Theorem 8. *Assume $\theta > 1/2$.*

(a) If

$$s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}, \quad \frac{2}{\min(1, \gamma(r))} \le q \le \infty \quad \text{and} \quad 2 \le r < \infty, \tag{3.9}$$

where $\gamma(r) = (n-1)\left(\frac{1}{2} - \frac{1}{r}\right)$, then

$$\|u\|_{L_t^q(L_x^r)} \le C \|u\|_{s,\theta}. \tag{3.10}$$

(b) If (3.9) holds, $(2/q, \gamma) \ne (1,1)$ and $0 \le a < 1 - 2/r$, then

$$\left\|D^{-a}(uv)\right\|_{L_t^{q/2}(L_x^{r/2})} \le C \|u\|_{s-a/2,\theta} \|v\|_{s-a/2,\theta}. \tag{3.11}$$

(c) If $n \ge 3$ and $0 \le a < (n-1)/2$, then

$$\left\|D^{-a}(u_+ v_+)\right\|_{L^2} \lesssim \|u_+\|_{(n-1)/4-a/2,\theta} \|v_+\|_{(n-1)/4-a/2,\theta}. \tag{3.12}$$

(d) If $n \ge 3$ and $0 \le a < (n-2)/2$, then

$$\left\|D^{-a}(u_+ v_-)\right\|_{L^2} \lesssim \|u_+\|_{(n-1)/4-a/2,\theta} \|v_-\|_{(n-1)/4-a/2,\theta}. \tag{3.13}$$

Note that (3.10) corresponds to the embedding

$$H^{s,\theta} \subseteq L_t^q(L_x^r),$$

which is the analog in the spaces $H^{s,\theta}$ of the Sobolev embedding

$$H^s(\mathbb{R}^n) \subseteq L^r(\mathbb{R}^n) \quad \text{for} \quad s = \frac{n}{2} - \frac{n}{r}, \ 2 \le r < \infty.$$

The analog of the embedding

$$H^s(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n) \quad \text{for} \quad s > \frac{n}{2} \tag{3.14}$$

is

$$H^{s,\theta}(\mathbb{R}^{1+n}) \subseteq L^\infty(\mathbb{R}^{1+n}) \quad \text{for} \quad s > \frac{n}{2}, \ \theta > \frac{1}{2}. \tag{3.15}$$

The latter is easily proved by direct estimation or by combining (3.14) with proposition 7(a).

Definition 2. *We let $S_\pm^\gamma$ be the symmetric bilinear operator given by*

$$\widehat{S_\pm^\gamma(f,g)}(\xi) = \int \Big| \big||\xi - \eta| \pm |\eta|\big| - |\xi| \Big|^\gamma \widehat{f}(\xi - \eta) \widehat{g}(\eta) \, d\eta. \tag{3.16}$$

Theorem 9. *Assume $\theta > 1/2$.*

(a) If $n = 2$ and $0 \le a < 1/2$, then

$$\left\| D^{-a} S_+^{1/4}(u_+, v_+) \right\|_{L^2} \lesssim \|u_+\|_{3/8-a/2,\theta} \|v_+\|_{3/8-a/2,\theta} . \tag{3.17}$$

(b) If $n = 2$ and $0 \le a < 1/4$, then

$$\left\| D^{-a} S_-^{1/4}(u_+, v_-) \right\|_{L^2} \lesssim \|u_+\|_{3/8-a/2,\theta} \|v_-\|_{3/8-a/2,\theta} . \tag{3.18}$$

(c) For any $n \ge 2$,

$$\left\| S_+^{1/2}(u_+, v_+) \right\|_{L^2} \le C \|u_+\|_{0,\theta} \|v_+\|_{n/2,\theta} , \tag{3.19a}$$

$$\left\| S_-^{1/2}(u_+, v_-) \right\|_{L^2} \le C \|u_+\|_{0,\theta} \|v_-\|_{n/2,\theta} . \tag{3.19b}$$

Proof. Observe that

$$D_-^{\gamma}(e^{itD} f \cdot e^{\pm itD} g) = S_\pm^{\gamma}(e^{itD} f, e^{\pm itD} g).$$

Since

$$\Big| \big| |\xi - \eta| \pm |\eta| \big| - |\xi| \Big| \le 2 \min\{ |\xi - \eta| , |\eta| \}, \tag{3.20}$$

it follows that

$$S_\pm^{1/4} : H^{3/8-a/2}(\mathbb{R}^n) \times H^{3/8-a/2}(\mathbb{R}^n) \longrightarrow H^{-1/2-a}(\mathbb{R}^n)$$

for $0 \le a < 1/2$, and

$$S_\pm^{1/2} : L^2(\mathbb{R}^n) \times H^{n/2}(\mathbb{R}^n) \longrightarrow H^{-1/2}(\mathbb{R}^n).$$

Therefore, applying proposition 8 to (2.12), (2.9) and (2.10), the theorem follows. □

Next, we define two bilinear operators which are ubiquitous in what follows. These operators are intimately connected with the bilinear operator $(u, v) \mapsto \Lambda_-(uv)$, and basically correspond to the case where u and v concentrate on the light cone, i.e., they are solutions of the homogeneous wave equation. Cf. lemma 5 below.

Definition 3. *Let R^γ be the symmetric bilinear operator given by*

$$R^\gamma(u, v) = S_+^\gamma(u_+, v_+) + S_+^\gamma(u_-, v_-) + S_-^\gamma(u_+, v_-) + S_-^\gamma(u_-, v_+). \tag{3.21}$$

Thus,

$$\widehat{R^\gamma(u, v)}(\tau, \xi) = \int r^\gamma(\tau - \lambda, \xi - \eta, \lambda, \eta) \widehat{u}(\tau - \lambda, \xi - \eta) \widehat{v}(\lambda, \eta) \, d\lambda \, d\eta,$$

where

$$r(\tau,\xi,\lambda,\eta) = \begin{cases} |\xi| + |\eta| - |\xi+\eta| & \text{if} \quad \tau\lambda \geq 0, \\ |\xi+\eta| - \big||\xi| - |\eta|\big| & \text{if} \quad \tau\lambda < 0. \end{cases} \tag{3.22}$$

Furthermore, let R_0^γ be the symmetric bilinear operator given by

$$\widehat{R_0^\gamma(u,v)}(\tau,\xi) = \int r_0^\gamma(\tau-\lambda,\xi-\eta,\lambda,\eta)\widehat{u}(\tau-\lambda,\xi-\eta)\widehat{v}(\lambda,\eta)\,d\lambda\,d\eta,$$

where $r_0(\tau,\xi,\lambda,\eta) = \big(r\chi_E\big)(\tau,\xi,\lambda,\eta)$ and χ_E is the characteristic function of the set

$$E = \big\{(\tau,\xi,\lambda,\eta) : w_-(\tau,\xi) + w_-(\lambda,\eta) \leq r(\tau,\xi,\lambda,\eta)\big\}. \tag{3.23}$$

An important feature of the norm $\|u\|_{s,\theta}$ is that it only depends on the absolute value of $\widehat{u}$. To avoid having to pass to frequency space every time we want to prove an estimate in this norm, we introduce some special notation.

Notation. *We write*

$$\lfloor u \rfloor = \mathcal{F}^{-1}|\widehat{u}|,$$

and

$$u \preceq v \quad \text{iff} \quad |\widehat{u}| \leq \widehat{v} \text{ a.e.}, \quad u \precsim v \quad \text{iff} \quad |\widehat{u}| \leq C\,\widehat{v} \text{ a.e.}$$

Lemma 5. *For any $\gamma > 0$, we have*

$$\Lambda^\gamma(uv) \preceq 3^\gamma\big(\Lambda^\gamma\lfloor u\rfloor \cdot \lfloor v\rfloor + \lfloor u\rfloor \cdot \Lambda^\gamma\lfloor v\rfloor\big), \tag{3.24}$$

$$\Lambda_+^\gamma(uv) \preceq 3^\gamma\big(\Lambda_+^\gamma\lfloor u\rfloor \cdot \lfloor v\rfloor + \lfloor u\rfloor \cdot \Lambda_+^\gamma\lfloor v\rfloor\big) \tag{3.25}$$

and

$$\Lambda_-^\gamma(uv) \precsim C^\gamma\big\{\Lambda_-^\gamma\lfloor u\rfloor \cdot \lfloor v\rfloor + \lfloor u\rfloor \cdot \Lambda_-^\gamma\lfloor v\rfloor + R_0^\gamma(\lfloor u\rfloor,\lfloor v\rfloor)\big\}, \tag{3.26}$$

where R_0^γ is as in definition 3.

Proof. The inequality

$$(1+|\xi|^2)^{\gamma/2} \leq 3^\gamma\big\{(1+|\xi-\eta|^2)^{\gamma/2} + (1+|\eta|^2)^{\gamma/2}\big\}$$

implies (3.24) and (3.25), whereas (3.26) follows from the inequality

$$w_-(\tau,\xi) \leq w_-(\tau-\lambda,\xi-\eta) + w_-(\lambda,\eta) + r(\tau-\lambda,\xi-\eta,\lambda,\eta), \tag{3.27}$$

where r is given by (3.22). □

Based on theorem 9, part (c), we derive the basic estimates satisfied by the operators R and R_0.

Proposition 9. *The operators R^γ and R_0^γ satisfy the following estimates.*

(a) *If* $\delta_1, \delta_2 \geq 0$ *and* $\delta = \delta_1 + \delta_2$, *then*

$$R^\gamma(u,v) \preceq 2^\delta R^{\gamma-\delta}(\Lambda^{\delta_1} \lfloor u \rfloor, \Lambda^{\delta_2} \lfloor v \rfloor) \qquad \text{for} \quad 0 < \delta < \gamma, \tag{3.28}$$

$$R_0^\gamma(u,v) \preceq R_0^{\gamma+\delta}(\Lambda_-^{-\delta_1} \lfloor u \rfloor, \Lambda_-^{-\delta_2} \lfloor v \rfloor) \qquad \text{for} \quad 0 < \gamma. \tag{3.29}$$

(b) *Assume* $\gamma \geq 1/2$. *If*

$$s_1, s_2 \geq 0, \quad s_1 + s_2 \geq \frac{n}{2} + \gamma - \frac{1}{2} \quad \text{and} \quad \theta > \frac{1}{2}, \tag{3.30}$$

then

$$\|R^\gamma(u,v)\|_{L^2} \leq C \|u\|_{s_1,\theta} \|v\|_{s_2,\theta}, \tag{3.31}$$

where C *depends on* γ, θ *and* n.

(c) *Assume* $0 < \gamma < 1/2$. *If*

$$s_1, s_2 \geq 0, \quad s_1 + s_2 \geq \frac{n}{2}, \quad \theta_1, \theta_2 > \gamma \quad \text{and} \quad \theta_1 + \theta_2 > \gamma + \frac{1}{2}, \tag{3.32}$$

then

$$\|R_0^\gamma(u,v)\|_{L^2} \leq C \|u\|_{s_1,\theta_1} \|v\|_{s_2,\theta_2}, \tag{3.33}$$

where C *depends on* γ, θ_1, θ_2 *and* n.

Proof. It suffices to prove these inequalities for all u and v such that $\widehat{u}, \widehat{v} \geq 0$.

The inequality (3.28) follows immediately from (3.20), whereas (3.29) follows from the fact that the Fourier symbol of R_0^γ is restricted to the set (3.23).

Assume that $\gamma \geq 1/2$ and (3.30) holds. Pick $0 \leq \varepsilon_1 \leq s_1$ and $0 \leq \varepsilon_2 \leq s_2$ such that $\gamma - 1/2 = \varepsilon_1 + \varepsilon_2$. Since $s_1 - \varepsilon_1 + s_2 - \varepsilon_2 \geq n/2$, it follows from (3.21) and (3.19) that

$$\left\| R^{1/2}(u,v) \right\|_{L^2} \leq C \|u\|_{s_1-\varepsilon_1,\theta} \|v\|_{s_2-\varepsilon_2,\theta}.$$

By (3.28),

$$R^\gamma(u,v) \preceq 2^{\gamma-1/2} R^{1/2}(\Lambda^{\varepsilon_1} u, \Lambda^{\varepsilon_2} v),$$

and we conclude that (3.31) holds.

Now assume that $0 < \gamma < 1/2$ and (3.32) holds. Pick $\varepsilon_1, \varepsilon_2 \geq 0$ such that

$$\theta_1 + \varepsilon_1 > 1/2, \quad \theta_2 + \varepsilon_2 > 1/2 \quad \text{and} \quad \varepsilon_1 + \varepsilon_2 = 1/2 - \gamma.$$

By (3.29),

$$R_0^\gamma(u,v) \preceq R_0^{1/2}(\Lambda_-^{-\varepsilon_1} u, \Lambda_-^{-\varepsilon_2} v) \preceq R^{1/2}(\Lambda_-^{-\varepsilon_1} u, \Lambda_-^{-\varepsilon_2} v),$$

so (3.33) also follows from (3.21) and (3.19). ∎

Proposition 10. *Let $s_j, \theta_j \geq 0$ for $1 \leq j \leq 3$. If*

$$s_1 + s_2 + s_3 > \frac{n}{2} \quad \text{and} \quad \theta_1 + \theta_2 + \theta_3 > \frac{1}{2},$$

then

$$\|uv\|_{-s_1,-\theta_1} \leq C \|u\|_{s_2,\theta_2} \|v\|_{s_3,\theta_3},$$

where C depends on $s_1 + s_2 + s_3$, $\theta_1 + \theta_2 + \theta_3$ and n. In fact, we can allow $s_1 + s_2 + s_3 = n/2$, provided $s_j \neq n/2$ for $1 \leq j \leq 3$. Similarly, we may take $\theta_1 + \theta_2 + \theta_3 = 1/2$, provided $\theta_j \neq 1/2$ for $1 \leq j \leq 3$.

Proof. By duality, the proposition is equivalent with the inequality

$$I \lesssim \|F\|_{L^2} \|G\|_{L^2} \|H\|_{L^2},$$

where

$$I = \int \frac{F(\tau - \lambda, \xi - \eta) G(\lambda, \eta) H(\tau, \xi)}{\langle \xi \rangle^{s_1} \langle \xi - \eta \rangle^{s_2} \langle \eta \rangle^{s_3} w_-^{\theta_1}(\tau, \xi) w_-^{\theta_2}(\tau - \lambda, \xi - \eta) w_-^{\theta_3}(\lambda, \eta)} \, d\tau \, d\lambda \, d\xi \, d\eta.$$

Applying the following lemma twice to the integral I, first in dimension $d = 1$ and then in dimension $d = n$, gives the desired inequality. □

Lemma 6. *(a) If $\varepsilon > 0$, then*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f(x)g(y)h(x+y)}{\langle x \rangle^{d/2+\varepsilon}} \, dx \, dy \leq C \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2},$$

where $C^2 = \int \langle x \rangle^{-d-2\varepsilon} \, dx$.

(b) If $0 < \delta < d/2$, then

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f(x)g(y)h(x+y)}{|x|^{\delta} |y|^{d/2-\delta}} \, dx \, dy \leq C \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2},$$

where C depends on δ and $d/2 - \delta$.

Proof. To prove (a), simply apply the Cauchy-Schwarz inequality twice, first w.r.t. dx and then dy. For (b), it suffices to prove

$$\int_{|x| \leq |y|} \frac{f(x)g(y)h(x+y)}{|x|^{\delta} |y|^{d-\delta}} \, dx \, dy \leq C \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}.$$

Since $\int_{|x| \leq |y|} |x|^{-2\delta} \, dx \lesssim |y|^{2(d-\delta)}$, we can apply Cauchy-Schwarz as indicated above. □

Next we prove a bilinear estimate which will be required later, but is of little interest in itself.

Proposition 11. *Assume $n \geq 2$, $s > n/2$. If*

$$1/2 < \theta < s - (n-1)/2, \quad \varepsilon < s - (n-1)/2 - \theta \quad \textit{and} \quad 1/4 \leq \delta \leq 1/2,$$

then

$$H^{1-\varepsilon,\theta} \times H^{s-1,-\delta} \longrightarrow H^{0,-\delta}. \tag{3.34}$$

Proof. By duality, this is equivalent to

$$H^{1-\varepsilon,\theta} \times H^{0,\delta} \longrightarrow H^{1-s,\delta},$$

so it suffices to prove the latter. By the usual reduction, it is enough to show

$$\left\| \Lambda^{1-s} R_0^\delta(u,v) \right\|_{L^2} \lesssim \|u\|_{1-\varepsilon,\theta} \|v\|_{0,\delta} \,. \tag{3.35}$$

Define

$$\begin{aligned} A &= \big\{ (\tau,\xi,\lambda,\eta) : \langle \xi+\eta \rangle < w_-(\lambda,\eta) \big\}, \\ B &= \big\{ (\tau,\xi,\lambda,\eta) : \langle \xi+\eta \rangle \geq w_-(\lambda,\eta) \big\}. \end{aligned}$$

Since

$$R_{0,A}^\delta(u,v) \precsim \Lambda^{-\delta}(\Lambda^\delta u \cdot \Lambda_-^\delta v),$$

it follows from proposition 10 that (3.35) holds with R_0^δ replaced by $R_{0,A}^\delta$. For $n \geq 3$, we have

$$\left\| \Lambda^{1-s} R_{0,B}^\delta(u,v) \right\|_{L^2} \precsim \left\| \Lambda^{1-s+\theta-\delta}(\Lambda^\delta u \cdot \Lambda_-^{\delta-\theta} v) \right\|_{L^2} .$$

Now we can apply (3.12) and (3.13), except in the off-diagonal case, where we use the estimate

$$\|uv\|_{L^2} \lesssim \|u\|_{(n-1)/2+\gamma,\theta} \|v\|_{0,\theta} \,, \tag{3.36}$$

which holds for $n \geq 3$ and $\gamma > 0$. By proposition 8, this estimate is a consequence of the inequality

$$\left\| e^{itD} f \cdot e^{\pm itD} g \right\|_{L^2} \lesssim \|f\|_{H^{(n-1)/2+\gamma}} \|v\|_{L^2} \,.$$

In fact, if $n \geq 4$, then we can take $\gamma = 0$; see [9].

If $n = 2$, then assuming—as we may—that $s \leq 1 + \theta - \delta$, we get

$$\begin{aligned} \left\| \Lambda^{1-s} R_{0,B}^\delta(u,v) \right\|_{L^2} &\precsim \left\| R_0^\delta(u, \Lambda_-^{1-s} v) \right\|_{L^2} \\ &\precsim \left\| R^{1+\theta-s}(u, \Lambda_-^{-\theta+\delta} v) \right\|_{L^2} \\ &\precsim \left\| R^{1/4}(\Lambda^{1+\theta-s-1/4} u, \Lambda_-^{-\theta+\delta} v) \right\|_{L^2} , \end{aligned}$$

so in the diagonal case we can apply (3.17) and (3.18). The off diagonal case reduces to

$$\left\| R^{1/4}(\Lambda^{\delta-1/4} u, \Lambda^{1-s} v) \right\|_{L^2} ,$$

and we apply

$$\left\| R^{1/4}(u,v) \right\|_{L^2} \lesssim \|u\|_{3/4-\gamma,\theta} \|v\|_{\gamma,\theta}\,, \tag{3.37}$$

valid for $n = 2$ and $\gamma > 0$. By proposition 8, this follows from

$$\left\| D_-^{1/4}(e^{itD} f \cdot e^{\pm itD} g) \right\|_{L^2} \lesssim \|f\|_{\dot{H}^{3/4-\gamma}} \|v\|_{\dot{H}^{\gamma}}\,,$$

which is just an asymmetric version of (2.9). □

3.4 The algebra property

A crucial property of the space $H^{s,\theta}$ is the algebra property. Recall that $H^s(\mathbb{R}^n)$ is an algebra when $s > n/2$. This fact has the following analog in the setting of the space $H^{s,\theta}$.

Theorem 10. *If*

$$n \geq 2, \quad s > n/2 \quad \text{and} \quad 1/2 < \theta \leq 1/2 + s - n/2, \tag{3.38}$$

then $H^{s,\theta}$ is an algebra, i.e.,

$$\|uv\|_{s,\theta} \leq C \|u\|_{s,\theta} \|v\|_{s,\theta} \tag{3.39}$$

for all $u, v \in H^{s,\theta}$.

This was proved in [18], although the case $n \geq 3$ was implicitly contained in [13].

Two remarks should be made at this point: First, in [18] the product inequality (3.39) was proved in the norm $\left\| \Lambda_+^s \Lambda_-^\theta u \right\|_{L^2}$. An inspection of the proof given in [18] shows that it works equally well, with some trivial modifications, for the norm $\|u\|_{s,\theta} = \left\| \Lambda^s \Lambda_-^\theta u \right\|_{L^2}$. Second, the proof actually gives a stronger inequality: if (3.38) holds and $\varepsilon = \theta - 1/2$, then

$$\|uv\|_{s,\theta} \leq C\big(\|u\|_{s,\theta} \|v\|_{n/2+\varepsilon,\theta} + \|u\|_{n/2+\varepsilon,\theta} \|v\|_{s,\theta} \big), \tag{3.40}$$

where C depends on s, θ and the space dimension n.

A further analogy between H^s and $H^{s,\theta}$ is that when these spaces are algebras, they are preserved not only under multiplication, but by any smooth nonlinear map leaving the origin fixed.

Theorem 11. *Assume that $F \in C^\infty(\mathbb{R}^d)$ and $F(0) = 0$. For any pair (s,θ) satisfying (3.38) and the additional condition $\theta \leq 1$, there exists a continuous function $f = f_{s,\theta} : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|F(u)\|_{s,\theta} \leq f\big(\|u\|_{n/2+\varepsilon,\theta} \big) \|u\|_{s,\theta}$$

for all real-valued $u \in \prod_1^d H^{s,\theta}$, where $\varepsilon = \theta - 1/2$.

The proof is inspired by an argument in [22], and relies on the next two results, the first of which is a generalization of the algebra inequality.

Proposition 12. *Assume that* (3.38) *holds and that* $0 \le \sigma \le s$, $0 \le \delta \le \theta$. *Let* a *be a real-valued function in* $H^{s,\theta}$, *and set* $A = \Lambda^\sigma \Lambda_-^\delta M_{ia} \Lambda^{-\sigma} \Lambda_-^{-\delta}$, *where* M_{ia} *is multiplication by* ia. *Then:*

(a) A *is bounded from* L^2 *to* L^2. *In fact,*

$$\|uv\|_{\sigma,\delta} \le C \|u\|_{s,\theta} \|v\|_{\sigma,\delta}, \tag{3.41}$$

where C *depends on* s, θ *and* n.

(b) ***(Smoothing)*** *If we impose the additional restriction* $\theta \le 1$, *and set* $\varepsilon = \frac{1}{2}(\theta - 1/2)$, *then*

$$\left|\langle (A + A^*)u, u \rangle\right| \le C\mathcal{E}_{\gamma,\sigma,\delta}(u) \|u\|_{L^2} \quad \textit{for all } \gamma \in [\sigma, s], \tag{3.42}$$

where

$$\begin{aligned}
\mathcal{E}_{\gamma,\sigma,0}(u) &= \|a\|_{\gamma,\theta} \|u\|_{n/2+\varepsilon-\gamma,0} + \|a\|_{n/2+2\varepsilon,\theta} \|u\|_{-\varepsilon,0}, \\
\mathcal{E}_{\gamma,0,\delta}(u) &= \|a\|_{n/2+2\varepsilon,\theta} \|u\|_{0,-\varepsilon}, \\
\mathcal{E}_{\gamma,\sigma,\delta}(u) &= \|a\|_{\gamma,\theta} \big(\|u\|_{n/2+\varepsilon-\gamma,0} + \|u\|_{n/2+2\varepsilon-\gamma,-\varepsilon}\big) \\
&\quad + \|a\|_{n/2+2\varepsilon,\theta} \big(\|u\|_{-\varepsilon,0} + \|u\|_{0,-\varepsilon}\big),
\end{aligned}$$

and the constant C *depends only on* s, θ *and* n.

Lemma 7. *Let* A *be a bounded linear operator on a Hilbert space* H. *If*

$$\langle (A + A^*)x, x \rangle \le 2F(x) \|x\| \quad \textit{for all } x \in H,$$

where $F \in C(H, \mathbb{R}_+)$, *then any solution* $x \in C^1(\mathbb{R}_+, H)$ *of the ODE*

$$x'(t) = Ax(t) + x_0, \quad x(0) = 0 \tag{3.43}$$

satisfies

$$\|x(t)\| \le \int_0^t F\big(x(s)\big)\, ds + \|x_0\| \, t \tag{3.44}$$

for all $t \ge 0$.

3.4.1 Proof of theorem 11

We split the proof into two steps.

Step 1 We show that it suffices to prove the special case

$$\left\| e^{iu\cdot\xi} - 1 \right\|_{s,\theta} \le |\xi| \, \|u\|_{s,\theta} \, P(|\xi| \, \|u\|_{n/2+2\varepsilon,\theta}), \tag{3.45}$$

where $P(x) = \sum_{j=0}^{N} c_j x^j$ and $\varepsilon = \frac{1}{2}(\theta - 1/2)$.

Assuming (3.45) holds, we can use Fourier inversion and the condition $F(0) = 0$ to write

$$F(u) = (\phi F)(u) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (e^{iu\cdot\xi} - 1) \widehat{\phi F}(\xi) \, d\xi$$

for any $\phi \in C_c^\infty(\mathbb{R}^d)$ which equals 1 on $B(0, \|u\|_{L^\infty})$. Define $R = 1 + \|u\|_{L^\infty}$, fix $\phi \in C_c^\infty\big(B(0,2)\big)$ such that $\phi = 1$ on $B(0,1)$, and set $\phi_R = \phi(\cdot/R)$. Then

$$\begin{aligned}
\|F(u)\|_{s,\theta} &\le \int_{\mathbb{R}^d} \left\| e^{iu\cdot\xi} - 1 \right\|_{s,\theta} \left| \widehat{\phi_R F}(\xi) \right| d\xi \\
&\le \sum_{j=0}^{N} c_j \, \|u\|_{n/2+2\varepsilon,\theta}^j \, \|u\|_{s,\theta} \int_{B(0,2R)} |\xi|^{j+1} \left| \widehat{\phi_R F}(\xi) \right| d\xi \\
&\le C \, \|u\|_{s,\theta} \left\{ \sum_{j=0}^{N} c_j \, \|u\|_{n/2+2\varepsilon,\theta}^j \right\} \left\{ (2R)^d \max_{|\alpha| \le N+1} \|\partial^\alpha(\phi_R F)\|_{L^1} \right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\|\partial^\alpha(\phi_R F)\|_{L^1} &\le \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \left\| \partial^\beta \phi_R \cdot \partial^\gamma F \right\|_{L^1} \\
&\le C \sum_{\beta+\gamma=\alpha} R^{d-|\beta|} \left\| \partial^\beta \phi \right\|_{L^1} \left\{ \sup_{B(0,2R)} |\partial^\gamma F| \right\}
\end{aligned}$$

and $R \lesssim 1 + \|u\|_{n/2+2\varepsilon,\theta}$, the theorem follows.

Step 2 We prove (3.45). Since $H^{s,\theta}$ is an algebra and

$$e^{iu\cdot\xi} - 1 = \prod_{j=1}^{d} e^{iu_j\xi_j} - 1 = \sum_J \prod_{j\in J} (e^{iu_j\xi_j} - 1),$$

where the sum is over all nonempty subsets J of $\{1, \dots, d\}$, we may assume $d = 1$. We therefore want to prove the existence of a polynomial P such that

$$\left\| e^{ita} - 1 \right\|_{s,\theta} \le t \, \|a\|_{s,\theta} \, P(t \, \|a\|_{n/2+2\varepsilon,\theta}) \tag{3.46}$$

for all $t \ge 0$ and all real-valued $a \in H^{s,\theta}$, $a = a(x_0, \dots, x_n)$.

Let N_1, N_2 and N_3 be the smallest positive integers such that

$$\varepsilon_1 \stackrel{\text{def}}{=} \frac{n/2 + 2\varepsilon}{N_1} \le \varepsilon, \quad \varepsilon_2 \stackrel{\text{def}}{=} \frac{s - n/2 - 2\varepsilon}{N_2} \le \varepsilon \quad \text{and} \quad \varepsilon_3 \stackrel{\text{def}}{=} \frac{\theta}{N_3} \le \varepsilon.$$

For $0 \le j \le N_1 + N_2$ and $0 \le k \le N_3$ we define $w_{j,k}(t) = T_{j,k}(e^{ita} - 1)$, where

$$T_{j,k} = \begin{cases} \Lambda^{j\varepsilon_1}\Lambda_-^{k\varepsilon_3} & \text{if} \quad j \le N_1, \\ \Lambda^{n/2+2\varepsilon+(j-N_1)\varepsilon_2}\Lambda_-^{k\varepsilon_3} & \text{if} \quad j > N_1. \end{cases}$$

We also set $w_{-1,k} = w_{j,-1} = 0$. Since $H^{s,\theta}$ is an algebra,

$$\left\|e^{ita} - 1\right\|_{s,\theta} \le e^{tC\|a\|_{s,\theta}} - 1 \quad \text{for} \quad t \ge 0,$$

so $w_{j,k}(t) \in L^2$. In fact, $w_{j,k}$ belongs to $C^1(\mathbb{R}_+, L^2)$ and solves the ODE

$$w_{j,k}'(t) = A_{j,k}w_{j,k} + T_{j,k}(ia), \quad w_{j,k}(0) = 0,$$

where $A_{j,k} = T_{j,k}M_{ia}T_{j,k}^{-1}$. Thus, by proposition 12 and lemma 7,

(3.47)
$$\|w_{j,k}(t)\|_{L^2} \le C\|a\|_{n/2+2\varepsilon,\theta}\int_0^t \left(1 + \|w_{j-1,k}(t')\|_{L^2} + \|w_{j,k-1}(t')\|_{L^2}\right) dt'$$

if $j \le N_1$, whereas

$$\begin{aligned} \|w_{j,k}(t)\|_{L^2} \le C\|a\|_{n/2+2\varepsilon,\theta}&\int_0^t \left(\|w_{j-1,k}(t')\|_{L^2} + \|w_{j,k-1}(t')\|_{L^2}\right) dt' \\ + C\|a\|_{s,\theta}&\int_0^t \left(1 + \|w_{N_1,k}(t')\|_{L^2}\right) dt' \end{aligned} \tag{3.48}$$

if $j > N_1$. In both these formulas, $t \ge 0$ and the constant C depends on s, θ and n.

By a nested induction argument on j and k, (3.47) implies

$$\|w_{j,k}(t)\|_{L^2} \le P_{j,k}(t\|a\|_{n/2+2\varepsilon,\theta}) \quad \text{for} \quad j \le N_1, \tag{3.49}$$

where $P_{j,k}$ is a polynomial of degree $j + k + 1$ with coefficients depending on s, θ and n. Inserting (3.49) in (3.48) yields

$$\begin{aligned} \|w_{j,k}(t)\|_{L^2} \le C\|a\|_{n/2+2\varepsilon,\theta}&\int_0^t \left(\|w_{j-1,k}(t')\|_{L^2} + \|w_{j,k-1}(t')\|_{L^2}\right) dt' \\ &+ Ct\|a\|_{s,\theta}P_k(t\|a\|_{n/2+2\varepsilon,\theta}) \end{aligned} \tag{3.50}$$

for $j > N_1$, where P_k has degree $N_1 + k + 1$. By the same induction scheme as before, but now starting at $j = N_1 + 1$, (3.50) and (3.49) imply (3.46), and we are done.

3.4.2 Proof of proposition 12

We may assume $\widehat{u}$, $\widehat{v} \ge 0$. By (3.24) and (3.26),

$$\begin{aligned} \|uv\|_{\sigma,\delta} &\lesssim \|\Lambda^\sigma u \cdot v\|_{0,\delta} + \|u \cdot \Lambda^\sigma v\|_{0,\delta} \\ &\lesssim \left\|\Lambda^\sigma\Lambda_-^\delta u \cdot v\right\|_{L^2} + \left\|\Lambda^\sigma u \cdot \Lambda_-^\delta v\right\|_{L^2} + \left\|\Lambda_-^\delta u \cdot \Lambda^\sigma v\right\|_{L^2} \\ &\quad + \left\|u \cdot \Lambda^\sigma\Lambda_-^\delta v\right\|_{L^2} + \left\|R_0^\delta(\Lambda^\sigma u, v)\right\|_{L^2} + \left\|R_0^\delta(u, \Lambda^\sigma v)\right\|_{L^2}. \end{aligned}$$

The first four terms on the right hand side of the last inequality are easily estimated by proposition 10, so it remains to estimate the last two terms, which only occur if $\delta > 0$. By (3.29),

$$R_0^\delta(\Lambda^\sigma u, v) \preceq 2^\theta R^\theta(\Lambda^\sigma u, \Lambda_-^{\delta-\theta} v), \quad R_0^\delta(u, \Lambda^\sigma v) \preceq 2^\theta R^\theta(u, \Lambda_-^{\delta-\theta}\Lambda^\sigma v),$$

and now we can apply (3.31).

We now prove (3.42). Since a is real-valued, $A^* = \Lambda^{-\sigma}\Lambda_-^{-\delta} M_{-ia}\Lambda^\sigma\Lambda_-^\delta$. Therefore,

$$\mathcal{F}\big\{(A + A^*)u\big\}(\tau, \xi) = \int K_{\sigma,\delta}(\tau, \xi, \lambda, \eta) i\, \widehat{a}(\tau - \lambda, \xi - \eta)\widehat{u}(\lambda, \eta)\, d\lambda\, d\eta,$$

where

$$K_{\sigma,\delta}(\tau, \xi, \lambda, \eta) = \frac{\langle\xi\rangle^\sigma\, w_-^\delta(\tau, \xi)}{\langle\eta\rangle^\sigma\, w_-^\delta(\lambda, \eta)} - \frac{\langle\eta\rangle^\sigma\, w_-^\delta(\lambda, \eta)}{\langle\xi\rangle^\sigma\, w_-^\delta(\tau, \xi)}.$$

Since we are assuming $\widehat{u} \geq 0$, it follows that

$$\begin{aligned} \big|\big\langle (A + A^*)u, u\big\rangle\big| &= \big|\big\langle \mathcal{F}\big\{(A + A^*)u\big\}, \mathcal{F}u\big\rangle\big| \\ &\leq \int |K_{\sigma,\delta}|\,(\tau, \xi, \lambda, \eta)\, |\widehat{a}|\,(\tau - \lambda, \xi - \eta)\widehat{u}(\tau, \xi)\widehat{u}(\lambda, \eta)\, d\tau\, d\xi\, d\lambda\, d\eta. \end{aligned}$$

We call this integral I. If E is a subset of $\mathbb{R}^{2+2n}$, we denote by I_E the restriction of I to E.

Case 1 Assume $0 < \sigma \leq \gamma \leq s$ and $\delta = 0$. Set

$$\begin{aligned} \Omega_1 &= \big\{(\tau, \xi, \lambda, \eta) : 2\langle\xi\rangle < \langle\eta\rangle\big\}, \\ \Omega_2 &= \big\{(\tau, \xi, \lambda, \eta) : 2\langle\eta\rangle < \langle\xi\rangle\big\}, \\ \Omega_3 &= \left\{(\tau, \xi, \lambda, \eta) : \frac{1}{2}\langle\eta\rangle \leq \langle\xi\rangle \leq 2\langle\eta\rangle\right\}. \end{aligned} \tag{3.51}$$

By symmetry, it suffices to estimate I_{Ω_2} and I_{Ω_3}. Since

$$|K_{\sigma,0}| \leq \frac{\langle\xi\rangle^\sigma}{\langle\eta\rangle^\sigma} \leq \frac{\langle\xi\rangle^\gamma}{\langle\eta\rangle^\gamma} \leq 2^\gamma \frac{\langle\xi - \eta\rangle^\gamma}{\langle\eta\rangle^\gamma} \quad \text{on} \quad \Omega_2, \tag{3.52}$$

we have, by proposition 10,

$$I_{\Omega_2} \leq 2^s \left\|\Lambda^\gamma \lfloor a\rfloor \cdot \Lambda^{-\gamma} u\right\|_{L^2} \|u\|_{L^2} \leq C\, \|a\|_{\gamma,\theta}\, \|u\|_{n/2+\varepsilon-\gamma,0}\, \|u\|_{L^2}.$$

It is readily verified that if $0 < \frac{x}{2} \leq y \leq 2x$ and $r > 0$, then

$$\left|\frac{x^r}{y^r} - \frac{y^r}{x^r}\right| \leq C^r \frac{|x^r - y^r|}{x^r} \leq C^r \frac{|x - y|}{x}. \tag{3.53}$$

Hence,

$$|K_{\sigma,0}| \leq C^s \frac{\langle\xi - \eta\rangle}{\langle\xi\rangle} \quad \text{on} \quad \Omega_3,$$

so proposition 10 yields

$$I_{\Omega_3} \leq C^s \left\|\Lambda \lfloor a\rfloor \cdot \Lambda^{-1} u\right\|_{L^2} \|u\|_{L^2} \leq C\, \|a\|_{n/2+2\varepsilon,\theta}\, \|u\|_{-\varepsilon,0}\, \|u\|_{L^2}.$$

Case 2 Assume $\sigma = 0$ and $0 < \delta \le \theta$. Set

$$\begin{aligned}\Gamma_1 &= \{(\tau,\xi,\lambda,\eta) : 2w_-(\tau,\xi) < w_-(\lambda,\eta)\},\\ \Gamma_2 &= \{(\tau,\xi,\lambda,\eta) : 2w_-(\lambda,\eta) < w_-(\tau,\xi)\},\\ \Gamma_3 &= \{(\tau,\xi,\lambda,\eta) : w_-(\lambda,\eta) \le w_-(\tau,\xi) \le 2w_-(\lambda,\eta)\},\\ \Gamma_4 &= \{(\tau,\xi,\lambda,\eta) : w_-(\tau,\xi) \le w_-(\lambda,\eta) \le 2w_-(\tau,\xi)\}.\end{aligned}$$

It suffices to estimate I_{Γ_2} and I_{Γ_3}. By (3.27),

$$w_-(\tau,\xi) \le 2w_-(\tau-\lambda,\xi-\eta) + 2r(\tau-\lambda,\xi-\eta,\lambda,\eta) \quad \text{on} \quad \Gamma_2,$$

whence

$$\begin{aligned}|K_{0,\delta}| &\le \frac{w_-^\theta(\tau,\xi)}{w_-^\theta(\lambda,\eta)}\\ &\lesssim \frac{w_-^\theta(\tau-\lambda,\xi-\eta)}{w_-^\theta(\lambda,\eta)} + \frac{r^\theta(\tau-\lambda,\xi-\eta,\lambda,\eta)}{w_-^\theta(\lambda,\eta)} \quad \text{on} \quad \Gamma_2.\end{aligned} \tag{3.54}$$

Therefore, by proposition 10 and proposition 9, we have

$$\begin{aligned}I_{\Gamma_2} &\lesssim \left\|\Lambda_-^\theta \lfloor a\rfloor \cdot \Lambda_-^{-\theta} u\right\|_{L^2} \|u\|_{L^2} + \left\|R^\theta(\lfloor a\rfloor, \Lambda_-^{-\theta} u)\right\|_{L^2} \|u\|_{L^2}\\ &\lesssim \|a\|_{n/2+2\varepsilon,\theta} \|u\|_{0,-\varepsilon} \|u\|_{L^2}.\end{aligned}$$

By (3.53) and (3.27),

$$\begin{aligned}|K_{0,\delta}| = |K_{0,\delta}|^{1-\theta} |K_{0,\delta}|^\theta &\le 2^{\delta(1-\theta)} C^{\delta\theta} \frac{|w_-(\tau,\xi) - w_-(\lambda,\eta)|^\theta}{w_-^\theta(\lambda,\eta)}\\ &\lesssim \frac{w_-^\theta(\tau-\lambda,\xi-\eta)}{w_-^\theta(\lambda,\eta)} + \frac{r^\theta(\tau-\lambda,\xi-\eta,\lambda,\eta)}{w_-^\theta(\lambda,\eta)} \quad \text{on} \quad \Gamma_3.\end{aligned}$$

Now proceed as for I_{Γ_2}.

Case 3 Assume $0 < \sigma \le \gamma \le s$ and $0 < \delta \le \theta$. It suffices to estimate I_{Ω_2} and I_{Ω_3}. We write $I_{\Omega_2} = I_{\Omega_2 \cap \Gamma_2} + I_{\Omega_2 \cap (\Gamma_1 \cup \Gamma_3 \cup \Gamma_4)}$. Since

$$K_{\sigma,\delta} = K_{\sigma,0} \frac{w_-^\delta(\tau,\xi)}{w_-^\delta(\lambda,\eta)} + \frac{\langle\eta\rangle^\sigma}{\langle\xi\rangle^\sigma} K_{0,\delta} = K_{\sigma,0} \frac{w_-^\delta(\lambda,\eta)}{w_-^\delta(\tau,\xi)} + \frac{\langle\xi\rangle^\sigma}{\langle\eta\rangle^\sigma} K_{0,\delta},$$

it is clear that

$$|K_{\sigma,\delta}| \le 2^\theta |K_{\sigma,0}| + 2^s |K_{0,\delta}| \quad \text{on } \Omega_2 \cap (\Gamma_1 \cup \Gamma_3 \cup \Gamma_4) \text{ and } \Omega_3,$$

so our previous estimates apply. By (3.52) and (3.54),

$$|K_{\sigma,\delta}| \le C^s \frac{\langle\xi-\eta\rangle^\gamma}{\langle\eta\rangle^\gamma} \left(\frac{w_-^\theta(\tau-\lambda,\xi-\eta)}{w_-^\theta(\lambda,\eta)} + \frac{r^\theta(\tau-\lambda,\xi-\eta,\lambda,\eta)}{w_-^\theta(\lambda,\eta)} \right)$$

on $\Omega_2 \cap \Gamma_2$, so proposition 10 and proposition 9 yield

$$\begin{aligned} I_{\Omega_2\cap\Gamma_2} &\lesssim \left\|\Lambda^\gamma \Lambda_-^\theta a \cdot \Lambda^{-\gamma}\Lambda_-^{-\theta} u\right\|_{L^2} \|u\|_{L^2} + \left\|R^\theta(\Lambda^\gamma \lfloor a \rfloor, \Lambda^{-\gamma}\Lambda_-^{-\theta} u)\right\|_{L^2} \|u\|_{L^2} \\ &\lesssim \|a\|_{\gamma,\theta} \|u\|_{n/2+2\varepsilon-\gamma,-\varepsilon} \|u\|_{L^2}. \end{aligned}$$

This concludes the proof of the lemma.

3.4.3 Proof of lemma 7

By (3.43), we have

$$\begin{aligned} \frac{d}{dt}\left(\|x\|^2\right) &= \langle x', x \rangle + \langle x, x' \rangle \\ &= \langle Ax, x \rangle + \langle x_0, x \rangle + \langle x, Ax \rangle + \langle x, x_0 \rangle \\ &= \langle (A + A^*)x, x \rangle + 2\Re \langle x_0, x \rangle \\ &\le 2F(x) \|x\| + 2 \|x_0\| \|x\|, \end{aligned}$$

whence

$$\frac{d\|x\|}{dt} \le F(x) + \|x_0\| \tag{3.55}$$

for all $t \ge 0$ such that $x(t) \ne 0$.

Fix $t \ge 0$. If $x(t) = 0$, (3.44) is trivially satisfied, so we assume $x(t) \ne 0$. Now set $a = \sup\{s \in [0, t] : x(s) = 0\}$. Since $x(0) = 0$, we have $a < t$. Moreover, $x(a) = 0$ and $x(s) \ne 0$ for $a < s \le t$. Integrating (3.55) from a to t, we obtain (3.44).

3.5 The space $\mathcal{X}^{s,\theta}$

Henceforth it will be assumed that $1/2 < \theta < 1$. We define

$$\mathcal{X}^{s,\theta} = \left\{u : u \in H^{s,\theta} \text{ and } \partial_t u \in H^{s-1,\theta}\right\},$$

and we equip this space with the norm

$$|u|_{s,\theta} = \|u\|_{s,\theta} + \|\partial_t u\|_{s-1,\theta}.$$

An equivalent definition is

$$\mathcal{X}^{s,\theta} = \left\{u \in \mathcal{S}' : \Lambda^{s-1}\Lambda_+\Lambda_-^\theta u \in L^2\right\},$$

and the corresponding norm $\left\|\Lambda^{s-1}\Lambda_+\Lambda_-^\theta u\right\|_{L^2}$ is equivalent with $|u|_{s,\theta}$. Thus $\left(\mathcal{X}^{s,\theta}, |\cdot|_{s,\theta}\right)$ is a Hilbert space containing the Schwartz class $\mathcal{S}$ as a dense subspace.

By propositions 7(a) and 2(b), we may identify $\mathcal{X}^{s,\theta}$ with a subspace of the Banach space

$$C_b(\mathbb{R}, H^s) \cap C_b^1(\mathbb{R}, H^{s-1}) \tag{3.56}$$

with norm

$$u \longmapsto \sup_{t\in\mathbb{R}} \big(\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} \big).$$

Remark. The space $\mathcal{X}^{s,\theta}$ is the basic space in which we will obtain solutions to nonlinear wave equations like the wave maps equation. The remark we want to make here concerns the relation of $\mathcal{X}^{s,\theta}$ to the slightly different space

$$X^{s,\theta} = \left\{ u \in \mathcal{S}' : \Lambda_+^s \Lambda_-^\theta u \in L^2 \right\},$$

with norm $\left\| \Lambda_+^s \Lambda_-^\theta u \right\|_{L^2}$. This space was used by Klainerman-Machedon [13, 14, 16, 17], Klainerman-Selberg [18] and Klainerman-Tataru [19] to prove existence for various nonlinear wave equations. It is clear that if $s \geq 1$, then $X^{s,\theta}$ embeds in $\mathcal{X}^{s,\theta}$, with equality iff $s = 1$.

The space $X^{s,\theta}$ has two major deficiencies, however. First, it embeds in the space (3.56) iff $s + \theta > 3/2$. In all the papers just mentioned, except [14], this condition is satisfied. In [14], however, s is arbitrarily close to $(n-2)/2$ and θ arbitrarily close to $1/2$, so that in space dimension $n = 3$, which is the lowest dimension considered in that paper, this condition may not be satisfied. In fact, the problem considered in [14] is a model equation derived from a certain formulation of the wave maps equation which is one of our main objects of study in this thesis.

Second, the time scaling argument used in section 3.6.5, which allows us to prove a genuine well-posedness result, does not work for $X^{s,\theta}$, unless s is close to 1. The failure of this argument means that, using contraction maps, one can only prove existence of solutions under the assumption that the norms of the data are small, and one cannot prove uniqueness in the space $\mathcal{X}^{s,\theta}$.

3.6 The linear wave equation and $\mathcal{X}^{s,\theta}$

Here we discuss how the space $\mathcal{X}^{s,\theta}$ relates to solutions of the linear wave equation. The culmination of this discussion will be the following theorem.

Theorem 12. *Consider the Cauchy problem* (1.1) *for the linear wave equation. Assume that* $F \in H^{s-1,\theta+\varepsilon-1}$, *where*

$$\frac{1}{2} < \theta < 1, \quad 0 \leq \varepsilon \leq 1 - \theta.$$

Then for any $0 < T < 1$ *there exists* $u = u_T \in \mathcal{X}^{s,\theta}$ *with the properties:*

(a) On the time interval $[0,T]$, *u agrees with the unique solution of* (1.1) *in the class* $C\big([0,T], H^s\big) \cap C^1\big([0,T], H^{s-1}\big)$;

(b) ***("Energy inequality")*** *u satisfies the estimate*

$$|u|_{s,\theta} \leq C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}} + T^{\varepsilon/2} \|F\|_{s-1,\theta+\varepsilon-1} \big),$$

where C only depends on θ.

The proof can be found in section 3.6.5, where we restate the theorem in a more precise form, giving an explicit formula for u. The problem is of course to find a suitable extension to $\mathbb{R}^{1+n}$ of the solution of (1.1) on $[0,T]\times\mathbb{R}^n$. Certainly, we cannot let u be the global solution of (1.1), since this solution fails to be in $\mathcal{X}^{s,\theta}$; see the discussion below. The natural thing to try is to cut the solution off smoothly outside the time interval $[0,T]$. Things are not quite that simple, however: one has to split the solution into different parts, some of which should be cut off and some of which should not. We start by splitting the solution of (1.1) into its homogeneous and inhomogeneous parts.

3.6.1 The homogeneous solution

We denote by u_0 the homogeneous part of the solution of (1.1)—that is, u_0 solves (1.1) with $F=0$. Since the Fourier transform of u_0 is not a tempered function—it is supported on the light cone, which has measure zero—we conclude that u_0 is not an element of $\mathcal{X}^{s,\theta}$. In fact, u_0 is given by

$$\begin{aligned} u_0(t) &= \cos(tD)\cdot f + D^{-1}\sin(tD)\cdot g \\ &= \frac{1}{2}\left(e^{itD}+e^{-itD}\right)f + \frac{1}{2i}\left(e^{itD}-e^{-itD}\right)D^{-1}g, \end{aligned} \tag{3.57}$$

whence

$$\widehat{u_0(t)}(\xi) = \frac{1}{2}\left(e^{it|\xi|}+e^{-it|\xi|}\right)\widehat{f}(\xi) + \frac{1}{2i}\left(e^{it|\xi|}-e^{-it|\xi|}\right)|\xi|^{-1}\,\widehat{g}(\xi),$$

and since the Fourier transform of

$$t \longmapsto e^{ita}, \quad \mathbb{R}\longrightarrow\mathbb{C}$$

is the measure $2\pi\delta(\tau-a)\,d\tau$, we conclude that the Fourier transform of u_0 is the measure

$$\begin{aligned} \widehat{u_0} &= \pi\big\{\delta(\tau-|\xi|)+\delta(\tau+|\xi|)\big\}\widehat{f}(\xi)\,d\tau\,d\xi \\ &\quad + \frac{\pi}{i}\big\{\delta(\tau-|\xi|)-\delta(\tau+|\xi|)\big\}\,|\xi|^{-1}\,\widehat{g}(\xi)\,d\tau\,d\xi, \end{aligned} \tag{3.58}$$

which is supported on the light cone $\big\{(\tau,\xi)\in\mathbb{R}^{1+n} : |\tau|=|\xi|\big\}$.

Locally in time, however, u_0 *does* belong to $\mathcal{X}^{s,\theta}$. That is, if $\chi\in C_c^\infty(\mathbb{R})$, then $\chi(t)u_0\in\mathcal{X}^{s,\theta}$. The reason is that the function

$$t \longmapsto \chi(t)e^{ita}, \quad \mathbb{R}\longrightarrow\mathbb{C}$$

has Fourier transform $\widehat{\chi}(\tau-a)$, so that when u_0 is replaced by $\chi(t)u_0$, the Dirac delta in (3.58) is in effect replaced by the Schwartz function $\widehat{\chi}$.

We have the following estimate for $\chi(t)u_0$.

Proposition 13. *If $\theta>1/2$ and $\chi\in C_c^\infty(\mathbb{R})$, the homogeneous solution u_0 of* (1.1) *satisfies*

$$\left|\chi(t)u_0\right|_{s,\theta} \le C\big(\|f\|_{H^s}+\|g\|_{H^{s-1}}\big),$$

where C only depends on χ and θ. More precisely,

$$C \simeq \|\chi\|_{H^\theta} + \|t\chi\|_{H^\theta} + \|\chi'\|_{H^\theta} + \|t\chi'\|_{H^\theta} .$$

The proof follows readily from the next proposition. Notice that in order to estimate the $\mathcal{X}^{s,\theta}$-norm of χu_0, we must estimate not only $\|\chi u\|_{s,\theta}$, but $\|\partial_t (\chi u)\|_{s,\theta}$ as well. By Leibniz' formula,

$$\partial_t (\chi u) = \chi' u + \chi \partial_t u.$$

Both these terms can of course be estimated using the next proposition—after all, $\partial_t u$ is a solution of (1.1) with Cauchy data $(g, \Delta f)$ at $t = 0$.

Proposition 14. *If $s \in \mathbb{R}$, $\theta > 1/2$, $\chi \in C_c^\infty(\mathbb{R})$ and $(f, g) \in H^s \times H^{s-1}$, then*

$$\left\|\chi(t) e^{\pm itD} f\right\|_{s,\theta} \le \|\chi\|_{H^\theta} \|f\|_{H^s} , \tag{3.59}$$

$$\|\chi(t)\cos(tD) \cdot f\|_{s,\theta} \le \|\chi\|_{H^\theta} \|f\|_{H^s} \tag{3.60}$$

and

$$\left\|\chi(t) D^{-1} \sin(tD) \cdot g\right\|_{s,\theta} \lesssim \left(\|\chi\|_{H^\theta} + \|t\chi\|_{H^\theta}\right) \|g\|_{H^{s-1}} . \tag{3.61}$$

If $|\rho| \le 1$ and $\operatorname{supp} \widehat{g} \subseteq \{\xi : |\xi| \le c\}$, then

$$\left\|\chi(t) e^{i\rho t D} g\right\|_{\sigma,\theta} \lesssim \left(c^\theta \|\chi\|_{L^2} + \|\chi\|_{H^\theta}\right) \|g\|_{H^\sigma} \quad \textit{for all } \sigma \in \mathbb{R}. \tag{3.62}$$

Proof. The Fourier transform of $\chi(t) e^{\pm itD} f$ is $\widehat{\chi}(\tau \mp |\xi|)\widehat{f}(\xi)$, and

$$\begin{aligned}
&\int (1 + |\xi|)^{2s} \left(1 + \big||\tau| - |\xi|\big|\right)^{2\theta} \left|\widehat{\chi}(\tau \mp |\xi|)\widehat{f}(\xi)\right|^2 d\tau\, d\xi \\
&\qquad \le \int \left(1 + \left|\tau \mp |\xi|\right|\right)^{2\theta} \left|\widehat{\chi}(\tau \mp |\xi|)\right|^2 (1 + |\xi|)^{2s} \left|\widehat{f}(\xi)\right|^2 d\tau\, d\xi \\
&\qquad = \|\chi\|_{H^\theta}^2 \|f\|_{H^s}^2 .
\end{aligned}$$

This proves (3.59), which in turn implies (3.60).

The proof of (3.62) is similar. We simply note that the Fourier transform of $\chi(t) e^{i\rho t D} g$ equals $\widehat{\chi}(\tau - \rho|\xi|)\widehat{g}(\xi)$, and that

$$\big||\tau| - |\xi|\big| \le \left|\tau - \rho|\xi|\right| + (1 - |\rho|)|\xi| \le \left|\tau - \rho|\xi|\right| + c$$

for $\xi \in \operatorname{supp} \widehat{g}$ and $|\rho| \le 1$.

To prove (3.61), we split $g = g_1 + g_2$, where $\widehat{g_1}$ is supported in the region $|\xi| < 1$ and $\widehat{g_2}$ is supported in $|\xi| \ge 1$. Since

$$D^{-1} \sin(tD) = t \int_0^1 e^{it(2\rho - 1)D}\, d\rho,$$

we have

$$\chi(t)D^{-1}\sin(tD)\cdot g_1 = \int_0^1 t\chi(t)e^{it(2\rho-1)D}g_1\,d\rho.$$

By (3.62),

$$\left\| t\chi(t)e^{it(2\rho-1)D}g_1 \right\|_{s,\theta} \lesssim \|t\chi\|_{H^\theta}\|g_1\|_{H^{s-1}} \quad \text{for} \quad 0\le\rho\le 1,$$

and by the dominated convergence theorem the map

$$\rho \longmapsto t\chi(t)e^{it(2\rho-1)D}g_1$$

belongs to $C([0,1],H^{s,\theta})$. Therefore, by theorem 1(a),

$$\left\|\chi(t)D^{-1}\sin(tD)\cdot g_1\right\|_{s,\theta} \le \int_0^1 \left\| t\chi(t)e^{it(2\rho-1)D}g_1\right\|_{s,\theta} d\rho \lesssim \|t\chi\|_{H^\theta}\|g_1\|_{H^{s-1}}.$$

This proves (3.61) with g replaced by its low frequency part g_1.

Since $\left\|D^{-1}g_2\right\|_{H^s} \le 2\|g\|_{H^{s-1}}$, the estimate (3.61) with g replaced by g_2 follows immediately from (3.59). ☐

3.6.2 The inhomogeneous solution

Now consider the inhomogeneous equation $\Box u = F$ with vanishing Cauchy data at $t=0$. Notice that $\Box$ is a bounded linear operator from $\mathcal{X}^{s,\theta}$ to $H^{s-1,\theta-1}$. Thus, it seems natural to assume $F \in H^{s-1,\theta-1}$. Assume also that $u \in \mathcal{S}'$ satisfies $\Box u = F$.

We first observe that u does not, in general, belong to $\mathcal{X}^{s,\theta}$. For if it does, then $\widehat{u}$ is a tempered function, and it follows from (1.1a) that

$$\widehat{u}(\tau,\xi) = \frac{\widehat{F}(\tau,\xi)}{\tau^2-|\xi|^2}. \tag{3.63}$$

But if, say, $\widehat{F}$ is nonzero and continuous at some point on the light cone, the function given by (3.63) is evidently not tempered, and we have a contradiction.

Nevertheless, if $\widehat{F}$ is supported in the complement of the neighborhood

$$\mathcal{N} = \left\{(\tau,\xi)\in\mathbb{R}^{1+n} : \big||\tau|-|\xi|\big| < 1\right\} \tag{3.64}$$

of the light cone, then clearly $u \in \mathcal{X}^{s,\theta}$ and

$$|u|_{s,\theta} \le C\|F\|_{s-1,\theta-1}.$$

This suggests writing

$$F = \phi(\Lambda_-)F + \big(1-\phi(\Lambda_-)\big)F = F_1 + F_2, \tag{3.65}$$

where

$$\phi \in C_c^\infty(\mathbb{R}), \quad \phi = 1 \quad \text{on} \quad [-2C_2, 2C_2], \quad \operatorname{supp}\phi \subseteq (-4C_2, 4C_2) \tag{3.66}$$

and the constant C_2 is as in (3.3). It is easily checked that

$$\operatorname{supp}\widehat{F_1} \subseteq \frac{4C_2}{C_1}\mathcal{N} \qquad \text{and} \qquad \operatorname{supp}\widehat{F_2} \subseteq \mathbb{R}^{1+n} \setminus \mathcal{N}, \tag{3.67}$$

where C_1 is the other constant in (3.3).

Note that since

$$F_1 \in H^{s-1,0} \subseteq L^1_{\mathrm{loc}}(\mathbb{R}, H^{s-1}),$$

we may use Duhamel's formula and define

$$u_1(t) = -\int_0^t D^{-1}\sin\big((t-t')D\big)\cdot F_1(t')\,dt'. \tag{3.68}$$

We will prove the following estimate for $\chi(t)u_1$.

Proposition 15. *Assume $1/2 < \theta < 1$ and $\chi \in C_c^\infty(\mathbb{R})$. If $F_1 \in H^{s-1,0}$ and*

$$2 + \big||\tau| - |\xi|\big| \le c \quad \textit{for} \quad (\tau,\xi) \in \operatorname{supp}\widehat{F_1},$$

then

$$|\chi u_1|_{s,\theta} \le C\,\|F_1\|_{s-1,0}\,,$$

where u_1 is given by (3.68) and

$$\begin{aligned} C \simeq{}& c^{1/2}\big(\|\chi\|_{\dot H^{\theta-1}} + \|t\chi\|_{H^\theta} + \|t\chi'\|_{\dot H^{\theta-1}} + \big\|t^2\chi'\big\|_{H^\theta}\big) \\ &+ \sum_{j=1}^\infty \bigg(\frac{c^{j+1/2}\,\big\|t^{j+1}\chi\big\|_{H^\theta}}{j!} + \frac{c^{j+1/2+\theta}\,\big\|t^{j+1}\chi\big\|_{L^2}}{j!} \\ &\qquad + \frac{c^{j-1/2}\,\big\|t^{j+1}\chi'\big\|_{H^\theta}}{j!} + \frac{c^{j-1/2+\theta}\,\big\|t^{j+1}\chi'\big\|_{L^2}}{j!} + \frac{c^{j-1/2}\,\big\|t^{j}\chi\big\|_{H^\theta}}{j!}\bigg). \end{aligned}$$

The proof, which is presented in section 3.6.4 below, relies on the following characterization of u_1.

Proposition 16. *Assume that $F_1 \in H^{s-1,0}$ and*

$$2 + \big||\tau| - |\xi|\big| \le c \quad \textit{for} \quad (\tau,\xi) \in \operatorname{supp}\widehat{F_1}, \tag{3.69}$$

and let u_1 be defined by (3.68). Then there exist $f_j^\pm \in H^s$, $g_j \in C([0,1], H^{s-1})$ for $j \ge 1$ such that

$$\begin{gathered} \operatorname{supp}\widehat{f_j^\pm} \subseteq \{\xi : |\xi| \ge c\}, \\ \operatorname{supp}\widehat{g_j(\rho)} \subseteq \{\xi : |\xi| < c\}, \\ \big\|f_j^\pm\big\|_{H^s}\,, \sup_{0\le\rho\le1} \|g_j(\rho)\|_{H^{s-1}} \lesssim c^{j-1/2}\,\|F_1\|_{s-1,0} \end{gathered}$$

and

$$u_1(t) = \sum_{j=1}^{\infty} \frac{t^{j+1}}{j!} \int_0^1 e^{it(2\rho-1)D} g_j(\rho)\, d\rho + \sum_{j=1}^{\infty} \frac{t^j}{j!} \left(e^{itD} f_j^+ + e^{-itD} f_j^- \right) + R_+(t) + R_-(t).$$

Here,

$$\widehat{R_+(t)}(\xi) = -\frac{\chi_{(c,\infty)}(|\xi|)}{4\pi\,|\xi|} \int_{-\infty}^{0} \frac{e^{it\tau} - e^{it|\xi|}}{|\tau| + |\xi|} \widehat{F_1}(\tau,\xi)\, d\tau,$$
$$\widehat{R_-(t)}(\xi) = -\frac{\chi_{(c,\infty)}(|\xi|)}{4\pi\,|\xi|} \int_{0}^{\infty} \frac{e^{it\tau} - e^{-it|\xi|}}{|\tau| + |\xi|} \widehat{F_1}(\tau,\xi)\, d\tau$$

and $\chi_{(c,\infty)}$ *is the characteristic function of the interval* (c,∞)*.*

Moreover, there exist $h_j^{\pm} \in H^{s-1}$ *for* $j \geq 1$ *such that*

$$\left\| h_j^{\pm} \right\|_{H^{s-1}} \lesssim c^{j-1/2} \left\| F_1 \right\|_{s-1,0}$$

and

$$\partial_t u_1(t) = \sum_{j=1}^{\infty} \frac{t^j}{j!} \left(e^{itD} h_j^+ + e^{-itD} h_j^- \right) - iDR_+(t) + iDR_-(t),$$

with $R_\pm$ *as above.*

The proof can be found in section 3.6.3 below.

In contrast with F_1, the distribution F_2 does not, in general, belong to $L^1_{\mathrm{loc}}(\mathbb{R}, H^{s-1})$. To see this, consider functions of the form $f(t)g(x)$. Pick f so that $\widehat{f} \in C_c^\infty \setminus \{0\}$, and choose $g \in H^{s+\theta-2} \setminus H^{s-1}$ with $\widehat{g}$ supported so far away from the origin that $|\xi| \geq 2\,|\tau|$ for all $\xi \in \operatorname{supp} \widehat{g}$ and $\tau \in \operatorname{supp} \widehat{f}$. Evidently, $f(t)g(x)$ belongs to $H^{s-1,\theta-1}$ but not to $L^1_{\mathrm{loc}}(\mathbb{R}, H^{s-1})$.

Thus, we cannot plug F_2 into Duhamel's formula. Instead we use the fact that on the support of $\widehat{F_2}$, the symbol of the wave operator is smooth and bounded away from zero. In fact, $\square^{-1}\big(1 - \phi(\Lambda_-)\big)$ is a bounded linear operator from $H^{s-1,\theta-1}$ to $\mathcal{X}^{s,\theta}$, where $\square^{-1}$ is the multiplier with symbol $\left(\tau^2 - |\xi|^2\right)^{-1}$, so we define

$$u_2 = \square^{-1} F_2. \tag{3.70}$$

Proposition 17. *Assume that* $F_2 \in H^{s-1,\theta-1}$ *and* $\operatorname{supp} \widehat{F_2} \subseteq \mathbb{R}^{1+n} \setminus \mathcal{N}$*, and let* u_2 *be defined as in* (3.70)*. Then*

$$|u_2|_{s,\theta} \lesssim \|F_2\|_{s-1,\theta-1}\,.$$

Moreover, u_2 *solves* $\square u_2 = F_2$ *with vanishing Cauchy data at* $t = 0$*.*

Proof. The only statement requiring a proof is the one about the data at $t = 0$. First, since $\mathcal{X}^{s,\theta}$ embeds in $C_b(\mathbb{R}, H^s) \cap C_b^1(\mathbb{R}, H^{s-1})$, the evaluation map

$$F \longmapsto \left(u_2\big|_{t=0}, \partial_t u_2\big|_{t=0}\right), \qquad H^{s-1,\theta-1} \longrightarrow H^s \times H^{s-1}$$

is bounded. Second, this map is the zero map on the dense subset $\mathcal{S}$ of $H^{s-1,\theta-1}$. For if $F \in \mathcal{S}$, then $F_2 \in \mathcal{S} \subseteq L^1_{\mathrm{loc}}(\mathbb{R}, H^{s-1})$, so we can define $u_2'(t) = -\int_0^t D^{-1} \sin\big((t-t')D\big) \cdot F_2(t')\,dt'$. Then u_2' has vanishing data at $t = 0$. Moreover, $\Box u_2' = F_2$, so u_2 and u_2' have identical Fourier transforms, whence $u_2 = u_2'$. $\square$

3.6.3 Proof of proposition 16

Both u_1 and $\partial_t u_1$ are linear combinations of $v_\pm(t) = \int_0^t e^{\pm i(t-t')D} F_1(t')\,dt'$. For a.e. ξ,

$$\begin{aligned}
\widehat{v_\pm(t)}(\xi) &= \int_0^t e^{\pm i(t-t')|\xi|} \widehat{F_1(t')}(\xi)\,dt' \\
&= \int_0^t e^{\pm i(t-t')|\xi|} \frac{1}{2\pi} \int e^{it'\tau} \widehat{F_1}(\tau,\xi)\,d\tau\,dt' \\
&= \frac{e^{\pm it|\xi|}}{2\pi} \int \left(\int_0^t e^{it'(\tau \mp |\xi|)}\,dt' \right) \widehat{F_1}(\tau,\xi)\,d\tau \\
&= \frac{e^{\pm it|\xi|}}{2\pi} \int \frac{e^{it(\tau\mp|\xi|)} - 1}{i(\tau \mp |\xi|)} \widehat{F_1}(\tau,\xi)\,d\tau.
\end{aligned} \tag{3.71}$$

Since

$$\frac{e^{it(\tau\mp|\xi|)} - 1}{i(\tau \mp |\xi|)} = \sum_{j=1}^\infty \frac{t^j}{j!} i^{j-1} (\tau \mp |\xi|)^{j-1},$$

it follows that

$$\begin{aligned}
\widehat{v_\pm(t)}(\xi) &= \frac{e^{\pm it|\xi|}}{2\pi} \int \sum_{j=1}^\infty \frac{t^j}{j!} i^{j-1} (\tau \mp |\xi|)^{j-1} \widehat{F_1}(\tau,\xi)\,d\tau \\
&= \frac{e^{\pm it|\xi|}}{2\pi} \sum_{j=1}^\infty \frac{t^j}{j!} \int i^{j-1} (\tau \mp |\xi|)^{j-1} \widehat{F_1}(\tau,\xi)\,d\tau
\end{aligned} \tag{3.72}$$

for a.e. ξ.

We write $F_1 = F_{1,1} + F_{1,2}$, where $\widehat{F_{1,1}}(\tau,\xi)$ and $\widehat{F_{1,2}}(\tau,\xi)$ are supported in the regions $|\xi| < c$ and $|\xi| \geq c$, respectively. Let $u_{1,j}$ be defined as in (3.68), but with F_1 replaced by $F_{1,j}$ for $j = 1, 2$.

Formula for $u_{1,1}$

By (3.72), we have

$$\begin{aligned}
&\widehat{u_{1,1}(t)}(\xi) \\
&\quad = \frac{1}{4\pi}\sum_{j=1}^{\infty}\frac{t^j}{j!}\int i^j\,|\xi|^{-1}\left(e^{it|\xi|}(\tau-|\xi|)^{j-1}-e^{-it|\xi|}(\tau+|\xi|)^{j-1}\right)\widehat{F_{1,1}}(\tau,\xi)\,d\tau \\
&\quad = \frac{1}{4\pi}\sum_{j=1}^{\infty}\frac{t^j}{j!}\int i^j\,|\xi|^{-1}\big(\alpha(|\xi|)-\alpha(-|\xi|)\big)\widehat{F_{1,1}}(\tau,\xi)\,d\tau \\
&\quad = \frac{1}{2\pi}\sum_{j=1}^{\infty}\frac{t^j}{j!}\int\int_0^1 i^j\alpha'\big((2\rho-1)\,|\xi|\big)\widehat{F_{1,1}}(\tau,\xi)\,d\rho\,d\tau \\
&\quad = \frac{1}{2\pi}\sum_{j=1}^{\infty}\frac{t^j}{j!}\int_0^1\int i^j\alpha'\big((2\rho-1)\,|\xi|\big)\widehat{F_{1,1}}(\tau,\xi)\,d\tau\,d\rho,
\end{aligned}$$

where $\alpha(r)=e^{itr}(\tau-r)^{j-1}$. Since

$$\alpha'(r)=ite^{itr}(\tau-r)^{j-1}-e^{itr}(j-1)(\tau-r)^{j-2},$$

where the second term only occurs for $j\geq 2$, we get

$$\begin{aligned}
u_{1,1}(t) &= \frac{1}{2\pi}\sum_{j=1}^{\infty}\frac{t^j}{j!}\int_0^1 i^{j+1}te^{it(2\rho-1)D}k_j(\rho)\,d\rho \\
&\quad -\frac{1}{2\pi}\sum_{j=2}^{\infty}\frac{t^j}{j!}\int_0^1 i^j(j-1)e^{it(2\rho-1)D}k_{j-1}(\rho)\,d\rho \\
&= \frac{1}{2\pi}\sum_{j=1}^{\infty}\frac{t^{j+1}}{j!}\int_0^1 i^{j+1}\Big(1-\frac{j}{j+1}\Big)e^{it(2\rho-1)D}k_j(\rho)\,d\rho,
\end{aligned}$$

where $k_j\in C(\mathbb{R},L^2)$ is given by $\widehat{k_j(\rho)}(\xi)=\int\big(\tau-(2\rho-1)\,|\xi|\big)^{j-1}\widehat{F_{1,1}}(\tau,\xi)\,d\tau$. Setting $g_j=(2\pi)^{-1}i^{j+1}\big(1-j/(j+1)\big)k_j$, we have

$$u_{1,1}(t)=\sum_{j=1}^{\infty}\frac{t^{j+1}}{j!}\int_0^1 e^{it(2\rho-1)D}g_j(\rho)\,d\rho. \tag{3.73}$$

Since $\big|\tau-(2\rho-1)\,|\xi|\big|\leq|\tau|+|\xi|\leq\big||\tau|-|\xi|\big|+2\,|\xi|\leq\big||\tau|-|\xi|\big|+2c\leq 3c$ for $\rho\in[0,1]$ and $\xi\in\operatorname{supp}\widehat{g_j(\rho)}$, it follows by the Cauchy-Schwarz inequality and (3.69) that $\|g_j(\rho)\|_{H^{s-1}}\lesssim c^{j-1/2}\,\|F_1\|_{s-1,0}$ for $\rho\in[0,1]$.

Formula for $u_{1,2}$

Combining (3.71) and (3.72), we see that

$$\begin{aligned}\widehat{u_{1,2}(t)}(\xi) &= \frac{e^{it|\xi|}}{4\pi\,|\xi|}\sum_{j=1}^{\infty}\frac{t^j}{j!}\int_0^{\infty} i^j(|\tau|-|\xi|)^{j-1}\widehat{F_{1,2}}(\tau,\xi)\,d\tau \\ &\quad - \frac{e^{-it|\xi|}}{4\pi\,|\xi|}\sum_{j=1}^{\infty}\frac{t^j}{j!}\int_{-\infty}^{0} i^j(|\xi|-|\tau|)^{j-1}\widehat{F_{1,2}}(\tau,\xi)\,d\tau \\ &\quad - \frac{1}{4\pi\,|\xi|}\int_{-\infty}^{0}\frac{e^{it\tau}-e^{it|\xi|}}{|\tau|+|\xi|}\widehat{F_{1,2}}(\tau,\xi)\,d\tau \\ &\quad - \frac{1}{4\pi\,|\xi|}\int_{0}^{\infty}\frac{e^{it\tau}-e^{-it|\xi|}}{|\tau|+|\xi|}\widehat{F_{1,2}}(\tau,\xi)\,d\tau\end{aligned}$$

for a.e. ξ. Hence,

$$u_{1,2}(t) = \sum_{j=1}^{\infty}\frac{t^j}{j!}\left(e^{itD}f_j^+ + e^{-itD}f_j^-\right) + R_+(t) + R_-(t),$$

where

$$\begin{aligned}\widehat{f_j^+}(\xi) &= (4\pi\,|\xi|)^{-1}\int_0^{\infty} i^j(|\tau|-|\xi|)^{j-1}\widehat{F_{1,2}}(\tau,\xi)\,d\tau, \\ \widehat{f_j^-}(\xi) &= -(4\pi\,|\xi|)^{-1}\int_{-\infty}^{0} i^j(|\xi|-|\tau|)^{j-1}\widehat{F_{1,2}}(\tau,\xi)\,d\tau, \\ \widehat{R_+(t)}(\xi) &= -\frac{1}{4\pi\,|\xi|}\int_{-\infty}^{0}\frac{e^{it\tau}-e^{it|\xi|}}{|\tau|+|\xi|}\widehat{F_{1,2}}(\tau,\xi)\,d\tau\end{aligned}$$

and

$$\widehat{R_-(t)}(\xi) = -\frac{1}{4\pi\,|\xi|}\int_{0}^{\infty}\frac{e^{it\tau}-e^{-it|\xi|}}{|\tau|+|\xi|}\widehat{F_{1,2}}(\tau,\xi)\,d\tau.$$

Formula for $\partial_t u_1$

Since $\partial_t u_1(t) = -\int_0^t \cos\bigl((t-t')D\bigr)\cdot F_1(t')\,dt'$, the stated formula for $\partial_t u_1$ follows by a straightforward modification of the derivation of the formula for $u_{1,2}$. Since the factor D^{-1} is not present, there is no need to consider separately low and high frequencies in this case.

3.6.4 Proof of proposition 15

We must prove that the expressions $\|\chi u_1\|_{s,\theta}$, $\|\chi' u_1\|_{s-1,\theta}$ and $\|\chi\partial_t u_1\|_{s-1,\theta}$ are all bounded by $C\,\|F_1\|_{s-1,0}$, with C as in the statement of the proposition. Since χ is just an arbitrary C_c^∞ function at this point, we will in fact estimate $\|\chi u_1\|_{s-1,\theta}$ rather than $\|\chi' u_1\|_{s-1,\theta}$.

For the purpose of estimating χu_1 it will be useful to split $u_1(t)$ into high and low frequency parts. Thus, as in the previous section we write $F_1 = F_{1,1} + F_{1,2}$, where $\widehat{F_{1,1}}(\tau,\xi)$ and $\widehat{F_{1,2}}(\tau,\xi)$ are supported in the regions $|\xi| < c$ and $|\xi| \geq c$, respectively. Let $u_{1,j}$ be defined as in (3.68), but with F_1 replaced by $F_{1,j}$ for $j = 1, 2$.

Estimates for $\|\chi u_{1,1}\|_{s,\theta}$ and $\|\chi u_{1,1}\|_{s-1,\theta}$

By proposition 16,

$$u_{1,1}(t) = \sum_{j=1}^{\infty} \frac{t^{j+1}}{j!} \int_0^1 e^{it(2\rho-1)D} g_j(\rho)\, d\rho.$$

Thus, by (3.62),

$$\|\chi u_{1,1}\|_{s,\theta} \leq C_1 \|F_1\|_{s-1,0},$$

where

$$C_1 \simeq \sum_{j=1}^{\infty} \frac{c^{j+1/2} \left\|t^{j+1}\chi\right\|_{H^\theta}}{j!} + \sum_{j=1}^{\infty} \frac{c^{j+1/2+\theta} \left\|t^{j+1}\chi\right\|_{L^2}}{j!}.$$

Similarly,

$$\|\chi u_{1,1}\|_{s-1,\theta} \leq C_2 \|F_1\|_{s-1,0},$$

where

$$C_2 \simeq \sum_{j=1}^{\infty} \frac{c^{j-1/2} \left\|t^{j+1}\chi\right\|_{H^\theta}}{j!} + \sum_{j=1}^{\infty} \frac{c^{j-1/2+\theta} \left\|t^{j+1}\chi\right\|_{L^2}}{j!}.$$

Estimate for $\|\chi u_{1,2}\|_{s,\theta}$

By proposition 16,

$$u_{1,2}(t) = \sum_{j=1}^{\infty} \frac{t^j}{j!} \left(e^{itD} f_j^+ + e^{-itD} f_j^-\right) + R_+(t) + R_-(t).$$

Since $\left\|f_j^\pm\right\|_{H^s} \leq c^{j-1/2} \|F_1\|_{s-1,0}$, it follows from (3.59) that

$$\left\| \chi \sum_{j=1}^{\infty} \frac{t^j}{j!} \left(e^{itD} f_j^+ + e^{-itD} f_j^-\right) \right\|_{s,\theta} \leq \left(\sum_{j=1}^{\infty} \frac{c^{j-1/2} \left\|t^j \chi\right\|_{H^\theta}}{j!} \right) \|F_1\|_{s-1,0}.$$

Next, since

$$\widehat{\chi R_+}(\tau,\xi) = -\frac{1}{4\pi|\xi|} \int_{-\infty}^0 \frac{\widehat{\chi}(\tau-\lambda) - \widehat{\chi}(\tau-|\xi|)}{|\lambda| + |\xi|} \widehat{F_{1,2}}(\lambda,\xi)\, d\lambda, \tag{3.74}$$

it follows from Minkowski's inequality that

$$\|\chi R_+\|_{s,\theta} \lesssim \int \left\| A(\lambda,\xi)(1+|\xi|)^{s-1} \widehat{F_{1,2}}(\lambda,\xi) \right\|_{L^2_\xi} d\lambda$$

where

$$A = \left\| \left(1 + \big||\tau| - |\xi|\big|\right)^{\theta} \frac{\widehat{\chi}(\tau - \lambda) - \widehat{\chi}(\tau - |\xi|)}{|\lambda| + |\xi|} \right\|_{L^2_\tau}.$$

We claim that $A \lesssim \|\chi\|_{\dot{H}^{\theta-1}} + \|t\chi\|_{H^\theta}$ for $(\lambda, \xi) \in \operatorname{supp} \widehat{F_{1,2}}$. This would give

$$\begin{aligned}
\|\chi R_+\|_{s,\theta} &\lesssim \left(\|\chi\|_{\dot{H}^{\theta-1}} + \|t\chi\|_{H^\theta}\right) \int \left\| (1+|\xi|)^{s-1} \widehat{F_{1,2}}(\lambda, \xi) \right\|_{L^2_\xi} d\lambda \\
&\lesssim c^{1/2} \left(\|\chi\|_{\dot{H}^{\theta-1}} + \|t\chi\|_{H^\theta}\right) \|F_1\|_{s-1,0},
\end{aligned}$$

where we used the Cauchy-Schwarz inequality and (3.69) to obtain the last inequality.

It remains to prove the claim. We have

$$\begin{aligned}
A &\lesssim \left\| \frac{\widehat{\chi}(\tau - \lambda) - \widehat{\chi}(\tau - |\xi|)}{|\lambda| + |\xi|} \right\|_{L^2_\tau} + \left\| \big||\tau| - |\xi|\big|^{\theta} \frac{\widehat{\chi}(\tau - \lambda) - \widehat{\chi}(\tau - |\xi|)}{|\lambda| + |\xi|} \right\|_{L^2_\tau(I)} \\
&\quad + \left\| \big||\tau| - |\xi|\big|^{\theta} \frac{\widehat{\chi}(\tau - \lambda) - \widehat{\chi}(\tau - |\xi|)}{|\lambda| + |\xi|} \right\|_{L^2_\tau(\mathbb{R}\setminus I)} \\
&= A_1 + A_2 + A_3,
\end{aligned}$$

where

$$I = I(\lambda, \xi) = \left\{ \tau \in \mathbb{R} : \big|\tau - |\xi|\big| < 2(|\lambda| + |\xi|) \right\}.$$

Since

$$\frac{\widehat{\chi}(\tau - \lambda) - \widehat{\chi}(\tau - |\xi|)}{|\lambda| + |\xi|} = \int_0^1 \widehat{\chi}'\left(\tau - |\xi| + \rho(|\lambda| + |\xi|)\right) d\rho, \tag{3.75}$$

an application of Minkowski's inequality yields

$$A_1 \leq \int_0^1 \|t\chi\|_{L^2} \, d\rho = \|t\chi\|_{L^2}.$$

Using (3.75) again, as well as the fact that

$$\big||\tau| - |\xi|\big| \leq \big|\tau - |\xi|\big| \leq 2\big|\tau - |\xi| + \rho(|\lambda| + |\xi|)\big|$$

for $\tau \in \mathbb{R} \setminus I$, we get $A_3 \lesssim \|t\chi\|_{H^\theta}$. Finally, since $|\tau - \lambda| \lesssim |\lambda| + |\xi|$ for $\tau \in I$, and since $\theta < 1$, we have

$$\begin{aligned}
A_2 &\lesssim \left\| (|\lambda| + |\xi|)^{\theta-1} \left(\widehat{\chi}(\tau - \lambda) - \widehat{\chi}(\tau - |\xi|)\right) \right\|_{L^2_\tau(I)} \\
&\lesssim \left\| (\tau - \lambda)^{\theta-1} \widehat{\chi}(\tau - \lambda) \right\|_{L^2_\tau(I)} + \left\| (\tau - |\xi|)^{\theta-1} \widehat{\chi}(\tau - |\xi|) \right\|_{L^2_\tau(I)} \\
&= 2\, \|\chi\|_{\dot{H}^{\theta-1}}.
\end{aligned}$$

This proves the claim.

By a similar argument,

$$\|\chi R_-\|_{s,\theta} \lesssim c^{1/2} \left(\|\chi\|_{\dot{H}^{\theta-1}} + \|t\chi\|_{H^\theta}\right) \|F_1\|_{s-1,0}.$$

Estimate for $\|\chi u_{1,2}\|_{s-1,\theta}$

For this estimate we do not use proposition 16. Instead, we use (3.71) to write

$$\widehat{u_{1,2}(t)}(\xi) = \frac{1}{4\pi|\xi|} \int \left\{ \frac{e^{it\tau} - e^{it|\xi|}}{\tau - |\xi|} - \frac{e^{it\tau} - e^{-it|\xi|}}{\tau + |\xi|} \right\} \widehat{F_{1,2}}(\tau,\xi)\, d\tau,$$

which gives

$$\begin{aligned} &\widehat{\chi u_{1,2}}(\tau,\xi) \\ &= \frac{1}{4\pi|\xi|} \int \left\{ \frac{\widehat{\chi}(\tau-\lambda) - \widehat{\chi}(\tau-|\xi|)}{\lambda - |\xi|} - \frac{\widehat{\chi}(\tau-\lambda) - \widehat{\chi}(\tau+|\xi|)}{\lambda+|\xi|} \right\} \widehat{F_{1,2}}(\lambda,\xi)\, d\lambda \\ &= -\frac{1}{4\pi|\xi|} \int\int_0^1 \left\{ \widehat{\chi}'(\tau-a) - \widehat{\chi}'(\tau-b) \right\} \widehat{F_{1,2}}(\lambda,\xi)\, d\rho\, d\lambda, \end{aligned}$$

where $a = |\xi| + \rho(\lambda + |\xi|)$ and $b = -|\xi| + \rho(\lambda+|\xi|)$. We distinguish two cases.

Case 1 Assume $|\tau| < 8|\xi|$. Then for $(\lambda,\xi) \in \operatorname{supp} \widehat{F_{1,2}}$,

$$\big||\tau| - |\xi|\big|, |\tau - a|, |\tau - b| \lesssim |\xi|,$$

and since $\theta < 1$, we obtain

$$\begin{aligned} &(1+|\xi|)^{s-1}\big(1 + \big||\tau|-|\xi|\big|\big)^{\theta} \left|\widehat{\chi u_{1,2}}(\tau,\xi)\right| \\ &\lesssim \int\int_0^1 \left\{ \frac{\left|\widehat{\chi}'(\tau-a)\right|}{|\tau-a|^{1-\theta}} + \frac{\left|\widehat{\chi}'(\tau-b)\right|}{|\tau-b|^{1-\theta}} \right\} (1+|\xi|)^{s-1} \left|\widehat{F_{1,2}}(\lambda,\xi)\right| d\rho\, d\lambda. \end{aligned}$$

Case 2 Assume $|\tau| \geq 8|\xi|$. In this case we write

$$\begin{aligned} &\widehat{\chi u_{1,2}}(\tau,\xi) \\ &\quad = \frac{1}{2\pi} \int\int_0^1 \int_0^1 \widehat{\chi}''\big(\tau - b + \sigma(b-a)\big)(1-\rho)\widehat{F_{1,2}}(\lambda,\xi)\, d\sigma\, d\rho\, d\lambda \end{aligned}$$

and use the fact that

$$\big|\tau - b + \sigma(b-a)\big| \geq |\tau| - |b| - |b-a| \geq |\tau| - 6|\xi| \gtrsim \big||\tau| - |\xi|\big|$$

to get

$$\begin{aligned} &(1+|\xi|)^{s-1}\big(1 + \big||\tau|-|\xi|\big|\big)^{\theta} \left|\widehat{\chi u_{1,2}}(\tau,\xi)\right| \\ &\quad \lesssim \int\int_0^1 \int_0^1 (1+|\tau-\alpha|)^{\theta} \left|\widehat{\chi}''(\tau-\alpha)\right| (1+|\xi|)^{s-1} \left|\widehat{F_{1,2}}(\lambda,\xi)\right| d\sigma\, d\rho\, d\lambda, \end{aligned}$$

where $\alpha = b + \sigma(a-b)$.

In both cases we conclude, by Minkowski's inequality, that

$$\|\chi u_{1,2}\|_{s-1,\theta} \lesssim c^{1/2}\big(\|t\chi\|_{\dot{H}^{\theta-1}} + \left\|t^2\chi\right\|_{H^{\theta}}\big) \|F_1\|_{s-1,0}\,.$$

Estimate for $\|\chi \partial_t u_1\|_{s-1,\theta}$

A straightforward modification of the argument used to estimate $\|\chi u_{1,2}\|_{s,\theta}$ shows that

$$\|\chi \partial_t u_1\|_{s-1,\theta} \lesssim c^{1/2}\big(\|\chi\|_{\dot{H}^{\theta-1}} + \|t\chi\|_{H^\theta}\big) \|F_1\|_{s-1,0} .$$

3.6.5 Proof of theorem 12

Let us restate the theorem in a more precise form.

Theorem 13. *Assume $s \in \mathbb{R}$, $\theta \in (1/2, 1)$, $\varepsilon \in (0, 1-\theta]$, $F \in H^{s-1,\theta+\varepsilon-1}$ and*

$$\chi \in C_c^\infty(\mathbb{R}), \quad \chi = 1 \quad \text{on} \quad [-1,1], \quad \operatorname{supp}\chi \subseteq (-2,2). \tag{3.76}$$

Let $0 < T < 1$ and define

$$u(t) = \chi(t)u_0 + \chi(t/T)u_1 + u_2,$$

where

$$\begin{aligned} u_0 &= \cos(tD)\cdot f + D^{-1}\sin(tD)\cdot g, \\ u_1 &= \int_0^t D^{-1}\sin\big((t-t')D\big)\cdot F_1(t')\,dt', \\ u_2 &= \Box^{-1}F_2, \\ F &= F_1 + F_2 = \phi(T^{1/2}\Lambda_-)F + \big(1-\phi(T^{1/2}\Lambda_-)\big)F \end{aligned}$$

and ϕ satisfies (3.66). *Then*

$$|u|_{s,\theta} \le C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}} + T^{\varepsilon/2}\|F\|_{s-1,\theta+\varepsilon-1}\big),$$

where C only depends on χ and θ. Moreover, u is the unique solution of (1.1) *on $[0,T]\times\mathbb{R}^n$ such that $u \in C\big([0,T],H^s\big)\cap C^1\big([0,T],H^{s-1}\big)$.*

Proof. By proposition 13,

$$|\chi(t)u_0|_{s,\theta} \le C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}}\big),$$

where C only depends on χ and θ.

Define $\chi_T(t) = \chi(t/T)$ for $t \in \mathbb{R}$. With C_1 and C_2 as in (3.3), and ϕ satisfying (3.66), it is easily checked that

$$\big||\tau| - |\xi|\big| \le \frac{4C_2}{C_1 T^{1/2}} \quad \text{for} \quad (\tau,\xi) \in \operatorname{supp}\widehat{F_1}, \tag{3.77}$$

whence (3.69) holds with $c = 2 + 4C_2C_1^{-1}T^{-1/2}$. Thus,

$$\|F_1\|_{s-1,0} \le c^{1-\theta}\|F\|_{s-1,\theta-1},$$

and it follows from proposition 15 that

$$|\chi u_1|_{s,\theta} \lesssim C_T \, \|F\|_{s-1,\theta-1} \, ,$$

where

$$\begin{aligned} C_T \simeq c^{3/2-\theta} &\big(\|\chi_T\|_{\dot H^{\theta-1}} + \|t\chi_T\|_{H^\theta} + \|t(\chi_T)'\|_{\dot H^{\theta-1}} + \left\|t^2(\chi_T)'\right\|_{H^\theta} \big) \\ &+ \sum_{j=1}^\infty \Bigg(\frac{c^{j+3/2-\theta} \left\|t^{j+1}\chi_T\right\|_{H^\theta}}{j!} + \frac{c^{j+3/2} \left\|t^{j+1}\chi_T\right\|_{L^2}}{j!} \\ &\quad + \frac{c^{j+1/2-\theta} \left\|t^{j+1}(\chi_T)'\right\|_{H^\theta}}{j!} + \frac{c^{j+1/2} \left\|t^{j+1}(\chi_T)'\right\|_{L^2}}{j!} \\ &\quad + \frac{c^{j+1/2-\theta} \left\|t^{j}\chi_T\right\|_{H^\theta}}{j!} \Bigg). \end{aligned}$$

Since

$$\|\chi_T\|_{H^\theta} \le C T^{1/2-\theta} \, \|\chi\|_{H^\theta} \quad \text{for} \quad 0 < T \le 1,$$

and

$$\|\chi_T\|_{\dot H^{\theta-1}} = T^{3/2-\theta} \, \|\chi\|_{\dot H^{\theta-1}} \quad \text{for} \quad \theta > 1/2,$$

we get

$$\begin{aligned} C_T \lesssim (cT)^{3/2-\theta} &\big(\|\chi\|_{\dot H^{\theta-1}} + \|t\chi\|_{H^\theta} + \|t\chi'\|_{\dot H^{\theta-1}} + \left\|t^2\chi'\right\|_{H^\theta} \big) \\ &+ \sum_{j=1}^\infty \Bigg(\frac{(cT)^{j+3/2-\theta} \left\|t^{j+1}\chi\right\|_{H^\theta}}{j!} + \frac{(cT)^{j+3/2} \left\|t^{j+1}\chi\right\|_{L^2}}{j!} \\ &\quad + \frac{(cT)^{j+1/2-\theta} \left\|t^{j+1}\chi'\right\|_{H^\theta}}{j!} + \frac{(cT)^{j+1/2} \left\|t^{j+1}\chi'\right\|_{L^2}}{j!} \\ &\quad + \frac{(cT)^{j+1/2-\theta} \left\|t^{j}\chi\right\|_{H^\theta}}{j!} \Bigg). \end{aligned}$$

Thus, since $c \lesssim T^{-1/2}$ and $\theta < 1$, we conclude that $C_T \le C_\chi T^{1/4}$, where

$$\begin{aligned} C_\chi \simeq &\|\chi\|_{\dot H^{\theta-1}} + \|t\chi'\|_{\dot H^{\theta-1}} \\ &+ \sum_{j=1}^\infty \frac{1}{j!} \left(\left\|t^{j+1}\chi\right\|_{H^\theta} + \left\|t^{j+1}\chi'\right\|_{H^\theta} + \left\|t^{j}\chi\right\|_{H^\theta} \right). \end{aligned}$$

Next, since it is readily verified that

$$\big||\tau| - |\xi|\big| > \frac{1}{T^{1/2}} \quad \text{for} \quad (\tau,\xi) \in \operatorname{supp} \widehat{F_2}, \tag{3.78}$$

we have

$$u_2 \precsim T^{\varepsilon/2} \Lambda_+^{-1} \Lambda_-^{\varepsilon-1} F,$$

whence $|u_2|_{s,\theta} \lesssim T^{\varepsilon/2} \, \|F\|_{s-1,\theta+\varepsilon-1}$.

Clearly, u solves (1.1) on the time interval $[0,T]$, and uniqueness follows from the proof of proposition 1, which works equally well with $\mathbb{R}$ replaced by $[0,T]$. □

3.7 The restriction space $\mathcal{X}_T^{s,\theta}$

If $T > 0$, we define the equivalence relation $\sim_T$ on $\mathcal{X}^{s,\theta}$ by

$$u \sim_T v \qquad \Longleftrightarrow \qquad u(t) = v(t) \quad \text{for} \quad t \in [0,T],$$

and we set $\mathcal{X}_T^{s,\theta} = \mathcal{X}^{s,\theta} / \sim_T$ and

$$|u|_{s,\theta,T} = \inf \left\{ |\widetilde{u}|_{s,\theta} : \widetilde{u} \in \mathcal{X}^{s,\theta},\, \widetilde{u} \sim_T u \right\}.$$

A trivial but useful observation is that for a given $u \in \mathcal{X}^{s,\theta}$, we have $|u|_{s,\theta,T} = \inf_{\widetilde{u} \in E_T(u)} |\widetilde{u}|_{s,\theta}$, where

$$E_T(u) = \left\{ \widetilde{u} \in \mathcal{X}^{s,\theta} : \widetilde{u} \sim_T u,\, |\widetilde{u}|_{s,\theta} \leq |u|_{s,\theta} \right\}. \tag{3.79}$$

By proposition 7(a),

$$\|u(t)\|_{H^s} \leq C \, |u|_{s,\theta,T} \quad \text{for} \quad t \in [0,T],$$

which implies that $|\cdot|_{s,\theta,T}$ is a norm on $\mathcal{X}_T^{s,\theta}$. Moreover, since $\mathcal{X}^{s,\theta}$ is complete, so is $\mathcal{X}_T^{s,\theta}$. We call $\mathcal{X}_T^{s,\theta}$ the *restriction space of* $\mathcal{X}^{s,\theta}$ *to time* T. By proposition 7(a), the restriction space embeds in $C_b\big([0,T], H^s\big) \cap C_b^1\big([0,T], H^{s-1}\big)$.

Chapter 4

Two Well-Posedness Theorems

Our purpose in this chapter is to provide a general framework for proving strong local well-posedness for nonlinear systems of wave equations of the form

$$\Box u = F(u, \partial u) \qquad (t,x) \in \mathbb{R}^{1+n} \tag{4.1a}$$

$$u\big|_{t=0} = f \in H^s, \quad \partial_t u\big|_{t=0} = g \in H^{s-1}, \tag{4.1b}$$

where ∂u is the space-time gradient of u and F is a smooth function satisfying $F(0) = 0$.

In section 4.1 we prove, using theorem 13, that well-posedness holds for data in $H^s \times H^{s-1}$ under the assumption that

$$u \longmapsto F(u, \partial u), \quad \mathcal{X}^{s,\theta} \longrightarrow H^{s-1,\theta+\varepsilon-1}$$

is bounded for some choice of $\theta > 1/2$ and $\varepsilon > 0$. We then apply this theorem to recover three well-known well-posedness results: the classical local existence theorem, a sharp local existence theorem of Ponce and Sideris, and the wave maps equation in local coordinates.

In section 4.2 we motivate the need for a more general version of this result, and we state an appropriate generalization.

4.1 First well-posedness theorem

Assume that for given

$$s \in \mathbb{R}, \quad \theta \in (1/2, 1) \quad \text{and} \quad \varepsilon \in (0, 1-\theta), \tag{4.2}$$

we have

$$\|F(u,\partial u)\|_{\sigma-1,\theta+\varepsilon-1} \le A_\sigma(|u|_{s,\theta})\,|u|_{\sigma,\theta} \quad \text{for all } \sigma \ge s \tag{4.3}$$

and

$$\|F(u,\partial u) - F(v,\partial v)\|_{s-1,\theta+\varepsilon-1} \le A_s(|u|_{s,\theta} + |v|_{s,\theta})\,|u-v|_{s,\theta} \tag{4.4}$$

for all $u, v \in \mathcal{X}^{s,\theta}$, where $A_\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and locally Lipschitz for every $\sigma \ge s$. Under these assumptions, (4.1) is locally well-posed, in the following precise sense.

Theorem 14. *If* (4.2), (4.3) *and* (4.4) *hold, there exists* $u \in \mathcal{X}^{s,\theta}$ *which solves* (4.1) *on* $[0,T] \times \mathbb{R}^n$, *where* $T = T(\|f\|_{H^s} + \|g\|_{H^{s-1}}) > 0$ *depends continuously on* $\|f\|_{H^s} + \|g\|_{H^{s-1}}$.

The solution is unique in the class $\mathcal{X}^{s,\theta}$, *in the sense that if* $u, v \in \mathcal{X}^{s,\theta}$ *are solutions of* (4.1) *on* $[0,T] \times \mathbb{R}^n$ *for some* $T > 0$, *then*

$$u(t) = v(t) \quad \textit{for} \quad t \in [0,T].$$

Moreover, the solution map

$$(f,g) \longmapsto u, \quad H^s \times H^{s-1} \longrightarrow \mathcal{X}^{s,\theta}$$

is locally Lipschitz, and if the data have the additional regularity

$$f \in H^\sigma, \quad g \in H^{\sigma-1}, \quad \textit{where} \quad \sigma > s,$$

then

$$u \in C\big([0,T], H^\sigma\big) \cap C^1\big([0,T], H^{\sigma-1}\big)$$

for any $T > 0$ *such that* u *solves* (4.1) *on* $[0,T] \times \mathbb{R}^n$. *In particular, if* $f, g \in \mathcal{S}$, *then* u *is* C^∞ *on* $[0,T] \times \mathbb{R}^n$.

Proof. The proof splits naturally into several steps.

Step 1: Existence Let χ be as in (3.76), and define

$$\begin{aligned}
\Phi u = {} & \chi(t)\big(\cos(tD)\cdot f + D^{-1}\sin(tD)\cdot g\big) \\
& - \chi(t/T)\int_0^t D^{-1}\sin\big((t-t')D\big)\cdot\big(\phi(T^{1/2}\Lambda_-)F(u,\partial u)\big)(t')\,dt' \\
& + \Box^{-1}\big(1-\phi(T^{1/2}\Lambda_-)\big)F(u,\partial u),
\end{aligned} \tag{4.5}$$

for $u \in \mathcal{X}^{s,\theta}$. Then by theorem 13,

$$\begin{gathered}
\Box\Phi u = F(u,\partial u) \quad \text{on} \quad [0,T]\times\mathbb{R}^n \\
\Phi u\big|_{t=0} = f, \quad \partial_t\Phi u\big|_{t=0} = g,
\end{gathered}$$

so any fixed point of Φ is a solution of (4.1) on $[0,T] \times \mathbb{R}^n$. Furthermore, combining theorem 13 with (4.3) and (4.4), we have

$$|\Phi u|_{s,\theta} \le C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}} + T^{\varepsilon/2}A_s(|u|_{s,\theta})\,|u|_{s,\theta}\big)$$

and

$$|\Phi u - \Phi v|_{s,\theta} \le CT^{\varepsilon/2} A_s(|u|_{s,\theta} + |v|_{s,\theta})\, |u-v|_{s,\theta} \,. \tag{4.6}$$

Therefore, if we let $T = T_s(\|f\|_{H^s} + \|g\|_{H^{s-1}})$, where

$$T_\sigma(r) = \min\left\{1, \left(2CA_\sigma(4Cr)\right)^{-2/\varepsilon}\right\} \quad \text{for} \quad r \ge 0, \tag{4.7}$$

then it follows that Φ is a contraction of the closed ball in $\mathcal{X}^{s,\theta}$ centered at 0 and with radius $2C(\|f\|_{H^s} + \|g\|_{H^{s-1}})$. Let us denote this ball by X. Since X is a complete metric space, Φ has a unique fixed point u in X.

Step 2: Uniqueness Assume that $u, v \in \mathcal{X}^{s,\theta}$ are solutions of (4.1) on $[0,T]\times \mathbb{R}^n$ for some $T > 0$. By a continuity argument, we see that it is enough to show (4.26) for arbitrarily small $T > 0$. This amounts to showing that $u = v$ in the restriction space $\mathcal{X}_T^{s,\theta}$. To this end, assume that $\widetilde{u} \sim_T u$ and $\widetilde{v} \sim_T v$. Then by theorem (13), $u \sim_T \Phi\widetilde{u}$ and $v \sim_T \Phi\widetilde{v}$, and it follows from (4.6) that

$$|u - v|_{s,\theta,T} \le |\Phi\widetilde{u} - \Phi\widetilde{v}|_{s,\theta} \le CT^{\varepsilon/2} A_s\big(|\widetilde{u}|_{s,\theta} + |\widetilde{v}|_{s,\theta}\big)\, |\widetilde{u} - \widetilde{v}|_{s,\theta} \,. \tag{4.8}$$

Recall that

$$|u - v|_{s,\theta,T} = \inf\left\{|\widetilde{w}|_{s,\theta} : w \in E_T(u-v)\right\}, \tag{4.9}$$

where $E_T(u-v)$ is defined in (3.79). Given $\widetilde{w} \in E_T(u-v)$, set $\widetilde{u} = u$ and $\widetilde{v} = \widetilde{u} - \widetilde{w}$. Then $|\widetilde{v}|_{s,\theta} \le |u|_{s,\theta} + |u-v|_{s,\theta}$, and since A_s is increasing, it follows from (4.8) that

$$|u - v|_{s,\theta,T} \le CT^{\varepsilon/2} A_s(2\,|u|_{s,\theta} + |u-v|_{s,\theta})\, |\widetilde{w}|_{s,\theta} \,.$$

Taking the infimum over $\widetilde{w} \in E_T(u-v)$ and using (4.9), we get

$$|u - v|_{s,\theta,T} \le CT^{\varepsilon/2} A_s(2\,|u|_{s,\theta} + |u-v|_{s,\theta})\, |u-v|_{s,\theta,T} \,.$$

Hence, if we choose $T > 0$ so small that

$$CT^{\varepsilon/2} A_s(2\,|u|_{s,\theta} + |u-v|_{s,\theta}) \le \frac{1}{2},$$

then we must have $|u-v|_{s,\theta,T} = 0$.

Step 3: Lipschitz continuity Here we prove that the dependence of the fixed point u on the data (f,g) is locally Lipschitz continuous. Let v be the fixed point corresponding to another set of data (f_*, g_*). We want to show that there exists a neighborhood U of (f,g) in $H^s \times H^{s-1}$ such that

$$|u - v|_{s,\theta} \lesssim \|f - f_*\|_{H^s} + \|g - g_*\|_{H^{s-1}} \quad \text{for} \quad (f_*, g_*) \in U.$$

Setting $T = T_s(\|f\|_{H^s} + \|g\|_{H^{s-1}})$ and $T_* = T_s(\|f_*\|_{H^s} + \|g_*\|_{H^{s-1}})$, where T_s is defined by (4.7), we have

$$\begin{aligned} u - v = {} & \chi(t)\big(\cos(tD)\cdot(f - f_*) + D^{-1}\sin(tD)\cdot(g - g_*)\big) \\ & - \chi(t/T_*)\int_0^t D^{-1}\sin\big((t-t')D\big) \\ & \qquad \times \big(\phi(T_*^{1/2}\Lambda_-)\{F(u,\partial u) - F(v,\partial v)\}\big)(t')\,dt' \\ & + \Box^{-1}\big(1 - \phi(T_*^{1/2}\Lambda_-)\big)\{F(u,\partial u) - F(v,\partial v)\} \\ & + \alpha(T_*) - \alpha(T) - \beta(T_*) + \beta(T), \end{aligned}$$

where

$$\alpha(T) = \chi(t/T)\int_0^t D^{-1}\sin\big((t-t')D\big)\cdot\big(\phi(T^{1/2}\Lambda_-)F(u,\partial u)\big)(t')\,dt'$$

and

$$\beta(T) = \Box^{-1}\big(1 - \phi(T^{1/2}\Lambda_-)\big)F(u,\partial u).$$

We claim that α and β are locally Lipschitz . Granting this, it follows from theorem 13 and (4.4) that

$$\begin{aligned} (4.10) \quad |u - v|_{s,\theta} \le {} & C\big(\|f - f_*\|_{H^s} + \|g - g_*\|_{H^{s-1}} \\ & + T_*^{\varepsilon/2}A(|u|_{s,\theta} + |v|_{s,\theta})\,|u - v|_{s,\theta}\big) + C'\,|T - T_*| \end{aligned}$$

for (f_*, g_*) sufficiently close to (f, g). Since

$$|u|_{s,\theta} \le 2C(\|f\|_{H^s} + \|g\|_{H^{s-1}}) \quad \text{and} \quad |v|_{s,\theta} \le 2C(\|f_*\|_{H^s} + \|g_*\|_{H^{s-1}}),$$

and since $CT^{\varepsilon/2}A_s\big(4C(\|f\|_{H^s} + \|g\|_{H^{s-1}})\big) \le 1/2$, it follows by the continuity of A_s and T_* that

$$CT_*^{\varepsilon/2}A_s(|u|_{s,\theta} + |v|_{s,\theta}) \le 3/4$$

for all (f_*, g_*) in some neighborhood U of (f, g). Hence, by (4.10),

$$|u - v|_{s,\theta} \lesssim 4C\big(\|f - f_*\|_{H^s} + \|g - g_*\|_{H^{s-1}}\big) + 4C'\,|T - T_*|$$

for $(f_*, g_*) \in U$. It remains to prove that by making U even smaller if necessary, we have

$$|T - T_*| \lesssim \|f - f_*\|_{H^s} + \|g - g_*\|_{H^{s-1}}\,.$$

This follows from the easily established fact that the function defined by (4.7) is locally Lipschitz.

We now prove that α and β are locally Lipschitz. We have

$$\begin{aligned} \alpha(T) - \alpha(T_*) = {} & \{\chi(t/T) - \chi(t/T_*)\} \\ & \qquad \times \int_0^t D^{-1}\sin\big((t-t')D\big)\cdot\big(\phi(T^{1/2}\Lambda_-)F(u,\partial u)\big)(t')\,dt' \\ & + \chi(t/T_*)\int_0^t D^{-1}\sin\big((t-t')D\big) \\ & \qquad \times \big(\{\phi(T^{1/2}\Lambda_-) - \phi(T_*^{1/2}\Lambda_-)\}F(u,\partial u)\big)(t')\,dt', \end{aligned}$$

and since

$$\chi(t/T) - \chi(t/T_*) = (T_* - T)\int_0^1 \chi'\left(\frac{t}{T_* + \rho(T - T_*)}\right)\frac{t}{\big(T_* + \rho(T - T_*)\big)^2}d\rho$$

and

$$(4.11)\quad \phi(T^{1/2}\Lambda_-) - \phi(T_*^{1/2}\Lambda_-) \\ = (T^{1/2} - T_*^{1/2})\int_0^1 \phi'\big(\{T_*^{1/2} + \rho(T^{1/2} - T_*^{1/2})\}\Lambda_-\big)\Lambda_-\,d\rho,$$

it follows from proposition 15 that

$$|\alpha(T) - \alpha(T_*)|_{s,\theta} \le |T - T_*|\,C\,\|F(u,\partial u)\|_{s-1,\theta-1}$$

for all T_* sufficiently close to T, where C depends on χ, ϕ and T.

Furthermore,

$$\beta(T_*) - \beta(T) = \square^{-1}\{\phi(T^{1/2}\Lambda_-) - \phi(T_*^{1/2}\Lambda_-)\}F(u,\partial u),$$

and using (4.11), we get

$$|\beta(T) - \beta(T_*)|_{s,\theta} \le |T - T_*|\,C\,\|F(u,\partial u)\|_{s-1,\theta-1}$$

for T_* close to T, with C again depending on χ, ϕ and T.

Step 4: Higher regularity If the data have the additional regularity $(f,g) \in H^\sigma \times H^{\sigma-1}$ for some $\sigma > s$, then by theorem 13 and (4.3),

$$|\Phi u|_{\sigma,\theta} \le C\big(\|f\|_{H^\sigma} + \|g\|_{H^{\sigma-1}} + T^{\varepsilon/2}A_\sigma(|u|_{s,\theta})\,|u|_{\sigma,\theta}\big).$$

Hence, if $T = \min\big\{T_s(\|f\|_{H^s} + \|g\|_{H^{s-1}}), T_\sigma(\|f\|_{H^s} + \|g\|_{H^{s-1}})\big\}$, then Φ is a contraction of the ball X defined above, and $\Phi(X_\sigma) \subseteq X_\sigma$, where

$$X_\sigma = X \cap \Big\{u : |u|_{\sigma,\theta} \le 2C\big(\|f\|_{H^\sigma} + \|g\|_{H^{\sigma-1}}\big)\Big\}.$$

Since X_σ is a closed subset of X, it follows that the fixed point of Φ must belong to X_σ.

Now let $u \in \mathcal{X}^{s,\theta}$ be a solution of (4.1) on $[0,T] \times \mathbb{R}^n$, where $T > 0$. By what we just showed, and using the translation invariance of the equation, it follows that for every $t_* \in [0,T]$ such that

$$\big(u(t_*), \partial_t u(t_*)\big) \in H^\sigma \times H^{\sigma-1},$$

there exists $u_* \in \mathcal{X}^{\sigma,\theta}$ which solves

$$\square u_* = F(u_*, \partial u_*) \quad \text{on} \quad (t_* - \delta_*, t_* + \delta_*) \times \mathbb{R}^n$$
$$u_*(t_*) = u(t_*), \quad \partial_t u_*(t_*) = \partial_t u(t_*),$$

where

$$\delta_* = \min\big\{T_s(\|u(t_*)\|_{H^s} + \|\partial_t u(t_*)\|_{H^{s-1}}), T_\sigma(\|u(t_*)\|_{H^s} + \|\partial_t u(t_*)\|_{H^{s-1}})\big\}.$$

Since $\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} \le C_\theta \left|u\right|_{s,\theta}$ for all t, we conclude that δ_* is bounded away from 0, and we set $\delta = \inf_{t_*} \delta_*$.

By the uniqueness statement proved above, we have

$$u_*(t) = u(t) \quad \text{for} \quad t \in [0,T] \cap (t_* - \delta, t_* + \delta).$$

Starting with $t_* = 0$ and then moving in steps of length $\delta/2$, say, we conclude that (4.27) holds. □

Remark. We make a general observation concerning (4.3) and (4.4) when the nonlinearity is multilinear, that is, $F(u, \partial u) = T(u, \dots, u)$, where T is a k-linear operator given by

$$\mathcal{F}\big\{T(u_1, \dots, u_k)\big\}(\tau, \xi) = \int \kappa(\tau_1, \xi_1, \dots, \tau_k, \xi_k)\widehat{u_1}(\tau_1, \xi_1) \cdots \widehat{u_k}(\tau_k, \xi_k)\, d\mu,$$

where $\tau_1 = \tau - \sum_2^k \tau_j$, $\xi_1 = \xi - \sum_2^k \xi_j$ and $d\mu = d\tau\, d\xi\, d\tau_2\, d\xi_2 \cdots d\tau_k\, d\xi_k$. We claim that (4.3) and (4.4) follow if we can show that

$$\| \lfloor T \rfloor (u_1, \dots, u_k)\|_{s-1,\theta+\varepsilon-1} \lesssim \left|u_1\right|_{s,\theta} \cdots \left|u_k\right|_{s,\theta} \tag{4.12}$$

for all $u_1, \dots, u_k \in \mathcal{X}^{s,\theta}$ such that $\widehat{u_j} \ge 0$, $1 \le j \le k$, where $\lfloor T \rfloor$ denotes the operator with symbol $|\kappa|$. First, since the norms on $H^{s,\theta}$ and $\mathcal{X}^{s,\theta}$ only depend on the absolute value of the Fourier transform, and since

$$T(u_1, \dots, u_k) \preceq \lfloor T \rfloor (\lfloor u_1 \rfloor, \dots, \lfloor u_k \rfloor),$$

it follows from (4.12) that

$$\|T(u_1, \dots, u_k)\|_{s-1,\theta+\varepsilon-1} \lesssim \left|u_1\right|_{s,\theta} \cdots \left|u_k\right|_{s,\theta} \quad \text{for all } u_1, \dots, u_k \in \mathcal{X}^{s,\theta}.$$

Therefore, (4.3) holds for $\sigma = s$, and (4.4) follows by multilinearity. To prove (4.3) for $\sigma > s$, we simply note that (cf. (3.24))

$$\Lambda^\gamma T(u_1, \cdots, u_k) \precsim \lfloor T \rfloor (\Lambda^\gamma u_1, u_2, \dots, u_k) + \cdots + \lfloor T \rfloor (u_1, \dots, u_{k-1}, \Lambda^\gamma u_k)$$

for $\gamma \ge 0$, assuming $\widehat{u_j} \ge 0$, $1 \le j \le k$. These facts will be used throughout the remainder of the dissertation, without further mention.

Let us look at some examples of equations to which theorem 14 can be applied.

4.1.1 The classical local existence theorem

Here we want to show that the classical local existence theorem for nonlinear hyperbolic equations, which states that (4.1) is well posed for $s > 1 + n/2$, can be proved using theorem 14.

Assume $s > 1 + n/2$, $1/2 < \theta < \min(1, s - 1/2 - n/2)$, $0 < \varepsilon < 1 - \theta$ and $n \geq 2$. Let F be any smooth function satisfying $F(0) = 0$. In this example we only consider real-valued $u \in \mathcal{X}^{s,\theta}$. By theorem 11, for any $\sigma \geq s$ there exists a continuous function $A_\sigma : \mathbb{R}_+ \to \mathbb{R}_+$, which we may assume is increasing and locally Lipschitz, such that

$$\|F(u, \partial u)\|_{\sigma-1,\theta} \leq A_\sigma(\|u, \partial u\|_{s-1,\theta}) \|u, \partial u\|_{\sigma-1,\theta} \quad \text{for all } \sigma \geq s,$$

and this implies that (4.3) holds. Similarly, since

$$\begin{aligned} &F(u, \partial u) - F(v, \partial v) \\ &= \int_0^1 \left\{ dF\big(v + \rho(u - v), \partial v + \rho(\partial u - \partial v)\big) - dF(0) \right\} \cdot (u - v, \partial u - \partial v)\, d\rho \\ &\quad + dF(0) \cdot (u - v, \partial u - \partial v), \end{aligned}$$

it follows from theorem 11 and the algebra property of $H^{s-1,\theta}$ that (4.4) also holds.

4.1.2 Sharp local existence

Here we reprove, using theorem 14, a sharp local well-posedness theorem of Ponce and Sideris for nonlinearities of the form

$$F(u, \partial u) = \Gamma(u)(\partial u)^\alpha, \tag{4.13}$$

where $\Gamma \in C^\infty$ and $\alpha = (\alpha_0, \ldots, \alpha_n)$ is a multi-index. We assume that $n = 3$ and $|\alpha| = \alpha_0 + \cdots + \alpha_n \geq 2$.

By the classical local existence theorem, (4.1) with F given by (4.13) is well posed for $s > 5/2$. We will show, using an asymmetric bilinear version of the L^4 Strichartz inequality in space dimension 3, that this can be improved to

$$s > \max\left\{2, \frac{5k - 7}{2k - 2}\right\},$$

where $k = |\alpha|$. This result was proved by Ponce and Sideris [21], and is sharp. Lindblad [20] proved that (4.1) is not well posed for $s = 2$ when $k = 2$, and the number $(5k - 7)/(2k - 2)$ is the critical exponent associated to (4.1) with $F = (\partial u)^\alpha$, so the problem is certainly not locally well posed for $s < (5k-7)/(2k-2)$.

In fact, if u is a solution of (4.1), then

$$u_\lambda(t, x) = \lambda^\beta u(\lambda t, \lambda x), \quad \text{where} \quad \beta = \frac{2 - k}{k - 1},$$

solves the same equation with data

$$u_\lambda\big|_{t=0} = \lambda^\beta f(\lambda\cdot), \quad \partial_t u_\lambda\big|_{t=0} = \lambda^{1+\beta} g(\lambda\cdot).$$

Since

$$\left\|\lambda^\beta f(\lambda\cdot)\right\|_{\dot{H}^s} = \lambda^{\beta+s-3/2} \|f\|_{\dot{H}^s}, \quad \left\|\lambda^{1+\beta} g(\lambda\cdot)\right\|_{\dot{H}^{s-1}} = \lambda^{\beta+s-3/2} \|g\|_{\dot{H}^{s-1}},$$

we see that the $\dot{H}^s \times \dot{H}^{s-1}$ norm of the data is invariant under this scaling iff $s = 3/2 - \beta = (5k-7)/(2k-2)$, and the $H^s \times H^{s-1}$ norm of the scaled data remains bounded as $\lambda \to \infty$ if $s \leq (5k-7)/(2k-2)$. On the other hand, if the lifespan of u is $T < \infty$, then u_λ has lifespan T/λ, which approaches 0 as $\lambda \to \infty$. If we take $\alpha = (k, 0, \ldots, 0)$, i.e., $F = (\partial_t u)^k$, then we can find smooth and compactly supported data such that (4.1a) blows up in finite time, by using the fact that the ODE $y' = y^k$ blows up at time $1/(k-1)$ for $k \geq 2$. Since the data are compactly supported, we can even produce a sequence (f_λ) of data supported in mutually disjoint balls, and add these up to produce data for which there is no local existence in a strip $[0, \varepsilon] \times \mathbb{R}^n$.

Assume $s > 2$, $1/2 < \theta < 1$, $0 < \varepsilon < 1 - \theta$ and $\theta + \varepsilon \leq 3/4$. We must prove (4.3) and (4.4). We first prove the special case $\Gamma = 1$, and then at the end e show how to reduce the general case to this.

We start by proving the result in the case $k = 2$. By the above remark, it suffices to prove

$$\|\Lambda_+ u \cdot \Lambda_+ v\|_{s-1,0} \lesssim |u|_{s,\theta} \, |v|_{s,\theta} \, .$$

By (3.24), this reduces to proving

$$\left\|\Lambda^{s-1}\Lambda_+ u \cdot \Lambda_+ v\right\|_{L^2} \lesssim |u|_{s,\theta} \, |v|_{s,\theta} \, ,$$

but this follows immediately from the estimate

$$\|uv\|_{L^2(\mathbb{R}^{1+3})} \lesssim \|u\|_{1+\gamma,\theta} \, \|v\|_{0,\theta} \, , \tag{4.14}$$

which holds for any $\gamma > 0$. This estimate is just a non-sharp, asymmetric bilinear version of the classical Strichartz estimate $\|u\|_{L^4(\mathbb{R}^{1+3})} \lesssim \|u\|_{1/2,\theta}$.

Now assume $k \geq 3$. We assume that

$$s > \frac{5k-7}{2k-2}, \quad 0 < 4\varepsilon < \min\left\{1, s - \frac{5k-7}{2k-2}\right\},$$

and we set $\theta = 1/2 + \varepsilon$. It suffices to prove

$$\left\|\Lambda^{s-1}\Lambda_+ u_1 \cdot \Lambda_+ u_2 \cdots \Lambda_+ u_k\right\|_{0,\theta+\varepsilon-1} \lesssim |u_1|_{s,\theta} \cdots |u_k|_{s,\theta} \, ,$$

which would follow from

$$\|u_1 u_2 \cdots u_k\|_{0,\theta+\varepsilon-1} \lesssim \|u_1\|_{0,\theta} \, \|u_2\|_{s-1,\theta} \cdots \|u_k\|_{s-1,\theta} \, . \tag{4.15}$$

Set $F_1(\tau, \xi) = w_-^\theta(\tau, \xi)\widehat{u_1}(\tau, \xi)$ and

$$F_j(\tau, \xi) = \langle\xi\rangle^{s-1} \, w_-^\theta(\tau, \xi)\widehat{u}_j(\tau, \xi) \quad \text{for} \quad 2 \geq j \leq k.$$

Then by the self-duality of L^2, (4.15) holds iff

$$\int \frac{G(\tau,\xi)F_1(\tau_1,\xi_1)F_2(\tau_2,\xi_2)\cdots F_k(\tau_k,\xi_k)}{w_-^{1-\theta-\varepsilon}(\tau,\xi)w_-^\theta(\tau_1,\xi_1)\,\langle\xi_2\rangle^{s-1}\,w_-^\theta(\tau_2,\xi_2)\cdots\langle\xi_k\rangle^{s-1}\,w_-^\theta(\tau_k,\xi_k)}d\mu$$
$$\leq C\,\|F_1\|_{L^2}\cdots\|F_k\|_{L^2} \quad \text{for all } G \geq 0 \text{ with } \|G\|_{L^2} \leq 1,$$

where $\tau_1 = \tau - \sum_2^k \tau_j$, $\xi_1 = \xi - \sum_2^k \xi_j$ and $d\mu = d\tau\, d\xi\, d\tau_2\, d\xi_2 \cdots d\tau_k\, d\xi_k$. Let us name this integral I.

By symmetry we may assume that I is restricted to the region

$$\langle \xi_2 \rangle \geq \langle \xi_3 \rangle \geq \cdots \geq \langle \xi_k \rangle . \tag{4.16}$$

We then write $I = I_1 + I_2$, where I_1 and I_2 are obtained by further restricting the domain of integration to the regions

$$w_-(\tau,\xi) \geq \langle \xi_2 \rangle \quad \text{and} \quad w_-(\tau,\xi) < \langle \xi_2 \rangle ,$$

respectively.

Since

$$s - 1 > 4\varepsilon + \frac{3k-5}{2k-2} = 1 + 4\varepsilon + \frac{k-3}{2k-2} \tag{4.17}$$

and $1 - \theta - \varepsilon = 1/2 - 2\varepsilon$, we have

$$\begin{aligned} s - 1 &+ \frac{1-\theta-\varepsilon}{k-2} + \frac{s-1-(1+\varepsilon)}{k-2} \\ &> 1 + 4\varepsilon + \frac{k-3}{2k-2} + \frac{1}{k-2}\left(\frac{1}{2} - 2\varepsilon\right) + \frac{1}{k-2}\left(\frac{k-3}{2k-2} + 2\varepsilon\right) \\ &= 1 + 4\varepsilon + \frac{k-3}{2k-2} + \frac{1}{k-1} = \frac{3}{2} + 4\varepsilon. \end{aligned}$$

Thus, on the domain of integration of I_1,

$$w_-^{1-\theta-\varepsilon}(\tau,\xi) \langle \xi_2 \rangle^{s-1} \cdots \langle \xi_k \rangle^{s-1} \geq \langle \xi_2 \rangle^{1+\varepsilon} \langle \xi_3 \rangle^{3/2+\varepsilon} \cdots \langle \xi_k \rangle^{3/2+\varepsilon} ,$$

so if we define $v_1, \ldots, v_k$ by

$$\widehat{v_1} = \frac{F_1}{w_-^\theta}, \quad \widehat{v_2} = \frac{F_2}{\langle\cdot\rangle^{1+\varepsilon} w_-^\theta} \quad \text{and} \quad \widehat{v_j} = \frac{F_j}{\langle\cdot\rangle^{3/2+\varepsilon} w_-^\theta} \quad \text{for} \quad 3 \leq j \leq k,$$

it follows from Hölder's inequality, the Strichartz inequality (4.14) and the embedding $H^{3/2+\varepsilon,\theta} \subseteq L^\infty(\mathbb{R}^{1+3})$ that

$$\begin{aligned} I_1 &\leq \|G\|_{L^2} \|v_1 v_2 \cdots v_k\|_{L^2} \\ &\leq \|v_1 v_2\|_{L^2} \|v_3\|_{L^\infty} \cdots \|v_k\|_{L^\infty} \\ &\leq C \|v_1\|_{0,\theta} \|v_2\|_{1+\varepsilon,\theta} \|v_3\|_{3/2+\varepsilon,\theta} \cdots \|v_k\|_{3/2+\varepsilon,\theta} \\ &= C \|F_1\|_{L^2} \cdots \|F_k\|_{L^2} . \end{aligned}$$

By (4.17), $s - 1 - 3\varepsilon > 1 + \varepsilon + (k-3)/(2k-2)$, so it follows from (4.16) that

$$\langle \xi_2 \rangle^{s-1-3\varepsilon} \langle \xi_3 \rangle^{s-1} \geq \langle \xi_2 \rangle^{1+\varepsilon} \langle \xi_3 \rangle^{1+\varepsilon} \left(\langle \xi_4 \rangle \cdots \langle \xi_k \rangle \right)^{1/(k-1)}.$$

Since $s-1+1/(k-1) > 3/2+\varepsilon$, it follows that on the domain of integration of I_2,

$$w_-^{1/2-2\varepsilon}(\tau,\xi)\,\langle\xi_2\rangle^{s-1}\cdots\langle\xi_k\rangle^{s-1} \geq w_-^{1/2+\varepsilon}\,\langle\xi_2\rangle^{1+\varepsilon}\,\langle\xi_3\rangle^{1+\varepsilon}\,\langle\xi_4\rangle^{3/2+\varepsilon}\cdots\langle\xi_k\rangle^{3/2+\varepsilon}.$$

Therefore, defining $v_0,\dots,v_k$ by

$$\mathcal{F}^{-1}v_0 = \frac{G}{w_-^\theta},\quad \widehat{v_1} = \frac{F_1}{w_-^\theta},\quad \widehat{v_j} = \frac{F_j}{\langle\cdot\rangle^{1+\varepsilon}\,w_-^\theta}\quad \text{for}\quad j=2,3$$

and

$$\widehat{v_j} = \frac{F_j}{\langle\cdot\rangle^{3/2+\varepsilon}\,w_-^\theta}\quad \text{for}\quad 4\leq j\leq k,$$

we get

$$\begin{aligned}
I_2 &\leq \int \mathcal{F}^{-1}v_0\cdot\mathcal{F}(v_1\cdots v_k)\,d\tau\,d\xi \\
&= \int v_0v_1\cdots v_k\,dt\,dx \\
&\leq \|v_0v_2\|_{L^2}\,\|v_1v_3\|_{L^2}\,\|v_4\|_{L^\infty}\cdots\|v_k\|_{L^\infty} \\
&\leq C\,\|v_0\|_{0,\theta}\,\|v_2\|_{1+\varepsilon,\theta}\,\|v_1\|_{0,\theta}\,\|v_3\|_{1+\varepsilon,\theta}\,\|v_4\|_{3/2+\varepsilon,\theta}\cdots\|v_k\|_{3/2+\varepsilon,\theta} \\
&= C\,\|G\|_{L^2}\,\|F_1\|_{L^2}\cdots\|F_k\|_{L^2}\,.
\end{aligned}$$

This concludes the proof in the special case $\Gamma=1$. We now show how to reduce the general case to this.

First note that

$$\|\Gamma(u)(\partial u)^\alpha\|_{\sigma-1,\theta+\varepsilon-1} \lesssim \|\Gamma_0(u)(\partial u)^\alpha\|_{\sigma-1,\theta+\varepsilon-1} + |\Gamma(0)|\,\|(\partial u)^\alpha\|_{\sigma-1,\theta+\varepsilon-1}\,,$$

where $\Gamma_0(u)=\Gamma(u)-\Gamma(0)$. By (3.24),

$$\begin{aligned}
\|\Gamma_0(u)(\partial u)^\alpha\|_{\sigma-1,\theta+\varepsilon-1}& \\
\lesssim \left\|\Lambda^{\sigma-1}\Gamma_0(u)\cdot(\partial u)^\alpha\right\|_{0,\theta+\varepsilon-1} &+ \left\|\Gamma_0(u)\cdot\Lambda^{\sigma-1}(\partial u)^\alpha\right\|_{0,\theta+\varepsilon-1}\,.
\end{aligned}\tag{4.18}$$

By part (a) of proposition 12,

$$H^{s-\varepsilon,\theta}\times H^{0,1-\theta-\varepsilon}\longrightarrow H^{0,1-\theta-\varepsilon},$$

and by duality this implies

$$H^{s-\varepsilon,\theta}\times H^{0,\theta+\varepsilon-1}\longrightarrow H^{0,\theta+\varepsilon-1}.\tag{4.19}$$

Applying this estimate to the second term on the right side of (4.18) and estimating the first term via proposition 11, we get

$$\begin{aligned}
&\|\Gamma_0(u)(\partial u)^\alpha\|_{\sigma-1,\theta+\varepsilon-1} \\
&\quad\leq \|\Gamma_0(u)\|_{\sigma,\theta}\,\|(\partial u)^\alpha\|_{s-1,\theta+\varepsilon-1} + \|\Gamma_0(u)\|_{s,\theta}\,\|(\partial u)^\alpha\|_{\sigma-1,\theta+\varepsilon-1} \\
&\quad\lesssim g(\|u\|_{s,\theta})\,\|u\|_{\sigma,\theta}\,\|(\partial u)^\alpha\|_{s-1,\theta+\varepsilon-1} + g(\|u\|_{s,\theta})\,\|(\partial u)^\alpha\|_{\sigma-1,\theta+\varepsilon-1}\,,
\end{aligned}$$

where we used theorem 11 to obtain the last inequality.

Since

$$\Gamma(u) - \Gamma(v) = \int_0^1 \left\{ d\Gamma\big(v + \rho(u-v)\big) - d\Gamma(0) \right\} \cdot (u-v)\, d\rho \\ + d\Gamma(0) \cdot (u-v),$$

we have

$$\|\Gamma(u) - \Gamma(v)\|_{s,\theta} \lesssim \left\{ h(\|u\|_{s,\theta} + \|v\|_{s,\theta}) + |d\Gamma(0)| \right\} \|u - v\|_{s,\theta},$$

and it follows that

$$\begin{aligned} &\|\Gamma(u)(\partial u)^\alpha - \Gamma(v)(\partial v)^\alpha\|_{s-1,\theta+\varepsilon-1} \\ &\quad\lesssim g(\|v\|_{s,\theta}) \|(\partial u)^\alpha - (\partial v)^\alpha\|_{s-1,\theta+\varepsilon-1} \\ &\qquad + \left\{ h(\|u\|_{s,\theta} + \|v\|_{s,\theta}) + |d\Gamma(0)| \right\} \|u-v\|_{s,\theta} \|(\partial u)^\alpha\|_{s-1,\theta+\varepsilon-1}, \end{aligned}$$

where h is continuous.

4.1.3 The wave map equation

In local coordinates on the target manifold N, the equation for a wave map u from Minkowski space $\mathbb{R}^{1+n}$ to N reads

$$\square u^I + \Gamma^I_{JK}(u) Q_0(u^J, u^K) = 0, \tag{4.20a}$$

where the Γ^I_{JK}'s are the Christoffel symbols on N, and Q_0 is the null form $Q_0(u,v) = \partial_\mu u \cdot \partial^\mu v = -\partial_t u \partial_t v + \sum_{i=1}^n \partial_i u \partial_i v$. We impose initial conditions

$$u\big|_{t=0} = f \in H^s, \quad \partial_t u\big|_{t=0} = g \in H^{s-1}. \tag{4.20b}$$

Note that u is now a vector in $\mathbb{R}^d$, where d is the dimension of the target manifold N. To simplify the notation, we drop the indices and treat the equation as a scalar equation.

By the result of Ponce and Sideris, (4.20) is well posed for $s > 2$ in dimension $n = 3$. Using the cancellation properties of the null form Q_0, Klainerman and Machedon [10] improved this to $s = 2$, and then in [13] they proved local existence for data with small $H^s \times H^{s-1}$-norm for $s > 3/2$. Klainerman and Selberg [18] extended this result to all dimensions $n \geq 2$, proving local existence for small data when $s > n/2$. Here we will prove that (4.3) and (4.4) are satisfied, thereby improving the results in [13], [18] to strong local well-posedness. Moreover, by applying theorem 11, we dispose of the assumption of analyticity of the Christoffel symbols which was made in [13] and [18].

Note that $n/2$ is the scaling limit for this equation. What happens in the exact limiting case $s = n/2$ is still open. One expects that data $(f,g) \in \dot{H}^{n/2} \times \dot{H}^{n/2-1}$ with small norm should give global existence, at least in space dimension two. A recent result in this direction can be found in Tataru [23].

We assume $n \geq 2$, $s > n/2$ and pick θ and ε satisfying

$$1/2 < \theta < 1, \quad 0 < \varepsilon < 1-\theta \quad \text{and} \quad \frac{n-1}{2} + \theta + \varepsilon < s.$$

By the analysis in the previous section, we may assume $\Gamma = 1$. Note that

$$\widehat{Q_0(u,v)}(\tau,\xi) \simeq \int q_0(\tau-\lambda, \xi-\eta, \lambda, \eta)\widehat{u}(\tau-\lambda,\xi-\eta)\widehat{v}(\lambda,\eta)\, d\lambda d\eta,$$

where $q_0(\tau,\xi,\lambda,\eta) = \tau\lambda - \xi\cdot\eta$. Since $\Gamma = 1$, we may assume $\widehat{u}$, $\widehat{v} \geq 0$. From the identity

$$\tau\lambda - \xi\cdot\eta = \frac{1}{2}\left\{(\tau+\lambda)^2 - |\xi+\eta|^2 - \tau^2 + |\xi|^2 - \lambda^2 + |\eta|^2\right\}$$

and the trivial estimate $|q_0(\tau,\xi,\lambda,\eta)| \leq 2(|\tau|+|\xi|)(|\lambda|+|\eta|)$, we conclude that

$$\begin{aligned} Q_0(u,v) &\precsim \Lambda_+^\gamma \Lambda_-^\gamma (\Lambda_+^{1-\gamma} u \cdot \Lambda_+^{1-\gamma} v) + \Lambda_+\Lambda_-^\gamma u \cdot \Lambda_+^{1-\gamma} v + \Lambda_+^{1-\gamma} u \cdot \Lambda_+ \Lambda_-^\gamma v \\ &= A_\gamma + B_\gamma + C_\gamma \end{aligned}$$

for all $0 \leq \gamma \leq 1$. The factors Λ_-^γ give cancellations on the null cone in frequency space, and this is why we can obtain more favorable estimates for $Q_0(u,v)$ than for a generic product $\partial_\mu u \partial_\nu v$, for which there is no such cancellation.

By symmetry, it suffices to estimate the terms $A_{1-\varepsilon}$ and $B_{1-\varepsilon}$. Using (3.24), (3.25) and (3.26), we get

$$\begin{aligned} \Lambda^{s-1}\Lambda_-^{\theta+\varepsilon-1} A_{1-\varepsilon} \precsim\ & \Lambda_-^\theta \Lambda^{s-1}\Lambda_+ u \cdot \Lambda_+^\varepsilon v + \Lambda_-^\theta \Lambda^{s-1}\Lambda_+^\varepsilon u \cdot \Lambda_+ v \\ &+ \Lambda^{s-1}\Lambda_+ u \cdot \Lambda_-^\theta \Lambda_+^\varepsilon v + \Lambda^{s-1}\Lambda_+^\varepsilon u \cdot \Lambda_-^\theta \Lambda_+ v \\ &+ R^\theta(\Lambda^{s-1}\Lambda_+ u, \Lambda_+^\varepsilon v) + R^\theta(\Lambda^{s-1}\Lambda_+^\varepsilon u, \Lambda_+ v) \\ &+ \textit{symmetric terms}, \end{aligned}$$

and we can apply propositions 10 and 9. By (3.24), (4.19) and (3.34),

$$\begin{aligned} \|B_{1-\varepsilon}\|_{s-1,\theta+\varepsilon-1} &\lesssim \left\|\Lambda^{s-1}\Lambda_+\Lambda_-^{1-\varepsilon} u \cdot \Lambda_+^\varepsilon v\right\|_{0,\theta+\varepsilon-1} \\ &\quad + \left\|\Lambda_+\Lambda_-^{1-\varepsilon} u \cdot \Lambda^{s-1}\Lambda_+^\varepsilon v\right\|_{0,\theta+\varepsilon-1} \\ &\lesssim \left\|\Lambda^{s-1}\Lambda_+\Lambda_-^{1-\varepsilon} u\right\|_{0,\theta+\varepsilon-1} \left\|\Lambda_+^\varepsilon v\right\|_{s-\varepsilon,\theta} \\ &\quad + \left\|\Lambda_+\Lambda_-^{1-\varepsilon} u\right\|_{s-1,\theta+\varepsilon-1} \left\|\Lambda^{s-1}\Lambda_+^\varepsilon v\right\|_{1-\varepsilon,\theta}. \end{aligned}$$

We conclude that $\|Q_0(u,v)\|_{s-1,\theta+\varepsilon-1} \lesssim |u|_{s,\theta}\, |v|_{s,\theta}$.

4.2 Second well-posedness theorem

Consider (4.1) as a system with nonlinear terms

$$F^I = Q^I_{JK}(u^J, u^K), \quad 1 \leq I \leq N,$$

where each Q^I_{JK} is a linear combination of the null forms

$$Q_{ij}(u,v) = \partial_i u \partial_j v - \partial_j u \partial_i v.$$

The scaling exponent for this problem is $n/2$, and Klainerman and Machedon [16] have proved local existence for $s > n/2$ in dimensions $n \geq 3$, under the assumption that the $H^s \times H^{s-1}$-norm of the data is small. However, the estimates (4.3) and (4.4) fail to hold for s close to $n/2$; see [13]. The same problem arises when we try to apply theorem 14 to hyperbolic model problems derived from the Maxwell-Klein-Gordon equations, the Yang-Mills equations and a certain coordinate-free formulation of the wave maps equation. Thus, among all the nonlinear field equations that interest us, it is only the wave maps equation in its local formulation which is amenable to analysis by the methods of the previous section.

The failure of (4.3) and (4.4) means that the solution operator Φ, defined in (4.5), is not a contraction map of $\mathcal{X}^{s,\theta}$. Note that proving existence by the contraction mapping principle amounts to showing that the sequence of Picard iterates (u_j) is Cauchy. The iterates are given by

$$u_{-1} = 0, \quad u_j = \Phi u_{j-1} \quad \text{for} \quad j \geq 0.$$

The first step is to show that the sequence is bounded, and to do this we have to be able to control the norm of u_j in terms of the norms of the previous iterates:

$$|u_j|_{s,\theta} \leq G\big(|u_{j-1}|_{s,\theta}, \dots, |u_0|_{s,\theta}\big).$$

If (4.3) holds, this means that we can control the norm of u_{j+1} in terms of just the norm of u_j. It should not be surprising if this is the exception rather than the rule. It is convenient to introduce the following terminology. We will say that the iteration argument for the Cauchy problem (4.1) can be *closed in k steps* if the $\mathcal{X}^{s,\theta}$-norm of the j-th iterate can be controlled in terms of the norms of the preceding k iterates, where k is independent of j. Of course, k will in general depend on the size of s, and it may be infinite. The smallest such k we will refer to as the *iteration depth*.

Even if the iteration argument cannot be closed in one step, it is often possible to cast the iteration in the form of a contraction argument. This requires that one can find a suitable subspace of $\mathcal{X}^{s,\theta}$ in which Φ is a contraction. For the above system, which we will refer to as the Q_{ij}-system, Klainerman and Machedon [16] managed, by a rather ingenious construction, to do just this. For this problem the iteration depth becomes unbounded as s approaches the scaling exponent $n/2$, but Klainerman and Machedon were able to construct a single subspace which works for the entire range $s > n/2$.

We now state the second well-posedness theorem, which is sufficiently general to handle the Q_{ij}-problem. In the next chapter, this theorem will be applied to a model problem for wave maps.

Given s, θ and ε satisfying (4.2), assume that $\|\cdot\|$ is a semi-norm on some subspace of $\mathcal{S}'$ containing $\mathcal{S}$, and define

$$X = \big\{u : \|u\|_X < \infty\big\}, \quad Y = \big\{F : \|F\|_Y < \infty\big\}, \tag{4.21}$$

where

$$\|u\|_X = |u|_{s,\theta} + \|u\|\,, \quad \|F\|_Y = \|F\|_{s-1,\theta+\varepsilon-1} + \left\|\Lambda^{-1}\Lambda_-^{-1+\varepsilon}F\right\|. \tag{4.22}$$

We assume that $(X, \|\cdot\|_X)$ is a complete space.

Assume that (cf. theorem 16 below)

$$\|u\| \le C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}} + T^{\varepsilon/2}\,\|F\|_Y\big) \tag{4.23}$$

for all $F \in Y$ and $0 < T < 1$, with u as in theorem 13. Combined with the estimate in theorem 13, (4.23) gives

$$\|u\|_X \le C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}} + T^{\varepsilon/2}\,\|F\|_Y\big).$$

Assume, moreover, that

$$\left\|\Lambda^{\sigma-s}F(u,\partial u)\right\|_Y \le A_\sigma(\|u\|_X)\left\|\Lambda^{\sigma-s}u\right\|_X \quad \text{for all } \sigma \ge s \tag{4.24}$$

and

$$\|F(u,\partial u) - F(v,\partial v)\|_Y \le A_s(\|u\|_X + \|v\|_X)\,\|u-v\|_X \tag{4.25}$$

for all $u, v \in X$, where $A_\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and locally Lipschitz for every $\sigma \ge s$.

Theorem 15. *If* (4.2),(4.23), (4.24) *and* (4.25) *hold, there exists* $u \in X$ *which solves* (4.1) *on* $[0,T] \times \mathbb{R}^n$, *where* $T = T(\|f\|_{H^s} + \|g\|_{H^{s-1}}) > 0$ *depends continuously on* $\|f\|_{H^s} + \|g\|_{H^{s-1}}$.

The solution is unique in the class X, *in the sense that if* u, $v \in X$ *are solutions of* (4.1) *on* $[0,T] \times \mathbb{R}^n$ *for some* $T > 0$, *then*

$$u(t) = v(t) \quad \textit{for} \quad t \in [0,T]. \tag{4.26}$$

Moreover, the solution map

$$(f,g) \longmapsto u, \quad H^s \times H^{s-1} \longrightarrow X$$

is locally Lipschitz, and if the data have the additional regularity

$$f \in H^\sigma, \quad g \in H^{\sigma-1}, \quad \textit{where} \quad \sigma > s,$$

then

$$u \in C\big([0,T], H^\sigma\big) \cap C^1\big([0,T], H^{\sigma-1}\big) \tag{4.27}$$

for any $T > 0$ *such that* u *solves* (4.1) *on* $[0,T] \times \mathbb{R}^n$. *In particular, if* $f, g \in \mathcal{S}$, *then* u *is* C^∞ *on* $[0,T] \times \mathbb{R}^n$.

The proof is a straightforward modification of the proof of theorem 14, and is therefore omitted.

The next result gives a sufficient condition for (4.23) to hold, and will prove quite useful later on.

Theorem 16. *Assume that the semi-norm $\|\cdot\|$ has the properties:*

(a) $u \preceq v$ *implies* $\|u\| \leq \|v\|$;

(b) There exists $1 < \gamma \leq \theta/2 + \varepsilon/2 + 5/4$ *such that for any* $v \in \mathcal{X}^{s,\theta}$ *satisfying* $u \preceq v$,

$$\|u\| \lesssim \left\| \langle \xi \rangle^s \, w_-^\gamma(\tau,\xi) \widehat{v}(\tau,\xi) \right\|_{L^2_\xi(L^\infty_\tau)}. \tag{4.28}$$

Then (4.23) *holds.*

The proof, which can be found in section 4.2.1, requires the following lemma. We define $\mathcal{D}^\gamma : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ to be the operator with Fourier symbol $\big(1 + |\tau|^2\big)^{\gamma/2}$.

Lemma 8. *If* $\chi \in C_c^\infty(\mathbb{R})$ *and* $(f,g) \in H^s \times H^{s-1}$, *then*

$$\left\| \chi(t) e^{\pm itD} f \right\| \leq \left\| \widehat{\mathcal{D}^\gamma \chi} \right\|_{L^\infty} \|f\|_{H^s}\,, \tag{4.29}$$

$$\|\chi(t)\cos(tD) \cdot f\| \leq \left\| \widehat{\mathcal{D}^\gamma \chi} \right\|_{L^\infty} \|f\|_{H^s} \tag{4.30}$$

and

$$\left\| \chi(t) D^{-1} \sin(tD) \cdot g \right\| \lesssim \left(\left\| \widehat{\mathcal{D}^\gamma \chi} \right\|_{L^\infty} + \left\| \widehat{\mathcal{D}^\gamma (t\chi)} \right\|_{L^\infty} \right) \|g\|_{H^{s-1}}\,. \tag{4.31}$$

Moreover, if $|\rho| \leq 1$ *and* $\operatorname{supp} \widehat{g} \subseteq \{\xi : |\xi| \leq c\}$, *then*

$$\left\| \chi(t) e^{i\rho t D} g \right\| \lesssim c \left(c^\gamma \|\widehat{\chi}\|_{L^\infty} + \left\| \widehat{\mathcal{D}^\gamma \chi} \right\|_{L^\infty} \right) \|g\|_{H^{s-1}} \tag{4.32}$$

Proof. Since the Fourier transform of $\chi(t)e^{\pm itD} f$ is $\widehat{\chi}(\tau \mp |\xi|)\widehat{f}(\xi)$, it follows from (4.28) that

$$\left\| \chi(t) e^{\pm itD} f \right\| \lesssim \left\| \widehat{\mathcal{D}^\gamma \chi}(\tau \mp |\xi|) \langle \xi \rangle^s \widehat{f}(\xi) \right\|_{L^2_\xi(L^\infty_\tau)} = \left\| \widehat{\mathcal{D}^\gamma \chi} \right\|_{L^\infty} \|f\|_{H^s}\,.$$

This proves (4.29), which in turn implies (4.30).

The proof of (4.32) is similar. We simply note that the Fourier transform of $\chi(t)e^{i\rho tD} g$ equals $\widehat{\chi}(\tau - \rho\,|\xi|)\widehat{g}(\xi)$, and that

$$\big| |\tau| - |\xi| \big| \leq \big| \tau - \rho\,|\xi| \big| + (1 - |\rho|)\,|\xi| \leq \big| \tau - \rho\,|\xi| \big| + c$$

for $\xi \in \operatorname{supp} \widehat{g}$ and $|\rho| \leq 1$.

To prove (4.31), we split $g = g_1 + g_2$, where $\widehat{g_1}$ is supported in the region $|\xi| < 1$ and $\widehat{g_2}$ is supported in $|\xi| \geq 1$. Since

$$D^{-1} \sin(tD) = t \int_0^1 e^{it(2\rho - 1)D} \, d\rho,$$

we have

$$\chi(t)D^{-1}\sin(tD)\cdot g_1 = \int_0^1 t\chi(t)e^{it(2\rho-1)D}g_1\,d\rho.$$

By (4.32),

$$\left\| t\chi(t)e^{it(2\rho-1)D}g_1 \right\| \lesssim \left\| \widehat{\mathcal{D}^\gamma(t\chi)} \right\|_{L^\infty} \|g_1\|_{H^{s-1}} \quad \text{for} \quad 0 \le \rho \le 1,$$

and it follows that

$$\left\| \chi(t)D^{-1}\sin(tD)\cdot g_1 \right\| \le \left\| \widehat{\mathcal{D}^\gamma(t\chi)} \right\|_{L^\infty} \|g_1\|_{H^{s-1}}.$$

This proves (4.31) with g replaced by its low frequency part g_1.

Since $\left\| D^{-1}g_2 \right\|_{H^s} \le 2\|g\|_{H^{s-1}}$, the estimate (4.31) with g replaced by g_2 follows immediately from (4.29). □

4.2.1 Proof of theorem 16

We must prove that

$$\|\chi(t)u_0 + \chi(t/T)u_1 + u_2\| \le C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}} + T^{\varepsilon/2}\|F\|_Y\big) \tag{4.33}$$

for all $F \in Y$ and $0 < T < 1$, where

$$\begin{aligned} u_0 &= \cos(tD)\cdot f + D^{-1}\sin(tD)\cdot g, \\ u_1 &= \int_0^t D^{-1}\sin\big((t-t')D\big)\cdot F_1(t')\,dt', \\ u_2 &= \Box^{-1}F_2 \end{aligned}$$

and

$$F = F_1 + F_2 = \phi(T^{1/2}\Lambda_-)F + \big(1-\phi(T^{1/2}\Lambda_-)\big)F.$$

Lemma 8 yields

$$\|\chi(t)u_0\| \lesssim \left(\left\| \widehat{\mathcal{D}^\gamma\chi} \right\|_{L^\infty} + \left\| \widehat{\mathcal{D}^\gamma(t\chi)} \right\|_{L^\infty} \right)\big(\|f\|_{H^s} + \|g\|_{H^{s-1}}\big). \tag{4.34}$$

By (3.78), $u_2 \precsim T^{\varepsilon/2}\Lambda_+^{-1}\Lambda_-^{-1+\varepsilon}F$, whence

$$\|u_2\| \le T^{\varepsilon/2}\left\| \Lambda_+^{-1}\Lambda_-^{-1+\varepsilon}F \right\|. \tag{4.35}$$

It remains to estimate $\|\chi(t/T)u_1\|$. By proposition 16,

$$\begin{aligned} u_1(t) &= \sum_{j=1}^\infty \frac{t^{j+1}}{j!}\int_0^1 e^{it(2\rho-1)D}g_j(\rho)\,d\rho \\ &\quad + \sum_{j=1}^\infty \frac{t^j}{j!}\big(e^{itD}f_j^+ + e^{-itD}f_j^-\big) + R_+(t) + R_-(t) \\ &= \Sigma_1 + \Sigma_2 + R_+(t) + R_-(t), \end{aligned}$$

where

$$\|g_j(\rho)\|_{H^{s-1}}, \left\|f_j^{\pm}\right\|_{H^s} \lesssim T^{\theta/2-j/2-1/4} \|F\|_{s-1,\theta-1} \tag{4.36}$$

and

$$\operatorname{supp} \widehat{g_j(\rho)} \subseteq \left\{\xi : |\xi| \lesssim T^{-1/2}\right\}. \tag{4.37}$$

Using lemma 8, (4.36), (4.37) and the fact that

$$\left\|\mathcal{F}\mathcal{D}^{\gamma}\left\{t^j\chi(t/T)\right\}\right\|_{L^\infty} \leq T^{1-\gamma+j}\left\|\widehat{\mathcal{D}^{\gamma}(t^j\chi)}\right\|_{L^\infty} \tag{4.38}$$

for $j \geq 0$ and $0 < T < 1$, we get

$$\|\chi(t/T)\,(\Sigma_1+\Sigma_2)\| \leq CT^{\theta/2+5/4-\gamma} \|F\|_{s-1,\theta-1}, \tag{4.39}$$

where

$$C \lesssim \sum_{j=1}^{\infty} \frac{\left\|\widehat{\mathcal{D}^{\gamma}(t^j\chi)}\right\|_{L^\infty}}{j!}.$$

To get the required decay, we must therefore have $2\gamma \leq \theta+\varepsilon+5/2$. In particular, $\gamma \leq 2$, and since χ is supported in $(-2,2)$, we have

$$\left\|\widehat{\mathcal{D}^{\gamma}(t^j\chi)}\right\|_{L^\infty} \leq \left\|\mathcal{D}^{\gamma}(t^j\chi)\right\|_{L^1} \lesssim j^2 2^j\left\{\|\chi\|_{L^\infty}+\|\chi'\|_{L^\infty}+\|\chi''\|_{L^\infty}\right\},$$

whence $C \lesssim \|\chi\|_{L^\infty}+\|\chi'\|_{L^\infty}+\|\chi''\|_{L^\infty}$.

Finally, notice that (3.75) implies

$$w_-^{\gamma}(\tau,\xi)\frac{\left|\widehat{\chi}(\tau-\lambda)-\widehat{\chi}(\tau-|\xi|)\right|}{|\lambda|+|\xi|} \lesssim \left\|\widehat{\mathcal{D}^{\gamma-1}\chi}\right\|_{L^\infty}+\left\|\widehat{\mathcal{D}^{\gamma}(t\chi)}\right\|_{L^\infty}, \tag{4.40}$$

so by (3.74), (3.77) and the Cauchy-Schwarz inequality,

$$\begin{aligned}&\langle\xi\rangle^s w_-^{\gamma}(\tau,\xi)\left|\widehat{\chi(t)R_+}(\tau,\xi)\right| \\ &\lesssim T^{\frac{\theta}{2}-\frac{3}{4}}\left(\left\|\widehat{\mathcal{D}^{\gamma-1}\chi}\right\|_{L^\infty}+\left\|\widehat{\mathcal{D}^{\gamma}(t\chi)}\right\|_{L^\infty}\right)\left(\int\left|\langle\xi\rangle^{s-1}\left(w_-^{\theta-1}\widehat{F}\right)(\lambda,\xi)\right|^2 d\lambda\right)^{\frac{1}{2}}.\end{aligned}$$

The same estimate holds for R_-. Hence, (4.28) yields

$$\|\chi(t)R_{\pm}\| \lesssim \left(\left\|\widehat{\mathcal{D}^{\gamma-1}\chi}\right\|_{L^\infty}+\left\|\widehat{\mathcal{D}^{\gamma}(t\chi)}\right\|_{L^\infty}\right)T^{\theta/2-3/4}\|F\|_{s-1,\theta-1}.$$

Using (4.38), we conclude that

$$\|\chi(t/T)R_{\pm}\| \lesssim \left(\left\|\widehat{\mathcal{D}^{\gamma-1}\chi}\right\|_{L^\infty}+\left\|\widehat{\mathcal{D}^{\gamma}(t\chi)}\right\|_{L^\infty}\right)T^{\theta/2+5/4-\gamma}\|F\|_{s-1,\theta-1}. \tag{4.41}$$

The estimates (4.34), (4.35), (4.39) and (4.41) collectively prove (4.33).

Chapter 5

A Coordinate-Free Formulation of Wave Maps

In this chapter we study the local existence properties of the system

$$\partial^\mu A_\mu = 0 \tag{5.1a}$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = [A_\nu, A_\mu], \tag{5.1b}$$

where A_μ is a Lie algebra-valued 1-form on the Minkowski space $(\mathbb{R}^{1+n}, g)$ with metric $g_{\mu\nu}$ equal to the diagonal matrix with entries $-1, 1, \ldots, 1$. We use standard coordinates $x^0, \ldots, x^n$ and set $t = x^0$. The summation convention is in effect, and Roman indices run from 1 to n, Greek indices from 0 to n.

We assume that A_μ is matrix-valued, and $[\cdot,\cdot]$ the matrix commutator. The initial condition is

$$A_\mu\big|_{t=0} = a_\mu \in H^s(\mathbb{R}^n), \tag{5.2}$$

where we must require that the compatibility condition

$$\partial_i a_j - \partial_j a_i = [a_j, a_i] \tag{5.3}$$

is satisfied.

Following Klainerman and Machedon [14], we define

$$\bar{A}_i = A_i + R_0 R_i A_0, \tag{5.4}$$

where $R_\mu = D^{-1}\partial_\mu$. Then it follows from (5.1),(5.2) that the 1-form $A_0 dx^0 + \bar{A}_i dx^i$ satisfies the system

$$\Box A_0 = \partial^i [A_0, \bar{A}_i - R_0 R_i A_0] \tag{5.5a}$$

$$\partial^i \bar{A}_i = 0 \tag{5.5b}$$

$$\partial_i \bar{A}_j - \partial_j \bar{A}_i = [\bar{A}_j - R_0 R_j A_0, \bar{A}_i - R_0 R_i A_0] \tag{5.5c}$$

with Cauchy data

$$A_0\big|_{t=0} = a_0 \in H^s(\mathbb{R}^n), \quad \partial_t A_0\big|_{t=0} = \partial^i a_i \in H^{s-1}(\mathbb{R}^n). \tag{5.6}$$

Note that $\bar{A}_i$ satisfies an elliptic Hodge system in the space variables, and hence no initial values are specified for $\bar{A}_i$. For this system the critical Sobolev exponent for the data is $s_c = (n-2)/2$.

Our main interest is the hyperbolic model problem obtained by setting $\bar{A}_i$ identically zero in the above system. In dimensions $n \geq 3$, this model problem was studied by Klainerman and Machedon [14]. Here we analyze the two-dimensional case, which has not been tackled before. We prove that the model problem is well posed for $s > 1/4$ in the two-dimensional case, and we write down some conjectures which would give well-posedness for $s > 0$.

We then extend the 3D result of Klainerman and Machedon to the full system (5.5). As one would expect, the estimates for the "elliptic" variable $\bar{A}_i$ are less delicate than those for A_0.

5.1 The connection with wave maps

Let G be a Lie group with a bi-invariant metric h, and let u be a wave map from $(\mathbb{R}^{1+n}, g)$ into (G, h), i.e., u is a critical point of the Lagrangian $\mathcal{L}[u] = \int_{\mathbb{R}^{1+n}} \langle du, du \rangle$.

Following the notation in Christodoulou and Tahvildar-Zadeh [2], we let $\{\Omega_I\}$ be an orthonormal basis of the Lie algebra of G, and $\{\omega^I\}$ the dual basis of left invariant 1-forms on G. Now define 1-forms ψ^I on $\mathbb{R}^{1+n}$ by

$$\psi^I_\mu = \omega^I_a(u)\partial_\mu u^a, \tag{5.7}$$

and set

$$A_\mu = \psi^I_\mu \Omega_I. \tag{5.8}$$

A computation (see [2, Section 3.1]) shows that the forms ψ^I satisfy a Hodge system, and when we express this system in terms of the Lie algebra-valued form A, we get

$$\partial^\mu A_\mu = [A^\mu, A_\mu], \quad \partial_\mu A_\nu - \partial_\nu A_\mu = [A_\nu, A_\mu].$$

Since $[A^\mu, A_\mu] = 0$, we obtain the system (5.1). This system is equivalent to the Euler-Lagrange equation for wave maps, which in local coordinates on G is the equation (4.20), in the sense that a given map $u : (\mathbb{R}^{1+n}, g) \to (G, h)$ is a wave map iff (5.1) holds.

The formulation (5.1) has the inherent advantage over (4.20) that it is global as opposed to local, and this fact was used by Christodoulou and Tahvildar-Zadeh in [2], where they establish the global regularity of spherically symmetric wave maps for smooth data of any size. A related system was also used by Freire, Müller and Struwe [3] to prove weak convergence of wave maps for $n = 2$, and by Helein [6] to prove regularity of weakly harmonic maps.

5.2 Hyperbolic model problem

Consider the Cauchy problem for the system

(5.9a) $$\Box u^I = a^I_{JK} Q(u^J, u^K) \qquad (t,x) \in \mathbb{R}^{1+n}$$

(5.9b) $$u\big|_{t=0} = f \in H^s, \quad \partial_t u\big|_{t=0} = g \in H^{s-1},$$

where the a^I_{JK}'s are constants, Q is the bilinear operator given by

(5.10) $$Q(u,v) = \sum_{i=1}^n \partial_i (R_0 R_i u \cdot v - u \cdot R_0 R_i v)$$

and $R_0 = D^{-1}\partial_t$, $R_i = D^{-1}\partial_i$. Notice that the Fourier symbol of Q is

$$q(\tau, \xi, \lambda, \eta) = (\xi + \eta) \cdot \left(\frac{\tau \xi}{|\xi|^2} - \frac{\lambda \eta}{|\eta|^2} \right).$$

The new result proved in this section is the following.

Theorem 17. *In dimension $n = 2$, the system* (5.9) *is locally well-posed for $s > 1/4$.*

Klainerman and Machedon [14] proved local existence for (5.9) when $s > s_c = (n-2)/2$ in dimensions $n \geq 3$. The new idea introduced in their proof is to use information from two previous Picard iterates to estimate the subsequent iterate. Applying this idea to the 2D problem we obtain the $s > 1/4$ result. In contrast with the higher dimensional case, however, we can show that it is not possible to go all the way to the scaling limit s_c using two iterates. In fact, we prove that if $s < 1/8$ one must use information from at least three previous iterates. This gives some indication of the difficulty of the problem in 2D. We then dicuss a strategy for proving well-posedness in 2D for all $s > 0$, subject to some conjectures. This is where the quadrilinear estimate proved in chapter two comes into the picture.

As a by-product of our approach to the two-dimensional problem, we also obtain a considerably simplified proof of the result of Klainerman and Machedon [14]. In dimensions $n \geq 3$ we have the $L^1_t(L^\infty_x)$ product estimate proved in [19], and this fact makes life much simpler than in the two-dimensional case, where no such estimate holds.

5.2.1 Outline of proof

The plan is to prove well-posedness for $s > 1/4$ in 2D and for $s > s_c$ in higher dimensions by applying theorem 15 with a suitably defined seminorm $\|\cdot\|$ which satisfies the properties:

(I) $\|Q(u,u)\|_{s-1,\theta+\varepsilon-1} \lesssim (|u|_{s,\theta} + \|u\|)^2$ for all $s > s_c$ and $n \geq 2$;

(II) $\left\|\Lambda^{-1}\Lambda_-^{-1+\varepsilon} Q(u,u)\right\| \lesssim |u|^2_{s,\theta}$ for $s > 1/4$ if $n = 2$ (resp. $s > s_c$ if $n \geq 3$).

The norm $\|\cdot\|$ depends on s, θ and ε, and the above properties hold for sufficiently small $\varepsilon > 0$ and $\theta > 1/2$, depending on s.

It is obvious that (I) and (II) imply

$$\|Q(u,u)\|_{s-1,\theta+\varepsilon-1} + \left\|\Lambda^{-1}\Lambda_-^{-1+\varepsilon}Q(u,u)\right\| \lesssim \left(|u|_{s,\theta} + \|u\|\right)^2$$

for $s > 1/4$ if $n = 2$ (resp. $s > s_c$ if $n \geq 3$). By theorem 15, this implies well-posedness.

In section 5.2.8 we prove that in dimension two, property (II) fails to hold for $s < 1/8$. This means that to prove well-posedness for $s < 1/8$, we must use at least three iterates.

We remark that if $s > s_c + 1/2$, $n \geq 2$, then the estimate in property (I) holds without the semi-norm $\|\cdot\|$ on the right hand side. This fact follows easily from Hölder's inequality and Sobolev embeddings; we omit the proof. It should be noted, however, that in two dimensions, even this last result requires the null structure of the bilinear operator Q, in contrast with the case of higher dimensions.

5.2.2 Strategy for proving well-posedness below $1/4$

Consider the two-dimensional case. The idea is to define a sequence of semi-norms $\|\cdot\|_j$, $j \geq 1$, where the norm corresponding to $j = 1$ coincides with the norm appearing in the previous section, and satisfying

(II) $\left\|\Lambda^{-1}\Lambda_-^{-1+\varepsilon}Q(u,u)\right\|_{j-1} \lesssim |u|_{s,\theta}^2$ for all $s > \frac{1}{2j}$, $j \geq 2$.

Since this property generalizes property (II) of the previous section, in the case $n = 2$, we denote them by the same Roman numeral.

Now assume we could prove that

(C_j) $\qquad \left\|\Lambda^{-1}\Lambda_-^{-1+\varepsilon}Q(u,u)\right\|_{j-1} \lesssim (|u|_{s,\theta} + \|u\|_j)^2 \quad$ for all $s > s_c$

holds for $2 \leq j \leq k$, and that we are given

$$s > \frac{1}{2(k+1)}.$$

Having chosen appropriate $\theta > 1/2$ and $\varepsilon > 0$ depending on s and k, we set

$$\|u\| = \|u\|_1 + \cdots + \|u\|_k .$$

The plan is to apply theorem 15 with this semi-norm. But it is immediate from properties (I), (II) and (C_j) that

$$\|Q(u,u)\|_{s-1,\theta+\varepsilon-1} + \left\|\Lambda^{-1}\Lambda_-^{-1+\varepsilon}Q(u,u)\right\| \lesssim \left(|u|_{s,\theta} + \|u\|\right)^2,$$

and this gives well-posedness for $s > \frac{1}{2(k+1)}$.

5.2.3 Three lemmas

Definition 4. *Let T_j, $j \geq 2$, be the sequence of operators given by*

$$T_2(u,v) = uv,$$
$$T_{j+1}(u_1,\dots,u_{j+1}) = u_1\Lambda^{-1}T_j(u_2,\dots,u_{j+1}).$$

The first lemma is crucial to the proof of property (II).

Lemma 9. *In space dimension $n = 2$,*

$$(5.11) \quad \left\|\Lambda^{-\sigma}T_{j+1}(u_1,\dots,u_j,v)\right\|_{L^2} \lesssim \|u_1\|_{s_1,\theta}\cdots\|u_{j-1}\|_{s_{j-1},\theta}\|u_j\|_{0,\theta}\|v\|_{L^2},$$

where σ and the s's are strictly positive, $\sigma + s_1 + \cdots + s_{j-1} > 1$, $\theta > 1/2$ and $j \geq 2$.

Proof. By duality, (5.11) is equivalent to

$$(5.12) \quad \|T_{j+1}(u_j,\dots,u_1,w)\|_{L^2} \lesssim \|u_1\|_{s_1,\theta}\cdots\|u_{j-1}\|_{s_{j-1},\theta}\|u_j\|_{0,\theta}\|w\|_{\sigma,0}.$$

We will prove (5.12) by induction. To prove the case $j = 2$, we note that proposition 10 yields

$$\left\|u_2\Lambda^{-1}(u_1w)\right\|_{L^2} \lesssim \|u_2\|_{0,\theta}\|\Lambda^{\varepsilon}(u_1w)\|_{L^2}$$

for any $\varepsilon > 0$, and since $s_1 + \sigma > 1$, (3.24) and proposition 10 give

$$\|\Lambda^{\varepsilon}(u_1w)\|_{L^2} \lesssim \|u_1\|_{s_1,\theta}\|w\|_{\sigma,0}$$

for ε sufficiently small.

Now assume (5.12) holds for some $j \geq 2$. We must show that it holds for $j+1$. Set $\omega = \Lambda^{-1}(u_1w)$. By (5.12),

$$\|T_{j+1}(u_{j+1}\dots,u_2,\omega)\|_{L^2} \lesssim \|u_{j+1}\|_{0,\theta}\|u_j\|_{s_j,\theta}\cdots\|u_2\|_{s_2,\theta}\|\omega\|_{\gamma,0},$$

where $1 - s_2 - \cdots - s_j < \gamma < \min\{1, s_1+\sigma\}$. Thus, it suffices to show

$$\|\omega\|_{\gamma,0} \lesssim \|u_1\|_{s_1,\theta}\|w\|_{\sigma,0}.$$

But this follows from proposition 10. □

To verify property (b) of theorem 16, we need the next two lemmas. We denote by $\mathcal{B}$ the space

$$\mathcal{F}^{-1}\{L^2_\xi(L^1_\tau)\}$$

with norm $\|u\|_{\mathcal{B}} = \|\widehat{u}(\tau,\xi)\|_{L^2_\xi(L^1_\tau)}$. Notice that $\|u\|_{\mathcal{B}} \lesssim \|u\|_{0,\theta}$ for $\theta > 1/2$.

Lemma 10. *In any dimension $n \geq 2$,*

$$\left\|\Lambda^{-n/2-\varepsilon}(uv)\right\|_{\mathcal{B}} \lesssim \|u\|_{\mathcal{B}}\|v\|_{\mathcal{B}}$$

for all $\varepsilon > 0$.

Proof. This reduces to the fact that

$$\|fg\|_{H^{-n/2-\varepsilon}} \lesssim \|f\|_{L^2} \|g\|_{L^2}$$

for any $\varepsilon > 0$, where f and g are functions on $\mathbb{R}^n$. □

Lemma 11. *In space dimension $n = 2$,*

$$\left\|\Lambda^{-1} T_{j+2}(u_1, \ldots, u_j, v_1, v_2)\right\|_{\mathcal{B}} \lesssim \|\Lambda^\varepsilon u_1\|_{\mathcal{B}} \cdots \|\Lambda^\varepsilon u_j\|_{\mathcal{B}} \|v_1\|_{\mathcal{B}} \|v_2\|_{\mathcal{B}} \tag{5.13}$$

for all $\varepsilon > 0$ and $j \geq 1$.

Proof. The proof is by induction. The case $j = 1$ follows from the estimate

$$\left\|\Lambda^{-1}\big(f \cdot \Lambda^{-1}(gh)\big)\right\|_{L^2} \lesssim \|f\|_{H^\varepsilon} \|g\|_{L^2} \|h\|_{L^2}, \tag{5.14}$$

where f, g and h are functions on $\mathbb{R}^2$. By duality, (5.14) is equivalent to

$$\left\|f \cdot \Lambda^{-1}(gh)\right\|_{L^2} \lesssim \|f\|_{L^2} \|g\|_{H^\varepsilon} \|h\|_{H^1}.$$

To prove the latter, we use lemma 6 to obtain

$$\left\|f \cdot \Lambda^{-1}(gh)\right\|_{L^2} \lesssim \|f\|_{L^2} \|gh\|_{H^{\varepsilon/2}} \lesssim \|f\|_{L^2} \|g\|_{H^\varepsilon} \|h\|_{H^1}.$$

Now assume (5.13) holds for some $j \geq 1$. If we can show that

$$\left\|\Lambda^{-1}(uv)\right\|_{\mathcal{B}} \lesssim \|\Lambda^\varepsilon u\|_{\mathcal{B}} \|v\|_{\mathcal{B}}, \tag{5.15}$$

then clearly it follows that (5.13) holds also for $j + 1$. But (5.15) is yet another trivial consequence of lemma 6. We omit the details. □

5.2.4 Definition of the semi-norms

Let (m_j), $j \geq 1$, be the sequence defined by

$$m_1 = 1, \quad m_{j+1} = m_j + j,$$

and set

$$E_j = \Lambda^{2\theta + m_j\varepsilon - 1} \Lambda_-^{j\varepsilon}.$$

Since θ will be close to $1/2$ and ε close to 0, the latter operator should be thought of as a small perturbation of the identity. Using (3.24) and the estimate

$$\Lambda_-^\varepsilon(uv) \precsim \Lambda^\varepsilon \Lambda_-^\varepsilon u \cdot \Lambda_-^\varepsilon v, \tag{5.16}$$

which is a consequence of (3.26) and (3.20), we find that

$$E_j(uv) \precsim E_{j+1} \Lambda_-^{-\varepsilon} u \cdot E_j v, \tag{5.17}$$

where it is assumed that u and v have non-negative Fourier transforms.

Now define

$$\|u\|_1 = \sup \left| \int_{\mathbb{R}^{1+n}} \Lambda^{2\theta+\varepsilon-2} \Lambda_- u \cdot vw \, dt \, dx \right|,$$

where the supremum is taken over all $v, w \in H^{0,\theta}$ with unit norms. In the case of space dimension $n = 2$ we also define, for $j \geq 2$,

$$\|u\|_j = \sup \left| \int_{\mathbb{R}^{1+2}} \Lambda^{-1} \Lambda_-^{1-\varepsilon} E_j u \cdot T_{j+1}(E_{j-1} w_1, \dots, E_1 w_{j-1}, w_j, w_{j+1}) \, dt \, dx \right|,$$

where the supremum is taken over all

$$w_1, \dots, w_{j-1} \in H^{s,\theta}, \quad w_j, w_{j+1} \in H^{0,\theta}$$

with unit norms.

Clearly, $\|u\|_j \leq \|v\|_j$ whenever $u \preceq v$, so we may assume that all functions have non-negative Fourier transform. Assuming

$$s \geq s_c + 2\theta + 2\varepsilon - 1,$$

it follows from lemma 10 that

$$u \preceq v \quad \Longrightarrow \quad \|u\|_1 \lesssim \left\| \mathcal{F}(\Lambda^s \Lambda_- v)(\tau, \xi) \right\|_{L^2_\xi(L^\infty_\tau)}.$$

Similarly, in the two-dimensional case, if $s > 0$ and $j \geq 2$ are given, choose $\theta > 1/2$ and $\varepsilon > 0$ so that

$$s \geq 2\theta + m_j \varepsilon - 1$$

and

$$E_k\big(H^{s,\theta}\big) \subseteq H^{0,1/2+\varepsilon} \quad \text{for} \quad k = 1, \dots, j-1.$$

It then follows from lemma 11 that

$$u \preceq v \quad \Longrightarrow \quad \|u\|_j \lesssim \left\| \mathcal{F}(\Lambda^s \Lambda_-^{1+(j-1)\varepsilon} v)(\tau, \xi) \right\|_{L^2_\xi(L^\infty_\tau)}.$$

Thus, the hypotheses of theorem 16 are satisfied.

Moreover, since in general

$$\|\widehat{u}\|_{L^2_\xi(L^\infty_\tau)} \leq \|u\|_{L^1_t(L^2_x)},$$

we conclude that

$$u \preceq v \quad \Longrightarrow \quad \|u\|_j \lesssim \left\| E_j \Lambda_-^{1-\varepsilon} v \right\|_{L^1_t(L^2_x)} \tag{5.18}$$

for $j \geq 1$, $n = 2$.

5.2.5 Proof of property (I)

Let us denote by u_L the low-frequency part of u, i.e., $u_L = \psi(\Lambda)u$, where $\psi \in C_c^\infty([-2,2])$ and $\psi = 1$ on $[-1,1]$. Let D_t be the multiplier with symbol $|\tau|$. We emphasize that in this section we consider all dimensions $n \geq 2$.

By [14, Lemmas 2.3 and 2.4],

$$\begin{aligned} Q(u,v) \precsim\ & D(D^{-1}D_t u_L \cdot v) + D(\Lambda^{-1}\Lambda_- u \cdot v) \\ & + DR_E(\Lambda^{-1}u, v) + \textit{symmetric terms}, \end{aligned} \tag{5.19}$$

where $E = \{(\tau,\xi,\lambda,\eta) : \langle\xi\rangle \leq \langle\eta\rangle\}$ and R is the operator defined by (3.21). For an explanation of the notation R_E we refer to p. 28.

Using (3.24), we thus obtain

$$\|Q(u,v)\|_{s-1,\theta+\varepsilon-1} \lesssim I_1 + I_2 + I_3 + I_4 + \textit{symmetric terms},$$

where

$$\begin{aligned} I_1 &= \left\| D^{-1}D_t u_L \cdot \Lambda^s v \right\|_{L^2}, \\ I_2 &= \left\| \Lambda_-^{\theta+\varepsilon-1}(\Lambda^{s-1}\Lambda_- u \cdot v)_{E^c} \right\|_{L^2}, \\ I_3 &= \left\| \Lambda_-^{\theta+\varepsilon-1}(\Lambda^{-1}\Lambda_- u \cdot \Lambda^s v) \right\|_{L^2}, \\ I_4 &= \left\| \Lambda_-^{\theta+\varepsilon-1} R(\Lambda^{-1}u, \Lambda^s v) \right\|_{L^2}. \end{aligned} \tag{5.20}$$

Notation. *If B is a subset of $\mathbb{R}^{2+2n}$, we denote by $I_{j,B}$ the expression obtained by replacing the multiplication operator in I_j by the restricted multiplication operator $(u,v) \mapsto (uv)_B$, defined on p. 28.*

Setting

$$B = \{(\tau,\xi,\lambda,\eta) : w_-(\tau+\lambda, \xi+\eta) \geq \langle\xi\rangle\}, \tag{5.21}$$

we write $I_j \leq I_{j,B} + I_{j,B^c}$ for $j = 2$, 3 and 4.

Estimates for $I_{2,B}$ and $I_{3,B}$ Since

$$I_{2,B} \leq \left\| \Lambda^{s+\varepsilon-2}\Lambda_+\Lambda_-^\theta u \cdot v \right\|_{L^2}, \quad I_{3,B} \leq \left\| \Lambda^{\varepsilon-2}\Lambda_+\Lambda_-^\theta u \cdot \Lambda^s v \right\|_{L^2}$$

it follows from proposition 10 that $I_{j,B} \lesssim |u|_{s,\theta}\, \|v\|_{s,\theta}$ for $j = 2$, 3, provided

$$s > s_c + \varepsilon = \frac{n-2}{2} + \varepsilon.$$

Estimate for $I_{4,B}$ By proposition 9,

$$I_{4,B} \lesssim \left\| R^{1/2}(\Lambda^{\theta+\varepsilon-3/2}u, \Lambda^s v) \right\|_{L^2} \lesssim \|u\|_{s,\theta}\, \|v\|_{s,\theta}.$$

Estimate for I_{2,B^c} Notice that

$$I_{2,B^c} \leq \left\| \Lambda_-^{-1/2-\varepsilon} (\Lambda^s \Lambda_- u \cdot \Lambda^{\theta+2\varepsilon-3/2} v) \right\|_{L^2}.$$

We claim that the right side is bounded by $\|u\|_{s,\theta} \|v\|_{s,\theta}$. This would follow from

$$H^{0,-1/2} \times H^{s-\theta-2\varepsilon+3/2,\theta} \longrightarrow H^{0,-1/2-\varepsilon},$$

which by duality is equivalent to

$$H^{0,1/2+\varepsilon} \times H^{s-\theta-2\varepsilon+3/2,\theta} \longrightarrow H^{0,1/2}.$$

The latter follows from part (a) of proposition 12.

Estimate for I_{4,B^c} Since (see [14, Corollary 1])

$$R^\gamma(u,v) \precsim \Lambda_-^\gamma(uv) + \Lambda_-^\gamma u \cdot v + u \cdot \Lambda_-^\gamma v, \tag{5.22}$$

we get

$$I_{4,B^c} \lesssim I_3 + \left\| R^{1/2}(\Lambda^{\theta+2\varepsilon-3/2} u, \Lambda^s v) \right\|_{L^2} + \left\| \Lambda_-^{-1/2-\varepsilon} (\Lambda^{\theta+2\varepsilon-3/2} u \cdot \Lambda^s \Lambda_- v) \right\|_{L^2},$$

and we just showed that the second and third terms are both bounded by $\|u\|_{s,\theta} \|v\|_{s,\theta}$.

Estimate for I_{3,B^c} It turns out that the estimate $I_{3,B^c} \lesssim |u|_{s,\theta} |v|_{s,\theta}$ fails for $s < s_c + 1/2$. Define

$$\|u\| = \sup_{v \neq 0} \frac{I_{3,B^c}}{\|v\|_{s,\theta}}.$$

Then the estimate $I_{3,B^c} \leq \|u\| \|v\|_{s,\theta}$ holds by definition, and since

$$I_{3,B^c} \leq \left\| \Lambda_-^{-\theta} (\Lambda^{2\theta-\varepsilon-2} \Lambda_- u \cdot \Lambda^s v) \right\|_{L^2},$$

a duality argument shows that $\|\cdot\| \leq \|\cdot\|_1$.

Estimate for I_1 By Hölder's inequality,

$$I_1 \leq \left\| D^{-1} \partial_t u_L \right\|_{L_t^2(L_x^\infty)} \|\Lambda^s v\|_{L_t^\infty(L_x^2)} \lesssim \left\| D^{-1} \partial_t u_L \right\|_{L_t^2(L_x^\infty)} \|v\|_{s,\theta}.$$

If $n \geq 3$, Sobolev embedding gives

$$\left\| D^{-1} \partial_t u_L \right\|_{L_t^2(L_x^\infty)} \lesssim \|\partial_t u\|_{s-1,\theta},$$

but this is no longer true in dimension two. However, we can certainly get

$$\left\| D^{-1} \partial_t u_L \right\|_{L_t^2(L_x^\infty)} \lesssim \left\| D^{-1/2} \partial_t u \right\|_{s-1/2,\theta}.$$

We therefore redefine the space $\mathcal{X}^{s,\theta}$, setting

$$|u|_{s,\theta} = \|u\|_{s,\theta} + \left\|D^{-1/2}\partial_t u\right\|_{s-1/2,\theta}.$$

Theorem 13 still holds, but the main estimate obviously changes to

$$|u|_{s,\theta} \lesssim \|f\|_{H^s} + \left\|D^{-1/2}g\right\|_{H^{s-1/2}} + T^{\varepsilon/2}\left\|D^{-1/2}F\right\|_{s-1/2,\theta+\varepsilon-1}.$$

Thus, we have to require that $D^{-1/2}g \in H^{s-1/2}$. This is not a limitation, however, since in the original problem one actually has $D^{-1}g \in H^s$; cf. (5.6). Moreover, since every term in (5.19) is of the form $DT(u,v)$, it is clear that

$$\left\|D^{-1/2}Q(u,v)\right\|_{s-1/2,\theta+\varepsilon-1} \lesssim I_1 + I_2 + I_3 + I_4 + \textit{symmetric terms},$$

with the I_j defined exactly as before.

5.2.6 Proof of property (II) when $n = 2$

In this section we assume $n = 2$, $j \geq 2$ and $s > \frac{1}{2j}$. By (5.19),

$$\left\|\Lambda^{-1}\Lambda_-^{-1+\varepsilon}Q(u,v)\right\|_{j-1} \lesssim J_1 + J_2 + J_3 + \textit{symmetric terms},$$

where

$$\begin{aligned} J_1 &= \left\|\Lambda_-^{-1+\varepsilon}(D^{-1}D_t u_L \cdot v)\right\|_{j-1}, \\ J_2 &= \left\|\Lambda_-^{-1+\varepsilon}(\Lambda^{-1}\Lambda_- u \cdot v)\right\|_{j-1}, \\ J_3 &= \left\|\Lambda_-^{-1+\varepsilon}R(\Lambda^{-1}u, v)\right\|_{j-1}. \end{aligned}$$

Estimate for J_1 By (5.17) and (5.18),

$$J_1 \lesssim \left\|D^{-1}D_t\Lambda_-^{(j-1)\varepsilon}u_L \cdot E_{j-1}v\right\|_{L_t^1(L_x^2)}.$$

Thus, applying Hölder's inequality and Sobolev embedding,

$$J_1 \lesssim \left\|D^{-1}D_t\Lambda_-^{(j-1)\varepsilon}u_L\right\|_{L_t^2(L_x^\infty)}\|E_{j-1}v\|_{L^2} \lesssim |u|_{s,\theta}\|v\|_{s,\theta}$$

for appropriate ε and θ.

Estimate for J_2 Applying (5.17), we see that

$$J_2 \lesssim \sup\left|\int_{\mathbb{R}^{1+2}} \Lambda^{-1}\Lambda_-^{1-\varepsilon}E_j u \cdot E_{j-1}v \cdot \Lambda^{-1}A_{j-1}\,dt\,dx\right|,$$

where

$$A_{j-1} = T_j(E_{j-2}w_1, \ldots, E_1 w_{j-2}, w_{j-1}, w_j)$$

and the supremum is taken over all

$$w_1, \ldots, w_{j-2} \in H^{s,\theta}, \quad w_{j-1}, w_j \in H^{0,\theta}$$

with unit norms.

For appropriate ε and θ,

$$\Lambda_-^{1-\varepsilon} E_j u \preceq \Lambda^{s-\frac{1}{2j}} \Lambda_+^{1/2} \Lambda_-^{\theta} u \preceq \Lambda^{1/2-\frac{1}{2j}} \Lambda^{s-1} \Lambda_+ \Lambda_-^{\theta} u,$$

whence

$$J_2 \lesssim \sup \left| \int_{\mathbb{R}^{1+2}} \Lambda^{s-1} \Lambda_+ \Lambda_-^{\theta} u \cdot \Lambda^{-1/2-\frac{1}{2j}} (E_{j-1} v \cdot \Lambda^{-1} A_{j-1}) \, dt \, dx \right|.$$

By lemma 9,

$$\Lambda^{-1/2-\frac{1}{2j}} (E_{j-1} v \cdot \Lambda^{-1} A_{j-1}) \in L^2,$$

and the Cauchy-Schwarz inequality yields $J_2 \lesssim |u|_{s,\theta} \, \|v\|_{s,\theta}$.

Estimate for J_3 With notation as above,

$$J_3 \lesssim \sup \left| \int_{\mathbb{R}^{1+2}} R(\Lambda^{-1} \Lambda_-^{-\varepsilon} E_j u, E_{j-1} v) \cdot \Lambda^{-1} A_{j-1} \, dt \, dx \right|.$$

By proposition 9,

$$J_3 \lesssim \sup \left| \int_{\mathbb{R}^{1+2}} \Lambda^{1/j-1/2} R^{1/2} (\Lambda^{-1/2} \Lambda_-^{-\varepsilon} E_j u, E_{j-1} v) \cdot \Lambda^{-1/2-1/j} A_{j-1} \, dt \, dx \right|$$

and

$$\Lambda^{1/j-1/2+\varepsilon} R^{1/2} (\Lambda^{-1/2} \Lambda_-^{-\varepsilon} E_j u, E_{j-1} v) \in L^2.$$

Moreover, lemma 9 (or proposition 10 if $j = 2$) implies that

$$\Lambda^{-\varepsilon-1/2-1/j} A_{j-1} \in L^2,$$

and we conclude that $J_3 \lesssim \|u\|_{s,\theta} \, \|u\|_{s,\theta}$.

5.2.7 Proof of property (II) when $n \geq 3$

In this section we assume $n \geq 3$ and $s > s_c$. For technical reasons we will use a slight modification of the norm $\|\cdot\|$. Define

$$\|u\| = \sup_{v \neq 0} \frac{I_3}{\|v\|_{s,\theta}}.$$

By duality,

$$\|u\| = \sup \left| \int \Lambda^{-1} \Lambda_- u \cdot v \cdot \Lambda_-^{\theta+\varepsilon-1} w \, dt \, dx \right|, \tag{5.23}$$

where the supremum is over all $v \in H^{0,\theta}$ and $w \in L^2$ with unit norms.

By Plancherel's theorem, Hölder's inequality and the Hausdorff-Young inequality,

$$\Lambda_-^{\theta+\varepsilon-1} : L_t^q(L_x^2) \longrightarrow L^2(\mathbb{R}^{1+n}) \quad \text{for} \quad (3/2-\theta-\varepsilon)^{-1} < q \le 2. \tag{5.24}$$

This implies

$$I_3 \lesssim \left\|\Lambda^{-1}\Lambda_- u \cdot \Lambda^s v\right\|_{L_t^q(L_x^2)} \le \left\|\Lambda^{-1+\varepsilon}\Lambda_- u\right\|_{L_t^q(L_x^r)} \left\|\Lambda^s v\right\|_{L_t^\infty(L_x^2)},$$

where

$$2-2\theta-\varepsilon \le \frac{1}{q} < \frac{3}{2}-\theta-\varepsilon \quad \text{and} \quad \frac{n}{\varepsilon} < r < \infty. \tag{5.25}$$

We conclude that

$$\|u\| \lesssim \left\|\Lambda^{-1+\varepsilon}\Lambda_- u\right\|_{L_t^q(L_x^r)}. \tag{5.26}$$

For later use, notice that the lower bound on $1/q$ implies

$$2s+1-3\varepsilon \ge n-n/r-1/q. \tag{5.27}$$

We now turn to the estimate for $\left\|\Lambda^{-1}\Lambda_-^{-1+\varepsilon}Q(u,v)\right\|$. In fact, this estimate requires no null structure, and we simply use the obvious fact that

$$Q(u,v) \precsim D(D^{-1}\partial_t u_L \cdot v) + D(u \cdot D^{-1}\partial_t v_L) + D(\Lambda^{-1}\Lambda_+ u \cdot \Lambda^{-1}\Lambda_+ v).$$

The low frequency terms are trivial to estimate, and we ignore them. In view of (5.26), we are thus left with the expression

$$\left\|\Lambda^{-1+\varepsilon}(UV)\right\|_{L_t^q(L_x^r)},$$

where $U = \Lambda^{-1+\varepsilon}\Lambda_+\Lambda_-^\varepsilon u$ and $V = \Lambda^{-1+\varepsilon}\Lambda_+\Lambda_-^\varepsilon v$. By (5.27) and (3.11), this expression is bounded by

$$\|U\|_{s-\varepsilon,\theta-\varepsilon} \|V\|_{s-\varepsilon,\theta-\varepsilon} \le |u|_{s,\theta} |v|_{s,\theta}.$$

5.2.8 A counterexample

In this section we prove that in dimension two, property (II) on p. 87 fails to hold for $s < 1/8$. In fact, we can produce bounded sequences (u_j) and (v_j) in $\mathcal{X}^{s,\theta}$ such that $\left\|\Lambda^{-1}\Lambda_-^{-1}Q(u_j,v_j)\right\|$ blows up as $j \to \infty$. Moreover, the Fourier supports of these functions will be such that $|u_j|_{s,\theta} \sim \|u_j\|_{s,\theta}$, $|v_j|_{s,\theta} \sim \|v_j\|_{s,\theta}$ and

$$\mathcal{F}Q(u_j,v_j) \sim \mathcal{F}\Lambda(\Lambda^{-1}\Lambda_- u_j \cdot v_j),$$

so in effect we are proving that the estimate $J_2 \lesssim \|u\|_{s,\theta} \|v\|_{s,\theta}$ fails.

A basic fact is that a product of two solutions of the wave equation with bounded L^2 data can concentrate in a null hyperplane in frequency space. The precise version of this statement that we use here is as follows:

Lemma 12. *Given a large positive parameter L, there exist $w_1, w_2 \in H^{0,\theta}$ such that $\|w_1\|_{0,\theta} = \|w_2\|_{0,\theta} = 1$ and*

$$\widehat{w_1 w_2}(\tau, \xi) \sim 1$$

for all (τ, ξ) satisfying

$$|\tau - \xi_1| \leq 1, \quad L/2 \leq |\xi| \leq 2L. \tag{5.28}$$

Proof. Let $\widehat{w_1}$ and $\widehat{w_2}$ be the characteristic functions of the regions

$$\big|\tau + |\xi|\big| \leq 30, \quad -4L^2 \leq \xi_1 \leq -L^2/4, \quad |\xi'| \leq 3L,$$

and

$$\big|\lambda - |\eta|\big| \leq 1, \quad L^2/2 \leq \eta_1 \leq 2L^2, \quad |\eta'| \leq L$$

respectively, where ξ' denotes $(\xi_2, \dots, \xi_n)$ for any $\xi \in \mathbb{R}^n$. Let A be the set determined by (5.28). We claim that if $(\lambda, \eta) \in \operatorname{supp} \widehat{w_2}$ and $(\tau, \xi) \in A$, then $(\tau - \lambda, \xi - \eta) \in \operatorname{supp} \widehat{w_1}$ for sufficiently large L. Clearly this would imply that

$$\widehat{w_1 w_2}(\tau, \xi) = |\operatorname{supp} \widehat{w_2}| \quad \text{for} \quad (\tau, \xi) \in A,$$

and since $|\operatorname{supp} \widehat{w_2}| \sim L^{n+1} \sim \|w_1\|_{0,\theta} \|w_2\|_{0,\theta}$, we only have to normalize w_1 and w_2 to have unit norms. The claim is easily checked. If $(\lambda, \eta) \in \operatorname{supp} \widehat{w_2}$ and $(\tau, \xi) \in A$, then clearly

$$L^2/4 \leq \eta_1 - \xi_1 \leq 4L^2, \quad |\xi' - \eta'| \leq 3L$$

for L sufficiently large. We have

$$\big|\tau - \lambda + |\xi - \eta|\big| \leq |\tau - \xi_1| + \big|\lambda - |\eta|\big| + |\eta| - \eta_1 + |\xi - \eta| - (\xi_1 - \eta_1),$$

and since

$$|\eta| - \eta_1 = \frac{|\eta'|^2}{|\eta| + \eta_1} \leq 1$$

and, similarly, $|\xi - \eta| - (\xi_1 - \eta_1) \leq 18$, it follows that $\big|\tau - \lambda + |\xi - \eta|\big| \leq 21$. □

Given any sufficiently large parameter L, we will construct functions u and v such that $|u|_{s,\theta} = |v|_{s,\theta} = 1$ and

$$\int_{\mathbb{R}^{1+2}} \Lambda^{2\theta-2} Q(u, v) \cdot \Lambda^{-1}(w_1 w_2)\, dt\, dx \gtrsim L^{\theta - 2s - 1/4},$$

where w_1 and w_2 are as above, except that in (5.28) we replace L by $100L$. If $s \leq 1/8$, the right hand side of the above inequality $\to \infty$ as $L \to \infty$, and this establishes the counterexample.

Let A be the set of all (τ, ξ) such that

$$|\tau - \xi_1| \leq 1, \quad 9L \leq \xi_1 \leq 10L, \quad 99L \leq \xi_2 \leq 100L.$$

By the choice of w_1, w_2,

$$\int_{\mathbb{R}^{1+2}} \Lambda^{2\theta-2} Q(u,v) \cdot \Lambda^{-1}(w_1 w_2)\, dt\, dx \gtrsim L^{2\theta-3} \int_A \widehat{Q(u,v)}(\tau,\xi)\, d\tau\, d\xi.$$

If we can find u, v such that $|u|_{s,\theta} = |v|_{s,\theta} = 1$ and

$$\widehat{Q(u,v)}(\tau,\xi) \sim L^{3/4-2s-\theta} \quad \text{for} \quad (\tau,\xi) \in A, \tag{5.29}$$

it follows that

$$L^{2\theta-3} \int_A \widehat{Q(u,v)}(\tau,\xi)\, d\tau\, d\xi \gtrsim L^{\theta-2s-1/4-2}\, |A| \sim L^{\theta-2s-1/4},$$

which is what we want.

We now construct the functions u and v. Let $\widehat{u}$ and $\widehat{v}$ be the characteristic functions of the regions

$$|\tau - \xi_1| \le 3, \quad 8L \le \xi_1 \le 10L, \quad 98L \le \xi_2 \le 100L,$$

and

$$\big|\lambda - |\eta|\big| \le 1, \quad \frac{999}{1000} L \le \eta_1 \le L, \quad 0 \le \eta_2 \le \sqrt{L}$$

respectively. If $(\tau,\xi) \in A$ and $(\lambda,\eta) \in \operatorname{supp} \widehat{v}$, then $(\tau-\lambda, \xi-\eta) \in \operatorname{supp} \widehat{u}$. Indeed,

$$|\tau - \lambda - (\xi_1 - \eta_1)| \le |\tau - \xi_1| + \big|\lambda - |\eta|\big| + |\eta| - \eta_1 \le 3,$$

and it is clear that

$$8L \le \xi_1 - \eta_1 \le 10L, \quad 98L \le \xi_2 - \eta_2 \le 100L.$$

Now we have to estimate the symbol of Q. Assume $(\tau,\xi) \in \operatorname{supp} \widehat{u}$ and $(\lambda,\eta) \in \operatorname{supp} \widehat{v}$. Then

$$\begin{aligned}
\left| (\xi+\eta) \cdot \left(\frac{\tau\xi}{|\xi|^2} - \frac{\lambda\eta}{|\eta|^2} \right) \right| &= \left| \tau - \lambda \left(1 + \frac{\xi_1\eta_1}{|\eta|^2} \right) - \lambda \frac{\xi_2\eta_2}{|\eta|^2} + \tau \frac{\xi\cdot\eta}{|\xi|^2} \right| \\
&\ge \left| \tau - \lambda \left(1 + \frac{\xi_1\eta_1}{|\eta|^2} \right) \right| - \lambda \frac{\xi_2\eta_2}{|\eta|^2} - \tau \frac{|\eta|}{|\xi|} \\
&= a - b - c.
\end{aligned}$$

We have

$$b \le 2L \frac{100L\sqrt{L}}{L^2/4} \le 800\sqrt{L}, \quad c \le (3+10L) \frac{2L}{98L} \le \frac{L}{4},$$

and

$$a \ge \lambda - |\tau - \xi_1| - \xi_1 \left| 1 - \frac{\lambda\eta_1}{|\eta|^2} \right|.$$

For L sufficiently large,

$$\frac{998}{1000}L \leq \lambda \leq \frac{1001}{1000}L, \quad \frac{999}{1000}L \leq |\eta| \leq \frac{1001}{1000}L,$$

which implies

$$\left|1 - \frac{\lambda\eta_1}{|\eta|^2}\right| \leq 1/100.$$

We conclude that $a \geq L/2$, whereas $b + c \leq L/3$ for L large enough. Hence,

$$\widehat{Q(u,v)}(\tau,\xi) \sim L\,|\operatorname{supp}\widehat{v}| \sim L^{5/2} \quad \text{for} \quad (\tau,\xi) \in A,$$

and since $|u|_{s,\theta} \sim L^{s+\theta}\,|\operatorname{supp}\widehat{u}|^{1/2} \sim L^{s+\theta+1}$ and $|v|_{s,\theta} \sim L^{s}\,|\operatorname{supp}\widehat{v}|^{1/2} \sim L^{s+3/4}$, we can simply normalize u and v to have unit norms, and (5.29) is proved.

Remark. The above counterexample does *not* show that the estimate

$$\left\|\Lambda^{-1}\Lambda_-^{-1}Q(u,v)\right\| \lesssim \|u\|\,\|v\|_{s,\theta}$$

fails. With u, v, w_1 and w_2 as in the previous section, it is clear that

$$\int \Lambda^{2\theta-2}\Lambda_- u \cdot w_1 w_2 \,dt\,dx \gtrsim L^{2\theta+1},$$

which implies $\|u\| \sim L^{2\theta+1}$, whereas $|u|_{s,\theta} \sim L^{s+\theta+1}$.

5.2.9 Remarks on the conjecture (C_j)

Proceeding as in section 5.2.6, we have to estimate J_2 and J_3.

Estimate for J_2 We have

$$J_2 \lesssim \sup\left|\int_{\mathbb{R}^{1+2}} \Lambda^{-1}\Lambda_-^{1-\varepsilon}E_j u \cdot E_{j-1}v \cdot \Lambda^{-1}A_{j-1}\,dt\,dx\right|,$$

where

$$A_{j-1} = T_j(E_{j-2}w_1, \ldots, E_1 w_{j-2}, w_{j-1}, w_j)$$

and the supremum is taken over all

$$w_1, \ldots, w_{j-2} \in H^{s,\theta}, \quad w_{j-1}, w_j \in H^{0,\theta}$$

with unit norms. It is therefore clear from the definition of $\|\cdot\|_j$ that

$$J_2 \lesssim \|u\|_j\,\|v\|_{s,\theta}\,.$$

Estimate for J_3 Let us take $j = 2$. Then, essentially,

$$J_3 \lesssim \sup \left| \int_{\mathbb{R}^{1+2}} R(\Lambda^{-1}u, v) \cdot \Lambda^{-1}(w_1 w_2) \, dt \, dx \right|,$$

where $u, v \in H^{s,\theta}$ and $w_1, w_2 \in H^{0,\theta}$. Denote by I the integral on the right hand side.

At first glance, one might think that the estimate

$$J_3 \lesssim \|u\|_{s,\theta} \|v\|_{s,\theta} \quad \text{for all } s > 0$$

follows from the quadrilinear estimate (2.26) via proposition 8. This is not so, however, since the absolute value is outside the integral I. Thus we would need the following generalization of (2.26): if $3/4 < a < 1$ and u_k, $1 \le k \le 4$, are solutions of $\Box u_k = 0$ on $\mathbb{R}^{1+2}$, with Cauchy data $u_k\big|_{t=0} = f_k$, $\partial_t u_k\big|_{t=0} = 0$, then

$$\left| \int_{\mathbb{R}^{1+2}} e^{it\rho} D^{-a} D_-(u_1 u_2) \cdot u_3 u_4 \, dt \, dx \right| \le C \|f_1\|_{\dot{H}^{2-a}} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}$$

with C independent of ρ. This estimate fails. However, if we restrict the region of integration in Fourier space (after applying Plancherel) so that ρ is small compared to the symbol of D_-, then our proof can be adapted, and we do indeed have the above estimate.

This means that if, in the integral I, all four functions are supported near the light cone in Fourier space, relative to the size of the symbol of R, we do have the estimate $|I| \lesssim \|u\|_{s,\theta} \|v\|_{s,\theta}$ for all $s > 0$.

The bad case is when either u or v not concentrated on the cone, but the remaining three functions are. The three functions which concentrate can be treated as solutions of the homogeneous wave equation. Thus, if we apply Cauchy-Schwarz, we get trilinear L^2 expressions of the form

$$\left\| D^{-a}\big(D^{-b} u_1 D^{-c}(u_2 u_3)\big) \right\|_{L^2(\mathbb{R}^{1+2})}$$

where $a+b+c = 3/2$ and the u_k are solutions of the homogeneous wave equation with L^2 initial data. Recall that the Strichartz estimate in 2D is L^6, so this makes sense. However, a simplified version of the counterexample in section 5.2.8 shows that no such estimate is true. The reason is that the product $u_2 u_3$ can concentrate on a null hyperplane in Fourier space. Thus a reduction to trilinear estimates is out of the question.

If u does not concentrate, we basically have

$$R(\Lambda^{-1}u, v) \sim \Lambda^{-1}\Lambda_- u \cdot v,$$

so in this case $J_3 \lesssim J_2$, and there is no problem. The remaining case is when v does not concentrate. Then we could have

$$R(\Lambda^{-1}u, v) \sim \Lambda^{-1}u \cdot \Lambda_- v,$$

and if v is at much higher frequency than u, we could be in trouble. However, if we go back to the bilinear operator $Q(u, v)$, which is what we actually have to estimate, we see that this potentially bad configuration can only occur when v concentrates near the cone and yet is supported far from the cone relative to the other functions. It should be relatively straightforward to determine how bad this can be, but the time limitation forces us to let this subject rest for now.

5.3 Preliminary analysis of (5.5)

Note that (5.5b), (5.5c) can be written

$$\delta \bar{A} = 0 \tag{5.30a}$$

$$d\bar{A} = F, \tag{5.30b}$$

where $\bar{A} = \bar{A}_i dx^i$, $F = F(A_0, \bar{A}) = -[\bar{A}_i - R_0 R_i A_0, \bar{A}_j - R_0 R_j A_0] dx^i \wedge dx^j$ and δ is the codifferential operator. If $\bar{A}$ solves (5.30), the 2-form F must be closed, whence $\mathcal{P}F = F$, where $\mathcal{P}$ is the projection onto the space of closed 2-forms. Note that $\mathcal{P} = -(-\Delta)^{-1} d\delta$ is a pseudodifferential operator of order zero.

We therefore replace (5.5) with the system

$$\Box A_0 = \partial^i [A_0, \bar{A}_i - R_0 R_i A_0] \tag{5.31a}$$

$$\delta \bar{A} = 0 \tag{5.31b}$$

$$d\bar{A} = \mathcal{P} F(A_0, \bar{A}). \tag{5.31c}$$

From now on we assume $n = 3$. Under the standard identification of 1-forms and 2-forms on $\mathbb{R}^3$ with vector fields on $\mathbb{R}^3$, we have $\delta \bar{A} = \operatorname{div} \bar{A}$, $d\bar{A} = \operatorname{curl} \bar{A}$ and

$$\mathcal{P} F = (-\Delta)^{-1} \operatorname{curl} \operatorname{curl} F.$$

Thus, $\mathcal{P}$ is simply the projection onto the space of divergence free vector fields, and acts on a vector $v = (v_1, v_2, v_3)$ according to the rule

$$(\mathcal{P} v)_i = v_i + D^{-2} \partial_i \partial^j v_j.$$

In particular, if $A_\mu dx^\mu$ solves (5.1), then $\bar{A}_i = (\mathcal{P} A)_i$. Of course, our forms are matrix-valued, so we must understand the vector operators as being applied at each entry of the matrix.

The first step in solving any nonlinear problem is to solve the corresponding linear problem. We must therefore find the solution operator S for the linearized version of (5.31b), (5.31c). Given a vector field v on $\mathbb{R}^3$, it is easily checked that the solution $u = Sv$ of the system

$$\operatorname{div} u = 0$$
$$\operatorname{curl} u = \mathcal{P} v$$

is given by $Sv = (-\Delta)^{-1} \operatorname{curl} v$. In terms of the Fourier transform,

$$\widehat{Sv}(\xi) = \frac{i}{|\xi|^2} (\xi_2 \widehat{v_3} - \xi_3 \widehat{v_2}, \xi_3 \widehat{v_1} - \xi_1 \widehat{v_3}, \xi_1 \widehat{v_2} - \xi_2 \widehat{v_1}).$$

We must determine the space in which the iteration can be carried out. First of all, notice that all the A_μ's satisfy nonlinear wave equations with data in $H^s \times H^{s-1}$. Hence, we should have $A_\mu \in \mathcal{X}^{s,\theta}$. Moreover, since $\partial_t A_0 = \partial^i A_i$, we should have $D^{-1}\partial_t A_0 \in H^{s,\theta}$. By (5.4) we must then have $\bar{A}_i \in H^{s,\theta}$.

To further determine the natural regularity properties of A_0 and $\bar{A}$, we must examine the structure of the nonlinearities. The following discussion is strictly informal. The details will be supplied later. Let us start with the right hand side of (5.31c). A typical entry of the matrix $[\bar{A}_j - R_0 R_j A_0, \bar{A}_i - R_0 R_i A_0]$ is of the form uv, with $u, v \in H^{s,\theta}$. Hence, the entries of $\bar{A}_i$ are schematically of the form $D^{-1}(uv)$, so the estimates (3.11), (3.12) and (3.13) show that $\bar{A}_i$ should have the regularity

$$\bar{A}_i \in L_t^q(L_x^\infty) \cap L_t^2(H_x^1) \cap H^{s,\theta},$$

where q is close to 1.

Thus, the entries of the matrix $\partial^i[A_0, \bar{A}_i - R_0 R_i A_0]$ are of the form $D(uw) + Q(u,v)$, where

$$u, v \in H^{s,\theta}, \quad w \in L_t^q(L_x^\infty) \cap L_t^2(H_x^1) \cap H^{s,\theta}$$

and Q is the null form given by (5.10). As we saw in section 5.2, the natural regularity assumption on A_0 is

$$A_0 \in \mathcal{X}^{s,\theta}, \quad \Lambda^{-1}\Lambda_- A_0 \in L_t^q(L_x^\infty), \tag{5.32}$$

where q again is close to 1.

5.4 Existence theorem

Before stating our result, we need a precise definition of the space in which the contraction argument will take place. First, we must redefine $\mathcal{X}^{s,\theta}$, setting

$$|u|_{s,\theta} = \|u\|_{s,\theta} + \left\|D^{-1}\partial_t u\right\|_{s,\theta}.$$

The results in section 5.2 remain valid for this redefined space, provided that the initial condition (5.9b) is changed to

$$u\big|_{t=0} = f, \quad \partial_t u\big|_{t=0} = Dg,$$

where both f and g are in H^s. Comparing with (5.6), we see that this is precisely the type of initial condition we have for A_0.

We assume

$$1/2 < \theta < 1, \quad s > 2\theta - 1/2. \tag{5.33}$$

Denote by $\|\cdot\|_1$ the norm defined in (5.23) with $\varepsilon = 0$, and set

$$X_0 = \{u : \|u\|_{X_0} < \infty\}, \quad Y_0 = \{G : \|G\|_{Y_0} < \infty\},$$

where

$$\|u\|_{X_0} = |u|_{s,\theta} + \|u\|_1\,, \quad \|G\|_{Y_0} = \|G\|_{s-1,\theta-1} + \left\|\Lambda^{-1}\Lambda_-^{-1}G\right\|_1.$$

Define

$$\|u\|_2 = \inf\Big\{\|v\|_{L_t^q(L_x^\infty)} : v \in H^{s+1/2,0},\, u \preceq v\Big\},$$

where q satisfies (5.25) and (5.27) with $r=\infty$ and $\varepsilon=0$. Now set

$$X_1 = \big\{u : \|u\|_{X_1} < \infty\big\}, \quad Y_1 = \big\{G : \|G\|_{Y_1} < \infty\big\},$$

where

$$\|u\|_{X_1} = \|u\|_{s,\theta} + \|u\|_{s+1/2,0} + \|u\|_2$$

and

$$\|G\|_{Y_1} = \left\|D^{-1}G\right\|_{s,\theta} + \left\|D^{-1}G\right\|_{s+1/2,0} + \left\|D^{-1}G\right\|_2.$$

Theorem 18. *If $n=3$, $s>1/2$ and (5.33) is satisfied, there exists $\varepsilon>0$ such that for any matrix-valued data $(a_0,a_1,a_2,a_3) \in H^s$ satisfying*

$$\sum_0^3 \|a_\mu\|_{H^s} \le \varepsilon,$$

there is a solution

$$(A_0, \bar{A}) \in X_0 \times X_1^3$$

of (5.31), (5.6) on $[0,1]\times\mathbb{R}^3$, and the solution map

$$(a_\mu) \longmapsto (A_0,\bar{A}), \quad B(0,\varepsilon) \subseteq H^s \longrightarrow X_0 \times X_1^3$$

is Lipschitz continuous.

5.5 Proofs of the bilinear estimates

In this section we aim to prove the estimates

$$\left\|\partial^i[A_0, \bar{A}_i - R_0R_iA_0]\right\|_{Y_0} \lesssim \|A_0\|_{X_0}\left\|\bar{A}\right\|_{X_1} + \|A_0\|_{X_0}^2 \tag{5.34}$$

and

$$\left\|[\bar{A}_j - R_0R_jA_0, \bar{A}_i - R_0R_iA_0]\right\|_{Y_1} \lesssim \left(|A_0|_{s,\theta} + \left\|\bar{A}\right\|_{s,\theta}\right)^2. \tag{5.35}$$

As usual, we may assume that all functions have non-negative Fourier transforms.

We first prove (5.34). By bilinearity,

$$\partial^i[A_0, \bar{A}_i - R_0R_iA_0] = \partial^i[A_0,\bar{A}_i] - \partial^i[A_0, R_0R_iA_0].$$

The entries of the matrix $\partial^i[R_0R_iA_0, A_0]$ are of the form $Q(u,v)$, where u and v are entries of A_0 and Q is the null form given by (5.10). In section 5.2 we proved that

$$\|Q(u,v)\|_{Y_0} \lesssim \|u\|_{X_0} \|v\|_{X_0} .$$

Any entry of the matrix $\partial_i[A_0, \bar{A}_i]$ is $\precsim D(uv)$, where u is an entry of A_0 and v is an entry of $\bar{A}_i$. Thus, $u \in X_0$ and $v \in X_1$. By (3.24),

$$\|D(uv)\|_{s-1,\theta-1} \lesssim \|\Lambda^s u \cdot v\|_{0,\theta-1} + \left\|u \cdot \Lambda^{s+1/2} v\right\|_{-1/2,\theta-1} = I_1 + I_2.$$

By (5.24) and the Hölder and Sobolev inequalities,

$$I_1 \lesssim \|\Lambda^s u \cdot v\|_{L_t^q(L_x^2)} \le \|\Lambda^s u\|_{L_t^\infty(L_x^2)} \|v\|_{L_t^q(L_x^\infty)} ,$$

and since we could have replaced v by any $w \in H^{s+1/2,0}$ such that $v \preceq w$, we conclude that $I_1 \lesssim \|u\|_{s,\theta} \|v\|_2$.

With notation as on p. 92, we write $I_2 \le I_{2,B} + I_{2,B^c}$, where B is defined in (5.21). Since

$$I_{2,B} \le \left\|\Lambda^{\theta-1} u \cdot \Lambda^{s+1/2} v\right\|_{-1/2,0} ,$$

it follows from proposition 10 that $I_{2,B} \lesssim \|u\|_{s,\theta} \|v\|_{s+1/2,0}$.

Next, we have

$$I_{2,B^c} \le \left\|\Lambda^{2\theta-1} u \cdot \Lambda^{s+1/2} v\right\|_{-1/2,-\theta} .$$

We now use the classical Strichartz inequality

$$H^{1/2,\theta} \times H^{1/2,\theta} \longrightarrow L^2, \tag{5.36}$$

which by duality is equivalent to

$$H^{1/2,\theta} \times L^2 \longrightarrow H^{-1/2,-\theta}.$$

Thus, $I_{2,B^c} \lesssim \|u\|_{s,\theta} \|v\|_{s+1/2,0}$.

Inequality (5.26) implies

$$\left\|\Lambda^{-1} \Lambda_-^{-1} D(uv)\right\|_1 \lesssim \left\|\Lambda^{-1}(uv)\right\|_{L_t^q(L_x^\infty)} .$$

By (5.27), (3.11) and the Sobolev inequality,

$$\left\|\Lambda^{-1}(uv)\right\|_{L_t^q(L_x^\infty)} \lesssim \|u\|_{s,\theta} \|v\|_{s,\theta} . \tag{5.37}$$

This concludes the proof of (5.34).

Proving (5.35) reduces to proving

$$\left\|D^{-1}(uv)\right\|_{s,\theta} \lesssim \|u\|_{s,\theta} \|v\|_{s,\theta} , \tag{5.38}$$

$$\left\|D^{-1}(uv)\right\|_{s+1/2,0} \lesssim \|u\|_{s,\theta} \|v\|_{s,\theta} , \tag{5.39}$$

$$\left\|D^{-1}(uv)\right\|_2 \lesssim \|u\|_{s,\theta} \|v\|_{s,\theta} . \tag{5.40}$$

First notice that (5.40) follows from (5.37).

By (3.24) and (3.26),

$$\left\|D^{-1}(uv)\right\|_{s,\theta} \lesssim \left\|D^{-1}(\Lambda^s\Lambda_-^\theta u \cdot v)\right\|_{L^2} + \left\|D^{-1}(\Lambda^s u \cdot \Lambda_-^\theta v)\right\|_{L^2} + \left\|D^{-1}\Lambda^s R^\theta(u,v)\right\|_{L^2} + \textit{symmetric terms}.$$

The first two terms on the right side of the inequality can be estimated directly by applying proposition 10. For the third term we estimate

$$\begin{aligned}\left\|D^{-1}\Lambda^s R^\theta(u,v)\right\|_{L^2} &\lesssim \left\|D^{-1}R^\theta(u,v)\right\|_{L^2} + \left\|D^{s-1}R^\theta(u,v)\right\|_{L^2} \\ &\lesssim \left\|D^{-1}(\Lambda^{\theta/2}u \cdot \Lambda^{\theta/2}v)\right\|_{L^2} + \left\|D^{s-1}(\Lambda^{\theta/2}u \cdot \Lambda^{\theta/2}v)\right\|_{L^2}.\end{aligned}$$

To bound the first term on the right side of the last inequality, we apply proposition 10, while for the second term we apply (3.12) and (3.13).

To prove (5.39), we estimate

$$\left\|D^{-1}(uv)\right\|_{s+1/2,0} \lesssim \left\|D^{-1}(uv)\right\|_{L^2} + \left\|D^{s-1/2}(uv)\right\|_{L^2}.$$

For the first term we apply proposition 10, and for the second term we use (3.24) and (5.36).

5.6 Proof of existence theorem

Here we prove theorem 18. Define

$$\Phi = (\Phi_0, \Phi_1) : X_0 \times X_1^3 \longrightarrow X_0 \times X_1^3$$

by

$$\begin{aligned}&\Phi_0(A_0, \bar{A}) \\ &\quad = \chi(t)\big(\cos(tD) \cdot a_0 + D^{-1}\sin(tD) \cdot \partial^i a_i\big) \\ &\quad - \chi(t)\int_0^t D^{-1}\sin\big((t-t')D\big) \cdot \big(\phi(\Lambda_-)\partial^i[A_0, \bar{A}_i - R_0R_iA_0]\big)(t')\,dt' \\ &\quad + \Box^{-1}\big(1-\phi(\Lambda_-)\big)\partial^i[A_0, \bar{A}_i - R_0R_iA_0]\end{aligned}$$

and

$$\Phi_1(A_0, \bar{A}) = S\big([\bar{A}_j - R_0R_jA_0, \bar{A}_i - R_0R_iA_0]\big).$$

To save space, we write $\left\|(A_0, \bar{A})\right\| = \|A_0\|_{X_0} + \left\|\bar{A}\right\|_{X_1}$. By the estimates proved in the previous section,

$$\left\|\Phi(A_0, \bar{A})\right\| \le C\Big(\sum \|a_\mu\|_{H^\sigma} + \left\|(A_0, \bar{A})\right\| \left\|(A_0, \bar{A})\right\|\Big) \tag{5.41}$$

and

$$\begin{aligned}&\left\|\Phi(A_0, \bar{A}) - \Phi(B_0, \bar{B})\right\| \\ &\qquad \le C\left(\left\|(A_0, \bar{A})\right\| + \left\|(B_0, \bar{B})\right\|\right)\left\|(A_0 - B_0, \bar{A} - \bar{B})\right\|.\end{aligned} \tag{5.42}$$

Set $\mathcal{E} = C \sum \|a_\mu\|_{H^s}$. It is clear from (5.41) and (5.42) that if $8C\mathcal{E} \leq 1$, then Φ is a contraction of the closed ball $\mathcal{N} = B(0, 2\mathcal{E})$ in $X_0 \times X_1^3$. Hence Φ has a fixed point $(A_0, \bar{A})$ in $\mathcal{N}$, and this fixed point is a solution of (5.31), (5.6) on $[0,1] \times \mathbb{R}^3$.

Next, we prove that the dependence of the solution on the data is Lipschitz. We denote by $\Phi_{(a_\mu)}$ the solution operator corresponding to data (a_μ). If $(A_0, \bar{A})$ and $(B_0, \bar{B})$ are fixed points of $\Phi_{(a_\mu)}$ and $\Phi_{(b_\mu)}$ respectively, where

$$\sum \|a_\mu\|_{H^\sigma}, \sum \|b_\mu\|_{H^\sigma} \leq \frac{1}{8C},$$

then

$$\begin{aligned}\big\|(A_0, \bar{A}) - (B_0, \bar{B})\big\| \leq C \sum \|a_\mu - b_\mu\|_{H^s} \\ + C \left(\big\|(A_0, \bar{A})\big\| + \big\|(B_0, \bar{B})\big\|\right) \big\|(A_0 - B_0, \bar{A} - \bar{B})\big\| .\end{aligned}$$

But $C\big(\big\|(A_0, \bar{A})\big\| + \big\|(B_0, \bar{B})\big\|\big) \leq 1/2$, whence

$$\big\|(A_0, \bar{A}) - (B_0, \bar{B})\big\| \leq 2C \sum \|a_\mu - b_\mu\|_{H^s} .$$

This concludes the proof of theorem 18.

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