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GLOBAL EXISTENCE FOR SYSTEMS OF NONLINEAR WAVE EQUATIONS IN 3D WITH MULTIPLE SPEEDS*

THOMAS C. SIDERIS† AND SHU-YI TU‡

Abstract. Global smooth solutions to the initial value problem for systems of nonlinear wave equations with multiple propagation speeds will be constructed in the case of small initial data and nonlinearities satisfying the null condition.

Key words. systems of nonlinear wave equations, global existence, null condition

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1. Introduction.

This paper is concerned with the Cauchy problem for coupled systems of quasi-linear wave equations in three space dimensions of the form

$$\partial_t^2 u^k - c_k^2 \triangle u^k = C^{jk}_{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u^j, \quad k = 1, \dots, m,$$

subject to suitably small initial conditions. We assume that the propagation speeds are distinct, and we refer to this situation as the nonrelativistic case. Here, ∂u stands for the full space-time gradient, and $C^{jk}_{\alpha\beta}(\xi) = O(|\xi|)$ are smooth functions near the origin in $\mathbb{R}^{4m}$. We shall construct a unique global classical solution, provided that the coefficients of the nonlinear terms satisfy the null condition, which permits only certain special nonlinear self-interactions of the kth component of the solution in the kth equation. This nonrelativistic system serves as a simplified model for wave propagation problems with different speeds, such as nonlinear elasticity, charged plasmas, and magneto-hydrodynamics.

The main difficulty in the nonrelativistic case is that the smaller symmetry group of the linear operator weakens the form of the invariant Klainerman inequality; see section 6. In order to obtain a viable $L^\infty - L^2$ estimate for solutions, we utilize an additional set of weighted L^2 estimates, as has been developed in [15], [19], [20]. The advantage of this method is the total avoidance of direct estimation of the fundamental solution for the linear problem as well as any type of asymptotic constructions. We treat nondivergence form nonlinearities which may contain both spatial and temporal derivatives.

In the three-dimensional (3D) relativistic (scalar) case, the null condition was first identified and shown to lead to global existence of small solutions by Christodoulou [3] and Klainerman [13]. Without it, small solutions remain smooth "almost globally" [8], [9], [12], but as examples show, arbitrarily small initial conditions can develop singularities in finite time [7], [18]. Small solutions always exist globally in higher dimensions [11], [17], [12]. The two-dimensional (2D) relativistic case is rather more complicated. The sharpest results are given in [1], [2], but other work appeared previously in [4], [10].

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†Department of Mathematics, University of California, Santa Barbara, CA 93106 (sideris@math.ucsb.edu). This author was supported in part by the National Science Foundation.
‡Department of Mathematics, St. Cloud State University, St. Cloud, MN 56301 (stu@stcloudstate.edu).

The case of nonrelativistic systems in 3D has also recently been considered by Yokoyama [21]. Under the same null condition as described below, Yokoyama establishes the existence of global small solutions. However, instead of expressing the smallness condition for the initial data in terms of a neighborhood of the origin in a Sobolev space, as we do here, Yokoyama considers only data of the form $(u, \partial_t u)|_{t=0} = \varepsilon(u_0, u_1)$ for fixed C^∞ functions with compact support. Another significant difference is that Yokoyama obtains decay of solutions through direct $L^\infty - L^\infty$ estimation of the fundamental solution for the linear wave equation. By avoiding such direct estimations, our $L^\infty - L^2$ approach is much simpler. Moreover, our estimates are sharper insofar as they do not require the logarithmic growth factors used in Proposition 3.1 in [21].

An early result for 3D nonrelativistic systems was obtained by Kovalyov [16] in the semilinear case under a strong *nonresonance condition* that ruled out all nonlinear self-interactions. The 2D case has been examined in [6] and [5] using an approach similar to [21].

The statement of the main result is given in section 3 after a summary of some standard notation. The rest of the paper presents the proof. To simplify the exposition, we truncate the nonlinearity at the quadratic level, but this entails no loss of generality since the higher-order terms do not affect the global behavior of small solutions [12].

2. Notation. Points in $\mathbb{R}^4$ will be denoted by $X = (x^0, x^1, x^2, x^3) = (t, x)$. Partial derivatives will be written as $\partial_k = \partial/\partial x^k$, $k = 0, \dots, 3$, with the abbreviations $\partial = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla)$. The angular-momentum operators are defined as

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla,$$

where $\wedge$ denotes the usual vector cross product in $\mathbb{R}^3$, and the scaling operator is defined by

$$S = t\partial_t + r\partial_r = x^\alpha \partial_\alpha. \tag{2.1}$$

The collection of these seven vector fields will be labeled as

$$\Gamma = (\Gamma_0, \dots, \Gamma_7) = (\partial, \Omega, S).$$

Instead of the usual multi-index notation, we will write $a = (a_1, \dots, a_\kappa)$ for a sequence of indices $a_i \in \{0, \dots, 7\}$ of length $|a| = \kappa$, and

$$\Gamma^a = \Gamma_{a_\kappa} \cdots \Gamma_{a_1}.$$

Suppose that b and c are disjoint subsequences of a. Then we will say $b + c = a$ if $|b| + |c| = |a|$, and $b + c < a$ if $|b| + |c| < |a|$.

The d'Alembertian will be used to denote the operator

$$\Box = \mathrm{Diag}(\Box_1, \dots, \Box_m) \quad \text{with} \quad \Box_k = \partial_t^2 - c_k^2 \triangle.$$

For convenience, we will assume that the speeds are distinct

$$c_1 > \cdots > c_m > 0.$$

It is also possible to treat the case where some of the speeds are the same; see the remark following the statement of Theorem 3.1. This operator acts on vector functions $u : \mathbb{R}^4 \to \mathbb{R}^m$. The standard energy is then defined as

$$E_1(u(t)) = \sum_{k=1}^m \int_{\mathbb{R}^3} [\, |\partial_t u^k(t,x)|^2 + c_k^2 \, |\nabla u^k(t,x)|^2 \,] \, dx,$$

and higher-order derivatives will be estimated through

$$E_\kappa(u(t)) = \sum_{|a|\le\kappa-1} E_1(\Gamma^a u(t)), \qquad \kappa = 2, 3, \dots. \tag{2.2a}$$

In order to describe the solution space, we introduce the time-independent vector fields $\Lambda = (\Lambda_1, \dots, \Lambda_7) = (\nabla, \Omega, r\partial_r)$. Define

$$H^\kappa_\Lambda(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3; \mathbb{R}^m) : \Lambda^a f \in L^2,\ |a| \le \kappa\}$$

with the norm

$$\|f\|_{H^\kappa_\Lambda} = \sum_{|a|\le\kappa} \|\Lambda^a f\|_{L^2}. \tag{2.2b}$$

Solutions will be constructed in the space $\dot{H}^\kappa_\Gamma(T)$ obtained by closing the set $C^\infty([0,T); C_0^\infty(\mathbb{R}^3, \mathbb{R}^m))$ in the norm $\sup_{0\le t<T} E_\kappa^{1/2}(u(t))$. Thus,

$$\dot{H}^\kappa_\Gamma(T) \subset \left\{ u(t,x) : \partial u(t,\cdot) \in \bigcap_{j=0}^{\kappa-1} C^j([0,T); H_\Lambda^{\kappa-1-j}) \right\}.$$

By (6.1) it will follow that $\dot{H}^\kappa_\Gamma(T) \subset C^{\kappa-2}([0,T) \times \mathbb{R}^3; \mathbb{R}^m)$.

An important intermediate role will be played by the weighted norm

$$\mathcal{X}_\kappa(u(t)) = \sum_{k=1}^m \sum_{|a|=2} \sum_{|b|\le\kappa-2} \|\langle c_k t - |x|\rangle \partial^a \Gamma^b u^k(t)\|_{L^2(\mathbb{R}^3)}, \tag{2.2c}$$

where we use the notation $\langle\rho\rangle = (1 + |\rho|^2)^{1/2}$.

3. Main result. Consider the initial value problem for a coupled nonlinear system of the form

$$\Box u = N(u, u) \tag{3.1}$$

in which the components of the quadratic nonlinearity depend on the form

$$N^k(u, v) = C^{ijk}_{\alpha\beta\gamma} \partial_\alpha u^i \partial_\beta \partial_\gamma v^j. \tag{3.2a}$$

Summation is performed over repeated indices regardless of their position, up or down. Greek indices range from 0 to 3 and Latin indices from 1 to m.

Existence of solutions depends on the energy method which requires the system to be symmetric:

$$C^{ijk}_{\alpha\beta\gamma} = C^{ikj}_{\alpha\beta\gamma} = C^{ijk}_{\alpha\gamma\beta}. \tag{3.2b}$$

The key assumption necessary for global existence is the following *null* condition which says that the self-interaction of each wave family is nonresonant:

$$C^{kkk}_{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma = 0 \quad \text{for all} \quad X \in \mathcal{N}_k, \quad k = 1, \dots, m, \tag{3.2c}$$

with the null cones

$$\mathcal{N}_k = \{X \in \mathbb{R}^4 : x_0^2 - c_k^2(x_1^2 + x_2^2 + x_3^2) = 0\}.$$

THEOREM 3.1. *Assume that the nonlinear terms in* (3.2a) *satisfy the symmetry and null conditions* (3.2b), (3.2c). *Then the initial value problem for* (3.1) *with initial data*

$$\partial_\alpha u(0) \in H^{\kappa-1}_\Lambda(\mathbb{R}^3), \quad \kappa \geq 9,$$

satisfying

$$E^{1/2}_{\kappa-2}(u(0)) \ \exp\ CE^{1/2}_\kappa(u(0)) < \varepsilon, \tag{3.3}$$

with ε sufficiently small, has a unique global solution $u \in \dot{H}^\kappa_\Gamma(T)$ for every $T > 0$. The solution satisfies the bounds

$$E^{1/2}_{\kappa-2}(u(t)) < 2\varepsilon \quad \textit{and} \quad E_\kappa(u(t)) \leq 4E_\kappa(u(0))\langle t\rangle^{C\varepsilon}.$$

Remark. We briefly discuss the case when some of the speeds are repeated. Suppose that only $\ell < m$ of the speeds $c_1 = c_{k_1} > c_{k_2} > \cdots > c_{k_\ell}$ are distinct. For $p = 1, \dots, \ell$, let $I_p = \{k : 1 \leq k \leq m,\ c_k = c_{k_p}\}$. The null condition is now extended to be

$$C^{ijk}_{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma = 0 \quad \text{for all} \quad X \in \mathcal{N}_{k_p},\ (i, j, k) \in I_p^3,\ p = 1, \dots, \ell.$$

The proof can easily be adjusted to handle this more general case.

4. Commutation and null forms. In preparation for the energy estimates, we need to consider the commutation properties of the vector fields Γ with respect to the nonlinear terms. It is necessary to verify that the null structure is preserved upon differentiation.

LEMMA 4.1. *Let u be solution u of* (3.1) *in $\dot{H}^\kappa_\Gamma(T)$. Assume that the null condition* (3.2c) *holds for the nonlinearity in* (3.2a). *Then for $|a| \leq \kappa - 1$,*

$$\Box \Gamma^a u = \sum_{b+c+d=a} N_d(\Gamma^b u, \Gamma^c u)$$

in which each N_d is a quadratic nonlinearity of the form (3.2a) *satisfying* (3.2c). *Moreover, if $b + c = a$, then $N_d = N$.*

Proof. First we note the well-known facts that

$$[\partial, \Box] = 0, \quad [\Omega, \Box] = 0, \quad [S, \Box] = -2\Box.$$

Recalling the definition (3.2a), we set

$$[\Gamma, N](u, v) = \Gamma N(u, v) - N(\Gamma u, v) - N(u, \Gamma v).$$

This is a quadratic nonlinearity of the form (3.2a). Thus, if $[\Gamma, N]$ is null for each Γ, then the result follows by induction. In fact, if $d = (d_1, \dots, d_k)$, then N_d is the k-fold commutator $N_d = [\Gamma_{d_k}, [\dots, [\Gamma_{d_1}, N]]]$.

A simple calculation shows that

$$[\partial, N](u, v) = 0 \quad \text{and} \quad [S, N](u, v) = -3N(u, v).$$

Thus, these commutators are null if N is null.

We can express the angular momentum operators as $\Omega_\lambda = \varepsilon_{\lambda\mu\nu} x_\mu \partial_\nu$, $\lambda = 1, 2, 3$, where $\varepsilon_{\lambda\mu\nu}$ is the tensor with value $+1$, -1 if $\lambda\mu\nu$ is an even, respectively, odd, permutation of 123, and with value 0 otherwise. Using this, we find that the kth component of $[\Omega_\lambda, N]$ is

$$[\Omega_\lambda, N]^k(u,v) = \widetilde{C}^{ijk}_{\alpha\beta\gamma}\partial_\alpha u^j \partial_\beta\partial_\gamma v^k$$

with

$$\widetilde{C}^{ijk}_{\alpha\beta\gamma} = [C^{ijk}_{\alpha\beta\nu}\varepsilon_{\lambda\gamma\nu} + C^{ijk}_{\nu\beta\gamma}\varepsilon_{\lambda\alpha\nu} + C^{ijk}_{\alpha\nu\gamma}\varepsilon_{\lambda\beta\nu}].$$

To see that this commutator is also null, write

$$h^k(X) = C^{kkk}_{\alpha\beta\gamma}X_\alpha X_\beta X_\gamma \quad \text{and} \quad \tilde{h}^k(X) = \widetilde{C}^{kkk}_{\alpha\beta\gamma}X_\alpha X_\beta X_\gamma.$$

Then $\tilde{h}^k(X) = -Dh^k(X)Y^\lambda$ with $Y^\lambda_\mu = \varepsilon_{\lambda\mu\nu}X_\nu$. Now the null condition says that $h^k(X) = 0$ for $X \in \mathcal{N}_k$. But since Y^λ is tangent to $\mathcal{N}_k$ at X, we have $\tilde{h}^k(X) = 0$ for $X \in \mathcal{N}_k$. This implies that $[\Omega_\lambda, N]$ is null. $\square$

5. Estimates for null forms. The utility of the null condition is captured in the next lemma. The presence of the terms with the weight $\langle c_k t - r\rangle$ in these inequalities is explained by the absence of the Lorentz rotations in our list of vector fields Γ.

LEMMA 5.1. *Suppose that the nonlinear form $N(u,v)$ defined in* (3.2a) *satisfies the null condition* (3.2c). *Set $c_0 = \min\{c_k/2 : k = 1,\dots,m\}$. For u, v, $w \in C^2([0,T]\times\mathbb{R}^3;\mathbb{R}^m)$, and $r \ge c_0 t$, we have at any point $X = (t,x)$*

$$\text{(5.1a)} \quad |C^{kkk}_{\alpha\beta\gamma}\partial_\alpha u^k \partial_\beta\partial_\gamma v^k| \le \frac{C}{\langle X\rangle}\Big[|\Gamma u^k||\partial^2 v^k| + |\partial u^k||\partial\Gamma v^k| + \langle c_k t - r\rangle|\partial u^k||\partial^2 v^k|\Big]$$

and

$$\text{(5.1b)} \quad |C^{kkk}_{\alpha\beta\gamma}\partial_\alpha u^k\partial_\beta v^k\partial_\gamma w^k| \le \frac{C}{\langle X\rangle}\Big[|\Gamma u^k||\partial v^k||\partial w^k| + |\partial u^k||\Gamma v^k||\partial w^k| + |\partial u^k||\partial v^k||\Gamma w^k| + \langle c_k t - r\rangle|\partial u^k||\partial v^k||\partial w^k|\Big]$$

in which $\langle X\rangle = (1+|X|^2)^{1/2}$.

Proof. Spatial derivatives have the decomposition

$$\nabla = \frac{x}{r}\partial_r - \frac{x}{r^2}\wedge\Omega.$$

So if we introduce the two operators $D^\pm_k = \frac{1}{2}(\partial_t \pm c_k\partial_r)$ and the null vectors $Y^\pm_k = (1, \pm x/c_k r) \in \mathcal{N}_k$, we obtain

$$\text{(5.2)} \qquad (\partial_t, \nabla) = (Y^-_k D^-_k + Y^+_k D^+_k) - \left(0, \frac{x}{r^2}\wedge\Omega\right).$$

On the other hand, if we write

$$D^+_k = \frac{c_k}{c_k t + r}S - \frac{c_k t - r}{c_k t + r}D^-_k,$$

the formula (5.2) can be transformed into

$$\partial = Y^-_k D^-_k - \frac{c_k t - r}{c_k t + r}Y^+_k D^-_k + \frac{c_k}{c_k t + r}Y^+_k S - \left(0, \frac{x}{r^2}\wedge\Omega\right).$$

Thus, we have

$$\partial \equiv Y_k^- D_k^- + R. \tag{5.3a}$$

Now, we may assume that $|X| \geq 1$, for otherwise the estimates are trivial. But then it follows that $1/r$ and $1/(c_k t + r)$ are bounded by $C/\langle X \rangle$, and as a consequence we have

$$|Ru| \leq C\langle X\rangle^{-1}[|\Gamma u| + \langle c_k t - r\rangle |\partial u|]. \tag{5.3b}$$

Using (5.3a), we have

$$\begin{aligned} C^{kkk}_{\alpha\beta\gamma}\partial_\alpha u^k \partial_\beta\partial_\gamma v^k = C^{kkk}_{\alpha\beta\gamma}[&Y^-_{k\alpha}Y^-_{k\beta}Y^-_{k\gamma}D^-_k u^k (D^-_k)^2 v^k + R_\alpha u^k \partial_\beta\partial_\gamma v^k \\ &+ Y^-_{k\alpha}D^-_k u^k R_\beta \partial_\gamma v^k + Y^-_{k\alpha}D^-_k u^k Y^-_{k\beta} D^-_k R_\gamma v^k]. \end{aligned} \tag{5.4}$$

The first term in (5.4) vanishes since N obeys the null condition, and by (5.3b) the remaining terms in (5.4) have the estimate (5.1a).

The proof of (5.1b) is similar. □

6. Sobolev inequalities. The following Sobolev inequalities involve only the angular momentum operators since we are in the nonrelativistic case. The weight $\langle ct - r\rangle$ compensates for this. We use the notation defined in (2.2a), (2.2b), (2.2c).

LEMMA 6.1. *Let $u \in \dot{H}^\kappa_\Gamma(T)$ with $\mathcal{X}_\kappa(u(t)) < \infty$.*

$$\langle r\rangle^{1/2}|\Gamma^a u(t,x)| \leq CE_\kappa^{1/2}(u(t)), \quad |a| + 2 \leq \kappa, \tag{6.1}$$

$$\langle r\rangle|\partial\Gamma^a u(t,x)| \leq CE_\kappa^{1/2}(u(t)), \quad |a| + 3 \leq \kappa, \tag{6.2}$$

$$\langle r\rangle\langle c_i t - r\rangle^{1/2}|\partial\Gamma^a u^i(t,x)| \leq C\Big[E_\kappa^{1/2}(u(t)) + \mathcal{X}_\kappa(u(t))\Big], \quad |a| + 3 \leq \kappa, \tag{6.3}$$

$$\langle r\rangle\langle c_i t - r\rangle|\partial^2\Gamma^a u^i(t,x)| \leq C\mathcal{X}_\kappa(u(t)), \quad |a| + 4 \leq \kappa. \tag{6.4}$$

Proof. This result is essentially Proposition 3.3 in [20] (see also [14]). □

7. Weighted decay estimates. The main extra step in the nonrelativistic case is to control the weighted norm $\mathcal{X}_\kappa(u(t))$. This will be accomplished in this section by a type of bootstrap argument.

LEMMA 7.1. *Let $u \in \dot{H}^\kappa_\Gamma(T)$. Then*

$$\mathcal{X}_\kappa(u(t)) \leq C\left[E_\kappa^{1/2}(u(t)) + \sum_{|a|\leq\kappa-2}\|(t+r)\,\Box\Gamma^a u(t)\|_{L^2}\right]. \tag{7.1}$$

Proof. Recall that the weighted norm involves derivatives in the form $\partial^2\Gamma^a u$. In the case when $\partial^2 = \nabla\partial$, the result was given in Lemma 3.1 of [15]. Otherwise, if $\partial^2 = \partial_t^2$, then the result is an immediate consequence of (2.10) in [15]. □

Now we assume that u solves the nonlinear PDE.

LEMMA 7.2. *Let $u \in \dot{H}^\kappa_\Gamma(T)$ be a solution of* (3.1). *Define $\kappa' = \left[\frac{\kappa-1}{2}\right] + 3$. Then for all $|a| \leq \kappa - 2$,*

$$\|(t+r)\Box\Gamma^a u(t)\|_{L^2} \leq C[\mathcal{X}_{\kappa'}(u(t))E_\kappa^{1/2}(u(t)) + \mathcal{X}_\kappa(u(t))E_{\kappa'}^{1/2}(u(t))]. \tag{7.2}$$

Proof. By Lemma 4.1, we must estimate terms of the form

$$\|(t+r)\partial\Gamma^b u^i \partial^2\Gamma^c u^j\|_{L^2},$$

but since $(t + r) \leq C\langle r\rangle\langle c_j t - r\rangle$, we will consider

$$\|\langle r\rangle\langle c_j t - r\rangle\partial\Gamma^b u^i \partial^2\Gamma^c u^j\|_{L^2} \tag{7.3}$$

with $b + c \leq a$ and $|a| \leq \kappa - 2$.

Let $m = \left[\frac{\kappa-1}{2}\right] = \kappa' - 3$. We separate two cases: either $|b| \leq m$ or $|c| \leq m - 1$. In the first case, (7.3) is estimated as follows using (6.2):

$$\|\langle r\rangle\partial\Gamma^b u^i\|_{L^\infty}\|\langle c_j t - r\rangle\partial^2\Gamma^c u^j\|_{L^2} \leq CE_{\kappa'}^{1/2}(u(t))\mathcal{X}_\kappa(u(t)).$$

Otherwise, we use (6.4) to estimate (7.3) by

$$\|\partial\Gamma^b u^i\|_{L^2}\|\langle r\rangle\langle c_j t - r\rangle\partial^2\Gamma^c u^j\|_{L^\infty} \leq CE_\kappa^{1/2}(u(t))\mathcal{X}_{\kappa'}(u(t)). \qquad \square$$

The next result gains control of the weighted norm by the energy. We distinguish two different energies, the smaller of which must remain small. In the next section, we will allow the larger energy to grow polynomially in time.

LEMMA 7.3. *Let $u \in \dot{H}^\kappa_\Gamma(T)$, $\kappa \geq 8$, be a solution of* (3.1). *Define $\mu = \kappa - 2$, and assume that*

$$\varepsilon_0 \equiv \sup_{0\leq t<T} E_\mu^{1/2}(u(t))$$

is sufficiently small. Then for $0 \leq t < T$,

$$\mathcal{X}_\mu(u(t)) \leq CE_\mu^{1/2}(u(t)) \tag{7.4a}$$

and

$$\mathcal{X}_\kappa(u(t)) \leq CE_\kappa^{1/2}(u(t)). \tag{7.4b}$$

Proof. Let $\mu' = \left[\frac{\mu-1}{2}\right] + 3$, $\mu = \kappa - 2$. Since $\mu \geq 6$, we have $\mu' \leq \mu$. Thus, by Lemmas 7.1 and 7.2, we find using our assumption that

$$\mathcal{X}_\mu(u(t)) \leq C[E_\mu^{1/2}(u(t)) + \varepsilon_0\mathcal{X}_\mu(u(t))].$$

Thus, if ε_0 is small enough, the bound (7.4a) results.

Again, since $\kappa \geq 8$, we have $\kappa' = \left[\frac{\kappa-1}{2}\right] + 3 \leq \mu = \kappa - 2$. From Lemmas 7.1 and 7.2 we now have

$$\mathcal{X}_\kappa(u(t)) \leq C[E_\kappa^{1/2}(u(t)) + \mathcal{X}_\mu(u(t))E_\kappa^{1/2}(u(t)) + \mathcal{X}_\kappa(u(t))E_\mu^{1/2}(u(t))].$$

If we apply (7.4a) and our assumption, then

$$\mathcal{X}_\kappa(u(t)) \leq C[E_\kappa^{1/2}(u(t)) + \varepsilon_0\mathcal{X}_\kappa(u(t))],$$

from which (7.4b) follows. $\square$

8. Energy estimates.

General energy method. In this section we shall complete the proof of Theorem 3.1. Assume that $u(t) \in \dot{H}^\kappa_\Gamma(T)$ is a local solution of the initial value problem for (3.1). Our task will be to show that $E_\kappa(u(t))$ remains finite for all $t \geq 0$. To do so, we will derive a pair of coupled differential inequalities for (modifications of) $E_\kappa(u(t))$ and $E_\mu(u(t))$ with $\mu = \kappa - 2$. If (3.3) holds, then $E_\mu^{1/2}(u(0)) < \varepsilon$. Suppose that T_0 is the largest time such that $E_\mu^{1/2}(u(t)) < 2\varepsilon$ for $0 \leq t < T_0$ with ε small enough so that Lemma 7.3 is valid. All of the following computations will be valid on this time interval.

Following the energy method, we have for any $\nu = 1, \dots, \kappa$,

$$E'_\nu(u(t)) = \sum_{|a| \leq \nu - 1} \int \langle \Box \Gamma^a u(t), \partial_t \Gamma^a u(t) \rangle dx,$$

and from Lemma 4.1, this takes the form

$$E'_\nu(u(t)) = \sum_{|a| \leq \nu-1} \sum_{b+c+d=a} \int \langle N_d(\Gamma^b u, \Gamma^c u), \partial_t \Gamma^a u \rangle dx. \tag{8.1}$$

Terms in (8.1) with $b = 0$, $c = a$, and $|a| = \nu - 1$ are handled with the aid of the symmetry condition (3.2b) which allows us to integrate by parts as follows. Recall that from Lemma 4.1, $N_d = N$ when $b + c = a$.

$$\begin{aligned}
\int \langle N(u, \Gamma^a u), \partial_t \Gamma^a u \rangle dx &= C^{ijk}_{\alpha\beta\gamma} \int \partial_\alpha u^i \partial_\beta \partial_\gamma \Gamma^a u^j \partial_t \Gamma^a u^k dx \\
&= C^{ijk}_{\alpha\beta\gamma} \int \partial_\gamma [\partial_\alpha u^i \partial_\beta \Gamma^a u^j \partial_t \Gamma^a u^k] dx \\
&\quad - C^{ijk}_{\alpha\beta\gamma} \int \partial_\alpha \partial_\gamma u^i \partial_\beta \Gamma^a u^j \partial_t \Gamma^a u^k dx \\
&\quad - C^{ijk}_{\alpha\beta\gamma} \int \partial_\alpha u^i \partial_\beta \Gamma^a u^j \partial_t \partial_\gamma \Gamma^a u^k dx \\
&= C^{ijk}_{\alpha\beta 0} \partial_t \int \partial_\alpha u^i \partial_\beta \Gamma^a u^j \partial_t \Gamma^a u^k dx \\
&\quad - C^{ijk}_{\alpha\beta\gamma} \int \partial_\alpha \partial_\gamma u^i \partial_\beta \Gamma^a u^j \partial_t \Gamma^a u^k dx \\
&\quad - \frac{1}{2} C^{ijk}_{\alpha\beta\gamma} \int \partial_\alpha u^i \partial_t [\partial_\beta \Gamma^a u^j \partial_\gamma \Gamma^a u^k] dx \\
&= \frac{1}{2} C^{ijk}_{\alpha\beta\gamma} \eta_{\gamma\delta} \; \partial_t \int \partial_\alpha u^i \partial_\beta \Gamma^a u^j \partial_\delta \Gamma^a u^k dx \\
&\quad - C^{ijk}_{\alpha\beta\gamma} \int \partial_\alpha \partial_\gamma u^i \partial_\beta \Gamma^a u^j \partial_t \Gamma^a u^k dx \\
&\quad + \frac{1}{2} C^{ijk}_{\alpha\beta\gamma} \int \partial_t \partial_\alpha u^i \partial_\beta \Gamma^a u^j \partial_\gamma \Gamma^a u^k dx,
\end{aligned}$$

using the symbol $\eta_{\gamma\delta} = \mathrm{Diag}[1, -1, -1, -1]$. The first term above can be absorbed into the energy as a lower order perturbation. Define

$$\widetilde{E}_\nu(u(t)) = E_\nu(u(t)) - \frac{1}{2} \sum_{|a| = \nu - 1} C^{ijk}_{\alpha\beta\gamma} \eta_{\gamma\delta} \int \partial_\alpha u^i \partial_\beta \Gamma^a u^j \partial_\delta \Gamma^a u^k dx.$$

The perturbation is bounded by $C\|\partial u\|_{L^\infty} E_\nu(u(t))$, but by (6.2), the maximum norm $\|\partial u\|_{L^\infty}$ is controlled by $E_3^{1/2}(u(t)) \le E_\mu^{1/2}(u(t)) < 2\varepsilon$. Thus, for small solutions we have

$$(1/2)E_\nu(u(t)) \le \widetilde{E}_\nu(u(t)) \le 2E_\nu(u(t)). \tag{8.2}$$

Returning to (8.1), we have derived the energy identity

$$\begin{aligned} \widetilde{E}_\nu'(u(t)) = &\sum_{|a|\le\nu-1} \sum_{\substack{b+c+d=a \\ |a|\ne\nu-1}} \int \langle N_d(\Gamma^b u, \Gamma^c u), \partial_t \Gamma^a u\rangle dx \\ &+ \sum_{|a|=\nu-1} \Bigg[\sum_{\substack{b+c=a \\ c\ne a}} \int \langle N(\Gamma^b u, \Gamma^c u), \partial_t\Gamma^a u\rangle dx. \\ &\qquad - C^{ijk}_{\alpha\beta\gamma} \int \partial_\alpha\partial_\gamma u^i \partial_\beta \Gamma^a u^j \partial_t \Gamma^a u^k dx \\ &\qquad + \frac{1}{2}\, C^{ijk}_{\alpha\beta\gamma} \int \partial_t\partial_\alpha u^i \partial_\beta\Gamma^a u^j \partial_\gamma \Gamma^a u^k dx \Bigg]. \end{aligned} \tag{8.3}$$

Higher energy. For the first series of estimates we take $\nu = \kappa$ in (8.3). We immediately obtain

$$\widetilde{E}_\kappa'(u(t)) \le C \sum_{i,j,k} \sum_{|a|\le\kappa-1} \sum_{\substack{b+c\le a \\ |c|\le\kappa-2}} \|\partial\Gamma^b u^i \partial^2\Gamma^c u^j\|_{L^2} \|\partial\Gamma^a u^k\|_{L^2}. \tag{8.4}$$

In some cases, the indices i and j have been interchanged. In the sum on the right-hand side of (8.4), we have either $|b| \le \kappa'$ or $|c| \le \kappa' - 1$ with $\kappa' = \left[\frac{\kappa}{2}\right]$. Note that since $\kappa \ge 9$, we have $\kappa' + 3 \le \kappa - 2 = \mu$. We will also use that $\langle t\rangle \le C\langle r\rangle\langle c_j t - r\rangle$.

In the first case, we estimate using (6.2) and (7.4b) as follows:

$$\begin{aligned} \|\partial\Gamma^b u^i \partial^2\Gamma^c u^j\|_{L^2} &\le C\langle t\rangle^{-1} \|\langle r\rangle \partial\Gamma^b u^i\|_{L^\infty} \|\langle c_j t - r\rangle \partial^2 \Gamma^c u^j\|_{L^2} \\ &\le C\langle t\rangle^{-1} E^{1/2}_{|b|+3}(u(t)) \mathcal{X}_\kappa(u(t)) \\ &\le C\langle t\rangle^{-1} E^{1/2}_\mu(u(t)) E^{1/2}_\kappa(u(t)). \end{aligned}$$

In the second case, we use (6.4) and then (7.4a):

$$\begin{aligned} \|\partial\Gamma^b u^i \partial^2\Gamma^c u^j\|_{L^2} &\le C\langle t\rangle^{-1} \|\partial\Gamma^b u^i\|_{L^2} \|\langle r\rangle\langle c_j t - r\rangle \partial^2\Gamma^c u^j\|_{L^\infty} \\ &\le C\langle t\rangle^{-1} E^{1/2}_\kappa(u(t)) \mathcal{X}_{|c|+4}(u(t)) \\ &\le C\langle t\rangle^{-1} E^{1/2}_\kappa(u(t)) \mathcal{X}_\mu(u(t)) \\ &\le C\langle t\rangle^{-1} E^{1/2}_\kappa(u(t)) E^{1/2}_\mu(u(t)). \end{aligned}$$

Going back to (8.4) and recalling (8.2), we have established the inequality

$$\begin{aligned} \widetilde{E}_\kappa'(u(t)) &\le C\langle t\rangle^{-1} E^{1/2}_\mu(u(t)) E_\kappa(u(t)) \\ &\le C\langle t\rangle^{-1} \widetilde{E}^{1/2}_\mu(u(t)) \widetilde{E}_\kappa(u(t)). \end{aligned} \tag{8.5}$$

Lower energy. The second series of energy estimates will exploit the null condition. We return to (8.3) now with $\nu = \mu = \kappa - 2$. The resulting integrals on the right-hand side of (8.3) will be subdivided into separate integrals over the regions $r \le c_0 t$ and $r \ge c_0 t$. Recall that the constant c_0 was defined in Lemma 5.1.

Inside the cones. On the region $r \le c_0 t$, we have that the right-hand side of (8.3) is bounded above by

$$\sum_{i,j,k} \sum_{|a| \le \mu - 1} \sum_{\substack{b+c \le a \\ |c| \le \mu - 2}} \|\partial \Gamma^b u^i \partial^2 \Gamma^c u^j \partial \Gamma^a u^k\|_{L^1(r \le c_0 t)}.$$

Since $r \le c_0 t$, we have that $\langle c_i t - r \rangle \ge C \langle t \rangle$ for each $i = 1, \ldots, m$. Thus, using (6.3), a typical term can be estimated by

$$\begin{aligned}
&C\langle t\rangle^{-3/2} \|\langle c_i t - r\rangle^{1/2} \partial \Gamma^b u^i \langle c_j t - r\rangle \partial^2 \Gamma^c u^j \partial \Gamma^a u^k\|_{L^1(r \le c_0 t)} \\
&\quad \le C\langle t\rangle^{-3/2} \|\langle c_i t - r\rangle^{1/2} \partial \Gamma^b u^i\|_{L^\infty} \|\langle c_j t - r\rangle \partial^2 \Gamma^c u^j\|_{L^2} \|\partial \Gamma^a u^k\|_{L^2} \\
&\quad \le C\langle t\rangle^{-3/2} \Big[E^{1/2}_{|b|+3}(u(t)) + \mathcal{X}_{|b|+3}(u(t))\Big] \mathcal{X}_{|c|+2}(u(t)) E^{1/2}_\mu(u(t)).
\end{aligned}$$

In the preceding, we have $|b| + 3 \le \kappa$, $|c| + 2 \le \mu$, and $|a| + 1 \le \mu$. With the aid of Lemma 7.3, we have achieved an upper bound of the form

$$C\langle t\rangle^{-3/2} E_\mu(u(t)) E^{1/2}_\kappa(u(t))$$

for the portion of the integrals over $r \le c_0 t$ on the right of (8.3).

Away from the origin. It remains to estimate the right-hand side of (8.3) for $r \ge c_0 t$.

First, we consider the nonresonant terms, i.e., those for which $(i, j, k) \ne (k, k, k)$. If $i \ne j$ and $r \ge c_0 t$, then $\langle t\rangle^{3/2} \le C\langle r\rangle \langle c_i t - r\rangle^{1/2} \langle c_j t - r\rangle$. Using (6.3) we have the estimate

$$\begin{aligned}
&\|\partial \Gamma^b u^i \partial^2 \Gamma^c u^j \partial \Gamma^a u^k\|_{L^1(r \ge c_0 t)} \\
&\quad \le C\langle t\rangle^{-3/2} \|\langle r\rangle \langle c_i t - r\rangle^{1/2} \partial \Gamma^b u^i\|_{L^\infty} \|\langle c_j t - r\rangle \partial^2 \Gamma^c u^j\|_{L^2} \|\partial \Gamma^a u^k\|_{L^2} \\
&\quad \le C\langle t\rangle^{-3/2} \Big[E^{1/2}_{|b|+3}(u(t)) + \mathcal{X}_{|b|+3}(u(t))\Big] \mathcal{X}_{|c|+2}(u(t)) E^{1/2}_{|a|+1}(u(t)) \\
&\quad \le C\langle t\rangle^{-3/2} E_\mu(u(t)) E^{1/2}_\kappa(u(t)).
\end{aligned}$$

Otherwise, if $j \ne k$, we pair the weight $\langle r\rangle \langle c_k t - r\rangle^{1/2}$ with $\partial \Gamma^a u^k$ in L^∞ to get the same upper bound.

We are left to consider the resonant terms in (8.3), i.e., $(i, j, k) = (k, k, k)$, in the region $r \ge c_0 t$. It is here, finally, where the null condition enters. An application of Lemma 5.1 yields the following upper bound for these terms:

$$\begin{aligned}
C\langle t\rangle^{-1} \sum_k \sum_{\substack{b+c=a \\ |c| \le \mu - 2}} \Big[&\|\Gamma^{b+1} u^k \partial^2 \Gamma^c u^k \partial \Gamma^a u^k\|_{L^1(r \ge c_0 t)} \\
&+ \|\partial \Gamma^b u^k \partial \Gamma^{c+1} u^k \partial \Gamma^a u^k\|_{L^1(r \ge c_0 t)} \\
&+ \|\langle c_k t - r\rangle \partial \Gamma^b u^k \partial^2 \Gamma^c u^k \partial \Gamma^a u^k\|_{L^1(r \ge c_0 t)}\Big].
\end{aligned}$$

We still need to squeeze out an additional decay factor of $\langle t\rangle^{-1/2}$.

Since $r \geq c_0 t$, we have $\langle r\rangle \geq C\langle t\rangle$. Thus, we have using (6.1) that

$$\begin{aligned}
&\|\Gamma^{b+1}u^k\partial^2\Gamma^c u^k\partial\Gamma^a u^k\|_{L^1(r\geq c_0t)}\\
&\qquad\leq C\langle t\rangle^{-1/2}\|\langle r\rangle^{1/2}\Gamma^{b+1}u^k\|_{L^\infty(r\geq c_0t)}\|\partial^2\Gamma^c u^k\|_{L^2}\|\partial\Gamma^a u^k\|_{L^2}\\
&\qquad\leq C\langle t\rangle^{-1/2}E^{1/2}_{|b|+3}(u(t))E_\mu(u(t))\\
&\qquad\leq C\langle t\rangle^{-1/2}E^{1/2}_{\kappa}(u(t))E_\mu(u(t)).
\end{aligned}$$

In a similar fashion, the second term is handled using (6.2):

$$\begin{aligned}
&\|\partial\Gamma^b u^k\partial\Gamma^{c+1}u^k\partial\Gamma^a u^k\|_{L^1(r\geq c_0t)}\\
&\qquad\leq C\langle t\rangle^{-1}\|\partial\Gamma^b u^k\|_{L^2}\|\langle r\rangle\partial\Gamma^{c+1}u^k\|_{L^\infty(r\geq c_0t)}\|\partial\Gamma^a u^k\|_{L^2}\\
&\qquad\leq C\langle t\rangle^{-1}E^{1/2}_{|c|+3}(u(t))E_\mu(u(t))\\
&\qquad\leq C\langle t\rangle^{-1}E^{1/2}_{\kappa}(u(t))E_\mu(u(t)).
\end{aligned}$$

The final set of terms are estimated using (6.2) again and (7.4a):

$$\begin{aligned}
&\|\langle c_kt-r\rangle\partial\Gamma^b u^k\partial^2\Gamma^c u^k\partial\Gamma^a u^k\|_{L^1(r\geq c_0t)}\\
&\qquad\leq C\langle t\rangle^{-1}\|\langle r\rangle\partial\Gamma^b u^k\|_{L^\infty(r\geq c_0t)}\|\langle c_kt-r\rangle\partial^2\Gamma^c u^k\|_{L^2}\|\partial\Gamma^a u^k\|_{L^2}\\
&\qquad\leq C\langle t\rangle^{-1}E^{1/2}_{|b|+3}(u(t))\mathcal{X}_{|c|+2}(u(t))E^{1/2}_\mu(u(t))\\
&\qquad\leq C\langle t\rangle^{-1}E^{1/2}_{\kappa}(u(t))E_\mu(u(t)).
\end{aligned}$$

Combining all the estimates in this subsection, we obtain, thanks to (8.2), the following inequality for the lower energy:

$$\begin{aligned}
\widetilde{E}'_\mu(u(t))&\leq C\langle t\rangle^{-3/2}E_\mu(u(t))E^{1/2}_\kappa(u(t))\\
&\leq C\langle t\rangle^{-3/2}\widetilde{E}_\mu(u(t))\widetilde{E}^{1/2}_\kappa(u(t)).
\end{aligned}\tag{8.6}$$

Conclusion of the proof. By (8.2), we have that the modified energy satisfies $\widetilde{E}^{1/2}_\mu(u(t))\leq C\varepsilon$ for $0\leq t<T_0$. So from (8.5), we find that

$$\widetilde{E}_\kappa(u(t))\leq\widetilde{E}_\kappa(u(0))\langle t\rangle^{C\varepsilon},$$

provided ε is small. Inserting this bound into (8.6) and using (8.2), we obtain

$$\begin{aligned}
(1/2)E_\mu(u(t))&\leq\widetilde{E}_\mu(u(t))\leq\widetilde{E}_\mu(u(0))\exp CI\widetilde{E}^{1/2}_\kappa(u(0))\\
&\leq 2E_\mu(u(0))\exp 2CIE^{1/2}_\kappa(u(0))<2\varepsilon^2
\end{aligned}$$

with $I=\int_0^\infty\langle s\rangle^{-3/2+C\varepsilon}ds$. With this we see that $E^{1/2}_\mu(u(t))$ remains strictly less than 2ε throughout the closed interval $0\leq t\leq T_0$. This shows that $E_\mu(u(t))$ is bounded for all time, which completes the proof of Theorem 3.1.

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