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A NOTE ON QUASILINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS

Dongbing Zha

School of Mathematical Sciences
Fudan University, Shanghai 200433, China

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Abstract. In this paper, we give an alternative proof of Alinhac's global existence result for the Cauchy problem of quasilinear wave equations with both null conditions in two space dimensions[S. Alinhac, The null condition for quasilinear wave equations in two space dimensions I, Invent. Math. 145 (2001) 597–618]. The innovation in our proof is that when applying the vector fields method to do the generalized energy estimates, we don't employ the Lorentz boost operator and only use the general space-time derivatives, spatial rotation and scaling operator.

1. Introduction.

We consider the following Cauchy problem for quasilinear wave equations in 2-D:

$$\partial_t^2 u - \Delta u = \Sigma_{0\le \alpha,\beta\le 2} g_{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u, \tag{1}$$

$$t = 0 : u = \varepsilon u_0, u_t = \varepsilon u_1. \tag{2}$$

Here $\partial = (\partial_t, \nabla), \nabla = (\partial_1, \partial_2), x_0 = t, x = (x_1, x_2)$. The coefficients $g_{\alpha\beta}$ are smooth real functions vanishing at the origin, i.e.,

$$g_{\alpha\beta}(\xi) = \Sigma g_{\alpha\beta\gamma}\xi_\gamma + \Sigma h_{\alpha\beta\gamma\delta}\xi_\gamma\xi_\delta + r_{\alpha\beta}(\xi),\ r_{\alpha\beta}(\xi) = \mathcal{O}(|\xi|^3). \tag{3}$$

In the initial conditions of (u, u_t), ε is a positive small parameter and $u_0, u_1 \in C_c^\infty(\mathbb{R}^2)$. Without loss of generality, we can assume that u_0 and u_1 are supported in $|x| \le 1$. We always assume that the following symmetric conditions hold:

$$g_{\alpha\beta\gamma} = g_{\beta\alpha\gamma} = g_{\gamma\beta\alpha}, \tag{4}$$

$$h_{\alpha\beta\gamma\delta} = h_{\beta\alpha\gamma\delta} = h_{\gamma\beta\alpha\delta} = h_{\delta\beta\gamma\alpha}. \tag{5}$$

The set $g = (g_{\alpha\beta\gamma})$ and $h = (h_{\alpha\beta\gamma\delta})$ are said to satisfy the *null* conditions if for any $x \in \mathbb{R}^2$, $\omega_0 = -1, \omega_i = x_i/r(i = 1, 2), r = |x|$, we have

$$g_{\alpha\beta\gamma}\omega_\alpha\omega_\beta\omega_\gamma = 0, \tag{6}$$

and

$$h_{\alpha\beta\gamma\delta}\omega_\alpha\omega_\beta\omega_\gamma\omega_\delta = 0, \tag{7}$$

respectively.

In Alinhac [2], for the Cauchy problem (1)–(2), the following global existence result is proved.

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Theorem 1.1. *If $g = (g_{\alpha\beta\gamma})$ and $h = (h_{\alpha\beta\gamma\delta})$ satisfy the null conditions, then for any given positive parameter ε small enough, the Cauchy problem (1)–(2) admits a unique global smooth solution.*

As pointed out in [2], Alinhac's proof of Theorem 1.1 relies on the construction of an approximate solution, combined with an energy integral method which displays the null conditions(the so called "ghost weight energy method"). In order to get suitable decay estimates, he used the Klainerman-Sobolev inequality which involves the full Lorentz invariance of the wave operator. So when applying the vector fields method, he used the general space-time derivatives, spatial rotation, scaling and Lorentz boost operator. The Lorentz boost operator is not suitable for wave systems which are not Lorentz invariant, such as nonrealistic system with multiple wave speeds(see Klainerman and Sideris [10], Sideris and Tu [16]), nonlinear wave equations on non-flat space-time (see Yang [17]), wave type equation with nonlocal nonlinear term(see Sideris and Thomases [15]) and exterior problem(see Metcalfe and Sogge [12]).

So how to remove the Lorentz boost operator in Alinhac's proof via a robust way seems interesting, and it should be the first step of treating the above problems in the 2-D case. In Hoshiga [8], a quasilinear system(containing semilinear terms) with different speeds of propagations is considered and global existence is proved under some suitable null conditions. In addition to Alinhac's ghost weight energy method, to get decay estimate for the solution of the wave equation and its derivatives, Hoshiga also used L^∞–L^∞ estimates which rely on the fundamental solution of the wave equation.

In this paper, we will give an alternative proof of Theorem 1.1. The proof does not rely on the construction of an approximate solution and the fundamental solution of the wave operator. And when applying the vector fields method, we do not employ the Lorentz boost operator. The key points of our proof are Alinhac's ghost weight energy estimates and some suitable decay estimates including Sobolev and Hardy type inequalities and Klainerman–Sideris estimates. The proof only involves energy type method and is more robust than Hoshiga's one(but the system (1) is simpler than the corresponding one in [8]). The author believe that the method employed in this manuscript will have potential applications in the problem of global existence of small solutions for nonlinear wave equations on non-flat space-time and outside a star shaped obstacle in 2-D under null conditions(in the 3-D case, corresponding problems have been treated in [17] and [12] respectively).

Global existence of small solutions to nonlinear wave equations with null condition has been a subject under active investigation for over 30 years. For the classical references in this field, we refer the reader to Klainerman [9], Christodoulou [4], Christodoulou and Klainerman [5], Alinhac [2], Agemi [1], Sideris [14], Sideris and Tu [16], and Lindblad and Rodnianski [11].

An outline of this paper is as follows. The remainder of this section will be devoted to the description of the basic notation which will be used in the sequel. In Section 2, some necessary tools used to prove Theorem 1.1 are introduced, including some properties of the null condition, Sobolev and Hardy inequalities, weighted decay estimates. Then, the proof of Theorem 1.1 will be presented in Section 3.

It is worth to point out that Sideris and Helms also obtained similar result independently and they have made significant progress on the problem of how to remove the compact support assumption for the initial data(Helms [6]).

1.1. **Notation.** Denote the spatial rotation and the scaling operator by

$$\Omega = x_1\partial_2 - x_2\partial_1 \text{ and } S = t\partial_t + x_1\partial_1 + x_2\partial_2, \tag{8}$$

respectively. We define the vector fields $Z = (\partial, \Omega, S) = (Z_1, \cdots, Z_5)$. For multi-indices $a = (a_1, \cdots, a_5)$, we denote $Z^a = Z_1^{a_1} \cdots Z_5^{a_5}$. $b \le a$ means for each $i = 1, \cdots, 5$, $b_i \le a_i$. It is easy to verify that we have the following commutation property $\big[Z, \partial\big] = \partial$. We also use the vector fields(the so called "good derivatives")

$$T_\alpha = \omega_\alpha \partial_t + \partial_\alpha, \tag{9}$$

where $\omega_0 = -1, \omega_i = x_i/r (i = 1, 2), r = |x|$. Denote $T = (T_0, T_1, T_2)$. It is easy to verify that for $r \ge t/2$,

$$|Tu| \le C\langle t + r\rangle^{-1}(|Zu| + \langle t - r\rangle|\partial u|), \tag{10}$$

where $\langle\sigma\rangle = (1 + |\sigma|^2)^{1/2}$.

The energy associated to the linear wave operator is

$$E_1(u(t)) = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2\, dx, \tag{11}$$

and the higher order energies are

$$E_k(u(t)) = \sum_{|a| \le k-1} E_1(Z^a u(t)). \tag{12}$$

We will also use the following weighted L^2 norm

$$M_k(u(t)) = \sum_{|a| \le k-2} ||\, \langle t - r\rangle\, \partial^2 Z^a u(t)||_{L^2(\mathbb{R}^2)}. \tag{13}$$

To simplify the exposition, we truncate the nonlinearity at the cubic level, but this entails no loss of generality since the higher order terms have no essential influence on the discussion of the global existence of solutions with small amplitude. The nonlinear equation which we consider can be written as the following form

$$\Box u = N_1(u, u) + N_2(u, u, u), \tag{14}$$

where

$$N_1(u, v) = g_{\alpha\beta\gamma}\partial_\gamma u \partial_\alpha\partial_\beta v, \tag{15}$$

$$N_2(u, v, w) = h_{\alpha\beta\gamma\delta}\partial_\gamma u\partial_\delta v\partial_\alpha\partial_\beta w. \tag{16}$$

2. Some preliminaries.

2.1. The null condition.

Lemma 2.1. *If $g = (g^{\alpha\beta\gamma})$ and $h = (h^{\alpha\beta\gamma\delta})$ satisfy the null conditions, then for any four smooth function u, v, w, z, we have*

$$|g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta v| \le C\big(|Tu||\partial^2 v| + |\partial u||T\partial v|\big), \tag{17}$$

$$|h_{\alpha\beta\gamma\delta}\partial_\gamma u\partial_\delta v\partial_\alpha\partial_\beta w| \le C\big(|Tu||\partial v||\partial^2 w| + |\partial u||Tv||\partial^2 w| + |\partial u||\partial v||T\partial w|\big), \tag{18}$$

$$|g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha v\partial_\beta w| \le C\big(|Tu||\partial v||\partial w| + |\partial u||Tv||\partial w| + |\partial u||\partial v||Tw|\big), \tag{19}$$

$$\begin{aligned}|h_{\alpha\beta\gamma\delta}\partial_\gamma u\partial_\delta v\partial_\alpha w\partial_\beta z| \le C\big(&|Tu||\partial v||\partial w||\partial z| + |\partial u||Tv||\partial w||\partial z| + |\partial u||\partial v||Tw||\partial z| \\ &+ |\partial u||\partial v||\partial w||Tz|\big).\end{aligned} \tag{20}$$

Proof. The above inequalities can be verified easily by the definition of good derivatives (9) and the null conditions (6) and (7). □

Combination of Lemma 2.1 and the decay estimates (10) gives

Lemma 2.2. *If $g = (g^{\alpha\beta\gamma})$ and $h = (h^{\alpha\beta\gamma\delta})$ satisfy the null conditions, then for any $r \geq t/2$,*

$$|g_{\alpha\beta\gamma}\partial_\gamma u \partial_\alpha \partial_\beta v| \leq C\langle t+r\rangle^{-1}\big(|Zu||\partial^2 v| + |\partial u||Z\partial v| + \langle t-r\rangle\, |\partial u||\partial^2 v|\big), \tag{21}$$

$$\begin{aligned} |h_{\alpha\beta\gamma\delta}\partial_\gamma u \partial_\delta v \partial_\alpha \partial_\beta w| \leq C\langle t+r\rangle^{-1}\big(&|Zu||\partial v||\partial^2 w| + |\partial u||Zv||\partial^2 w| + |\partial u||\partial v||Z\partial w| \\ &+ \langle t-r\rangle\, |\partial u||\partial v||\partial^2 w|\big), \end{aligned} \tag{22}$$

$$\begin{aligned} |g_{\alpha\beta\gamma}\partial_\gamma u \partial_\alpha v \partial_\beta w| \leq C\langle t+r\rangle^{-1}\big(&|Zu||\partial v||\partial w| + |\partial u||Zv||\partial w| + |\partial u||\partial v||Zw| \\ &+ \langle t-r\rangle\, |\partial u||\partial v||\partial w|\big), \end{aligned} \tag{23}$$

$$\begin{aligned} |h_{\alpha\beta\gamma\delta}\partial_\gamma u \partial_\delta v \partial_\alpha w \partial_\beta z| \leq C\langle t+r\rangle^{-1}\big(&|Zu||\partial v||\partial w||\partial z| + |\partial u||Zv||\partial w||\partial z| \\ &+ |\partial u||\partial v||Zw||\partial z| + |\partial u||\partial v||\partial w||Zz| \\ &+ \langle t-r\rangle\, |\partial u||\partial v||\partial w||\partial z|\big). \end{aligned} \tag{24}$$

Lemma 2.3. *Let u be a solution of (14) and assume that the null conditions (6), (7) hold for the nonlinearities (15), (16) respectively. Then for any multi-indices $a = (a_1, \cdots, a_5)$, we have*

$$\Box Z^a u = \sum_{b+c+d=a} N_{1d}(Z^b u, Z^c u) + \sum_{b+c+d+e=a} N_{2e}(Z^b u, Z^c u, Z^d u), \tag{25}$$

where each N_{1d} is a quadratic nonlinearity of the form (15) satisfying the null condition (6) and each N_{2e} is a cubic nonlinearity of the form (16) satisfying the null condition (7). Moreover, if $b + c = a$, then $N_{1d} = N_1$; if $b + c + d = a$, then $N_{2e} = N_2$.

Proof. See Hörmander [7], Lemma 6.6.5 and Sideris [16], Lemma 4.1. □

2.2. **Sobolev and Hardy inequalities.**

Lemma 2.4. *For $u \in C_c^\infty(\mathbb{R}^2)$, $r = |x|$, we have the following weighted Sobolev inequalities:*

$$r^{1/2}|u(x)| \leq C \sum_{|a|\leq 1} ||\Omega^a u||_{L^2(\mathbb{R}^2)} + C \sum_{|a|\leq 1} ||\partial_r \Omega^a u||_{L^2(\mathbb{R}^2)}, \tag{26}$$

$$r^{1/2}\langle t-r\rangle^{1/2}|u(x)| \leq C \sum_{|a|\leq 1} ||\Omega^a u||_{L^2(\mathbb{R}^2)} + C \sum_{|a|\leq 1} ||\langle t-|y|\rangle\, \partial_r \Omega^a u||_{L^2_y(\mathbb{R}^2)}, \tag{27}$$

$$\begin{aligned} r^{1/2}\langle t-r\rangle\, |u(x)| \leq C &\sum_{|a|\leq 1} ||\langle t-|y|\rangle\, \Omega^a u||_{L^2_y(\mathbb{R}^2)} \\ &+ C \sum_{|a|\leq 1} ||\langle t-|y|\rangle\, \partial_r \Omega^a u||_{L^2_y(\mathbb{R}^2)}. \end{aligned} \tag{28}$$

Proof. See Sideris [13], Lemma 1. □

Now we give the higher order version of Lemma 2.4.

Lemma 2.5. *For $u \in C^\infty([0,T] \times \mathbb{R}^2)$, we have*

$$\langle r\rangle^{1/2}|\partial Z^a u(t,x)| \leq C E_k^{1/2}(u(t)),\ |a| + 3 \leq k, \tag{29}$$

$$\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}|\partial Z^a u(t,x)| \leq C(E_k^{1/2}(u(t)) + M_k(u(t))),\ |a| + 3 \leq k, \tag{30}$$

$$\langle r\rangle^{1/2}\langle t-r\rangle\, |\partial^2 Z^a u(t,x)| \leq C M_k(u(t)),\ |a| + 4 \leq k. \tag{31}$$

Proof. For $r \geq 1$, (29), (30) and (31) result from application of (26), (27) and (28) respectively to $\partial Z^a u$ and the commutation property $[Z, \partial] = \partial$. For $r \leq 1$, (29) comes directly from the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. Now we consider (30) for $r \leq 1$. It suffices to prove

$$\langle t-r\rangle^{1/2} |\partial Z^a u(t,x)| \leq C(E_k^{1/2}(u(t)) + M_k(u(t))). \tag{32}$$

If $t \leq 4$, then (32) is an immediate consequence of the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. If $t > 4$, then $\langle t-r\rangle \sim \langle t\rangle$ holds. Define a smooth cut-off function

$$\chi(r) = \begin{cases} 1, & r < 1, \\ 0, & r > 2. \end{cases} \tag{33}$$

It follows from the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and Poincaré inequality that

$$\begin{aligned} \langle t-r\rangle^{1/2} |\partial Z^a u(t,x)| &\leq C \langle t\rangle^{1/2} \chi(r) |\partial Z^a u(t,x)| \\ &\leq C \langle t\rangle^{1/2} ||\chi \partial Z^a u||_{H^2(\mathbb{R}^2)} \\ &\leq C \langle t\rangle^{1/2} \big(||\partial Z^a u||_{L^\infty_x(1\leq|x|\leq 2)} + \sum_{|b|\leq 1} ||\nabla \nabla^b \partial Z^a u||_{L^2(|x|\leq 2)}\big) \\ &\leq C||r^{1/2} \langle t-r\rangle^{1/2} \partial Z^a u||_{L^\infty(1\leq|x|\leq 2)} \\ &\quad + C \sum_{|b|\leq 1} || \langle t-r\rangle \nabla \nabla^b \partial Z^a u||_{L^2(|x|\leq 2)} \\ &\leq C(E_k^{1/2}(u(t)) + M_k(u(t))), \end{aligned} \tag{34}$$

where we have employed (27) at the last inequality of (34). (31) for $r \leq 1$ can be proved similarly. The proof is completed. $\square$

Lemma 2.6. *If $u = u(t,x)$ is smooth and supported in $|x| \leq t+1$, then we have*

$$|| \langle t-r\rangle^{-1} u(t,x)||_{L^2_x(\mathbb{R}^2)} \leq C||\partial_r u||_{L^2(\mathbb{R}^2)}, \tag{35}$$

and

$$|| \langle t-r\rangle^{-1} u(t,x)||_{L^\infty_x(\mathbb{R}^2)} \leq C\big(\sum_{|a|\leq 1} ||\nabla \Omega^a u||_{L^2(\mathbb{R}^2)} + ||\nabla^2 u||_{L^2(\mathbb{R}^2)}\big). \tag{36}$$

Proof. For the Hardy type inequality (35), see Alinhac [3], Lemma 2.2. For (36), on the region $r \geq 1$, it follows from (26) and (35) that

$$\begin{aligned} &|| \langle t-r\rangle^{-1} u(t,x)||_{L^\infty(r\geq 1)} \\ &\leq ||r^{1/2} \langle t-r\rangle^{-1} u(t,x)||_{L^\infty(r\geq 1)} \\ &\leq C \sum_{|a|\leq 1} \big(|| \langle t-r\rangle^{-1} \Omega^a u||_{L^2(\mathbb{R}^2)} + ||\partial_r(\langle t-r\rangle^{-1} \Omega^a u)||_{L^2(\mathbb{R}^2)}\big) \\ &\leq C \sum_{|a|\leq 1} ||\partial_r \Omega^a u||_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{37}$$

On the region $r \leq 1$, by the Sobolev embedding $H^2(B_1) \hookrightarrow L^\infty(B_1)$ and (35), we have

$$\begin{aligned}
&|| \langle t-r\rangle^{-1} u(t,x)||_{L^\infty(r\leq 1)} \\
&\leq C(2+t)^{-1}||u||_{L^\infty(r\leq 1)} \\
&\leq C(2+t)^{-1}\big(||u||_{L^2(r\leq 1)} + ||\nabla u||_{L^2(r\leq 1)} + ||\nabla^2 u||_{L^2(r\leq 1)}\big) \\
&\leq C|| \langle t-r\rangle^{-1} u(t,x)||_{L^2_x(\mathbb{R}^2)} + C\big(||\nabla u||_{L^2} + ||\nabla^2 u||_{L^2}\big) \\
&\leq C\big(||\nabla u||_{L^2} + ||\nabla^2 u||_{L^2}\big). \qquad (38)
\end{aligned}$$

□

The higher order version of Lemma 2.6 is the following:

Lemma 2.7. *For $u \in C^\infty([0,T]\times\mathbb{R}^2)$ which is supported in $|x| \leq t+1$, we have*

$$|| \langle t-r\rangle^{-1} Z^a u(t,x)||_{L^2_x(\mathbb{R}^2)} \leq CE_k^{1/2}(u(t)),\ |a|+1 \leq k, \qquad (39)$$

$$|| \langle t-r\rangle^{-1} Z^a u(t,x)||_{L^\infty_x(\mathbb{R}^2)} \leq CE_k^{1/2}(u(t)),\ |a|+2 \leq k. \qquad (40)$$

2.3. **Weighted decay estimates.** In this section, we will give the control of the weighted L^2 norm $M_k(u(t))$.

Lemma 2.8. *(Klainerman–Sideris Estimates) Let $u \in C^\infty([0,T]\times\mathbb{R}^2)$. Then*

$$M_k\big(u(t)\big) \leq C\big(E_k^{1/2}\big(u(t)\big) + \sum_{|a|\leq k-2} ||(t+r)\Box Z^a u(t)||_{L^2}\big). \qquad (41)$$

Proof. See Klainerman and Sideris [10], Lemma 3.1. Note that their proof is obvious valid for all spatial dimension $n \geq 2$. □

Next we estimate the nonlinear terms on the right-hand side of (41).

Lemma 2.9. *If $u \in C^\infty([0,T]\times\mathbb{R}^2)$ satisfies (14), then we have*

$$\begin{aligned}
M_k(u(t)) \leq & CE_k^{1/2}(u(t)) + C\Big(E_{k'}(u(t))E_k^{1/2}(u(t)) + M_{k'}(u(t))E_k^{1/2}(u(t)) \\
&\qquad + M_k(u(t))E_{k'}^{1/2}(u(t))\Big) \\
&+ C\Big(M_{k'}(u(t))E_{k'}^{1/2}(u(t))E_k^{1/2}(u(t)) + M_k(u(t))E_{k'}(u(t))\Big), \qquad (42)
\end{aligned}$$

where $k' = [\frac{k-2}{2}] + 5$.

Proof. For all $|a| \leq k-2$, it follows from Lemma 2.3 that

$$\begin{aligned}
||(t+r)\Box Z^a u(t)||_{L^2} \leq & C \sum_{b+c+d=a} ||(t+r)N_{1d}(Z^b u, Z^c u)||_{L^2} \\
&+ C \sum_{b+c+d+e=a} ||(t+r)N_{2e}(Z^b u, Z^c u, Z^d u)||_{L^2}. \qquad (43)
\end{aligned}$$

We first estimate the first collection on the right hand side of (43). On the region $r \leq t/2$, noting that $\langle t-r\rangle \sim \langle t\rangle$, we have

$$||(t+r)N_{1d}(Z^b u, Z^c u)||_{L^2(r\leq t/2)} \leq C|| \langle t-r\rangle \partial Z^b u \partial^2 Z^c u||_{L^2}. \qquad (44)$$

Let $m = [\frac{k-2}{2}] + 1 = k' - 4$. We have either $|b| \leq m$ or $|c| \leq m$. In the first case, (44) can be estimated as follows using (29):

$$|| \langle r\rangle^{1/2} \partial Z^b u||_{L^\infty} || \langle t-r\rangle \partial^2 Z^c u||_{L^2} \leq CE_{k'}^{1/2}(u(t))M_k(u(t)). \qquad (45)$$

Otherwise, we use (31) to estimate (44) by

$$||\partial Z^b u||_{L^2} || \langle r\rangle^{1/2} \langle t-r\rangle \partial^2 Z^c u||_{L^\infty} \le C E_k^{1/2}(u(t)) M_{k'}(u(t)). \tag{46}$$

On the region $r \ge t/2$, it follows from Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned} &||(t+r)N_{1d}(Z^b u, Z^c u)||_{L^2(r\ge t/2)} \\ &\le C||Z^{b+1}u\partial^2 Z^c u||_{L^2(r\ge t/2)} + C||\partial Z^b u \partial Z^{c+1} u||_{L^2(r\ge t/2)} \\ &+ C|| \langle t-r\rangle \partial Z^b u \partial^2 Z^c u||_{L^2(r\ge t/2)}. \end{aligned} \tag{47}$$

For the first term on the right hand side of (47), when $|b| \le m$, it follows from (40) that

$$\begin{aligned} ||Z^{b+1}u\partial^2 Z^c u||_{L^2(r\ge t/2)} &\le C|| \langle t-r\rangle^{-1} Z^{b+1}u||_{L^\infty} || \langle t-r\rangle \partial^2 Z^c u||_{L^2} \\ &\le C E_{k'}^{1/2}(u(t)) M_k(u(t)). \end{aligned} \tag{48}$$

When $|c| \le m$, by (39), we have

$$\begin{aligned} ||Z^{b+1}u\partial^2 Z^c u||_{L^2} &\le C|| \langle t-r\rangle^{-1} Z^{b+1}u||_{L^2} || \langle t-r\rangle \partial^2 Z^c u||_{L^\infty} \\ &\le C E_k^{1/2}(u(t)) M_{k'}(u(t)). \end{aligned} \tag{49}$$

For the second and third term on the right hand side of (47), similar dichotomy argument can derive the following bound:

$$E_k^{1/2}(u(t)) E_{k'}^{1/2}(u(t)) + E_k^{1/2}(u(t)) M_{k'}(u(t)) + E_{k'}^{1/2}(u(t)) M_k(u(t)). \tag{50}$$

Now we estimate the second collection on the right hand side of (43). We must estimate terms of the form

$$||(t+r)\partial Z^b u \partial Z^c u \partial^2 Z^d u||_{L^2}. \tag{51}$$

Noting that $(t+r) \le C \langle r\rangle \langle t-r\rangle$, we have that

$$\begin{aligned} &||(t+r)\partial Z^b u \partial Z^c u \partial^2 Z^d u||_{L^2} \le || \langle r\rangle \langle t-r\rangle \partial Z^b u \partial Z^c u \partial^2 Z^d u||_{L^2} \\ &\le C \begin{cases} || \langle r\rangle^{1/2} \partial Z^b u||_{L^\infty} || \langle r\rangle^{1/2} \partial Z^c u||_{L^\infty} || \langle t-r\rangle \partial^2 Z^d u||_{L^2}, & |b|, |c| \le m, \\ || \langle r\rangle^{1/2} \partial Z^b u|||_{L^\infty} ||\partial Z^c u||_{L^2} || \langle r\rangle^{1/2} \langle t-r\rangle \partial^2 Z^d u||_{L^\infty}, & |b|, |d| \le m, \\ ||\partial Z^b u||_{L^2} || \langle r\rangle^{1/2} \partial Z^c u||_{L^\infty} || \langle r\rangle^{1/2} \langle t-r\rangle \partial^2 Z^d u||_{L^\infty}, & |c|, |d| \le m. \end{cases} \end{aligned} \tag{52}$$

In the first case, owing to (29), we have the upper bound

$$E_{k'}(u(t)) M_k(u(t)). \tag{53}$$

In the second and the third case, the combination of (29) with (31) gives the upper bound

$$E_{k'}^{1/2}(u(t)) E_k^{1/2}(u(t)) M_{k'}(u(t)). \tag{54}$$

□

Lemma 2.10. *Let $u \in C^\infty([0,T] \times \mathbb{R}^2)$ be a solution of system (14) and $k \ge 10$. Define $\mu = k-3$, and assume that*

$$\varepsilon_0 \equiv \sup_{0\le t<T} E_\mu^{1/2}(u(t)) \tag{55}$$

is sufficiently small. Then for $0 \le t \le T$,

$$M_\mu(u(t)) \le C E_\mu^{1/2}(u(t)), \tag{56}$$

$$M_k(u(t)) \le C E_k^{1/2}(u(t)). \tag{57}$$

Proof. See the proof of Lemma 7.3 in Sideris and Tu [16]. □

3. **Proof of Theorem 1.1.** In this section we shall complete the proof of Theorem 1.1. Assume that $u(t)$ is a local smooth solution of the initial value problem (14)–(2) on $[0, T]$. Let $k \ge 20, \mu = k - 3$. We will prove that for any $t \in [0, T]$, it holds that (i) $E_\mu(u(t) \le A_1\varepsilon^2$ and (ii) $E_k(u(t) \le A_2\varepsilon^2 \langle t\rangle^{\widetilde{c}\varepsilon}$, where the constant $A_1, A_2, \widetilde{c}$ will be determined later. There are two key steps in the proof. First, under the assumption (i) for the lower order energy, we will prove higher order energy estimate (ii). Second, under the assumption (i) and (ii), we will show that (i) holds with A_1 replaced by $A_1/2$. To accomplish this bootstrap argument, we will derive a pair of coupled differential inequalities for the lower order energy $E_\mu(u(t)$ and the (modified) higher order energy $E_k(u(t)$.

3.1. **High order energy.** Following Alinhac [2], we will use the ghost weight energy method. Let $\sigma = t - r$, $q(\sigma) = \arctan\sigma, q'(\sigma) = \frac{1}{1+\sigma^2} = \langle t - r\rangle^{-2}$. It follows from Lemma 2.3 that

$$\begin{aligned}
&\sum_{|a|\le k-1} \int \langle e^{-q(\sigma)}\partial_t Z^a u, \Box Z^a u\rangle\, dx \\
&= \sum_{|a|\le k-1} \sum_{b+c+d=a} \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle\, dx \\
&+ \sum_{|a|\le k-1} \sum_{b+c+d+e=a} \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u\rangle\, dx.
\end{aligned} \tag{58}$$

As we are now treating quasilinear system, for the right hand side of (58), $|a| = k-1$, special attention should be paid to terms with $b = a$ or $c = a$ for the first collection of terms and $d = a$ for the second collection of terms. So we can rewrite (58) as

$$\begin{aligned}
&\sum_{|a|\le k-1} \int \langle e^{-q(\sigma)}\partial_t Z^a u, \Box Z^a u\rangle\, dx \\
&= \sum_{|a|= k-1} \Big[\sum_{\substack{b+c+d=a\\ b,c\ne a}} \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle\, dx \\
&+ \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_1(Z^a u, u)\rangle\, dx \\
&+ \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_1(u, Z^a u)\rangle\, dx \Big] \\
&+ \sum_{|a|\le k-2} \sum_{b+c+d=a} \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle\, dx \\
&+ \sum_{|a|= k-1} \Big[\sum_{\substack{b+c+d=a\\ d\ne a}} \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u)\rangle\, dx
\end{aligned}$$

$$+ \int \langle e^{-q(\sigma)} \partial_t Z^a u, N_2(u, u, Z^a u) \rangle \, dx \Big]$$
$$+ \sum_{|a| \le k-2} \sum_{b+c+d+e=a} \int \langle e^{-q(\sigma)} \partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u) \rangle \, dx. \tag{59}$$

By the integration by parts argument, we have

$$\int \langle e^{-q(\sigma)} \partial_t Z^a u, \Box Z^a u \rangle \, dx$$
$$= \frac{1}{2} \frac{d}{dt} \int e^{-q(\sigma)} |\partial Z^a u|^2 \, dx + \frac{1}{2} \int e^{-q(\sigma)} \langle t - r \rangle^{-2} |T Z^a u|^2 \, dx. \tag{60}$$

Since q is bounded, there exists $c > 1$ such that

$$c^{-1} \le e^{-q(\sigma)} \le c. \tag{61}$$

Denote

$$\overline{E}_k(u(t)) = \frac{1}{2} \sum_{|a| \le k-1} \int_{\mathbb{R}^2} e^{-q(\sigma)} |\partial Z^a u(t, x)|^2 \, dx. \tag{62}$$

We have

$$c^{-1} E_k(u(t)) \le \overline{E}_k(u(t)) \le c E_k(u(t)). \tag{63}$$

It follows from the symmetric conditions (4) and the integration by parts argument that

$$\begin{aligned}
\int \langle e^{-q(\sigma)} \partial_t Z^a u, N_1(u, Z^a u) \rangle \, dx &= g_{\alpha\beta\gamma} \int e^{-q(\sigma)} \partial_t Z^a u \partial_\alpha u \partial_\beta \partial_\gamma Z^a u \, dx \\
&= g_{\alpha\beta\gamma} \int \partial_\gamma (e^{-q(\sigma)} \partial_t Z^a u \partial_\alpha u \partial_\beta Z^a u) \, dx \\
&\quad - g_{\alpha\beta\gamma} \int e^{-q(\sigma)} \partial_t Z^a u \partial_\alpha \partial_\gamma u \partial_\beta Z^a u \, dx \\
&\quad - g_{\alpha\beta\gamma} \int e^{-q(\sigma)} \partial_t \partial_\gamma Z^a u \partial_\alpha u \partial_\beta Z^a u \, dx \\
&\quad + g_{\alpha\beta\gamma} \int e^{-q(\sigma)} q'(\sigma) \partial_\gamma \sigma \partial_t Z^a u \partial_\alpha u \partial_\beta Z^a u \, dx \\
&= g_{\alpha\beta 0} \partial_t \int e^{-q(\sigma)} \partial_t Z^a u \partial_\alpha u \partial_\beta Z^a u \, dx \\
&\quad - g_{\alpha\beta\gamma} \int e^{-q(\sigma)} \partial_t Z^a u \partial_\alpha \partial_\gamma u \partial_\beta Z^a u \, dx \\
&\quad - \frac{1}{2} g_{\alpha\beta\gamma} \partial_t \int e^{-q(\sigma)} \partial_\gamma Z^a u \partial_\alpha u \partial_\beta Z^a u \, dx \\
&\quad + \frac{1}{2} g_{\alpha\beta\gamma} \int e^{-q(\sigma)} \partial_\gamma Z^a u \partial_\alpha \partial_t u \partial_\beta Z^a u \, dx \\
&\quad - \frac{1}{2} g_{\alpha\beta\gamma} \int e^{-q(\sigma)} q'(\sigma) \partial_\gamma Z^a u \partial_\alpha u \partial_\beta Z^a u \, dx
\end{aligned}$$

$$
\begin{aligned}
&+ g_{\alpha\beta\gamma}\int e^{-q(\sigma)}q'(\sigma)\partial_\gamma\sigma\partial_t Z^a u\partial_\alpha u\partial_\beta Z^a u\,dx\\
&= \frac{1}{2}g_{\alpha\beta\gamma}\eta_{\gamma\nu}\partial_t\int e^{-q(\sigma)}\partial_\nu Z^a u\partial_\alpha u\partial_\beta Z^a u\,dx\\
&- g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_t Z^a u\partial_\alpha\partial_\gamma u\partial_\beta Z^a u\,dx\\
&+ \frac{1}{2}g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_\gamma Z^a u\partial_\alpha\partial_t u\partial_\beta Z^a u\,dx\\
&- \frac{3}{2}g_{\alpha\beta\gamma}\int e^{-q(\sigma)}q'(\sigma)\partial_\gamma Z^a u\partial_\alpha u\partial_\beta Z^a u\,dx\\
&+ g_{\alpha\beta\gamma}\int e^{-q(\sigma)}q'(\sigma)T_\gamma Z^a u\partial_\alpha u\partial_\beta Z^a u\,dx,
\end{aligned}
\tag{64}
$$

where the symbol $\eta_{\gamma\nu} = \mathrm{diag}[1,-1,-1]$. Similarly, via the symmetric conditions (5), we can get

$$
\begin{aligned}
\int\langle e^{-q(\sigma)}\partial_t Z^a u, N_2(u,u,Z^a u)\rangle\,dx &= h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}\partial_t Z^a u\partial_\alpha u\partial_\beta u\partial_\gamma\partial_\delta Z^a u\,dx\\
&= \frac{1}{2}h_{\alpha\beta\gamma\delta}\eta_{\gamma\nu}\partial_t\int (e^{-q(\sigma)}\partial_\nu Z^a u\partial_\alpha u\partial_\beta u\partial_\delta Z^a u)\,dx\\
&- h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}\partial_t Z^a u\partial_\gamma(\partial_\alpha u\partial_\beta u)\partial_\delta Z^a u\,dx\\
&+ \frac{1}{2}h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}\partial_\gamma Z^a u\partial_t(\partial_\alpha u\partial_\beta u)\partial_\delta Z^a u\,dx\\
&- \frac{3}{2}h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}q'(\sigma)\partial_\gamma Z^a u\partial_\alpha u\partial_\beta u\partial_\delta Z^a u\,dx\\
&+ h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}q'(\sigma)T_\gamma Z^a u\partial_\alpha u\partial_\beta u\partial_\delta Z^a u\,dx.
\end{aligned}
\tag{65}
$$

Define the perturbed energy

$$
\begin{aligned}
&\widetilde{E}_k(u(t))\\
&= \sum_{|a|\le k-1}\int e^{-q(\sigma)}|\partial Z^a u|^2\,dx - \frac{1}{2}\sum_{|a|=k-1}\int g_{\alpha\beta\gamma}\eta_{\gamma\nu}e^{-q(\sigma)}\partial_\nu Z^a u\partial_\alpha u\partial_\beta Z^a u\,dx\\
&- \frac{1}{2}\sum_{|a|=k-1}\int h_{\alpha\beta\gamma\delta}\eta_{\gamma\nu}e^{-q(\sigma)}\partial_\nu Z^a u\partial_\alpha u\partial_\beta u\partial_\delta Z^a u\,dx.
\end{aligned}
\tag{66}
$$

Noting that $||\partial u||_{L^\infty} \le CE_3^{1/2}(u(t))$, for small solutions, by (63) we have

$$
(2c)^{-1}E_k(u(t)) \le \widetilde{E}_k(u(t)) \le 2cE_k(u(t)). \tag{67}
$$

Noting the symmetric condition (4), we have

$$
\begin{aligned}
\int\langle e^{-q(\sigma)}\partial_t Z^a u, N_1(Z^a u,u)\,dx &= g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_t Z^a u\partial_\alpha Z^a u\partial_\beta\partial_\gamma u\,dx\\
&= g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_t Z^a u\partial_\alpha\partial_\gamma u\partial_\beta Z^a u\,dx.
\end{aligned}
\tag{68}
$$

Returning to (59), we have derived the following energy identity:

$$
\begin{aligned}
\frac{d}{dt}\widetilde{E}_k(u(t)) &+ \frac{1}{2}\sum_{|a|\le k-1} ||e^{-q(\sigma)/2}\langle t-r\rangle^{-1} TZ^a u||_{L^2}^2 \\
&= \sum_{|a|=k-1}\Big[\sum_{\substack{b+c+d=a\\ b,c\neq a}}\int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle\, dx \\
&+ \frac{1}{2}g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_\gamma Z^a u \partial_\alpha \partial_t u \partial_\beta Z^a u\, dx \\
&- \frac{3}{2}g_{\alpha\beta\gamma}\int e^{-q(\sigma)}q'(\sigma)\partial_\gamma Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx \\
&+ g_{\alpha\beta\gamma}\int e^{-q(\sigma)}q'(\sigma)T_\gamma Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx\Big] \\
&+ \sum_{|a|\le k-2}\sum_{b+c+d=a}\int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle\, dx \\
&+ \sum_{|a|=k-1}\Big[\sum_{\substack{b+c+d+e=a\\ d\neq a}}\int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u)\rangle\, dx \\
&- h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}\partial_t Z^a u \partial_\gamma(\partial_\alpha u \partial_\beta u)\partial_\delta Z^a u\, dx \\
&+ \frac{1}{2}h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}\partial_\gamma Z^a u \partial_t(\partial_\alpha u \partial_\beta u)\partial_\delta Z^a u\, dx \\
&- \frac{3}{2}h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}q'(\sigma)\partial_\gamma Z^a u \partial_\alpha u \partial_\beta u \partial_\delta Z^a u\, dx \\
&+ h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}q'(\sigma)T_\gamma Z^a u \partial_\alpha u \partial_\beta u \partial_\delta Z^a u\, dx\Big] \\
&+ \sum_{|a|\le k-2}\sum_{b+c+d+e=a}\int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u)\rangle\, dx. \qquad (69)
\end{aligned}
$$

The second term on the left hand side of (69) is called the ghost weight energy, which play a key role in the control of the highest generalized derivatives in the nonlinearity after applying the null condition.

Now we can estimate the terms on the right hand side of (69). All terms corresponding to the cubic nonlinearity are bounded above by

$$
\sum_{|a|\le k-1}\sum_{\substack{b+c+d\le a\\ |d|\le k-2}} ||\partial Z^a u \partial Z^b u \partial Z^c u \partial^2 Z^d u||_{L^1} + ||\langle t-r\rangle^{-1}\partial Z^a u \partial u||_{L^2}^2. \qquad (70)
$$

Note that $\langle t\rangle \le C\langle r\rangle\langle t-r\rangle$. Let $m = [\frac{k}{2}]+1$. It follows from (30) that

$$
\begin{aligned}
&||\partial Z^a u \partial Z^b u \partial Z^c u \partial^2 Z^d u||_{L^1} \le C\langle t\rangle^{-1}\cdot \\
&\begin{cases}
||\partial Z^a u||_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}\partial Z^b u||_{L^\infty}||\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}\partial Z^c u||_{L^\infty}||\partial^2 Z^d u||_{L^2}, |b|,|c|\le m,\\
||\partial Z^a u||_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}\partial Z^b u||_{L^\infty}||\partial Z^c u||_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}\partial^2 Z^d u||_{L^\infty}, |b|,|d|\le m,\\
||\partial Z^a u||_{L^2}||\partial Z^b u||_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}\partial Z^c u||_{L^\infty}||\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}\partial^2 Z^d u||_{L^\infty}, |c|,|d|\le m,
\end{cases}\\
&\le C\langle t\rangle^{-1}.
\end{aligned}
$$

$$\begin{cases} E_k(u(t))\big(E^{1/2}_{|b|+3}(u(t))+M_{|b|+3}(u(t))\big)\big(E^{1/2}_{|c|+3}(u(t))+M_{|c|+3}(u(t))\big), |b|,|c|\le m,\\ E_k(u(t))\big(E^{1/2}_{|b|+3}(u(t))+M_{|b|+3}(u(t))\big)\big(E^{1/2}_{|d|+4}(u(t))+M_{|d|+4}(u(t))\big), |b|,|d|\le m,\\ E_k(u(t))\big(E^{1/2}_{|c|+3}(u(t))+M_{|c|+3}(u(t))\big)\big(E^{1/2}_{|d|+4}(u(t))+M_{|d|+4}(u(t))\big), |c|,|d|\le m,\end{cases}$$
$$\le C\langle t\rangle^{-1}E_\mu(u(t))E_k(u(t)). \tag{71}$$

Similarly, we have

$$\begin{aligned} ||\langle t-r\rangle^{-1}\partial Z^a u\partial u||^2_{L^2} &\le C\langle t\rangle^{-1}||\partial Z^a u||^2_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}\partial u||^2_{L^\infty}\\ &\le C\langle t\rangle^{-1}E_\mu(u(t))E_k(u(t)). \end{aligned}\tag{72}$$

To estimate the terms on the right hand side of (69) corresponding to the quadratic nonlinearity, we need to exploit the null condition. So we will separate integrals over the regions $r\le t/2$ and $r\ge t/2$.

Inside the cone. On the region $r\le t/2$, all terms on the right hand side of (69) corresponding to the quadratic nonlinearity are bounded above by

$$\sum_{|a|\le k-1}\sum_{\substack{b+c\le a\\ |c|\le k-2}}||\partial Z^a u\partial Z^b u\partial^2 Z^c u||_{L^1(r\le t/2)}+||\langle t-r\rangle^{-2}\partial Z^a u\partial u\partial Z^a u||_{L^1(r\le t/2)}. \tag{73}$$

Since $r\le t/2$, we have $\langle t-r\rangle\sim\langle t\rangle$. For the first part of (73), it follows from (29) and (31) that

$$\begin{aligned} &||\partial Z^a u\partial Z^b u\partial^2 Z^c u||_{L^1(r\le t/2)}\\ &\quad\le C\langle t\rangle^{-1}\begin{cases}||\partial Z^a u||_{L^2}||\partial Z^b u||_{L^2}||\langle t-r\rangle\partial^2 Z^c u||_{L^\infty},\ |c|\le m,\\ ||\partial Z^a u||_{L^2}||\partial Z^b u||_{L^\infty}||\langle t-r\rangle\partial^2 Z^c u||_{L^2},\ |b|\le m,\end{cases}\\ &\quad\le C\langle t\rangle^{-1}\begin{cases}E_k^{1/2}(u(t))E^{1/2}_{|b|+1}(u(t))M_{|c|+4},\ |c|\le m,\\ E_k^{1/2}(u(t))E^{1/2}_{|b|+3}(u(t))M_{|c|+2},\ |b|\le m.\end{cases} \end{aligned}\tag{74}$$

It follows from Lemma 2.10 that we have achieved an upper bound of the form

$$\langle t\rangle^{-1}E_\mu^{1/2}(u(t))E_k(u(t)). \tag{75}$$

For the second part in (73), by (29) we have

$$\begin{aligned} &||\langle t-r\rangle^{-2}\partial Z^a u\partial u\partial Z^a u||_{L^1(r\le t/2)}\\ &\le C\langle t\rangle^{-1}||\partial Z^a u||^2_{L^2}||\partial u||_{L^\infty}\\ &\le C\langle t\rangle^{-1}E_\mu^{1/2}(u(t))E_k(u(t)). \end{aligned}\tag{76}$$

Away from the origin. Now we consider the terms on the right hand side of (69) corresponding to the quadratic nonlinearity on the region $r\ge t/2$. For $|a|=k-1, b+c+d=a,\ b,c\ne a$ or $|a|\le k-2, b+c+d=a$, an application of Lemma 2.2 yields

$$\begin{aligned} &||\langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle||_{L^1(r\ge t/2)}\\ &\le C\langle t\rangle^{-1}\Big[||\partial Z^a u Z^{b+1}u\partial^2 Z^c u||_{L^1(r\ge t/2)}\\ &+||\partial Z^a u\partial Z^b u\partial Z^{c+1}u||_{L^1(r\ge t/2)}\\ &+||\langle t-r\rangle\partial Z^a u\partial Z^b u\partial^2 Z^c u||_{L^1(r\ge t/2)}\Big]. \end{aligned}\tag{77}$$

Denote $m = [\frac{k-1}{2}] + 1$. It follows from (31), (39) and (40) that

$$\begin{aligned}
&||\partial Z^a u Z^{b+1} u \partial^2 Z^c u||_{L^1(r\geq t/2)}\\
&\leq C \begin{cases} ||\partial Z^a u||_{L^2} ||\langle t-r\rangle^{-1} Z^{b+1} u||_{L^\infty} ||\langle t-r\rangle \partial^2 Z^c u||_{L^2},\ |b| \leq m,\\ ||\partial Z^a u||_{L^2} ||\langle t-r\rangle^{-1} Z^{b+1} u||_{L^2} ||\langle t-r\rangle \partial^2 Z^c u||_{L^\infty},\ |c| \leq m,\end{cases}\\
&\leq C \begin{cases} E_k^{1/2}(u(t)) E_{|b|+2}^{1/2}(u(t)) M_{|c|+2}(u(t)),\ |b| \leq m,\\ E_k^{1/2}(u(t)) E_{|b|+2}^{1/2}(u(t)) M_{|c|+4}(u(t)),\ |c| \leq m,\end{cases}\\
&\leq C E_k(u(t)) E_\mu^{1/2}(u(t)). \qquad (78)
\end{aligned}$$

The second term can be handled using (29):

$$\begin{aligned}
&||\partial Z^a u \partial Z^b u \partial Z^{c+1} u||_{L^1(r\geq t/2)}\\
&\leq C \begin{cases} ||\partial Z^a u||_{L^2} ||\partial Z^b u||_{L^\infty} ||\partial Z^{c+1} u||_{L^2},\ |b| \leq m,\\ ||\partial Z^a u||_{L^2} ||\partial Z^b u||_{L^2} ||\partial Z^{c+1} u||_{L^\infty},\ |c| \leq m,\end{cases}\\
&\leq C \begin{cases} E_k^{1/2}(u(t)) E_{|b|+3}^{1/2}(u(t)) E_{|c|+2}(u(t)),\ |b| \leq m,\\ E_k^{1/2}(u(t)) E_{|b|+1}^{1/2}(u(t)) E_{|c|+4}(u(t)),\ |c| \leq m,\end{cases}\\
&\leq C E_k(u(t)) E_\mu^{1/2}(u(t)). \qquad (79)
\end{aligned}$$

The final term can be estimated using (29), (31), (39) and (40):

$$\begin{aligned}
&||\langle t-r\rangle \partial Z^a u \partial Z^b u \partial^2 Z^c u||_{L^1(r\geq t/2)}\\
&\leq C \begin{cases} ||\partial Z^a u||_{L^2} ||\partial Z^b u||_{L^\infty} ||\langle t-r\rangle \partial^2 Z^c u||_{L^2},\ |b| \leq m,\\ ||\partial Z^a u||_{L^2} ||\partial Z^b u||_{L^2} ||\langle t-r\rangle \partial^2 Z^c u||_{L^\infty},\ |c| \leq m,\end{cases}\\
&\leq C \begin{cases} E_k^{1/2}(u(t)) E_{|b|+3}^{1/2}(u(t)) M_{|c|+2}(u(t)),\ |b| \leq m,\\ E_k^{1/2}(u(t)) E_{|b|+1}^{1/2}(u(t)) M_{|c|+4}(u(t)),\ |c| \leq m,\end{cases}\\
&\leq C E_k(u(t)) E_\mu^{1/2}(u(t)). \qquad (80)
\end{aligned}$$

It follows from (19), the Cauchy–Schwartz inequality, (10), (29), (31) and (56) that

$$\begin{aligned}
&||g_{\alpha\beta\gamma} e^{-q(\sigma)} \partial_\gamma Z^a u \partial_\alpha \partial_t u \partial_\beta Z^a u||_{L^1(r\geq t/2)}\\
&\leq C||T Z^a u \partial^2 u \partial Z^a u||_{L^1(r\geq t/2)} + C||\partial Z^a u T \partial u \partial Z^a u||_{L^1(r\geq t/2)}\\
&\leq \frac{1}{8} ||e^{-q(\sigma)/2} \langle t-r\rangle^{-1} T Z^a u||^2_{L^2(r\geq t/2)}\\
&\quad + C\langle t\rangle^{-1} ||\partial Z^a u||^2_{L^2(r\geq t/2)} ||\langle r\rangle^{1/2} \langle t-r\rangle \partial^2 u||^2_{L^\infty(r\geq t/2)}\\
&\quad + C\langle t\rangle^{-1} ||\partial Z^a u||_{L^2(r\geq t/2)} ||\partial Z^a u||_{L^2(r\geq t/2)} \big(||Z\partial u||_{L^\infty(r\geq t/2)} + ||\langle t-r\rangle \partial^2 u||_{L^\infty(r\geq t/2)}\big)\\
&\leq \frac{1}{8} ||e^{-q(\sigma)/2} \langle t-r\rangle^{-1} T Z^a u||^2_{L^2} + C\langle t\rangle^{-1} E_k(u(t)) M_4^2(u(t))\\
&\quad + C\langle t\rangle^{-1} E_k(u(t)) (E_3^{1/2}(u(t)) + M_4(u(t)))\\
&\leq \frac{1}{8} ||e^{-q(\sigma)/2} \langle t-r\rangle^{-1} T Z^a u||^2_{L^2} + C\langle t\rangle^{-1} E_k(u(t)) E_\mu(u(t))\\
&\quad + C\langle t\rangle^{-1} E_k(u(t)) E_\mu^{1/2}(u(t)). \qquad (81)
\end{aligned}$$

The first term on the right hand side of (81) can be absorbed into the ghost weight energy term on the left hand side of (69).

Now we estimate the terms involve $q'(\sigma) = \langle t-r\rangle^{-2}$. Owing to (19), the Cauchy–Schwartz inequality, (10), (29) and (40), we have

$$\begin{aligned}
&||g_{\alpha\beta\gamma}e^{-q(\sigma)}q'(\sigma)\partial_\gamma Z^a u\partial_\alpha u\partial_\beta Z^a u||_{L^1(r\geq t/2)}\\
&\quad + ||g_{\alpha\beta\gamma}e^{-q(\sigma)}q'(\sigma)T_\gamma Z^a u\partial_\alpha u\partial_\beta Z^a u||_{L^1(r\geq t/2)}\\
&\leq C||\langle t-r\rangle^{-2}TZ^a u\partial u\partial Z^a u||_{L^1(r\geq t/2)} + C||\langle t-r\rangle^{-2}\partial Z^a uTu\partial Z^a u||_{L^1(r\geq t/2)}\\
&\leq \frac{1}{8}||e^{-q(\sigma)/2}\,\langle t-r\rangle^{-1}\,TZ^a u||^2_{L^2(r\geq t/2)}\\
&\quad + C\langle t\rangle^{-1}||\langle r\rangle^{1/2}\partial u||^2_{L^\infty(r\geq t/2)}||\partial Z^a u||^2_{L^2(r\geq t/2)}\\
&\quad + C\langle t\rangle^{-1}||\partial Z^a u||^2_{L^2(r\geq t/2)}(||\langle t-r\rangle^{-2}Zu||_{L^\infty(r\geq t/2)} + ||\langle t-r\rangle^{-1}\partial u||_{L^\infty(r\geq t/2)})\\
&\leq \frac{1}{8}||e^{-q(\sigma)/2}\,\langle t-r\rangle^{-1}\,TZ^a u||^2_{L^2} + C\langle t\rangle^{-1}E_k(u(t))E_\mu(u(t))\\
&\quad + C\langle t\rangle^{-1}E_k(u(t))E_\mu^{1/2}(u(t)).
\end{aligned} \tag{82}$$

3.2. **Low order energy.** Following the general energy method, we have

$$\begin{aligned}
\frac{d}{dt}E_\mu(u(t)) &= \sum_{|a|\leq\mu-1}\int\langle\partial_t Z^a u, \Box Z^a u\rangle\,dx\\
&= \sum_{|a|\leq\mu-1}\sum_{b+c+d=a}\int\langle\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle\,dx\\
&\quad + \sum_{|a|\leq\mu-1}\sum_{b+c+d+e=a}\int\langle\partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u)\rangle\,dx.
\end{aligned} \tag{83}$$

Inside the cone. On the region $r\leq t/2$, the right hand side of (83) is bounded above by

$$\begin{aligned}
&\sum_{|a|\leq\mu-1}\sum_{b+c\leq a}||\partial Z^a u\partial Z^b u\partial^2 Z^c u||_{L^1(r\leq t/2)}\\
&\quad + \sum_{|a|\leq\mu-1}\sum_{b+c+d\leq a}||\partial Z^a u\partial Z^b u\partial Z^c u\partial^2 Z^d u||_{L^1(r\leq t/2)}.
\end{aligned} \tag{84}$$

Since $r\leq t/2$, we have $\langle t-r\rangle\sim\langle t\rangle$. It follows from (30) that a typical term in the first part of (84) can be estimated as the following:

$$\begin{aligned}
&||\partial Z^a u\partial Z^b u\partial^2 Z^c u||_{L^1(r\leq t/2)}\\
&\leq C\langle t\rangle^{-3/2}||\partial Z^a u\langle t-r\rangle^{1/2}\partial Z^b u\langle t-r\rangle\partial^2 Z^c u||_{L^1(r\leq t/2)}\\
&\leq C\langle t\rangle^{-3/2}||\partial Z^a u||_{L^2}||\langle t-r\rangle^{1/2}\partial Z^b u||_{L^\infty}||\langle t-r\rangle\partial^2 Z^c u||_{L^2}\\
&\leq C\langle t\rangle^{-3/2}E_\mu^{1/2}(u(t))M_{|c|+2}(u(t))\big[E^{1/2}_{|b|+3}(u(t)) + M_{|b|+3}(u(t))\big].
\end{aligned} \tag{85}$$

Noting that $b+c\leq a, |a|\leq\mu-1, \mu=k-3,\ k\geq 20$. We have either $|b|+3\leq\mu$ and $|c|+2\leq k$, or $|b|+3\leq k$ and $|c|+2\leq\mu$. By Lemma 2.10, we have achieved an upper bound of the form

$$\langle t\rangle^{-3/2}E_\mu(u(t))E_k^{1/2}(u(t)). \tag{86}$$

Similar argument gives the following upper bound for the second part of (84):

$$\langle t\rangle^{-2}E_\mu(u(t))E_k(u(t)). \tag{87}$$

Away from the origin. On the region $r \geq t/2$, we must use the null condition. An application of Lemma 2.2 gives

$$\begin{aligned}
&||\langle \partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle||_{L^1(r\geq t/2)} \\
&\quad \leq C\langle t\rangle^{-1}\Big[||\partial Z^a u Z^{b+1} u \partial^2 Z^c u||_{L^1(r\geq t/2)} \\
&\quad + ||\partial Z^a u \partial Z^b u \partial Z^{c+1} u||_{L^1(r\geq t/2)} \\
&\quad + ||\langle t-r\rangle \partial Z^a u \partial Z^b u \partial^2 Z^c u||_{L^1(r\geq t/2)}\Big]. \qquad (88)
\end{aligned}$$

We still need to squeeze out an additional decay factor of $\langle t\rangle^{-1/2}$. Since $r \geq t/2$, we have $\langle r\rangle \geq C\langle t\rangle$. It follows from (31), (39), (56) and (57) that

$$\begin{aligned}
&||\partial Z^a u Z^{b+1} u \partial^2 Z^c u||_{L^1(r\geq t/2)} \\
&\quad \leq C\langle t\rangle^{-1/2}||\partial Z^a u||_{L^2}||\langle t-r\rangle^{-1} Z^{b+1} u||_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle \partial^2 Z^c u||_{L^\infty} \\
&\quad \leq C\langle t\rangle^{-1/2} E_\mu^{1/2}(u(t)) E_{|b|+2}^{1/2}(u(t)) M_{|c|+4}(u(t)) \\
&\quad \leq C\langle t\rangle^{-1/2} E_\mu(u(t)) E_k^{1/2}(u(t)). \qquad (89)
\end{aligned}$$

The second term can be handled similarly by using (29):

$$\begin{aligned}
&||\partial Z^a u \partial Z^b u \partial Z^{c+1} u||_{L^1(r\geq t/2)} \\
&\quad \leq C\langle t\rangle^{-1/2}||\partial Z^a u||_{L^2}||\langle r\rangle^{1/2}\partial Z^b u||_{L^\infty}||\partial Z^{c+1} u||_{L^2} \\
&\quad \leq C\langle t\rangle^{-1/2} E_\mu^{1/2}(u(t)) E_{|b|+3}^{1/2}(u(t)) E_{|c|+2}^{1/2}(u(t)) \\
&\quad \leq C\langle t\rangle^{-1/2} E_\mu(u(t)) E_k^{1/2}(u(t)). \qquad (90)
\end{aligned}$$

Owing to (29), (56) and (57), the final term can be estimated as the following:

$$\begin{aligned}
&||\langle t-r\rangle \partial Z^a u \partial Z^b u \partial^2 Z^c u||_{L^1(r\geq t/2)} \\
&\quad \leq C||\partial Z^a u||_{L^2}||\langle r\rangle^{1/2}\partial Z^b u||_{L^\infty}||\langle t-r\rangle \partial^2 Z^c u||_{L^2} \\
&\quad \leq C\langle t\rangle^{-1/2} E_\mu^{1/2}(u(t)) E_{|b|+3}^{1/2}(u(t)) M_{|c|+2}(u(t)) \\
&\quad \leq C\langle t\rangle^{-1/2} E_\mu(u(t)) E_k^{1/2}(u(t)). \qquad (91)
\end{aligned}$$

So we have

$$\begin{aligned}
&\sum_{|a|\leq \mu-1}\sum_{b+c+d=a} ||\langle \partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle||_{L^1(r\geq t/2)} \\
&\qquad \leq C\langle t\rangle^{-3/2} E_\mu(u(t)) E_k^{1/2}(u(t)). \qquad (92)
\end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{aligned}
&||\langle \partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u)\rangle||_{L^1(r\geq t/2)} \\
&\quad \leq C\langle t\rangle^{-1}\Big[||\partial Z^a u Z^{b+1} u \partial Z^c u \partial^2 Z^d u||_{L^1(r\geq t/2)} \\
&\quad + ||\partial Z^a u \partial Z^b u Z^{c+1} u \partial^2 Z^d u||_{L^1(r\geq t/2)} \\
&\quad + ||\partial Z^a u \partial Z^b u \partial Z^b u \partial Z^c u \partial Z^{d+1} u||_{L^1(r\geq t/2)} \\
&\quad + ||\langle t-r\rangle \partial Z^a u \partial Z^b u \partial Z^c u \partial^2 Z^d u||_{L^1(r\geq t/2)}\Big]. \qquad (93)
\end{aligned}$$

Similar argument of treating (88) can give

$$\sum_{|a|\le\mu-1}\sum_{b+c+d+e=a}||\langle\partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u)||_{L^1(r\ge t/2)} \le C\langle t\rangle^{-2}E_\mu(u(t))E_k(u(t)). \tag{94}$$

3.3. **Conclusion of the proof.** Now we have get a pair of coupled differential inequalities for the lower order energy $E_\mu(u(t)$ and the modified higher order energy $\widetilde{E}_k(u(t)$:

$$\frac{d}{dt}\widetilde{E}_k(u(t)) \le C_1\langle t\rangle^{-1}\widetilde{E}_k(u(t))E_\mu^{1/2}(u(t)) + C_1\langle t\rangle^{-1}\widetilde{E}_k(u(t))E_\mu(u(t)), \tag{95}$$

$$\frac{d}{dt}E_\mu(u(t)) \le C_2\langle t\rangle^{-3/2}E_\mu(u(t))\widetilde{E}_k^{1/2}(u(t)) + C_2\langle t\rangle^{-2}E_\mu(u(t))\widetilde{E}_k(u(t)). \tag{96}$$

It follows from (63), (i), (95) and the Gronwall inequality that

$$E_k(u(t)) \le 2c\widetilde{E}_k(u(t)) \le 2c\bar{c}\varepsilon^2\langle t\rangle^{C_1(A_1^{1/2}\varepsilon + A_1\varepsilon^2)}. \tag{97}$$

Take $A_2 = 2c\bar{c}$, $\widetilde{c} = 2C_1A_1^{1/2}$. If $A_1^{1/2}\varepsilon \le 1$, then we have

$$E_k(u(t) \le A_2\varepsilon^2\,\langle t\rangle^{\widetilde{c}\varepsilon}\,, \tag{98}$$

so (ii) holds. Now by (i), (ii) and (96), we have

$$E_\mu(u(t)) \le C_3\varepsilon^2 + C_4A_1\varepsilon^2(A_2^{1/2}\varepsilon + A_2\varepsilon^2). \tag{99}$$

where

$$C_4 = 2cC_2I, I = \int_0^{+\infty}(\langle t\rangle^{-3/2+\widetilde{c}\varepsilon/2} + \langle t\rangle^{-2+\widetilde{c}\varepsilon})\,dt. \tag{100}$$

Take $A_1 = 4C_3$. If $4C_4(A_2^{1/2}\varepsilon + A_2\varepsilon^2) \le 1$, then we have

$$E_\mu(u(t)) \le A_1\varepsilon^2/2, \tag{101}$$

which completes the proof.

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REFERENCES

[1] R. Agemi, Global existence of nonlinear elastic waves, *Invent. Math.*, **142** (2000), 225–250.

[2] S. Alinhac, The null condition for quasilinear wave equations in two space dimensions I, *Invent. Math.*, **145** (2001), 597–618.

[3] S. Alinhac, Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux, *Invent. Math.*, **111** (1993), 627–670.

[4] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data, *Comm. Pure Appl. Math.*, **39** (1986), 267–282.

[5] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space*, vol. 41 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1993.

[6] J. Helms, Private communication via E-mail, Oct. 2014.

[7] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Mathématiques & Applications (Berlin) [Mathematics & Applications], 26, Springer-Verlag, Berlin, 1997.

[8] A. Hoshiga, The existence of global solutions to systems of quasilinear wave equations with quadratic nonlinearities in 2-dimensional space, *Funkcial. Ekvac.*, **49** (2006), 357–384.

[9] S. Klainerman, The null condition and global existence to nonlinear wave equations, in *Nonlinear Systems of Partial Differential Equations in Applied Mathematics, Part 1* (Santa Fe, N.M., 1984), Lectures in Appl. Math., 23, Amer. Math. Soc., Providence, RI, 1986, 293–326.

[10] S. Klainerman and T. C. Sideris, On almost global existence for nonrelativistic wave equations in 3D, *Comm. Pure Appl. Math.*, **49** (1996), 307–321.

[11] H. Lindblad and I. Rodnianski, The global stability of Minkowski space-time in harmonic gauge, *Ann. of Math. (2)*, **171** (2010), 1401–1477.

[12] J. Metcalfe and C. D. Sogge, Global existence of null-form wave equations in exterior domains, *Math. Z.*, **256** (2007), 521–549.

[13] T. C. Sideris, Delayed singularity formation in 2D compressible flow, *Amer. J. Math.*, **119** (1997), 371–422.

[14] T. C. Sideris, Nonresonance and global existence of prestressed nonlinear elastic waves, *Ann. of Math. (2)*, **151** (2000), 849–874.

[15] T. C. Sideris and B. Thomases, Global existence for three-dimensional incompressible isotropic elastodynamics, *Comm. Pure Appl. Math.*, **60** (2007), 1707–1730.

[16] T. C. Sideris and S.-Y. Tu, Global existence for systems of nonlinear wave equations in 3D with multiple speeds, *SIAM J. Math. Anal.*, **33** (2001), 477–488.

[17] S. Yang, Global solutions of nonlinear wave equations in time dependent inhomogeneous media, *Arch. Ration. Mech. Anal.*, **209** (2013), 683–728.

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E-mail address: ZhaDongbing@fudan.edu.cn