J. Math. Anal. Appl. 439 (2016) 419–435

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A note on quasilinear wave equations in two space dimensions II: Almost global existence of classical solutions

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ARTICLE INFO

Article history:
Received 8 January 2016
Available online 10 March 2016
Submitted by X. Zhang

Keywords:
Quasilinear wave equations
Two space dimensions
Null condition
Almost global existence

ABSTRACT

We give an alternative proof of Alinhac's almost global existence result for the Cauchy problem of quasilinear wave equations with quadratic nonlinearity satisfying the null condition in 2-D (S. Alinhac, 2001, [3]). The main innovation of our proof is that the Lorentz boost operator is not employed in the generalized energy estimates.

1. Introduction

This manuscript is a continuation of the second author's previous work [29]. We consider the following Cauchy problem for quasilinear wave equation in 2-D:

$$\partial_t^2 u - \Delta u = \Sigma_{0\leq\alpha,\beta\leq 2} g_{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta u, \tag{1.1}$$

$$t = 0 : u = \varepsilon u_0, u_t = \varepsilon u_1, \tag{1.2}$$

where $\partial = (\partial_t, \nabla)$, $\nabla = (\partial_1, \partial_2)$, $x_0 = t$, $x = (x_1, x_2)$, and the coefficients $g_{\alpha\beta}$ are smooth real functions vanishing at the origin, i.e.,

$$g_{\alpha\beta}(\xi) = \Sigma g_{\alpha\beta\gamma}\xi_\gamma + \Sigma h_{\alpha\beta\gamma\delta}\xi_\gamma\xi_\delta + r_{\alpha\beta}(\xi),\ r_{\alpha\beta}(\xi) = \mathscr{O}(|\xi|^3). \tag{1.3}$$

In the initial conditions of (u, u_t), ε is a positive small parameter and $u_0, u_1 \in C_c^\infty(\mathbb{R}^2)$. Without loss of generality, we can assume that u_0 and u_1 are supported in $|x| \leq 1$. We always assume that the following symmetric conditions hold:

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http://dx.doi.org/10.1016/j.jmaa.2016.03.011

$$g_{\alpha\beta\gamma} = g_{\beta\alpha\gamma} = g_{\gamma\beta\alpha}, \tag{1.4}$$

$$h_{\alpha\beta\gamma\delta} = h_{\beta\alpha\gamma\delta} = h_{\gamma\beta\alpha\delta} = h_{\delta\beta\gamma\alpha}. \tag{1.5}$$

Following Alinhac [3], the set $g = (g_{\alpha\beta\gamma})$ is said to satisfy the *null* conditions if for any $x \in \mathbb{R}^2$, $\omega_0 = -1$, $\omega_i = x_i/r (i = 1, 2)$ with $r = |x|$, it holds that

$$g_{\alpha\beta\gamma}\omega_\alpha\omega_\beta\omega_\gamma = 0; \tag{1.6}$$

and $h = (h_{\alpha\beta\gamma\delta})$ is said to satisfy the *null* conditions if

$$h_{\alpha\beta\gamma\delta}\omega_\alpha\omega_\beta\omega_\gamma\omega_\delta = 0. \tag{1.7}$$

In Alinhac [3], for the Cauchy problem (1.1)–(1.2), the following global and almost global existence results are proved.

Theorem 1.1. *If $g = (g_{\alpha\beta\gamma})$ satisfies the null condition (1.6) and $h = (h_{\alpha\beta\gamma\delta})$ satisfies the null condition (1.7), then for any given positive parameter ε small enough, the Cauchy problem (1.1)–(1.2) admits a unique global smooth solution $u \in C^\infty([0, +\infty) \times \mathbb{R}^2)$.*

Theorem 1.2. *If $g = (g_{\alpha\beta\gamma})$ satisfies the null condition (1.6), then for any given positive parameter ε small enough, the Cauchy problem (1.1)–(1.2) admits a unique smooth solution $u \in C^\infty([0, T_\varepsilon) \times \mathbb{R}^2)$ with*

$$T_\varepsilon \geq \exp(\frac{c}{\varepsilon^2}), \tag{1.8}$$

where c is a positive constant independent of ε.

Remark 1.1. When $g = (g_{\alpha\beta\gamma})$ satisfies the null condition (1.6) and $h = (h_{\alpha\beta\gamma\delta})$ violates the null condition (1.7), the sharpness of lifespan estimate (1.8) is proved in Alinhac [4].

In Alinhac [3], in order to get suitable decay estimates, he used the Klainerman–Sobolev inequality which involves the full Lorentz invariance of the wave operator, therefore, the general space–time derivatives, spatial rotation, scaling and Lorentz boost operator are all used in the generalized energy estimates. However, the Lorentz boost operator is not suitable for some wave systems which are not Lorentz invariant, such as nonrealistic system with multiple wave speeds (see Klainerman and Sideris [14], Sideris and Tu [24]), nonlinear wave equations on non-flat space–time (see Luk [17], Wang and Yu [26], Yang [27,28]), wave type equation with nonlocal nonlinear term (see Sideris and Thomases [22,23]) and exterior problem (see Metcalfe and Sogge [18,19]), etc. So how to remove the Lorentz boost operator in Alinhac's proof of Theorem 1.1 and Theorem 1.2 via a robust way seems interesting.

In [29], the second author gave a new proof of Theorem 1.1, in which the Lorentz boost operator was not employed and only the energy type method was used. In the previous work [10], Hoshiga also considered a quasilinear wave system with different propagation speeds and proved global existence of classical solutions under some suitable null conditions. To get decay estimates for the solution of the wave equation and its derivatives, Hoshiga used some L^∞–L^∞ estimates which rely on the fundamental solution of the wave equation.

Naturally we also want to remove the Lorentz boost operator in Alinhac's proof of Theorem 1.2. But in fact the method in [10] or [29] can only give the lifespan estimate $T_\varepsilon \geq \exp(\frac{c}{\varepsilon})$, which is not sharp in contrast with Alinhac's original result. To get the sharp lifespan estimate (1.8), more delicate analysis is needed.

In this manuscript, we will give an alternative proof of Theorem 1.2. When applying the vector fields method to do the generalized energy estimates, we do not employ the Lorentz boost operator. Besides the techniques used in [29] such as Alinhac's ghost weight energy estimates and some suitable decay estimates, we also introduce some new decay estimates to get enough decay in time on the region $r \le t/2$. On the region $r \ge t/2$, the analysis is also more delicate than the corresponding part in [29] in order to obtain enough decay in time.

Lifespan estimate of small solutions to nonlinear wave equations has been a subject under active investigation for over 30 years. For classical references in this field, we refer the reader to Klainerman [12,13], Christodoulou [6], John [11], Hörmander [8], Sogge [25], Christodoulou and Klainerman [7], Klainerman and Sideris [14], Alinhac [3,5], Agemi [1], Sideris [21], Sideris and Tu [24] and Lindblad and Rodnianski [16]. A systematic summary on this topic can be found in Li and Zhou [15].

An outline of this paper is as follows. The remainder of this section will be devoted to the description of some basic notations which will be used in the sequel. In Section 2, we will introduce some necessary tools used to prove Theorem 1.2, including some properties of the null condition, Sobolev and Hardy inequalities and weighted decay estimates. The proof of Theorem 1.2 will be given in Section 3.

1.1. Notations

Denote the spatial rotation

$$\Omega = x_1\partial_2 - x_2\partial_1 \tag{1.9}$$

and the scaling operator

$$S = t\partial_t + r\partial_r, \tag{1.10}$$

where $r = |x|$ and $\partial_r = \frac{x}{r}\cdot\nabla$.

Let $Z = (Z_1, \cdots, Z_5) = (\partial, \Omega, S)$ be called the generalized derivatives. The commutator of Z and ∂ is in the span of ∂, schematically, we write

$$\big[Z, \partial\big] = \partial. \tag{1.11}$$

For any given multi-index $a = (a_1, \cdots, a_5)$, we denote $Z^a = Z_1^{a_1}\cdots Z_5^{a_5}$. Moreover $b \le a$ means $b_i \le a_i$ for each $i = 1, \cdots, 5$.

We also use the good derivatives

$$T_\alpha = \omega_\alpha\partial_t + \partial_\alpha, \tag{1.12}$$

where $\omega_0 = -1$, $\omega_i = x_i/r (i = 1, 2)$ with $r = |x|$. Denote $T = (T_0, T_1, T_2)$.

The energy associated to the linear wave operator is

$$E_1(u(t)) = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2\, dx, \tag{1.13}$$

and the higher order energies are

$$E_k(u(t)) = \sum_{|a|\le k-1} E_1(Z^a u(t)). \tag{1.14}$$

We will also use the following weighted L^2 norm

$$M_k(u(t)) = \sum_{|a|\le k-2} ||\langle t-r\rangle \partial^2 Z^a u(t)||_{L^2(\mathbb{R}^2)}, \tag{1.15}$$

where $\langle \sigma\rangle = (1+|\sigma|^2)^{1/2}$.

To simplify the presentation and without loss of generality, we truncate the nonlinearity at the cubic level, since the higher order terms have no essential influence on the discussion of the global existence of solutions with small amplitude. Thus, the nonlinear equation which we will consider can be written in the following form:

$$\Box u = N_1(u,u) + N_2(u,u,u), \tag{1.16}$$

where

$$N_1(u,v) = g_{\alpha\beta\gamma}\partial_\gamma u \partial_\alpha\partial_\beta v, \tag{1.17}$$

$$N_2(u,v,w) = h_{\alpha\beta\gamma\delta}\partial_\gamma u\partial_\delta v\partial_\alpha\partial_\beta w. \tag{1.18}$$

2. Some preliminaries

2.1. The null condition

Lemma 2.1. *The following pointwise decay estimate holds:*

$$|Tu| \le C\langle t\rangle^{-1}(|Zu| + \langle t-r\rangle|\partial u|), \tag{2.1}$$

here and hereafter C denotes a positive constant.

Proof. Since we have

$$T_1 = t^{-1}\big(\omega_1 S + \omega_2\Omega + (t-r)\partial_1\big) \tag{2.2}$$

and

$$T_2 = t^{-1}\big(\omega_2 S - \omega_1\Omega + (t-r)\partial_2\big), \tag{2.3}$$

(2.1) follows from the obvious estimate $|Tu| \le 2|\partial u|$. □

Lemma 2.2. *If $g = (g_{\alpha\beta\gamma})$ satisfies the null condition (1.6), then for any three smooth functions u, v and w, we have*

$$|g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta v| \le C\big(|Tu||\partial^2 v| + |\partial u||T\partial v|\big), \tag{2.4}$$

$$|g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha v\partial_\beta w| \le C\big(|Tu||\partial v||\partial w| + |\partial u||Tv||\partial w| + |\partial u||\partial v||Tw|\big). \tag{2.5}$$

Proof. By the definition (1.12) of good derivatives, we have the following pointwise equality:

$$\begin{aligned} g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha v\partial_\beta w &= g_{\alpha\beta\gamma}T_\gamma u\partial_\alpha v\partial_\beta w - g_{\alpha\beta\gamma}\omega_\gamma\partial_t u T_\alpha v\partial_\beta w \\ &\quad + g_{\alpha\beta\gamma}\omega_\gamma\partial_t u\omega_\alpha\partial_t v T_\beta w - g_{\alpha\beta\gamma}\omega_\alpha\omega_\beta\omega_\gamma\partial_t u\partial_t v\partial_t w, \end{aligned} \tag{2.6}$$

thus (2.5) follows from (2.6) and the null condition (1.6). (2.4) can be similarly proved. □

The combination of Lemma 2.1 and Lemma 2.2 gives

Lemma 2.3. *If $g = (g_{\alpha\beta\gamma})$ satisfies the null condition (1.6), then for any $r \geq t/2$,*

$$\begin{aligned}|g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta v| &\leq C\langle t+r\rangle^{-1}\big(|Zu||\partial^2 v| + |\partial u||Z\partial v| + \langle t-r\rangle|\partial u||\partial^2 v| \\ &\quad + \langle t-r\rangle|\partial u||\partial v||\partial^2 w|\big), \qquad (2.7)\\ |g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha v\partial_\beta w| &\leq C\langle t+r\rangle^{-1}\big(|Zu||\partial v||\partial w| + |\partial u||Zv||\partial w| + |\partial u||\partial v||Zw| \\ &\quad + \langle t-r\rangle|\partial u||\partial v||\partial w|\big). \qquad (2.8)\end{aligned}$$

Lemma 2.4. *Let u be a solution of (1.16) and assume that the null condition (1.6) holds for the nonlinearity (1.17). Then for any multi-indices $a = (a_1, \cdots, a_5)$, we have*

$$\Box Z^a u = \sum_{b+c+d=a} N_{1d}(Z^b u, Z^c u) + \sum_{b+c+d+e=a} N_{2e}(Z^b u, Z^c u, Z^d u), \qquad (2.9)$$

where each N_{1d} is a quadratic nonlinearity of the form (1.17) satisfying the null condition (1.6) and each N_{2e} is a cubic nonlinearity of the form (1.18). Moreover, if $b + c = a$, then $N_{1d} = N_1$; if $b + c + d = a$, then $N_{2e} = N_2$.

Proof. See Hörmander [9] (Lemma 6.6.5) and Sideris [24] (Lemma 4.1). □

2.2. Sobolev and Hardy inequalities

Lemma 2.5. *For $u \in C_c^\infty(\mathbb{R}^2)$, $r = |x|$, we have the following weighted Sobolev inequalities:*

$$r^{1/2}|u(x)| \leq C\sum_{|a|\leq 1}||\Omega^a u||_{L^2(\mathbb{R}^2)} + C\sum_{|a|\leq 1}||\partial_r\Omega^a u||_{L^2(\mathbb{R}^2)}, \qquad (2.10)$$

$$r^{1/2}\langle t-r\rangle^{1/2}|u(x)| \leq C\sum_{|a|\leq 1}||\Omega^a u||_{L^2(\mathbb{R}^2)} + C\sum_{|a|\leq 1}||\langle t-|y|\rangle\partial_r\Omega^a u||_{L^2_y(\mathbb{R}^2)}, \qquad (2.11)$$

$$r^{1/2}\langle t-r\rangle|u(x)| \leq C\sum_{|a|\leq 1}||\langle t-|y|\rangle\Omega^a u||_{L^2_y(\mathbb{R}^2)} + C\sum_{|a|\leq 1}||\langle t-|y|\rangle\partial_r\Omega^a u||_{L^2_y(\mathbb{R}^2)}. \qquad (2.12)$$

Proof. See Sideris [20] (Lemma 1). □

The higher order version of Lemma 2.5 is the following:

Lemma 2.6. *For $u \in C^\infty([0, T] \times \mathbb{R}^2)$, we have*

$$\langle r\rangle^{1/2}|\partial Z^a u(t, x)| \leq CE^{1/2}_{|a|+3}(u(t)), \qquad (2.13)$$

$$\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}|\partial Z^a u(t, x)| \leq C\big(E^{1/2}_{|a|+3}(u(t)) + M_{|a|+3}(u(t))\big), \qquad (2.14)$$

$$\langle r\rangle^{1/2}\langle t-r\rangle|\partial^2 Z^a u(t, x)| \leq CM_{|a|+4}(u(t)). \qquad (2.15)$$

Proof. See Sideris [20] (Lemma 1) and Lemma 2.5 in [29]. □

Lemma 2.7. *If $u = u(t, x)$ is smooth and supported in $|x| \leq t + 1$, then we have*

$$||\langle t-r\rangle^{-1}u||_{L^2(\mathbb{R}^2)} \leq C||\partial_r u||_{L^2(\mathbb{R}^2)}, \qquad (2.16)$$

and

$$||\langle r\rangle^{1/2}\langle t-r\rangle^{-1}u||_{L^\infty(\mathbb{R}^2)} \le C\big(\sum_{|a|\le 1}||\nabla\Omega^a u||_{L^2(\mathbb{R}^2)} + ||\nabla^2 u||_{L^2(\mathbb{R}^2)}\big). \tag{2.17}$$

Proof. For the Hardy type inequality (2.16), see Alinhac [2] (Lemma 2.2). For (2.17), on the region $r \ge 1$, it follows from (2.10) and (2.16) that

$$\begin{aligned} &||\langle r\rangle^{1/2}\langle t-r\rangle^{-1}u||_{L^\infty(r\ge 1)}\\ &\le C||r^{1/2}\langle t-r\rangle^{-1}u||_{L^\infty(r\ge 1)}\\ &\le C\sum_{|a|\le 1}\big(||\langle t-r\rangle^{-1}\Omega^a u||_{L^2(\mathbb{R}^2)} + ||\partial_r(\langle t-r\rangle^{-1}\Omega^a u)||_{L^2(\mathbb{R}^2)}\big)\\ &\le C\sum_{|a|\le 1}||\partial_r\Omega^a u||_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{2.18}$$

On the region $r \le 1$, by the Sobolev embedding $H^2(B_1) \hookrightarrow L^\infty(B_1)$ and (2.16), we have

$$\begin{aligned} &||\langle r\rangle^{1/2}\langle t-r\rangle^{-1}u||_{L^\infty(r\le 1)}\\ &\le C(2+t)^{-1}||u||_{L^\infty(r\le 1)}\\ &\le C(2+t)^{-1}\big(||u||_{L^2(r\le 1)} + ||\nabla u||_{L^2(r\le 1)} + ||\nabla^2 u||_{L^2(r\le 1)}\big)\\ &\le C||\langle t-r\rangle^{-1}u||_{L^2(\mathbb{R}^2)} + C\big(||\nabla u||_{L^2} + ||\nabla^2 u||_{L^2}\big)\\ &\le C\big(||\nabla u||_{L^2} + ||\nabla^2 u||_{L^2}\big). \end{aligned} \tag{2.19}$$

Then (2.17) follows from (2.18) and (2.19). □

Now we give the higher order version of Lemma 2.7 as follows.

Lemma 2.8. *For $u \in C^\infty([0,T]\times\mathbb{R}^2)$ which is supported in $|x| \le t+1$, we have*

$$||\langle t-r\rangle^{-1}Z^a u)||_{L^2(\mathbb{R}^2)} \le CE^{1/2}_{|a|+1}(u(t)), \tag{2.20}$$

$$||\langle r\rangle^{1/2}\langle t-r\rangle^{-1}Z^a u||_{L^\infty(\mathbb{R}^2)} \le CE^{1/2}_{|a|+2}(u(t)). \tag{2.21}$$

Lemma 2.9. *For $u \in C_c^\infty(\mathbb{R}^2)$, $r = |x|$, $0 < \alpha < 1$, we have the following Hardy type inequality:*

$$||r^{-(1-\alpha)}u||_{L^2(\mathbb{R}^2)} \le C||r^\alpha\nabla u||_{L^2(\mathbb{R}^2)}. \tag{2.22}$$

Proof. By the fundamental theorem of calculus and Hölder inequality, we can get

$$\begin{aligned} &||r^{-(1-\alpha)}u||^2_{L^2(\mathbb{R}^2)}\\ &= \int_{\mathbb{R}^2}|x|^{-2+2\alpha}|u(x)|^2dx\\ &= \int_{S^1}\int_0^{+\infty}|u(rw)|^2r^{-1+2\alpha}drdw\\ &\le C\int_{S^1}\int_0^{+\infty}|u(rw)||\partial_r u(rw)|r^{2\alpha}drdw \end{aligned}$$

$$\begin{aligned}&\le C\|r^{2\alpha-1}u\nabla u\|_{L^1(\mathbb{R}^2)}\\&\le C\|r^{-(1-\alpha)}u\|_{L^2(\mathbb{R}^2)}\|r^{\alpha}\nabla u\|_{L^2(\mathbb{R}^2)},\end{aligned}\tag{2.23}$$

which proves (2.22). □

Lemma 2.10. *For $u \in C^\infty([0,T]\times\mathbb{R}^2)$ which is supported in $|x| \le t+1$, we have*

$$||\langle r\rangle^{-1/2}\langle t-r\rangle^{1/2}u||_{L^2(r\le t/2)} \le C\big(||u||_{L^2} + ||\langle t-r\rangle\nabla u||_{L^2}\big).\tag{2.24}$$

Consequently, the following higher order version holds:

$$||\langle r\rangle^{-1/2}\langle t-r\rangle^{1/2}\partial Z^a u||_{L^2(r\le t/2)} \le C\big(E^{1/2}_{|a|+1}(u(t)) + M_{|a|+2}(u(t))\big).\tag{2.25}$$

Proof. We only prove (2.24). (2.25) can be obtained directly from (2.24). Note that for $r \le t/2$, $\langle t-r\rangle \sim t$. It follows from the Hardy type inequality (2.22) and the pointwise estimate $|\nabla(\langle t-r\rangle)| \le C$ that

$$\begin{aligned}&||\langle r\rangle^{-1/2}\langle t-r\rangle^{1/2}u||_{L^2(r\le t/2)}\\&\quad\le C\langle t\rangle^{-1/2}||\langle r\rangle^{-1/2}\langle t-r\rangle u||_{L^2}\\&\quad\le C\langle t\rangle^{-1/2}||\langle r\rangle^{1/2}\nabla(\langle t-r\rangle u)||_{L^2(r\le t+1)}\\&\quad\le C\big(||\nabla(\langle t-r\rangle)u||_{L^2} + ||\langle t-r\rangle\nabla u||_{L^2}\big)\\&\quad\le C\big(||u||_{L^2} + ||\langle t-r\rangle\nabla u||_{L^2}\big).\qquad\square\end{aligned}\tag{2.26}$$

Lemma 2.11. *For $u \in C^\infty([0,T]\times\mathbb{R}^2)$, the following pointwise estimate holds:*

$$|\partial_t\partial u| \le C\langle t\rangle^{-1}(|\partial u| + |\partial S u| + \langle r\rangle|\partial^2 u|).\tag{2.27}$$

Proof. Noting that the scaling operator $S = t\partial_t + r\partial_r$, we have

$$\partial_t\partial u = t^{-1}(S\partial u - r\partial_r\partial u).\tag{2.28}$$

We also have the following commutation property $\big[S,\partial\big] = \partial$. So it holds that

$$|\partial_t\partial u| \le Ct^{-1}(|\partial u| + |\partial S u| + r|\partial^2 u|).\tag{2.29}$$

It is obvious that

$$|\partial_t\partial u| \le |\partial^2 u|.\tag{2.30}$$

The combination of (2.29) and (2.30) gives (2.27). □

2.3. Weighted decay estimates

In this subsection, we will estimate the weighted L^2 norm $M_k(u(t))$.

Lemma 2.12 *(Klainerman–Sideris estimates). Let $u \in C^\infty([0,T]\times\mathbb{R}^2)$. Then*

$$M_k\big(u(t)\big) \le C\big(E_k^{1/2}\big(u(t)\big) + \sum_{|a|\le k-2}||(t+r)\Box Z^a u(t)||_{L^2}\big).\tag{2.31}$$

Proof. See Klainerman and Sideris [14] (Lemma 3.1). Note that their proof is obvious valid for all spatial dimension $n \geq 2$. □

Next, by estimating the nonlinear terms on the right-hand side of (2.31), we get

Lemma 2.13. *If $u \in C^\infty([0,T] \times \mathbb{R}^2)$ satisfies (1.16), then we have*

$$\begin{aligned} M_k(u(t)) \leq{} & C E_k^{1/2}(u(t)) + C\big(E_{k'}(u(t))E_k^{1/2}(u(t)) + M_{k'}(u(t))E_k^{1/2}(u(t)) + M_k(u(t))E_{k'}^{1/2}(u(t))\big) \\ & + C\big(M_{k'}(u(t))E_{k'}^{1/2}(u(t))E_k^{1/2}(u(t)) + M_k(u(t))E_{k'}(u(t))\big), \end{aligned} \tag{2.32}$$

where $k' = [\frac{k-2}{2}] + 5$.

Proof. See Lemma 2.9 in [29]. □

Lemma 2.14. *Let $u \in C^\infty([0,T] \times \mathbb{R}^2)$ be a solution of system (1.16) and $k \geq 10$. Assume that*

$$\varepsilon_0 \equiv \sup_{0 \leq t < T} E_k^{1/2}(u(t)) \tag{2.33}$$

is sufficiently small. Then for $0 \leq t \leq T$,

$$M_k(u(t)) \leq C E_k^{1/2}(u(t)). \tag{2.34}$$

Proof. Note that for $k \geq 10$, $k' \leq k$. It follows from (2.32) that

$$M_k(u(t)) \leq C E_k^{1/2}(u(t)) + C\varepsilon_0 E_k^{1/2}(u(t)) + C\varepsilon_0 M_k(u(t)) + C\varepsilon_0^2 M_k(u(t)). \tag{2.35}$$

If ε_0 is sufficiently small, then the last two terms on the right-hand side of (2.35) can be absorbed into the left-hand side. So we can get (2.34). □

3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2 by a bootstrap argument. Assume that $u = u(t,x)$ is a local smooth solution of the initial value problem (1.16) and (1.2) on $[0,T]$. Let $k \geq 10$. We will show that there exist positive constants c and A such that for any $T \leq \exp(\frac{c}{\varepsilon^2})$, we have $\sup_{0 \leq t \leq T} E_k^{1/2}(u(t)) \leq A\varepsilon$ under the assumption that $\sup_{0 \leq t \leq T} E_k^{1/2}(u(t)) \leq 2A\varepsilon$, if ε is sufficiently small.

3.1. Generalized energy estimates

Following Alinhac [3], we will use the ghost weight energy method. Let $\sigma = t - r$, $q(\sigma) = \arctan \sigma$, and $q'(\sigma) = \frac{1}{1+\sigma^2} = \langle t - r\rangle^{-2}$. By Lemma 2.4, we have

$$\begin{aligned} & \sum_{|a| \leq k-1} \int \langle e^{-q(\sigma)} \partial_t Z^a u, \Box Z^a u\rangle \, dx \\ & = \sum_{|a| \leq k-1} \sum_{b+c+d=a} \int \langle e^{-q(\sigma)} \partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle \, dx \\ & \quad + \sum_{|a| \leq k-1} \sum_{b+c+d+e=a} \int \langle e^{-q(\sigma)} \partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u\rangle \, dx. \end{aligned} \tag{3.1}$$

Because we are now treating a quasilinear system, for the right-hand side of (3.1), special attention should be paid on terms with $b = a$ or $c = a$ with $|a| = k - 1$ for the first collection of terms and $d = a$ with $|a| = k - 1$ for the second collection of terms. We rewrite (3.1) as

$$\begin{aligned}
&\sum_{|a|\le k-1} \int \langle e^{-q(\sigma)}\partial_t Z^a u, \Box Z^a u\rangle\, dx \\
&= \sum_{|a|=k-1} \Big[\sum_{\substack{b+c+d=a \\ b,c\neq a}} \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle\, dx \\
&\qquad + \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_1(Z^a u, u)\rangle\, dx \\
&\qquad + \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_1(u, Z^a u)\rangle\, dx \Big] \\
&\quad + \sum_{|a|\le k-2} \sum_{b+c+d=a} \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle\, dx \\
&\quad + \sum_{|a|=k-1} \Big[\sum_{\substack{b+c+d=a \\ d\neq a}} \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u)\rangle\, dx \\
&\qquad + \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_2(u, u, Z^a u)\rangle\, dx \Big] \\
&\quad + \sum_{|a|\le k-2} \sum_{b+c+d+e=a} \int \langle e^{-q(\sigma)}\partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u)\rangle\, dx.
\end{aligned} \tag{3.2}$$

For the left-hand side of (3.2), by integration by parts we have

$$\begin{aligned}
&\int \langle e^{-q(\sigma)}\partial_t Z^a u, \Box Z^a u\rangle\, dx \\
&= \frac{1}{2}\frac{d}{dt}\int e^{-q(\sigma)}|\partial Z^a u|^2\, dx + \frac{1}{2}\int e^{-q(\sigma)}\langle t-r\rangle^{-2}|TZ^a u|^2\, dx.
\end{aligned} \tag{3.3}$$

Since q is bounded, there exists $c_1 > 1$ such that

$$c_1^{-1} \le e^{-q(\sigma)} \le c_1. \tag{3.4}$$

Denote

$$\overline{E}_k(u(t)) = \frac{1}{2}\sum_{|a|\le k-1}\int_{\mathbb{R}^2} e^{-q(\sigma)}|\partial Z^a u(t,x)|^2\, dx. \tag{3.5}$$

We have

$$c_1^{-1}E_k(u(t)) \le \overline{E}_k(u(t)) \le c_1 E_k(u(t)). \tag{3.6}$$

For the right-hand side of (3.2), it follows from the symmetric conditions (1.4) and integration by parts that

$$
\begin{aligned}
\int \langle e^{-q(\sigma)}\partial_t Z^a u, N_1(u, Z^a u)\rangle\, dx &= g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_t Z^a u \partial_\alpha u \partial_\beta \partial_\gamma Z^a u\, dx \\
&= g_{\alpha\beta\gamma}\int \partial_\gamma (e^{-q(\sigma)}\partial_t Z^a u \partial_\alpha u \partial_\beta Z^a u)\, dx \\
&\quad - g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_t Z^a u \partial_\alpha \partial_\gamma u \partial_\beta Z^a u\, dx \\
&\quad - g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_t \partial_\gamma Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx \\
&\quad + g_{\alpha\beta\gamma}\int e^{-q(\sigma)} q'(\sigma)\partial_\gamma \sigma \partial_t Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx \\
&= g_{\alpha\beta 0}\partial_t \int e^{-q(\sigma)}\partial_t Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx \\
&\quad - g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_t Z^a u \partial_\alpha \partial_\gamma u \partial_\beta Z^a u\, dx \\
&\quad - \frac{1}{2} g_{\alpha\beta\gamma}\partial_t \int e^{-q(\sigma)}\partial_\gamma Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx \\
&\quad + \frac{1}{2} g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_\gamma Z^a u \partial_\alpha \partial_t u \partial_\beta Z^a u\, dx \\
&\quad - \frac{1}{2} g_{\alpha\beta\gamma}\int e^{-q(\sigma)} q'(\sigma)\partial_\gamma Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx \\
&\quad + g_{\alpha\beta\gamma}\int e^{-q(\sigma)} q'(\sigma)\partial_\gamma \sigma \partial_t Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx \\
&= \frac{1}{2} g_{\alpha\beta\gamma}\eta_{\gamma\nu}\partial_t \int e^{-q(\sigma)}\partial_\nu Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx \\
&\quad - g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_t Z^a u \partial_\alpha \partial_\gamma u \partial_\beta Z^a u\, dx \\
&\quad + \frac{1}{2} g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_\gamma Z^a u \partial_\alpha \partial_t u \partial_\beta Z^a u\, dx \\
&\quad + \frac{1}{2} g_{\alpha\beta\gamma}\int e^{-q(\sigma)} q'(\sigma)\partial_\gamma Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx \\
&\quad - g_{\alpha\beta\gamma}\int e^{-q(\sigma)} q'(\sigma) T_\gamma Z^a u \partial_\alpha u \partial_\beta Z^a u\, dx,
\end{aligned} \tag{3.7}
$$

where the symbol $\eta_{\gamma\nu} = \mathrm{diag}[1, -1, -1]$. Similarly, via the symmetric conditions (1.5), we can get

$$
\begin{aligned}
\int \langle e^{-q(\sigma)}\partial_t Z^a u, N_2(u, u, Z^a u)\rangle\, dx &= h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}\partial_t Z^a u \partial_\alpha u \partial_\beta u \partial_\gamma \partial_\delta Z^a u\, dx \\
&= \frac{1}{2} h_{\alpha\beta\gamma\delta}\eta_{\gamma\nu}\partial_t \int (e^{-q(\sigma)}\partial_\nu Z^a u \partial_\alpha u \partial_\beta u \partial_\delta Z^a u)\, dx \\
&\quad - h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}\partial_t Z^a u \partial_\gamma (\partial_\alpha u \partial_\beta u)\partial_\delta Z^a u\, dx \\
&\quad + \frac{1}{2} h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}\partial_\gamma Z^a u \partial_t (\partial_\alpha u \partial_\beta u)\partial_\delta Z^a u\, dx \\
&\quad + \frac{1}{2} h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)} q'(\sigma)\partial_\gamma Z^a u \partial_\alpha u \partial_\beta u \partial_\delta Z^a u\, dx \\
&\quad - h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)} q'(\sigma) T_\gamma Z^a u \partial_\alpha u \partial_\beta u \partial_\delta Z^a u\, dx.
\end{aligned} \tag{3.8}
$$

Define the perturbed energy

$$\widetilde{E}_k(u(t)) = \sum_{|a|\le k-1} \int e^{-q(\sigma)}|\partial Z^a u|^2\,dx - \frac{1}{2}\sum_{|a|=k-1}\int g_{\alpha\beta\gamma}\eta_{\gamma\nu}e^{-q(\sigma)}\partial_\nu Z^a u\partial_\alpha u\partial_\beta Z^a u\,dx$$
$$-\frac{1}{2}\sum_{|a|=k-1}\int h_{\alpha\beta\gamma\delta}\eta_{\gamma\nu}e^{-q(\sigma)}\partial_\nu Z^a u\partial_\alpha u\partial_\beta u\partial_\delta Z^a u\,dx. \tag{3.9}$$

Noting that $||\partial u||_{L^\infty} \le CE_3^{1/2}(u(t))$, by (3.6), for small solutions we have

$$(2c_1)^{-1}E_k(u(t)) \le \widetilde{E}_k(u(t)) \le 2c_1E_k(u(t)). \tag{3.10}$$

Noting the symmetric condition (1.4), we have

$$\int \langle e^{-q(\sigma)}\partial_t Z^a u, N_1(Z^a u,u)\,dx = g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_t Z^a u\partial_\alpha Z^a u\partial_\beta\partial_\gamma u\,dx$$
$$= g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_t Z^a u\partial_\alpha\partial_\gamma u\partial_\beta Z^a u\,dx. \tag{3.11}$$

Returning to (3.2), we have derived the following energy identity:

$$\begin{aligned}
&\frac{d}{dt}\widetilde{E}_k(u(t)) + \frac{1}{2}\sum_{|a|\le k-1}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}TZ^a u||^2_{L^2}\\
&= \sum_{|a|=k-1}\Big[\sum_{\substack{b+c+d=a\\ b,c\ne a}}\int\langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle\,dx\\
&\quad + \frac{1}{2}g_{\alpha\beta\gamma}\int e^{-q(\sigma)}\partial_\gamma Z^a u\partial_\alpha\partial_t u\partial_\beta Z^a u\,dx\\
&\quad + \frac{1}{2}g_{\alpha\beta\gamma}\int e^{-q(\sigma)}q'(\sigma)\partial_\gamma Z^a u\partial_\alpha u\partial_\beta Z^a u\,dx\\
&\quad - g_{\alpha\beta\gamma}\int e^{-q(\sigma)}q'(\sigma)T_\gamma Z^a u\partial_\alpha u\partial_\beta Z^a u\,dx\Big]\\
&\quad + \sum_{|a|\le k-2}\sum_{b+c+d=a}\int\langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle\,dx\\
&\quad + \sum_{|a|=k-1}\Big[\sum_{\substack{b+c+d+e=a\\ d\ne a}}\int\langle e^{-q(\sigma)}\partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u)\rangle\,dx\\
&\quad - h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}\partial_t Z^a u\partial_\gamma(\partial_\alpha u\partial_\beta u)\partial_\delta Z^a u\,dx\\
&\quad + \frac{1}{2}h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}\partial_\gamma Z^a u\partial_t(\partial_\alpha u\partial_\beta u)\partial_\delta Z^a u\,dx\\
&\quad + \frac{1}{2}h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}q'(\sigma)\partial_\gamma Z^a u\partial_\alpha u\partial_\beta u\partial_\delta Z^a u\,dx\\
&\quad - h_{\alpha\beta\gamma\delta}\int e^{-q(\sigma)}q'(\sigma)T_\gamma Z^a u\partial_\alpha u\partial_\beta u\partial_\delta Z^a u\,dx\Big]\\
&\quad + \sum_{|a|\le k-2}\sum_{b+c+d+e=a}\int\langle e^{-q(\sigma)}\partial_t Z^a u, N_{2e}(Z^b u, Z^c u, Z^d u)\rangle\,dx.
\end{aligned} \tag{3.12}$$

The second term on the left-hand side of (3.12) is called the ghost weight energy, which will play a key role in the control of the terms involving the highest generalized derivatives on the right-hand side of (3.12) after applying the null condition.

Now we are ready to estimate the terms on the right-hand side of (3.12). It is easy to verify that all terms corresponding to the cubic nonlinearity are bounded above by

$$\sum_{|a|\le k-1}\sum_{\substack{b+c+d\le a\\ |d|\le k-2}} ||\partial Z^a u\partial Z^b u\partial Z^c u\partial^2 Z^d u||_{L^1} + ||\langle t-r\rangle^{-1}\partial Z^a u\partial u||^2_{L^2}. \tag{3.13}$$

Note that $\langle t\rangle \le C\langle r\rangle\langle t-r\rangle$. Let $m=[\frac{k-1}{2}]+1$. For $|a|\le k-1$, $b+c+d\le a$, $|d|\le k-2$, if $|b|,|c|\le m$, then it follows from (2.14) and Lemma 2.14 that

$$\begin{aligned}
&||\partial Z^a u\partial Z^b u\partial Z^c u\partial^2 Z^d u||_{L^1}\\
&\le C\langle t\rangle^{-1}||\partial Z^a u||_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}\partial Z^b u||_{L^\infty}||\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}\partial Z^c u||_{L^\infty}||\partial^2 Z^d u||_{L^2}\\
&\le C\langle t\rangle^{-1}E_k(u(t))\big(E^{1/2}_{|b|+3}(u(t))+M_{|b|+3}(u(t))\big)\big(E^{1/2}_{|c|+3}(u(t))+M_{|c|+3}(u(t))\big)\\
&\le C\langle t\rangle^{-1}E_k^2(u(t)).
\end{aligned}\tag{3.14}$$

Other cases can be handed by the same method. For the second part in (3.13), similarly, by (2.14) and Lemma 2.14 we have

$$\begin{aligned}
||\langle t-r\rangle^{-1}\partial Z^a u\partial u||^2_{L^2} &\le C\langle t\rangle^{-1}||\partial Z^a u||^2_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle^{1/2}\partial u||^2_{L^\infty}\\
&\le C\langle t\rangle^{-1}E_k^2(u(t)).
\end{aligned}\tag{3.15}$$

So it follows from (3.13)–(3.15) that all terms corresponding to the cubic nonlinearity on the right-hand side of (3.12) are bounded above by

$$C\langle t\rangle^{-1}E_k^2(u(t)). \tag{3.16}$$

To estimate the terms corresponding to the quadratic nonlinearity, i.e., the first to fifth parts on the right-hand side of (3.12), we will separate integrals over the regions $r\le t/2$ and $r\ge t/2$, respectively. To get enough decay in time on the region $r\ge t/2$, we need to exploit the null condition.

Inside the cone. On the region $r\le t/2$, all terms corresponding to the quadratic nonlinearity on the right-hand side of (3.12) are bounded above by

$$\begin{aligned}
&\sum_{|a|\le k-1}\sum_{\substack{b+c\le a\\ |b|,|c|\le k-2}} ||\partial Z^a u\partial Z^b u\partial^2 Z^c u||_{L^1(r\le t/2)}\\
&\quad+\sum_{|a|=k-1}||\partial Z^a u\partial_t\partial u\partial Z^a u||_{L^1(r\le t/2)}+\sum_{|a|=k-1}||\langle t-r\rangle^{-2}\partial Z^a u\partial u\partial Z^a u||_{L^1(r\le t/2)}.
\end{aligned}\tag{3.17}$$

Since $r\le t/2$, we have $\langle t-r\rangle\sim\langle t\rangle$. Now we estimate the first part in (3.17). For any $|a|\le k-1$, $b+c\le a$, $|b|,|c|\le k-2$, when $|b|\le m$, it follows from (2.14) and Lemma 2.14 that

$$\begin{aligned}
&||\partial Z^a u\partial Z^b u\partial^2 Z^c u||_{L^1(r\le t/2)}\\
&\le C\langle t\rangle^{-3/2}||\partial Z^a u||_{L^2}||\langle t-r\rangle^{1/2}\partial Z^b u||_{L^\infty}||\langle t-r\rangle\partial^2 Z^c u||_{L^2}\\
&\le C\langle t\rangle^{-3/2}E_k^{1/2}(u(t))E^{1/2}_{|b|+3}(u(t))M_{|c|+2}\\
&\le C\langle t\rangle^{-3/2}E_k^{3/2}(u(t)).
\end{aligned}\tag{3.18}$$

When $|c| \le m$, by (2.25), (2.15) and Lemma 2.14 we have

$$\begin{aligned}
&||\partial Z^a u \partial Z^b u \partial^2 Z^c u||_{L^1(r\le t/2)} \\
&\quad \le C\langle t\rangle^{-3/2}||\partial Z^a u||_{L^2(r\le t/2)}||\langle r\rangle^{-1/2}\langle t-r\rangle^{1/2}\partial Z^b u||_{L^2(r\le t/2)}||\langle r\rangle^{1/2}\langle t-r\rangle\partial^2 Z^c u||_{L^\infty(r\le t/2)} \\
&\quad \le C\langle t\rangle^{-3/2}E_k^{1/2}(u(t))\big(E_{|b|+1}^{1/2}(u(t)) + M_{|b|+2}(u(t))\big)M_{|c|+4} \\
&\quad \le C\langle t\rangle^{-3/2}E_k^{3/2}(u(t)).
\end{aligned} \tag{3.19}$$

Now we estimate the second part in (3.17). By (2.27), (2.14), (2.15) and Lemma 2.14, we have

$$\begin{aligned}
&||\partial Z^a u \partial_t \partial u \partial Z^a u||_{L^1(r\le t/2)} \\
&\quad \le C||\partial Z^a u||_{L^2}^2||\partial_t \partial u||_{L^\infty(r\le t/2)} \\
&\quad \le C\langle t\rangle^{-3/2}||\partial Z^a u||_{L^2}^2(||\partial u||_{L^\infty(r\le t/2)} + ||\partial S u||_{L^\infty(r\le t/2)} + ||\langle r\rangle\partial^2 u||_{L^\infty(r\le t/2)}) \\
&\quad \le C\langle t\rangle^{-3/2}||\partial Z^a u||_{L^2}^2\big(||\langle t-r\rangle^{1/2}\partial u||_{L^\infty} + ||\langle t-r\rangle^{1/2}\partial S u||_{L^\infty}\big) \\
&\qquad + C\langle t\rangle^{-2}||\partial Z^a u||_{L^2}^2||\langle r\rangle\langle t-r\rangle\partial^2 u||_{L^\infty(r\le t/2)} \\
&\quad \le C\langle t\rangle^{-3/2}||\partial Z^a u||_{L^2}^2\big(||\langle t-r\rangle^{1/2}\partial u||_{L^\infty} + ||\langle t-r\rangle^{1/2}\partial S u||_{L^\infty} + ||\langle r\rangle^{1/2}\langle t-r\rangle\partial^2 u||_{L^\infty}\big) \\
&\quad \le C\langle t\rangle^{-3/2}E_k(u(t))\big(E_3^{1/2}(u(t)) + M_4(u(t))\big) \\
&\quad \le C\langle t\rangle^{-3/2}E_k^{3/2}(u(t)).
\end{aligned} \tag{3.20}$$

For the third part in (3.17), by (2.13) we have

$$\begin{aligned}
&||\langle t-r\rangle^{-2}\partial Z^a u \partial u \partial Z^a u||_{L^1(r\le t/2)} \\
&\quad \le C\langle t\rangle^{-2}||\partial Z^a u||_{L^2}^2||\partial u||_{L^\infty} \\
&\quad \le C\langle t\rangle^{-2}E_k^{3/2}(u(t)).
\end{aligned} \tag{3.21}$$

So it follows from (3.17)–(3.21) that on the region $r \le t/2$, all terms corresponding to the quadratic nonlinearity on the right-hand side of (3.12) are bounded above by

$$C\langle t\rangle^{-3/2}E_k^{3/2}(u(t)). \tag{3.22}$$

Away from the origin. Now we consider the terms corresponding to the quadratic nonlinearity on the right-hand side of (3.12) on the region $r \ge t/2$. We first estimate the lower order terms, i.e., the first and fifth parts on the right-hand side of (3.12). For $|a| = k-1$, $b+c+d = a$, $b, c \ne a$ or $|a| \le k-2$, $b+c+d = a$, an application of Lemma 2.3 yields

$$\begin{aligned}
&||\langle e^{-q(\sigma)}\partial_t Z^a u, N_{1d}(Z^b u, Z^c u)\rangle||_{L^1(r\ge t/2)} \\
&\quad \le C\langle t\rangle^{-1}\Big[||\partial Z^a u Z^{b+1} u \partial^2 Z^c u||_{L^1(r\ge t/2)} \\
&\qquad + ||\partial Z^a u \partial Z^b u \partial Z^{c+1} u||_{L^1(r\ge t/2)} \\
&\qquad + ||\langle t-r\rangle\partial Z^a u \partial Z^b u \partial^2 Z^c u||_{L^1(r\ge t/2)}\Big].
\end{aligned} \tag{3.23}$$

It follows from (2.15), (2.20), (2.21) and Lemma 2.14 that

$$\begin{aligned}
&||\partial Z^a u Z^{b+1} u \partial^2 Z^c u||_{L^1(r\geq t/2)} \\
&\leq C\langle t\rangle^{-1/2} \begin{cases} ||\partial Z^a u||_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle^{-1} Z^{b+1}u||_{L^\infty}||\langle t-r\rangle\partial^2 Z^c u||_{L^2}, \ |b|\leq m, \\ ||\partial Z^a u||_{L^2}||\langle t-r\rangle^{-1} Z^{b+1}u||_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle\partial^2 Z^c u||_{L^\infty}, \ |c|\leq m, \end{cases} \\
&\leq C\langle t\rangle^{-1/2} \begin{cases} E_k^{1/2}(u(t))E_{|b|+2}^{1/2}(u(t))M_{|c|+2}(u(t)), \ |b|\leq m, \\ E_k^{1/2}(u(t))E_{|b|+2}^{1/2}(u(t))M_{|c|+4}(u(t)), \ |c|\leq m, \end{cases} \\
&\leq C\langle t\rangle^{-1/2}E_k^{3/2}(u(t)). \qquad (3.24)
\end{aligned}$$

The second term can be handled using (2.13):

$$\begin{aligned}
&||\partial Z^a u \partial Z^b u \partial Z^{c+1} u||_{L^1(r\geq t/2)} \\
&\leq C\langle t\rangle^{-1/2} \begin{cases} ||\partial Z^a u||_{L^2}||\langle r\rangle^{1/2}\partial Z^b u||_{L^\infty}||\partial Z^{c+1} u||_{L^2}, \ |b|\leq m, \\ ||\partial Z^a u||_{L^2}||\partial Z^b u||_{L^2}||\langle r\rangle^{1/2}\partial Z^{c+1} u||_{L^\infty}, \ |c|\leq m, \end{cases} \\
&\leq C\langle t\rangle^{-1/2} \begin{cases} E_k^{1/2}(u(t))E_{|b|+3}^{1/2}(u(t))E_{|c|+2}(u(t)), \ |b|\leq m, \\ E_k^{1/2}(u(t))E_{|b|+1}^{1/2}(u(t))E_{|c|+4}(u(t)), \ |c|\leq m, \end{cases} \\
&\leq C\langle t\rangle^{-1/2}E_k^{3/2}(u(t)). \qquad (3.25)
\end{aligned}$$

The final term can be estimated using (2.13), (2.15), (2.20), (2.21) and Lemma 2.14:

$$\begin{aligned}
&||\langle t-r\rangle \partial Z^a u \partial Z^b u \partial^2 Z^c u||_{L^1(r\geq t/2)} \\
&\leq C\langle t\rangle^{-1/2} \begin{cases} ||\partial Z^a u||_{L^2}||\langle r\rangle^{1/2}\partial Z^b u||_{L^\infty}||\langle t-r\rangle\partial^2 Z^c u||_{L^2}, \ |b|\leq m, \\ ||\partial Z^a u||_{L^2}||\partial Z^b u||_{L^2}||\langle r\rangle^{1/2}\langle t-r\rangle\partial^2 Z^c u||_{L^\infty}, \ |c|\leq m, \end{cases} \\
&\leq C\langle t\rangle^{-1/2} \begin{cases} E_k^{1/2}(u(t))E_{|b|+3}^{1/2}(u(t))M_{|c|+2}(u(t)), \ |b|\leq m, \\ E_k^{1/2}(u(t))E_{|b|+1}^{1/2}(u(t))M_{|c|+4}(u(t)), \ |c|\leq m, \end{cases} \\
&\leq C\langle t\rangle^{-1/2}E_k^{3/2}(u(t)). \qquad (3.26)
\end{aligned}$$

So it follows from (3.23)–(3.26) that on the region $r\geq t/2$, the first and fifth parts on the right-hand side of (3.12) are bounded above by

$$C\langle t\rangle^{-3/2}E_k^{3/2}(u(t)). \qquad (3.27)$$

Now we estimate the terms involving the highest order generalized derivatives, i.e., the second, third and fourth parts on the right-hand side of (3.12). For the second part on the right-hand side of (3.12), it follows from (2.5), the Cauchy–Schwartz inequality, (2.1), (2.13), (2.15) and Lemma 2.14 that

$$\begin{aligned}
&||g_{\alpha\beta\gamma}e^{-q(\sigma)}\partial_\gamma Z^a u \partial_\alpha \partial_t u \partial_\beta Z^a u||_{L^1(r\geq t/2)} \\
&\leq C||TZ^a u \partial^2 u \partial Z^a u||_{L^1(r\geq t/2)} + C||\partial Z^a u T\partial u \partial Z^a u||_{L^1(r\geq t/2)} \\
&\leq \frac{1}{8}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}TZ^a u||^2_{L^2(r\geq t/2)} + C\langle t\rangle^{-1}||\partial Z^a u||^2_{L^2(r\geq t/2)}||\langle r\rangle^{1/2}\langle t-r\rangle\partial^2 u||^2_{L^\infty(r\geq t/2)} \\
&\quad + C\langle t\rangle^{-3/2}||\partial Z^a u||_{L^2(r\geq t/2)}||\partial Z^a u||_{L^2(r\geq t/2)}\big(||\langle r\rangle^{1/2}Z\partial u||_{L^\infty(r\geq t/2)} + ||\langle r\rangle^{1/2}\langle t-r\rangle\partial^2 u||_{L^\infty(r\geq t/2)}\big) \\
&\leq \frac{1}{8}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}TZ^a u||^2_{L^2} + C\langle t\rangle^{-1}E_k(u(t))M_4^2(u(t)) \\
&\quad + C\langle t\rangle^{-3/2}E_k(u(t))\big(E_3^{1/2}(u(t)) + M_4(u(t))\big) \\
&\leq \frac{1}{8}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}TZ^a u||^2_{L^2} + C\langle t\rangle^{-1}E_k^2(u(t)) + C\langle t\rangle^{-3/2}E_k^{3/2}(u(t)). \qquad (3.28)
\end{aligned}$$

For the third and fourth parts on the right-hand side of (3.12), owing to (2.5), the Cauchy–Schwartz inequality, (2.1), (2.13) and (2.21), we have

$$\begin{aligned}
&||g_{\alpha\beta\gamma}e^{-q(\sigma)}q'(\sigma)\partial_\gamma Z^a u\partial_\alpha u\partial_\beta Z^a u||_{L^1(r\geq t/2)} + ||g_{\alpha\beta\gamma}e^{-q(\sigma)}q'(\sigma)T_\gamma Z^a u\partial_\alpha u\partial_\beta Z^a u||_{L^1(r\geq t/2)} \\
&\quad \leq C||\langle t-r\rangle^{-2}TZ^a u\partial u\partial Z^a u||_{L^1(r\geq t/2)} + C||\langle t-r\rangle^{-2}\partial Z^a u T u\partial Z^a u||_{L^1(r\geq t/2)} \\
&\quad \leq \frac{1}{8}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}TZ^a u||^2_{L^2(r\geq t/2)} + C\langle t\rangle^{-1}||\langle r\rangle^{1/2}\partial u||^2_{L^\infty(r\geq t/2)}||\partial Z^a u||^2_{L^2(r\geq t/2)} \\
&\qquad + C\langle t\rangle^{-3/2}||\partial Z^a u||^2_{L^2(r\geq t/2)}\big(||\langle r\rangle^{1/2}\langle t-r\rangle^{-2}Zu||_{L^\infty(r\geq t/2)} + ||\langle r\rangle^{1/2}\langle t-r\rangle^{-1}\partial u||_{L^\infty(r\geq t/2)}\big) \\
&\quad \leq \frac{1}{8}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}TZ^a u||^2_{L^2} + C\langle t\rangle^{-1}E_k^2(u(t)) + C\langle t\rangle^{-3/2}E_k^{3/2}(u(t)).
\end{aligned} \tag{3.29}$$

So it follows from (3.28)–(3.29) that on the region $r \geq t/2$, the second, third and fourth parts on the right-hand side of (3.12) are bounded above by

$$\frac{1}{8}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}TZ^a u||^2_{L^2} + C\langle t\rangle^{-1}E_k^2(u(t)) + C\langle t\rangle^{-3/2}E_k^{3/2}(u(t)). \tag{3.30}$$

We note that the first term in (3.30) can be absorbed into the ghost weight energy term on the left-hand side of (3.12).

3.2. Conclusion of the proof

Now we have got that for any integer $k \geq 10$, it holds that

$$\frac{d}{dt}\widetilde{E}_k(u(t)) \leq C\langle t\rangle^{-3/2}E_k^{3/2}(u(t)) + C\langle t\rangle^{-1}E_k^2(u(t)). \tag{3.31}$$

Assume that

$$\widetilde{E}_k(u(0)) \leq \bar{c}\varepsilon^2. \tag{3.32}$$

Noting (3.10), by the assumption

$$\sup_{0\leq t\leq T} E_k^{1/2}(u(t)) \leq 2A\varepsilon, \tag{3.33}$$

we can get

$$\sup_{0\leq t\leq T} E_k(u(t)) \leq 2c_1\widetilde{E}_k(u(t)) \leq 2c_1\bar{c}\varepsilon^2 + C_1A^3\varepsilon^3 + C_1\log(2+T)A^4\varepsilon^4. \tag{3.34}$$

Take $A^2 = 8c_1\bar{c}$ and $\varepsilon > 0$ sufficiently small such that

$$C_1A\varepsilon \leq \frac{1}{4}, \tag{3.35}$$

and

$$C_1\log(2+T)A^2\varepsilon^2 \leq \frac{1}{4}, \tag{3.36}$$

then we have

$$\sup_{0\le t\le T} E_k^{1/2}(u(t)) \le A\varepsilon. \tag{3.37}$$

Consequently, it follows from (3.36) that we can get the lifespan estimate of classical solutions to the Cauchy problem (1.16) and (1.2):

$$T_\varepsilon \ge \exp(\frac{c}{\varepsilon^2}), \tag{3.38}$$

where c a positive constant independent of ε.

Acknowledgments

The authors would like to express their sincere gratitude to Prof. Tatsien Li and Prof. Yi Zhou for their helpful suggestions and encouragements. The first author is supported by NSFC (grant No. 11471215), Shanghai Leading Academic Discipline Project (grant No. XTKX2012).

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