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Global and almost global existence for general quasilinear wave equations in two space dimensions

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A B S T R A C T

For 2-D quasilinear wave equations of the form $\Box u = F(u, Du, D^2u)$, we show that the corresponding Cauchy problem admits a unique global classical solution with small initial data, if both the quadratic nonlinearity and the cubic nonlinearity satisfy the corresponding null conditions. We also prove that the classical solution is almost global, i.e., the lifespan of solution $T_\varepsilon \geq \exp(c\varepsilon^{-2})$, where $\varepsilon > 0$ denotes the amplitude of the initial data and c is a positive constant, provided that the quadratic nonlinearity satisfies the null condition and $\partial_u^\beta F(0, 0, 0) = 0$, $\beta = 3, 4$. These results extend Alinhac's seminal work Alinhac (2001) [9] to the general case in a unified way.
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R É S U M É

Pour les équations d'onde quasilinéaires 2D de la forme $\Box u = F(u, Du, D^2u)$, nous montrons que le problème de Cauchy correspondant admet une solution classique globale unique avec de petites données initiales, si la non-linéarité quadratique et la non-linéarité cubique vérifie les conditions nulles correspondantes. Nous prouvons aussi que la solution classique est presque globale, ie, la durée de vie de la solution $T_\varepsilon \geq \exp(c\varepsilon^{-2})$, où $\varepsilon > 0$ dénote l'amplitude des données initiales et c est une constante positive, à condition que la non-linéarité quadratique satisfasse la condition nulle et que $\partial_u^\beta F(0, 0, 0) = 0$, $\beta = 3, 4$. Ces résultats étendent le travail séminal d'Alinhac Alinhac (2001) [9] au cas général d'une manière unifiée.
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1. Introduction

We consider the following Cauchy problem for 2-D quasilinear wave equations:

$$\begin{cases}\Box u(t,x) = F(u, Du, D^2u),\ (t,x)\in \mathbb{R}^+\times\mathbb{R}^2,\\ t=0: u=\varepsilon f, u_t=\varepsilon g,\ x\in\mathbb{R}^2,\end{cases} \tag{1}$$

where $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ is the 2-D wave operator, $(t,x) = (x_0, x_1, x_2)$, $\partial_\alpha = \frac{\partial}{\partial x_\alpha}(\alpha = 0,1,2)$, $D = (\partial_0, \partial_1, \partial_2)$, $\varepsilon > 0$ is a small parameter, and $f, g \in C_c^\infty(\mathbb{R}^2)$. Without loss of generality, we assume that f and g are supported in $|x| \le 1$. Let

$$\widehat{\lambda} = (\lambda; (\lambda_\alpha), \alpha = 0,1,2; (\lambda_{\alpha\beta}), \alpha,\beta = 0,1,2). \tag{2}$$

Suppose that in a neighborhood of $\widehat{\lambda} = 0$, say for $|\widehat{\lambda}| \le 1$, the nonlinear term $F = F(\widehat{\lambda})$ is a sufficiently smooth function satisfying

$$F(\widehat{\lambda}) = \mathscr{O}(|\widehat{\lambda}|^{1+\theta}), \tag{3}$$

where $\theta \ge 1$ is an integer, and is affine with respect to $\lambda_{\alpha\beta}$ $(\alpha,\beta = 0,1,2)$.

Our aim is to study the lifespan of classical solutions to Cauchy problem (1). By definition, the lifespan T_ε is the supremum of all $T > 0$, such that there exists a classical solution to (1) on $0 \le t \le T$, i.e.,

$$T_\varepsilon \overset{\text{def}}{=} \sup\big\{T > 0 : (1) \text{ has a unique classical solution on } [0,T]\big\}. \tag{4}$$

Under condition (3), complete results on lifespan estimates of classical solutions to Cauchy problem (1) have been established as follows:

$\theta =$	1	2	3,4,...
$T_\varepsilon \ge$	$be(\varepsilon)$ $b\varepsilon^{-1}$, if $\int g dx = 0$ $b\varepsilon^{-2}$, if $\partial_u^2 F(0,0,0) = 0$	$b\varepsilon^{-6}$ $b\varepsilon^{-18}$, if $\partial_u^3 F(0,0,0) = 0$ $\exp(b\varepsilon^{-2})$, if $\partial_u^\beta F(0,0,0), \beta = 3,4$	$+\infty$

Here $e(\varepsilon)$ satisfies

$$\varepsilon^2 e(\varepsilon)\ln(1 + e(\varepsilon)) = 1, \tag{5}$$

and b is positive constant independent of ε. The results corresponding to $\theta = 1$ were first established by Li and Zhou [1]; for the results on $\theta = 2$, the first and third ones were shown by Li and Zhou [2] and the second one was proved by Katayama [3]; the result in the case $\theta \ge 3$ can be found in Li and Zhou [4]. We point out that all the above results are sharp (see Li and Zhou [5]).

On the other hand, if we add some null conditions (see Definition 1.1) on the nonlinearity, the lifespan can be improved significantly. In the case $\theta = 2$, if the cubic nonlinearity satisfies the null condition, the global existence of classical solutions was first proved by Zhou [6] (see also [5]), then by Katayama [7]. The special case that F is independent of u was considered by Hoshiga [8].

For the most important and difficult case $\theta = 1$, in the special case

$$F(u, Du, D^2u) = g_{\alpha\beta}(Du)\partial_\alpha\partial_\beta u, \tag{6}$$

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here and hereafter, we always use the summation convention that repeated indices are summed, Alinhac [9] proved that if both the quadratic nonlinearity and the cubic nonlinearity satisfy the corresponding null conditions, then Cauchy problem (1) admits a unique global classical solution with small initial data; moreover, if only the quadratic nonlinearity satisfies the null condition, then the classical solution is almost global, i.e., the lifespan $T_\varepsilon \geq \exp(c\varepsilon^{-2})$.

Lifespan estimates of classical solutions to the Cauchy problem of nonlinear wave equations have been an active subject for over 30 years since John and Klainerman's pioneering works. In other spatial dimensions, classical references can be found in John [10,11], John and Klainerman [12], Klainerman [13–16], Christodoulou [17], Hörmander [18–20], Lindblad [21], Li and Chen [22,23], Li and Yu [24], Li and Zhou [25], etc. Especially, Klainerman's commutative vector field method [14] offers the basic tool for treating this kind of problem. A systematic summary on this topic can be found in Li and Zhou's recent monograph [5], which is based on the global iterative method.

A natural problem for 2-D Cauchy problem (1) is how to extend Alinhac's work [9] to the general case, which was suggested as an open problem in Li and Zhou's monograph [5]. This paper is intended to solve this problem. Specifically speaking, we will show that in the case $\theta = 1$, without the additional assumption (6), Cauchy problem (1) admits a unique global classical solution with small initial data, if both the quadratic nonlinearity and the cubic nonlinearity satisfy the corresponding null conditions; and the classical solution is almost global, i.e., the lifespan $T_\varepsilon \geq \exp(c\varepsilon^{-2})$, provided that the quadratic nonlinearity satisfies the null condition and $\partial_u^\beta F(0,0,0) = 0, \beta = 3, 4$.

Now we will precisely formulate our problem. We write

$$F(u, Du, D^2u) = Q(u, Du, D^2u) + C(u, Du, D^2u) + H(u, Du, D^2u), \tag{7}$$

where Q is the quadratic nonlinearity, C stands for the cubic nonlinearity, and H denotes the higher order nonlinearities, i.e., $H(\widehat{\lambda}) = \mathscr{O}(|\widehat{\lambda}|^4)$.

Following Klainerman [15], we will give the definition of null condition. First for $y = (y_0, y_1, y_2)$, if $y_0^2 - y_1^2 - y_2^2 = 0$, then y is called to be a null vector.

Definition 1.1. The quadratic nonlinearity $Q(u, Du, D^2u)$ satisfies the first null condition, if for any given null vector y and any given constants p, q, r, we have

$$Q(p, qy, ryy) = 0. \tag{8}$$

Similarly, the cubic nonlinearity $C(u, Du, D^2u)$ satisfies the second null condition, if for any given null vector y and any given constants p, q, r, we have

$$C(p, qy, ryy) = 0. \tag{9}$$

Remark 1.1. In [6], Zhou first observed that the first null condition (8) just means that if we insert any given plane wave solution of the homogeneous linear wave equation into the quadratic nonlinearity of the nonlinear equation, it will vanish identically. The second null condition (9) can be explained in the same way. See also Li and Zhou [5].

The main results of this paper are the following

Theorem 1.1. *For Cauchy problem (1) with $\theta = 1$, assume that the quadratic nonlinearity satisfies the first null condition (8) and the cubic nonlinearity satisfies the second null condition (9). Then for any given positive parameter ε small enough, (1) admits a unique global classical solution u.*

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Theorem 1.2. *For Cauchy problem (1) with $\theta = 1$, assume that the quadratic nonlinearity satisfies the first null condition (8) and $\partial_u^\beta F(0,0,0) = 0, \beta = 3,4$. Then for any given positive parameter ε small enough, (1) admits a unique classical solution u with the lifespan*

$$T_\varepsilon \geq \exp(c\varepsilon^{-2}), \tag{10}$$

where c is positive constant independent of ε.

Remark 1.2. Besides [4], [6], [7] and [9], some global existence results on physically meaningful equations in two space dimensions which fall into the form of quasilinear wave equations, such as timelike minimal surface [26], Faddeev model [27], curl free Chaplygin gas [28] and radially symmetric elastic waves [29], can be also deduced from Theorem 1.1. While, Theorem 1.2 generalizes the corresponding results in [2] and [9] simultaneously.

Remark 1.3. The proof of Theorem 1.1 and Theorem 1.2 is also valid for some systems of quasilinear wave equations. Because a fully nonlinear wave equation can be reduced to a system of quasilinear wave equations (see [5]), the results in Theorem 1.1 and Theorem 1.2 can be extended to the fully nonlinear case. See also Remark 2.1.

Remark 1.4. Due to the special form (6) of the nonlinearity, the proof in [9] only involves the energy type method including general energy estimates and ghost weight energy estimates. To treat general nonlinearity, the main difficult is how to design and estimate the norms concerning the solution itself in the bootstrap argument, which is also the main innovation of this paper. For this purpose, we will take advantage of Hörmander's L^1–L^∞ estimates [30] and Li and Zhou's L^2 and L^p estimates [1,4]. Together with sufficient using of the null condition, we can close our bootstrap argument. We should also point out that our work is inspired by previous works Li and Zhou [1,2,4], Zhou [6], Katayama [7] and Alinhac [9].

An outline of this paper is as follows. The remainder of this introduction will be devoted to the description of some notations which will be used in the sequel. In the next Section, some necessary tools used to prove Theorem 1.1 and Theorem 1.2 are introduced. Then the proof of Theorem 1.1 and Theorem 1.2 will be given in Section 3 and Section 4, respectively.

1.1. Notation

As in Klainerman [14], we first introduce some vectors fields. Denote the spatial rotation $\Omega = x_1\partial_2 - x_2\partial_1$, the scaling operator $S = t\partial_t + x_1\partial_1 + x_2\partial_2$, and the Lorentz boost operators $L_i = t\partial_i + x_i\partial_t$, $i = 1,2$. Define the vector fields $\Gamma = (D, \Omega, S, L_1, L_2) = (\Gamma_1, \dots, \Gamma_7)$. For any given multi-index $a = (a_1, \dots, a_7)$, we denote $\Gamma^a = \Gamma_1^{a_1} \cdots \Gamma_7^{a_7}$. It can be verified that (see [21])

$$|Du| \leq C\langle t - r\rangle^{-1}|\Gamma u|, \tag{11}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. We will also use the good derivatives (see [9])

$$T_\alpha = \omega_\alpha \partial_t + \partial_\alpha, \tag{12}$$

where $\omega_0 = -1, \omega_i = x_i/r$ $(i = 1,2), r = |x|$. Denote $T = (T_0, T_1, T_2)$. Compared with (11), we get the following decay estimate:

$$|Tu| \leq C(1 + t)^{-1}|\Gamma u|. \tag{13}$$

As in Li and Yu [24], we introduce the Banach space $L^{p,q}(\mathbb{R}^2)$ equipped with the norm

$$\|f\|_{L^{p,q}(\mathbb{R}^2)} = \|f(r\omega)r^{\frac{1}{p}}\|_{L^p_r(0,\infty;L^q_\omega(S^1))}, \tag{14}$$

where $1 \le p, q \le +\infty$. It is easy to see that if $p = q$, then $L^{p,q}(\mathbb{R}^2)$ becomes the usual Lebesgue space $L^p(\mathbb{R}^2)$. For any given integer N, define

$$\|u(t,\cdot)\|_{\Gamma,N,p,q,\chi} = \sum_{|k|\le N} ||\chi(t,\cdot)\Gamma^k u(t,\cdot)||_{L^{p,q}(\mathbb{R}^2)}, \tag{15}$$

where $\chi = \chi(t, x)$ is the characteristic function of any given set in $\mathbb{R}^+ \times \mathbb{R}^2$. In what follows, we also briefly denote

$$\|u(t,\cdot)\|_{\Gamma,N,p,q,\chi} = \begin{cases} \|u(t,\cdot)\|_{\Gamma,N,p,\chi}, & \text{if } p = q; \\ \|u(t,\cdot)\|_{\Gamma,N,p,q}, & \text{if } \chi \equiv 1; \\ \|u(t,\cdot)\|_{\Gamma,N,p}, & \text{if } p = q,\ \chi \equiv 1; \\ \|u(t,\cdot)\|_{p,q,\chi}, & \text{if } N = 0; \\ \|u(t,\cdot)\|_{L^{p,q}(\mathbb{R}^2)}, & \text{if } N = 0,\ \chi \equiv 1. \end{cases} \tag{16}$$

2. Preliminaries

2.1. Commutation relations

The following lemma concerning the commutation relation between the wave operator and the vector fields Γ was first established by Klainerman [14].

Lemma 2.1. *For any given multi-index $k = (k_1, \dots, k_7)$, we have*

$$[\Box, \Gamma^k]u = \sum_{|a|\le|k|-1} C_{ak}\Gamma^a \Box u, \tag{17}$$

where $[\cdot,\cdot]$ stands for the Poisson's bracket, i.e., $[A, B] = AB - BA$, and C_{ak} are constants.

2.2. Null conditions

Lemma 2.2. *The quadratic nonlinearity $Q(u, Du, D^2u)$ satisfies the first null condition, if and only if*

$$Q(u, Du, D^2u) = g_{\alpha\beta\gamma}\partial_\gamma u \partial_\alpha\partial_\beta u + g_{\alpha\beta}\partial_\alpha u \partial_\beta u, \tag{18}$$

and

$$g_{\alpha\beta\gamma}y_\alpha y_\beta y_\gamma = g_{\alpha\beta}y_\alpha y_\beta = 0 \tag{19}$$

for any given null vector y. The cubic nonlinearity $C(u, Du, D^2u)$ satisfies the second null condition, if and only if

$$\begin{aligned} C(u, Du, D^2u) = {} & h_{\alpha\beta\gamma\delta}\partial_\delta u \partial_\gamma u \partial_\alpha\partial_\beta u + h_{\alpha\beta\gamma}u\partial_\gamma u \partial_\alpha\partial_\beta u \\ & + \overline{h}_{\alpha\beta\gamma}\partial_\alpha u \partial_\beta u \partial_\gamma u + h_{\alpha\beta}u\partial_\alpha u \partial_\beta u, \end{aligned} \tag{20}$$

and

$$h_{\alpha\beta\gamma\delta}y_\alpha y_\beta y_\gamma y_\delta = h_{\alpha\beta\gamma}y_\alpha y_\beta y_\gamma = \overline{h}_{\alpha\beta\gamma}y_\alpha y_\beta y_\gamma = h_{\alpha\beta}y_\alpha y_\beta = 0 \tag{21}$$

for any given null vector y.

Proof. For the quadratic nonlinearity, the 3-D analogue of (18) and (19) can be found in Klainerman [15], Christodoulou [17] and Li and Zhou [5]. As to the proof of (20) and (21) for the cubic nonlinearity, the details can be found in Li and Zhou [5]. □

Remark 2.1. In what follows we will always assume that (18) and (19) hold. Without loss of generality, we can further assume $g_{\alpha\beta} = 0$. In fact, we have

$$g_{\alpha\beta}\partial_\alpha u\partial_\beta u = c(|\nabla u|^2 - |u_t|^2), \tag{22}$$

where c is a constant. If $c \neq 0$, the transformation $v = e^{cu} - 1$ can be used to cancel (22), and the properties satisfied by other kinds of terms will not be effected. In the fully nonlinear case, the semilinear term in the quadratic part also takes the form (22), so it can be canceled by the same way. Such transformation was first used for Nirenberg's example in [13] (see also [31]).

Denote the bilinear form

$$N(u, v) = g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta v, \tag{23}$$

where for any given null vector y, we have

$$g_{\alpha\beta\gamma}y_\alpha y_\beta y_\gamma = 0. \tag{24}$$

Lemma 2.3. *For any given multi-index* $k = (k_1, \dots, k_7)$*, we have*

$$\Gamma^k N(u, v) = \sum_{b+c+d=k} N_d(\Gamma^b u, \Gamma^c v), \tag{25}$$

where each N_d *is a quadratic nonlinearity of the form* (23) *satisfying* (24)*. Moreover, if* $b + c = k$*, then* $N_d = N$.

Proof. See Hörmander [20]. □

Lemma 2.4. *For any given three smooth functions* u, v, w*, we have*

$$|g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta v| \leq C\big(|Tu||D^2v| + |Du||TDv|\big), \tag{26}$$

$$|g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha v\partial_\beta w| \leq C\big(|Tu||Dv||Dw| + |Du||Tv||Dw| + |Du||Dv||Tw|\big). \tag{27}$$

Here and in what follows C *denotes a positive constant.*

Proof. By definition (12) of good derivatives, we have the following pointwise equality:

$$\begin{aligned} g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha v\partial_\beta w &= g_{\alpha\beta\gamma}T_\gamma u\partial_\alpha v\partial_\beta w - g_{\alpha\beta\gamma}\omega_\gamma\partial_t uT_\alpha v\partial_\beta w \\ &\quad + g_{\alpha\beta\gamma}\omega_\gamma\partial_t u\omega_\alpha\partial_t vT_\beta w - g_{\alpha\beta\gamma}\omega_\alpha\omega_\beta\omega_\gamma\partial_t u\partial_t v\partial_t w, \end{aligned} \tag{28}$$

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thus (27) follows directly from (28) and (24). (26) can be similarly proved. The details can be found in Alinhac [9]. □

Lemma 2.5. *If the quadratic nonlinearity $Q(u, Du, D^2u)$ satisfies the first null condition (8), then we have*

$$|\Gamma^k Q(u, Du, D^2u)| \le C(1+t)^{-1} \sum_{|k_1|+|k_2|\le|k|} \big(|D\Gamma^{k_1}u||D\Gamma^{k_2+1}u| + |\Gamma^{k_1+1}u||D^2\Gamma^{k_2}u|\big). \tag{29}$$

Proof. *Noting that $Q(u, Du, D^2u) = N(u, u)$, (29) follows from Lemma 2.3, Lemma 2.4 and (13).* □

Similarly, we have

Lemma 2.6. *If the cubic nonlinearity $C(u, Du, D^2u)$ satisfies the second null condition (9), then we have*

$$|\Gamma^k C(u, Du, D^2u)| \le C(1+t)^{-1} \sum_{|k_1|+|k_2|+|k_3|\le|k|+3} |\Gamma^{k_1}u||\Gamma^{k_2}u||D\Gamma^{k_3}u|. \tag{30}$$

2.3. Sobolev and Hardy type inequalities

Lemma 2.7. *If $u = u(t, x)$ is a smooth function with sufficient decay at the infinity, then we have*

$$|u(t, x)| \le C\langle t + |x|\rangle^{-\frac{1}{2}}\langle t - |x|\rangle^{-\frac{1}{2}}\|u(t, \cdot)\|_{\Gamma,2,2}. \tag{31}$$

Proof. See Klainerman [16]. □

Lemma 2.8. *If $u = u(t, x)$ is a smooth function supported in $|x| \le t + 1$, then we have the following Hardy type inequality:*

$$\|\langle t - r\rangle^{-1}u\|_{L^2(\mathbb{R}^2)} \le C\|\partial_r u\|_{L^2(\mathbb{R}^2)}. \tag{32}$$

Proof. See Alinhac [32] or Lindblad [21]. □

2.4. Estimates for linear wave equations

Consider the Cauchy problems

$$\begin{cases} \Box u(t, x) = F(t, x), \ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ t = 0 : u = 0, u_t = 0, \ x \in \mathbb{R}^2 \end{cases} \tag{33}$$

and

$$\begin{cases} \Box u(t, x) = \partial_\alpha G_\alpha(t, x), \ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ t = 0 : u = 0, u_t = 0, \ x \in \mathbb{R}^2, \end{cases} \tag{34}$$

where $\alpha = 0, 1, 2$. We also denote $G = (G_0, G_1, G_2)$.

The following two lemmas on L^1–L^∞ estimates can be found in Hörmander [30].

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Lemma 2.9. *Let u satisfy* (33). *We have*

$$\|\langle t+|\cdot|\rangle^{\frac{1}{2}}\langle t-|\cdot|\rangle^{l}u(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le C\int_0^t (1+\tau)^{-\frac{1}{2}+l}\|F(\tau,\cdot)\|_{\Gamma,1,1}d\tau, \tag{35}$$

where $0 \le l \le \frac{1}{2}$.

Lemma 2.10. *Let u satisfy* (34). *We have*

$$\begin{aligned}&(1+t)^{\frac{1}{2}}\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\\ &\le C\int_0^t (1+\tau)^{\frac{1}{2}}\|G(\tau,\cdot)\|_{L^\infty(\mathbb{R}^2)} + (1+\tau)^{-\frac{3}{2}}\|G(\tau,\cdot)\|_{\Gamma,3,1}d\tau.\end{aligned} \tag{36}$$

Lemma 2.11. *Let u satisfy* (33). *We have*

$$\|u(t,\cdot)\|_{L^p(\mathbb{R}^2)} \le C\int_0^t \|F(\tau,\cdot)\|_{L^q(\mathbb{R}^2)}d\tau, \tag{37}$$

where

$$2 < p < +\infty, \quad \frac{1}{q} = \frac{1}{2} + \frac{1}{p}. \tag{38}$$

Proof. By (38), we can take s $(0 < s < 1)$ such that

$$\frac{2}{p} = -s + 1 = -1 + \frac{2}{q}. \tag{39}$$

Then it follows from Sobolev inequalities and the generalized energy inequality that

$$\|u(t,\cdot)\|_{L^p(\mathbb{R}^2)} \le C\|u(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^2)} \le C\int_0^t \|F(\tau,\cdot)\|_{\dot{H}^{s-1}(\mathbb{R}^2)}d\tau \le C\int_0^t \|F(\tau,\cdot)\|_{L^q(\mathbb{R}^2)}d\tau. \tag{40}$$

The details of the proof can be found in Li and Zhou [4]. □

Lemma 2.12. *Let u satisfy* (33). *We have*

$$(1+t)^{-\delta}\|u(t,\cdot)\|_{L^2(\mathbb{R}^2)} \le C\int_0^t \|F(\tau,\cdot)\|_{q,\chi_1} + (1+\tau)^{-\delta}\|F(\tau,\cdot)\|_{1,2,\chi_2}d\tau, \tag{41}$$

where $0 < \delta < \frac{1}{2}$ *and*

$$\frac{1}{q} = 1 - \frac{\delta}{2}. \tag{42}$$

In (41), χ_1 *stands for the characteristic function of the set* $\{(t,x) : |x| \le \frac{1+t}{2}\}$, $\chi_2 = 1 - \chi_1$.

Proof. See Li and Zhou [1]. □

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Lemma 2.13. *Let u satisfy (34). We have*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^2)} \le C(\log(2+t))^{1/2}\|(1+|\cdot|)^2 G_0(0,\cdot)\|_{L^2(\mathbb{R}^2)} + C\int_0^t \|G(\tau,\cdot)\|_{L^2(\mathbb{R}^2)}d\tau. \tag{43}$$

Proof. Let u_α $(\alpha=0,1,2)$ satisfy

$$\begin{cases} \Box u_\alpha(t,x) = G_\alpha(t,x), \ (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \\ t=0: u_\alpha = 0, \partial_t u_\alpha = 0, \ x\in\mathbb{R}^2 \end{cases} \tag{44}$$

and v satisfy

$$\begin{cases} \Box v(t,x) = 0, \ (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \\ t=0: v=0, v_t = G_0(0,x), \ x\in\mathbb{R}^2. \end{cases} \tag{45}$$

It is easy to verify that

$$u = \partial_\alpha u_\alpha - v. \tag{46}$$

By the general energy estimate, we get

$$\|Du_\alpha(t,\cdot)\|_{L^2(\mathbb{R}^2)} \le C\int_0^t \|G(\tau,\cdot)\|_{L^2(\mathbb{R}^2)}d\tau. \tag{47}$$

We also have (see [20] and [5])

$$\|v(t,\cdot)\|_{L^2(\mathbb{R}^2)} \le C(\log(2+t))^{1/2}\|(1+|\cdot|)^2 G_0(0,\cdot)\|_{L^2(\mathbb{R}^2)}. \tag{48}$$

Thus, (43) follows from (46)–(48). □

2.5. Some estimates on product functions and composite functions

Lemma 2.14. *Assume that u and v are smooth functions supported in $|x|\le t+1$. If the multi-indices k_1,k_2 satisfy $|k_1|+|k_2|\le m$,*

$$\frac{1}{p_1} = \frac{1}{2}+\frac{1}{p_2}, \ \frac{1}{q_1} = \frac{1}{2}+\frac{1}{q_2}, \tag{49}$$

then we have

$$\|\Gamma^{k_1}uD\Gamma^{k_2}v\|_{L^{p_1,q_1}(\mathbb{R}^2)} \le C\big(\|u\|_{\Gamma,[\frac{m}{2}]+2,p_2,q_2}\|Dv\|_{\Gamma,m,2} + \|Du\|_{\Gamma,m,2}\|v\|_{\Gamma,[\frac{m}{2}]+2,p_2,q_2}\big), \tag{50}$$

where $[n]$ stands for the integer part of the real number n.

Proof. The details of the proof of Lemma 2.14 can be found in Li and Zhou [5]. Because the technique in the proof will be frequently used in what follows, we will write it down here briefly. Note that $|k_1|+|k_2|\le m$. If $|k_1|\le[\frac{m}{2}]+1$, it follows from Hölder inequality that

$$\|\Gamma^{k_1}uD\Gamma^{k_2}v\|_{L^{p_1,q_1}(\mathbb{R}^2)} \le \|\Gamma^{k_1}u\|_{L^{p_2,q_2}(\mathbb{R}^2)}\|D\Gamma^{k_2}v\|_{L^2(\mathbb{R}^2)} \le C\|u\|_{\Gamma,[\frac{m}{2}]+1,p_2,q_2}\|Dv\|_{\Gamma,m,2}. \tag{51}$$

If $|k_2| \le [\frac{m}{2}]+1$, by Hölder inequality, (11) and the Hardy type inequality (32), we have

$$\begin{aligned}
&\|\Gamma^{k_1}uD\Gamma^{k_2}v\|_{L^{p_1,q_1}(\mathbb{R}^2)}\\
&\le \|\langle t-r\rangle^{-1}\Gamma^{k_1}u\|_{L^2(\mathbb{R}^2)}\|\langle t-r\rangle D\Gamma^{k_2}v\|_{L^{p_2,q_2}(\mathbb{R}^2)}\\
&\le \|D\Gamma^{k_1}u\|_{L^2(\mathbb{R}^2)}\|\Gamma^{k_2+1}v\|_{L^{p_2,q_2}(\mathbb{R}^2)} \le C\|Du\|_{\Gamma,m,2}\|v\|_{\Gamma,[\frac{m}{2}]+2,p_2,q_2}.
\end{aligned} \tag{52}$$

The above discussion gives (50). □

Similarly, we have

Lemma 2.15. *Assume that u, v and w are smooth functions supported in $|x| \le t+1$. If the multi-indices k_1, k_2, k_3 satisfy $|k_1|+|k_2|+|k_3| \le m$,*

$$\frac{1}{p_1} = \frac{1}{2} + \frac{1}{p_2} + \frac{1}{p_3}, \tag{53}$$

then we have

$$\begin{aligned}
&\|\Gamma^{k_1}u\Gamma^{k_2}vD\Gamma^{k_3}w\|_{L^{p_1}(\mathbb{R}^2)}\\
&\le C\big(\|u\|_{\Gamma,[\frac{m}{2}]+2,p_2}\|v\|_{\Gamma,[\frac{m}{2}]+2,p_3}\|Dw\|_{\Gamma,m,2} + \|u\|_{\Gamma,[\frac{m}{2}]+2,p_2}\|Dv\|_{\Gamma,m,2}\|w\|_{\Gamma,[\frac{m}{2}]+2,p_3}\\
&\quad + \|Du\|_{\Gamma,m,2}\|v\|_{\Gamma,[\frac{m}{2}]+2,p_2}\|w\|_{\Gamma,[\frac{m}{2}]+2,p_3}\big).
\end{aligned} \tag{54}$$

Lemma 2.16. *Suppose that $H = H(w)$ is a sufficiently smooth function of $w = (w_1, \cdots, w_M)$ with*

$$H(w) = \mathscr{O}(|w|^{1+\beta}), \tag{55}$$

where $\beta \ge 0$ is an integer. For any given integer $S \ge 0$, if a vector function $w = w(t,x)$ satisfies

$$\|w(t,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \le \nu_0, \tag{56}$$

where ν_0 is a positive constant, then, when $\beta = 0$, we have

$$\|H(w(t,\cdot))\|_{\Gamma,S,p,q,\chi} \le C(\nu_0)\|w(t,\cdot)\|_{\Gamma,S,p,q,\chi}, \tag{57}$$

and when $\beta \ge 1$, we have

$$\|H(w(t,\cdot))\|_{\Gamma,S,p,q,\chi} \le C(\nu_0)\prod_{i=1}^{\beta}\|w(t,\cdot)\|_{\Gamma,[\frac{S}{2}],p_i,q_i,\chi}\|w(t,\cdot)\|_{\Gamma,S,p_0,q_0,\chi}, \tag{58}$$

where $1 \le p, q, p_i, q_i \le \infty$ $(i = 0, 1, \cdots, \beta)$,

$$\frac{1}{p} = \sum_{i=0}^{\beta}\frac{1}{p_i},\ \ \frac{1}{q} = \sum_{i=0}^{\beta}\frac{1}{q_i}, \tag{59}$$

and $C(\nu_0)$ is a constant depending on ν_0.

Proof. We have the following pointwise estimate (see [5]):

$$|\Gamma^k G(w(t,x))| \le C(\nu_0) \sum_{\substack{|l_0|+\cdots+|l_\beta|\le|k| \\ 1\le i_j\le M (j=0,\cdots,\beta)}} \prod_{j=0}^{\beta} |\Gamma^{l_j} w_{i_j}(t,x)|. \tag{60}$$

(57) is a direct consequence of (60), while, (58) follows from (60) and Hölder inequality. □

Lemma 2.17. *Suppose that $H = H(w)$ is a sufficiently smooth function of $w = (w_1, \cdots, w_M)$ with*

$$H(w) = \mathscr{O}(|w|^{1+\beta}), \tag{61}$$

where $\beta \ge 1$ is an integer. If a vector function $w = w(t,x)$ is supported in $|x| \le t+1$ and satisfies (56), then we have

$$\|H(w(t,\cdot))\|_{\Gamma,S,2} \le C(\nu_0)\|Dw(t,\cdot)\|_{\Gamma,S,2} \sum_{|k|\le[\frac{S}{2}]+2} \|\langle t-\cdot\rangle^{\frac{1}{\beta}} \Gamma^k w(t,\cdot)\|_{L^\infty}^{\beta} \tag{62}$$

and

$$\|H(w(t,\cdot))\|_{\Gamma,S,2} \le C(\nu_0)(1+t)\|Dw(t,\cdot)\|_{\Gamma,S,2}\|w(t,\cdot)\|^{\beta}_{\Gamma,[\frac{S}{2}]+2,\infty}. \tag{63}$$

Proof. Noting the support property of $w = w(t,x)$, (62) and (63) follow from (60), Hölder inequality and the Hardy type inequality (32). □

3. Proof of Theorem 1.1

In this section we shall prove Theorem 1.1 by a bootstrap argument. Assume that $u = u(t,x)$ is a local smooth solution to the Cauchy problem (1) on $[0,T]$. Let

$$\begin{aligned} D_{N,T}(u) = & \sup_{0\le t\le T} (1+t)^{-\sigma}\|Du(t,\cdot)\|_{\Gamma,N,2} + \sup_{0\le t\le T} \|Du(t,\cdot)\|_{\Gamma,N-4,2} \\ & + \sup_{0\le t\le T} \sum_{|k|\le[\frac{N}{2}]+4} \|\langle t+|\cdot|\rangle^{\frac{1}{2}} \langle t-|\cdot|\rangle^{\frac{1}{3}} \Gamma^k u(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \\ & + \sup_{0\le t\le T} (1+t)^{-\sigma}\|u(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,p}, \end{aligned} \tag{64}$$

where $N \ge 30$ is an integer, σ is a suitable small number (for example we can take $\sigma = \frac{1}{1000}$), and p $(2<p<3)$ satisfies

$$\frac{1}{p} > \sigma + \frac{1}{3}. \tag{65}$$

The design of norms in (64) is inspired by [9] and [7]. We will show that there exists a positive constant A such that for any given $T > 0$, we have $D_{N,T}(u) \le A\varepsilon$ under the assumption $D_{N,T}(u) \le 2A\varepsilon$, if ε is sufficiently small.

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3.1. High order energy estimates

First we will give the estimate on the higher order energy $\|Du(t,\cdot)\|_{\Gamma,N,2}$. For this purpose, it is necessary to introduce some notations. Denote

$$H(u,Du,D^2u)=b_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta u+H_1(u,Du). \tag{66}$$

Let $\tilde\lambda=(\lambda;(\lambda_i),i=0,1,2)$. We have $b_{\alpha\beta}(\tilde\lambda)=\mathscr{O}(|\tilde\lambda|^3)$ and $H(\tilde\lambda)=\mathscr{O}(|\tilde\lambda|^4)$. We also denote

$$C(u,Du,D^2u)+H(u,Du,D^2u)=a_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta u+F_1(u,Du). \tag{67}$$

Noting that $a_{\alpha\beta}(\tilde\lambda)=\mathscr{O}(|\tilde\lambda|^2), F_1(\tilde\lambda)=\mathscr{O}(|\tilde\lambda|^3)$ and $\partial_u^3F_1(0,0)=0$, we can rewrite our equation as

$$\Box u=g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta u+a_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta u+F_1(u,Du). \tag{68}$$

Thanks to Lemma 2.1 and Lemma 2.3, $\Gamma^k u$ satisfies

$$\begin{aligned}\Box\Gamma^k u&=g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta\Gamma^k u+g_{\alpha\beta\gamma}\partial_\gamma\Gamma^k u\partial_\alpha\partial_\beta u+a_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta\Gamma^k u\\&\quad+\sum_{\substack{b+c+d=k\\ b,c\neq k}}N_d(\Gamma^b u,\Gamma^c u)+\big[\Gamma^k,a_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta\big]u+\Gamma^kF_1(u,Du).\end{aligned} \tag{69}$$

Following Alinhac [9], we will use the ghost weight energy method. We will take the weight used in [33]. Let $\sigma=t-r$, $q(\sigma)=\arctan\sigma$, $q'(\sigma)=\frac{1}{1+\sigma^2}=\langle t-r\rangle^{-2}$. By the integration by parts argument, we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\langle e^{-q(\sigma)}\partial_t\Gamma^k u,\Box\Gamma^k u\rangle\,dx\\&=\frac12\frac{d}{dt}\int_{\mathbb{R}^2}e^{-q(\sigma)}|D\Gamma^k u|^2\,dx+\frac12\int_{\mathbb{R}^2}e^{-q(\sigma)}\,\langle t-r\rangle^{-2}\,|T\Gamma^a u|^2\,dx,\end{aligned} \tag{70}$$

$$\begin{aligned}&\int_{\mathbb{R}^2}\langle e^{-q(\sigma)}\partial_t\Gamma^k u,g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta\Gamma^k u\rangle\,dx\\&=\frac12 g_{\alpha\beta\gamma}\eta_{\alpha\nu}\partial_t\int_{\mathbb{R}^2}e^{-q}\partial_\nu\Gamma^k u\partial_\gamma u\partial_\beta\Gamma^k udx\\&-g_{\alpha\beta\gamma}\int_{\mathbb{R}^2}e^{-q}\partial_t\Gamma^k u\partial_\alpha\partial_\gamma u\partial_\beta\Gamma^k udx+\frac12 g_{\alpha\beta\gamma}\int_{\mathbb{R}^2}e^{-q}\partial_\alpha\Gamma^k u\partial_t\partial_\gamma u\partial_\beta\Gamma^k udx\\&-g_{\alpha\beta\gamma}\int_{\mathbb{R}^2}e^{-q}q'(\sigma)T_\alpha\Gamma^k u\partial_\gamma u\partial_\beta\Gamma^k udx+\frac12 g_{\alpha\beta\gamma}\int_{\mathbb{R}^2}e^{-q}q'(\sigma)\partial_\alpha\Gamma^k u\partial_\gamma u\partial_\beta\Gamma^k udx\end{aligned} \tag{71}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^2}\langle e^{-q(\sigma)}\partial_t\Gamma^k u,a_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta\Gamma^k u\rangle\,dx\\&=\frac12\eta_{\alpha\nu}\partial_t\int_{\mathbb{R}^2}e^{-q}a_{\alpha\beta}(u,Du)\partial_\nu\Gamma^k u\partial_\beta\Gamma^k udx\end{aligned}$$

$$-\int_{\mathbb{R}^2} e^{-q}\partial_\alpha(a_{\alpha\beta}(u,Du))\partial_t\Gamma^k u\partial_\beta\Gamma^k udx + \frac{1}{2}\int_{\mathbb{R}^2} e^{-q}\partial_t(a_{\alpha\beta}(u,Du))\partial_\alpha\Gamma^k u\partial_\beta\Gamma^k udx$$
$$-\int_{\mathbb{R}^2} e^{-q}q'(\sigma)a_{\alpha\beta}(u,Du)T_\alpha\Gamma^k u\partial_\beta\Gamma^k udx + \frac{1}{2}\int_{\mathbb{R}^2} e^{-q}q'(\sigma)a_{\alpha\beta}(u,Du)\partial_\alpha\Gamma^k u\partial_\beta\Gamma^k udx, \tag{72}$$

where $\eta_{\alpha\nu} = \mathrm{diag}[1,-1,-1]$. Denote

$$E_k(u(t)) = \frac{1}{2}\int_{\mathbb{R}^2} e^{-q(\sigma)}|D\Gamma^k u|^2\,dx - \frac{1}{2}g_{\alpha\beta\gamma}\eta_{\alpha\nu}\int_{\mathbb{R}^2} e^{-q}\partial_\nu\Gamma^k u\partial_\gamma u\partial_\beta\Gamma^k udx$$
$$-\frac{1}{2}\eta_{\alpha\nu}\int_{\mathbb{R}^2} e^{-q}a_{\alpha\beta}(u,Du)\partial_\nu\Gamma^k u\partial_\beta\Gamma^k udx. \tag{73}$$

Since q is bounded, there exists $c_1 > 1$, such that

$$c_1^{-1} \le e^{-q(\sigma)} \le c_1. \tag{74}$$

Thus there exists $c_2 > 1$, such that

$$c_2^{-1}\|D\Gamma^k u\|_{L^2}^2 \le E_k(u(t)) \le c_2\|D\Gamma^k u\|_{L^2}^2. \tag{75}$$

It follows from (69)–(73) that

$$\frac{d}{dt}E_k(u(t)) + \frac{1}{2}\int_{\mathbb{R}^2} e^{-q(\sigma)}\langle t-r\rangle^{-2}|T\Gamma^a u|^2\,dx$$
$$= -g_{\alpha\beta\gamma}\int_{\mathbb{R}^2} e^{-q}\partial_t\Gamma^k u\partial_\alpha\partial_\gamma u\partial_\beta\Gamma^k udx + g_{\alpha\beta\gamma}\int_{\mathbb{R}^2} e^{-q}\partial_t\Gamma^k u\partial_\alpha\partial_\beta u\partial_\gamma\Gamma^k udx$$
$$+\frac{1}{2}g_{\alpha\beta\gamma}\int_{\mathbb{R}^2} e^{-q}\partial_\alpha\Gamma^k u\partial_t\partial_\gamma u\partial_\beta\Gamma^k udx - g_{\alpha\beta\gamma}\int_{\mathbb{R}^2} e^{-q}q'(\sigma)T_\alpha\Gamma^k u\partial_\gamma u\partial_\beta\Gamma^k udx$$
$$+\frac{1}{2}g_{\alpha\beta\gamma}\int_{\mathbb{R}^2} e^{-q}q'(\sigma)\partial_\alpha\Gamma^k u\partial_\gamma u\partial_\beta\Gamma^k udx + \sum_{\substack{b+c+d=k\\ b,c\neq k}}\int_{\mathbb{R}^2}\partial_t\Gamma^k uN_d(\Gamma^b u,\Gamma^c u)dx$$
$$-\int_{\mathbb{R}^2} e^{-q}\partial_\alpha(a_{\alpha\beta}(u,Du))\partial_t\Gamma^k u\partial_\beta\Gamma^k udx + \frac{1}{2}\int_{\mathbb{R}^2} e^{-q}\partial_t(a_{\alpha\beta}(u,Du))\partial_\alpha\Gamma^k u\partial_\beta\Gamma^k udx$$
$$-\int_{\mathbb{R}^2} e^{-q}q'(\sigma)a_{\alpha\beta}(u,Du)T_\alpha\Gamma^k u\partial_\beta\Gamma^k udx + \frac{1}{2}\int_{\mathbb{R}^2} e^{-q}q'(\sigma)a_{\alpha\beta}(u,Du)\partial_\alpha\Gamma^k u\partial_\beta\Gamma^k udx$$
$$+\int_{\mathbb{R}^2}\partial_t\Gamma^k u\left[\Gamma^k, a_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta\right]udx + \int_{\mathbb{R}^2}\partial_t\Gamma^k u\Gamma^k F_1(u,Du)dx. \tag{76}$$

The second term on the left-hand side of (76) is called the ghost weight energy, which will play a key role in the control of the terms involving the highest generalized derivatives on the right-hand side of (76) after having applied the null condition.

Now we shall estimate all terms on the right-hand side of (76). First, by Lemma 2.4, (11), (13) and the Cauchy–Schwarz inequality, we have

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$$\begin{aligned}
&\|g_{\alpha\beta\gamma}e^{-q}\partial_t\Gamma^k u\partial_\alpha\partial_\gamma u\partial_\beta\Gamma^k u\|_{L^1}+\|g_{\alpha\beta\gamma}e^{-q}\partial_t\Gamma^k u\partial_\alpha\partial_\beta u\partial_\gamma\Gamma^k u\|_{L^1}\\
&+\|g_{\alpha\beta\gamma}e^{-q}\partial_\alpha\Gamma^k u\partial_t\partial_\gamma u\partial_\beta\Gamma^k u\|_{L^1}\\
&\le C||T\Gamma^k uD^2uD\Gamma^k u||_{L^1}+C||D\Gamma^k uTDuD\Gamma^k u||_{L^1}\\
&\le \frac{1}{8}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}T\Gamma^k u||^2_{L^2}+C||D\Gamma^k u||^2_{L^2}||\langle t-r\rangle D^2u||^2_{L^\infty}\\
&+||D\Gamma^k u||^2_{L^2}||TDu||_{L^\infty}\\
&\le \frac{1}{8}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}T\Gamma^k u||^2_{L^2}+C||D\Gamma^k u||^2_{L^2}(||\Gamma Du||^2_{L^\infty}+(1+t)^{-1}||\Gamma Du||_{L^\infty}).
\end{aligned}\tag{77}$$

It follows from Lemma 2.4, (13) and the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\|g_{\alpha\beta\gamma}e^{-q}q'(\sigma)T_\alpha\Gamma^k u\partial_\gamma u\partial_\beta\Gamma^k u\|_{L^1}+\|g_{\alpha\beta\gamma}e^{-q}q'(\sigma)\partial_\alpha\Gamma^k u\partial_\gamma u\partial_\beta\Gamma^k u\|_{L^1}\\
&\le C\|\langle t-r\rangle^{-2}T\Gamma^k uDuD\Gamma^k u\|_{L^1}+C\|\langle t-r\rangle^{-2}D\Gamma^k uTuD\Gamma^k u\|_{L^1}\\
&\le \frac{1}{8}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}T\Gamma^k u||^2_{L^2}+C||D\Gamma^k u||^2_{L^2}||Du||^2_{L^\infty}\\
&+||D\Gamma^k u||^2_{L^2}||Tu||_{L^\infty}\\
&\le \frac{1}{8}||e^{-q(\sigma)/2}\langle t-r\rangle^{-1}T\Gamma^k u||^2_{L^2}+C||D\Gamma^k u||^2_{L^2}(||Du||^2_{L^\infty}+(1+t)^{-1}||\Gamma u||_{L^\infty}).
\end{aligned}\tag{78}$$

We note that the first term on the right-hand side of (77) and (78) can be absorbed into the ghost weight energy term on the left-hand side of (76).

By Hölder inequality, we have

$$\sum_{\substack{b+c+d=k\\ b,c\neq k}}\|\partial_t\Gamma^k uN_d(\Gamma^b u,\Gamma^c u)\|_{L^1}\le C\|\partial_t\Gamma^k u\|_{L^2}\sum_{\substack{b+c+d=k\\ b,c\neq k}}\|N_d(\Gamma^b u,\Gamma^c u)\|_{L^2}.\tag{79}$$

Owning to Lemma 2.4 and Lemma 2.3, we have

$$\|N_d(\Gamma^b u,\Gamma^c u)\|_{L^2}\le C(1+t)^{-1}\big(\|D\Gamma^b uD\Gamma^{c+1}u\|_{L^2}+\|\Gamma^{b+1}uD^2\Gamma^c u\|_{L^2}\big).\tag{80}$$

Applying Hölder inequality gives

$$\|D\Gamma^b uD\Gamma^{c+1}u\|_{L^2}\le C\|u\|_{\Gamma,[\frac{|k|}{2}]+2,\infty}\|Du\|_{\Gamma,|k|,2}.\tag{81}$$

Thanks to Hölder inequality, (11) and the Hardy type inequality (32), we can see that

$$\begin{aligned}
&\|\Gamma^{b+1}uD^2\Gamma^c u\|_{L^2}\\
&\le C\begin{cases}\|\Gamma^{b+1}u\|_{L^\infty}\|D^2\Gamma^c u\|_{L^2}, & |b|\le[\frac{|k|}{2}]+1,\\ \|\langle t-r\rangle^{-1}\Gamma^b u\|_{L^2}\|\langle t-r\rangle D^2\Gamma^c u\|_{L^\infty}, & |c|\le[\frac{|k|}{2}]+1,\end{cases}\\
&\le C\begin{cases}\|\Gamma^{b+1}u\|_{L^\infty}\|D^2\Gamma^c u\|_{L^2}, & |b|\le[\frac{|k|}{2}]+1,\\ \|D\Gamma^b u\|_{L^2}\|D\Gamma^{c+1}u\|_{L^\infty}, & |c|\le[\frac{|k|}{2}]+1,\end{cases}\\
&\le C\|u\|_{\Gamma,[\frac{|k|}{2}]+2,\infty}\|Du\|_{\Gamma,|k|,2}.
\end{aligned}\tag{82}$$

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By (79)–(82), we get

$$\sum_{\substack{b+c+d=k\\ b,c\neq k}} \|\partial_t\Gamma^k u N_d(\Gamma^b u,\Gamma^c u)\|_{L^1} \le C(1+t)^{-1}\|u\|_{\Gamma,[\frac{|k|}{2}]+2,\infty}\|Du\|^2_{\Gamma,|k|,2}. \tag{83}$$

Noting that $a(\tilde{\lambda}) = \mathscr{O}(|\tilde{\lambda}|^2)$, by Hölder inequality and the estimates on composite functions given in Lemma 2.16, we have

$$\begin{aligned}
&\|e^{-q}\partial_\alpha(a_{\alpha\beta}(u,Du))\partial_t\Gamma^k u\partial_\beta\Gamma^k u\|_{L^1} + \|e^{-q}\partial_t(a_{\alpha\beta}(u,Du))\partial_\alpha\Gamma^k u\partial_\beta\Gamma^k u\|_{L^1}\\
&+\|e^{-q}q'(\sigma)a_{\alpha\beta}(u,Du)T_\alpha\Gamma^k u\partial_\beta\Gamma^k u\|_{L^1} + \|e^{-q}q'(\sigma)a_{\alpha\beta}(u,Du)\partial_\alpha\Gamma^k u\partial_\beta\Gamma^k u\|_{L^1}\\
&\le C\big(\|a(u,Du)\|_{L^\infty} + \|D(a(u,Du))\|_{L^\infty}\big)\|D\Gamma^k u\|^2_{L^2}\\
&\le C\|u\|^2_{\Gamma,2,\infty}\|D\Gamma^k u\|^2_{L^2}.
\end{aligned} \tag{84}$$

By Hölder inequality, we get

$$\|\partial_t\Gamma^k u\big[\Gamma^k, a_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta\big]u\|_{L^1} \le C\|\partial_t\Gamma^k u\|_{L^2}\|\big[\Gamma^k, a_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta\big]u\|_{L^2}. \tag{85}$$

We have

$$\|\big[\Gamma^k, a_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta\big]u\|_{L^2} \le C\sum_{\substack{|k_1|+|k_2|\le|k|\\ |k_2|\le|k|-1}}\|\Gamma^{k_1}a(u,Du)D^2\Gamma^{k_2}u\|_{L^2}. \tag{86}$$

For k_1, k_2 with $|k_1|+|k_2|\le|k|$ and $|k_2|\le|k|-1$, if $|k_1|\le[\frac{|k|}{2}]+1$, then we have

$$\begin{aligned}
&\|\Gamma^{k_1}a(u,Du)D^2\Gamma^{k_2}u\|_{L^2} \le C\|\Gamma^{k_1}a(u,Du)\|_{L^\infty}\|D^2\Gamma^{k_2}u\|_{L^2}\\
&\le C\|\Gamma^{k_1}a(u,Du)\|_{L^\infty}\|D^2\Gamma^{k_2}u\|_{L^2} \le C\|a(u,Du)\|_{\Gamma,[\frac{|k|}{2}]+1,\infty}\|Du\|_{\Gamma,|k|,2}\\
&\le C\|u\|^2_{\Gamma,[\frac{|k|}{2}]+2,\infty}\|Du\|_{\Gamma,|k|,2}.
\end{aligned} \tag{87}$$

While, if $|k_2|\le[\frac{|k|}{2}]+1$, we have

$$\begin{aligned}
&\|\Gamma^{k_1}a(u,Du)D^2\Gamma^{k_2}u\|_{L^2}\\
&\le C\sum_{|l_1|+|l_2|\le|k_1|}\big(\|\Gamma^{l_1}u\Gamma^{l_2}uD^2\Gamma^{k_2}u\|_{L^2} + \|\Gamma^{l_1}uD\Gamma^{l_2}uD^2\Gamma^{k_2}u\|_{L^2}\\
&\qquad + \|D\Gamma^{l_1}u\Gamma^{l_2}uD^2\Gamma^{k_2}u\|_{L^2} + \|D\Gamma^{l_1}uD\Gamma^{l_2}uD^2\Gamma^{k_2}u\|_{L^2}\big).
\end{aligned} \tag{88}$$

The combination of Hölder inequality, (11) and the Hardy type inequality (32) gives

$$\begin{aligned}
&\|\Gamma^{l_1}u\Gamma^{l_2}uD^2\Gamma^{k_2}u\|_{L^2}\\
&\le C\begin{cases}\|\Gamma^{l_1}u\langle t-r\rangle^{-1}\Gamma^{l_2}u\langle t-r\rangle D^2\Gamma^{k_2}u\|_{L^2}, |l_1|\le[\frac{k_1}{2}]+1,\\ \|\langle t-r\rangle^{-1}\Gamma^{l_1}u\Gamma^{l_2}u\langle t-r\rangle D^2\Gamma^{k_2}u\|_{L^2}, |l_2|\le[\frac{k_1}{2}]+1,\end{cases}\\
&\le C\begin{cases}\|\Gamma^{l_1}u\|_{L^\infty}\|D\Gamma^{l_2}u\|_{L^2}\|D\Gamma^{k_2+1}u\|_{L^\infty}, |l_1|\le[\frac{k_1}{2}]+1,\\ \|D\Gamma^{l_1}u\|_{L^2}\|\Gamma^{l_2}u\|_{L^\infty}\|D\Gamma^{k_2+1}u\|_{L^\infty}, |l_2|\le[\frac{k_1}{2}]+1,\end{cases}\\
&\le C\|u\|^2_{\Gamma,[\frac{|k|}{2}]+2,\infty}\|Du\|_{\Gamma,|k|,2}.
\end{aligned} \tag{89}$$

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Other terms on the right-hand side of (88) can be estimated by a similar way. Thus when $|k_2| \le [\frac{|k|}{2}] + 1$, we have

$$\|\Gamma^{k_1} a(u, Du) D^2 \Gamma^{k_2} u\|_{L^2} \le C\|u\|^2_{\Gamma,[\frac{|k|}{2}]+2,\infty} \|Du\|_{\Gamma,|k|,2}. \tag{90}$$

The above discussion on the two different cases gives

$$\|\big[\Gamma^k, a_{\alpha\beta}(u, Du)\partial_\alpha\partial_\beta\big]u\|_{L^2} \le C\|u\|^2_{\Gamma,[\frac{|k|}{2}]+2,\infty} \|Du\|^2_{\Gamma,|k|,2}. \tag{91}$$

Now we will estimate the last term on the right-hand side of (76). First we have

$$\|\partial_t \Gamma^k u \Gamma^k F_1(u, Du)\|_{L^1} \le C\|\partial_t \Gamma^k u\|_{L^2} \|\Gamma^k F_1(u, Du)\|_{L^2}. \tag{92}$$

We rewrite

$$\begin{aligned} F_1(u, Du) &= F_1(u, 0) + \big(F_1(u, Du) - F(u, 0)\big) \\ &= \widehat{F}(u) + \widehat{F}_1(u, Du)Du, \end{aligned} \tag{93}$$

where

$$\widehat{F}(u) = F_1(u, 0) = \mathscr{O}(|u|^4), \tag{94}$$

$$\widehat{F}_1(u, Du) = F_1(u, Du) - F(u, 0) \tag{95}$$

and

$$\widehat{F}_1(\widetilde{\lambda}) = \mathscr{O}(|\widetilde{\lambda}|^2). \tag{96}$$

Noting (94), by the estimates on composite functions (62) given in Lemma 2.17, we have

$$\|\Gamma^k \widehat{F}(u)\|_{L^2} \le C\|Du\|_{\Gamma,|k|,2} \sum_{|a|\le[\frac{|k|}{2}]+2} \|\langle t-r\rangle^{\frac{1}{3}} \Gamma^a u\|^3_{L^\infty}. \tag{97}$$

Noting (96), similarly to the proof of (90), we have

$$\|\Gamma^k(\widehat{F}_1(u, Du)Du)\|_{L^2} \le C\|u\|^2_{\Gamma,[\frac{|k|}{2}]+1,\infty} \|Du\|_{\Gamma,|k|,2}. \tag{98}$$

Thanks to (92), (93), (97) and (98), we get

$$\begin{aligned} &\|\partial_t \Gamma^k u \Gamma^k F_1(u, Du)\|_{L^1} \\ &\le C\|Du\|^2_{\Gamma,|k|,2} \big(\|u\|^2_{\Gamma,[\frac{|k|}{2}]+1,\infty} + \sum_{|a|\le[\frac{|k|}{2}]+2} \|\langle t-r\rangle^{\frac{1}{3}} \Gamma^a u\|^3_{L^\infty}\big). \end{aligned} \tag{99}$$

It follows from (75), (76), (77), (78), (83), (84), (91) and (99) that if ε is sufficiently small, then we have

$$\|D\Gamma^k u(t,\cdot)\|^2_{L^2} \le C\varepsilon^2 + CA^2\varepsilon^2 \int_0^t (1+\tau)^{-1} \|Du(\tau,\cdot)\|^2_{\Gamma,|k|,2} d\tau. \tag{100}$$

Hence we have

$$\|Du(t,\cdot)\|^2_{\Gamma,N,2} \le C\varepsilon^2 + CA^2\varepsilon^2 \int_0^t (1+\tau)^{-1}\|Du(\tau,\cdot)\|^2_{\Gamma,N,2}d\tau$$
$$\le C\varepsilon^2 + CA^4\varepsilon^4 \int_0^t (1+\tau)^{-1+2\sigma}d\tau \le C\varepsilon^2 + CA^4\varepsilon^4(1+t)^{2\sigma}. \tag{101}$$

Consequently, we get

$$(1+t)^{-\sigma}\|Du(t,\cdot)\|_{\Gamma,N,2} \le C\varepsilon + CA^2\varepsilon^2. \tag{102}$$

3.2. Low order energy estimates

Now we will estimate the low order energy $\|Du(t,\cdot)\|_{\Gamma,N-4,2}$. By the general energy estimate, we have

$$\|Du(t,\cdot)\|_{\Gamma,N-4,2} \le C\varepsilon + C\int_0^t \|F(u,Du,D^2u)\|_{\Gamma,N-4,2}d\tau. \tag{103}$$

In order to estimate $\|F(u,Du,D^2u)\|_{\Gamma,N-4,2}$, we first estimate $\|Q(u,Du,D^2u)\|_{\Gamma,N-4,2}$. By Lemma 2.5, we have

$$|\Gamma^k Q(u,Du,D^2u)| \le C(1+\tau)^{-1} \sum_{|k_1|+|k_2|\le|k|+2} |\Gamma^{k_1}u||D\Gamma^{k_2}u|. \tag{104}$$

It follows from Lemma 2.14 that

$$\|\Gamma^{k_1}uD\Gamma^{k_2}u\|_{L^2} \le C\|u\|_{\Gamma,[\frac{N}{2}]+4,\infty}\|Du\|_{\Gamma,N,2}. \tag{105}$$

Due to (104) and (105), we get that

$$\|Q(u,Du,D^2u)\|_{\Gamma,N-4,2} \le C(1+\tau)^{-1}\|u\|_{\Gamma,[\frac{N}{2}]+4,\infty}\|Du\|_{\Gamma,N,2} \le C(1+\tau)^{-\frac{3}{2}+\sigma}A^2\varepsilon^2. \tag{106}$$

For the estimate of $\|C(u,Du,D^2u)\|_{\Gamma,N-4,2}$, we first note that Lemma 2.6 gives

$$|\Gamma^k C(u,Du,D^2u)| \le C(1+\tau)^{-1} \sum_{|k_1|+|k_2|+|k_3|\le|k|+3} |\Gamma^{k_1}u||\Gamma^{k_2}u||D\Gamma^{k_3}u|. \tag{107}$$

By Lemma 2.15, we have

$$\|\Gamma^{k_1}u\Gamma^{k_2}uD\Gamma^{k_3}u\|_{L^2} \le C\|u\|^2_{\Gamma,[\frac{N}{2}]+4,\infty}\|Du\|_{\Gamma,N,2}. \tag{108}$$

Then it follows from (107) and (108) that

$$\|C(u,Du,D^2u)\|_{\Gamma,N-4,2} \le C(1+\tau)^{-1}\|u\|^2_{\Gamma,[\frac{N}{2}]+4,\infty}\|Du\|_{\Gamma,N,2} \le C(1+\tau)^{-2+\sigma}A^3\varepsilon^3. \tag{109}$$

Now we estimate $\|H(u,Du,D^2u)\|_{\Gamma,N-4,2}$. It follows from the estimates on composite functions (62) given in Lemma 2.17 that

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$$\begin{aligned}
&\|H(u,Du,D^2u)\|_{\Gamma,N-4,2}\\
&\le C\sum_{|k|\le[\frac{N}{2}]+4}\|\langle\tau-|\cdot|\rangle^{\frac{1}{3}}\Gamma^k u(\tau,\cdot)\|^3_{L^\infty(\mathbb{R}^2)}\|Du\|_{\Gamma,N,2}\\
&\le C(1+\tau)^{-\frac{3}{2}+\sigma}A^4\varepsilon^4.
\end{aligned}\tag{110}$$

The combination of (103), (106), (109) and (110) gives that if ε is sufficiently small, then

$$\|Du(t,\cdot)\|_{\Gamma,N-4,2}\le C\varepsilon+CA^2\varepsilon^2.\tag{111}$$

3.3. Weighted L^∞ estimates

Now we estimate $\sum_{|k|\le[\frac{N}{2}]+4}\|\langle t+|\cdot|\rangle^{\frac{1}{2}}\langle t-|\cdot|\rangle^{\frac{1}{3}}\Gamma^k u(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}$. It follows from Lemma 2.1 and Lemma 2.9 that

$$\begin{aligned}
&\sum_{|k|\le[\frac{N}{2}]+4}\|\langle t+|\cdot|\rangle^{\frac{1}{2}}\langle t-|\cdot|\rangle^{\frac{1}{3}}\Gamma^k u(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\\
&\le C\varepsilon+\int_0^t(1+\tau)^{-\frac{1}{6}}\|F(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}d\tau
\end{aligned}\tag{112}$$

In order to estimate $\|F(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}$, we first estimate $\|Q(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}$. It follows from Lemma 2.5 that

$$|\Gamma^kQ(u,Du,D^2u)|\le C(1+\tau)^{-1}\sum_{|k_1|+|k_2|\le|k|}\big(|D\Gamma^{k_1}u||D\Gamma^{k_2+1}u|+|\Gamma^{k_1+1}u||D^2\Gamma^{k_2}u|\big).\tag{113}$$

By Hölder inequality, it is easy to see that

$$\|D\Gamma^{k_1}uD\Gamma^{k_2+1}u\|_{L^1(\mathbb{R}^2)}\le C\|Du\|^2_{\Gamma,N-4,2}.\tag{114}$$

It follows from Lemma 2.14 that

$$\|\Gamma^{k_1+1}uD^2\Gamma^{k_2}u\|_{L^1(\mathbb{R}^2)}\le C\|Du\|^2_{\Gamma,N-4,2}.\tag{115}$$

Thanks to (113)–(115), we have

$$\|Q(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}\le C(1+\tau)^{-1}\|Du\|^2_{\Gamma,N-4,2}\le C(1+\tau)^{-1}A^2\varepsilon^2.\tag{116}$$

For the estimate of $\|C(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}$, we first note that Lemma 2.6 gives

$$|\Gamma^kC(u,Du,D^2u)|\le C(1+\tau)^{-1}\sum_{|k_1|+|k_2|+|k_3|\le|k|+3}|\Gamma^{k_1}u||\Gamma^{k_2}u||D\Gamma^{k_3}u|.\tag{117}$$

Applying Lemma 2.15, we get

$$\|\Gamma^{k_1}u\Gamma^{k_2}uD\Gamma^{k_3}u\|_{L^1(\mathbb{R}^2)}\le C\|u\|^2_{\Gamma,[\frac{N}{2}]+4,4}\|Du\|_{\Gamma,N-4,2}.\tag{118}$$

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Moreover, we have

$$\begin{aligned}&\|u(\tau,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,4}\\&\le C(1+\tau)^{-\frac{1}{2}}\|\langle\tau-r\rangle^{-\frac{1}{3}}\|_{L^4(|x|\le\tau+1)}\sum_{|k|\le[\frac{N}{2}]+4}\|\langle\tau+|\cdot|\rangle^{\frac{1}{2}}\langle\tau-|\cdot|\rangle^{\frac{1}{3}}\Gamma^k u(\tau,\cdot)\|_{L^\infty(\mathbb{R}^2)}\\&\le C(1+\tau)^{-\frac{1}{4}}A\varepsilon.\end{aligned}\tag{119}$$

It follows from (117)–(119) that

$$\|C(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}\le C(1+\tau)^{-1}\|u\|^2_{\Gamma,[\frac{N}{2}]+4,4}\|Du\|_{\Gamma,N-4,2}\le C(1+\tau)^{-\frac{3}{2}}A^3\varepsilon^3.\tag{120}$$

For the estimate of $\|H(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}$, by the estimates on composite functions given in Lemma 2.16, we have

$$\|H(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}\le C\|u\|_{\Gamma,[\frac{N}{2}]+8,p}\|u\|^3_{\Gamma,[\frac{N}{2}]+4,3p'}\tag{121}$$

Similarly to (119), we have

$$\begin{aligned}&\|u(\tau,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,3p'}\\&\le C(1+\tau)^{-\frac{1}{2}}\|\langle\tau-r\rangle^{-\frac{1}{3}}\|_{L^{3p'}(|x|\le\tau+1)}\sum_{|k|\le[\frac{N}{2}]+4}\|\langle\tau+|\cdot|\rangle^{\frac{1}{2}}\langle\tau-|\cdot|\rangle^{\frac{1}{3}}\Gamma^k u(\tau,\cdot)\|_{L^\infty(\mathbb{R}^2)}\\&\le C(1+\tau)^{-\frac{1}{2}+\frac{1}{3p'}}A\varepsilon.\end{aligned}\tag{122}$$

Then, it follows from (121) and (122) that

$$\|H(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}\le C(1+\tau)^{-\frac{1}{2}-\frac{1}{p}+\sigma}A^4\varepsilon^4.\tag{123}$$

Noting (65), by (112), (116), (120) and (123), we have that if ε is sufficiently small, then

$$\sum_{|k|\le[\frac{N}{2}]+4}\|\langle t+|\cdot|\rangle^{\frac{1}{2}}\langle t-|\cdot|\rangle^{\frac{1}{3}}\Gamma^k u(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\le C\varepsilon+CA^2\varepsilon^2.\tag{124}$$

3.4. L^p estimates

Finally we estimate $\|u(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,p}$. It follows from Lemma 2.1 and Lemma 2.11 that

$$\|u(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,p}\le C\varepsilon+C\int_0^t\|F(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,q}d\tau,\tag{125}$$

where $\frac{1}{q}=\frac{1}{2}+\frac{1}{p}$.

In order to estimate $\|F(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,q}$, we first estimate $\|Q(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,q}$. By Lemma 2.5, we have

$$|\Gamma^k Q(u,Du,D^2u)|\le C(1+\tau)^{-1}\sum_{|k_1|+|k_2|\le|k|+2}|\Gamma^{k_1}u||D\Gamma^{k_2}u|.\tag{126}$$

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By Lemma 2.14, we have

$$\|\Gamma^{k_1} u D\Gamma^{k_2} u\|_{L^q} \le C\|u\|_{\Gamma,[\frac{N}{2}]+8,p}\|Du\|_{\Gamma,N-4,2}. \tag{127}$$

Owing to (126) and (127), we have

$$\|Q(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,q} \le C(1+\tau)^{-1}\|u\|_{\Gamma,[\frac{N}{2}]+8,p}\|Du\|_{\Gamma,N-4,2} \le C(1+\tau)^{-1+\sigma}A^2\varepsilon^2. \tag{128}$$

For the estimate of $\|C(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,q}$, we first note that by Lemma 2.6, we have

$$|\Gamma^k C(u,Du,D^2u)| \le C(1+\tau)^{-1} \sum_{|k_1|+|k_2|+|k_3|\le|k|+3} |\Gamma^{k_1}u||\Gamma^{k_2}u||D\Gamma^{k_3}u|. \tag{129}$$

It follows from Lemma 2.15 that

$$\|\Gamma^{k_1}u\Gamma^{k_2}uD\Gamma^{k_3}u\|_{L^q(\mathbb{R}^2)} \le C\|u\|_{\Gamma,[\frac{N}{2}]+4,\infty}\|u\|_{\Gamma,[\frac{N}{2}]+8,p}\|Du\|_{\Gamma,N-4,2}. \tag{130}$$

The combination of (129) and (130) gives

$$\begin{aligned}
&\|C(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,q}\\
&\le C(1+\tau)^{-1}\|u\|_{\Gamma,[\frac{N}{2}]+4,\infty}\|u\|_{\Gamma,[\frac{N}{2}]+8,p}\|Du\|_{\Gamma,N-4,2}\\
&\le C(1+\tau)^{-\frac{3}{2}+\sigma}A^3\varepsilon^3.
\end{aligned} \tag{131}$$

For the estimate of $\|H(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,q}$, it follows from the estimates on composite functions given in Lemma 2.16 that

$$\|H(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,q} \le C\|u\|_{\Gamma,[\frac{N}{2}]+8,p}\|u\|^3_{\Gamma,[\frac{N}{2}]+4,6}. \tag{132}$$

Similarly to (119), we have

$$\begin{aligned}
&\|u(\tau,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,6}\\
&\le C(1+\tau)^{-\frac{1}{2}}\|\langle\tau-r\rangle^{-\frac{1}{3}}\|_{L^6(|x|\le\tau+1)} \sum_{|k|\le[\frac{N}{2}]+4} \|\langle\tau+|\cdot|\rangle^{\frac{1}{2}}\langle\tau-|\cdot|\rangle^{\frac{1}{3}}\Gamma^k u(\tau,\cdot)\|_{L^\infty(\mathbb{R}^2)}\\
&\le C(1+\tau)^{-\frac{1}{3}}A\varepsilon.
\end{aligned} \tag{133}$$

By (132) and (133), we have

$$\|H(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,q} \le C(1+\tau)^{-1+\sigma}A^4\varepsilon^4. \tag{134}$$

Due to (125), (128), (131) and (134), if ε is sufficiently small, then

$$(1+t)^{-\sigma}\|u(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,p} \le C\varepsilon + CA^2\varepsilon^2. \tag{135}$$

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3.5. *Conclusion of the proof*

Thanks to (102), (111), (124) and (135), for any given $T>0$,

$$D_{N,T}(u) \le C_1\varepsilon + C_2A^2\varepsilon^2. \tag{136}$$

Assume that

$$D_{N,0}(u) \le C_3\varepsilon. \tag{137}$$

Taking $A=\max\{2C_1,2C_3\}$ and ε so small that

$$C_2A\varepsilon \le \frac{1}{2}, \tag{138}$$

we get

$$D_{N,T}(u) \le A\varepsilon, \tag{139}$$

which completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

In this section we shall prove Theorem 1.2 by a bootstrap argument. Assume that $u=u(t,x)$ is a local smooth solution to Cauchy problem (1) on $[0,T]$. Let

$$\begin{aligned} D_{N,T}(u) = {} & \sup_{0\le t\le T} \|Du(t,\cdot)\|_{\Gamma,N,2} + \sup_{0\le t\le T}(1+t)^{\frac12}\|u(t,\cdot)\|_{\Gamma,[\frac N2]+4,\infty} \\ & + \sup_{0\le t\le T}(1+t)^{-\frac12}\|u(t,\cdot)\|_{\Gamma,[\frac N2]+8,2}, \end{aligned} \tag{140}$$

where $N\ge 30$ is an integer. We will show that there exist positive constants c and A such that for any $T\le \exp(c\varepsilon^{-2})$, we have $D_{N,T}(u)\le A\varepsilon$ under the assumption $D_{N,T}(u)\le 2A\varepsilon$, if ε is sufficiently small.

4.1. *Energy estimates*

Denote

$$C(u,Du,D^2u)+H(u,Du,D^2u) = a_{\alpha\beta}(u,Du)\partial_\alpha\partial_\beta u + F_1(u,Du), \tag{141}$$

where $a_{\alpha\beta}(\tilde\lambda)=\mathscr{O}(|\tilde\lambda|^2)$, $F_1(\tilde\lambda)=\mathscr{O}(|\tilde\lambda|^3)$ and

$$\partial_u^3F_1(0,0)=\partial_u^4F_1(0,0)=0. \tag{142}$$

For the estimate of $\|Du(t,\cdot)\|_{\Gamma,N,2}$, the terms on the right-hand side of (76) can be estimated by the same way as in Section 3.1 except the last term (note that in Section 3.1 we do not use the second null condition of the cubic nonlinearity). For the last term on the right-hand side of (76), we first have

$$\|\partial_t\Gamma^k u\Gamma^k F_1(u,Du)\|_{L^1} \le C\|\partial_t\Gamma^k u\|_{L^2}\|\Gamma^k F_1(u,Du)\|_{L^2}. \tag{143}$$

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Similarly to (93), we rewrite

$$F_1(u, Du) = \widehat{F}(u) + \widehat{F}_1(u, Du)Du, \tag{144}$$

where

$$\widehat{F}(u) = F_1(u, 0) = \mathscr{O}(|u|^5), \tag{145}$$

$$\widehat{F}_1(u, Du) = F_1(u, Du) - F(u, 0) \tag{146}$$

and

$$\widehat{F}_1(\widetilde{\lambda}) = \mathscr{O}(|\widetilde{\lambda}|^2). \tag{147}$$

Noting (145), by the estimates on composite functions (63) given in Lemma 2.17, we have

$$\|\Gamma^k \widehat{F}(u)\|_{L^2} \leq C(1+t)\|Du\|_{\Gamma,|k|,2}\|u\|^4_{\Gamma,[\frac{|k|}{2}]+1,\infty}. \tag{148}$$

Similarly to (98), we have

$$\|\Gamma^k(\widehat{F}_1(u, Du)Du)\|_{L^2} \leq C\|u\|^2_{\Gamma,[\frac{|k|}{2}]+1,\infty}\|Du\|_{\Gamma,|k|,2}. \tag{149}$$

Due to (143), (144), (148) and (149), we get

$$\begin{aligned}&\|\partial_t\Gamma^k u\Gamma^k F_1(u, Du)\|_{L^1}\\ &\leq C\|Du\|^2_{\Gamma,|k|,2}\big(\|u\|^2_{\Gamma,[\frac{|k|}{2}]+1,\infty} + (1+t)\|u\|^4_{\Gamma,[\frac{|k|}{2}]+1,\infty}\big).\end{aligned} \tag{150}$$

It follows from (75), (76), (77), (78), (83), (84), (91) and (150) that if ε is sufficiently small, then we have

$$\|D\Gamma^k u(t,\cdot)\|^2_{L^2} \leq C\varepsilon^2 + CA^2\varepsilon^2\int_0^t (1+\tau)^{-1}\|Du(\tau,\cdot)\|^2_{\Gamma,|k|,2}d\tau. \tag{151}$$

Hence we have

$$\begin{aligned}\|Du(t,\cdot)\|^2_{\Gamma,N,2} &\leq C\varepsilon^2 + CA^2\varepsilon^2\int_0^t (1+\tau)^{-1}\|Du(\tau,\cdot)\|^2_{\Gamma,N,2}d\tau\\ &\leq C\varepsilon^2 + CA^4\varepsilon^4\log(2+t),\end{aligned} \tag{152}$$

then

$$\|Du(t,\cdot)\|_{\Gamma,N,2} \leq C\varepsilon + CA^2\varepsilon^2(\log(2+t))^{1/2}. \tag{153}$$

4.2. L^∞ estimates

Noting that

$$\begin{aligned}&a_{\alpha\beta}(u, Du)\partial_\alpha\partial_\beta u\\ &= a_{\alpha\beta}(u,0)\partial_\alpha\partial_\beta u + \big(a_{\alpha\beta}(u, Du) - a_{\alpha\beta}(u,0)\big)\partial_\alpha\partial_\beta u\\ &= \partial_\alpha\big(a_{\alpha\beta}(u,0)\partial_\beta u\big) - \partial_\alpha\big(a_{\alpha\beta}(u,0)\big)\partial_\beta u + \big(a_{\alpha\beta}(u, Du) - a_{\alpha\beta}(u,0)\big)\partial_\alpha\partial_\beta u\end{aligned} \tag{154}$$

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and

$$\begin{aligned}F_1(u,Du) &= F_1(u,0) + \big(F_1(u,Du) - F(u,0)\big)\\ &= F_1(u,0) + \widetilde{F}_1(u,Du)Du\\ &= F_1(u,0) + \widetilde{F}_1(u,0)Du + \big(\widetilde{F}_1(u,Du) - \widetilde{F}_1(u,0)\big)Du\\ &= F_1(u,0) + \widetilde{F}_{1\alpha}(u,0)\partial_\alpha u + \big(\widetilde{F}_1(u,Du) - \widetilde{F}_1(u,0)\big)Du\\ &= F_1(u,0) + \partial_\alpha\big(G_\alpha(u,0)\big) + \big(\widetilde{F}_1(u,Du) - \widetilde{F}_1(u,0)\big)Du, \end{aligned} \tag{155}$$

where $G_\alpha(u,0)$ is the primitive function of $\widetilde{F}_{1\alpha}(u,0)$, and using (142), we rewrite

$$\begin{aligned}F(u,Du,D^2u) = {}& g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta u + \partial_\alpha\widehat{G}_\alpha(u,Du) + \widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u + \widehat{B}_{\alpha\beta}(u,Du)\partial_\alpha u\partial_\beta u\\ &+ \widehat{C}_{\alpha\beta\gamma}(u,Du)\partial_\gamma u\partial_\alpha\partial_\beta u + \widehat{F}(u),\end{aligned} \tag{156}$$

where

$$\widehat{F}(u) = F_1(u,0) = \mathscr{O}(|u|^5), \tag{157}$$

$$\widehat{A}_{\alpha\beta}(u) = F_1(u,0) = \mathscr{O}(|u|), \tag{158}$$

$$\widehat{G}_\alpha(\widetilde{\lambda}) = \mathscr{O}(|\widetilde{\lambda}|^3),\ \alpha = 0,1,2, \tag{159}$$

$\widehat{G}_\alpha(\widetilde{\lambda})$ is affine with respect to $\lambda_\alpha(\alpha = 0,1,2)$, and

$$\widehat{C}_{\alpha\beta\gamma}(\widetilde{\lambda}),\ \widehat{B}_{\alpha\beta} = \mathscr{O}(|\widetilde{\lambda}|),\ \alpha,\beta,\gamma = 0,1,2. \tag{160}$$

Following [2], we write the solution u to Cauchy problem (1) as

$$u = u_0 + u_1 + u_2 + u_3, \tag{161}$$

where u_0 satisfies

$$\Box u_0 = g_{\alpha\beta\gamma}\partial_\gamma u\partial_\alpha\partial_\beta u \tag{162}$$

with the homogeneous initial data, u_1 satisfies

$$\Box u_1 = \partial_\alpha\widehat{G}_\alpha(u,Du) \tag{163}$$

with the homogeneous initial data, u_2 satisfies

$$\Box u_2 = \widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u + \widehat{B}_{\alpha\beta}(u,Du)\partial_\alpha u\partial_\beta u + \widehat{C}_{\alpha\beta\gamma}(u,Du)\partial_\gamma u\partial_\alpha\partial_\beta u \tag{164}$$

with the same initial data as u, and u_3 satisfies

$$\Box u_3 = F(u) \tag{165}$$

with the homogeneous initial data.

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It follows from Lemma 2.1 and Lemma 2.9 that

$$\begin{aligned}&(1+t)^{\frac{1}{2}}\|u_0(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,\infty}\\&\le C\varepsilon+C\int_0^t(1+\tau)^{-\frac{1}{2}}\|Q(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}d\tau.\end{aligned}\tag{166}$$

By a similar argument as in the proof of (116), the combination of Lemma 2.5, Lemma 2.14 and Hölder inequality gives

$$\|Q(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+5,1}\le C(1+\tau)^{-1}\|Du\|^2_{\Gamma,N,2}\le C(1+\tau)^{-1}A^2\varepsilon^2.\tag{167}$$

Due to (168) and (167), we have

$$(1+t)^{\frac{1}{2}}\|u_0(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,\infty}\le C\varepsilon+CA^2\varepsilon^2.\tag{168}$$

By Lemma 2.1 and Lemma 2.10, we have

$$\begin{aligned}&(1+t)^{\frac{1}{2}}\|u_1(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,\infty}\\&\le C\varepsilon+C\sum_{\alpha=0}^{2}\int_0^t(1+\tau)^{\frac{1}{2}}\|\widehat{G}_\alpha(u,Du)\|_{\Gamma,[\frac{N}{2}]+4,\infty}d\tau\\&\quad+C\sum_{\alpha=0}^{2}\int_0^t(1+\tau)^{-\frac{3}{2}}\|\widehat{G}_\alpha(u,Du)\|_{\Gamma,[\frac{N}{2}]+7,1}d\tau.\end{aligned}\tag{169}$$

Noting (159) and the assumption that $\widehat{G}_\alpha(u,Du)$ is affine with respect to Du, by the estimates on composite functions given in Lemma 2.16, we have

$$\|\widehat{G}_\alpha(u,Du)\|_{\Gamma,[\frac{N}{2}]+4,\infty}\le C\|u\|^2_{\Gamma,[\frac{N}{2}]+4,\infty}\|(u,Du)\|_{\Gamma,[\frac{N}{2}]+4,\infty}\le C(1+\tau)^{-\frac{3}{2}}A^3\varepsilon^3.\tag{170}$$

Similarly, we have

$$\|\widehat{G}_\alpha(u,Du)\|_{\Gamma,[\frac{N}{2}]+7,1}\le C\|u\|_{\Gamma,[\frac{N}{2}]+4,\infty}\|u\|_{\Gamma,[\frac{N}{2}]+7,2}\|(u,Du)\|_{\Gamma,[\frac{N}{2}]+7,2}\le C(1+\tau)^{\frac{1}{2}}A^3\varepsilon^3.\tag{171}$$

The combination of (169)–(171) gives

$$(1+t)^{\frac{1}{2}}\|u_1(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,\infty}\le C\varepsilon+CA^3\varepsilon^3\log(2+t).\tag{172}$$

Similarly to (166), we have

$$\begin{aligned}&(1+t)^{\frac{1}{2}}\|u_2(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,\infty}\\&\le C\varepsilon+C\int_0^t(1+\tau)^{-\frac{1}{2}}\Big(\|\widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+5,1}+\|\widehat{B}_{\alpha\beta}(u,Du)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+5,1}\\&\quad+\|\widehat{C}_{\alpha\beta\gamma}(u,Du)\partial_\gamma u\partial_\alpha\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+5,1}\Big)d\tau.\end{aligned}\tag{173}$$

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Noting (158) and using Lemma 2.14, we get

$$\begin{aligned}&\|\widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+5,1} \le C\|uDu\|_{\Gamma,[\frac{N}{2}]+5,2}\|Du\|_{\Gamma,[\frac{N}{2}]+5,2}\\ &\le C\|u\|_{\Gamma,[\frac{N}{4}]+3,\infty}\|Du\|^2_{\Gamma,[\frac{N}{2}]+5,2} \le C(1+\tau)^{-\frac{1}{2}}A^3\varepsilon^3. \end{aligned} \tag{174}$$

Similarly, we have

$$\|\widehat{B}_{\alpha\beta}(u,Du)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+5,1} + \|\widehat{C}_{\alpha\beta\gamma}(u,Du)\partial_\gamma u\partial_\alpha\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+5,1} \le C(1+\tau)^{-\frac{1}{2}}A^3\varepsilon^3. \tag{175}$$

Owing to (173)–(175), we have

$$(1+t)^{\frac{1}{2}}\|u_2(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,\infty} \le C\varepsilon + CA^3\varepsilon^3\log(2+t). \tag{176}$$

Similarly to (166), we also have

$$(1+t)^{\frac{1}{2}}\|u_3(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,\infty} \le C\varepsilon + C\int_0^t (1+\tau)^{-\frac{1}{2}}\|\widehat{F}(u)\|_{\Gamma,[\frac{N}{2}]+5,1}d\tau. \tag{177}$$

Noting (157), by the estimates on composite functions given in Lemma 2.16, we get

$$\|\widehat{F}(u)\|_{\Gamma,[\frac{N}{2}]+5,1} \le C\|u\|^2_{\Gamma,[\frac{N}{2}]+5,2}\|u\|^3_{\Gamma,[\frac{N}{4}]+3,\infty} \le C(1+\tau)^{-\frac{1}{2}}A^5\varepsilon^5. \tag{178}$$

It follows from (177) and (178) that

$$(1+t)^{\frac{1}{2}}\|u_3(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,\infty} \le C\varepsilon + CA^5\varepsilon^5\log(2+t). \tag{179}$$

Combining (168), (172), (176) and (179), and using (161), if ε is sufficiently small, then we have

$$(1+t)^{\frac{1}{2}}\|u(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+4,\infty} \le C\varepsilon + CA^3\varepsilon^3\log(2+t). \tag{180}$$

4.3. L^2 estimates

By Lemma 2.1 and Lemma 2.12 we have

$$\begin{aligned}&(1+t)^{-\frac{1}{3}}\|u_0(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,2}\\ &\le C\varepsilon + C\int_0^t \|Q(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1}d\tau\\ &+ C\int_0^t (1+\tau)^{-\frac{1}{3}}\|Q(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2}d\tau. \end{aligned} \tag{181}$$

It follows from Lemma 2.5 that for any given k with $|k| \le [\frac{N}{2}]+8$,

$$|\Gamma^k Q(u,Du,D^2u)| \le C(1+\tau)^{-1}\sum_{|k_1|+|k_2|\le|k|+2}|\Gamma^{k_1}u||D\Gamma^{k_2}u|. \tag{182}$$

By Lemma 2.14 we have

$$\|\Gamma^{k_1} u D\Gamma^{k_2} u\|_{L^{\frac{6}{5}}(\mathbb{R}^2)} \le C\|u\|_{\Gamma,[\frac{N}{2}]+4,3}\|Du\|_{\Gamma,N,2}. \tag{183}$$

By Hölder inequality, we have

$$\|u\|_{\Gamma,[\frac{N}{2}]+4,3} \le C\|u\|^{\frac{2}{3}}_{\Gamma,[\frac{N}{2}]+4,2}\|u\|^{\frac{1}{3}}_{\Gamma,[\frac{N}{2}]+4,\infty}. \tag{184}$$

Then, it follows from (182)–(184) that

$$\begin{aligned}
&\|Q(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1} \\
&\le C(1+\tau)^{-1}\|u\|^{\frac{2}{3}}_{\Gamma,[\frac{N}{2}]+8,2}\|u\|^{\frac{1}{3}}_{\Gamma,[\frac{N}{2}]+4,\infty}\|Du\|_{\Gamma,N,2} \\
&\le C(1+\tau)^{-\frac{5}{6}}A^2\varepsilon^2.
\end{aligned} \tag{185}$$

Thanks to Lemma 2.14 and the Sobolev embedding on the unit sphere: $H^1(S^1) \hookrightarrow L^\infty(S^1)$, we have

$$\|\Gamma^{k_1} u D\Gamma^{k_2} u\|_{L^{1,2}(\mathbb{R}^2)} \le C\|u\|_{\Gamma,[\frac{N}{2}]+4,2,\infty}\|Du\|_{\Gamma,N,2} \le C\|u\|_{\Gamma,[\frac{N}{2}]+8,2}\|Du\|_{\Gamma,N,2}. \tag{186}$$

Owing to (182) and (186), we have

$$\|Q(u,Du,D^2u)\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2} \le C(1+\tau)^{-1}\|u\|_{\Gamma,[\frac{N}{2}]+8,2}\|Du\|_{\Gamma,N,2} \le C(1+\tau)^{-\frac{1}{2}}A^2\varepsilon^2. \tag{187}$$

By (181), (185) and (187), we have

$$(1+t)^{-\frac{1}{2}}\|u_0(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,2} \le C\varepsilon + CA^2\varepsilon^2. \tag{188}$$

It follows from Lemma 2.1 and Lemma 2.13 that

$$\|u_1(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,2} \le C\varepsilon^2(\log(2+t))^{1/2} + C\sum_{\alpha=0}^{2}\int_0^t \|\widehat{G}_\alpha(u,Du)\|_{\Gamma,[\frac{N}{2}]+8,2}d\tau. \tag{189}$$

Noting (159) and the assumption that $\widehat{G}_\alpha(u,Du)$ is affine with respect to Du, by the estimates on composite functions given in Lemma 2.16, we have

$$\begin{aligned}
&\|\widehat{G}_\alpha(u,Du)\|_{\Gamma,[\frac{N}{2}]+8,2} \\
&\le C\|u\|^2_{\Gamma,[\frac{N}{4}]+5,\infty}\|u\|_{\Gamma,[\frac{N}{2}]+8,2} + \|u\|^2_{\Gamma,[\frac{N}{4}]+5,\infty}\|Du\|_{\Gamma,[\frac{N}{2}]+8,2} \\
&+ C\|u\|_{\Gamma,[\frac{N}{4}]+5,\infty}\|Du\|_{\Gamma,[\frac{N}{4}]+5,\infty}\|u\|_{\Gamma,[\frac{N}{2}]+8,2}
\end{aligned} \tag{190}$$

By the Klainerman–Sobolev inequality (31), we get

$$\|Du\|_{\Gamma,[\frac{N}{4}]+5,\infty} \le C(1+\tau)^{-\frac{1}{2}}\|Du\|_{\Gamma,N,2}. \tag{191}$$

The combination of (190) and (191) gives

$$\|\widehat{G}_\alpha(u,Du)\|_{\Gamma,[\frac{N}{2}]+8,2} \le C(1+\tau)^{-\frac{1}{2}}A^3\varepsilon^3. \tag{192}$$

Noting (189) and (192), we have

$$(1+t)^{-\frac{1}{2}}\|u_1(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,2} \le C\varepsilon^2 + CA^3\varepsilon^3. \tag{193}$$

Similarly to (181), we have

$$\begin{aligned}
&(1+t)^{-\frac{1}{3}}\|u_2(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,2} \\
&\le C\varepsilon + C\int_0^t \big(\|\widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1} + \|\widehat{B}_{\alpha\beta}(u,Du)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1} \\
&\qquad + \|\widehat{C}_{\alpha\beta\gamma}(u,Du)\partial_\gamma u\partial_\alpha\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1}\big)d\tau \\
&+ C\int_0^t (1+\tau)^{-\frac{1}{3}}\big(\|\widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2} + \|\widehat{B}_{\alpha\beta}(u,Du)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2} \\
&\qquad + \|\widehat{C}_{\alpha\beta\gamma}(u,Du)\partial_\gamma u\partial_\alpha\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2}\big)d\tau.
\end{aligned} \tag{194}$$

It follows from Hölder inequality that

$$\begin{aligned}
&\|\widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1} \\
&\le C\big(\|uDu\|_{\Gamma,[\frac{N}{4}]+5,3,\chi_1}\|Du\|_{\Gamma,[\frac{N}{2}]+8,2} + \|uDu\|_{\Gamma,[\frac{N}{2}]+8,2}\|Du\|_{\Gamma,[\frac{N}{4}]+5,3,\chi_1}\big).
\end{aligned} \tag{195}$$

We have

$$\|uDu\|_{\Gamma,[\frac{N}{4}]+5,3,\chi_1} \le C\|u\|_{\Gamma,[\frac{N}{4}]+5,\infty}\|Du\|_{\Gamma,[\frac{N}{4}]+5,3,\chi_1}) \tag{196}$$

and it follows from Lemma 2.14 that

$$\|uDu\|_{\Gamma,[\frac{N}{2}]+8,2} \le C\|u\|_{\Gamma,[\frac{N}{4}]+6,\infty}\|Du\|_{\Gamma,[\frac{N}{2}]+8,2}). \tag{197}$$

Thanks to (195)–(197), we have

$$\|\widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1} \le C\|u\|_{\Gamma,[\frac{N}{2}]+4,\infty}\|Du\|_{\Gamma,N,2}\|Du\|_{\Gamma,[\frac{N}{4}]+6,3,\chi_1}. \tag{198}$$

It follows from Hölder inequality and the Klainerman–Sobolev inequality (31) that

$$\|Du\|_{\Gamma,[\frac{N}{4}]+6,3,\chi_1} \le C\|Du\|^{\frac{2}{3}}_{\Gamma,[\frac{N}{4}]+6,2}\|Du\|^{\frac{1}{3}}_{\Gamma,[\frac{N}{4}]+6,\infty,\chi_1} \le C(1+\tau)^{-\frac{1}{3}}\|Du\|_{\Gamma,N,2}. \tag{199}$$

The combination of (198) and (199) gives

$$\|\widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{5}{6}}A^3\varepsilon^3. \tag{200}$$

Similar argument leads to

$$\|\widehat{B}_{\alpha\beta}(u,Du)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{5}{6}}A^3\varepsilon^3 \tag{201}$$

and

$$\|\widehat{C}_{\alpha\beta\gamma}(u,Du)\partial_\gamma u\partial_\alpha\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1} \le C(1+\tau)^{-\frac{5}{6}}A^3\varepsilon^3. \tag{202}$$

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Due to Hölder inequality and the Sobolev embedding on the unit sphere: $H^1(S^1) \hookrightarrow L^\infty(S^1)$, we have

$$\begin{aligned}
&\|\widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2} \\
&\le C\big(\|uDu\|_{\Gamma,[\frac{N}{4}]+5,2,\infty,\chi_2}\|Du\|_{\Gamma,[\frac{N}{2}]+8,2} + \|uDu\|_{\Gamma,[\frac{N}{2}]+8,2}\|Du\|_{\Gamma,[\frac{N}{4}]+5,2,\infty,\chi_2}\big) \\
&\le C\big(\|uDu\|_{\Gamma,[\frac{N}{4}]+5,2,\infty,\chi_2} + \|uDu\|_{\Gamma,[\frac{N}{2}]+8,2}\big)\|Du\|_{\Gamma,N,2}.
\end{aligned} \tag{203}$$

By the Sobolev embedding on the unit sphere: $H^1(S^1) \hookrightarrow L^\infty(S^1)$, we have

$$\|uDu\|_{\Gamma,[\frac{N}{4}]+5,2,\infty,\chi_2} \le C\|u\|_{\Gamma,[\frac{N}{4}]+5,\infty}\|Du\|_{\Gamma,[\frac{N}{4}]+5,2,\infty,\chi_2} \le C\|u\|_{\Gamma,[\frac{N}{2}]+4,\infty}\|Du\|_{\Gamma,N,2}. \tag{204}$$

Owing to (203), (204) and (197), we have

$$\|\widehat{A}_{\alpha\beta}(u)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2} \le C\|u\|_{\Gamma,[\frac{N}{2}]+4,\infty}\|Du\|^2_{\Gamma,N,2} \le C(1+\tau)^{-\frac{1}{2}}A^3\varepsilon^3. \tag{205}$$

Similarly, we have

$$\|\widehat{B}_{\alpha\beta}(u,Du)\partial_\alpha u\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2} \le C(1+\tau)^{-\frac{1}{2}}A^3\varepsilon^3 \tag{206}$$

and

$$\|\widehat{C}_{\alpha\beta\gamma}(u,Du)\partial_\gamma u\partial_\alpha\partial_\beta u\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2} \le C(1+\tau)^{-\frac{1}{2}}A^3\varepsilon^3. \tag{207}$$

Putting (194), (200)–(202), (205)–(207) together, we get

$$(1+t)^{-\frac{1}{2}}\|u_2(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,2} \le C\varepsilon + CA^3\varepsilon^3. \tag{208}$$

Similarly to (181), we have

$$\begin{aligned}
&(1+t)^{-\frac{1}{3}}\|u_3(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,2} \\
&\le C\varepsilon + C\int_0^t \|\widehat{F}(u)\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1}d\tau + C\int_0^t (1+\tau)^{-\frac{1}{3}}\|\widehat{F}(u)\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2}d\tau.
\end{aligned} \tag{209}$$

Noting (157), by using the estimates on composite functions given in Lemma 2.16 and (184), we have

$$\begin{aligned}
&\|\widehat{F}(u)\|_{\Gamma,[\frac{N}{2}]+8,\frac{6}{5},\chi_1} \le C\|u\|^3_{\Gamma,[\frac{N}{4}]+5,\infty}\|u\|_{\Gamma,[\frac{N}{4}]+5,3,\chi_1}\|u\|_{\Gamma,[\frac{N}{2}]+8,2} \\
&\le C\|u\|^{\frac{10}{3}}_{\Gamma,[\frac{N}{2}]+4,\infty}\|u\|^{\frac{5}{3}}_{\Gamma,N,2} \le C(1+\tau)^{-\frac{5}{6}}A^5\varepsilon^5.
\end{aligned} \tag{210}$$

By the estimates on composite functions given in Lemma 2.16 and the Sobolev embedding on the unit sphere: $H^1(S^1) \hookrightarrow L^\infty(S^1)$, we have

$$\begin{aligned}
&\|\widehat{F}(u)\|_{\Gamma,[\frac{N}{2}]+8,1,2,\chi_2} \le C\|u\|^3_{\Gamma,[\frac{N}{4}]+5,\infty}\|u\|_{\Gamma,[\frac{N}{4}]+5,2,\infty,\chi_2}\|u\|_{\Gamma,[\frac{N}{2}]+8,2} \\
&\le C\|u\|^3_{\Gamma,[\frac{N}{2}]+4,\infty}\|u\|^2_{\Gamma,[\frac{N}{2}]+8,2} \le C(1+\tau)^{-\frac{1}{2}}A^5\varepsilon^5.
\end{aligned} \tag{211}$$

Due to (209)–(211), we get

$$(1+t)^{-\frac{1}{2}}\|u_3(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,2} \le C\varepsilon + CA^5\varepsilon^5. \tag{212}$$

Combining (188), (193), (208) and (212), and using (161), if ε is sufficiently small, then we have

$$(1+t)^{-\frac{1}{2}}\|u(t,\cdot)\|_{\Gamma,[\frac{N}{2}]+8,2} \leq C\varepsilon + CA^2\varepsilon^2. \tag{213}$$

4.4. Conclusion of the proof

By (153), (180) and (213), we have

$$D_{N,T}(u) \leq C_1\varepsilon + C_2A^2\varepsilon^2 + C_3A^2\varepsilon^2(\log(2+t))^{1/2} + C_4A^3\varepsilon^3\log(2+t). \tag{214}$$

Assume that

$$D_{N,0}(u) \leq C_5\varepsilon. \tag{215}$$

Taking $A = \max\{2C_1, 2C_5\}$ and ε sufficiently small such that

$$C_2A\varepsilon, C_3A\varepsilon(\log(2+t))^{1/2}, C_4A^2\varepsilon^2\log(2+t) \leq \frac{1}{6}, \tag{216}$$

we have

$$D_{N,T}(u) \leq A\varepsilon, \tag{217}$$

then, it follows from (216) that the lifespan of classical solutions to Cauchy problem (1) possesses the following lower bound estimate:

$$T_\varepsilon \geq \exp(c\varepsilon^{-2}), \tag{218}$$

where c is a positive constant independent of ε. Thus the proof of Theorem 1.2 is complete.

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References

[1] T. Li, Y. Zhou, Life-span of classical solutions to nonlinear wave equations in two space dimensions, J. Math. Pures Appl. (9) 73 (3) (1994) 223–249.

[2] T. Li, Y. Zhou, Life-span of classical solutions to nonlinear wave equations in two-space-dimensions. II, J. Partial Differ. Equ. 6 (1) (1993) 17–38.

[3] S. Katayama, Lifespan of solutions for two space dimensional wave equations with cubic nonlinearity, Commun. Partial Differ. Equ. 26 (1–2) (2001) 205–232, https://doi.org/10.1081/PDE-100001753.

[4] T. Li, Y. Zhou, Nonlinear stability for two-space-dimensional wave equations with higher-order perturbations, Nonlinear World 1 (1) (1994) 35–58.

[5] T. Li, Y. Zhou, Nonlinear Wave Equations, Series in Contemporary Mathematics, vol. 2, Springer, 2017.

[6] Y. Zhou, Lifespan of Classical Solutions to the Cauchy Problem of Fully Nonlinear Wave Equations with Small Initial Data (in Chinese), Ph.D. thesis, Fudan University, 1992.

[7] S. Katayama, Global existence for systems of nonlinear wave equations in two space dimensions. II, Publ. Res. Inst. Math. Sci. 31 (4) (1995) 645–665, https://doi.org/10.2977/prims/1195163919.

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[8] A. Hoshiga, The initial value problems for quasi-linear wave equations in two space dimensions with small data, Adv. Math. Sci. Appl. 5 (1) (1995) 67–89.

[9] S. Alinhac, The null condition for quasilinear wave equations in two space dimensions I, Invent. Math. 145 (3) (2001) 597–618, https://doi.org/10.1007/s002220100165.

[10] F. John, Blow-up for quasilinear wave equations in three space dimensions, Commun. Pure Appl. Math. 34 (1) (1981) 29–51, https://doi.org/10.1002/cpa.3160340103.

[11] F. John, Nonlinear Wave Equations, Formation of Singularities, University Lecture Series, vol. 2, American Mathematical Society, Providence, RI, 1990, seventh Annual Pitcher Lectures delivered at Lehigh University, Bethlehem, Pennsylvania, April 1989.

[12] F. John, S. Klainerman, Almost global existence to nonlinear wave equations in three space dimensions, Commun. Pure Appl. Math. 37 (4) (1984) 443–455, https://doi.org/10.1002/cpa.3160370403.

[13] S. Klainerman, Global existence for nonlinear wave equations, Commun. Pure Appl. Math. 33 (1) (1980) 43–101, https://doi.org/10.1002/cpa.3160330104.

[14] S. Klainerman, Uniform decay estimates and the Lorentz invariance of the classical wave equation, Commun. Pure Appl. Math. 38 (3) (1985) 321–332, https://doi.org/10.1002/cpa.3160380305.

[15] S. Klainerman, The null condition and global existence to nonlinear wave equations, in: Nonlinear Systems of Partial Differential Equations in Applied Mathematics, Part 1, Santa Fe, N.M., 1984, in: Lectures in Appl. Math., vol. 23, Amer. Math. Soc., Providence, RI, 1986, pp. 293–326.

[16] S. Klainerman, Remarks on the global Sobolev inequalities in the Minkowski space $\mathbf{R}^{n+1}$, Commun. Pure Appl. Math. 40 (1) (1987) 111–117, https://doi.org/10.1002/cpa.3160400105.

[17] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data, Commun. Pure Appl. Math. 39 (2) (1986) 267–282, https://doi.org/10.1002/cpa.3160390205.

[18] L. Hörmander, The lifespan of classical solutions of nonlinear hyperbolic equations, in: Pseudodifferential Operators, Oberwolfach, 1986, in: Lecture Notes in Math., vol. 1256, Springer, Berlin, 1987, pp. 214–280.

[19] L. Hörmander, On the fully nonlinear Cauchy problem with small data. II, in: Microlocal Analysis and Nonlinear Waves, Minneapolis, MN, 1988–1989, in: IMA Vol. Math. Appl., vol. 30, Springer, New York, 1991, pp. 51–81.

[20] L. Hörmander, Lectures on Nonlinear Hyperbolic Differential Equations, Mathématiques & Applications (Berlin) (Mathematics & Applications), vol. 26, Springer-Verlag, Berlin, 1997.

[21] H. Lindblad, On the lifespan of solutions of nonlinear wave equations with small initial data, Commun. Pure Appl. Math. 43 (4) (1990) 445–472, https://doi.org/10.1002/cpa.3160430403.

[22] T. Li, Y. Chen, Initial value problems for nonlinear wave equations, Commun. Partial Differ. Equ. 13 (4) (1988) 383–422, https://doi.org/10.1080/03605308808820547.

[23] T. Li, Y. Chen, Global Classical Solutions for Nonlinear Evolution Equations, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 45, Longman Scientific & Technical, Harlow, 1992; copublished in the United States with John Wiley & Sons, Inc., New York.

[24] T. Li, X. Yu, Life-span of classical solutions to fully nonlinear wave equations, Commun. Partial Differ. Equ. 16 (6–7) (1991) 909–940, https://doi.org/10.1080/03605309108820785.

[25] T. Li, Y. Zhou, A note on the life-span of classical solutions to nonlinear wave equations in four space dimensions, Indiana Univ. Math. J. 44 (4) (1995) 1207–1248, https://doi.org/10.1512/iumj.1995.44.2026.

[26] H. Lindblad, A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time, Proc. Am. Math. Soc. 132 (4) (2004) 1095–1102, https://doi.org/10.1090/S0002-9939-03-07246-0.

[27] Z. Lei, F.H. Lin, Y. Zhou, Global solutions of the evolutionary Faddeev model with small initial data, Acta Math. Sin. Engl. Ser. 27 (2) (2011) 309–328, https://doi.org/10.1007/s10114-011-0465-1.

[28] D. Kong, K. Liu, Y. Wang, Global existence of smooth solutions to two-dimensional compressible isentropic Euler equations for Chaplygin gases, Sci. China Math. 53 (3) (2010) 719–738, https://doi.org/10.1007/s11425-010-0060-4.

[29] D. Zha, Remarks on nonlinear elastic waves in the radial symmetry in 2-D, Discrete Contin. Dyn. Syst. 36 (7) (2016) 4051–4062, https://doi.org/10.3934/dcds.2016.36.4051.

[30] L. Hörmander, L^1, L^∞ estimates for the wave operator, in: Analyse Mathématique et Applications, Gauthier-Villars, Montrouge, 1988, pp. 211–234.

[31] P. Godin, Lifespan of solutions of semilinear wave equations in two space dimensions, Commun. Partial Differ. Equ. 18 (1993) 895–916.

[32] S. Alinhac, Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux, Invent. Math. 111 (3) (1993) 627–670, https://doi.org/10.1007/BF01231301.

[33] Z. Lei, T.C. Sideris, Y. Zhou, Almost global existence for 2-D incompressible isotropic elastodynamics, Trans. Am. Math. Soc. 367 (11) (2015) 8175–8197.

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