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FINDING SIMPLE CURVES IN SURFACE COVERS IS UNDECIDABLE

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Abstract. It is shown that various questions about the existence of simple closed curves in normal subgroups of surface groups are undecidable.

Contents

1. Introduction

Let S be a pointed hyperbolic surface, possibly with boundary. Here $\tilde{S}$ denotes a regular covering space of S. Given a word in $\pi_1(S)$ there are numerous algorithms that will determine whether this word represents a simple curve on S, for example [1], [2], [13] and [16]. However, if we want to decide whether (a) a given word representing a simple curve is contained in $\pi_1(\tilde{S})$, or (b) whether there exists a word in $\pi_1(\tilde{S})$ representing a simple curve in S, it turns out that these two questions are undecidable. A brief discussion of the notion of undecidability in given in Section 2. The first claim is a consequence of undecidability of the word problem for finitely presented groups, and follows easily from Corollary 4. The second claim is proven in this paper, by finding a reduction of the word problem for finitely presented groups to problem (b).

Theorem 1. *There does not exist an algorithm that determines whether a normal subgroup $\pi_1(\tilde{S}) \subset \pi_1(S)$ contains a word representing a simple curve on S.*

When $\pi_1(\tilde{S})$ does not contain a word representing a simple curve in $\pi_1(S)$, we will say that the cover $\tilde{S}$ does not have any simple curves. Note that "simpleness" here refers to S, not to $\tilde{S}$.

Remark 2. *Most questions about groups only depend on the groups up to isomorphism. However, Theorem 1 is not just about the group $\pi_1(\tilde{S})$, but depends on how $\pi_1(\tilde{S})$ is embedded in the larger group $\pi_1(S)$.*

Theorem 1 arose out of an attempt to devise an algorithm for testing the simple loop conjecture. Let $f : S \to M$ be a 2-sided immersion of a closed surface S in a 3-manifold M, and denote by $f_\sharp : \pi_1(S) \to \pi_1(M)$ the induced map on the fundamental group. The simple loop conjecture states that if the normal subgroup $ker(f_\sharp)$ is nontrivial, then there exists an essential simple closed curve in $ker(f_\sharp)$. The simple loop conjecture has been proven in many special cases, for example Seifert fibered 3-manifolds, [4], graph 3-manifolds, [14] and 3-manifolds with geometric structures modelled on Sol, [15]. Since writing the first draft of this paper, a proof for hyperbolic 3-manifolds has been announced, [10].

There is a sense in which covering spaces not containing simple curves are generic. One reason for this is that any cover factoring through another cover that does not contain simple curves, will also not contain simple curves; some examples will be given in Section 4. The simple loop conjecture is therefore making a very strong claim about maps of surfaces to 3-manifolds, and this paper raises doubts about whether the conjecture is verifiable algorithmically.

Outline of paper. In Section 2, some basic notation is introduced, and some background on computability theory is given. Section 3 will revise some known properties of covering spaces, and Section 4 will construct examples of covering spaces that do not have simple curves. The techniques used in these examples will be used in Section 5 to give a proof of Theorem 1, by modifying Rabin's construction.

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2. Notation, background and assumptions

The 2-sphere has no nontrivial covers, and for the torus, homological methods give a simple algorithm for finding all simple curves on a covering space. When the surface S has empty boundary, it will therefore be assumed that there is a connected component of S with genus g at least 2, otherwise S is assumed to have a connected component with genus at least one, or a 3-holed sphere.

When S is connected with nonempty boundary, $\pi_1(S)$ is a free group. As is often the case when working with surface groups, the proof of theorem 1 is considerably easier when $\pi_1(S)$ is free. A proof of Theorem 1 will first be given in the case that S is connected, orientable, and with empty boundary. This case is important for many topological applications, and can be generalised to the unorientable case, and simplified to the case with free fundamental group and at least two generators. A notable special case is that of the 3-holed sphere. On a 3-holed sphere, any simple curve is freely homotopic to one of the three boundary curves.

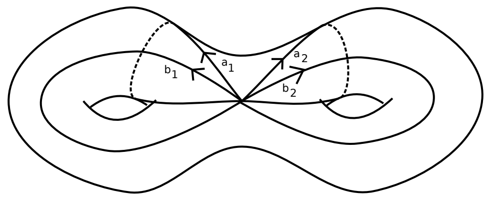

FIGURE 1. A set of simple curves representing a basis for $\pi_1(S)$.

Undecidablity of problem (a) therefore implies undecidability of problem (b).

When S is orientable, connected and with empty boundary, the usual presentation $\langle a_1, \dots, a_g, b_1, \dots, b_g \mid \Pi_{i=1}^{i=g}[a_i, b_i] \rangle$ is assumed, where the curves $\{a_i, b_i\}$ are shown in Figure 1 in the case $g = 2$.

A *curve* on a surface is the image of a map of S^1 into the surface. In this convention, a curve is necessarily closed and connected. A curve γ is said to be *simple* if γ is freely homotopic to an embedding of S^1.

All covering spaces will be assumed to have finitely presented deck transformation groups.

Decidability. Even before the development of modern computers, it was understood that there are many innocuous seeming computational problems that computers can not - both in theory and in practice- solve. This body of knowledge originated in the work of Church and Turing, who developed the theoretical concept of a universal computation machine, that can simulate any other computational device. There is a large number of textbooks covering this material, for example, Chapters 2 and 3 of [6]. The next few paragraphs outline what this theory means for the problems dealt with in this paper. Italicised words have technical definitions, that can be found for example in [6].

In the context of this paper, the *languages* in question are sets of words representing elements of fundamental groups of surfaces. Suppose we want to know whether a given word w_0 representing a simple curve belongs to a normal subgroup $\pi_1(\tilde{S}) \subset \pi_1(S)$. This is done theoretically by selecting a Turing machine that *accepts* the language in question. The word w_0 represents an *initial state* for such a Turing machine. If the question (a) from the introduction were *decidable*, there would exist a Turing machine T accepting the language, such that for any possible initial state w_0, T would perform some finite number of steps, after which it then either *accepts* the input (if w_0 is a word in $\pi_1(\tilde{S})$) or *rejects* the input (if w_0 is not a word in $\pi_1(\tilde{S})$). Corollary 6 claims that problem (a) is not decidable, hence no such Turing machine exists. Similarly for problem (b). Loosely speaking, a question is decidable, if for any allowable input, there exists an algorithm that is guaranteed to terminate after finitely many steps, and give a correct answer, either "yes" or "no".

In practice, a question is shown to be undecidable by showing that a solution to it solves a known (model) question. A *reduction* of a question A to a question B is a *computable* function f from the inputs of A to the inputs of B such that solving B on $r(x)$ solves A on x. It was shown by Novikov, [11], that the word problem is undecidable for finitely presented groups. As is common in group theory, the questions in this paper are shown to be undecidable by finding a reduction of the word problem for finitely presented groups to these questions.

Verifying positive answers. It is possible to develop algorithms that verify positive answers to questions (a) and (b) in finite numbers of steps. Dehn's algorithm, [3], solves the word problem for surface groups, by putting words into a unique, reduced normal form. Since the group $\pi_1(\tilde{S})$ is countable, it is possible to enumerate the reduced words representing the elements of $\pi_1(\tilde{S})$. If the word w_0 is contained in $\pi_1(\tilde{S})$, by comparing the reduced word w_0 with the list of elements of $\pi_1(\tilde{S})$, an algorithm is obtained that will, eventually, verify this. There is an algorithm for testing whether a word represents a simple curve, so, similarly, there exists an algorithm for verifying positive answers to problem (b).

To understand how the algorithms outlined in the previous paragraph do not contradict the claims that the two questions are undecidable is slightly subtle. This has to do with the fact that there is no algorithm for enumerating the elements of $\pi_1(\tilde{S})$ by using, for example, word length. To see one reason for this, first note that the fundamental group of the covering space $\pi_1(\tilde{S})$ will often be infinitely generated, therefore a free group with a countably infinite set of generators. In this case, there is no useful notion of word length, unless we consider word length in $\pi_1(S)$, of which $\pi_1(\tilde{S})$ is a subgroup. It is not possible to devise an algorithm that will enumerate the elements of $\pi_1(\tilde{S})$ in such a way that any element is only preceded by other elements with the same or shorter word length in $\pi_1(S)$. Even if it were possible to use Lemma 3 to enumerate the generators of $\pi_1(\tilde{S})$ in this way, if $\pi_1(S)$ is not a free group, a short word in $\pi_1(S)$ can occur as a product of long generators of $\pi_1(\tilde{S})$, or as a long word in the generators of $\pi_1(\tilde{S})$.

3. Elementary properties of covers

This section recalls some properties of covering spaces. For this section and the next, it will be assumed that S is orientable.

Given a regular cover $\tilde{S}$ of S, there is the following exact sequence, where D denotes the deck transformation group of the cover.

$$1 \to \pi_1(\tilde{S}) \to \pi_1(S) \xrightarrow{\phi} D \to 1$$

The image under ϕ of the generating set $\{a_i, b_i\}$ is a generating set for D.

Lemma 3. *Consider the presentation of D given by*

$$\langle \phi(a_1), \dots, \phi(a_g), \phi(b_1), \dots, \phi(b_g) \mid r_1, \dots, r_s \rangle \tag{1}$$

The subgroup $\pi_1(\tilde{S})$ of $\pi_1(S)$ is generated by the words $\{r_i'\}$ and their conjugates by elements of $\pi_1(S)$ that map to nontrivial elements of D. Here r_i' is the word in the generating set of $\pi_1(S)$ obtained from r_i by replacing each instance of $\phi(a_j)$ by a_j and $\phi(b_j)$ by b_j.

Proof. This claim seems to be well-known. A proof is given in Lemma 4 of [7]. □

When discussing algorithmic properties of groups, it is typical to describe the groups concretely in terms of generators and relations, whenever possible. In this paper, we are not only discussing the surface groups themselves, but an embedding of one surface group inside another. By Lemma 3, this extra structure makes it possible to describe the elements of infinitely generated surface groups in terms of a presentation of the deck transformation group. This is how normal subgroups will be described in this paper. An equivalent means of approaching the problem is to describe normal subgroups as kernels of homomorphisms. This approach also involves performing computations in the deck transformation group, however, in addition, it is necessary to explicitly describe the homomorphisms, which can be difficult.

Corollary 4 (Corollary of Lemma 3)**.** *Any finitely presented group G can be embedded in the deck transformation group of a cover of a closed surface of genus g.*

Proof. Suppose $G := \langle x_1, \ldots, x_s \mid r_1(x_1, \ldots, x_s), \ldots, r_q(x_1, \ldots, x_s)\rangle$, where s is less than $2g$. Let $\{g_1, \ldots, g_{2g}\}$ be a generating set of $\pi_1(S)$. Then by Lemma 3, G is the deck transformation group of the covering space corresponding to the subgroup of $\pi_1(S)$ normally generated by the words $\{r'_i, g_{s+1}, \ldots, g_{2g}\}$. Here r'_i is the word obtained from r_i by replacing every instance of x_1 by the first generator, g_1, of $\pi_1(S)$, every instance of x_2 by the second generator of $\pi_1(S)$, etc. up to x_s. By the Freiheitssatz, $\{g_1, \ldots, g_s\}$ generates a free group, so all we are doing here is writing G as a quotient of a free group in the usual way.

Now suppose G has n generators, where n is larger than or equal to $2g$. It follows from the theory of HNN extensions that every countably generated group can be embedded in a group with two generators; the proof of Theorem 3.1 of [8] gives an algorithm for doing this. Embed G in a two generator group T. Then T is isomorphic to the deck transformation group of a cover of closed surface of genus g, as explained in the previous paragraph. □

Remark 5. *The algorithm from the proof of Theorem 3.1 of [8] allows one to choose a generator of G, and then construct the embedding in a 2-generator group in such a way that this generator is mapped to a generator of the 2-generator group.*

Corollary 6. *Let c be simple curve represented by the conjugacy class of a word in $\pi_1(S)$, and $\tilde{S} \to S$ a regular cover of S. There are pairs of simple curves and covers for which there does not exist an algorithm to determine whether c represents a word in the cover.*

Proof. Let G be a finitely presented group, for which there is no algorithm to decide whether or not G is trivial. Fix a generator g of G. As mentioned in Remark 5, the embedding of G in the deck transformation group D can be done in such a way that g is mapped to the image in D of a simple curve a_1. If G is the trivial group, then the image of a_1 is trivial in D, and hence by Lemma 3, the simple curve a_1 represents a word in $\pi_1(\tilde{S})$. However, if G is nontrivial, g is a nontrivial element of G, and by Lemma 3, a_1 does not represent a curve in the cover. This problem is therefore undecidable. □

4. Covering spaces without simple curves

This section uses homological arguments and properties of normal subgroups of $[\pi_1(S), \pi_1(S)]$ to construct covering spaces without simple curves.

Necessary conditions for curves to be simple. A primitive homology class is an element of $H_1(S;\mathbb{Z})$ that is either trivial or can not be written as a multiple $k \in (\mathbb{Z} \setminus \{\pm 1\})$ of another element of $H_1(S;\mathbb{Z})$. On an orientable surface, a necessary condition for a curve to be simple is that it is a representative of a primitive homology class. For curves represented by words in $[\pi_1(S), \pi_1(S)]$, as proven in [9], a necessary condition for being simple is that the word is not contained in $[\pi_1(S), [\pi_1(S), \pi_1(S)]]$.

Covers without simple curves. An example of a covering space without simple curves is a covering with deck transformation group that can not be generated by any proper subset of $\{\phi(a_1), \dots, \phi(a_g), \phi(b_1), \dots, \phi(b_g)\}$, and for which the words $\{r_i'\}$ in $\pi_1(S)$ from Lemma 3 are all in $[\pi_1(S), [\pi_1(S), \pi_1(S)]]$.

Another source of covers without simple curves comes from Section 3.5 of [7]. In [7], examples of covering spaces are given, for which connected components of pre-images of simple curves do not span the integer homology of the covering space. These covering spaces are constructed by iterating mod m homology covers, as will now be explained briefly.

The mod m homology cover is defined by

$$1 \to \pi_1(\tilde{S}) \to \pi_1(S) \xrightarrow{\phi} H_1(S;\mathbb{Z}_m) \to 1. \tag{2}$$

Due to the fact that ϕ factors through the Hurewicz homomorphism, it follows from Lemma 3 that $[\pi_1(S), \pi_1(S)] \lhd \pi_1(\tilde{S})$. The commutator subgroup of $\pi_1(S)$ contains simple curves, so mod m homology covers do have simple curves. However, the only simple curves in mod m homology covers are contained in $[\pi_1(S), \pi_1(S)]$, because by Lemma 3, none of the other elements of $\pi_1(\tilde{S})$ are contained in primitive homology classes in $H_1(S;\mathbb{Z})$. As shown in Lemma 3 of [7], none of the simple curves in $[\pi_1(S), \pi_1(S)]$ are contained in $[\pi_1(\tilde{S}), \pi_1(\tilde{S})]$, so any simple curves in a mod m homology cover $\tilde{S}$ are nonseparating in $\tilde{S}$. Taking a mod m homology cover $\tilde{\tilde{S}}$ of $\tilde{S}$ therefore kills off any simple curves that were in $\tilde{S}$. It follows that $\tilde{\tilde{S}}$ does not contain simple curves.

A characteristic covering space of S is a cover invariant under the induced action of the mapping class group on $\pi_1(S)$. The homomorphism ϕ from Equation (2) is composed of two homomorphisms; the Hurewicz homomorphism, and the map from $H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z}_2)$. Since both of these maps are invariant under the induced action of the mapping class group of S, it follows that mod m homology covers are characteristic. Composing two mod m homology covers to obtain $\tilde{\tilde{S}}$ therefore gives a regular cover of S.

5. Proof of undecidability

This section starts with a discussion of properties of mod 2 homology covers. These will then be combined with the simpleness criteria from the previous section, and ideas coming from Rabin's construction [12], to prove Theorem 1.

5.1. **Mod 2 homology covers.** Some specific properties of mod 2 homology covers will now be discussed.

Lemma 7. *Let $\tilde{S} \to S$ be the mod 2 homology cover of S. Then $\pi_1(\tilde{S})$ is generated by $2^{2g}-1$ elements $\{g_1 = c_1^2, \ldots, g_n = c_n^2\}$ and their conjugates by elements of $\pi_1(S)$ mapping to nontrivial elements of D. Here c_i is a word representing a simple, nonseparating curve on S.*

Proof. We first claim that a complete set of relations for the deck transformation group D is given by $\{d_i^2, i = 1, \ldots, 2^{2g}-1\}$, where d_i is one of the $2^{2g}-1$ nontrivial elements of D. Note that this is special to the case $m = 2$, as it is only for $m = 2$ that every element of $H_1(S;\mathbb{Z}_m)$ is self inverse, which implies the commutation relations:

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba$$

$H_1(S;\mathbb{Z}_2)$ is by definition the abelian group with $2g$ generators, for which every group element has order two. This proves the claim.

Using Lemma 3, it remains to prove that each of the nontrivial elements of $H_1(S;\mathbb{Z}_2)$ has a simple curve representative. Fix an element h of $H_1(S;\mathbb{Z}_2)$. Then h has as representative a subset of the set of curves $\{a_i, b_j\}$. If this subset contains both a_i and b_i for some i, replace the curves a_i and b_i by the simple curve obtained by Dehn twisting a_i around b_i. Therefore, h is represented by a set of at most g simple, pairwise disjoint curves. It is therefore possible to take a connected sum to obtain a simple curve representing h. A set of simple curves representing the elements of $H_1(S;\mathbb{Z}_2)$ is denoted by $\{c_1, \ldots, c_n\}$. □

5.2. **Tower of covers.** We now have all the ingredients to prove the main theorem of this paper.

Theorem. *There does not exist an algorithm that determines whether a normal subgroup $\pi_1(\tilde{S}) \lhd \pi_1(S)$ contains a word representing a simple curve on S.*

From now on, for simplicity of notation, the same symbols will be used to denote elements of $\pi_1(S)$, and their images in the deck transformation group of the cover.

Proof. Start off by assuming S is connected, orientable and with empty boundary. Denote by S_2 the mod 2 homology cover of S. In what follows, a tower of covers

$$\tilde{S}_2 \to S_2' \to S_2 \to S$$

will be constructed. The most difficult part is to obtain the cover S_2' of S_2; $\tilde{S}_2$ is then the smallest regular cover of S that factors through S_2'. To describe the cover $S_2' \to S$, let $H = \langle h_1, h_2, \ldots h_m \mid R_1, \ldots, R_s\rangle$ be a finitely presented group with unsolvable word problem. The group H will be embedded in the deck transformation group of the cover $\tilde{S}_2 \to S$.

This will be done in such a way that the word problem is reduced to problem (b) from the introduction. More specifically, the letter w will be used to denote the image under this embedding of a word in H for which there is no algorithm to determine whether or not it is equal to the identity. The covering space $S_2' \to S_2$ will depend on the choice of w. When w is the identity, it will be shown that $\pi_1(S_2') = \pi_1(S_2)$. However, when w is not the identity, the construction will give a cover whose fundamental group does not contain any simple curves from $\pi_1(S)$.

It will now be explained how to embed the group $H * \langle x \rangle$ in a two generator group U. Recall the definition of $\{g_i\}$ from Lemma 7. This subset of a generating set of $\pi_1(S_2)$ will be mapped to generators (denoted by the same symbols) of the deck transformation group of the cover S_2' of S_2. By the Freiheitssatz, the $2^{2g} - 1$ elements $\{g_i\}$ generate a free subgroup of $\pi_1(S_2)$. A cover of S_2 will now be constructed, with deck transformation group generated by the images of the group elements $\{g_i\}$.

Let $a := [g_1, [g_1, g_2]]$ and $b := [g_2, [g_1, g_2]]$. The commutator subgroup of $\pi_1(S_2)$ is a free group, so the subgroup $[\pi_1(S_2), [\pi_1(S_2), \pi_1(S_2)]]$ of the commutator subgroup is also free, and a and b freely generate a free subgroup of this group. Similarly, the set

$$\{a, b^{-1}ab, b^{-2}ab^2, \ldots, b^{-(m-1)}ab^{m-1}\}$$

freely generates a free subgroup of $\langle a, b \rangle$.

Let ψ be a homomorphism with trivial kernel, embedding $H * \langle x \rangle$ in a quotient of $\langle g_1, g_2 \rangle$, where ψ is determined by $\psi(h_i) = b^{-(i-1)}ab^{i-1}$ and $\psi(x) = g_1$. Let U be the group with presentation

$$U = \langle g_1, g_2 \mid R_1', \ldots, R_s' \rangle$$

where R_i' is the word in the generators g_1 and g_2 obtained by taking R_i and replacing every instance of h_j with the word in $\langle g_1, g_2 \rangle$ given by $\psi(h_j)$.

Remark 8. *It is necessary here that $\langle g_1, g_2, \ldots, g_{2^{2g}-1} \rangle$ generates a free group. As a result, the above construction gives an embedding of H in a larger group, in the usual way. The same is true for the embeddings of groups that follow.*

Now let

$$J := \langle U, g_3, g_4 \mid g_3 g_1 g_3^{-1} = g_1^2, \quad g_4^{-1} g_2 g_4 = g_2^2 \rangle$$

$$K_0 := \langle J, g_5 \mid g_5^{-1} g_3 g_5 = g_3^2, \quad g_5^{-1} g_4 g_5 = g_4^2 \rangle$$

$$K_1 := \langle K_0, g_6 \mid g_6^{-1} g_5 g_6 = g_5^2 \rangle$$

$$\vdots$$

$$K_{2^{2g}-9} := \langle K_{2^{2g}-10}, \quad g_{2^{2g}-4} \mid g_{2^{2g}-4}^{-1} g_{2^{2g}-5} g_{2^{2g}-4} = g_{2^{2g}-5}^2 \rangle$$

$$Q := \langle g_{2^{2g}-3}, \quad g_{2^{2g}-2}, \quad g_{2^{2g}-1} \mid g_{2^{2g}-2}^{-1} g_{2^{2g}-3} g_{2^{2g}-2} = g_{2^{2g}-3}^2, \quad g_{2^{2g}-1}^{-1} g_{2^{2g}-2} g_{2^{2g}-1} = g_{2^{2g}-2}^2 \rangle$$

$$D_w := \langle K_{2^{2g}-9} * Q \mid g_{2^{2g}-3} = g_{2^{2g}-4}, \quad g_{2^{2g}-1} = [w, g_1] \rangle$$

where w is the image of a word in the generators of H. Note that, given a presentation for H, it is possible to write down a presentation $D_w = \langle g_1, \ldots, g_{2^{2g}-1} \mid r_1, \ldots, r_M \rangle$ for D_w. Recall from earlier that w represents a word for which there is no algorithm to determine whether or not it is the identity.

Consider the group

$$\langle g_1, \ldots, g_{2^{2g}-1}, d_i(g_1), \ldots, d_i(g_{2^{2g}-1}) \mid r_k(g_1, \ldots, g_{2^{2g}-1}), d_j(g_1), d_j(g_2), \ldots, d_j(g_{2^{2g}-1}) \rangle$$

where $i = 1, \ldots, 2^{2g} - 1$, $j = 1, \ldots, 2^{2g} - 1$, $k = 1, \ldots, M$, and d_i is an element of the deck transformation group of the cover $S_2 \to S$. This group can be seen to be the deck transformation group of a cover of S_2, by comparing with the form of the deck transformation groups discussed in Lemma 3. The relations $\{d_j(g_1), d_j(g_2), \ldots, d_j(g_{2^{2g}-1})\}$ were added to ensure that the deck transformation group of the cover is isomorphic to D_w. By abuse of notation, this deck transformation group will also be referred to as D_w. The cover of S_2 with deck transformation group D_w will be called S_2'.

Claim: The following are equivalent:

(1) $w = 1 \in H$
(2) $S_2' \to S_2$ is the trivial cover.

To show that 1 implies 2, note that by construction, $[w, x]$ is only trivial in the deck transformation group when w is the trivial word in H. When w is the trivial word in H, this implies $g_{2^{2g}-1}$ is trivial, from which it then follows that $g_{2^{2g}-2}$ is trivial, and hence $g_{2^{2g}-3}, \ldots, g_1$ are all trivial. In this case S_2^1 is the trivial cover of S_2.

To show that 2 implies 1, when w is not the trivial word in H, $D_w = \langle g_1, \ldots, g_{2^{2g}-1} \mid r_1, \ldots, r_M \rangle$ is a nontrivial group. This follows from normal form theorems for HNN extensions and free products with amalgamation arising from the work of [5]. A reference is Section 2 of Chapter IV of [8]. These normal form theorems also imply that none of the generators $g_1, \ldots, g_{2^{2g}-1}$ are trivial in D_w. This proves the claim.

Now please note that S_2' is a regular cover of S_2, but $\pi_1(S_2')$ is not a normal subgroup of $\pi_1(S)$. We want to find a regular cover $\tilde{S}_2$ of S factoring through S_2'. A necessary condition is that the deck transformation group of the cover $\tilde{S}_2 \to S_2$ is invariant under the induced action of the deck transformation group of the cover $S_2 \to S$. Without this condition, it is not possible to find a deck transformation group, hence the cover could not be regular. We therefore consider the covering spaces $S_{2,i}$ of S_2, where $S_{2,i}$ has deck transformation group $\tilde{D}_{w,i}$ given by

$$\langle d_j(g_1), \ldots, d_j(g_{2^{2g}-1}), d_i(g_1), \ldots, d_i(g_{2^{2g}-1}) \mid r_k(d_i(g_1), \ldots, d_i(g_{2^{2g}-1})), d_j(g_1), \ldots, d_j(g_{2^{2g}-1}) \rangle$$

where i is a fixed natural number $0 \leq i \leq 2^{2g}-1$, the index $j = 0, 1, \ldots, i-1, i+1, \ldots, 2^{2g}-1$, the index $k = 1, \ldots, M$, and d_i, d_j are elements of the deck transformation group of the cover $S_2 \to S$, where $d_0 = I$. Here $\tilde{D}_{w,0}$ coincides with what we were previously calling D_w.

The group $\pi_1(\tilde{S}_2)$ is defined to be the largest normal subgroup of $\pi_1(S)$ in the intersection of all the groups $\pi_1(S_{2,i})$. Denote by N the intersection in $\pi_1(S)$ of the groups $\pi_1(S_{2,i})$, i.e.

$$N := \cap_{i=0}^{2^{2g}-1} \pi_1(S_{2,i})$$

To show that $\pi_1(\tilde{S}_2)$ is nontrivial, we first show that N is nonempty. This is a consequence of Lemma 3, as words of the form $r_k(d_i(g_1), \ldots, d_i(g_{2^{2g}-1}))$, for $i = 0, 1, \ldots, 2^{2g} - 1$ and $k = 1, \ldots, M$ are in the intersection.

To show that N contains a nontrivial normal subgroup, first note that N is invariant under conjugation by elements $\{a_i, b_j\}$ that are mapped to generators of the deck transformation group. This is because conjugation by such elements merely permutes the groups whose intersection is N. Since N is invariant under conjugation by the generators of $\pi_1(S)$, it follows that N is normal in $\pi_1(S)$, and hence $N = \pi_1(\tilde{S}_2)$.

Given a presentation for D_w, a presentation for the deck transformation group of the cover $\tilde{S}_2 \rightarrow S$ can be obtained. The relations of this presentation consist of the relations of D_w, the conjugates by $\{a_i, b_j\}$ of the relations of D_w, and the image of the relation $\Pi_{i=1,\ldots,g}[a_i, b_i]$ in $\pi_1(S)$.

By construction, $\tilde{S}_2$ is a nontrivial cover of S_2 if and only if w is a nontrivial word in H. When w is trivial, $\tilde{S}_2 \rightarrow S$ is the cover $S_2 \rightarrow S$, which we have seen does contain simple curves. We claim that when w is nontrivial in H, none of the words in $\pi_1(\tilde{S}_2)$ represent simple curves on S.

To prove the claim, recall that words representing nonprimitive homology classes, or elements of $[\pi_1(S), [\pi_1(S), \pi_1(S)]]$, can not be simple. By construction, elements of $\pi_1(\tilde{S}_2)$ are representatives of homology classes of $2H_1(S; \mathbb{Z})$. Words in $\pi_1(\tilde{S}_2)$ representing nontrivial homology classes in S are therefore all nonsimple. All other words are normally generated by $\{g_i\}$ and have the property that for any i, the sum of the indices of g_i are zero. Please note that the relations in the definition of D_w set each g_i equal to a commutator. It follows that words in $\pi_1(\tilde{S}_2)$ representing trivial homology classes in S must be in $[[\pi_1(S), \pi_1(S)], [\pi_1(S), \pi_1(S)]] \subset [\pi_1(S), [\pi_1(S), \pi_1(S)]]$. This proves the claim.

The cover $\tilde{S}_2 \rightarrow S$ therefore has simple curves if and only if w is a trivial word in H. An algorithm deciding whether covers have simple curves or not could therefore be used to solve the word problem for finitely presented groups. For connected, orientable surfaces with empty boundary, the theorem follows by contradiction.

When S has nonempty boundary and $\pi_1(S)$ has at least two generators, an identical construction proves the theorem, with one small difference being that a generating set for $\pi_1(S)$ would then consist of curves $\{a_i, b_j\}$ as before, as well as some curves freely homotopic to boundary curves. Also, by first factoring through a double cover, the above construction applies when S is not orientable. Similarly, the proof can be generalised in the obvious way to disconnected surfaces, for which the theorem applies to at least one connected component. □

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