

SIMPLE MODULES FOR AFFINE VERTEX ALGEBRAS IN THE MINIMAL NILPOTENT ORBIT

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ABSTRACT. We explicitly construct, in terms of Gelfand–Tsetlin tableaux, a new family of simple positive energy representations for simple affine vertex algebra $V_k(\mathfrak{sl}_{n+1})$ in the minimal nilpotent orbit of \mathfrak{sl}_{n+1} . These representations are quotients of induced modules over the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_{n+1}$ and include in particular all admissible simple highest weight modules previously described by Arakawa, and all simple modules induced from \mathfrak{sl}_2 . Any such simple module in the minimal nilpotent orbit has bounded weight multiplicities.

1. INTRODUCTION

Relaxed highest weight modules for affine Kac-Moody algebras attract a considerable interest due to their connection to the representation theory of conformal vertex algebras and conformal field theories, cf. [FST88], [AM95], [AM07], [AP08], [AM09], [A16], [ACD18] and references therein. Using the Zhu’s functor the study of positive energy representations of simple affine vertex algebras reduces to the representations of underlined finite-dimensional Lie algebra. This allows to construct new families of simple representations of admissible vertex algebras. This has been exploited in [AFR17], where new families of simple modules were constructed for the affine vertex algebra of \mathfrak{sl}_n and a complete classification of all simple relaxed highest weight representations with finite-dimensional weight spaces in the case $n = 3$ was obtained. Further on, this approach was developed in [KD19-1] and [KD19-2] using Fernando-Mathieu classification of simple weight representations with finite weight multiplicities of finite-dimensional simple Lie algebras. However, explicit construction of such representations is rather difficult. We refer to the recent paper [?], where the localization and the Wakimoto functors were used to construct *relaxed Wakimoto modules* for the universal and simple affine vertex algebras.

In [AFR17], the Gelfand–Tsetlin tableau realization was used to describe admissible families in type A . Such realization provides a basis and explicit formulas for the action of the Lie algebra. In this paper we describe several new families of simple positive energy weight representations of the admissible affine vertex algebra $V_k(\mathfrak{sl}_n)$. These families include all simple highest weight modules in the minimal nilpotent orbit described in [Ara15]. They also include simple modules in the minimal nilpotent orbit induced from an \mathfrak{sl}_2 -subalgebra. All constructed modules have bounded weight multiplicities. Our approach is based on the theory of *relation* Gelfand–Tsetlin modules developed in [FRZ19],

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where explicit tableau basis was constructed for different classes of simple Gelfand–Tsetlin modules for \mathfrak{gl}_n .

Our first main result is Theorem 4.11, which provides a tableau realization for simple highest weight modules, namely we give necessary and sufficient conditions for such modules to be relation modules with respect to the standard Gelfand–Tsetlin subalgebra. In Section 5 we apply twisted localization to relation Gelfand–Tsetlin modules. We establish the following result (cf. Theorem 5.4).

Theorem 1.1. *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, Γ the standard Gelfand–Tsetlin subalgebras (corresponding to the chain of embeddings starting from the upper left corner), $f = E_{21}$, M a Γ -relation module with an injective action of f . Then the twisted localization $D_f^x M$ of M is a Γ -relation Gelfand–Tsetlin module for any $x \in \mathbb{C}$.*

Using the theorem above we obtain our second main result - Theorem 5.6, where we classify all simple Γ -relation Gelfand–Tsetlin modules induced from \mathfrak{sl}_2 -subalgebra generated by E_{12} and E_{21} . Also, in Theorem 5.8 we obtain an explicit construction of a family of simple \mathfrak{sl}_{n+1} -modules, which are parabolically induced from cuspidal \mathfrak{a} -modules, where $\mathfrak{a} \simeq \mathfrak{sl}_m$ is part of the Γ -flag, $m = 2, \dots, n$. Further, localization of simple highest weight relation modules is considered in Proposition 5.12 for a multiplicative set generated by any number of commuting generators of the form E_{k1} .

Finally, in Section 6 we establish our third main results - explicit description of simple admissible highest weight \mathfrak{sl}_{n+1} -modules and modules induced from \mathfrak{sl}_2 subalgebra in the unique minimal non-trivial nilpotent orbit \mathbb{O}_{min} of \mathfrak{sl}_{n+1} . Let

$$k + n = p/q - 1, \quad p > n, \quad q \geq 1 \text{ and } (p, q) = 1,$$

$a \in \{1, 2, \dots, q-1\}$ and $\lambda_a = \lambda - \frac{ap}{q}\varpi_1$, where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a weight of \mathfrak{sl}_{n+1} with $\lambda_i \in \mathbb{Z}_{\geq 0}$, for all $i = 1, \dots, n$, $\lambda_1 + \dots + \lambda_n < p - n$. Then we have (cf. Theorem 6.3 and Theorem 6.11):

Theorem 1.2. (i) *If Γ is the standard Gelfand–Tsetlin subalgebra then the simple highest weight module $L(\lambda_a)$ is a bounded Γ -relation Gelfand–Tsetlin \mathfrak{sl}_{n+1} -module. Moreover, all simple admissible highest weight modules in the minimal nilpotent orbit are bounded Γ -relation Gelfand–Tsetlin modules.*

(ii) *Let β be a root of \mathfrak{g} , \mathfrak{b} a Borel subalgebra of \mathfrak{g} for which β is a positive root, ρ_β the half-sum of positive (with respect to \mathfrak{b}) roots. Let $L_{\mathfrak{b}}(\lambda)$ be an admissible simple \mathfrak{b} -highest weight \mathfrak{g} -module in the minimal orbit, such that $\langle \lambda + \rho_\beta, \beta^\vee \rangle \notin \mathbb{Z}$, and $f = f_\beta$. Denote by $A_{\mathfrak{b}, \beta}$ the set of all $x \in \mathbb{C} \setminus \mathbb{Z}$ for which $x + \langle \lambda + \rho_\beta, \beta^\vee \rangle \notin \mathbb{Z}$. Then the twisted localization modules $D_{f_\beta}^x L_{\mathfrak{b}}(\lambda)$, $x \in A_{\mathfrak{b}, \beta}$, exhaust all simple \mathfrak{sl}_2 -induced admissible modules in the minimal orbit. Moreover, there exists a flag \mathcal{F} such that $D_{\mathcal{F}}^x L_{\mathfrak{b}}(\lambda)$ is $\Gamma_{\mathcal{F}}$ -relation Gelfand–Tsetlin \mathfrak{g} -module.*

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2. PRELIMINARIES

2.1. Weight modules. Let \mathfrak{g} be a simple complex finite dimensional Lie algebra, \mathfrak{h} a fixed Cartan subalgebra, $U(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} , and W the Weyl group of \mathfrak{g} . By Δ we denote the root system of \mathfrak{g} and by Δ_+ the set of positive roots with respect to a fixed basis π of Δ . Put $Q = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha$ for the root lattice and $Q^\vee = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha^\vee$ for the coroot lattice, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Let ρ to be the half sum of positive roots.

By \mathfrak{b}_{st} we denote the standard Borel subalgebra of \mathfrak{g} . In addition to the standard Borel subalgebra of \mathfrak{g} we also consider the standard parabolic subalgebras of \mathfrak{g} . For a subset Σ of π denote by Δ_Σ the root subsystem in \mathfrak{h}^* generated by Σ . Then the standard parabolic subalgebra \mathfrak{p}_Σ of \mathfrak{g} associated to Σ is defined as $\mathfrak{p}_\Sigma = \mathfrak{l}_\Sigma \oplus \mathfrak{u}_\Sigma^+$ with nilradical $\mathfrak{u}_\Sigma^+ := \bigoplus_{\alpha \in \Delta_+ \setminus \Delta_\Sigma} \mathfrak{g}_\alpha$, opposite nilradical $\mathfrak{u}_\Sigma^- := \bigoplus_{\alpha \in \Delta_+ \setminus \Delta_\Sigma} \mathfrak{g}_{-\alpha}$, and Levi subalgebra \mathfrak{l}_Σ defined by

$$\mathfrak{l}_\Sigma := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_\Sigma} \mathfrak{g}_\alpha.$$

Moreover, we have the corresponding triangular decomposition $\mathfrak{g} = \mathfrak{u}_\Sigma^- \oplus \mathfrak{l}_\Sigma \oplus \mathfrak{u}_\Sigma^+$. Note that if $\Sigma = \emptyset$ then $\mathfrak{p}_\Sigma = \mathfrak{b}_{st}$, and if $\Sigma = \pi$ then $\mathfrak{p}_\Sigma = \mathfrak{g}$.

Let V be a simple weight \mathfrak{l}_Σ -module. Consider V as a $\mathfrak{p} - \mathfrak{p}_\Sigma$ module with trivial action of the nilradical \mathfrak{u}_Σ^+ . The *generalized Verma* \mathfrak{g} -module $M_{\mathfrak{p}}^{\mathfrak{g}}(\Sigma, V)$ is the induced module

$$(1) \quad M_{\mathfrak{p}}^{\mathfrak{g}}(\Sigma, V) = \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Sigma)} V.$$

The module $M_{\mathfrak{p}}^{\mathfrak{g}}(\Sigma, V)$ has a unique maximal submodule and a unique simple quotient $L_{\mathfrak{p}}^{\mathfrak{g}}(\Sigma, V)$. We write $M(\lambda)$ for the Verma module $M_{\mathfrak{b}_{st}}^{\mathfrak{g}}(\emptyset, \mathbb{C}v_\lambda)$ and $L(\lambda)$ for $L_{\mathfrak{b}_{st}}^{\mathfrak{g}}(\emptyset, \mathbb{C}v_\lambda)$.

Recall that a \mathfrak{g} -module M is called *weight* if \mathfrak{h} is diagonalizable on M . For $\lambda \in \mathfrak{h}^*$ the subspace M_λ of those $v \in V$ such that $hv = \lambda(h)v$ is the *weight subspace* of weight λ . The dimension of M_λ is the *multiplicity* of weight λ . A weight \mathfrak{g} -module M is *torsion free* if for any nonzero weight subspace M_λ and any root α , a nonzero root vector $X \in \mathfrak{g}_\alpha$ defines an isomorphism between M_λ and $M_{\lambda+\alpha}$. A module M is *cuspidal* if M is finitely generated torsion free module with finite weight multiplicities.

Proposition 2.1. [Mat00, Corollary 1.4] *For a simple weight \mathfrak{g} -module M with finite weight multiplicities, the following assertions are equivalent:*

- a) M is cuspidal;
- b) M is torsion free;
- c) The support of M is exactly one Q -coset.

A weight module satisfying Proposition 2.1, c) is called *dense*.

2.2. Annihilators. Recall that $\lambda \in \mathfrak{h}^*$ is *dominant* if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{<0}$, for all $\alpha \in \Delta^+$. Similarly λ *antidominant* if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Delta^+$. Also, λ is *regular* if $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Delta$. If \mathcal{Z} is the center of $U(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$, then $\chi_\lambda : \mathcal{Z} \rightarrow \mathbb{C}$ stands for the central character of $L(\lambda)$.

For an associative algebra A and an A -module M , the *annihilator* of M in A is the ideal $\{a \in A \mid am = 0 \text{ for all } m \in M\}$ and will be denoted by $\text{Ann}_A(M)$.

Proposition 2.2. [Dix77, 8.5.8] *If $\lambda \in \mathfrak{h}^*$ is dominant, then the annihilator of $L(\lambda)$ is the unique maximal two-sided ideal of $U(\mathfrak{g})$ containing $U(\mathfrak{g}) \ker \chi_\lambda$.*

If λ is regular dominant, then the correspondence $I \mapsto IM(\lambda)$ gives an order-preserving bijection between two sided ideals of $U(\mathfrak{g})$ containing $U(\mathfrak{g}) \ker \chi_\lambda$ and submodules of $M(\lambda)$ [Jos79, Corollaries 4.3 and 4.8]. As a consequence of Proposition 2.2 we immediately have the following well known assertion.

Corollary 2.3. *Let $\lambda, \mu \in \mathfrak{h}^*$ be dominant. If $\mu = w \cdot \lambda$ for some $w \in W$ then $\text{Ann}_{U(\mathfrak{g})} L(\lambda) = \text{Ann}_{U(\mathfrak{g})} L(\mu)$. Moreover, the converse holds if λ is regular.*

2.3. Affine vertex algebras. Consider the non-twisted affine Kac-Moody algebra associated with \mathfrak{g} : $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K$, where K is a central element. Denote by $\widehat{\Delta}$ the set of roots of $\widehat{\mathfrak{g}}$. Then $\widehat{\Delta}^{re} = \{\alpha + m\delta \mid \alpha \in \Delta, m \in \mathbb{Z}\}$ is the set of real roots and $\widehat{\Delta}_+^{re} = \{\alpha + m\delta \mid \alpha \in \Delta_+, m \in \mathbb{Z}_{\geq 0}\} \sqcup \{-\alpha + n\delta \mid \alpha \in \Delta_+, m \in \mathbb{Z}_{\geq 1}\}$ is the set of positive real roots with respect to the basis $\pi \cup \{-\theta + \delta\}$, where θ is the maximal positive root of \mathfrak{g} and δ is the minimal indivisible positive imaginary root. By $\widetilde{W} = W \times Q^\vee$ we denote the extended affine Weyl group of $\widehat{\mathfrak{g}}$.

For $k \in \mathbb{C}$ denote by $V^k(\mathfrak{g})$ the *universal affine vertex algebra* associated with \mathfrak{g} at level k [Kac98, FB04]:

$$(2) \quad V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}})_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}v_k,$$

where $\mathfrak{g}[t]v_k = 0$ and $Kv_k = kv_k$.

The *simple affine vertex algebra* associated with \mathfrak{g} at level k is the unique simple graded quotient of $V^k(\mathfrak{g})$, which will be denoted by $V_k(\mathfrak{g})$. In the conformal case there is a one-to-one correspondence between simple positive energy representations of $V_k(\mathfrak{g})$ and simple $A(V_k(\mathfrak{g}))$ -modules [Zhu96], where

$$(3) \quad A(V_k(\mathfrak{g})) \cong U(\mathfrak{g})/I_k$$

is the *Zhu's algebra* of $V_k(\mathfrak{g})$ and I_k is some two-sided ideal of $U(\mathfrak{g})$.

For $\lambda \in \mathfrak{h}^*$ and $k \in \mathbb{C}$ denote by $\widehat{L}_k(\lambda)$ the simple highest weight $\widehat{\mathfrak{g}}$ -module with the highest weight $\widehat{\lambda} := \lambda + k\Lambda_0 \in \widehat{\mathfrak{h}}^*$, where $\Lambda_0(K) = 1$, and $\Lambda_0(h) = 0$ for $h \in \mathfrak{h}$. Following [KW89], the module $\widehat{L}_k(\lambda)$, and the highest weight λ , will be called *admissible* if

- (1) $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \widehat{\Delta}_+^{re}$;
- (2) $\mathbb{Q}\widehat{\Delta}(\lambda) = \mathbb{Q}\widehat{\Delta}^{re}$,

where $\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{re} \mid \langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\}$. The level k of $\widehat{\mathfrak{g}}$ is *admissible* if the $\widehat{\mathfrak{g}}$ -module $V_k(\mathfrak{g}) \cong \widehat{L}_k(0)$ is admissible.

In this paper we are interested in type A only, hence from now on we fix $n \geq 2$ and assume that \mathfrak{g} is of type A_n . In this case the description of admissible levels is as follows.

Proposition 2.4 (Proposition 1.2 in [KW08]). *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. The number k is admissible if and only if*

$$k + n + 1 = \frac{p}{q} \quad \text{with } p, q \in \mathbb{N}, (p, q) = 1, p \geq n + 1.$$

2.4. Relaxed highest weight modules. Fix an admissible number k . Following [AFR17], we say that a \mathfrak{g} -module M is *admissible* of level k if M is an $A(V_k(\mathfrak{g}))$ -module. The corresponding simple quotient of the induced module for $\widehat{\mathfrak{g}}$ is a *relaxed highest weight module*. We have a one-to-one correspondence between the set of isomorphism classes of simple admissible \mathfrak{g} -modules of level k and the isomorphism classes of simple positive energy representations of $V_k(\mathfrak{g})$.

If k has the denominator q then

$$Pr_k = \bigcup_{\substack{y \in \widetilde{W} \\ w(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}_+^{re}}} Pr_{k,y}, \quad Pr_{k,y} := w \cdot Pr_{k,\mathbb{Z}},$$

is the set of admissible weights λ such that there exists $y \in \widetilde{W}$ satisfying $\widehat{\Delta}(\lambda) = y(\widehat{\Delta}(k\Lambda_0))$, where $Pr_{k,\mathbb{Z}} = Pr_k \cap \widehat{P}_k$ and

$$\widehat{P}_k = \{\lambda \in \widehat{\mathfrak{h}}^* \mid \lambda(K) = k, \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } i = 1, \dots, l\}.$$

where l is the rank of \mathfrak{g} ([KW89]).

Let

$$\overline{Pr}_k = \{\bar{\lambda} \mid \lambda \in Pr_k\} \subset \mathfrak{h}^*,$$

where $\bar{\lambda}$ is the projection of λ to \mathfrak{h}^* . The importance of this set comes from the following Arakawa's result.

Theorem 2.5. [Ara15, Theorem 9.1] *Let k be admissible, $\lambda \in \mathfrak{h}^*$. Then $\widehat{L}_k(\lambda)$ is a module over $V_k(\mathfrak{g})$ if and only if $\lambda \in \overline{Pr}_k$.*

For $\lambda \in \mathfrak{h}^*$ denote $J_\lambda = \text{Ann}_{U(\mathfrak{g})} L(\lambda)$ the corresponding primitive ideal. By Theorem 2.5, a simple \mathfrak{g} -module M is admissible of level k if and only if $\text{Ann}_{U(\mathfrak{g})} M = J_\lambda$ for some $\lambda \in \overline{Pr}_k$.

Now as a consequence of Proposition 2.2 and Corollary 2.3, we have immediately the following statement.

Proposition 2.6. [AFR17, Proposition 2.4] *For $\lambda \in \overline{Pr}_k$, the primitive ideal J_λ is the unique maximal two-sided ideal of $U(\mathfrak{g})$ containing $U(\mathfrak{g}) \ker \chi_\lambda$. In particular, $J_\lambda = J_\mu$ for $\lambda, \mu \in \overline{Pr}_k$ if and only if there exists $w \in W$ such that $\mu = w \cdot \lambda$.*

For a two-sided ideal I of $U(\mathfrak{g})$, denote $\text{Var}(I)$ the *associated variety* of I (the zero locus of the associated graded ideal grI in \mathfrak{g}^* , with respect to the PBW filtration). If I is primitive then $\text{Var}(I) = \overline{\mathbb{O}}$ for some nilpotent orbit \mathbb{O} of \mathfrak{g} [Jos85, Theorem 3.10].

Definition 2.7. A simple \mathfrak{g} -module M is in the orbit \mathbb{O} if $\text{Var}(\text{Ann}_{U(\mathfrak{g})} M) = \overline{\mathbb{O}}$.

We have

Theorem 2.8. [Ara15, Corollary 9.2 and Theorem 9.5] A simple \mathfrak{g} -module M in the orbit \mathbb{O} is admissible of level k if and only if $\text{Ann}_{U(\mathfrak{g})} M = J_\lambda$ for some $\lambda \in [\overline{Pr}_k^\mathbb{O}]$. Moreover, there exists a nilpotent orbit \mathbb{O}_q that depends only on q such that

$$\text{Var}(I_k) = \overline{\mathbb{O}_q}.$$

Hence, $\text{Var}(J_\lambda) \subset \text{Var}(I_k) = \overline{\mathbb{O}_q}$ for any $\lambda \in \overline{Pr}_k$. Moreover, by Theorems 2.5 and 2.8, we have

$$\overline{Pr}_k = \bigsqcup_{\mathbb{O} \subset \overline{\mathbb{O}_q}} \overline{Pr}_k^\mathbb{O},$$

where \mathbb{O} is a nilpotent orbit of \mathfrak{g} and $\overline{Pr}_k^\mathbb{O} = \{\lambda \in \overline{Pr}_k \mid \text{Var}(J_\lambda) = \overline{\mathbb{O}}\}$.

Set $[\overline{Pr}_k] = \overline{Pr}_k / \sim$, where $\lambda \sim \mu$ if and only if there exists $w \in W$ such that $\mu = w \cdot \lambda$. By Proposition 2.6, the ideal J_λ depends only on the class of $\lambda \in \overline{Pr}_k$ in $[\overline{Pr}_k]$. Therefore,

$$[\overline{Pr}_k] = \bigsqcup_{\mathbb{O} \subset \overline{\mathbb{O}_q}} [\overline{Pr}_k^\mathbb{O}],$$

where $[\overline{Pr}_k^\mathbb{O}]$ is the image of $\overline{Pr}_k^\mathbb{O}$ in $[\overline{Pr}_k]$.

3. TWISTED LOCALIZATION

In this section $F := \{f_1, \dots, f_r\}$ will denote any set of locally ad-nilpotent commuting elements of $U(\mathfrak{g})$, \mathbf{x} any element of \mathbb{C}^r .

By $D_F U(\mathfrak{g})$ we denote the localization of $U(\mathfrak{g})$ relative to the multiplicative set $\langle F \rangle$ generated by F . Similarly, for a \mathfrak{g} -module M denote by $D_F M = D_F U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$ the localization of M relative to $\langle F \rangle$. We will consider $D_F M$ both as a $U(\mathfrak{g})$ -module and as a $D_F U(\mathfrak{g})$ -module.

Proposition 3.1. Let M be an F -bijective \mathfrak{g} -module, L, N submodules of M , such that L is not F -bijective, and M/L is simple. If L is isomorphic to a submodule of $D_F N$ then $D_F N \simeq M$.

Proof. As D_F is an exact functor, then $D_F N$ is a submodule of $D_F M \simeq M$. Since L is a proper submodule of $D_F N$, the statement follows from the maximality of L in M . \square

For $\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{C}^r$ consider the automorphism $\Theta_F^\mathbf{x}$ of $D_F U(\mathfrak{g})$ such that

$$\Theta_F^\mathbf{x}(u) := \sum_{i_1, \dots, i_r \geq 0} \binom{x_1}{i_1} \dots \binom{x_r}{i_r} \text{ad}(f_1)^{i_1} \dots \text{ad}(f_r)^{i_r}(u) f_1^{-i_1} \dots f_r^{-i_r},$$

for $u \in D_F U(\mathfrak{g})$, where $\binom{x}{i} := x(x-1)\dots(x-i+1)/i!$ for $x \in \mathbb{C}$, $i \in \mathbb{Z}_{>0}$ and $\binom{x}{0} := 1$ ([Mat00, Section 4]).

Definition 3.2. For a $D_F U(\mathfrak{g})$ -module N , we will denote by $\Phi_F^{\mathbf{x}} N$ the $D_F U(\mathfrak{g})$ -module on N twisted by $\Theta_F^{\mathbf{x}}$, where the new action is given by

$$u \cdot v^{\mathbf{x}} := (\Theta_F^{\mathbf{x}}(u) \cdot v)^{\mathbf{x}},$$

for $u \in D_F U(\mathfrak{g})$, $v \in N$. Here $v^{\mathbf{x}}$ stands for the element v considered as an element of $\Phi_F^{\mathbf{x}} N$.

If M is a \mathfrak{g} -module and $\mathbf{x} \in \mathbb{C}^r$, then $D_F^{\mathbf{x}} M := \Phi_F^{\mathbf{x}} D_F M$ is the *twisted localization* of M relative to F and \mathbf{x} .

Remark 3.3. Note that for $\mathbf{x} \in \mathbb{Z}^r$, we have $\Theta_F^{\mathbf{x}}(u) = \mathbf{f}^{\mathbf{x}} u \mathbf{f}^{-\mathbf{x}}$, where $\mathbf{f}^{\mathbf{x}} := f_1^{x_1} \dots f_r^{x_r}$. Moreover, $D_F^{\mathbf{x}} M$ and $D_F M$ are isomorphic for any \mathfrak{g} -module M .

The following important property of the twisted localization is a consequence of [GP18, Lemma 2.8].

Proposition 3.4. Let M be a weight \mathfrak{g} -module with finite dimensional weight spaces and assume that F is injective on M . Then

$$\text{Ann}_{U(\mathfrak{g})} M = \text{Ann}_{U(\mathfrak{g})} D_F M \subset \text{Ann}_{U(\mathfrak{g})} D_F^{\mathbf{x}} M,$$

for any $\mathbf{x} \in \mathbb{C}^r$. Moreover,

$$\text{Ann}_{U(\mathfrak{g})} M \subset \text{Ann}_{U(\mathfrak{g})} N,$$

for any subquotient N of $D_F^{\mathbf{x}} M$.

Applying Proposition 3.4 and Proposition 2.2 to simple highest weight modules with dominant highest weights we obtain the following statement.

Corollary 3.5. Let $\lambda \in \mathfrak{h}^*$ be dominant, $\mathbf{x} \in \mathbb{C}^r$ and F is injective on $L(\lambda)$. If $N \neq 0$ is a subquotient of $D_F^{\mathbf{x}} L(\lambda)$ then

$$\text{Ann}_{U(\mathfrak{g})} L(\lambda) = \text{Ann}_{U(\mathfrak{g})} D_F^{\mathbf{x}} L(\lambda) = \text{Ann}_{U(\mathfrak{g})} N.$$

4. RELATION GELFAND–TSETLIN MODULES

4.1. Gelfand–Tsetlin modules. For any flag $\mathcal{F} : \mathfrak{gl}_1 \subset \dots \subset \mathfrak{gl}_{n+1}$ we have an induced flag $U_1 \subset \dots \subset U_{n+1}$ of the corresponding universal enveloping algebras. Let \mathcal{Z}_m be the center of U_m . Following [DFO94], we call the subalgebra $\Gamma_{\mathcal{F}}$ of U generated by $\{\mathcal{Z}_m \mid m = 1, \dots, n+1\}$ the *Gelfand–Tsetlin subalgebra* of U (with respect to \mathcal{F}). If the flag is given by left-upper corner inclusions, the corresponding Gelfand–Tsetlin subalgebra will be called the *standard Gelfand–Tsetlin subalgebra* and will be denoted by Γ_{st} . Consider a standard basis E_{ij} , $i, j = 1, \dots, n+1$ of \mathfrak{gl}_{n+1} . Then the standard flag of \mathfrak{gl}_{n+1} consists of \mathfrak{gl}_k , $k = 1, \dots, n+1$, with \mathfrak{gl}_s generated by E_{ij} , $i, j = 1, \dots, s$ for all s .

Definition 4.1. A finitely generated U -module M is called a $\Gamma_{\mathcal{F}}$ -Gelfand–Tsetlin module if M splits into a direct sum of $\Gamma_{\mathcal{F}}$ -modules:

$$M = \bigoplus_{\mathfrak{m} \in \text{Specm} \Gamma_{\mathcal{F}}} M(\mathfrak{m}),$$

where

$$M(\mathfrak{m}) = \{v \in M \mid \mathfrak{m}^k v = 0 \text{ for some } k \geq 0\}.$$

Identifying \mathfrak{m} with the homomorphism $\chi : \Gamma_{\mathcal{F}} \rightarrow \mathbb{C}$ with $\text{Ker}\chi = \mathfrak{m}$, we will call \mathfrak{m} a *Gelfand–Tsetlin character* of M if $M(\mathfrak{m}) \neq 0$. The set of all Gelfand–Tsetlin characters of M is the *Gelfand–Tsetlin support* of M . Also $\dim M(\mathfrak{m})$ will be called the *Gelfand–Tsetlin multiplicity* of \mathfrak{m} .

Remark 4.2. Let $\tau : \mathfrak{gl}_{n+1} \rightarrow \mathfrak{sl}_{n+1}$ be a natural projection which extends to a homomorphism $\bar{\tau} : U(\mathfrak{gl}_{n+1}) \rightarrow U(\mathfrak{sl}_{n+1})$. If Γ is a Gelfand–Tsetlin subalgebra of \mathfrak{gl}_{n+1} then $\bar{\tau}(\Gamma)$ of Γ is called a Gelfand–Tsetlin subalgebra of \mathfrak{sl}_{n+1} .

Remark 4.3. Since all flags are conjugated by the Weyl group then any $\Gamma_{\mathcal{F}}$ is obtained from Γ_{st} by conjugation. We will write $\Gamma_{\mathcal{F}} = w\Gamma_{st}$ for $w \in W$.

Unless otherwise is stated, from now on we will assume that $\Gamma_{\mathcal{F}} = \Gamma_{st}$ which we simply denote by Γ . We will refer to Γ -Gelfand–Tsetlin modules simply as Gelfand–Tsetlin modules.

4.2. Relation modules. A class of *relation* Gelfand–Tsetlin modules was constructed in [FRZ19]. These modules have properties similar to finite dimensional modules [GT50] and to generic Gelfand–Tsetlin modules [DFO94]. We recall their construction here since it will be used later on.

Set $\mathfrak{V} := \{(i, j) \mid 1 \leq j \leq i \leq n+1\}$, and $\mathcal{R} := \mathcal{R}^- \cup \mathcal{R}^0 \cup \mathcal{R}^+ \subset \mathfrak{V} \times \mathfrak{V}$, where

- (4) $\mathcal{R}^+ := \{((i, j); (i-1, t)) \mid 2 \leq j \leq i \leq n+1, 1 \leq t \leq i-1\}$
- (5) $\mathcal{R}^- := \{((i, j); (i+1, s)) \mid 1 \leq j \leq i \leq n, 1 \leq s \leq i+1\}$
- (6) $\mathcal{R}^0 := \{((n+1, i); (n+1, j)) \mid 1 \leq i \neq j \leq n+1\}$

Any subset $\mathcal{C} \subseteq \mathcal{R}$ will be called a *set of relations*. With any $\mathcal{C} \subseteq \mathcal{R}$ we associate a directed graph $G(\mathcal{C})$ with the set of vertices \mathfrak{V} , that has an arrow from vertex (i, j) to (r, s) if and only if $((i, j); (r, s)) \in \mathcal{C}$. For convenience, we will picture the set \mathfrak{V} as a triangular tableau with $n+1$ rows, where the k -th row is $\{(k, 1), \dots, (k, k)\}$, $k = 1, \dots, n+1$.

For $v \in \mathbb{C}^{\frac{(n+1)(n+2)}{2}}$ denote by $T(v)$ the image of v via the natural isomorphism between $\mathbb{C}^{\frac{(n+1)(n+2)}{2}}$ and $\mathbb{C}^{n+1} \times \dots \times \mathbb{C}^1$. If $T(v) = (v^{(n+1)}, \dots, v^{(1)})$ then we refer to $v^{(k)} = (v_{k1}, \dots, v_{kk})$ as the k -th row of $T(v)$. Hence, we can picture $T(v)$ as a triangular tableau, a *Gelfand–Tsetlin tableau* of height $n+1$. Finally, by $\mathbb{Z}_0^{\frac{n(n+1)}{2}}$ we will denote the set of vectors v in $\mathbb{Z}^{\frac{(n+1)(n+2)}{2}}$ such that $v^{(n+1)} = \mathbf{0}$.

Definition 4.4. [FRZ19, Definition 4.2] *Let \mathcal{C} be a set of relations and $T(L)$ any Gelfand–Tsetlin tableau, where $L = (l_{ij}) \in \mathbb{C}^{\frac{(n+1)(n+2)}{2}}$.*

- a) *We say that $T(L)$ satisfies \mathcal{C} if:*
 - (i) $l_{ij} - l_{rs} \in \mathbb{Z}_{\geq 0}$ for any $((i, j); (r, s)) \in \mathcal{C} \cap (\mathcal{R}^+ \cup \mathcal{R}^0)$;
 - (ii) $l_{ij} - l_{rs} \in \mathbb{Z}_{> 0}$ for any $((i, j); (r, s)) \in \mathcal{C} \cap \mathcal{R}^-$.
- b) *We say that $T(L)$ is a \mathcal{C} -realization if $T(L)$ satisfies \mathcal{C} and for any $1 \leq k \leq n$ we have, $l_{ki} - l_{kj} \in \mathbb{Z}$ if only if (k, i) and (k, j) in the same connected component of $G(\mathcal{C})$.*
- c) *We call \mathcal{C} noncritical if for any \mathcal{C} -realization $T(L)$ one has $l_{ki} \neq l_{kj}$, $1 \leq k \leq n$, $i \neq j$, if (k, i) and (k, j) are in the same connected component of $G(\mathcal{C})$.*

d) Suppose that $T(L)$ satisfies \mathcal{C} . Then $\mathcal{B}_{\mathcal{C}}(T(L))$ denotes the set of all tableaux of the form $T(L+z)$, $z \in \mathbb{Z}_0^{\frac{(n+1)(n+2)}{2}}$ satisfying \mathcal{C} . Also, $V_{\mathcal{C}}(T(L))$ denotes the complex vector space spanned by $\mathcal{B}_{\mathcal{C}}(T(L))$.

Remark 4.5. Note that if $T(L)$ satisfies \mathcal{C} , and \mathcal{C} is the maximal set of relations satisfied by $T(L)$, then $T(L)$ is a \mathcal{C} -realization.

Definition 4.6. [FRZ19, Definition 4.4] Let \mathcal{C} be a set of relations. We call \mathcal{C} admissible if for any \mathcal{C} -realization $T(L)$, $V_{\mathcal{C}}(T(L))$ has a structure of a \mathfrak{sl}_{n+1} -module, endowed with the following action of the generators of \mathfrak{sl}_{n+1} :

$$(7) \quad \begin{aligned} E_{k,k+1}(T(w)) &= - \sum_{i=1}^k \left(\frac{\prod_{j=1}^{k+1} (w_{ki} - w_{k+1,j})}{\prod_{j \neq i}^k (w_{ki} - w_{kj})} \right) T(w + \delta^{ki}), \\ E_{k+1,k}(T(w)) &= \sum_{i=1}^k \left(\frac{\prod_{j=1}^{k-1} (w_{ki} - w_{k-1,j})}{\prod_{j \neq i}^k (w_{ki} - w_{kj})} \right) T(w - \delta^{ki}), \\ H_k(T(w)) &= \left(2 \sum_{i=1}^k w_{ki} - \sum_{i=1}^{k-1} w_{k-1,i} - \sum_{i=1}^{k+1} w_{k+1,i} - 1 \right) T(w). \end{aligned}$$

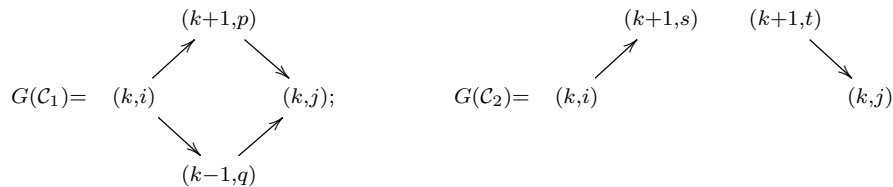
Suppose that \mathcal{C} is a noncritical set of relations whose associated graph $G = G(\mathcal{C})$ satisfies the following conditions:

- (i) G is reduced;
- (ii) If there is a path in G connecting (k, i) and (k, j) with tail (k, i) and head (k, j) , then $i < j$ (in particular G does not contain loops);
- (iii) If the graph G contains an arrow connecting (k, i) and $(k+1, t)$ then $(k+1, s)$ and (k, j) with $i < j$, $s < t$ are not connected in G .

By

Theorem 4.7. [FRZ19, Theorem 4.33], the set \mathcal{C} is admissible if and only if $G(\mathcal{C})$ is a union of disconnected graphs satisfying the following

\diamond -**Condition:** For every adjoining pair (k, i) and (k, j) , $1 \leq k \leq n$, there exist p, q such that $\mathcal{C}_1 \subseteq \mathcal{C}$ or, there exist $s < t$ such that $\mathcal{C}_2 \subseteq \mathcal{C}$, where the graphs associated to \mathcal{C}_1 and \mathcal{C}_2 are as follows



From now on we will assume that \mathcal{C} is an admissible set of relations and consider the \mathfrak{sl}_{n+1} -module $V_{\mathcal{C}}(T(L))$. The simplicity criterion for $V_{\mathcal{C}}(T(L))$ is as follows.

Theorem 4.8. [FRZ19, Theorem 5.6] *The Gelfand-Tsetlin module $V_{\mathcal{C}}(T(L))$ is simple if and only if \mathcal{C} is the maximal (admissible) set of relations satisfied by $T(L)$.*

We will call module $V_{\mathcal{C}}(T(L))$ a Γ -relation Gelfand–Tsetlin module. Similar construction can be made for any Gelfand–Tsetlin subalgebra $\Gamma_{\mathcal{F}}$. If $\Gamma_{\mathcal{F}} = w\Gamma_{st}$ for some $w \in W$, then applying w to the formulas (7) we obtain explicit action of \mathfrak{sl}_{n+1} on $\Gamma_{\mathcal{F}}$ -relation Gelfand–Tsetlin modules.

4.3. Classification of highest weight relation modules. The goal of this section is to describe all simple highest weight modules that can be realized as $V_{\mathcal{C}}(T(L))$ for some admissible set of relations \mathcal{C} . From now on we fix $\mathfrak{g} := \mathfrak{sl}_{n+1}$, the set of simple roots $\pi = \{\alpha_1, \dots, \alpha_n\}$, and set $\alpha_{r,s} := \alpha_r + \dots + \alpha_s$ for $1 \leq r \leq s \leq n$. We use elements E_{ij} , $i, j = 1, \dots, n+1$, $i \neq j$, $H_k := E_{kk} - E_{k+1,k+1}$, $k = 1, \dots, n$, as a basis of \mathfrak{g} .

From [FRZ19, Proposition 5.9], we have the following statement.

Lemma 4.9. *If $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \Delta^+ \setminus \{\alpha_{k,n} \mid k = 1, \dots, n\}$, then the simple highest weight module $L(\lambda)$ is a Γ -relation module.*

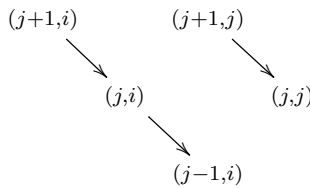
Remark 4.10. The module $L(-\rho)$ is not a Γ -relation module.

The following assertion classifies all simple highest weight Γ -relation modules.

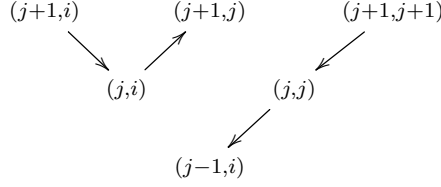
Theorem 4.11. *The simple highest weight module $L(\lambda)$ is a Γ -relation module if and only if one of the following conditions holds:*

- a) $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$, for all $\alpha \in \Delta^+ \setminus \{\alpha_{k,n} \mid k = 1, \dots, n\}$.
- b) *There exist unique i, j with $1 \leq i \leq j < n$ such that:*
 - i) $\langle \lambda + \rho, \alpha_k^\vee \rangle \in \mathbb{Z}_{>0}$ for each $k > j$,
 - ii) $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \Delta^+ \setminus \{\alpha_{i,k} \mid k \geq j\}$,
 - iii) $\langle \lambda + \rho, \alpha_{i,n}^\vee \rangle \in \mathbb{Z}_{\leq 0}$.

Proof. Suppose that $L(\lambda)$ is a Γ -relation Gelfand–Tsetlin module. Then $L(\lambda) \simeq V_{\mathcal{C}}(T(v))$ for some tableau $T(v)$. As $L(\lambda)$ is simple, we can assume without loss of generality that $T(v)$ is a highest weight vector of weight λ and \mathcal{C} is the maximal set of relations satisfied by $T(v)$ (cf. Theorem 4.8). Then $E_{k,k+1}(T(v)) = 0$ for all $k = 1, \dots, n$, which implies $v_{ij} = v_{kj}$ for all $1 \leq i, k < j \leq n+1$ (see (7)), and $\mathcal{C} \supset \{(i+1, j); (i, j) \mid 1 \leq j \leq i \leq n\}$. Set $v_j := v_{n+1,j}$, and note that $v_i - v_{j+1} = \langle \lambda + \rho, \alpha_{i,j}^\vee \rangle$ for all $1 \leq i \leq j \leq n$. Now suppose that for some $\alpha \in \Delta^+ \setminus \{\alpha_{k,n} \mid k = 1, \dots, n\}$ we have $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\leq 0}$. Hence, there exists a pair (r, s) with $1 \leq r \leq s < n$ such that $v_r - v_{s+1} = \langle \lambda + \rho, \alpha_{r,s}^\vee \rangle \in \mathbb{Z}_{<0}$. This shows that $I = \{(r, s) \mid v_s - v_r \in \mathbb{Z}_{>0} \text{ and } 1 \leq r < s < n\} \neq \emptyset$. Now choose $i = \min\{r \mid (r, s) \in I\}$ and $j = \min\{s \mid (i, s) \in I\}$. Then (j, i) and (j, j) form an adjoining pair where the associated graph looks as follows:



On the other hand, \mathcal{C} is an admissible set of relations and, hence, by the \diamond -Condition the associated graph must satisfy



Therefore, $v_j - v_{j+1} \in \mathbb{Z}_{>0}$ and $v_{j+1} - v_i \in \mathbb{Z}_{>0}$. Repeating the same argument, we conclude that $v_k - v_{k+1} \in \mathbb{Z}_{>0}$ for all $k = j, \dots, n$ and $v_{n+1} - v_i \in \mathbb{Z}_{\geq 0}$. Consequently we have $v_r - v_s \in \mathbb{Z}_{>0}$ for all $j \leq r < s \leq n+1$. On the other hand, from the choice of i we have that $v_r - v_s \notin \mathbb{Z}_{\leq 0}$ for each $1 \leq r < i$ and $s > r$. From the definition of j we also have $v_i - v_s \notin \mathbb{Z}_{\leq 0}$ for each $i < s < j$. Given that $v_s - v_i \in \mathbb{Z}_{\geq 0}$ for all $s \geq j$ and $v_i - v_r \notin \mathbb{Z}_{\leq 0}$ for all $i < r < j$, we obtain $v_s - v_r \notin \mathbb{Z}_{\leq 0}$ for all s, r such that $i < r < j \leq s \leq n+1$. Finally, $v_s - v_r \notin \mathbb{Z}_{\leq 0}$ for all s, r such that $i < r < s < j$. Indeed, assume that $v_s - v_r \in \mathbb{Z}_{<0}$ for some $i < r < s < j$. Then there exists $r < j' \leq s$ such that $v_k - v_{k+1} \in \mathbb{Z}_{>0}$ for all $k = j', \dots, n$. In particular, $v_s - v_j \in \mathbb{Z}_{>0}$, which is a contradiction.

Conversely, from item a) and Lemma 4.9 we have that $L(\lambda)$ is a Γ -relation module. Now assume b) and let $v_s - v_{s+1} = \langle \lambda + \rho, \alpha_s^\vee \rangle$ for each $1 \leq s \leq n$ such that $\sum_{s=1}^{n+1} v_s = -n - 1$.

Then for some $1 \leq i \leq j < n$ the following conditions are satisfied:

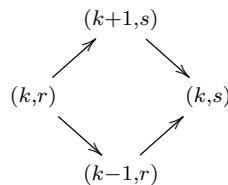
- $v_{n+1} - v_i \in \mathbb{Z}_{\geq 0}$,
- $v_r - v_s \in \mathbb{Z}_{>0}$ for all $j+1 \leq r < s \leq n+1$,
- $v_r - v_s \notin \mathbb{Z}_{\leq 0}$ for all $1 \leq r \leq j, r < s \leq n+1$ and $r, s \neq i$,
- $v_r - v_i \in \mathbb{Z}_{>0}$ for all $j+1 \leq r \leq n$,
- $v_r - v_i \notin \mathbb{Z}_{\leq 0}$ for all $1 \leq r \leq j$ and $r \neq i$.

Let $T(v)$ be a Gelfand–Tsetlin tableau with entries

$$v_{rs} = \begin{cases} v_s, & \text{if } 1 \leq s < i \text{ or } i \leq r \leq j, \\ v_{s+j-i+1}, & \text{if } i \leq s < r+i-j, \\ v_{s-r+j}, & \text{if } s \geq r+i-j, \end{cases}$$

for $1 \leq s \leq r \leq n+1$, and let \mathcal{C} be the maximal set of relations satisfied by $T(v)$. To prove that \mathcal{C} is admissible we consider the following cases.

Case I: Suppose that (k, r) and (k, s) form an adjoining pair for some $1 \leq k < j$. Then we have an indecomposable subset \mathcal{C}' of \mathcal{C} with the associated graph

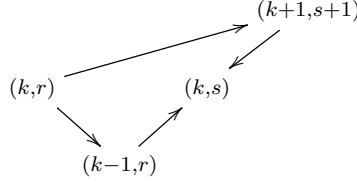


The same happens in the following cases:

- $j \leq k \leq n$ and $1 < s < i$;

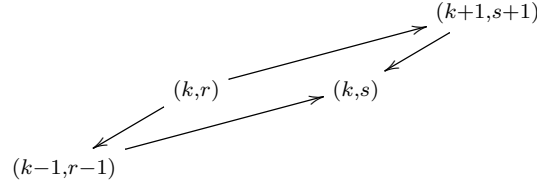
- $j < k \leq n$ and $1 \leq r < s = i$;
- $j + 1 < k \leq n$ and $i < s < n + i - j$ where $r = s - 1$.

Case II: Suppose $j \leq k \leq n$, $1 \leq r < i$ and $s \geq k + i - j$. If (k, r) and (k, s) form an adjoining pair, then we have an indecomposable subset \mathcal{C}' of \mathcal{C} with the following associated graph:

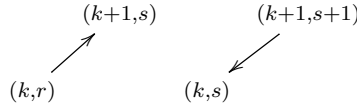


The graph is the same in the case $k = j$ and $r \geq i$.

Case III: Fix $j < k \leq n$ with $r \geq k + i - j$. If (k, r) and (k, s) form an adjoining pair, then the associated graph of the indecomposable subset \mathcal{C}' of \mathcal{C} is as follows:



Case IV: Finally, let $j < k \leq n$, $s = k + i - j$ and $r = s - 1$. In this case (k, r) and (k, s) is an adjoining pair with the associated graph as follows:



The same happens in the case: $j < k \leq n$ and $r = k + i - j - 1$ for each adjoining pair (k, r) and (k, s) .

Therefore \mathcal{C} is admissible and $V_{\mathcal{C}}(T(v))$ is a simple module by Theorem 4.8. Hence, $V_{\mathcal{C}}(T(v)) = U(\mathfrak{g})T(v)$, $E_{k, k+1}(T(v)) = 0$ and $H_k(T(v)) = \langle \lambda, \alpha_k^{\vee} \rangle$ for all $k = 1, \dots, n$. \square

In particular, applying Theorem 4.11 and [Mat00, Proposition 8.5] we get the following criterion for infinite dimensional simple highest weight modules to be Γ -relation modules.

Corollary 4.12. *The simple highest weight module $L(\lambda)$ is a bounded infinite-dimensional Γ -relation module if and only if one of the following conditions holds:*

- $\langle \lambda + \rho, \alpha_n^{\vee} \rangle \notin \mathbb{Z}_{>0}$ and $\langle \lambda + \rho, \alpha_k^{\vee} \rangle \in \mathbb{Z}_{>0}$ for all $k < n$;
- $\langle \lambda + \rho, \alpha_1^{\vee} \rangle \notin \mathbb{Z}$ and $\langle \lambda + \rho, \alpha_k^{\vee} \rangle \in \mathbb{Z}_{>0}$ for all $k > 1$;
- $\langle \lambda + \rho, \alpha_1^{\vee} \rangle \in \mathbb{Z}_{<0}$ and $\langle \lambda + \rho, \alpha_{1n}^{\vee} \rangle \in \mathbb{Z}_{\leq 0}$ and $\langle \lambda + \rho, \alpha_k^{\vee} \rangle \in \mathbb{Z}_{>0}$ for all $k > 1$;
- There exists a unique $i \in \{2, \dots, n-1\}$ such that $\langle \lambda + \rho, \alpha_i^{\vee} \rangle \in \mathbb{Z}_{<0}$, $\langle \lambda + \rho, \alpha_{i-1, i}^{\vee} \rangle \in \mathbb{Z}_{>0}$, $\langle \lambda + \rho, \alpha_{in}^{\vee} \rangle \in \mathbb{Z}_{\leq 0}$, and $\langle \lambda + \rho, \alpha_k^{\vee} \rangle \in \mathbb{Z}_{>0}$ for all $k \neq i$;
- There exists a unique $i \neq n$ such that $\langle \lambda + \rho, \alpha_i^{\vee} \rangle \notin \mathbb{Z}$, $\langle \lambda + \rho, \alpha_{i+1}^{\vee} \rangle \notin \mathbb{Z}$, $\langle \lambda + \rho, \alpha_{i, i+1}^{\vee} \rangle \in \mathbb{Z}_{>0}$, and $\langle \lambda + \rho, \alpha_k^{\vee} \rangle \in \mathbb{Z}_{>0}$ for all $k \neq i, i+1$.

Remark 4.13. For $i = 1, 2, \dots, n$, the highest weight module $L(-\omega_i)$ is bounded but not a Γ -relation module.

Corollary 4.14. *Let $\lambda \in \mathfrak{h}^*$. The following conditions are equivalent*

- a) *The Verma module $M(\lambda)$ is a simple Γ -relation module;*
- b) *$\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}$, for any $\alpha \in \Delta^+ \setminus \{\alpha_{k,n} \mid k = 1, \dots, n\}$, and $\langle \lambda + \rho, \alpha_{k,n}^\vee \rangle \notin \mathbb{Z}_{>0}$ for $k = 1, \dots, n$.*

Remark 4.15. By Corollary 4.14, $M(\lambda)$ is a simple Γ -relation module if and only if $M(\lambda) \simeq V_{\mathcal{C}}(T(v))$, where $T(v)$ is a generic tableau and \mathcal{C} is the maximal set of relations satisfied by $T(v)$. This generalizes [Maz98, Proposition 1] (cf. [AFR17, Corollary 3.7] and [FRZ19, Example 5.10]).

Since $L(\lambda)$ has finite-dimensional weight subspaces, then it is a $\Gamma_{\mathcal{F}}$ -Gelfand–Tsetlin module for any choice of the flag \mathcal{F} . Denote by \widetilde{W} the extension of the Weyl group of \mathfrak{g} by the symmetries of the root system. Let $w \in \widetilde{W}$ and $\mathfrak{b} = w\mathfrak{b}_{st}$. Then, clearly, $L_{\mathfrak{b}}(w\lambda)$ is a $\Gamma_{\mathcal{F}}$ -relation Gelfand–Tsetlin module if and only if $w\lambda$ satisfies the conditions of Corollary 4.12 for $w\alpha_k^\vee$, $k = 1, \dots, n$. In particular, we obtain

Corollary 4.16. *Let $\lambda \in \mathfrak{h}^*$ and $\langle \lambda + \rho, \beta_j^\vee \rangle \in \mathbb{Z}_{>0}$ for some simple roots β_j , $j = 1, \dots, t$. Consider the minimal $w \in W$ such that $\langle w(\lambda + \rho), \beta_j^\vee \rangle \in \mathbb{Z}_{\leq 0}$ for all $j = 1, \dots, t$, and set $\Gamma_{\mathcal{F}} = w\Gamma$. Then the simple highest weight module $L(\lambda)$ is a bounded infinite-dimensional $\Gamma_{\mathcal{F}}$ -relation Gelfand–Tsetlin module if and only if λ satisfies one of the conditions of Corollary 4.12.*

Proof. Since $L_{w\mathfrak{b}_{st}}(w\lambda) \simeq L(\lambda)$, it is sufficient to consider the weight $w\lambda$ and apply Corollary 4.12. \square

Corollary 4.16 shows that $L(\lambda)$ can have different realizations via Gelfand–Tsetlin tableaux as a relation module.

Corollary 4.17. *Suppose λ satisfies Corollary 4.12 and $\langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{Z}_{>0}$ for some simple root β . Then $L_{s_{\beta}\mathfrak{b}_{st}}(\lambda + \beta)$ is a bounded $s_{\beta}\Gamma$ -relation Gelfand–Tsetlin module.*

5. INDUCED RELATION MODULES

In this section we fix $f = E_{21}$ and $\Gamma = \Gamma_{st}$.

5.1. Localization of relation modules.

Lemma 5.1. *Suppose that $T(v)$ is a \mathcal{C} -realization for some admissible set of relations \mathcal{C} .*

- a) *f is injective on $V_{\mathcal{C}}(T(v))$ if and only if $((1, 1); (2, 1)) \notin \mathcal{C}$ and $((1, 1); (2, 2)) \notin \mathcal{C}$.*
- b) *f is surjective on $V_{\mathcal{C}}(T(v))$ if and only if $((2, 1); (1, 1)) \notin \mathcal{C}$ and $((2, 2); (1, 1)) \notin \mathcal{C}$.*

Proof. For any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ we have $f(T(w)) = T(w - \delta^{11}) \in \mathcal{B}_{\mathcal{C}}(T(v))$, since $((1, 1); (2, 1)) \notin \mathcal{C}$ and $((1, 1); (2, 2)) \notin \mathcal{C}$. Suppose that $0 \neq u = \sum_{i \in I} a_i T(w^i) \in V_{\mathcal{C}}(T(v))$.

Then

$$f(u) = f\left(\sum_{i \in I} a_i T(w^i)\right) = \sum_{i \in I} a_i T(w^i - \delta^{11}) \neq 0,$$

as $a_i \neq 0$, $T(w^i - \delta^{11}) \in V_{\mathcal{C}}(T(v))$ and $T(w^i - \delta^{11}) \neq T(w^j - \delta^{11})$ for all $i \neq j \in I$. On the other hand, if $((1, 1); (2, 1)) \in \mathcal{C}$ or $((1, 1); (2, 2)) \in \mathcal{C}$ then there exists $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$

such that $w_{11} - w_{21} = 1$ or $w_{11} - w_{22} = 1$, and hence $f(T(w)) = T(w - \delta^{11}) = 0$, as $T(w - \delta^{11}) \notin \mathcal{B}_{\mathcal{C}}(T(v))$. This shows item (a).

If $((2, 1); (1, 1)) \notin \mathcal{C}$ and $((2, 2); (1, 1)) \notin \mathcal{C}$ then for any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$, we have $T(w + \delta^{11}) \in \mathcal{B}_{\mathcal{C}}(T(v))$ and $f(T(w + \delta^{11})) = T(w)$. Hence, for any $u \in V_{\mathcal{C}}(T(v))$,

$$u = \sum_{i \in I} a_i T(w^i) = \sum_{i \in I} a_i f(T(w^i + \delta^{11})) = f\left(\sum_{i \in I} a_i T(w^i + \delta^{11})\right).$$

On the other hand, assume that $((2, 1); (1, 1)) \in \mathcal{C}$ or $((2, 2); (1, 1)) \in \mathcal{C}$ and f is surjective on $V_{\mathcal{C}}(T(v))$. Then there exists $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ with $w_{11} = w_{21}$ or $w_{11} = w_{22}$ and $u \in V_{\mathcal{C}}(T(v))$, such that

$$f(u) = \sum_{i \in I} a_i T(w^i - \delta^{11}) = T(w).$$

Hence, $T(w) = T(w^i - \delta^{11})$ for some $i \in I$. As $T(w^i) \in \mathcal{B}_{\mathcal{C}}(T(v))$, we have $0 = w_{21} - w_{11} = w_{21}^i - w_{11}^i + 1 \in \mathbb{Z}_{>0}$ or $0 = w_{22} - w_{11} = w_{22}^i - w_{11}^i + 1 \in \mathbb{Z}_{>0}$. This implies (b). \square

Next we consider the twisted localization of $V_{\mathcal{C}}(T(v))$ with respect to f and its tableaux realization.

Lemma 5.2. *Let \mathcal{C} be an admissible set of relations such that*

$$\mathcal{C} \cap \{((1, 1); (2, 1)), ((1, 1); (2, 2))\} = \emptyset$$

and $T(v)$ a \mathcal{C} -realization. If $\mathcal{D} = \mathcal{C} \setminus \{((2, 1); (1, 1)), ((2, 2); (1, 1))\}$ then the localized module $D_f(V_{\mathcal{C}}(T(v)))$ is isomorphic to $V_{\mathcal{D}}(T(v))$.

Proof. Set $M = V_{\mathcal{C}}(T(v))$ and $N = V_{\mathcal{D}}(T(v))$. By Lemma 5.1, the action of f on M is injective but not bijective. Hence M is a proper submodule of $D_f M$. On the other hand, consider the set of relations $\mathcal{C}_1 = \mathcal{D} \cup \{((1, 1); (2, 1))\}$. By Theorem 4.8, $V_{\mathcal{C}_1}(T(v + \delta^{11}))$ is a simple module isomorphic to N/M and hence, M is a maximal submodule of N . Finally, since f is bijective on N , by Proposition 3.1 we have $N \simeq D_f M$. \square

Under the assumptions of Lemma 5.2 the following is straightforward.

Lemma 5.3. *For any tableau $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ and any $x \in \mathbb{C}$ denote by $T(w)^x$ the image of $T(w)$ as an element of the twisted module $D_f^x(V_{\mathcal{C}}(T(v)))$. Then we have*

$$\begin{aligned} E_{12} \cdot T(w)^x &= -(w_{11} + x - w_{21})(w_{11} + x - w_{22})T(w + \delta^{11})^x. \\ E_{k,k+1} \cdot T(w)^x &= -\sum_{i=1}^k \left(\frac{\prod_{j=1}^{k+1} (w_{ki} - w_{k+1,j})}{\prod_{j \neq i}^k (w_{ki} - w_{kj})} \right) T(w + \delta^{ki})^x, \text{ for all } k = 2, \dots, n. \\ E_{21} \cdot T(w)^x &= T(w - \delta^{11})^x. \\ E_{32} \cdot T(w)^x &= \frac{w_{21} - (w_{11} + x)}{w_{21} - w_{22}} T(w - \delta^{21})^x + \frac{w_{22} - (w_{11} + x)}{w_{22} - w_{21}} T(w - \delta^{22})^x. \\ E_{k+1,k} \cdot T(w)^x &= \sum_{i=1}^k \left(\frac{\prod_{j=1}^{k-1} (w_{ki} - w_{k-1,j})}{\prod_{j \neq i}^k (w_{ki} - w_{kj})} \right) T(w - \delta^{ki})^x, \text{ for all } k = 3, \dots, n. \\ H_1 \cdot T(w)^x &= (2(w_{11} + x) - (w_{21} + w_{22}) - 1)T(w)^x. \\ H_2 \cdot T(w)^x &= (2(w_{21} + w_{22}) - (w_{31} + w_{32} + w_{33}) - (w_{11} + x) - 1)T(w)^x. \end{aligned}$$

$$H_k \cdot T(w)^x = \left(2 \sum_{i=1}^k w_{ki} - \sum_{i=1}^{k+1} w_{k+1,i} - \sum_{i=1}^{k-1} w_{k-1,i} - 1 \right) T(w)^x, \text{ for all } k = 3, \dots, n.$$

Theorem 5.4. *Let M be a Γ -relation module with an injective action of f . Then $D_f^x M$ is a Γ -relation Gelfand–Tsetlin module.*

Proof. By hypothesis, $M \simeq V_{\mathcal{C}}(T(v))$ for some \mathcal{C} -realization $T(v)$. By Lemma 5.2, the set $\{T(v + \ell\delta^{11}) \mid \ell \in \mathbb{Z}\}$ is a basis of the localized module $D_f M$. Define the linear isomorphism ϕ from $D_f^x(V_{\mathcal{C}}(T(v)))$ to $V_{\mathcal{D}}(T(v + x\delta^{11}))$, such that $\phi(T(w)) := T(w + x\delta^{11}) = T(w)^x \in \mathcal{B}_{\mathcal{D}}(T(v + x\delta^{11}))$ for any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$. Comparing the twisted action of \mathfrak{g} from Lemma 5.3 with the Gelfand–Tsetlin formulas, we have

$$\phi(g \cdot T(w)^x) = g\phi(T(w)^x)$$

for any $g \in \mathfrak{g}$. Hence, ϕ is an isomorphism of modules, which completes the proof. \square

Corollary 5.5. *Let M be a simple Γ -relation Gelfand–Tsetlin module.*

- a) *If f is bijective on M , then M is isomorphic to $D_f^x(N)$ for some simple Γ -relation Gelfand–Tsetlin module N with an injective action of f and $x \in \mathbb{C} \setminus \mathbb{Z}$.*
- b) *If f is surjective on M but not injective, then M is isomorphic to $D_f(N)/N$ for some simple Γ -relation Gelfand–Tsetlin module N with an injective action of f .*

Proof. As M is a simple relation module, then $M \simeq V_{\mathcal{C}}(T(v))$ for some Gelfand–Tsetlin tableau and a maximal admissible set of relations \mathcal{C} satisfied by $T(v)$.

Part (a): Suppose that f is bijective on M . By Lemma 5.1, we have $v_{11} - v_{21} \notin \mathbb{Z}$ and $v_{11} - v_{22} \notin \mathbb{Z}$. Without loss of generality we assume that $v_{21} - v_{22} \notin \mathbb{Z}$. In this case, let $x = v_{11} - v_{21}$ and consider the Gelfand–Tsetlin tableau $T(v') = T(v + x\delta^{11})$. Then $\mathcal{C}' = \mathcal{C} \cup \{((2, 1); (1, 1))\}$ is the maximal admissible set of relations satisfied by $T(v')$. Hence, by Theorem 4.8, $N = V_{\mathcal{C}'}(T(v'))$ is a simple Γ -relation Gelfand–Tsetlin module. By Lemma 5.1, f is injective on N but not surjective. Then $D_f^x N \simeq M$ by Theorem 5.4.

Part (b): Suppose that f is surjective on M , but not injective. By Lemma 5.1, we can assume that $v_{11} - v_{21} \in \mathbb{Z}_{>0}$ and $v_{11} - v_{22} \notin \mathbb{Z}$. Set again $x = v_{11} - v_{21}$ and $T(v') = T(v + x\delta^{11})$. Then $\mathcal{C}' = \left(\mathcal{C} \setminus \{((1, 1); (2, 1))\} \right) \cup \{((2, 1); (1, 1))\}$ is the maximal admissible set of relations satisfied by $T(v')$. Hence, $N = V_{\mathcal{C}'}(T(v'))$ is a simple relation Gelfand–Tsetlin module by Theorem 4.8. From Lemma 5.1 we have that f is injective on N , and that the localized module $D_f(N)$ is isomorphic to $V_{\mathcal{C} \setminus \{((1, 1); (2, 1))\}}(T(v'))$ (cf. Lemma 5.2). Moreover, N is a maximal submodule of $D_f(N)$, and hence $D_f(N)/N \simeq V_{\mathcal{C}}(T(v))$ is a simple module. We conclude that $D_f(N)/N \simeq V_{\mathcal{C}}(T(v))$. \square

5.2. \mathfrak{sl}_2 -induced relation modules. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra of \mathfrak{g} with the Levi factor isomorphic to $\mathfrak{a}_{\alpha} = \mathfrak{sl}_2 + \mathfrak{h}$ based on the root $\alpha = \alpha_1 \in \pi$. Let $V = V(\gamma, \mu)$ be a simple cuspidal weight \mathfrak{a}_{α} -module, where $\mu \in \mathfrak{h}^*$ is such that $V_{\mu} \neq 0$, and $\gamma \in \mathbb{C}$ is the eigenvalue of the Casimir element $c_{\alpha} = (E_{11} - E_{22} + 1)^2 + 4E_{21}E_{12}$ of \mathfrak{a}_{α} . Denote by $L(\gamma, \mu) = L_{\mathfrak{p}}^{\mathfrak{g}}(\{\alpha\}, V)$ the unique simple quotient of the induced module $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\{\alpha\}, V)$

with the trivial action of the radical of \mathfrak{p} on V . Let Γ_α be any Gelfand–Tsetlin subalgebra corresponding to the flag containing \mathfrak{a}_α .

We have

Theorem 5.6. *Let $n > 1$. The module $L(\gamma, \mu)$ is a Γ_α -relation Gelfand–Tsetlin \mathfrak{g} -module if and only if $L(\lambda)$ is a Γ_α -relation highest weight \mathfrak{g} -module, where $\langle \mu - \lambda, \alpha_1^\vee \rangle = 2x$, $\langle \mu - \lambda, \alpha_2^\vee \rangle = -x$, $\langle \mu - \lambda, \alpha_i^\vee \rangle = 0$ for each $i = 3, \dots, n$, $\langle \lambda + \rho, \alpha_1^\vee \rangle \notin \mathbb{Z}_{\geq 0}$ and $(2x - \mu_1 - 1)^2 = \gamma$. In this case $L(\gamma, \mu) \simeq D_f^x(L(\lambda))$ and x satisfies the condition $x - \langle \mu + \rho, \alpha_1^\vee \rangle \notin \mathbb{Z}$. Moreover,*

- a) $L(\gamma, \mu)$ is bounded if and only if $L(\lambda)$ is bounded;
- b) If λ is dominant then $s_1 \cdot \lambda$ is dominant and $\gamma \neq m^2$ for all $m \in \mathbb{Z}_{\geq 0}$. In this case, $\text{Ann}_{U(\mathfrak{g})} L(\gamma, \mu) = \text{Ann}_{U(\mathfrak{g})} L(\lambda) = \text{Ann}_{U(\mathfrak{g})} L(s_1 \cdot \lambda)$;
- c) If $\gamma = m^2$ for some $m \in \mathbb{Z}_{> 0}$ then $L(\lambda)$ is bounded and the weight $w \cdot \lambda$ is integral, for all $w \in W$. In this case, $\text{Ann}_{U(\mathfrak{g})} L(\lambda) \subset \text{Ann}_{U(\mathfrak{g})} L(\gamma, \mu) \subset \text{Ann}_{U(\mathfrak{g})} L(w \cdot \lambda)$, if $w \cdot \lambda$ is a dominant.

Proof. Since V is a simple dense \mathfrak{sl}_2 -module, then $\gamma \neq (\mu_1 - 2k + 1)^2$ for all $k \in \mathbb{Z}$, and hence $x \notin \mathbb{Z}$. On the other hand, $\gamma = m^2$ for some $m \in \mathbb{Z}_{\geq 0}$ if and only if $\langle \lambda + \rho, \alpha_1^\vee \rangle \in \mathbb{Z}$. Consider the following two cases:

Case 1: Let $\gamma \neq m^2$ for all $m \in \mathbb{Z}$. Assume that $L(\gamma, \mu)$ is a Γ_α -relation Gelfand–Tsetlin \mathfrak{g} -module. Then $L(\gamma, \mu) \simeq V_{\mathcal{C}}(T(v))$ for some set of relations \mathcal{C} and a tableau $T(v) = T(v_{ij})$, such that

$$v_{ij} = \begin{cases} v_1 + x, & \text{if } i = 1 \\ v_1, & \text{if } i > j = 1 \\ v_j, & \text{if } j \geq 2, \end{cases}$$

with $v_1 - v_2 = \langle \mu + \rho, \alpha_1^\vee \rangle - 2x$, $v_2 - v_3 = \langle \mu + \rho, \alpha_2^\vee \rangle + x$, $v_j - v_{j+1} = \langle \mu + \rho, \alpha_j^\vee \rangle$ for each $3 \leq j \leq n$ and $\sum_{j=1}^{n+1} v_j = -n - 1$. On the other hand, f is bijective on $V_{\mathcal{C}}(T(v))$ if and only if $x - \langle \mu, \alpha_1^\vee \rangle \notin \mathbb{Z}$, in which case

$$\mathcal{C} \cap \{((1, 1); (2, 1)), ((1, 1); (2, 2)), ((2, 1); (1, 1)), ((2, 2); (1, 1))\} = \emptyset.$$

Consider the tableau $T(v') = T(v'_{ij})$ with entries $v'_{11} = v_{21}$ and $v'_{ij} = v_{ij}$ for $i \neq 1$. Then $\mathcal{D} = \mathcal{C} \cup \{((2, 1); (1, 1))\}$ is the maximal set of relations satisfied by $T(v')$, since $\langle \lambda + \rho, \alpha_1^\vee \rangle \notin \mathbb{Z}$. Hence the module $V_{\mathcal{D}}(T(v'))$ is simple by Theorem 4.8. Then $V_{\mathcal{D}}(T(v')) \simeq U(\mathfrak{g})T(v')$, $E_{k, k+1}(T(v')) = 0$ and $H_k(T(v)) = \lambda_k T(v)$ for all $k = 1, \dots, n$, $\lambda_k = \langle \lambda, \alpha_k^\vee \rangle$. We get $V_{\mathcal{D}}(T(v')) \simeq L(\lambda)$. Moreover, $c_\alpha(T(v')) = (\lambda_1 + 1)^2 T(v') = \gamma T(v')$. Applying Theorem 5.4, we conclude that $D_f^x L(\lambda) \simeq L(\gamma, \mu)$.

Conversely, let $L(\lambda)$ be a Γ_α -relation Gelfand–Tsetlin module. Then $L(\lambda) \simeq V_{\mathcal{C}}(T(v))$, where $T(v) = T(v_{ij})$ is the Gelfand–Tsetlin tableau such that $v_{ij} = v_j$ with $v_j - v_{j+1} = \langle \lambda + \rho, \alpha_j^\vee \rangle$ for each $1 \leq j \leq n$, and $\sum_{j=1}^{n+1} v_j = -n - 1$. Note that \mathcal{C} is the maximal set of relations satisfied by $T(v)$. Without loss of generality we assume that $((2, 1); (1, 1)) \in \mathcal{C}$ and $((1, 1); (2, 1)) \notin \mathcal{C}$. Hence, the localized module $D_f^x(L(\lambda))$ is isomorphic to $V_{\mathcal{D}}(T(v) +$

$x\delta^{11}$), where $\mathcal{D} = \mathcal{C} \setminus \{((2, 1); (1, 1))\}$ by Theorem 5.4. Given that $x + v_{11} - v_{22} \notin \mathbb{Z}$, then $V_{\mathcal{D}}(T(v + x\delta^{11}))$ is a simple module. Further, the \mathfrak{sl}_2 -module $\text{span}_{\mathbb{C}}\{T(v + (x + \ell)\delta^{11}) \mid \ell \in \mathbb{Z}\}$ is isomorphic to V . By (7), we have $E_{k, k+1}(T(v + (x + \ell)\delta^{11})) = 0$ for all $k = 2, \dots, n$ and $\ell \in \mathbb{Z}$. Hence, we get an epimorphism of $U(\mathfrak{g})$ -modules

$$\phi : M_{\mathfrak{p}}^{\mathfrak{g}}(\{\alpha\}, V) \rightarrow V_{\mathcal{D}}(T(v + x\delta^{11})),$$

such that $\phi(u \otimes T(v)) = uT(v)$ for all $u \in U(\mathfrak{g})$. Therefore, $L(\gamma, \mu) \simeq V_{\mathcal{D}}(T(v + x\delta^{11}))$.

Case 2: Let $\gamma = m^2$. In this case $\langle \lambda + \rho, \alpha_1^{\vee} \rangle \in \mathbb{Z}_{<0}$ and the construction is similar. We leave the details out.

Note that if $\langle \lambda + \rho, \alpha_1^{\vee} \rangle = 0$ then $L(\gamma, \mu)$ is not a relation module. The statement a) is clear from the construction, while b) and c) follows from Corollary 3.5 and definition of λ . \square

5.3. Family of induced relation modules. In this section we give an explicit construction of a family of parabolically induced bounded Γ -relation modules.

Fix complex $\{u_i\}_{i=1, \dots, n+1}$ and $\{v_i\}_{i=1, \dots, n}$ satisfying conditions:

- a) $u_i - v_1 \notin \mathbb{Z}$ for any $1 \leq i \leq n$.
- b) $v_j - v_{j+1} \in \mathbb{Z}_{>0}$ for any $1 \leq j < n$.

Let $T(v)$ be a Gelfand–Tsetlin tableau with entries

$$(8) \quad v_{ij} = \begin{cases} u_i, & \text{if } j = 1 \\ v_{j-1}, & \text{if } j \neq 1 \end{cases}$$

for $1 \leq j \leq i \leq n+1$.

Consider the set of relations $\mathcal{Q} = \mathcal{Q}^+ \cup \mathcal{Q}^-$, where

$$\begin{aligned} \mathcal{Q}^+ &:= \{((i+1, j); (i, j)) \mid 2 \leq j \leq i \leq n\} \\ \mathcal{Q}^- &:= \{((i, j); (i+1, j+1)) \mid 2 \leq j \leq i \leq n\}. \end{aligned}$$

Lemma 5.7. *Let $T(v)$ be a Gelfand–Tsetlin tableau (8). Then $V_{\mathcal{Q}}(T(v))$ is a bounded dense \mathfrak{g} -module. Moreover, $V_{\mathcal{Q}}(T(v))$ is simple if and only if $u_i - u_{i+1} \notin \mathbb{Z}$ for all $1 \leq i \leq n$.*

Proof. Follows from [Maz03, Lemmas 3.1, 3.2, 3.3]. \square

If \mathcal{C} is any admissible set of relations containing \mathcal{Q} and $T(v)$ is a \mathcal{C} -realization, then $V_{\mathcal{C}}(T(v))$ is a submodule of $V_{\mathcal{Q}}(T(v))$ and hence a bounded module. In particular, $V_{\mathcal{C}}(T(v))$ has finite length [Fer90].

For $m \in \{2, \dots, n\}$ consider the tableau $T^m(v)$ as in (8) satisfying the conditions

$$(9) \quad u_i = u_m \text{ for } i = m+1, \dots, n+1, \text{ and } u_i - u_{i+1} \notin \mathbb{Z} \text{ for } i = 1, 2, \dots, m-1.$$

Let $\mathcal{C}^m = \mathcal{Q} \cup \{((i+1, 1); (i, 1)) \mid m \leq i \leq n\}$. Then $T^m(v)$ is a \mathcal{C}^m -realization and the relation module $V_{\mathcal{C}^m}(T^m(v))$ is simple.

Consider the subset $\Sigma_m = \{\alpha_1, \dots, \alpha_{m-1}\} \subset \pi$ of simple roots and the corresponding parabolic subalgebra $\mathfrak{p}_m = \mathfrak{p}_{\Sigma_m} \subset \mathfrak{g}$. Then \mathfrak{p}_m has the Levi subalgebra isomorphic to $\mathfrak{sl}_m + \mathfrak{h}$. We have

Theorem 5.8. *For $m \in \{2 \dots, n\}$, the module $V_{\mathcal{C}^m}(T^m(v))$ is isomorphic to $L_{\mathfrak{p}_m}^{\mathfrak{g}}(\Sigma_m, V)$ for some simple cuspidal \mathfrak{sl}_m -module V .*

Proof. By construction, \mathcal{C}^m is the maximal set of admissible relations satisfied by $T^m(v)$. Hence, $V_{\mathcal{C}^m}(T^m(v))$ is simple Gelfand–Tsetlin \mathfrak{g} -module by Theorem 4.8. Let $\mathcal{D}^m = \mathcal{D}^+ \cup \mathcal{D}^-$, where

$$\begin{aligned}\mathcal{D}^+ &= \{((i+1, j); (i, j)) \mid 2 \leq j \leq i \leq m-1\} \\ \mathcal{D}^- &= \{((i, j); (i+1, j+1)) \mid 2 \leq j \leq i \leq m-1\}.\end{aligned}$$

Then $V = V_{\mathcal{D}^m}(T^m(v)) = \text{span}_{\mathbb{C}} \mathcal{B}_{\mathcal{D}^m}(T^m(v))$ is a simple cuspidal \mathfrak{sl}_m -module by Lemma 5.7, where $\mathcal{B}_{\mathcal{D}^m}(T^m(v))$ denotes the set of \mathcal{D}^m -realizations of the form $T^m(v+z)$, $z \in \mathbb{Z}_0^{\frac{m(m+1)}{2}}$. It follows from (7) that V is a \mathfrak{p}_m -module with the trivial action of $\mathfrak{u}_{\Sigma_m}^+$. Hence, we have a homomorphism

$$\phi : M_{\mathfrak{p}}^{\mathfrak{g}}(\Sigma, V) \rightarrow V_{\mathcal{C}^m}(T^m(v))$$

of $U(\mathfrak{g})$ -modules, such that $\phi(u \otimes T^m(v)) = uT^m(v)$ for all $u \in U(\mathfrak{g})$. Since $V_{\mathcal{C}^m}(T^m(v))$ is a simple \mathfrak{g} -module, then ϕ is surjective, and $L_{\mathfrak{p}}^{\mathfrak{g}}(\Sigma, V) \simeq V_{\mathcal{C}^m}(T^m(v))$. \square

5.4. Localization of highest weight modules with respect to E_{m1} . For any $2 < m \leq n+1$ and any $k \leq m-1$ fix $i_k \in \{1, \dots, k\}$. Associated with the set $\{i_1, \dots, i_{m-1}\}$ define

$$\varepsilon(i_1, \dots, i_{m-1}) := -\delta^{1i_1} - \delta^{2i_2} - \delta^{3i_3} - \dots - \delta^{m-1, i_{m-1}} \in \mathbb{Z}_0^{\frac{(n+1)(n+2)}{2}}.$$

Suppose that $T(v)$ is a \mathcal{C} -realization for some admissible set of relations \mathcal{C} . For each $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ and any $1 \leq i_k \leq k \leq m-1$ define

$$(10) \quad a(w, i_1, \dots, i_{m-1}) := \begin{cases} 0, & \text{if } T(w + \varepsilon(i_1, \dots, i_{m-1})) \notin \mathcal{B}_{\mathcal{C}}(T(v)) \\ \prod_{s=2}^{m-1} \frac{\prod_{t \neq i_{s-1}}^{s-1} (w_{si_s} - w_{s-1,t})}{\prod_{t \neq i_s} (w_{si_s} - w_{st})}, & \text{if } T(w + \varepsilon(i_1, \dots, i_{m-1})) \in \mathcal{B}_{\mathcal{C}}(T(v)). \end{cases}$$

One can easily check by direct computation the following analog of [FGR16, Proposition 3.13].

Proposition 5.9. *Let \mathcal{C} be an admissible set of relations and $T(v)$ a \mathcal{C} -realization. If $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ then*

$$(11) \quad E_{m1}(T(w)) = \sum_{\substack{k=1, \dots, m-1 \\ (i_1, \dots, i_{m-1}) \in \{1, \dots, k\}^{m-1}}} a(w, i_1, \dots, i_{m-1}) T(w + \varepsilon(i_1, \dots, i_{m-1})),$$

for $m \in \{3, \dots, n+1\}$.

Theorem 5.10. *Let $T(v)$ be the Gelfand–Tsetlin tableau satisfying (8), \mathcal{C} an admissible set of relations containing \mathcal{Q} , for which $T(v)$ is a \mathcal{C} -realization. Then*

- a) E_{m1} is injective on $V_{\mathcal{C}}(T(v))$ if and only if $((m-1, 1); (m, 1)) \notin \mathcal{C}$;
b) E_{m1} is surjective on $V_{\mathcal{C}}(T(v))$ if and only if $((m, 1); (m-1, 1)) \notin \mathcal{C}$.

Proof. a) For any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ we have

$$T(w + \varepsilon(1, \dots, 1)) = T(w - \delta^{11} - \delta^{21} - \delta^{31} - \dots - \delta^{m-1,1}) \in \mathcal{B}_{\mathcal{C}}(T(v)).$$

In fact, $(w_{i+1,1} - 1) - (w_{i1} - 1) = w_{i+1,1} - w_{i1}$ for all $i = 1, \dots, m-2$, $w_{m1} - (w_{m-1,1} - 1) = w_{m1} - w_{m-1,1} + 1 \notin \mathbb{Z}_{\leq 1}$, and $(w_{i1} - 1) - w_{i+1,2} = w_{i1} - w_{i+1,2} - 1 \notin \mathbb{Z}$ for all $i = 1, \dots, m-1$. On the other hand, for any $s \in \{3, \dots, m-1\}$ given that $w_{s1} - w_{s-1,2} \notin \mathbb{Z}$ and $w_{s-1,2} - w_{s-1,t} \in \mathbb{Z}$ for all $t \in \{2, \dots, s-1\}$, we have $w_{s1} - w_{s-1,t} \notin \mathbb{Z}$ for all $t \in \{2, \dots, s-1\}$. Hence $a(w, 1, \dots, 1) \neq 0$. Using the fact that $T(w + \varepsilon(1, \dots, 1)) \neq T(w + \varepsilon(i_1, \dots, i_{m-1}))$ for all $(i_1, \dots, i_{m-1}) \neq (1, \dots, 1)$, we obtain (cf. formula (11))

$$\begin{aligned} E_{m1}(T(w)) &= a(w, 1, \dots, 1) T(w + \varepsilon(1, \dots, 1)) + \\ &+ \sum_{k=2}^{m-1} \sum_{(i_1, \dots, i_{m-1}) \neq (1, \dots, 1)} a(w, i_1, \dots, i_{m-1}) T(w + \varepsilon(i_1, \dots, i_{m-1})) \neq 0. \end{aligned}$$

Now, suppose that $0 \neq u = \sum_{i \in I} c_i T(w^i) \in V_{\mathcal{C}}(T(v))$ with $c_i \neq 0$ for all $i \in I$. Then

$$E_{m1}(u) = \sum_{\substack{i \in I, k=1, \dots, m-1 \\ (i_1, \dots, i_{m-1}) \in \{1, \dots, k\}^{m-1}}} c_i a(w^i, i_1, \dots, i_{m-1}) T(w^i - \varepsilon(i_1, \dots, i_{m-1})) \neq 0.$$

In fact, for any $i \in I$ we have $c_i a(w^i, 1, \dots, 1) T(w^i + \varepsilon(1, \dots, 1)) \neq 0$ and $T(w^i + \varepsilon(1, \dots, 1)) \neq T(w^i + \varepsilon(i_1, \dots, i_{m-1}))$ for all $(i_1, \dots, i_{m-1}) \neq (1, \dots, 1)$. Also, as $T(w^i) \neq T(w^j)$ we obtain $T(w^i + \varepsilon(1, \dots, 1)) \neq T(w^j + \varepsilon(1, \dots, 1))$ for all $i \neq j \in I$. Finally, suppose that for any $i \in I$ there exists $j \in I$ such that $j \neq i$ and $T(w^i + \varepsilon(1, \dots, 1)) = T(w^j + \varepsilon(i_1, \dots, i_{m-1}))$ for some $(i_1, \dots, i_{m-1}) \neq (1, \dots, 1)$. Then there exists $s \in \{2, \dots, m-1\}$ with $w_{s1}^i - 1 = w_{s1}^j$, and hence

$$\sum_{j \in I} w_{s1}^j = \sum_{i \in I} (w_{s1}^i - 1) = \sum_{i \in I} w_{s1}^i - \#I = \sum_{j \in I} w_{s1}^j - \#I,$$

which is a contradiction. Thus, there exists $i \in I$ such that $T(w^i + \varepsilon(1, \dots, 1)) \neq T(w^j + \varepsilon(i_1, \dots, i_{m-1}))$ for all $j \neq i$ and for all $(i_1, \dots, i_{m-1}) \neq (1, \dots, 1)$.

Conversely, let $((m-1, 1); (m, 1)) \in \mathcal{C}$. By the hypothesis,

$$\{((m-1, j); (m, j+1)) \mid j = 2, \dots, m-1\} \subset \mathcal{C}$$

and hence there exists $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ such that $w_{m-1,1} - w_{m1} = w_{m-1,j} - w_{m,j+1} = 1$ for each $j = 2, \dots, m-1$. Therefore $T(w + \varepsilon(i_1, \dots, i_{m-1})) \notin \mathcal{B}_{\mathcal{C}}(T(v))$ for any $1 \leq i_k \leq k \leq m-1$ and $E_{m1}(T(w)) = 0$.

- b) First, note that for any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ we have

$$T(w') := T(w - \varepsilon(1, \dots, 1)) \in \mathcal{B}_{\mathcal{C}}(T(v)),$$

since $((m, 1); (m-1, 1)) \notin \mathcal{C}$.

Using (11) we obtain

$$E_{m_1}(T(w')) = b(w, 1, \dots, 1) T(w) + \sum_{\substack{k=2, \dots, m-1 \\ (i_2, \dots, i_{m-1}) \neq (1, \dots, 1)}} b(w, i_2, \dots, i_{m-1}) T(w + \varepsilon'(i_2, \dots, i_{m-1})),$$

where

$$\varepsilon'(i_2, \dots, i_{m-1}) := \varepsilon(i_1, \dots, i_{m-1}) - \varepsilon(1, \dots, 1)$$

and

$$b(w, i_2, \dots, i_{m-1}) := \begin{cases} 0, & \text{if } T(w + \varepsilon'(i_2, \dots, i_{m-1})) \notin \mathcal{B}_C(T(v)) \\ \prod_{s=2}^{m-1} \frac{\prod_{t \neq i_{s-1}}^{s-1} (w_{si_s} - w_{s-1,t} + \delta_{1i_s} - \delta_{1t})}{\prod_{t \neq i_s} (w_{si_s} - w_{st} + \delta_{1i_s} - \delta_{1t})}, & \text{if } T(w + \varepsilon'(i_2, \dots, i_{m-1})) \in \mathcal{B}_C(T(v)). \end{cases}$$

In particular, for any $s \in \{3, \dots, m-1\}$ we have $w_{s1} - w_{s-1,t} \notin \mathbb{Z}$ for any $t \in \{2, \dots, s-1\}$, and hence

$$b(w, 1, \dots, 1) = \prod_{s=2}^{m-1} \frac{\prod_{t=2}^{s-1} (w_{s1} - w_{s-1,t})}{\prod_{t=2} (w_{s1} - w_{st} + 1)} \neq 0.$$

As $T(w) \in \mathcal{B}_C(T(v))$, we have $w_{ij} - w_{i+1,j+1} \in \{1, 2, \dots, d_{ij}\}$, where $d_{ij} := v_{j-1} - v_{n+j-i} - n + i \in \mathbb{Z}_{>0}$. We consider the following cases:

Case I: $w_{22} - w_{33} \in \{1, 2, \dots, d_{22}\}$ and $w_{ij} - w_{i+1,j+1} = 1$ for all $3 \leq i \leq n$ and $2 \leq j \leq i$. In this case, $T(w + \varepsilon'(i_2, \dots, i_{m-1})) \notin \mathcal{B}_C(T(v))$ for all (i_2, \dots, i_{m-1}) such that $i_k > 1$ for some $k = 3, \dots, m-1$, since $(w_{ij} - 1) - w_{i+1,j+1} = 0$ for all $3 \leq i \leq n$ and $2 \leq j \leq i$. Therefore:

$$E_{m_1}(T(w')) = b(w, 1, 1, \dots, 1)T(w) + b(w, 2, 1, \dots, 1)T(w + \delta^{21} - \delta^{22})$$

If $((1, 1); (2, 1)) \in \mathcal{C}$ or $((3, 1); (2, 1)) \in \mathcal{C}$, then $T(w + \delta^{21} - \delta^{22}) \notin \mathcal{B}_C(T(v))$, when $w_{11} - w_{21} = 1$ or $w_{31} - w_{21} = 0$. Hence in these cases $b(w, 2, 1, \dots, 1) = 0$, $E_{m_1}(T(w')) = b(w, 1, 1, \dots, 1)T(w)$.

Suppose now that $((1, 1); (2, 1)) \notin \mathcal{C}$ and $((3, 1); (2, 1)) \notin \mathcal{C}$ and consider the following cases:

a) $w_{22} - w_{33} = 1$.

In this case $T(w + \delta^{21} - \delta^{22}) \notin \mathcal{B}_C(T(v))$ implying $E_{m_1}(T(w')) = b(w, 1, 1, \dots, 1)T(w)$.

b) $w_{22} - w_{33} = i \in \{2, 3, 4, \dots, d_{22}\}$.

In this case $T(w + \delta^{21} - \delta^{22}) \in \mathcal{B}_C(T(v))$ as $(w_{22} - 1) - w_{33} = w_{22} - w_{33} - 1 = i - 1$ and $w_{32} - (w_{22} - 1) = w_{32} - w_{22} + 1 \in \mathbb{Z}_{>0}$. By the induction on i we have $T(w + \delta^{21} - \delta^{22}) = E_{m_1}(T(w_1))$, where $T(w_1)$ is a tableau in $V_C(T(v))$. So,

$E_{m_1}(T(w')) = b(w, 1, 1, \dots, 1)T(w) + b(w, 2, 1, \dots, 1)T(w + \delta^{21} - \delta^{22})$. Consequently, $T(w) = E_{m_1}(T(w_0))$ for some $T(w_0) \in V_{\mathcal{C}}(T(v))$.

Case II: $w_{22} - w_{33} \in \{1, 2, \dots, d_{22}\}$, $w_{32} - w_{43} = j \in \{2, 3, 4, \dots, d_{32}\}$ and $w_{ij} - w_{i+1, j+1} = w_{33} - w_{44} = 1$ for all $4 \leq i \leq n$ and $2 \leq j \leq i$.

In this case, $T(w + \varepsilon'(i_2, \dots, i_{m-1})) \notin \mathcal{B}_{\mathcal{C}}(T(v))$ for all (i_2, \dots, i_{m-1}) such that $i_3 = 3$ or $i_k > 1$ for any $k = 4, \dots, m-1$, as $(w_{ij} - 1) - w_{i+1, j+1} = (w_{33} - 1) - w_{44} = 0$ for all $4 \leq i \leq n$ and $2 \leq j \leq i$.

- a) If $w_{22} - w_{33} = 1$ then $T(w + \delta^{21} - \delta^{22})$ and $T(w + \delta^{21} - \delta^{22} + \delta^{31} - \delta^{32})$ do not belong $\mathcal{B}_{\mathcal{C}}(T(v))$. On the other hand, $T(w + \delta^{31} - \delta^{32}) \in \mathcal{B}_{\mathcal{C}}(T(v))$, as $w_{42} - (w_{32} - 1) \in \mathbb{Z}_{>0}$, $(w_{32} - 1) - w_{43} = j - 1 \geq 2$ and $(w_{32} - 1) - w_{22} = w_{43} - w_{44} + j - 3 \in \mathbb{Z}_{>0}$. Following the case I-(a), we conclude that there exists $T(w_1) \in V_{\mathcal{C}}(T(v))$ such that $T(w + \delta^{31} - \delta^{32}) = E_{m_1}(T(w_1))$. Therefore, $E_{m_1}(T(w')) = b(w, 1, 1, \dots, 1)T(w) + b(w, 1, 2, \dots, 1)E_{m_1}(T(w_1))$.
- b) Assume $w_{22} - w_{33} = i \in \{2, 3, \dots, d_{22}\}$. We immediately get that $T(w + \delta^{21} - \delta^{22})$ and $T(w + \delta^{21} - \delta^{22} + \delta^{31} - \delta^{32})$ belong to $\mathcal{B}_{\mathcal{C}}(T(v))$. On the other hand, $T(w + \delta^{31} - \delta^{32}) \in \mathcal{B}_{\mathcal{C}}(T(v))$ if and only if $w_{43} - w_{44} \geq i - j + 2$, since $w_{42} - (w_{32} - 1) \in \mathbb{Z}_{>0}$ and $(w_{32} - 1) - w_{22} = w_{43} - w_{44} + j - i - 2$. Then, following the cases I and II-a), there exists $T(w_k) \in V_{\mathcal{C}}(T(v))$ for each $k = 1, 2, 3$ such that $T(w + \delta^{21} - \delta^{22}) = E_{m_1}(T(w_1))$, $T(w + \delta^{21} - \delta^{22} + \delta^{31} - \delta^{32}) = E_{m_1}(T(w_2))$ and $T(w + \delta^{31} - \delta^{32}) = E_{m_1}(T(w_3))$ (if $w_{43} - w_{44} < i - j + 2$, then $T(w_3) = 0$).

Hence,

$$\begin{aligned} E_{m_1}(T(w')) = & b(w, 1, 1, \dots, 1)T(w) + b(w, 2, 1, \dots, 1)E_{m_1}(T(w_1)) \\ & + b(w, 2, 2, \dots, 1)E_{m_1}(T(w_2)) + b(w, 1, 2, \dots, 1)E_{m_1}(T(w_3)). \end{aligned}$$

Repeating the process, after finitely many steps we obtain that for any $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ there exists $T(w_0) \in V_{\mathcal{C}}(T(v))$ such that $T(w) = E_{m_1}(T(w_0))$ which implies the surjectivity of E_{m_1} .

Conversely, assume that E_{m_1} is surjective on $V_{\mathcal{C}}(T(v))$ but $((m, 1); (m-1, 1)) \in \mathcal{C}$. Choose $T(w) \in \mathcal{B}_{\mathcal{C}}(T(v))$ such that $w_{m-1, 1} = w_{m, 1}$. As $((m, j); (m-1, j)) \in \mathcal{C}$ for all $j = 2, \dots, m-1$, we can assume without loss of generality that $w_{m-1, j} = w_{m, j}$ for all $j = 2, \dots, m-1$. On the other hand, there exists $u \in V_{\mathcal{C}}(T(v))$ such that

$$E_{m_1}(u) = \sum_{\substack{i \in I, k=1, \dots, m-1 \\ (i_1, \dots, i_{m-1}) \in \{1, \dots, k\}^{m-1}}} c_i a(w^i, i_1, \dots, i_{m-1}) T(w^i - \varepsilon(i_1, \dots, i_{m-1})) = T(w).$$

Hence $T(w) = T(w^i - \varepsilon(i_1, \dots, i_{m-1}))$ for some $i \in I$ and $1 \leq i_k \leq k \leq m-1$. Then there exists $j = 1, \dots, m-1$ such that $0 = w_{mj} - w_{m-1, j} = w_{mj}^i - w_{m-1, j}^i + 1 > 0$. Therefore, $((m, 1); (m-1, 1)) \notin \mathcal{C}$. □

Remark 5.11. Note that in Theorem 5.10 the set of relations \mathcal{C} need not to be indecomposable and need not to be the maximal set of relations satisfied by $T(v)$. This is the case, for example, when $u_{i+1} - u_i \in \mathbb{Z}_{\geq 0}$ for any $1 \leq i < n$ and $\mathcal{C} = \{((i+1, 1); (i, 1)) \mid 1 \leq i \leq n\}$ or $u_{i+1} - u_i \notin \mathbb{Z}$ for any $1 \leq i < n$ and $\mathcal{C} = \emptyset$.

Let $F := \{E_{m_i 1} \mid i = 1, \dots, k\}$ such that, $m_i \in \{2, \dots, n+1\}$ for each $i = 1, \dots, k$. We have

Proposition 5.12. *Let $\lambda \in \mathfrak{h}^*$.*

- a) *If λ satisfies conditions b), c) or e) for $i = 1$ of Corollary 4.12, then $D_F L(\lambda)$ is a bounded relation Gelfand-Tsetlin module.*
- b) *If $M = L(\lambda)$ and $F_m = \{E_{m_1}\}$ for some $m \in \{2, \dots, n+1\}$ such that E_{m_1} acts injectivity on M , then $D_{F_m}(M)/M$ is simple.*

Proof. Suppose that λ satisfies the condition b) of Corollary 4.12 (the proof of other cases is similar). Then $L(\lambda) \simeq V_{\mathcal{C}}(T(v))$ where $T(v)$ is the Gelfand–Tsetlin tableau (8) such that $u_i = u_1$ for all $i = 2, 3, \dots, n+1$ and $\mathcal{C} = \mathcal{Q} \cup \{((i+1, 1); (i, 1)) \mid 1 \leq i \leq n\}$. Set $M = V_{\mathcal{C}}(T(v))$ and $N = V_{\mathcal{D}}(T(v))$, where $\mathcal{D} = \mathcal{C} \setminus \{((m_i, 1); (m_i - 1, 1)) \mid i = 1, \dots, k\}$. Then $M \subset D_F M \subset N$ by Theorem 5.10.

Suppose first that $k = 1$. Then $F = F_m = \{E_{m_1}\}$. Since E_{m_1} is injective but not bijective on M , we conclude by Theorem 5.10 that M is a proper submodule of $D_{F_m} M$. On the other hand, consider the set of relations $\mathcal{D}_m = \mathcal{D} \cup \{((m-1, 1); (m, 1))\}$. Then $V_{\mathcal{D}_m}(T(v + \delta^{m-1, 1}))$ is a simple module by Theorem 4.8, and $V_{\mathcal{D}_m}(T(v + \delta^{m-1, 1})) \simeq N/M$. Hence, M is a maximal submodule of N . Finally, given that E_{m_1} acts bijectively on N we get that $N \simeq D_{F_m} M$ by Proposition 3.1. This completes the proof in the case $k = 1$.

Now, suppose that statement holds for all subsets of F with $k-1$ elements. Define $F_i := F \setminus \{E_{m_i 1}\}$ and $\mathcal{D}_i := \mathcal{D} \cup \{((m_i, 1); (m_i - 1, 1))\}$ for any $i = 1, \dots, k$. Then $D_{F_i} M \simeq V_{\mathcal{D}_i}(T(v))$. Set $L = V_{\mathcal{D}_1}(T(v)) + \dots + V_{\mathcal{D}_k}(T(v))$. As $D_{F_i} M \subset D_{\{E_{m_i 1}\}} D_{F_i} M \simeq D_F M$, we have $L \subset D_F M$. Since L is not F -bijective, then L is a proper submodule of $D_F M$. On the other hand, let $T(w) = T(v + i_1 \delta^{m_1-1, 1} + (i_1 - 1) \delta^{m_2-1, 1} + \dots + \delta^{m_{i_s}-1, 1} + \dots + (k - i_s + 1) \delta^{m_{i_s+1}-1, 1} + \dots + \delta^{m_k-1, 1})$. Consider $\mathcal{A} = \mathcal{D} \cup \{((m_i - 1, 1); (m_i, 1)) \mid i = 1, \dots, k\}$, where $\{m_i \mid i = 1, 2, \dots, k\} = \{m_1, \dots, m_{i_1}\} \cup \dots \cup \{m_{i_s}, \dots, m_k\}$ is a disjoint union of sets with consecutive elements. Then \mathcal{A} is a maximal set of relations satisfied by $T(w)$ and $V_{\mathcal{A}}(T(w))$ is a simple module by Theorem 4.8. As $N/L \simeq V_{\mathcal{A}}(T(w))$, we conclude that L is a maximal submodule of N and $D_F M \simeq N$ by Proposition 3.1. □

6. SIMPLE MODULES IN THE MINIMAL NILPOTENT ORBIT

In this section $\mathfrak{g} = \mathfrak{sl}_{n+1}$.

6.1. Minimal nilpotent orbit. Let k an admissible number for $\widehat{\mathfrak{g}}$ with denominator $q \in \mathbb{N}$. In this section we discuss explicit construction of simple admissible highest weight \mathfrak{g} -modules and modules induced from \mathfrak{sl}_2 subalgebra of \mathfrak{g} in the minimal nilpotent orbit. Recall that \mathbb{O}_{min} denotes the unique minimal non-trivial nilpotent orbit of \mathfrak{g} , $\dim \mathbb{O}_{min} = 2n$. We have the following description of $[\overline{Pr}_k^{\mathbb{O}_{min}}]$:

Proposition 6.1. [AFR17, Proposition 2.10] *Then*

$$[\overline{Pr}_k^{\mathbb{O}_{min}}] = \bigsqcup_{a=1}^{q-1} \{[\bar{\lambda} - \frac{ap}{q} \varpi_1] \mid \lambda \in \widehat{P}_+^{p-n-1}\},$$

where \widehat{P}_+^{p-n-1} is the set of level $p - n - 1$ integral dominant weights of $\widehat{\mathfrak{g}}$ and ϖ_1 is the first fundamental weight.

To describe simple admissible \mathfrak{g} -modules in the minimal orbit \mathbb{O}_{min} of level k we need to find those simple \mathfrak{g} -modules V for which $\text{Ann}_{U(\mathfrak{g})}V = J_\lambda$, for each $\lambda \in [\overline{Pr}_k^{\mathbb{O}_{min}}]$. We start with the highest weight modules.

6.2. Explicit realization of highest weight modules. Let

$$k + n = p/q - 1, \quad p > n, \quad q \geq 1 \text{ and } (p, q) = 1.$$

By Proposition 6.1, an element of $[\overline{Pr}_k^{\mathbb{O}_{min}}]$ has the form

$$\lambda' = \bar{\lambda} - \frac{ap}{q}\varpi_1 = \left(\lambda_1 - \frac{ap}{q}, \lambda_2, \lambda_3, \dots, \lambda_{n-1}, \lambda_n \right),$$

where $\lambda_i \in \mathbb{Z}_{\geq 0}$, for all $i = 1, \dots, n$ are such that $\lambda_1 + \dots + \lambda_n < p - n$ and $a \in \{1, 2, \dots, q - 1\}$.

Example 6.2. Let $\mathfrak{g} = \mathfrak{sl}_3$. As $\Lambda_1 = (\lambda_1 - \frac{ap}{q}, \lambda_2)$ is regular dominant, by Corollary 2.3, the simple admissible highest weight modules in the minimal orbit are $L(\Lambda_i)$, $i = 1, 2, 3$, where:

- $\Lambda_2 = s_1 \cdot \Lambda_1 = (\frac{ap}{q} - \lambda_1 - 2, \lambda_1 + \lambda_2 - \frac{ap}{q} + 1)$;
- $\Lambda_3 = s_2 s_1 \cdot \Lambda_1 = (\lambda_2, \frac{ap}{q} - \lambda_1 - \lambda_2 - 3)$.

These modules have weight multiplicities bounded by $\lambda_2 + 1$.

Applying Corollary 4.12, b), we have

Theorem 6.3. *For λ' as above, the simple highest weight module $L(\lambda')$ is a bounded Γ -relation Gelfand–Tsetlin \mathfrak{g} -module. Moreover, all simple admissible highest weight modules in the minimal nilpotent orbit are bounded Γ -relation Gelfand–Tsetlin modules.*

Example 6.4. Let $\mathfrak{g} = \mathfrak{sl}_4$ and $\Lambda_1 = (\lambda_1 - \frac{ap}{q}, \lambda_2, \lambda_3)$. Then the simple admissible highest weight modules in the minimal orbit are $L(\Lambda_i)$, $i = 1, 2, 3, 4$, where:

- $\Lambda_2 = s_1 \cdot \Lambda_1 = (\frac{ap}{q} - \lambda_1 - 2, \lambda_1 - \frac{ap}{q} + \lambda_2 + 1, \lambda_3)$;
- $\Lambda_3 = s_2 s_1 \cdot \Lambda_1 = (\lambda_2, \frac{ap}{q} - \lambda_1 - \lambda_2 - 3, \lambda_1 + \lambda_2 + \lambda_3 - \frac{ap}{q} + 2)$;
- $\Lambda_4 = s_3 s_2 s_1 \cdot \Lambda_1 = (\lambda_2, \lambda_3, \frac{ap}{q} - \lambda_1 - \lambda_2 - \lambda_3 - 4)$.

These modules are bounded, e.g. the weight multiplicities of $L(\Lambda_1)$ are bounded by $\frac{1}{2}(\lambda_2 + 1)(\lambda_3 + 1)(\lambda_2 + \lambda_3 + 2)$.

Let $F := \{E_{m_i 1} \mid i = 1, \dots, k\}$ such that $m_i \in \{2, \dots, n + 1\}$ for each $i = 1, \dots, k$. From Corollary 3.5 and Proposition 5.12 we immediately obtain

Corollary 6.5. *Let $n \geq 2$. Then all simple subquotients of $D_F L(\lambda')$ are admissible bounded Γ -relation Gelfand–Tsetlin modules of level k in the minimal orbit.*

Remark 6.6. (i) Corollary 6.5 for $\mathfrak{sl}(3)$ was shown in [AFR17, Theorem 5.6].

(ii) All simple modules in Corollary 6.5 are highest weight modules (with respect to some Borel subalgebra) with bounded weight multiplicities.

We also have from Corollaries 4.16 and 4.17:

Corollary 6.7. *Let $\lambda \in \mathfrak{h}^*$.*

- (i) *Let $L(\lambda)$ be an admissible highest weight module in the minimal orbit, $\langle \lambda + \rho, \beta_j^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for some simple roots β_j , $j = 1, \dots, t$. Let $w \in W$ be such that $\langle w(\lambda + \rho), \beta_j^\vee \rangle \in \mathbb{Z}_{< 0}$ for all $j = 1, \dots, t$. Then $L(\lambda) \simeq L_{w\mathfrak{b}_{st}}(w\lambda)$ is a $\Gamma_{\mathcal{F}}$ -relation Gelfand–Tsetlin module, where $\Gamma_{\mathcal{F}} = w\Gamma$.*
- (ii) *Suppose that $L(\lambda)$ is admissible highest weight module in the minimal orbit and $\langle \lambda, \beta^\vee \rangle \notin \mathbb{Z}_{\geq 0}$ for some simple root β . Then $L_{s_\beta \mathfrak{b}_{st}}(\lambda + \beta)$ is an admissible $s_\beta \Gamma$ -relation Gelfand–Tsetlin module in the minimal orbit.*

6.3. Classification of \mathfrak{sl}_2 -induced modules. Let $k = p/q - n - 1$ be an admissible number for $\widehat{\mathfrak{g}}$. In this section we complete the classification of all simple admissible \mathfrak{g} -modules in the minimal orbit, which are the quotients of modules induced from parabolic subalgebras with the Levi factor isomorphic to $\mathfrak{sl}_2 + \mathfrak{h}$. We will also show that all such modules are relation modules and, hence, obtain their explicit tableaux realization.

Suppose M is an admissible \mathfrak{g} -module of level k . Consider a parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_\Sigma = \mathfrak{l}_\Sigma + \mathfrak{n}_\Sigma$ of \mathfrak{g} , where Σ consists of one simple root β . Denote by M^n the subspace of all \mathfrak{n}_Σ -invariants, which is an \mathfrak{l}_Σ -module. Suppose $M = L_{\mathfrak{p}}^{\mathfrak{g}}(\Sigma, N)$ for some simple weight \mathfrak{l}_Σ -module N with $\mathfrak{n}_\Sigma N = 0$. Then $M^n \simeq N$ is admissible \mathfrak{l}_Σ -module of level $k_\beta = \frac{2}{\langle \beta, \beta \rangle} (k + n + 1) - 2$ by [AFR17, Theorem 2.12].

The following is straightforward.

Proposition 6.8. *Let $\mathfrak{a} = \mathfrak{sl}_2$, $V = V(\gamma, \mu)$ a simple dense weight \mathfrak{a} -module, $\gamma, \mu \in \mathbb{C}$. Then V is admissible of level k in the minimal orbit if and only if $\mu = \lambda - \frac{ap}{q} + 2x$*

and $\gamma = \left(\lambda - \frac{ap}{q} + 1 \right)^2$, where $\lambda \in \{0, 1, \dots, p-2\}$, $a \in \{1, \dots, q-1\}$, $x \in \mathbb{C} \setminus \mathbb{Z}$ and $x - \frac{ap}{q} \notin \mathbb{Z}$.

Remark 6.9. In the proposition above we have $V = V_{\mathcal{C}}(T(v))$, where $\mathcal{C} = \emptyset$, $T(v) = T(v_{ij})$ is the Gelfand–Tsetlin tableau with height 2, such that $v_{11} = \frac{\mu}{2}$, $v_{21} = \frac{1}{2} \left(\lambda - \frac{ap}{q} \right)$ and $v_{22} = \frac{1}{2} \left(\frac{ap}{q} - \lambda - 2 \right)$ (cf. Theorem 5.6). The set $\mathcal{B}_{\mathcal{C}}(T(v)) = \{T(v + \ell \delta^{11}) \mid \ell \in \mathbb{Z}\}$ is a basis of V .

Applying Proposition 6.8, Theorem 2.8 and Theorem 5.6 we obtain the following statement.

Corollary 6.10. *Let $n > 1$, $\gamma \in \mathbb{C}$, $\mu = (\mu_1, \dots, \mu_n) \in \mathfrak{h}^*$, such that $V \simeq V(\gamma, \mu_1)$ is a simple dense weight \mathfrak{sl}_2 -module and $\mu_i = \langle \mu, \alpha_i^\vee \rangle$, $i = 1, \dots, n$. Let $L(\gamma, \mu) = L_{\mathfrak{p}}^{\mathfrak{g}}(\{\alpha\}, V)$. Then $L(\gamma, \mu)$ is admissible of level k in the minimal orbit if and only if $\mu_1 = \lambda_1 - \frac{ap}{q} + 2x$,*

$\mu_2 = \lambda_2 - x$, $\mu_j = \lambda_j$ for all $j = 3, \dots, n$ and $\gamma = \left(\lambda_1 - \frac{ap}{q} + 1 \right)^2$ with $\{\lambda_i\}_{i=1, \dots, n} \subset \mathbb{Z}_{\geq 0}$ such that $\lambda_1 + \dots + \lambda_n < p - n$, $a \in \{1, 2, \dots, q - 1\}$, $x \in \mathbb{C} \setminus \mathbb{Z}$ and $x - \frac{ap}{q} \notin \mathbb{Z}$.

Theorem 6.11. *Let β be a root of \mathfrak{g} , \mathfrak{b} a Borel subalgebra of \mathfrak{g} for which β is a positive root, ρ_β the half-sum of positive (with respect to \mathfrak{b}) roots. Let $L_{\mathfrak{b}}(\lambda)$ be an admissible simple \mathfrak{b} -highest weight \mathfrak{g} -module in the minimal orbit, such that $\langle \lambda + \rho_\beta, \beta^\vee \rangle \notin \mathbb{Z}$, and $f = f_\beta$. Denote by $A_{\mathfrak{b}, \beta}$ the set of all $x \in \mathbb{C} \setminus \mathbb{Z}$ such that $x + \langle \lambda + \rho_\beta, \beta^\vee \rangle \notin \mathbb{Z}$.*

- (i) *The \mathfrak{g} -module $D_{f_\beta}^x L_{\mathfrak{b}}(\lambda)$ is admissible in the minimal orbit for any $x \in A_{\mathfrak{b}, \beta}$;*
- (ii) *Modules $D_{f_\beta}^x L_{\mathfrak{b}}(\lambda)$, where $\mathfrak{g}_\beta \subset \mathfrak{b}$, $x \in A_{\mathfrak{b}, \beta}$, $\langle \lambda + \rho_\beta, \beta^\vee \rangle \notin \mathbb{Z}$ and $L_{\mathfrak{b}}(\lambda)$ is admissible in the minimal orbit, exhaust all simple \mathfrak{sl}_2 -induced admissible modules in the minimal orbit. All such modules have bounded weight multiplicities;*
- (iii) *There exists a flag \mathcal{F} such that $D_{f_\beta}^x L_{\mathfrak{b}}(\lambda)$ is $\Gamma_{\mathcal{F}}$ -relation Gelfand–Tsetlin \mathfrak{g} -module.*

Proof. We will assume that \mathfrak{b} is the standard Borel subalgebra of \mathfrak{g} . For arbitrary Borel subalgebra the argument is analogous. Let $\beta = \alpha_r + \dots + \alpha_t$ for some consecutive simple roots α_i , $i = r, \dots, t$. Since λ is admissible in the minimal orbit then we have two possibilities: either there exists only one simple root i such that $\langle \lambda + \rho, \alpha_i^\vee \rangle \notin \mathbb{Z}$, or there are only two such roots which are consecutive. Assume the first case. Then either $i = r = 1$ or $i = t = n - 1$. Without loss of generality we may assume that $i = 1$, and hence $\langle \lambda + \rho, \alpha_j^\vee \rangle \in \mathbb{Z}_{>0}$, $j = 2, \dots, t$. Take the following basis of the root system

$$\pi' = \{\beta, -\alpha_t, \dots, -\alpha_2, \alpha_2 + \dots + \alpha_{t+1}, \alpha_{t+2}, \dots, \alpha_{n-1}\}.$$

Let $w \in W$ be such that $w\pi = \pi'$. Then w satisfies Corollary 6.7, hence $L(\lambda) \simeq L_{w\mathfrak{b}_{st}}(w\lambda)$ is a $w\Gamma$ -relation Gelfand–Tsetlin module. Then the statement follows from Theorem 5.4.

Consider now the second case. Again, without loss of generality we may assume that $\lambda_i \notin \mathbb{Z}$ for $i = 1, 2$. If $\beta = \alpha_1$ the statement follows from Theorem 5.4. Let $\beta = \alpha_2 + \dots + \alpha_t$. Take minimal $w \in W$ such that $w\pi$ contains β and $-\alpha_3, \dots, -\alpha_t$ (such w clearly exists). Then $L(\lambda) \simeq L_{w\mathfrak{b}_{st}}(w\lambda)$. Hence, we can assume that β is a simple root, $\beta = \alpha_2$. We will show how to realize the localization of $L(\lambda)$ with respect to $-\alpha_2$. Let s be the reflection in root α_2 . By Corollary 6.7, (ii), the module $L_{s\mathfrak{b}_{st}}(\lambda + \alpha_2)$ is an admissible $s\Gamma$ -relation Gelfand–Tsetlin module in the minimal orbit. Consider the following basis of the root system

$$\pi' = \{-\alpha_2, \dots, -\alpha_{n-1}, \alpha_1 + \dots + \alpha_{n-1}\}.$$

Let $w \in W$ be such that $ws\pi = \pi'$. Then $L_{s\mathfrak{b}_{st}}(\lambda + \alpha_2) \simeq L_{ws\mathfrak{b}_{st}}(w(\lambda + \alpha_2))$ is an admissible $ws\Gamma$ -relation Gelfand–Tsetlin module in the minimal orbit. Moreover, by Theorem 5.4, $D_e^x L_{s\mathfrak{b}_{st}}(\lambda + \alpha_2)$ is admissible $ws\Gamma$ -relation Gelfand–Tsetlin \mathfrak{g} -module in the minimal orbit for any $x \in \mathbb{C}$ and $e = e_{\alpha_2}$. Since $D_e^x L_{s\mathfrak{b}_{st}}(\lambda + \alpha_2)$ and $D_f^x L(\lambda)$ are isomorphic if $x \in \mathbb{C} \setminus \mathbb{Z}$ by Theorem 5.6. Hence, statements (i) and (ii) follow. Now, Theorem 5.6 and statement (i) imply (iii). \square

We note that Theorem 6.11 was initially proved for $\mathfrak{g} = \mathfrak{sl}(3)$ in [AFR17, Theorem 5.6].

Let β be a root of \mathfrak{g} and \mathfrak{b} be a Borel subalgebra of \mathfrak{g} for which β is a first simple root (such \mathfrak{b} always exists either by the conjugation by the Weyl group or by the symmetry of Dynkin diagram). Let $\mathfrak{p} = \mathfrak{a}_\beta \oplus \mathfrak{n}$ be a parabolic subalgebra of \mathfrak{g} containing \mathfrak{b} with the Levi

factor $\mathfrak{a}_\beta \simeq \mathfrak{sl}_2 + \mathfrak{h}$ based on the root β . Let $V = V(\gamma_\beta, \mu)$, $\gamma_\beta \in \mathbb{C}$, $\mu = \mu_1\varpi_1 + \dots + \mu_n\varpi_n \in \mathfrak{h}^*$, where γ_β is the eigenvalue of the Casimir element of \mathfrak{a}_β , $\langle \text{varpi}_i, \beta_j^\vee \rangle = \delta_{ij}$, $i = 1, \dots, n$.

Corollary 6.12. *The module $L_{\mathfrak{b},\beta}(\gamma_\beta, \mu) := L_{\mathfrak{p}}^{\mathfrak{g}}(V)$ is admissible in the minimal orbit if and only if $\langle \mu - \lambda, \beta_1^\vee \rangle = 2x$, $\langle \mu - \lambda, \beta_2^\vee \rangle = -x$, $\langle \mu - \lambda, \beta_i^\vee \rangle = 0$ for each $i = 3, \dots, n$, and $\gamma_\beta = \langle \lambda + \rho_{\mathfrak{b}}, \beta_1^\vee \rangle^2$ with $\langle \lambda + \frac{ap}{q}\varpi_1 + \rho_{\mathfrak{b}}, \beta_1^\vee \rangle \in \mathbb{Z}_{>0}$, $\langle \lambda + \rho_{\mathfrak{b}}, \beta_i^\vee \rangle \in \mathbb{Z}_{>0}$ for all $i \in \{2, \dots, n\}$, $\langle \lambda + \frac{ap}{q}\varpi_1 + \rho_{\mathfrak{b}}, \beta_{1,n}^\vee \rangle < p$, $a \in \{1, 2, \dots, q-1\}$, $x \in \mathbb{C} \setminus \mathbb{Z}$ and $x + \langle \lambda + \rho_{\mathfrak{b}}, \beta_1^\vee \rangle \notin \mathbb{Z}$.*

Remark 6.13. Theorem 6.11 provides an algorithm how to list all simple \mathfrak{sl}_2 -induced admissible modules in the minimal orbit:

- Consider all possible Borel subalgebras of \mathfrak{g} containing \mathfrak{h} ;
- For each Borel subalgebra \mathfrak{b} describe $\lambda \in \mathfrak{h}^*$ for which $L_{\mathfrak{b}}(\lambda)$ is admissible using Arakawa's classification for the standard Borel and applying the Weyl group;
- Choose any positive (with respect to \mathfrak{b}) root β such that $\langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{Z}$;
- Define $D_f^x L_{\mathfrak{b}}(\lambda)$ for any $x \in \mathbb{C} \setminus \mathbb{Z}$ such that $x + \langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{Z}$, where $f = f_\beta$.

Obtained modules exhaust all simple \mathfrak{sl}_2 -induced admissible modules in the minimal orbit. Moreover, the proof of Theorem 6.11 explains how to define the flag \mathcal{F} for which $L_{\mathfrak{b},\beta}(N)$ is a $\Gamma_{\mathcal{F}}$ -relation Gelfand–Tsetlin \mathfrak{g} -module, and hence to obtain explicit tableaux realization for all such modules.

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